Persistent Homoclinic Orbits
for a Perturbed Nonlinear Schrödinger Equation

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Abstract

The persistence of homoclinic orbits for certain perturbations of the integrable nonlinear Schrödinger equation under even periodic boundary conditions is established. More specifically, the existence of a symmetric pair of homoclinic orbits is established for the perturbed NLS equation through an argument that combines Melnikov analysis with a geometric singular perturbation theory for the PDE. © 1996 John Wiley & Sons, Inc.

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1 Introduction and Summary of Results

In recent years there has been a considerable amount of work done on proving the existence of chaotic behavior in deterministic dynamical systems. In finite-dimensional systems there are well-developed techniques to show the existence of chaos (e.g., see [25, 53, 61]); in particular, the Melnikov method has proven to be very useful. This method establishes the transversal intersection of the stable and unstable manifolds of a saddle type invariant set, which is an important step towards establishing the existence of an invariant set for which the dynamics is topologically equivalent to the Bernoulli shift.

In infinite dimensions some of these techniques were successfully used to establish chaotic behavior [31]. More specifically, for perturbation of the integrable Hamiltonian system the Melnikov function was used in [31] to establish the existence of the transversal intersection of the stable and unstable manifolds when one of them is one-dimensional. When the perturbation introduces new, slow saddle directions (i.e., when the problem is singular), none of these methods are sufficient to establish the transversal intersection of these manifolds.

In this paper we establish the existence of a nontrivial symmetric pair of homoclinic orbits for a perturbed nonlinear Schrödinger equation (PNLS)

\[
\begin{align*}
  i q_t &= q_{xx} + 2 \left[ q q - \omega^2 \right] q + i \epsilon \left[ \hat{D} q - \Gamma \right], \\
  \hat{D} &\text{ is a bounded dissipative operator and } \epsilon > 0.
\end{align*}
\]

These results were announced at the ICM in Zürich [50]. Our methods are based on infinite-dimensional versions of invariant manifolds, geometric singular perturbation theory, near-identity normal forms transformation, as well as Melnikov analysis. In addition to providing the existence of nontrivial homoclinic orbits for PDEs, this work establishes a dynamical systems framework for the study of perturbations of conservative PDEs, especially when the temporal behavior is singular.

An outline of the historical development and the background leading to this result is given in Section 2, while the remainder of this introduction describes in detail the methods and results of this manuscript.

In Section 3 we study solutions of equation (1.1) that are independent of \(x\). The "plane of constants" \(\Pi_\omega\) is an invariant plane for this equation on which the orbits of the perturbed and unperturbed (\(\epsilon = 0\)) equations behave very differently. The situation can be analyzed with phase-plane methods. The features (discussed in Section 3) are the following:
When $\epsilon = 0$, $S_\omega$, the circle of radius $\omega$, is a circle of fixed points. For $\epsilon > 0$, only two of these fixed points persist, a sink $P$ and a saddle $Q$.

At $S_\omega$, expansion and contraction rates on the plane are of order $O(\nu = \sqrt{\epsilon})$.

Global representations (in the plane $\Pi_c$) of the stable manifold of the saddle $Q$, $C^s_\epsilon(Q)$, are developed in terms of polar coordinates, $q = \sqrt{1} \exp i\theta$

$$I = \omega^2 + J = \omega^2 + \nu j;$$

the dynamics in a neighborhood of $C^s_\epsilon$ is represented in terms of $(r, \beta)$, where $r$ is a measure of normal distance off $C^s_\epsilon$ and $\beta$ is a measure of length on $C^s_\epsilon$.

At this point one sees the need for singular perturbation theory. First, when one studies the linear stability of the circle of fixed points, $S_\omega$, one recovers the small $O(\nu = \sqrt{\epsilon})$ growth rates on $\Pi_c$ and, in addition, a fast $O(\epsilon^0)$ growth rate off the plane. Thus one sees the existence of two distinct time scales. Moreover, as will be described in detail in the text, one sees a potential obstruction to the persistence of homoclinic orbits: The integrable orbits that are homoclinic to the circle $S_\omega$ are actually heteroclinic to pairs of fixed points on $S_\omega$. Thus, any deformation of an integrable homoclinic orbit that emerges from the saddle $Q$ experiences a phase shift upon returning near the plane $\Pi_c$; hence, typically the orbit does not "land" near a perturbed fixed point. Some kind of return along the plane is required (see Figure 1.1). Because of the small contraction rate on the plane, this return must be on the slow time scale, and the need for singular perturbation methods is apparent.

In Section 4 we set up the equations in a full function space neighborhood of the circle $S_\omega$. First, we introduce coordinates $(J, \theta, f)$ given by

$$q := [\rho(t) + f(x, t)] \exp i\theta(t)$$

$$\rho = \sqrt{J + \omega^2 - \langle f \hat{f} \rangle},$$

where $\rho$ and $\theta$ are polar coordinates on the plane $\Pi_c$ and has $f$ spatial mean zero. The main mathematical result in Section 4.2 is Proposition 4.1, which establishes the existence of a transformation to normal form coordinates,

$$g = f + \mathcal{K}(f, f),$$

where $\mathcal{K}(\cdot, \cdot)$ denotes a bounded bilinear map on mean zero functions that is chosen to eliminate the leading order quadratic nonlinearities from the $f$
equation. This normal form transformation for the PDE is one of the key features of this work. It is used in Section 5 to provide detailed information about the stable manifold $W^s(Q)$, information that is needed in the second measurement of Section 7.

The need for the normal form transformation can be illustrated by the following simple example: The length of the stable manifold of $x = 0$ for the equation $\dot{x} = -\epsilon x + |x|x$ is $2\epsilon$; on the other hand, for the equation $\dot{x} = -\epsilon x + x^3$ the length of the stable manifold is $2\sqrt{\epsilon}$. For a certain estimate that we shall call the “second measurement” to be valid, we need the stable manifold of $Q$ to be of order bigger than $O(\epsilon)$. This size restriction makes the normal form transformation necessary. Section 4 concludes with the equations for the complex coordinates $(J, \theta, g)$ and for the real coordinates $(J, \theta, u)$, where $u \in \mathbb{R}^2$ and $g = u_1 + iu_2$.

In Section 5 we introduce a cutoff function that serves to localize the dynamics about $S_w$. We establish the existence of several invariant manifolds in a neighborhood of $S_w$, as well as a “fiber representation” of these manifolds that is useful for singular perturbation calculations. Because of the dichotomy of expansion and contraction rates, a codimension 2 “slow manifold” $M_\epsilon$ exists as a deformation of a codimension 2 center manifold. Again because this of dichotomy, the manifold $M_\epsilon$ is normally hyperbolic, with persistent codimension 1 center-stable and center-unstable manifolds $W_{cs}^\epsilon$ and $W_{cu}^\epsilon$, respectively. These existence results are described in Theorem 5.2.

Although the invariant manifolds $M_\epsilon$, $W_{cs}^\epsilon$, and $W_{cu}^\epsilon$ are differentiable in $\epsilon$, their representations as unions over orbits mask this differentiability due to the dynamics on the slow time scale $\sqrt{\epsilon} t$. In a finite-dimensional setting, Fenichel introduced a representation of these manifolds in terms of “fibers” that are differentiable several times in $\epsilon$ and therefore are very useful for singular perturbation calculations [18, 19, 20, 21]. In Section 5.2 we develop the fibration for the PDE (1.1). The result is described in Theorem 5.4. In contrast to the geometric graph transform approach used in the finite-dimensional literature, our proofs use an analytic approach which sets up integral equations that define a transformation of coordinates to a normal form that nearly decouples the flow into slow and fast variables. The fibers are then defined in terms of these coordinates, which we call “fiber coordinates.” Section 5 concludes with the use of the normal form of the PNLS equation obtained in Section 4.2 to yield detailed information about the stable manifold of the saddle point $Q$, $W^s(Q)$. This information is needed in the second measurement, which is described below.

In Section 6 we present a (nearly) self-contained summary of the integrable theory of NLS that is used in this paper. More detailed information may
be found in [44, 51, 43]. Specifically, we describe the Lax pair [63], the Floquet spectral theory of the Zakharov-Shabat differential operator, formulas for the homoclinic orbits and whiskered tori (Theorem 6.3), and an extremely important constant of the motion $F$. The Melnikov function will be constructed from this constant of motion; hence, explicit formulas for the grad $F$ evaluated on the homoclinic orbit are given by equation (6.20).

Actually, for this work very little integrable theory is needed. All one needs is an expression for the homoclinic orbit and a constant of motion that is sufficiently general to permit construction of a Melnikov function. However, we set up the construction in a manner natural for generalization to more interesting whiskered tori. Integrable theory provides representations of very general homoclinic orbits (Theorem 6.3). In addition, the constant of motion $F$ is the natural one to use in constructing the Melnikov function because this functional is critical on the singular tori [44]; moreover, the Hessian of this functional detects the saddle structure of the critical tori. This criticality of $F$ causes the integrand of the Melnikov function to vanish exponentially as $t \to \infty$ and ensures convergence of the integrals even when homoclinic to a full collection of critical tori—a property that most constants of motion do not possess.

In Section 7 we prove the existence of a symmetric pair of persistent homoclinic orbits, provided the external parameters are constrained to lie on a codimension 1 set in parameter space (Theorem 7.5). Each of these orbits has either a center or edge spatial structure, and both are asymptotic as $t \to \pm \infty$ to the saddle $Q$. The proof proceeds in two steps, which we refer to as the first and second measurements. Each measurement is set up to involve an orbit that hits or misses a codimension 1 manifold. Hence, one-dimensional shooting arguments are applicable.

The unstable manifold $W^u(Q)$ is two-dimensional, with a slow direction in the plane of constants $\Pi_c$ and a fast escape that is tangent to a $\cos x$ direction in the function space. The first measurement begins by considering an orbit in $W^u(Q)$ that leaves $Q$ and slowly creeps over the curve in the plane $W^u(Q) \cap \Pi_c$ until it finally takes off away from the plane. With the fibration, the takeoff point can be parameterized by an angle $\theta_b$ on the circle $S_\omega$, which provides a one-parameter labeling of the takeoff orbits. The first measurement selects a discrete set of takeoff points $\theta_b$ for which the orbit asymptotically returns to the slow manifold $M_\epsilon$ in forward time.

The second measurement is more delicate. While the first measurement ensures that an orbit exists that starts on $W^u(Q)$ (but off $\Pi_c$) and is asymptotic in forward time to the slow manifold $M_\epsilon$, it provides no information about the fate of that orbit on $M_\epsilon$. That is the task of the second measurement,
which is designed to determine if the orbit actually returns asymptotically to
the saddle $Q$. In the four-dimensional work it was observed that this return
to $Q$ does not happen without some extra freedom in parameter space [52]. It
seems natural to introduce a two-parameter form of dissipation, damping and
diffusion, where the dissipative operator $\hat{D}$ is assumed to take the form

\begin{equation}
\hat{D}q = -\alpha q + \beta q_{xx}
\end{equation}

for positive constants $\alpha$ and $\beta$. However, we need the flow to be defined for
all time; for this reason we smooth the dissipation at short wavelengths:

\begin{equation}
\hat{D}q = -\alpha q - \beta \hat{B}q,
\end{equation}

where $\hat{B}$ has symbol

\begin{equation}
b(k) = \begin{cases} 
k^2 & k < \kappa 
0 & k \geq \kappa.
\end{cases}
\end{equation}

The second measurement is then carried out for this particular type of
bounded damping. Inside $\mathcal{M}_e$, the stable manifold $W^s(Q)$ is codimension
1, which permits a shooting argument. Moreover, the normal form for the
PDE of Section 4.2 enables us to establish in Section 5.3 that $W^s(Q)$ is tall
enough to allow a successful shooting argument to be carried out. In this
manner Theorem 7.5 is established, and we have the existence of a symmetric
pair of homoclinic orbits to the saddle $Q$ that are singular perturbations of the
integrable whiskers.
Although existence of an orbit homoclinic to $Q$ requires an additional dissipation term (such as the $\beta$ term), very irregular temporal behavior has been observed numerically with only simple damping [51]. Some other mechanism must be responsible for this chaotic behavior. For example, one can use the first measurement only to construct a connection from the saddle $Q$ to the sink $P$. In finite dimensions Haller has shown, with a somewhat different measurement technique, that there exist connections with complicated (but finite in number) center-edge symbols [29]. Although these long transients are not chaotic, they indicate the complicated dynamics that is supported by the system. Currently Haller [28] is extending his construction to the level of the PDE, using the local invariant manifold and fiber construction as a starting point.

2 Background

The proof of existence of homoclinic orbits for perturbed integrable PDEs is a natural and important step in a mathematical program that some of the authors have developed and worked on for several years. The historical development that led to this work is very extensive. Here we give a short description.

2.1 Numerical Experiments: Chaotic Behavior for PNLS

When PNLS (1.1) is studied numerically, one finds solutions that, at large time $t \gg 1$, consist of very regular spatial patterns that oscillate irregularly, maybe even chaotically, in time $t$ [51]. These numerical experiments proceed by fixing the form of the dissipation operator $\hat{D}$ and varying $\Gamma$, the amplitude of the external driving force. As this “bifurcation parameter” $\Gamma$ is increased, the long-time behavior of solutions changes from one with regular patterns in both space and time to regular spatial patterns that evolve chaotically in time. Typical spatial–temporal profiles of the states are depicted in Figures 2.1 and 2.2. For example, the solution can possess one exponentially localized “solitary wave” in each basic period $x \in [0, 2\pi)$ whose temporal oscillations appear to be chaotic (as measured by a broad-band power spectrum, positive Lyapunov exponents, “spattered” Poincaré sections, fractional Lyapunov dimensions, etc.). These numerical experiments are described in detail in [51]. Here we emphasize only two points:

1. While the chaotic behavior seems to require sufficiently large $\Gamma$, $\epsilon \Gamma$ is still small, and the system may still be considered as a perturbation of the integrable equation.
2. Under even symmetry, the single solitary wave can reside only at two locations—centered at $x = 0$ or at $x = \pi$. One type of chaos is accompanied by the solitary wave jumping irregularly between the $x = 0$ and $x = \pi$ locations. (See Figure 2.2.)

2.2 Homoclinic Orbits and Whiskered Tori for the NLS

Since equation (1.1) is a perturbation of the integrable, focusing, nonlinear Schrödinger equation (NLS), a natural question to consider is: Which integrable instabilities provide the sources of sensitivity required for chaotic behavior? The simultaneous level sets of the infinite family of commuting NLS constants of motion are generically infinite-dimensional tori $T^\infty = S \times S \times S \times \cdots$. Singular tori arise when one or more of these circles are pinched off and their associated degrees of freedom (angle variables) are no longer free to oscillate. These singular tori can be unstable. Inverse spectral theory [44, 51, 16] of soliton mathematics can be used to represent these singular tori, to determine their instabilities, and to represent their stable and unstable manifolds as unions of homoclinic orbits. These orbits are homoclinic to unstable singular tori. The unstable singular tori are called “whiskered tori” [1] with the homoclinic orbits constituting the whiskers. Solutions residing on whiskered tori can have quasi-periodic temporal behavior and periodic spatial behavior. Such solutions provide many rich examples of nontrivial homoclinic orbits.

The simplest example begins with the temporally periodic solution of NLS that is independent of $x$:

\[
q = c \exp \left[ -i \left( 2(e^2 - \omega^2)t - \gamma \right) \right] = ce^{i\theta}
\]

where $c$ and $\gamma$ denote real constants. This solution resides on a torus for which all but one of the circles have been pinched off; that is, the singular torus is a circle $S$. A simple linearized stability calculation shows that, for $c \approx \omega$, this circle is unstable—with one unstable mode, one stable mode, and the remaining infinite number of center modes. Using Bäcklund (or Darboux) transformations from soliton mathematics, one can “exponentiate” the linearized unstable mode to obtain an exact solution of NLS,

\[
q_h(t) = \left\{ \cos 2p - i \sin 2p \tanh \tau \pm \sin p \sech \tau \cos x \over 1 \mp \sin p \sech \tau \cos x \right\} q ,
\]

where $\tan p = \sqrt{4e^2 - 1}$ and $\tau = \tan p(t + t_o)$ [16, 44]. This solution, $q_h$, is homoclinic to the circle $S$ and provides an explicit representation of its whiskers. It is the solution that we will use in studying the perturbed system.
2.3 The Connection of Chaotic Behavior in PNLS to Integrable Whiskered Tori

With the aid of the spectral transform, one can understand the neighborhood of these whiskered tori. First, consider a four-dimensional invariant manifold obtained by pinching off all but two circles. This manifold is defined by setting the radii of the $j^{th}$ circle $r_j = 0, j \neq 0, 1$. The four-dimensional submanifold that remains is stratified by the whiskered circle, together with nested families of two-tori.

Topologically, this stratification by two-tori is equivalent to the “trousers” [22, 16, 44] depicted in Figure 2.3. Note that the saddle point $\times S$ depicts the whiskered circle, with the figure eight depicting the symmetric pair of whiskers. The remainder of the diagram consists of three pieces: two “legs” $\ell_1$ and $\ell_2$ for levels above critical and a “torso” $T$ for levels below critical. Each of these pieces is composed of a family of circles (indexed by their location on the diagram) $\times S$. In this manner, the diagram depicts a nested family of two-tori together with the unstable manifold of one critical circle—a whiskered circle. The full infinite-dimensional neighborhood of this whiskered
circle can be constructed by opening each radius $0 \leq r_j \leq \delta_j$, $\forall j \neq 0, 1$, that is, by opening a countable number of center directions.

These integrable trouser diagrams provide an intuitive description of one origin of chaotic behavior when NLS is perturbed by damping and driving. For this intuition, first realize that $\ell_1$ and $\ell_2$ represent a distinct class of spatial excitations. Either through explicit analytical formulas or through numerical representations, one learns [49, 44] that $\ell_1$ and $\ell_2$ represent oscillations centered about $x = 0$ and about $x = \pi$, respectively, while $T$ represents standing waves that consist of oscillations between $x = 0$ and $x = \pi$. In the integrable case, initial conditions fix one circle (say, in $\ell_1$), and the motion in time oscillates around that fixed circle forever. Under a very small damped-driven perturbation, one can imagine motion which, as it oscillates many times around $\ell_1$, slowly creeps up $\ell_1$ until it crosses the figure eight onto $T$. After many oscillations around $T$ (during which it loses “memory” about which piece it originated from), the dissipation causes it to drop back across the figure eight onto either $\ell_1$ or $\ell_2$, after which the process repeats aperiodically. The instability associated with the whiskered circle would provide the sensitivity to data needed to produce chaotic dynamics; moreover, the correlation of $\ell_1$ with the center $x = 0$ and of $\ell_2$ with the edge $x = \pi$ spatial excitation provides natural candidates for a symbol dynamics based upon two symbols.

But is this intuitive picture really relevant to chaotic behavior in PNLS? First, we emphasize that direct numerical measurements with the spectral transform establish that the chaotic time series for the PNLS resides in a small function space neighborhood of the whiskered circle, a neighborhood in which all but two of the radii are small ($r_j \ll 1; r_j \neq 0, 1$) [8, 51]. This fact is established through a direct measurement of the action variables associated with
the radii \( r_j \), using a numerical realization of the spectral transform ([8, 51]). Moreover, other numerical simulations of a very-low-dimensional model system have produced a movie that confirms this type of aperiodic oscillatory behavior in the perturbed system, as well as its relationship to an integrable trouser diagram [37].

In the theory of finite-dimensional dynamical systems, one standard method for converting such an intuitive picture of chaotic dynamics into a rigorous mathematical description is to construct a Smale horseshoe in a neighborhood of a symmetric pair of homoclinic orbits [58, 53, 25, 61], and then to use this horseshoe to establish the existence of an invariant set on which the motion is topologically equivalent to a Bernoulli shift on two symbols. Such constructions begin with the existence, persistence, and breaking of homoclinic orbits, which in turn are obtained through Melnikov arguments [32, 33, 61].

2.4 Homoclinic Orbits

The analysis, for existence of homoclinic orbits in this setting, was first carried out for finite-dimensional versions of the original equation—first, a four-dimensional Fourier truncation [38, 42, 52, 41, 29] and then a finite \((2N + 2)\)-dimensional finite difference discretization of PNLS [43, 45, 47]. In both of these finite-dimensional cases, the analysis proceeded in five steps:

1. Numerical studies were performed in which chaotic behavior was seen, monitored, and analyzed.

2. Invariant manifolds were constructed.

3. Fiber representations of these invariant manifolds were developed that were suitable for singular perturbation calculations.

4. The existence (persistence) of homoclinic orbits was established by using two measurements, one of which was a Melnikov measurement.

5. Horseshoes were constructed, as were the accompanying symbol dynamics.

To carry out the analysis for the four-dimensional model, the need for fibration from geometric perturbation theory became apparent. For the \((2N + 2)\)-dimensional model, additional techniques were required. Integrable soliton theory was used to construct the unperturbed homoclinic orbits, and estimates were required to control the additional center directions. These additional center directions introduced an oscillatory behavior into the homoclinic orbits
that is reminiscent of Shilnikov behavior and that significantly altered the construction of horseshoes and symbol dynamics.

The use of integrals for measuring the splitting of manifolds for perturbed Hamiltonian systems dates back to the work of Poincaré and Arnold. A systematic development of Melnikov methods for high-dimensional dynamical systems, not necessarily Hamiltonian, is described in [61]. The early work of Holmes and Marsden was influential in our initial study of this problem [32, 33]. Their work on the beam and sine-Gordon equations was related but very different [30, 31]. There the homoclinic structures are deformations of planar homoclinic orbits with one-dimensional unstable manifolds, and only a standard Melnikov function construction was used since the behavior was assumed to be nonsingular.

2.5 Invariant Manifolds and Fiber Coordinates

In dynamical systems theory there are several methods to prove the existence of invariant manifolds. Of these we distinguish between the geometric graph transform method of Hadamard [26] and the analytic method of Perron [54]. Recently, in works on finite-dimensional geometric singular perturbation theory [18, 19, 20, 21, 34, 62, 39, 42, 43, 40, 41, 52, 45], it has become standard to use versions of Hadamard's graph transform. To extend this work to the infinite-dimensional setting of a PDE, the use of compactness (which is so natural in finite dimensions) must be circumvented. This can be done [46, 5]. However, since we are centering the motion on the plane of constants $H_c$, it is natural for this work to use variation of parameters to write the partial differential equations in integral equation form and to adapt a version of Perron's method. Although perhaps less intuitive than a more geometric approach, this analytical method has the advantage of being very explicit.

There is a large body of literature concerning invariant manifolds in infinite dimensions and for PDEs. One example is "inertial manifolds" for strongly dissipative PDEs [60, 13], in particular for the complex Ginzburg-Landau equation [12, 14, 3]. (Another is work on the sine-Gordon equation that focuses upon attractors [23, 7].) However, we emphasize that inertial manifolds are for strongly dissipative systems, where the dynamics is governed by a finite-dimensional system on the finite-dimensional inertial manifold. Here, the dynamics is reversible or nearly so. As such, the natural objects are deformations of infinite-dimensional center manifolds. Literature closely related to our work on invariant manifolds includes [2, 4, 11, 10]).

For normally hyperbolic manifolds in finite dimensions, Fenichel introduced fibers in the context of geometric singular perturbation theory [18].
Detailed descriptions of the fibration in finite dimensions may be found in [34, 56, 62]. Simple examples of these fibers that help promote an intuitive understanding may be found in [51, 52].

In the finite-dimensional case, Jones and Kopell [35, 36, 34] realized the importance of fiber coordinates in understanding the Fenichel construction. They used the existence of fibers derived by Fenichel to define fiber coordinates that simplified the description of the flow close to an invariant manifold.

There is very little in the literature on this fibration for PDEs. For nonlinear diffusion equations, there is the work of [11, 10]. In his thesis, Li [43] began to study the PNLS equation with a graph transform approach. This work is completed in [46]. Recently Bates and Lu have constructed a fibration for PDEs [5].

3 Preliminary Results

In this section we present some elementary results on existence and uniqueness of solutions to equation (3.1). We also describe the perturbed flow for the $x$-independent solutions. The main points of this section are the growth rates given by equations (3.5) and equation (3.15), which describes the flow in a $\sqrt{\epsilon}$-neighborhood of the circle $S_\omega$.

3.1 Existence and Regularity of Solutions

Consider a perturbed nonlinear Schrödinger equation

\[ iq_t = q_{xx} + 2[q\bar{q} - \omega^2]q + i\epsilon [\hat{D}q - 1], \]

where the constant $\omega \in (\frac{1}{2}, 1)$, $\epsilon$ is a small positive constant, and $\hat{D}$ is a bounded dissipative linear operator on the Sobolev space $H^1_{e,p}$ of even, $2\pi$-periodic functions that are square integrable with square integrable first derivative. The PDE is well posed in $H^1_{e,p}$ as the following theorem states:

**Theorem 3.1 (Cauchy Problem)** For all $q_0 \in H^1_{e,p}$ and for all $t \in (-\infty, \infty)$, there exists a unique solution $q(t, q_0; \epsilon)$, continuous in $t$ with values in $H^1_{e,p}$ of equation (3.1) such that $q|_{t=t_0} = q_0$; moreover, $q(t, \cdot; \cdot)$ depends smoothly on $q_0$ and $\epsilon$.

The proof of this theorem is a standard application of the energy method; for example, see [59] and [9]. Thus the flow $F^t(q_0; \epsilon) = q(t, q_0; \epsilon)$ can be viewed as a smooth dynamical system on $H^1_{e,p}$. 
3.2 Analysis of Space-Independent Solutions

Motion on the Invariant Plane

The plane of constants $\Pi_c$,

$$\Pi_c := \{ q(x, t) : \partial_x q(x, t) \equiv 0 \},$$

is an invariant plane for equation (3.1), and on $\Pi_c$ the equation takes the form

$$i q_t = 2[qq - \omega^2]q - it[\alpha q + 1].$$

where it is assumed that the dissipation operator $\hat{D}$ acts invariantly on $\Pi_c$ as

$$\hat{D}q = -\alpha q$$

for $\alpha$ a positive constant. Equivalently, in terms of polar coordinates,

$$q := \sqrt{I} \exp i\theta,$$

these equations take the form

$$I_t = -2t[\alpha I + \sqrt{I} \cos \theta]$$

(3.3)

$$\theta_t = -2(I - \omega^2) + \frac{\epsilon}{\sqrt{I}} \sin \theta.$$  

When $\epsilon = 0$, the unperturbed orbits on $\Pi_c$ are nested circles with $S_\omega$ a circle of fixed points given by $I = \omega^2$. For positive $\epsilon$, the perturbed orbits on $\Pi_c$ are very different (see Figure 3.1). First, only three fixed points exist: $O$, which is a deformation of the origin; $Q$, a saddle that deforms from the circle

![Figure 3.1. Phase-plane diagram of the ODE.](image)
$S_\omega$; and $P$, a spiral sink that also deforms from the circle $S_\omega$. Formulas for these fixed points, together with their associated growth rates, are given by

\[
I_\omega = \epsilon^2 \left[ \frac{1}{4\omega^4} \right] + O(\epsilon^4)
\]
\[
\theta_\omega = -\frac{\pi}{2} - \left( \frac{\alpha}{2\omega^2} + O(\epsilon^2) \right)
\]
\[
I_p = \omega^2 + \frac{\epsilon}{2\omega} \sqrt{(1 - \alpha^2\omega^2)} + O(\epsilon^2)
\]
\[
\theta_p = -\tan^{-1} \left( \frac{\sqrt{1 - \alpha^2\omega^2}}{\alpha \omega} \right) - \pi + O(\epsilon)
\]
\[
I_q = \omega^2 - \frac{\epsilon}{2\omega} \sqrt{(1 - \alpha^2\omega^2)} + O(\epsilon^2)
\]
\[
\theta_q = \tan^{-1} \left( \frac{\sqrt{1 - \alpha^2\omega^2}}{\alpha \omega} \right) - \pi + O(\epsilon)
\]

\[
\sigma_\omega = \pm i - \epsilon \alpha + O(\epsilon^2)
\]
\[
\sigma_p = \pm 2i \sqrt{\epsilon \omega} \left[ 1 - \alpha^2 \omega^2 \right]^{\frac{1}{4}} - \epsilon \alpha + O(\epsilon^\frac{3}{2})
\]
\[
\sigma_q = \pm 2 \sqrt{\epsilon \omega} \left[ 1 - \alpha^2 \omega^2 \right]^{\frac{1}{4}} - \epsilon \alpha + O(\epsilon^\frac{3}{2}).
\]

Although the circle of fixed points $S_\omega$ for the unperturbed ($\epsilon = 0$) problem does not persist as a circle of fixed points, motion near $S_\omega$ remains slow for small positive $\epsilon$. By introducing the variable $J$

\[
J = I - \omega^2
\]
equation (3.3) can be written as

\[
J_t = -2\epsilon \left[ \alpha (J + \omega^2) + \sqrt{J + \omega^2 \cos \theta} \right]
\]
\[
\theta_t = -2J + \frac{\epsilon}{\sqrt{J + \omega^2}} \sin \theta.
\]

**The Stable Manifold at $Q$ in $\Pi_c$**

In order to describe the flow close to the stable manifold of the point $Q$, we rescale the coordinates

\[
\tau = \nu t
\]
\[
J = \nu j
\]
where $\nu = \sqrt{e}$. This rescaling is suggested by the expressions (3.5) for the growth rates $\sigma_p$ and $\sigma_q$.

In these scaled coordinates, the equations (3.3) on the plane $\Pi_c$ take the form

$$
\begin{align*}
  j_r &= -2 \left[ \alpha(\omega^2 + \nu j) + (\omega^2 + \nu j)^{1/2} \cos \theta \right] \\
  \theta_r &= -2j + \nu(\omega^2 + \nu j)^{-1/2} \sin \theta,
\end{align*}
$$

which can be written in terms of the variable $y := (j, \theta)^T$ as

$$
y_r = Y(y; \nu)
$$

where $Y = (Y_1, Y_2)^T$ is defined by equation (3.8). In terms of these variables, the point $Q$ has coordinates $y_q = (j_q, \theta_q)$ where

$$
\begin{align*}
  j_q &= -\nu \frac{(1 - \alpha^2 \omega^2)^{1/2}}{2\omega} + O(\nu^3) \\
  \theta_q &= \tan^{-1} \left( \frac{\sqrt{1 - \alpha^2 \omega^2}}{\alpha \omega} \right) - \pi + O(\nu^2)
\end{align*}
$$

Linearizing the equation around $y_q$ and writing $\tilde{y} = y - y_q$, we obtain

$$
\tilde{y}_r = Y'(y_q; \nu) \tilde{y} + O(\tilde{y}^2)
$$

where $Y'$ is the $2 \times 2$ matrix whose entries are

$$
\begin{bmatrix}
  -\nu (2\alpha + (\omega^2 + \nu j_q)^{-1/2} \cos \theta_q) & 2(\omega^2 + \nu j_q)^{1/2} \sin \theta_q \\
  -2 - \frac{1}{2} \epsilon (\omega^2 + \nu j_q)^{-3/2} \sin \theta_q & \nu(\omega^2 + \nu j)^{-1/2} \cos \theta_q
\end{bmatrix}.
$$

Thus we see that the eigenvalues of $Y'(y_q; \nu)$ are

$$
\begin{align*}
  \lambda &= -2\sqrt{\omega}(1 - \alpha^2 \omega^2)^{1/4} - \nu \alpha + O(\nu^2) \\
  \mu &= +2\sqrt{\omega}(1 - \alpha^2 \omega^2)^{1/4} - \nu \alpha + O(\nu^2)
\end{align*}
$$

with eigenvectors $\epsilon_1(\nu)$ and $\epsilon_2(\nu)$, respectively. The eigenvectors depend smoothly on $\nu$ and

$$
\begin{align*}
  \epsilon_1(0) &= \left( -\frac{\sigma}{2}, 1 \right)^T \\
  \epsilon_2(0) &= \left( \frac{\sigma}{2}, 1 \right)^T \\
  \sigma &= 2\sqrt{\omega}(1 - \alpha^2 \omega^2)^{1/4}
\end{align*}
$$
From regular perturbation theory and the stable manifold theorem we have, for $\epsilon$ sufficiently small, an open neighborhood $U$ of $Q$, independent of $\epsilon$, such that the stable manifold of $Q$ in $U$ is a smooth function of $\nu$ [27]. Therefore, a portion of the local stable manifold can be parametrized in terms of

\begin{align}
    y &= y_*(s; \nu) \\
    s &= \exp[\lambda \tau]
\end{align}

(3.11)

for $0 \leq s \leq s_* = \exp[\lambda \tau_*]$, with $s_*$ small and independent of $\epsilon$. To understand the stable manifold away from $Q$, we note that $Q$ is an order $\nu$ perturbation of the point $Q_0$

\begin{align*}
    j_0 &= 0 \\
    \theta_0 &= \tan^{-1}\left(\frac{\sqrt{1 - \alpha^2 \omega^2}}{\alpha \omega}\right) - \pi
\end{align*}

and that equations (3.8) are an $O(\nu)$ perturbation of the conservative system

\begin{align*}
    j_\tau &= -2(\alpha \omega^2 + \omega \cos \theta) \\
    \theta_\tau &= -2j.
\end{align*}

Thus, it is useful to introduce the energy of the above system,

\begin{equation}
    E(j, \theta) \equiv \frac{1}{2} j^2 - \omega (\sin \theta + \alpha \omega \theta),
\end{equation}

where the curve $E(j, \theta) = E(j_0, \theta_0)$ is the stable manifold of the conservative system, which we denote by $C^s_0 : y_0(\tau) = (j_0(\tau), \theta_0(\tau))$. If we fix a $\tau_0 < \tau_*$, we have from regular perturbation theory that for $\tau_0 < \tau < \tau_*$ the stable manifold of $Q$ is given by

\begin{align*}
    y &= y_0(\tau) + O(\nu).
\end{align*}

Therefore, if we denote by $C^s_\epsilon$ the portion of the stable manifold corresponding to $\tau \in [\tau_0, \infty)$, then $C^s_\epsilon$ can be parametrized by $s \in [0, s_0]$

\begin{align*}
    y &= y_*(s; \nu) \\
    s &= \exp[\lambda \tau]
\end{align*}

where $y_*$ is a smooth function of $s$ and $\nu$. The curve $y_*$ is order $O(\nu)$ close to the $y_0$ for $\tau_0 < \tau$ and satisfies the equation

$$
(\lambda s)y_{*,s} = Y_*
$$
where \( Y_* = Y(y_*; \nu) \) (see Figure 3.2). Since \(|y_{0,s}|\) is bounded away from 0, we have

\[ m(s; \nu) := |y_{*,s}(s; \nu)| \geq m_0, \quad 0 \leq s \leq s_0. \]

where \( m_0 \) is a positive constant independent of \( \epsilon \) and the unit tangent vector to the curve is given by

\[ t = \frac{y_{*,s}}{m} = \frac{Y_*}{(\lambda s)m}. \]

Figure 3.2. Phase-plane diagram of the ODE in the \( \theta \)-coordinates.

To describe the flow near \( C^s_* \), it is convenient to introduce coordinates \((r, s)\) where \( r \) is a measure of distance in the normal direction to the curve \( C^s_* \). These coordinates are given by

\[ y = y_* (s; \nu) + r n(s; \nu) \]

where \( n(s; \nu) \) is the unit normal vector to the curve \( y_* (s; \nu) \). The unit normal vector and the unit tangent vector are related by the equation

\[ n_s = \kappa(s, \nu)t \]

where \( \kappa \) is a smooth function. We can rewrite equation (3.9) in terms of \((r, s)\) by observing that

\[ y_r = (m + \kappa r)t s_r + r, n_r, \]

which leads to the equations

\[ r_r = Y \cdot n, \quad s_r = \frac{Y \cdot t}{m + \kappa r}. \]
Expanding the above equation in \( r \) we obtain
\[
Y \cdot n = \left[ (Y_r \cdot n) \cdot n \right] r + O(r^2)
\]
\[
\frac{Y \cdot t}{m + \kappa r} = \frac{(Y_r \cdot t)}{m} + \left[ \frac{(Y_r \cdot n) \cdot t}{m} - \frac{\kappa |Y_r|}{m^2} \right] r + O(r^2).
\]

Note that from equation (3.13) we have
\[
\frac{Y_r \cdot t}{m} = \lambda s
\]
and the equations for \((r, s)\) are
\[
\begin{align*}
    r_t &= a(s; \nu) r + O(r^2) \\
    s_r &= \lambda s + b(s, \nu) r + O(r^2)
\end{align*}
\]
where \(a\) and \(b\) are smooth functions in \((s, \nu)\) for \(0 < s < s_0\). In these coordinates \(C_0^s\) corresponds to \(r = 0\), and the flow on \(C_0^s\) is given by
\[
s_r = \lambda s.
\]
Moreover, since \(c_1\) is tangent to \(C_0^s\) at \(Q\), we have from (3.10)
\[
a(0, \nu) = (Y_0(y, \nu) n) \cdot n = \mu.
\]

Although the \((r, s)\) equations are simpler than the original equations (3.6), they are still not in a form in which the flow near \(C_0^s\) can be described well. The reason is the presence of the linear term that couples \(r\) to the equation for \(s_r\), that is, \(b(s, \nu) r\). To see how this term drastically changes the \(s\) flow depending on whether \(r = 0\) or \(r \neq 0\), it is sufficient to consider the case when \(a = \mu\) and \(b\) is a constant. In this case the flow of \(s\) changes from exponential decay at a rate \(\lambda\) to exponential growth at a rate \(\mu\). Even if \(b(0, \nu) = 0\), we still have a change from fast decay to slow decay. For this reason we will introduce a new variable \(\beta\) whose linear flow will not be affected by the \(r\) equation as \(\tau \to \infty\), that is, for small \(s\). Let
\[
s = \beta + (h_0 + h_1 \beta) r
\]
where the constants \(h_0\) and \(h_1\) are chosen to be
\[
\begin{align*}
h_0 &= \frac{b_0}{\mu - \lambda} \\
h_1 &= \frac{b_1 - h_0 a_1}{\mu}
\end{align*}
\]
and where \( a = \mu + a_1 s + O(s^2) \), and \( b = b_0 + b_1 s + O(s^2) \). In terms of \((r, \beta)\) the equations are

\[
(3.17) \quad 
\begin{align*}
    r_r &= a(\beta, \nu) r + O(r^2) \\
    \beta_r &= \lambda \beta + c(\beta, \nu) r + O(r^2)
\end{align*}
\]

where \(|c(\beta, \nu)| \leq c_0|\beta|^2\) on \(C_\nu^q\); that is, \(0 \leq \beta \leq s_0\).

Equation (3.17) has the advantage that the linearized flow around \(C_\nu^q\), that is, \(r = 0\) and \(\beta_* = \beta_0 e^{\lambda \tau}\),

\[
\begin{align*}
    \delta r_r &= a_* \delta r \\
    \delta \beta_r &= \lambda \delta \beta + c_* \delta r
\end{align*}
\]

has expansion and contraction rates similar to those obtained by linearizing the equation around \(Q\). This will be shown in Section 5.3.

4 The Equations in a Neighborhood of \(S_\omega\)

In this section we show that in a neighborhood of \(S_\omega\), equation (3.1) can be written in a simplified form given by equations (4.9) or (4.10).

4.1 Basic Equations

In order to study the dynamics of solutions to the nonlinear problem in a neighborhood of the circle of fixed points \(S_\omega\), we write (3.1) in terms of coordinates that are suited for this purpose. This entails introducing coordinates \((J, \theta, f)\) where \(J\) is a measure of distance from \(S_\omega\) on the plane \(\Pi\), \(\theta\) is the angle on \(S_\omega\), and \(f\) is in the orthogonal complement of \(\Pi\). These coordinates are determined in the following manner:

First, write \(q\) as

\[
(4.1) \quad q \equiv [\rho(t) + f(x, t)] e^{i \theta(t)} ,
\]

where \(\rho\) and \(\theta\) are polar coordinates on the plane \(\Pi\) and \(f \in \Pi^{\perp}_\epsilon\); that is, \(f\) has spatial mean zero.

The \(L^2\)-norm is a constant of motion for the unperturbed \((\epsilon = 0)\) flow; therefore, it will be used as a coordinate instead of \(\rho\):

\[
(4.2) \quad I := \frac{1}{2\pi} \int_0^{2\pi} q q d\theta = \rho^2 + \langle f f \rangle .
\]
Finally, since we are working in a neighborhood of the circle of fixed points $S_\alpha$ that corresponds to $I = \omega^2$, it will be convenient to introduce the variable $J$ defined by

\begin{equation}
J = I - \omega^2.
\end{equation}

In terms of these variables, equation (3.1) takes the form

\begin{equation}
\begin{aligned}
J_t &= -2\epsilon[\alpha(\omega^2 + J) + \rho \cos \theta] + \epsilon Q_1(f) \\
\theta_t &= -2J - \frac{1}{\rho} \sin \theta + Q_2(f) + \frac{1}{\rho} C_2(f) \\
in_t &= \mathcal{L}_e f + W_e f + \rho Q_3(f) + C_3(f) + \frac{1}{\rho} C_2(f) f
\end{aligned}
\end{equation}

where

\begin{align*}
\rho &= \sqrt{J + \omega^2 - \langle ff \rangle} \\
\mathcal{L}_e f &= f_{xx} + i\epsilon \hat{D} f + 2\omega^2 (f + \bar{f}) \\
W_e f &= 2J (f + \bar{f}) + \epsilon \frac{\sin \theta}{\rho} f
\end{align*}

and where

\begin{align*}
Q_1(f) &= 2\langle \hat{f}(\hat{D} - \alpha)f \rangle \\
Q_2(f) &= -\langle (f + \bar{f})^2 \rangle \\
Q_3(f) &= 4(f\bar{f} - \langle f\bar{f} \rangle) + 2(f^2 - \langle f^2 \rangle) \\
C_2(f) &= -\langle f\bar{f}(f + \bar{f}) \rangle \\
C_3(f) &= 2\langle f\bar{f}f - \langle f\bar{f}f \rangle \rangle - \langle f^2 + \bar{f}^2 + 6ff \rangle / 2
\end{align*}

In these equations, $\langle \cdot \rangle$ denotes the spatial mean over one period.

For $J$ and $f$ in a small but otherwise fixed neighborhood of $0$, equation (4.4) can be considered as a perturbation of (3.6):

\begin{equation}
\begin{aligned}
J_t &= -2\epsilon[\alpha(J + \omega^2) + \sqrt{J + \omega^2} \cos \theta] + \mathcal{E}_1(J, \theta, f; \epsilon) \\
\theta_t &= -2J + \epsilon(J + \omega^2)^{-1/2} \sin \theta + \mathcal{E}_2(J, \theta, f; \epsilon) \\
in_t &= \mathcal{L}_e f + W_e f + \omega Q_3(f) + \mathcal{E}_3(J, \theta, f; \epsilon)
\end{aligned}
\end{equation}
where

\[ W_f = 2J(f + f') + \epsilon \frac{\sin \theta}{\sqrt{J(\omega^2 + J)}} f \]

and where \( E_k \) are \( 2\pi \)-periodic functions in \( \theta \) of order

\[ E_1(J, \theta, f; \epsilon) = O(\epsilon f^2) \]
\[ E_2(J, \theta, f; \epsilon) = O(f^2) \]
\[ E_3(J, \theta, f; \epsilon) = O(J f^2 + f^3) \]

for small \( J \) and \( f \).

4.2 Normal Forms

The point \( Q \) is a critical point for equation (4.4). In constructing a homoclinic orbit to \( Q \), we need to estimate the size of the local stable manifold of \( Q \) (see Section 5.5). The size of the variables \((J, \theta)\) is determined from the curve \( C'_s \), which is the intersection of the stable manifold and the plane of constants \( \Pi_c \). To estimate the size of \( f \), we have to use equation (4.4), which has a troublesome quadratic term \( Q_3(f) \). However, this term is nonresonant and can be removed using a normal form transformation, as will be demonstrated below.

Quadratic Resonance

In order to eliminate the quadratic term \( Q_3(f) \), we must analyze the quadratic resonances of the linear equation

\[ if_t = f_{xx} + 2\omega^2 (f + f), \]

which corresponds to the linear part of the \( f \) equation (4.4) evaluated at \( \epsilon = 0 \).

Substituting \( f = e^{i(k x + \lambda t)} \), we find that the dispersion relation of the linear equation is given by:

\[ \lambda = \pm ik\sqrt{k^2 - 4\omega^2}, \quad k = 1, 2, \ldots. \]

Quadratic resonance means that there exist nonzero integers \( k_1, k_2, \) and \( k_3 \) such that

\[ k_1 + k_2 = k_3 \]
\[ \lambda_1 \pm \lambda_2 = \pm \lambda_3. \]
We check the above resonance conditions by squaring the $\lambda$-equation and substituting the $k$-equation to obtain

$$2k_1^2 + 3k_1k_2 + 2k_2^2 - 4\omega^2 = \pm \sqrt{k_1^2 - 4\omega^2} \sqrt{k_2^2 - 4\omega^2}.$$

By squaring the above equation once more, we obtain

$$[(k_1 + k_2)^2 + k_1^2 + k_2^2 - 6\omega^2] (k_1 + k_2)^2 = 0,$$

which is only possible if $k_1 + k_2 = k_3 = 0$, since $\omega \in (\frac{1}{2}, 1)$. However, since we are considering the equation in the space $\Pi_\varphi^1$, we have that $k_3 \neq 0$. Therefore the linear equation (4.7) has no quadratic resonance in the space $\Pi_\varphi^1$.

**Normal Form Transformation**

To compute the normal form transformation, we start by writing $f$ in terms of a Fourier expansion:

$$f(x) = \sum_{k \neq 0} \hat{f}(k) e^{ikx}.$$

The quadratic term $Q_2(f)$ is made up of two terms:

$$f^2(x) - \langle f^2 \rangle = \sum_{\ell + k \neq 0} \hat{f}(k)\hat{f}(\ell) e^{i(k+\ell)x},$$

$$|f(x)|^2 - \langle |f|^2 \rangle = \sum_{\ell + k \neq 0} \hat{f}(k)\overline{\hat{f}(\ell)} e^{i(k+\ell)x}.$$

Now a general quadratic near-identity map that is translation-invariant with respect to $x$ can be written as

$$(4.8) \quad g = f + K(f, f)$$

$$K(f, h) := K_{11}(f, h) + K_{11}(f, h) + K_{11}(f, h) + K_{11}(\hat{f}, \hat{h}),$$

where $K$ are bounded bilinear maps $\Pi_\varphi^1 \times \Pi_\varphi^1 \to \Pi_\varphi^1$.

$$K_{11}(f, h) = \iiint K_{11}(x - y_1, x - y_2) f(y_1) h(y_2) dy_1 dy_2$$

$$K_{11}(\hat{f}, \hat{h}) = \iiint K_{11}(x - y_1, x - y_2) \hat{f}(y_1) \hat{h}(y_2) dy_1 dy_2.$$
with similar expressions for $K_{11}$ and $K_{11}$. In terms of Fourier expansions these bilinear maps can be written as

$$K_{11}(f, h) = \sum_{k+\ell \neq 0} \hat{K}_{11}(k, \ell) \hat{f}(k) \hat{h}(\ell) e^{i(k+\ell)x}$$

$$K_{11}(f, h) = \sum_{k+\ell \neq 0} \hat{K}_{11}(k, \ell) \hat{f}(k) \hat{h}(-\ell) e^{i(k+\ell)x}$$

**PROPOSITION 4.1** There exists a near-identity quadratic map of the form (4.8) that transforms the equation

$$i\partial_t f = f_{xx} + 2\omega^2(f + \dot{f}) + \omega Q_3(f)$$

into an equation with a cubic nonlinearity

$$i\partial_t g = g_{xx} + 2\omega^2(g + \dot{g}) + O(g^3)$$

**PROOF:** Compute

$$Sg := i\partial_t g - \partial_x^2 g - 2\omega^2(g + \dot{g})$$

where $g$ is given by (4.8). We obtain

$$Sg = Sf + H_{11}(f, f) + H_{11}(f, \dot{f}) + H_{11}(f, f) + H_{11}(f, \dot{f}) + C'(f)$$

where

$$\hat{H}_{11} = 2(k\ell + \omega^2)\hat{K}_{11} - 2\omega^2\hat{K}_{11} - 2\omega^2\hat{K}_{11} - 2\omega^2\hat{K}_{11}$$

$$\hat{H}_{11} = 2\omega^2\hat{K}_{11} + 2(\ell^2 + k\ell - \omega^2)\hat{K}_{11} - 2\omega^2\hat{K}_{11} - 2\omega^2\hat{K}_{11}$$

$$\hat{H}_{11} = 2\omega^2\hat{K}_{11} - 2\omega^2\hat{K}_{11} + 2(k^2 + k\ell - \omega^2)\hat{K}_{11} - 2\omega^2\hat{K}_{11}$$

$$\hat{H}_{11} = -2\omega^2\hat{K}_{11} + 2\omega^2\hat{K}_{11} + 2\omega^2\hat{K}_{11} + 2(k^2 + \ell^2 + k\ell - 3\omega^2)\hat{K}_{11}$$

and where $C'(f)$ consists of terms of the form $K(f, Sf)$. Substituting the equation for $f$ in the above $Sf = \omega Q_3(f)$, we obtain

$$Sg = \omega Q_3(f) + H_{11}(f, f) + H_{11}(f, \dot{f}) + H_{11}(f, f) + H_{11}(f, \dot{f}) + C'(f)$$

where $C'(f)$ is a cubic term.

Therefore, to eliminate the $Q_3$ term from the $g$ equation, we need

$$\hat{H}_{11} = -2\omega, \quad \hat{H}_{11} = -2\omega, \quad \hat{H}_{11} = -2\omega, \quad \hat{H}_{11} = 0.$$
for all the $k$ and $\ell$ such that $k + \ell \neq 0$. Since $H_{ab} = \hat{H}_{ab}$, we deduce that $K_{ab} = \hat{K}_{ab}$ for $a$ and $b \in \{1, \bar{1}\}$. To find the four-vector $K$, we have to solve the linear equation

$$UK = H$$

for the given vector $H = (-2\omega, -2\omega, -2\omega, 0)^T$. Since there are no quadratic resonances, $\det U \neq 0$ and these equations have a unique solution

$$\hat{K}_{11}(k, \ell) = -\frac{\omega}{k\ell},$$

$$\hat{K}_{1\bar{1}}(k, \ell) = -\frac{\omega}{\ell(k + \ell)},$$

$$\hat{K}_{\bar{1}}(k, \ell) = -\frac{\omega}{k(k + \ell)},$$

$$\hat{K}_{1\bar{1}}(k, \ell) = 0.$$

Note that $k \neq 0$, $\ell \neq 0$, and $\ell + k \neq 0$ since we are in the space $\Pi^\perp$. Moreover, since

$$\sum |\hat{K}_{ab}(\ell, k)|^2 < \infty$$

we have $K \in L^2(S^1 \times S^1)$, which implies that $K$ is a bounded bilinear map on $\Pi^\perp$

$$\|K(f, f)\|_{H^1} \leq C\|f\|_{H^1}^2$$

for all $f \in \Pi^\perp$. Finally, we can invert the equation

$$g = f + K(f, f)$$

for $f$ in a neighborhood of the zero to obtain

$$f = g + K(g)$$

where $K$ is of order $O(g^2)$. Therefore $C(f)$ is a cubic term in $g$.

4.3 Local Equations

The equations in a neighborhood of $S_\omega$ were given in (4.5) in terms of the variables $(J, \theta, f)$:

$$J_t = -2\epsilon \left[ \alpha(J + \omega^2) + \sqrt{J + \omega^2} \cos \theta \right] + \mathcal{E}_1(J, \theta, f; \epsilon)$$

$$\theta_t = -2J + \epsilon(J + \omega^2)^{-1/2} \sin \theta + \mathcal{E}_2(J, \theta, f; \epsilon)$$

$$if_t = \mathcal{L}_\epsilon f + W_\epsilon f + \omega Q_3(f) + \mathcal{E}_3(J, \theta, f; \epsilon).$$
Local Equations with Complex Coordinates

To eliminate the term $\omega Q_3(f)$ from equation (4.6) we apply the transformation

$$g = f + K(f, f).$$

This procedure will eliminate the $Q_3$ term, but it will introduce new quadratic terms in the equation for $g$ that have $\epsilon$ coefficients, such as

$$\epsilon \hat{D}K(f, f)$$

due to the presence of terms such as $\epsilon \hat{D}f$ in the $f$ equation. Therefore, using $(J, \theta, g)$ as coordinates, the equations near $S_\omega$ are written as

$$N_1(J, \theta, g; \epsilon) = 0(\epsilon g^2)$$

$$N_2(J, \theta, g; \epsilon) = O(g^2)$$

$$N_3(J, \theta, g; \epsilon) = O(J g^2 + \epsilon g^2 + g^3)$$

Local Equations with Real Coordinates

Since we will be working with invariant real manifolds in a neighborhood of the circle of fixed points $S_\omega$, it will be convenient to introduce a real coordinate system:

$$u = (\text{Re}(g), \text{Im}(g))^T.$$

In terms of these variables the above equation takes the form

$$N_1(J, \theta, g; \epsilon) = 0(\epsilon g^2)$$

$$N_2(J, \theta, g; \epsilon) = O(g^2)$$

$$N_3(J, \theta, g; \epsilon) = O(J g^2 + \epsilon g^2 + g^3)$$

Here $N_3$ is interpreted as a two-vector,

$$L_\epsilon = \mathcal{J} \partial_x^2 - 4\omega^2 S + \epsilon \hat{D}$$

$$V_\epsilon = -4JS + \epsilon \frac{\sin \theta}{\sqrt{J + \omega^2}} \mathcal{J}.$$
where

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \]

5 Invariant Manifolds

In this section we prove the persistence of invariant manifolds in a neighborhood of \( S_\omega \) and fiber these manifolds in the strong contracting or expanding directions. We also estimate the size of the stable manifold at the point \( Q \) by using equations (4.10) derived above, and we show that the stable manifold is at least of size \( O(\epsilon^\mu) \), \( \mu < 1 \), in the \( u \)-direction (i.e., off the plane of constants).

5.1 Existence of Local Invariant Manifolds

Analysis of the Linear Equations

In a neighborhood of \( S_\omega \), equations (4.10) can be viewed as a perturbation of the linear system

\[
\begin{align*}
J_t &= 0 \\
\theta_t &= -2J \\
u_t &= L_\epsilon u.
\end{align*}
\]

(5.1)

In order to study the local behavior of solutions to the nonlinear equations (4.10), we have to analyze the spectrum of the operator \( L_\epsilon \); therefore, we consider the eigenvalue problem

\[ L_\epsilon e = \lambda e \]

for the eigenpairs \( \{e(x), \lambda\} \). Using Fourier expansions, one finds a quadratic expression for the eigenvalue \( \lambda \):

\[
(\lambda - \epsilon d(j))^2 + j^2(j^2 - 4\omega^2) = 0, \quad j = 1, 2, \ldots,
\]

where \( d(j) \) denotes the symbol of \( \widehat{\sigma} \). Since \( \omega \in (\frac{1}{2}, 1) \), we have for \( j = 1 \)

\[
e_{s,u} = \frac{1}{2\sqrt{\pi\omega}}(1, \mp \sigma)^T \cos x,
\]

\[
\sigma_{s,u}^T = \pm \sigma - \epsilon d(1),
\]

(5.2)
For \( j \geq 2 \), the eigenvalues come in complex conjugate pairs with negative real part:

\[
\lambda_j = i\Omega_j - \epsilon d(j),
\]

where

\[
\Omega_j = j \sqrt{j^2 - 4\omega^2} > 0.
\]

In terms of this eigenbasis, the mean zero function \( u \) can be written as

\[
u(x) = v_u e_u(x) + v_s e_s(x) + v_o(x)
\]

where \( v_u \) and \( v_s \) are real scalars and where \( v_o(x) \in [\text{span}\{I_c, e_u, e_s\}]^L \).

In terms of these variables, linear equations (5.1) split into

\[
\begin{align*}
J_t &= 0 \\
\theta_t &= -2J \\
v_{u,t} &= \sigma'_u v_u \\
v_{s,t} &= -\sigma'_s v_s \\
v_{o,t} &= L_r v_o
\end{align*}
\]

Thus, we explicitly see that, for \( \epsilon = 0 \), the linear equations have one unstable direction \( (e_u) \), one stable direction \( (e_s) \), and an infinite number of center directions \( (J, \theta, v_o) \). Combining these center variables as \( v_c = (J, \theta, v_o)^T \), equation (5.6) can be written as

\[
\begin{align*}
v_{u,t} &= \sigma'_u v_u \\
v_{s,t} &= -\sigma'_s v_s \\
v_{c,t} &= A v_c
\end{align*}
\]

where \( A \) is defined from equations (5.6).

In a \( \delta \)-neighborhood of the circle of fixed points \( S_\omega \), the nonlinear equation (4.10) can be viewed as a perturbation of the linear equation (5.1). Under the flow of this linear equation and for \( \epsilon = 0 \), \( S_\omega \) has one-dimensional stable and unstable manifolds, together with a codimension 2 center manifold. We focus
our attention on the center manifold $E^c(S_\omega)$, together with the center-stable $E^{cs}(S_\omega)$ and center-unstable $E^{cu}(S_\omega)$ manifolds:

$$E^{cs}(S_\omega) = \text{span}\{e_u\}^\perp$$
$$E^{cu}(S_\omega) = \text{span}\{e_s\}^\perp$$
$$E^c(S_\omega) = \text{span}\{e_u, e_s\}^\perp$$

An important feature of the linear equation (5.6) is that the growth rates on the invariant manifolds are separated by a wide gap. To see this, we note that for $\epsilon = 0$ the spectrum of the operator has real part $\pm \sigma$ and 0. Thus for any integer $n$ and for $\epsilon < \sigma/4n$ we have

$$\|\exp[At]\| \leq nC \exp \left[ \frac{\sigma|t|}{n} \right]$$

and the invariant manifolds $E^{cs}$, $E^{cu}$, and $E^c$ can be described by solutions whose growth rates are bounded by $\exp[\sigma t/n]$ for $t > 0$, $\exp[-\sigma t/n]$ for $t < 0$, and $\exp[\sigma |t|/n]$ for all $t$, respectively. Solutions that belong to the stable or unstable manifolds, that is, $\text{span}\{e_s\}$ and $\text{span}\{e_u\}$, respectively, will have growth rates of order $\sigma$. This gap of order $n$ in the growth rates will be the main ingredient in proving that the nonlinear equation (4.4) has locally invariant smooth manifolds that can be foliated by smooth fibers.

**Localized Equations**

We start by fixing an integer $n_0$ sufficiently large and a localization parameter $\delta = a/n_0^2$, where $a$ is a constant that is independent of $\epsilon$ that will be specified later. We introduce a localization function $\psi_\delta$,

$$\psi_\delta : \mathbb{R} \to \mathbb{R}, \quad \psi_\delta(s) = \psi(s/\delta),$$

where $\psi$ is $C^\infty$ and satisfies

$$\psi(s) = \begin{cases} 
1, & |s| \leq 1 \\
0, & |s| \geq 2.
\end{cases}$$

The localization function localizes the equations (4.10):

$$J_t = -2\epsilon \left[ \alpha(J_\delta + \omega^2) + \sqrt{J_\delta + \omega^2} \cos \theta \right]$$
$$+ N_1(J_\delta, \theta, u_\delta; \epsilon)$$

$$\theta_t = -2J + \epsilon \left[ (J_\delta + \omega^2)^{-1/2} \sin \theta \right]$$
$$+ N_2(J_\delta, \theta, u_\delta; \epsilon)$$

$$u_t = L_\epsilon u + [V_\epsilon u_\delta + N_3(J_\delta, \theta, u_\delta; \epsilon)],$$

(5.8)
where for any variable $s$ we denote $s_\delta = s\psi(s/\delta)$.

Note that we do not cut off the variable $\theta$; thus, the function $\psi_\delta(J, f)$ has the effect of cutting off the right-hand sides whenever the phase point lies outside a neighborhood $U_\delta$ of the circle $S_\omega$. Moreover, all the nonlinear terms in equation (5.8) are either multiplied by $\epsilon$ or are at least quadratic in $(J, u)$ and localized in a $\delta$-neighborhood of 0. This implies that equation (5.8) has a global Lipschitz constant of order $O(\epsilon + \delta)$.

This localization has the effect of keeping the flow unchanged in a $\delta$-neighborhood of $S_\omega$ while changing the nonlinear equations (4.10) to become a global $\delta$-perturbation of a linear constant-coefficient system. Using $v := (v_u, v_s, v_c)\top$ as variables and the operator $A$ defined in equation (4.7), we can write equations (5.8) as

\begin{align}
v_{u,t} &= \sigma^s_u v_u + R^\delta_u(v; \epsilon) \\
v_{s,t} &= -\sigma^s_s v_s + R^\delta_s(v; \epsilon) \\
v_{c,t} &= Av_c + R^\delta_c(v; \epsilon),
\end{align}

where $R^\delta(v; \epsilon)$ and its first derivatives are of order $O(\delta + \epsilon)$.

We will show that the localized equation has $C^\ell$-invariant manifolds that deform smoothly from $E^{cs}$, $E^{cu}$, and $E^c$. In turn, for the original equations, these manifolds will be locally invariant in a $\delta$-neighborhood of $S_\omega$.

Statement and Proof of the Persistence Theorem

First we define local invariance.

**DEFINITION 5.1** Given an open set $O$, a manifold $M$ is called *locally invariant* (in $O$) under a flow $F^t$ if, for every open interval $I$ such that $F^t(q) \subset O$, $F^{t*}(q) \in M$ for some $t_* \in I \implies F^t(q) \in M$, $\forall t \in I$.

Now we state the persistence theorem:

**THEOREM 5.2** There exist a $\delta$-neighborhood $U_\delta$ of $S_\omega$, an $\epsilon_0(\delta) > 0$, and an integer $\ell > 3$ such that $\forall \epsilon \in [0, \epsilon_0)$, equation (4.10) has a locally invariant (in $U_\delta$) manifold of codimension 1,

\begin{align}
W^{cs}_\epsilon &= \{v \in H^1 : v_u = h_u(v_s, v_c; \epsilon)\},
\end{align}

where the function $h_u$ is $C^\ell$ in all of its arguments and $2\pi$-periodic in $\theta$. Moreover, for $\epsilon = 0$, $W^{cs}_0$ intersects $E^{cs}$ tangentially along $S_\omega$.

Similarly, we have a locally invariant manifold given by

\begin{align}
W^{cu}_\epsilon &= \{v \in H^1 : v_s = h_s(v_u, v_c; \epsilon)\}
\end{align}
where the function \( h_s \) is \( C^\ell \) in all of its arguments and \( 2\pi \)-periodic in \( \theta \). Moreover, for \( \epsilon = 0 \), \( W_0^{cu} \) intersects \( E^{cu} \) tangentially along \( S_\omega \).

The existence of a codimension 2 "slow manifold" \( \mathcal{M}_\epsilon \) is then given by the following:

**Corollary 5.3** Let \( \mathcal{M}_\epsilon \) denote the intersection

\[
\mathcal{M}_\epsilon = W_{\epsilon}^{cs} \bigcap W_{\epsilon}^{cu}.
\]

Then \( \mathcal{M}_\epsilon \) is a locally invariant (in \( U_\delta \)) manifold of codimension 2,

\[
\mathcal{M}_\epsilon = \{ v \in H^1 : \psi_u = h_u^c(v_c; \epsilon), \psi_s = h_s^c(v_c; \epsilon) \}
\]

where the functions \( h_u^c, h_s^c \) are \( C^\ell \) in their arguments and \( 2\pi \)-periodic in \( \theta \). Moreover, for \( \epsilon = 0 \), \( \mathcal{M} \) intersects \( E^c \) tangentially along \( S_\omega \).

**Remark.** The flow on \( \mathcal{M}_\epsilon \) is given by the equations

\[
J_t = -2\epsilon \left[ \alpha (J_\delta + \omega^2) + \sqrt{J_\delta + \omega^2 \cos \theta} \right] \\
\theta_t = -2J + \epsilon (J_\delta + \omega^2)^{-1/2} \sin \theta \\
v_{o,t} = \mathcal{L}_\epsilon v_o + [ V_{\epsilon} v_{o\delta} + \hat{N}_3(J_\delta, \theta, v_{o\delta}; \epsilon) ]
\]

where, as before, for any variable \( s \) we denote \( s_\delta = s \psi(s/\delta) \) and where \( \hat{N} \) are the restrictions of \( N \) to \( \mathcal{M}_\epsilon \) given by the functions \( h_u^c \) and \( h_s^c \).

**Proof of Theorem 5.2:** Once the integral equations are set up, this proof is just a standard application of a fixed-point argument. We include it here for the sake of completeness. First, we rewrite equation (5.9) in integral form:

\[
v_u(t) = \exp \left[ \sigma_u^c(t - t_u) \right] v_u(t_u) + \int_{t_u}^t \exp \left[ \sigma_u^c(t - s) \right] R_u^\delta(v(s); \epsilon) \, ds \\
v_s(t) = \exp \left[ -\sigma_s^c(t - t_s) \right] v_s(t_s) + \int_{t_s}^t \exp \left[ -\sigma_s^c(t - s) \right] R_s^\delta(v(s); \epsilon) \, ds \\
v_c(t) = \exp[A t] v_c(0) + \int_0^t \exp[A(t - s)] R_c^\delta(v(s); \epsilon) \, ds.
\]
Because of the gap in the growth rates, we will characterize the invariant manifolds $W^\text{cs}_\epsilon$ and $W^\text{cu}_\epsilon$ by

\begin{align}
W^\text{cs}_\epsilon &= \left\{ v \in H^1 : \sup_{t \geq 0} \left( \exp \left[ -\sigma t \right] \| F^t(v;\epsilon) \|_{H^1} \right) < \infty \right\}, \\
W^\text{cu}_\epsilon &= \left\{ v \in H^1 : \sup_{t \leq 0} \left( \exp \left[ \sigma t \right] \| F^t(v;\epsilon) \|_{H^1} \right) < \infty \right\},
\end{align}

where $F^t(v;\epsilon)$ is the flow of equations (5.9). Focusing our attention upon $W^\text{cs}_\epsilon$, for $\tilde{v}$ in a ball $B$ of arbitrary radius $\rho$ we introduce the norm

$$
\| v \|_{\lambda} = \sup_{v \leq \| v \|_{H^1}} \exp \left\{ -\frac{\sigma t}{\lambda} \right\}.
$$

From the definition of $W^\text{cs}_\epsilon$ we have for $v \in W^\text{cs}_\epsilon$

$$
\exp[-\sigma t]\|v_u(t_u)\| \to 0 \quad \text{as} \quad t_u \to \infty.
$$

Therefore for solutions on $W^\text{cs}_\epsilon$, the integral equation can be written as

\begin{align}
v_u(t) &= \int_{+\infty}^{t} \exp[\sigma_s(t-s)] R^\delta_u(v(s);\epsilon) \, ds \\
v_s(t) &= \exp[-\sigma_s t]v_s + \int_{0}^{t} \exp[-\sigma_s(t-s)] R^\delta_s(v(s);\epsilon) \, ds \\
v_c(t) &= \exp[A(t)]v_c + \int_{0}^{t} \exp[A(t-s)] R^\delta_c(v(s);\epsilon) \, ds.
\end{align}

To show existence of $W^\text{cs}_\epsilon$ we use Newton’s iterations: Let $\nu^0 = 0$ and

\begin{align}
\nu_{u}^{k+1}(t) &= \int_{+\infty}^{t} \exp[\sigma_s(t-s)] R^\delta_u(v^k(s);\epsilon) \, ds \\

\nu_{s}^{k+1}(t) &= \exp[-\sigma_s t]v_s + \int_{0}^{t} \exp[-\sigma_s(t-s)] R^\delta_s(v^k(s);\epsilon) \, ds \\

\nu_{c}^{k+1}(t) &= \exp[A(t)]v_c + \int_{0}^{t} \exp[A(t-s)] R^\delta_c(v^k(s);\epsilon) \, ds,
\end{align}

This generates a well-defined sequence of functions. For if $\| v^k \|_{n_0} \leq C$, then

$$
\| v^{k+1}(t) \|_{H^1} \leq n_0 C \exp \left[ \frac{\sigma t}{n_0} \right] (\| v_s \|_{H^1} + \| v_c \|_{H^1})
+ \int_{t}^{\infty} \exp \left[ \frac{\sigma}{2} (t-s) \right] \| R^\delta_u(\nu^k(s);\epsilon) \|_{H^1} \, ds
.$$
\[ + \int_0^t \exp \left[ -\frac{\sigma}{2} (t-s) \right] \left\| R_s^\delta (v^k(s); \epsilon) \right\|_{H^1} ds \]

\[ + \int_0^t n_0 C \exp \left[ \frac{\sigma}{2 n_0} (t-s) \right] \left\| R_s^\delta (v^k(s); \epsilon) \right\|_{H^1} ds. \]

By (5.8) we note that \( R_s^\delta (w, \epsilon) \) is a smooth function whose terms are either linear with coefficient \( \epsilon \) or nonlinear and localized in a \( \delta \)-neighborhood of \( S_\omega \). Therefore if we let \( R' \) be the derivative of \( R_s^\delta \), we have the estimate

\[ \| R^\delta (w, \epsilon) \|_{H^1} \leq \| R' \| \| w \|_{H^1} + \epsilon, \]

where \( \| R' \| \) is the supremum of the magnitude of \( R' \), which is equal to \( C(\delta + \epsilon) \) because of the localization function. Here the \( \epsilon \) term is needed because of terms such as \( \epsilon \rho \sin(\theta) \) in (5.8). This implies that

\[ \| v^{k+1}(t) \|_{H^1} \leq n_0 C \exp \left[ \frac{\sigma}{n_0} t \right] \left( \| v_s \|_{H^1} + \| v_c \|_{H^1} + \epsilon \right) \]

\[ + \int_0^\infty \exp \left[ \frac{\sigma}{2} (t-s) \right] C(\epsilon + \delta) \| v^k(s) \|_{H^1} ds \]

\[ + \int_0^t \exp \left[ \frac{\sigma}{2 n_0} (t-s) \right] n_0 C(\epsilon + \delta) \| v^k(s) \|_{H^1} ds. \]

By using the bound on \( v^k \) we obtain

\[ \| v^{k+1}(t) \|_{H^1} \leq C \left[ \| v_s \|_{H^1} + \| v_c \|_{H^1} + \epsilon \right] \]

\[ + n_0^2 (\epsilon + \delta) \| v^k \|_{n_0} \exp \left[ \frac{\sigma}{n_0 t} \right] \]

where the constant \( C \) is independent of \( n_0, \epsilon, \) and \( \delta \). Now by fixing \( \delta = \frac{a}{n_0} \) where \( a = \frac{1}{4} C \), we obtain for all \( \epsilon < a/n_0^2 \)

\[ \| v^{k+1} \|_{n_0} \leq C(\rho) + \frac{1}{2} \| v^k \|_{n_0}. \]

Therefore the sequence \( v^k \) is well-defined and \( \| v^k \|_{n_0} \leq 2C(\rho) \). Since the nonlinear term is smooth, we have a similar estimate for the difference

\[ \| v^{k+1} - v^k \|_{n_0} \leq \frac{1}{2} \| v^k - v^{k-1} \|_{n_0}, \]

which implies that \( v^k \to v \), a continuous function in \( t \) and \( \bar{v} \) with values in \( H^1 \), and that

\[ \| v \|_{n_0} \leq 2C(\epsilon + \| v_s \|_{H^1} + \| v_c \|_{H^1}). \]
To show smoothness of $v$ with respect to $v_s$, $v_c$, and $\epsilon$, we note that all terms in equation (5.16) are smooth, which implies that the sequence $\{v^k\}$ is differentiable. The derivative $Dv^k$ satisfies

$$
\|Dv^{k+1}(t)\|_{H^1} \leq C\exp\left[\frac{\sigma}{2 n_0} t\right] + C \int_t^\infty \exp\left[\frac{\sigma}{2} (t - s)\right] \|R'Dv^k\|_{H^1} ds \nonumber
$$

$$
+ C \int_0^t \exp\left[\frac{\sigma}{2 n_0} (t - s)\right] \|R'Dv^k\|_{H^1} ds.
$$

By using the bounds on $R'$, we obtain

$$
\|Dv^{k+1}\|_{n_0} \leq C + \frac{1}{2} \|Dv^k\|_{n_0},
$$

which implies $\|Dv^k\|_{n_0} \leq 2C$. To estimate the difference between two terms in the sequence, we note that by the mean value theorem

$$
\left\| \left[ R'(v^k) - R'(v^{k-1}) \right] w \right\|_{H^1} \leq C \left\| v^k - v^{k-1} \right\|_{H^1} \|w\|_{H^1},
$$

where the constant in the above equation depends on the norm of $R''$. Letting $\delta v^k = v^k - v^{k-1}$ we have

$$
\|D\delta v^{k+1}(t)\|_{H^1} \leq C \int_t^\infty \exp\left[\frac{\sigma}{2} (t - s)\right] \|R'(D\delta v^k)\|_{H^1} ds \nonumber
$$

$$
+ \left( \|Dv^k\|_{H^1} + \|Dv^{k-1}\|_{H^1} \right) \|\delta v^k\|_{H^1} ds
$$

(5.21)

$$
+ C \int_0^t \exp\left[\frac{\sigma}{2 n_0} (t - s)\right] \|R'(D\delta v^k)\|_{H^1} ds \nonumber
$$

$$
+ \left( \|Dv^k\|_{H^1} + \|Dv^{k-1}\|_{H^1} \right) \|\delta v^k\|_{H^1} ds.
$$

The quadratic terms in the equation lead to an increase in the growth

$$
\left\| v^k - v^{k-1} \right\|_{H^1} \|Dv^k\|_{H^1} \leq \exp\left[\frac{2\sigma}{n_0} t\right] \left\| v^k - v^{k-1} \right\|_{n_0} \|Dv^k\|_{n_0}.
$$

This increase in the growth rate restricts the estimate of the difference of the first derivative to

$$
\|Dv^{k+1} - Dv^k\|_{n_0/2} \leq C \left\| v^k - v^{k-1} \right\|_{n_0} + \frac{1}{2} \|Dv^k - Dv^{k-1}\|_{n_0}.
$$

Therefore sequence $\{v^k\}$ converges in $C^1$ using the $\| \cdot \|_{n_0/2}$-norm. This procedure can be repeated to obtain bounds on $\{D^jv^k\}$ in the $\| \cdot \|_{n_j}$-norm.
provided \( n/j > 2 \). Passing to the limit we obtain \( v \in C^\ell \) for \( \ell \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \). With these estimates we define

\begin{equation}
(5.22) \quad h_u(v_s, v_c; \epsilon) = v_u(0) = \int_0^\infty \exp \left[ - \sigma_u s \right] R_\delta(v(s); \epsilon) \, ds
\end{equation}

which is a \( C^\ell \) functional with \( \|Dh_u\| \leq \frac{1}{2} \), and

\[ W_{r_{cs}} = \{ v \in H^1 : v_u = h_u(v_s, v_c; \epsilon) \} \]

is a \( C^\ell \) manifold. The invariance of \( W_{r_{cs}} \) follows from the definition of \( h_u \) and from the invariance of the equation under time translation.

For \( W_{r_{cs}} \) to be a locally invariant manifold for equations (3.1), the function \( h_u \) has to be \( 2\pi \)-periodic in \( \theta \). This is obviously so since the integral equations are \( 2\pi \)-periodic in \( \theta \) and have a unique solution for every \( \theta \in \mathbb{R} \). Finally, the tangency of \( W_{r_{cs}} \) to \( S_\omega \) follows from observing that for \( \epsilon = 0 \) \( R_\delta \) is at least quadratic in \( u \) and \( J \) for all \( \theta \in [0, 2\pi] \).

An identical argument establishes the existence of a \( C^\ell \) function \( h_s \) such that \( \|Dh_s\| < \frac{1}{2} \) and of a \( C^\ell \) locally invariant manifold for equation (3.1) given by

\[ W_{r_{cu}} = \{ v \in H^1 \mid v_s = h_s(v_u, v_c; \epsilon) \} \]

\[ \Box \]

**Proof of the Corollary 5.3:** The intersection of \( W_{r_{cu}} \) and \( W_{r_{cs}} \) can be described by the solution of the system of equations

\begin{align*}
&v_u = h_u(v_s, v_c; \epsilon) \\
&v_s = h_s(v_u, v_c; \epsilon)
\end{align*}

Note that \( Dh_u \) and \( Dh_s \) have magnitude less than \( \frac{1}{2} \) by (5.22) and estimate (5.20). By the implicit function theorem the above system has a unique solution given by

\begin{align*}
&v_u = h_u^c(v_c; \epsilon) \\
&v_s = h_s^c(v_c; \epsilon)
\end{align*}

where \( h_{u,s}^c \) are \( C^\ell \) functions and

\[ \mathcal{M}_\epsilon = \{ v \in H^1 : v_u = h_u^c(v_c; \epsilon), \ v_s = h_s^c(v_c; \epsilon) \} \]

\[ \Box \]

**Remark.** The normal form transformation is not needed for this proof or for the proof in the next section.
5.2 The Fibration

Because the linearized problem has small growth rates, the global construction of homoclinic orbits for \( t \in (-\infty, \infty) \) is a singular perturbation problem. The equations under consideration basically have the following model structure:

\[
\begin{align*}
\dot{\eta} &= \left[-1 + \epsilon \Omega(\eta, v)\right] \eta \\
\dot{v} &= \epsilon \left[ v + S(\eta, v) \right],
\end{align*}
\]

where \( \Omega \) is of order \( O(|\eta| + |v|) \) and \( S \) of order \( O(\eta^2 + v^2) \) for small \( (\eta, v) \) (compare with equation (5.32)). Note that for equation (5.23) \( \eta = 0 \) is an invariant manifold \( \mathcal{M} \) on which the motion is slow. This slow motion is very different when \( \epsilon > 0 \) as opposed to \( \epsilon = 0 \). This difference is the origin of the singular nature of the problem. If the system were completely uncoupled, then the long-time behavior of the initial value problem would be completely described by the solution of the \( v \) equation. For the coupled system, it is unclear which particular solution of the \( v \) equation is approached asymptotically.

Although a complete decoupling would be ideal, it is too much to expect. However, it is sufficient to decouple the slow motion. For the moment, assume there exists a smooth change of variables

\[
(\eta, v) \rightarrow (\eta, \eta_c), \quad v = f(\eta, \eta_c; \epsilon), \quad f(0, \eta_c; \epsilon) = \eta_c,
\]

which partially decouples the flow by placing equation (5.23) into the following form:

\[
\begin{align*}
\dot{\eta} &= \left[-1 + \epsilon \Omega(\eta, v)\right] \eta \\
\dot{\eta}_c &= \epsilon \left[ \eta_c + \epsilon S(\eta_c) \right].
\end{align*}
\]

With this form of the equations, it is clear that as \( t \to \infty \)

\[
(\eta(t; \eta_0), \eta_c(t; \eta_0)) \rightarrow (0, \eta_c(t; \eta_0))
\]

exponentially fast as \( \exp(-t) \), where \( \eta_0 = (\eta(0), \eta_c(0)) \) and \( \eta_0 = (0, \eta_c(0)) \). Thus long-time motion through an arbitrary point \( (\eta(0), \eta_c(0)) \) can be tracked by following motion through the point \( (0, \eta_c(0)) \) on the slow manifold.

Following the work of Fenichel [21], it has become the practice in the literature to describe this partial decoupling in a coordinate-free manner. One introduces a family of curves indexed by the slow manifold \( \mathcal{M} \):

\[
\mathcal{F}_v : [-1, 1] \rightarrow W^{cs}(\mathcal{M}) \quad \forall v \in \mathcal{M},
\]
which satisfy $F_v(0) = v$. These curves are $C^2$ in $(\eta, v; \epsilon)$ and are characterized by the requirement that a point $(\eta, \tilde{v})$ lies on the curve $F_v^s$ if and only if

$$\| F'(\eta, \tilde{v}; \epsilon) - F'(0, v; \epsilon) \| \to 0$$

at a fast rate as $t \to \infty$. The curve $F_v$ is called a Fenichel stable fiber through base point $v \in M$. Clearly, in our partially decoupled coordinate system, the fiber $F_v$ is given by

$$F_v = \{ (\eta, \tilde{v}) : \tilde{v} = f(\eta, v; \epsilon) \}$$

where $f(\eta, v; \epsilon)$ is the coordinate transformation. We will refer to the coordinates of the partial decoupling as “fiber coordinates.” Although the geometric role of fibers is perhaps clearer in the coordinate-free description, each of their properties becomes immediately apparent in the fiber coordinate system. Moreover, the proof of existence of the fibration reduces to the analytical construction of a coordinate transformation.

Statement of the Fibration Theorem

**Theorem 5.4** For all $\epsilon \in [0, \epsilon_0]$ the $C^\ell$ manifold $W_\epsilon^{cu}$ admits a $C^{\ell-2}$ coordinate system

\[
\begin{align*}
  v_u &= \eta_u, \quad \eta_u \in [-\eta_0, \eta_0], \\
  v_c &= f^u(\eta_u, \eta_c; \epsilon), \quad \eta_u \in E^c
\end{align*}
\]

such that the submanifold $\mathcal{M}_\epsilon$ corresponds to $\eta_u = 0$, and the flow on $W_\epsilon^{cu}$ decouples in the following manner:

\[
\begin{align*}
  \dot{\eta}_u &= [\sigma^* + \Gamma^\delta(\eta_u, \eta_c; \epsilon)]\eta_u \\
  \dot{\eta}_c &= A\eta_c + S^\delta_c(\eta_c; \epsilon)
\end{align*}
\]

where $\eta, \Gamma^\delta, S^\delta_c$, and their first derivatives are of order $O(\epsilon + \delta)$. A similar statement holds for $W_\epsilon^{cs}$.

Setup and Proof of the Fibration Theorem

The manifold $\mathcal{M}_\epsilon$ is a codimension 1 submanifold of $W_\epsilon^{cu}$, and this implies that locally $W_\epsilon^{cu}$ can be viewed as a product $\mathcal{M}_\epsilon \times (-z_0, z_0)$ with local coordinates $(v_c, \eta_u)$, where $v_c$ are coordinates on $\mathcal{M}_\epsilon$ and $\eta_u$ is the coordinate on an interval $(-z_0, z_0)$. The setup to prove the fibration of $W_\epsilon^{cu}$ over the slow manifold $\mathcal{M}_\epsilon$ begins by writing the flow on $W_\epsilon^{cu}$ in terms of these local coordinates.
The manifold $W^{cu}_r$ is given as a graph over $(v_u, v_c)$

$$v_s = h_s(v_u, v_c; \epsilon) ,$$

and the flow on $W^{cu}_r$ is given by restricting equations (5.9) to this graph:

$$\begin{align*}
\dot{v}_u &= \sigma^\delta u v_u + S^\delta u (v_u, v_c; \epsilon) \\
\dot{v}_c &= Av_c + S^\delta_c (v_u, v_c; \epsilon)
\end{align*}$$

where $S^\delta u (v_u, v_c; \epsilon) := R^\delta u (v_u, h_s(v_u, v_c; \epsilon), v_c; \epsilon)$ are the restrictions of $R^\delta u$ to the graph.

Similarly, using $(v_u, v_c)$ as local coordinates on $W^{cu}_r$, the submanifold $M_r$ is given as a graph over $v_c$,

$$v_u = h^r_u(v_c; \epsilon) ,$$

and the flow on $M_r$ is given by

$$\dot{v}_c = Av_c + S^\delta_c (v_c; \epsilon)$$

where $S^\delta_c (v_c; \epsilon) = S^\delta_c (h^r_u(v_c; \epsilon), v_c; \epsilon)$ is the restriction of $S^\delta_c$ to the graph.

Since $M_r$ is an invariant submanifold of $W^{cu}_r$, we introduce coordinates given by $(v_c, \eta_u)$ where $v_c$ are coordinates on $M_r$ and

$$\eta_u = v_u - h^r_u(v_c; \epsilon) .$$

In terms of these coordinates the submanifold $M_r$ is given by $\eta_u = 0$.

To derive the equations for the flow on $W^{cu}_r$ in terms of $(v_u, v_c)$, we have to differentiate $\eta_u$ in equation (5.27) along solutions of (5.25). This gives rise to a technical difficulty, since solutions are only continuous in time with values in $H^1$, not $C^1$. (Solutions are $C^1$ in the sense of distributions.) However, if we start with initial data in $H^3$, we have solutions that are $C^1$ in time with values in $H^1$. We also have that solutions to equation (5.25) have continuous dependence on the initial data in the space $H^1$. Therefore we can derive the equation for the flow on $W^{cu}_r$ by assuming that our initial data are in $H^3$ and then using continuous dependence on the initial data to conclude that the equations hold in the sense of distributions for initial data in $H^1$.

Proceeding with the derivation, differentiate (5.27) with respect to $t$,

$$\dot{\eta}_u = \dot{v}_u - Dh^r_u(v_c; \epsilon) \dot{v}_c ,$$
where $Dh^c_u$ denotes the derivative of $h^c_u$ with respect to $v_c$. By equation (5.25) we can write the above equation as

$$\dot{h}_u = \sigma'_u v_u + S^\delta_u - Dh^c_u (Av_u + S^\delta_c).$$

This equation can be simplified by using an identity for $h^c_u$ that comes from the invariance of $\mathcal{M}_e$ as a submanifold of $W^{cu}_v$. To derive this identity, we consider the flow on $\mathcal{M}_e$ given by $(h^c_u(v_c(t)), v_c(t))$. Differentiate with respect to $t$ and substitute into equation (5.25) to obtain

$$Dh^c_u \dot{v}_c = \sigma'_u h^c_u + S^\delta_u (h^c_u, v_c; \epsilon)$$

$$\dot{v}_c = Av_c + S^\delta_c (h^c_u, v_c; \epsilon).$$

These equations imply that $h^c_u$ satisfies

$$Dh^c_u (Av_c + \tilde{S}_c^\delta) = \sigma'_u h^c_u + \tilde{S}_u^\delta$$

where

$$\tilde{S}_u^\delta (v_c; \epsilon) = S^\delta_u (h^c_u, v_c; \epsilon).$$

Using this identity for $h^c_u$, we can simplify equation (5.28) and describe the flow on $W^{cu}_v$ in terms of $(v_c, \eta_u)$ equations:

$$\dot{\eta}_u = \sigma'_u \eta_u + \Omega_u (\eta_u, v_c; \epsilon)$$

$$\dot{v}_c = Av_c + \Omega_c (\eta_u, v_c; \epsilon)$$

where

$$\Omega_u = S^\delta_u (\eta_u + h^c_u, v_c; \epsilon) - S^\delta_u - Dh^c_u \left[ S^\delta_c (\eta_u + h^c_u, v_c; \epsilon) - \tilde{S}_c^\delta \right]$$

$$\Omega_c = S^\delta_c (\eta_u + h^c_u, v_c; \epsilon)$$

are $C^{\ell-1}$. Note that $\Omega_u(0, v_c; \epsilon) = 0$, $\Omega_c(0, v_c; \epsilon) = \tilde{S}_c^\delta(v_c; \epsilon)$, and that all the terms in $\Omega_u,c$ and their first derivatives are either of order $\epsilon$ or $\delta$ due to the localization. This implies that equations (5.30) can be written as

$$\dot{\eta}_u = [\sigma'_u + \Omega_u (\eta_u, v_c; \epsilon)] \eta_u$$

$$\dot{v}_c = Av_c + \Omega_c (\eta_u, v_c; \epsilon)$$

where $\tilde{\Omega}_u, \Omega_c$, and first derivatives of $\Omega_c$ are of order $O(\epsilon + \delta)$. 


The above equations will be used to prove the fibration of $W^{cu}_{r}$ over $\mathcal{M}_{r}$, which is equivalent to finding another coordinate system $(\eta_{c}, \eta_{u})$ where the above system decouples in the following manner:

\begin{equation}
\dot{\eta}_{u} = [\sigma' + \Gamma(\eta_{u}, \eta_{c}; \epsilon)]\eta_{u}
\end{equation}
\begin{equation}
\dot{\eta}_{c} = A\eta_{c} + S_{c}^{\delta}(\eta_{c}; \epsilon).
\end{equation}

Here $\Gamma := \Omega_{u}$ is evaluated in the new coordinates.

If this change of variable exists, then $\eta_{u}(t)$ approaches 0 at a fast rate, at least $\exp[\sigma t/2]$ as $t \to -\infty$. This implies that the equation for $v_{c}$ approaches the equation for $\eta_{c}$ exponentially fast. Thus, we let $\gamma_{c} = v_{c} - \eta_{c}$ and have

$$
\dot{\gamma}_{c} = A\gamma_{c} + \Omega_{c}(\eta_{u}, \gamma_{c} + \eta_{c}; \epsilon) - \Omega_{c}(0, \eta_{c}; \epsilon),
$$
which can be integrated from $-\infty$ to $t$,

$$
v_{c}(t) - \eta_{c}(t) = \int_{-\infty}^{t} \exp[-A(t - s)] \left( \Omega_{c}(\eta_{u}, \gamma_{c} + \eta_{c}; \epsilon) - \Omega_{c}(0, \eta_{c}; \epsilon) \right) ds.
$$

The coordinate change is given by $v_{c}(0)$:

\begin{equation}
f^{u}(\eta_{u}(0), \eta_{c}(0); \epsilon) := \eta_{c}(0) + \int_{-\infty}^{0} \exp[-As] \left( \Omega_{c} - S_{c}^{\delta} \right) ds,
\end{equation}

where the right-hand side depends implicitly on $f^{u}$. Since the terms under the integral sign are of $O(\epsilon + \delta)$, we can find $f^{u}$ as the fixed point of the above equation with the property that

\begin{equation}
f^{u}(\eta_{u}, \eta_{c}; \epsilon) = \eta_{c} + \tilde{f}^{u}(\eta_{u}, \eta_{c}; \epsilon),
\end{equation}

where $\tilde{f}^{u}$ is a $C^{r-2}$ function whose first derivatives are of order $O(\delta)$ and $\tilde{f}(0, \eta_{c}; \epsilon) = 0$. This is the basic idea of the proof that will be presented below.

\textbf{Remark.} The fibers are then given by

$$
v_{u} = \eta_{u}
v_{c} = f^{u}(\eta_{u}, \eta_{c}; \epsilon)
$$

\textbf{Proof of Theorem 5.4:} Fix an orbit $\eta_{c}$ on $\mathcal{M}_{r}$ and consider the system

$$
\dot{\eta}_{u} = [\sigma' + \Omega_{u}(\eta_{u}, \gamma_{c} + \eta_{c}; \epsilon)] \eta_{u}
\end{equation}
\begin{equation}
\dot{\gamma}_{c} = A\gamma_{c} + \Omega_{c}(\eta_{u}, \gamma_{c} + \eta_{c}; \epsilon) - \Omega_{c}(0, \eta_{c}; \epsilon)
\end{equation}$$

$$
$$
where $\gamma_c = v_c - \eta_c$. Set $a(\eta_c; \epsilon) = \Omega_u(0, \eta_c; \epsilon)$; then the above equation can be written as

$$
\dot{\eta}_u = [\sigma_u^e + a(\eta_c; \epsilon) + \Psi(\eta_u, \gamma_c, \eta_c; \epsilon)] \eta_u
$$

$$
\dot{\gamma}_c = A\gamma_c + \Phi(\eta_u, \gamma_c, \eta_c; \epsilon)
$$

where $\Psi(0, 0, \eta_c; \epsilon) = 0$, $\Phi(0, 0, \eta_c; \epsilon) = 0$, and $a$, $\Psi$, and $\Phi$ are of order $O(\epsilon + \delta)$.

The proof now proceeds exactly as in the proof of Theorem 5.2, where we defined a norm on continuous functions with values in $H^1$:

$$
\beta = (\gamma_c, \eta_u)
$$

$$
\|\beta\|_\lambda = \sup_{t \leq 0} \{\exp[-\sigma t/\lambda]\|\beta(t)\|_{H^1}\}
$$

and we set up a Newton’s iteration given by

$$
\eta_u^{k+1}(t) = G(t, 0) \eta_u(0) + \int_0^t G(t, s) \Psi(\eta_u^k, \gamma_c^k, \eta_c; \epsilon) \eta_u^k ds
$$

$$
\gamma_c^{k+1} = \int_{-\infty}^t \exp[A(t - s)] \Phi(\eta_u^k, \gamma_c^k, \eta_c; \epsilon) ds
$$

where $G(t, s) = \exp[\int_s^t (\sigma_u^e + a)]$.

The convergence and smoothness of the sequence $\beta^k = (\gamma_c^k, \eta_u^k)$ follows, as in the proof of Theorem 5.2, from the following:

- For $|\eta_u(0)| \leq \delta$ we have a well-defined sequence. Assume that for some $k$ we have $\|\beta^k\|_{m_0} \leq C \delta$ where $m_0 = n_0/\ell$; then the equation for $\beta^{k+1}$ implies

$$
|\eta_u^{k+1}(t)| \leq C(\delta + \epsilon) \int_0^t \exp \left[ \frac{\sigma}{2} (t - s) + \frac{\sigma}{m_0} s \right] \|\beta^k\|_{m_0} ds
$$

$$
+ c\delta \exp[\sigma t/2]
$$

$$
\|\gamma_c^{k+1}(t)\|_{H^1} \leq C(\delta + \epsilon) \int_{-\infty}^t m_0 \exp \left[ \frac{\sigma}{n_0} (t - s) + \frac{\sigma}{m_0} s \right] \|\beta^k\|_{m_0} ds
$$

and for $\epsilon \leq \epsilon_0$, defined in Theorem 5.2, we have

$$
\|\beta^{k+1}\|_{m_0} \leq C \delta + C \delta m_0 n_0 \|\beta^k\|_{m_0}.
$$

Since $\delta$ was chosen to be $\delta = 1/(C n_0^2)$, where $C$ is a large constant, we have

$$
\|\beta^{k+1}\|_{m_0} \leq C \delta + \frac{1}{2} \|\beta^k\|_{m_0}
$$

which implies that $\beta^k$ is a bounded sequence.
• The sequence $\beta^k$ converges. The functions $\Phi$ and $\Psi$ are $C^{\ell-1}$, and all the terms in these expressions are either localized in a region of size $\delta$ or have a linear growth with a coefficient of order $\epsilon$. Therefore, the above argument implies that

$$
||\beta^{k+1} - \beta^k||_{m_0} \leq \frac{1}{2} ||\beta^k - \beta^{k-1}||_{m_0},
$$

which gives the convergence of $\beta^k \to \beta$, a continuous function in $t$ with values in $H^1$, and $||\beta||_{m_0} \leq c\delta$.

• $\beta$ is smooth with respect to $\eta_c(0), \eta_u(0)$, and $\epsilon$. The difficulty here is that $||\eta_c(t)||_{H^1}$ can grow for $t < 0$ at a rate $\exp[-\sigma t/n_0]$ and that $a, \Phi,$ and $\Psi$ depend on $\eta_c(t)$. If we differentiate $\beta^{k+1}$ with respect to $\eta_c(0)$, we obtain

$$
\eta^{tk+1}_u(t) = \eta_u(0) G(t, 0) \int_0^t a' \left[ G(t, s) \Psi \eta^{tk}_u ds \right ] + \int_0^t G(t, s) \left[ \left( \Psi' \eta'_c + \Psi' \eta'^{tk}_u + \Psi' \gamma^{tk}_c \right) \eta^{tk}_u + \Psi \eta'^{tk}_u \right ] ds
$$

and

$$
\gamma^{tk+1}_c(t) = \int_{-\infty}^t \exp[A(t-s)] \left[ \Phi' \eta'_c + \Phi' \eta'^{tk}_u + \Phi' \gamma^{tk}_c \right ] ds
$$

where the primes denote various derivatives.

The above equation cannot be estimated with the $|| \cdot ||_{m_0}$-norm because of $a'$ and $\eta'_c$, which grow at a rate $\exp[-\sigma t/n_0]$ for $t < 0$. However, we can estimate the above equation with respect to the $|| \cdot ||_{2m_0}$-norm, which has a slower rate of decay,

$$
||\beta^{tk+1}(t)||_{H^1} \leq C\delta \exp[(\sigma/2 - \sigma/n_0)t] + C\delta \int_0^t \exp[\sigma(t-s)/2] \left[ \exp[(\sigma/m_0 - \sigma/n_0)s] + ||\beta^{tk+1}(s)|| \right ] ds
$$

and

$$
+ C\delta \int_{-\infty}^t \exp[\sigma/n_0(t-s)] \left[ \exp[(\sigma/m_0 - \sigma/n_0)s] + ||\beta^{tk}(s)|| \right ] ds.
$$

Furthermore, since $2m_0 > n_0$, the integrals converge and give

$$
||\beta^{tk+1}(t)||_{H^1} \leq C\delta \exp[\sigma t/2m_0] (1 + ||\beta^k||_{2m_0}).
$$
which implies the boundedness of $\|\beta_j\|_{2m_0}$. Similarly, we can estimate the $(j - 1)$-derivative of $\beta^k$ in the $\|\cdot\|_{2m_0}$ norm, and all the integrals will converge provided $jm_0 < n_0$. Since all the terms in equation (5.36) are $C^{\epsilon-1}$, this procedure can be continued to obtain $\beta$ in $C^{\epsilon-2}$.

Now we can define the fibration of the manifold $W_{c}^{cu}$ by setting

$$f^u(\eta_c(0), \eta_u(0); \epsilon) = \eta_c(0) + \gamma_c(0),$$
$$v_c(0) = \eta_c(0) + \int_{-\infty}^{0} \exp[-As] [\Omega_c(\eta_u, \gamma_c + \eta_c; \epsilon) - \Omega_c(0, \eta_c; \epsilon)] \, ds.$$

The invariance of the function $f^u$ under the flow follows from the definition of $f$ and from reparametrizing time $t \to t + \tau$. Therefore

$$v_c(t) = f^u(\eta_c(t), \eta_u(t); \epsilon),$$

which is well-defined provided the initial data $\eta_u(0)$ is sufficiently small. This invariance implies that if we use $(\eta_c, \eta_u)$ as a coordinate system instead of $(v_c, \eta_u)$, we’ll obtain a decoupled system of equations (5.33) that describe the flow on $W_{c}^{cu}$,

$$\dot{\eta}_u = \left[\sigma'_u + \Gamma_c(\eta_u, \eta_c; \epsilon)\right] \eta_u$$
$$\dot{\eta}_c = A\eta_c + S'_c(\eta_c; \epsilon).$$

### 5.3 Stable Manifold to $Q$ in $M_c$

For $\alpha < 1/\omega$, the point $Q$, which is stationary under the flow (2.1), is given in terms $(J, \theta, u)$ by

$$J_q = -\frac{\epsilon}{2\omega} \sqrt{1 - \alpha^2 \omega^2} + O(\epsilon^2)$$
$$\theta_q = \arctan \left( \sqrt{\frac{1 - \alpha^2 \omega^2}{\alpha \omega}} \right) - \pi + O(\epsilon)$$
$$u = 0.$$

By linearizing equation (5.8) around $Q$ we obtain that $Q$ is a saddle point with a two-dimensional unstable manifold and a codimension 2 stable manifold. The unstable manifold intersects the plane of constants $\Pi_c$ along the curve $C_{c}^{u}$, and intersects $W_{c}^{cu}$ along a curve tangent to the $v_u$-direction. The local
The Setup

Recall that the stable manifold of $Q$ in the plane $\Pi_c$ is parametrized by

$$C^s_r = \{ y = (j, \theta) : y = y_*(s; \nu) \}$$

where $\nu = \sqrt{r}$ and $y_*$ is given in equation (3.11),

$$y_{*, r} = Y_1(j_*, \theta_*; \nu)$$
$$\theta_{*, r} = Y_2(j_*, \theta_*; \nu)$$

and where $s = \exp[\lambda \tau]$. The flow on $\mathcal{M}_c$ in terms of $(j, \theta, v_\circ)$ is derived from equations (5.13) by substituting $J = \nu j$:

$$j_t = \nu Y_1(j, \theta; \nu) + \tilde{N}_1(j, \theta, v_\circ; \nu)$$
$$\theta_t = \nu Y_2(j, \theta; \nu) + \tilde{N}_2(j, \theta, v_\circ; \nu)$$
$$v_{\circ t} = L_\circ v_\circ + V_\circ v_\circ + \tilde{N}_3(j, \theta, v_\circ; \nu).$$

To estimate the size of $\mathcal{W} = W^s(Q) \cap \mathcal{M}_c$, we use the coordinates $(\beta, r)$ on the plane $\Pi_c$ defined in (3.14) and (3.16). In terms of these variables, the flow on $\mathcal{M}_c$ in a neighborhood of $C^s_r$ is given by the equations

$$\dot{r} = \nu a(\beta, j_\circ) r + O(\nu r^2 + v_\circ^2)$$
$$\dot{\beta} = \nu \lambda \beta + \nu c(\beta, \nu) r + O(\nu r^2 + v_\circ^2)$$
$$\dot{v}_\circ = L_\circ v_\circ + V_\circ v_\circ + O(\nu r v_\circ + \nu v_\circ^2 + v_\circ^3)$$

where

$$V_\circ = -4 \nu j_*(s, \nu) S + \frac{\nu^2 \sin(\theta_*(s; \nu))}{\sqrt{\omega^2 + \nu j_*(s; \nu)}} J,$$

and where $a$ and $c$ are smooth functions in $(\beta, \nu)$. Finally, to construct a local stable manifold to $Q$ that contains $C^s_r$, we linearize the flow along the flow on $C^s_r$,

$$\beta_*(t; \nu) = \beta_0 \exp[\nu \lambda t]$$
where \(0 \leq \beta_0 \leq s_0\) and \(t \geq 0\). Introducing the variable
\[
\gamma = \beta - \beta_*(t, \nu)
\]
equations (5.38) can be written as
\[
\begin{align*}
\dot{\gamma} &= \nu \lambda \gamma + \nu c_*(t, \nu) r + N_{*1}(t, r, \gamma, \nu_0; \nu) \\
\dot{\nu}_0 &= L_c \gamma + V_*(t, \nu, \beta_0) \nu_0 + N_{*3}(t, r, \gamma, \nu_0; \nu)
\end{align*}
\]
where
\[
\begin{align*}
N_{*1} &= O(\nu r^2 + \nu \gamma^2 + \nu_0^2) \\
N_{*2} &= O(\nu r^2 + \nu \gamma^2 + \nu_0^2) \\
N_{*3} &= O(\nu r^2 + \nu \gamma^2 + \nu \nu_0^2 + \nu_0^3)
\end{align*}
\]

**Estimate on the Linear Flow**

The linear part of equation (5.39) consists of a coupled system of ODEs and a PDE with a time-dependent coefficient.

**ODE Estimates.** The fundamental solution of the ODEs is given by the
\[
2 \times 2 \text{ matrix}
\begin{bmatrix}
A_*(t, s; \beta_0, \nu) & 0 \\
\Gamma(t, s; \beta_0, \nu) & \exp[\nu \lambda (t - s)]
\end{bmatrix}
\]
where
\[
A_*(t, s; \beta_0, \nu) = \exp \left[ \int_s^t \nu a_* ds' \right]
\]
\[
\Gamma(t, s; \beta_0, \nu) = \nu \int_s^t \exp[\nu \lambda (t - \alpha)] c_*(\alpha) A_*(\alpha, s) d\alpha.
\]

To estimate \(A_*\) we note that \(a_*\) can be written
\[
a_* = \mu + \bar{a}
\]
\[
|\bar{a}| \leq C \beta_0 \exp[\nu \lambda t] \leq C s_0 \exp[\nu \lambda t]
\]
where \(\mu\) is given in equation (3.10). This implies the following estimate on \(A_*\) for \(t, s \geq 0\):
\[
c_1 \exp[\nu \mu (t - s)] \leq A_* \leq c_2 \exp[\nu \mu (t - s)],
\]
PERSISTENT HOMOCLINIC ORBITS

where $c_1$ and $c_2$ are constants independent of $\epsilon$.

To obtain a bound on $\Gamma$ we recall that

$$|c_*| \leq CS_0 \exp[2\nu \lambda t]$$

which implies for $t, s \geq 0$

$$|\Gamma| \leq C \nu \left| \int_s^t \exp(\nu \lambda(t-x) + 2\lambda x + \mu(x-s)) \, dx \right|$$

for $s, t \geq 0$. Since

$$\lambda + \mu = -2\alpha_\nu + O(\nu^2) < 0,$$

we can bound the above integral for $s, t \geq 0$ as follows:

$$|\Gamma| \leq C \nu |t - s| \exp[\nu \lambda(t - s)]$$

where $C$ is independent of $\epsilon$.

**PDE Estimates.** The difficulty in this estimate is the time dependence of the linear operator. To estimate the growth rate of the fundamental solution of the PDE, it is easiest to represent the linear operator in terms of its Fourier coefficients $L(k)$ given by

$$L(k) = \begin{bmatrix} -\epsilon d(k) & -k^2 + \epsilon \alpha_1 \\ k^2 - 4\omega^2 - \epsilon \alpha_1 - \nu \alpha_2 & -\epsilon d(k) \end{bmatrix}$$

where $\alpha_1$ and $\alpha_2$ are smooth functions of $\exp[\nu \lambda t]$ and $\nu$, $d(k) \geq \alpha$ (the symbol of $\hat{D}$), and where $k = 2, 3, \ldots$. The operator $L(k)$ has eigenvalues given by

$$\lambda_{1,2} = -\epsilon d(k) \pm iD(k)$$

$$D(k) = \sqrt{(k^2 - \epsilon \alpha_1)(k^2 - 4\omega^2 - \epsilon \alpha_1 - \nu \alpha_2)}$$

and can be diagonalized by the matrix $U(k)$

$$U(k) = \begin{bmatrix} 1 & 1 \\ -k^2 + \epsilon \alpha_1 & k^2 - \epsilon \alpha_1 \end{bmatrix}$$

giving $U^{-1}AU = \Lambda$, the diagonal matrix $\text{diag} \{\lambda_1, \lambda_2\}$. Note that $U$ is a bounded operator on the space $[\text{span} \{\Pi_c, e_{\omega}, e_{\epsilon}\}]^1$; thus if we change variables from $v_o$ to $w$ where $\hat{v}_o = U(k) \hat{w}(k)$, we obtain

$$\hat{w}(k) = \Lambda \hat{w}(k) - U^{-1} \hat{U} \hat{w}(k).$$
The term $U^{-1} \dot{U}$ can be bounded as follows:

$$\left| U^{-1} \dot{U} \right| \leq \frac{C}{k^2} \epsilon \exp[\nu \lambda t]$$

where $C$ is independent of $k$ and $\epsilon$. This implies that the solution, which is given by the integral equation

$$\tilde{\omega}(k) = F(t, s; k) \tilde{\omega}_0(k) + \int_s^t F(t, s'; k) U^{-1} \dot{U} \tilde{\omega} ds',$$

can be estimated for $t \geq s \geq 0$ by

$$|\tilde{\omega}(k)| \leq C \exp[-\epsilon d(k)(t-s)] |\tilde{\omega}_0(k)| + C \int_s^t \epsilon \exp[-\epsilon d(k)(t-s') + \nu \lambda s'] |\tilde{\omega}(k)| ds'.$$

Since $\lambda < 0$ this implies that for $t \geq s \geq 0$

$$|\tilde{\omega}(k)| \leq C \exp[-\epsilon d(k)(t-s)] |\tilde{\omega}_0(k)| .$$

Finally, if we denote the fundamental solution of the PDE by $U(t, s)$, we have the estimate

$$\|U(t, s)v_{io}\|_{H^1} \leq C \exp[-\epsilon \alpha(t-s)] \|v_{io}\|_{H^1}$$

since $d(k) \geq \alpha$.

**Size of the Stable Manifold**

**THEOREM 5.5** The point $Q$ has a $C^1$ local stable manifold in $\mathcal{M}_e$ that can be parametrized by $(\beta, v_{io})$,

$$\mathcal{W} = \{(r, \beta, v_{io}) : r = f(\beta, v_{io})\} ,$$

for all $\beta \in [0, s_0]$ and $\|v_{io}\|_{H^1} \in [0, \epsilon^{3/4}]$. Moreover, $f(\beta, 0) = 0$ and $|r| \leq c\epsilon$.

**PROOF:** Given the linear estimates and the normal form transformation, the proof of this theorem becomes standard. From equations (5.39) we set up integral equations for $t \geq 0$:

$$r = \int_t^\infty A_*(t, s) N_{*1} ds$$

$$\gamma = \int_0^t \left[ \exp[\nu \lambda(t-s)] N_{*2} + \Gamma(t, s) N_{*1} \right] ds$$

$$v_o = U(t, 0)v_{io} + \int_0^t U(t, s) N_{*3} ds .$$
Using the linear estimates on $A_\ast$, $\Gamma$, and $U$ and the order of magnitude of $N_\ast$, we have:

$$|r| \leq C \int_t^\infty \exp[\nu \mu(t-s)] [\nu r^2 + \nu \gamma^2 + \|v_o\|_{H^1}^2] \, ds$$

$$|\gamma| \leq C \int_0^t \exp[\nu \lambda(t-s)] [1 + \nu \lambda(t-s)][\nu r^2 + \nu \gamma^2 + \|v_o\|_{H^1}^2] \, ds$$

$$\|v_o\|_{H^1} \leq C \int_0^t \exp[-\nu \alpha(t-s)] \left[ \nu (r^2 + \gamma^2 + \|v_o\|_{H^1}^2) + \|v_o\|_{H^1}^3 \right] \, ds$$

Scaling $(r, \gamma, v_o)$ by $\sqrt{\epsilon}$ yields $O(\sqrt{\epsilon})$ a priori estimates on solutions. Since we only need $O(\epsilon^\mu)$, $\mu < 1$, estimates, these inequalities imply the following for $t \geq 0$:

$$|r| \leq C \epsilon \exp[-\epsilon \alpha t]$$

$$|\gamma| \leq C \epsilon \exp[-\epsilon \alpha t]$$

$$\|v_o\|_{H^1} \leq C \epsilon^{3/4} \exp[-\epsilon \alpha t]$$

provided $\|v_{i.o}\| \leq C \epsilon^{3/4}$. This a priori estimate allows us to apply Newton’s iteration to the integral equations to show that they have a unique solution that satisfies the same estimate. Define

$$(5.40) \quad f(\beta_0, v_{i.o}) = \int_0^\infty A_\ast(0, s)N_{*1} \, ds$$

where the dependence on $\beta_0$ is implicit in $A_\ast$ as well as $N_{*1}$. The differentiability of the function $f$ follows from the differentiability of $A_\ast$ and $N_{*1}$.

6 Global Integrable Theory

In this section we briefly survey results from integrable theory. In order to prove the persistence of homoclinic orbits, all that is needed from this section are the analytic expression of the homoclinic orbit given in equation (6.11) and the analytic expression of the normal to the center stable manifold along the homoclinic orbit given in Section 6.8.

6.1 Lax Pair

The unperturbed ($\epsilon = 0$) NLS equation is a Hamiltonian system on the function space $H_{\epsilon,p}^1$,

$$(6.1) \quad -iq_t = \frac{\delta}{\delta q} H,$$
with the Hamiltonian $H$ given by
\[
H = \int_0^{2\pi} \left[ q_x \bar{q}_x - (q \bar{q})^2 + 2\omega^2 q \bar{q} \right] dx.
\]

It is well-known that this is a completely integrable Hamiltonian system, a fact whose verification begins with the Lax pair

\[
\begin{align*}
\varphi_t &= U^{(\lambda)} \varphi \\
\varphi_x &= V^{(\lambda)} \varphi ,
\end{align*}
\]

where
\[
U^{(\lambda)} := i\lambda \sigma_3 + i \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix},
\]
\[
V^{(\lambda)} := i [2\lambda^2 - (q \bar{q} - \omega^2)] \sigma_3 + \begin{pmatrix} 0 & 2i\lambda q + q_x \\ 2i\lambda q - q_x & 0 \end{pmatrix},
\]

and $\sigma_3$ denotes the third Pauli matrix $\sigma_3 := \text{diag}(1, -1)$. This overdetermined system is compatible ($\partial_t \varphi_x = \partial_x \varphi_t$) if and only if the coefficient $q$ satisfies the NLS equation. Consequently, one can use this linear system to develop representations of solutions $q(x, t)$ of the NLS.

This observation is the starting point for the inverse spectral representation of $q$, which forms the basis of completely integrable soliton mathematics. An overview in the NLS context can be found in [51].

### 6.2 The Zakharov-Shabat Spectral Problem

We now focus on the "spatial flow" (6.2). The integration of the NLS equation is accomplished by employing the spectral theory of the differential operator $\widehat{L} = \widehat{L}(q)$,
\[
\widehat{L} := -i\sigma_3 \frac{d}{dx} - \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix},
\]

which is viewed as an operator on $L^2(\mathbb{R})$ with dense domain $H^1$. In this $L^2$ setting, the spectrum $\sigma(\widehat{L})$ is defined as the closure of the set of complex $\lambda$ for which there exists a solution of

\[
\widehat{L} \psi = \lambda \psi ,
\]

which is bounded for all $x \in (-\infty, +\infty)$. Since the coefficient $q$ is a periodic function of $x$, Floquet theory can be used to characterize this spectrum.
Floquet theory begins from the fundamental matrix \( M = M(x; \lambda; q) \), which is defined as the \( 2 \times 2 \) matrix valued solution of the linear problem (6.3) whose initial value at \( x = 0 \) is the identity matrix. Next, one introduces the transfer matrix \( T \),

\[
T(\lambda; q) = M(2 \pi; \lambda; q).
\]

Then the spectrum \( \sigma(\hat{L}) \) can be characterized as the set of all \( \lambda \) for which the \( 2 \times 2 \) matrix \( T \) has eigenvalues on the unit circle. Since \( \det T = 1 \), this is in turn determined by a single scalar function called the Floquet discriminant:

\[
\Delta : \mathbb{C} \times H^1_{c,p} \to \mathbb{C} \quad \text{by} \quad \Delta(\lambda; q) = \text{tr}[T(\lambda; q)].
\]

In terms of \( \Delta \), the spectrum is given by

\[
\sigma(\hat{L}(q)) = \{ \lambda \in \mathbb{C} : \Delta(\lambda, q) \text{ is real and } -2 \leq \Delta \leq +2 \}.
\]

**Proposition 6.1**

i. The Floquet discriminant \( \Delta(\lambda; q, \bar{q}) \),

\[
\Delta : \mathbb{C} \times H^1_{c,p} \times H^1_{c,p} \to \mathbb{C}.
\]

is entire in \( \lambda, q, \) and \( \bar{q} \).

ii. The first variation admits the following representation:

\[
\delta \Delta(\lambda; q, \bar{q}) = \int_0^{2\pi} \left[ \frac{\delta \Delta}{\delta q(x)} \delta q(x) + \frac{\delta \Delta}{\delta \bar{q}(x)} \delta \bar{q}(x) \right] dx.
\]

where

\[
\frac{\delta}{\delta q(x)} \Delta(\lambda; q, \bar{q}) = -\frac{i}{2} \text{tr} \left[ M^{-1}(x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} M(x + 2\pi) \right],
\]

\[
\frac{\delta}{\delta \bar{q}(x)} \Delta(\lambda; q, \bar{q}) = -\frac{i}{2} \text{tr} \left[ M^{-1}(x) \begin{pmatrix} 0 & 0 \\ iq(x) & 0 \end{pmatrix} M(x + 2\pi) \right].
\]

Here \( M(x) = M(x; \lambda, q, \bar{q}) \) denotes the fundamental matrix. A similar representation exists for \( \frac{d}{d\lambda} \Delta \).

To prove part (i) of this proposition, one writes the linear differential equation (1.4) for \( M \) as the integral equation

\[
M(x) = \exp(i \sigma_3 \lambda x) + \int_0^x \exp[i \sigma_3 \lambda(x - y)] \begin{pmatrix} 0 & iq(y) \\ iq(y) & 0 \end{pmatrix} M(y) dy.
\]
Iteration produces a formal series representation, each term of which consists of polynomials in $q$ and $q$ of the form

$$q(y_1) \cdots q(y_n)q(y_{n+1}) \cdots q(y_m).$$

The series is shown to converge uniformly [55, 24, 44]; hence, $\Delta$ is entire in $q$ and $q$. Analyticity in $\lambda$ is established similarly.

The first variation in part (ii) of the theorem is computed as follows [51]:

$$(\hat{L} - \lambda)M = 0, \quad M(0) = I$$

$$(\hat{L} - \lambda)\delta M = \begin{pmatrix} 0 & \delta q \\ -\delta q & 0 \end{pmatrix} M, \quad \delta M(0) = 0.$$

One then solves for $\delta M(x)$ by variation of parameters, which together with the definition

$$\delta \Delta = \text{tr} \delta M(1),$$

produces the representation.

**Remark.** This proposition is quite important. In addition to providing explicit formulas for the first variation, it shows that the Floquet discriminant is much smoother in $q$ and $q$ than, for example, the Hamiltonian $H$ is. Moreover, its proof makes the reason for this smoothness completely transparent. The next proposition shows that the Floquet discriminant generates an infinite family of constants of the motion for the NLS equation.

**Proposition 6.2**

i. Floquet discriminants Poisson-commute:

$$\{\Delta(\lambda; q, \bar{q}), \Delta(\lambda'; q, \bar{q})\} = 0 \quad \forall \lambda, \lambda',$$

where the Poisson bracket is defined as

$$\{F, G\} = \int_0^{2\pi} i \left( \frac{\delta F}{\delta q} \frac{\delta G}{\delta q} - \frac{\delta F}{\delta \bar{q}} \frac{\delta G}{\delta \bar{q}} \right) dx.$$

ii. $\Delta(\lambda; q, \bar{q})$ is a constant of the motion for the NLS equation since its Poisson bracket with the Hamiltonian vanishes:

$$\{\Delta(\lambda; q, \bar{q}), H(q, \bar{q})\} = 0 \quad \forall \lambda.$$

Thus, $\Delta(\lambda; \bar{q})$ generates an infinite family of NLS constants of motion, one for each $\lambda$. 
The proof of this proposition, which can be found in the survey [51], begins by using the representation (6.4) of the gradients to express the Poisson brackets. From representation (6.4) we can note that these gradients actually consist of quadratic products of eigenfunctions; consequently, the integrals can be explicitly evaluated. This evaluation is possible because quadratic products of eigenfunctions themselves satisfy ordinary differential equations; this fact can be used to show that the integrands are derivatives. Finally, (ii) follows from the asymptotic behavior of $\Delta(\lambda'; q, q)$ as $\lambda' \to \infty$.

The difficulty with the spectral theory of $\hat{L}$ is that this operator is not self-adjoint. Nevertheless, certain properties of its spectrum follow as in the standard Floquet theory of Hill's operator [48]. The spectrum occurs in bands, not necessarily real, which terminate at periodic or antiperiodic eigenvalues $\lambda_j$ for which $\Delta(\lambda_j) = \pm 2$.

Next we define critical points and multiple points: First, critical points are defined by the condition

$$
\frac{d\Delta(\lambda; q)}{d\lambda} \bigg|_{\lambda'(q)} = 0,
$$

while a multiple point, denoted $\lambda^m$, is a critical point for which

$$
\Delta(\lambda^m; q) = \pm 2.
$$

The algebraic multiplicity of $\lambda^m$ is defined as the order of the zero of $\Delta(\lambda) \mp 2$. Usually it is 2, but it can exceed 2; when it does equal 2, we call the multiple point a double point and denote it by $\lambda^d$. The geometric multiplicity of $\lambda^m$ is defined as the dimension of the eigenspace of $\hat{L}$ at $\lambda^m$ and is either 1 or 2 (since $\hat{L}$ is a first-order differential operator for a two-vector).

The real axis is a subset of the spectrum, $\mathbb{R} \subset \sigma(\hat{L})$. Turning to properties of the spectrum of $\hat{L}$ that are rather directly related to the non-self-adjointness of $\hat{L}$, we consider a critical point $\lambda^c$ at which $-2 < \Delta(\lambda^c) < 2$. Such a critical point is a point of bifurcation of the spectrum. (Many spectral figures may be found in [51].) Furthermore, asymptotic behavior for large real $\lambda$ shows that there are a countable number of such critical points on the real axis that approach $j/2$ as $j \to \infty$ and at which a short “spine” of spectrum bifurcates from the real axis into the complex $\lambda$-plane. In addition to these short spines of spectrum that are connected (through spectrum) to the real axis, examples related to solitons show that there can also exist curves of spectrum in the complex $\lambda$-plane that are not connected (through spectrum) to the real axis.

This spectral theory is discussed in detail in [44]. However, for the present work such generality is not needed, and it is sufficient to understand $\sigma(\hat{L}(q))$.
for $q$ near the circle of fixed points $S_\omega$. This situation is discussed in the next subsection.

### 6.3 The Basic Example

Consider the example of $q(x, t)$ constant and independent of $x$:

$$q(x, t) = c \exp \{-i[2(c^2 - \omega^2)t - \gamma]\}.$$

In this case, two linearly independent solutions of the Lax pair are given by

$$\begin{pmatrix}
\psi_1^{(\pm)} \\
\psi_2^{(\pm)}
\end{pmatrix} = \exp \{\pm i[\kappa(\lambda)(x + 2\lambda t)]\}$$

$$\times \begin{pmatrix}
c \exp \{-i[2(c^2 - \omega^2)t - \gamma]/2\} \\
(\pm \kappa(\lambda) - \lambda) \exp \{i[2(c^2 - \omega^2)t - \gamma]/2\}
\end{pmatrix}. $$

In these formulas, $\kappa(\lambda)$ is given by

$$\kappa(\lambda) = \sqrt{\lambda^2 + c^2}.$$

This example of $q(x, t)$ constant and independent of $x$ is very useful for illustrating several crucial points. First, the spectrum of the linear operator $\hat{L}$ for coefficients independent of $x$ is easily computed from the Floquet discriminant

$$\Delta[\lambda; q(\cdot, t; c, \gamma)] = 2 \cos 2\pi \kappa(\lambda)$$

$$= 2 \cos [2\pi(\lambda^2 + c^2)^{1/2}],$$

from which we see that $\lambda_j$ is given by

$$\kappa(\lambda_j) = \frac{j}{2}.$$

Notice that the continuous spectrum consists of the real axis together with one band of spectrum on the imaginary axis. All critical points except the origin are double points. For $c \simeq \omega$, one double point lies in the upper half-plane on the band of spectrum. Its conjugate is also a double point. All other double points (countable in number) reside on the real axis.

Finally, let $q$ denote a small perturbation

$$q(x, t) \simeq c \exp \{-i[2(c^2 - \omega^2)t - \gamma]\}.$$

Then by regular perturbation theory one can establish the following facts about the spectrum $\sigma[\hat{L}(q)]$:...
1. Generically, at each real double point, a small band of complex spectrum emerges.

2. Generically, at the two complex double points, a bifurcation occurs. Specifically, either a gap or a cross occurs in the spectral band.

In this last case, the additional constraint at the complex double point

\[ \chi^\nu_1 = \nu, \]

\[ \text{grad} \Delta(\nu, q) = 0, \]

will ensure that the complex double point remains double [44].

Finally, as described in [17, 44], in a four-dimensional submanifold defined by the constraints that all of the real critical points are double, the phase portraits (restricted to the neighborhood \( U_{\delta} \)) are topological “trousers” (see Figure 2.3).

### 6.4 Instabilities

In studies of dynamical systems such as this work, one is often primarily interested in the temporal stability or instability of a particular solution \( q(x, t) \). The complete integrability of NLS allows one to address this issue in unusual generality. Although we will need these results only in the special case when \( q(x, t) \) is independent of \( x \), an overview of the more general situation will be useful.

Fix a solution \( q(x, t) \) of the NLS equation that is periodic in \( x \) and quasi-periodic in \( t \); that is, fix \( q \) on a torus. Linearizing NLS about \( q \) yields a linearized equation that can be solved with quadratic products of solutions of the Lax pair. Actually, these quadratic products generate a basis of solutions of the linearization [15, 6]. With this basis one can assess the linear stability properties of the solution \( q \). First, in the absence of higher-order multiple points, the basis splits into two parts, one labeled by simple eigenvalues and one labeled by double points. There is no exponential growth in that part of the basis associated with the simple eigenvalues or in that part associated with real double points. The only possible exponential instabilities are labeled by complex multiple points, which are at most finite in number. For each complex double point there can be one exponentially growing and one exponentially decaying linearized solution. At present, each complex double point must be investigated individually for instability. Those linearized solutions that do behave exponentially generate a basis for the (finite-dimensional) tangent spaces at \( q \) of the stable and unstable manifolds (for the NLS flow) of the torus.
6.5 Homoclinic Orbits and Whiskered Tori

Using Bäcklund (Darboux) transformations one can exponentiate these linearized solutions to obtain global solutions (homoclinic orbits) of the NLS equation. Fix a periodic solution $q$ of an NLS that is quasi-periodic in $t$ and for which the linear operator $\hat{L}$ has a complex double point $\nu$ of geometric multiplicity 2 associated with an NLS instability. We denote two linearly independent solutions of the Lax pair at $\lambda = \nu$ by $(\tilde{\phi}^+, \tilde{\phi}^-)$. Thus, a general solution of the linear system at $(q, \nu)$ is given by

$$\tilde{\phi}(x, t; \nu; c_+, c_-) = c_+ \tilde{\phi}^+ + c_- \tilde{\phi}^-.$$  

(6.6)

We use $\tilde{\phi}$ to define a transformation matrix $[57] G$ by

$$G = G(\lambda; \nu; \tilde{\phi}) \equiv N \begin{pmatrix} \lambda - \nu & 0 \\ 0 & \lambda - \nu \end{pmatrix} N^{-1},$$  

(6.7)

where

$$N \equiv \begin{bmatrix} \phi_1 & -\tilde{\phi}_2 \\ \phi_2 & \tilde{\phi}_1 \end{bmatrix}.$$  

(6.8)

Then we define $Q$ and $\tilde{\Psi}$ by

$$Q(x, t) := q(x, t) + 2(\nu - \tilde{\nu}) \frac{\phi_1 \phi_2}{\phi_1 \phi_1 + \phi_2 \phi_2}$$  

(6.9)

and

$$\tilde{\Psi}(x, t; \lambda) \equiv G(\lambda; \nu; \tilde{\phi}) \tilde{\psi}(x, t; \lambda)$$  

(6.10)

where $\tilde{\psi}$ solves the Lax pair at $(q, \lambda)$. Formulas (6.9) and (6.10) are the Bäcklund transformations for the potential and eigenfunctions, respectively.

We have the following theorem:

**Theorem 6.3** Let $q(x, t)$ denote a periodic solution of NLS, which is linearly unstable with an exponential instability associated with a complex double point $\nu$ in $\sigma(\hat{L}(q))$. Furthermore, assume that the complex double point $\nu$ has geometric multiplicity 2. Let $(\tilde{\phi}^+, \tilde{\phi}^-)$ denote an eigenbasis for the Lax pair at $(q, \nu)$, and define $Q(x, t)$ and $\tilde{\Psi}(x, t; \lambda)$ by (6.9) and (6.10). Then

i. $Q(x, t)$ is a solution of NLS with spatial period $2\pi$;

ii. $\sigma(\hat{L}(Q)) = \sigma(\hat{L}(q))$;
iii. \( Q(x, t) \) is homoclinic to \( q(x, t) \) in the sense that \( Q(x, t) \to q_{\theta_{\pm}}(x, t) \) exponentially as \( \exp(-\sigma_{\nu}|t|) \) as \( t \to \pm \infty \). Here \( q_{\theta_{\pm}} \) is a "torus translate" of \( q \), \( \sigma_{\nu} \) is the nonvanishing growth rate associated with the complex double point \( \nu \), and explicit formulas can be developed for this growth rate and for the translation parameters \( \theta_{\pm} \):

iv. \( \Psi(x, t; \lambda) \) solves the linear system (4.1) at \((Q, \lambda)\).

REMARK. This theorem is quite general, constructing homoclinic solutions from a wide class of starting solutions \( q(x, t) \). In references [49] and [51], several qualitative features of these homoclinic orbits are emphasized: (1) \( Q(x, t) \) is homoclinic to a torus that possesses rather complicated spatial and temporal structure and is not just a fixed point; (2) nevertheless, the homoclinic orbit typically has still more complicated spatial structure than its "target torus." (3) When there are several complex double points, each with nonvanishing growth rate, one can iterate the Bäcklund transformations to generate more complicated homoclinic manifolds. (4) The number of complex double points with nonvanishing growth rates counts the dimension of the unstable manifold of the critical torus in that two unstable directions are coordinatized by the complex ratio \( c_+/c_- \). Under even symmetry only one real dimension satisfies the constraint of evenness. (5) These Bäcklund formulas provide coordinates for the stable and unstable manifolds of the critical tori; thus, they provide explicit representations of the critical level sets that consist of whiskered tori.

The proof of this theorem proceeds by direct verification following the sine-Gordon model [16]. Specifically, one defines \( \Psi \) by (6.10), calculates \( \partial_x \Psi \) and \( \partial \xi \Psi \) using the original Lax pair at \((q, \lambda)\), and shows that \( \Psi \) solves a Lax pair at \((Q, \lambda)\) provided \( Q \) is defined by (6.9). Compatibility of this Lax pair then demands that \( Q \) solve NLS. Periodicity in \( x \) is achieved by the choice of a double point \( \nu \) as the transformation parameter, and the other properties follow by direct inspection.

The Basic Example

Let \( \eta \) be independent of \( x \),

\[
q = c \exp\{-i[2(c^2 - \omega^2)t - \gamma]\} = ce^{-i\theta}.
\]

Then, as described above,

\[
\Delta(\lambda; q) = 2 \cos [2\pi \kappa(\lambda)], \quad \kappa(\lambda) = \sqrt{\lambda^2 + c^2};
\]

\[
\kappa(\nu) = 1/2 \implies \nu = \frac{i}{2} \sqrt{4c^2 - 1}.
\]
Furthermore, from (5.1) we know that $q$ is unstable, with linearized growth rate given by

$$
\sigma = \sqrt{4c^2 - 1} = 2|\nu|.
$$

Using these formulas, as well as the eigenfunctions (6.5), specializes the general formula for the homoclinic orbit $Q_H$ to

$$
q_h^\pm = \left\{ \frac{\cos 2p - i \sin 2p \tanh \tau \pm \sin p \sech \tau \cos x}{1 \mp \sin p \sech \tau \cos x} \right\} q,
$$

where

$$
\tau = \sigma(t + t_0), \quad e^{ip} = \frac{1 + i\sigma}{2c}.
$$

Here $\pm$ denotes the two lobes of the figure eight in the trouser diagram of Figure 2.3. Notice that $-\cos x = \cos(x + \pi)$, which shows that one lobe $(\pm)$ represents an excitation centered at $x = 0$, while the other $(-)$ an excitation centered at $x = \pi$. Thus, (6.11) provides an explicit representation of a “whiskered circle,” while from another viewpoint, it provides an explicit representation of the unstable manifold $W_u(S) = W^s(S) = \bigcup_{\gamma, t_0, \pm} q_h^\pm(t; \gamma, t_0, c)$.

### 6.6 An Important Invariant

In this subsection, following material in [44], we introduce a very important and useful constant of the motion. Fix a potential $q_o \in \mathcal{H}_c^1$, that has (for simplicity) either a purely real or a purely imaginary critical point $\lambda^c$,

$$
\frac{\partial}{\partial \lambda} \Delta(\lambda; q_o)\bigg|_{\lambda^c} = 0.
$$

Let $N_b = N_b(q_o)$ denote a small neighborhood of $q_o$ in $\mathcal{H}_c^1$, and consider the critical point as a functional on this neighborhood, $\lambda^c = \lambda^c(q)$:

$$
\frac{\partial}{\partial \lambda} \Delta(\lambda; q)\bigg|_{\lambda^c(q)} = 0; \quad \lambda^c(q_o) = \lambda^c.
$$

In terms of this purely real (or purely imaginary) critical point, we introduce an important invariant $F : N_b \to \mathbb{R}$ given by

$$
F := \Delta(\lambda^c(q); q).
$$
PROPOSITION 6.4 \( F : N_b \to \mathbb{R} \) is smooth, provided \( \frac{d^2}{d\lambda^2} \Delta(\lambda, q) \neq 0 \) for all \( q \in N_b \).

To prove this, one calculates
\[
\frac{\delta F}{\delta q} = \frac{\delta}{\delta q} \Delta(\lambda^t(q); q) = \Delta'(\lambda^t(q); q) \frac{\delta \lambda^t}{\delta q} + \delta \frac{\delta \Delta}{\delta q} = \left. \frac{\delta \Delta}{\delta q} (\lambda; q) \right|_{\lambda = \lambda^t(q)}
\]
provided \( \lambda^t(q) \) is differentiable. But \( \lambda^t(q) \) is smooth, as the following calculation shows:
\[
\Delta'(\lambda^t(q); q) = 0 \\
\Delta''(\lambda^t(q); q) \frac{\delta \lambda^t}{\delta q} + \delta \frac{\delta \lambda'}{\delta q} = 0.
\]
If \( \Delta''(\lambda^t(q); q) \neq 0 \), one can continue to differentiate to any order.

**Remark.** The functions \( q_b \in H^1_{c,p} \) at which \( \Delta''[\lambda^t(q_b); q_b] = 0 \) are branch points for the functional \( F(q) \). This branching presents challenging obstacles to a global theory. When working with the functional \( F(q) \), it is judicious to avoid a branch point \( q = q_b \) with the condition \( \Delta''[\lambda^t(q); q] \neq 0 \).

The critical points of the functional \( F \) will play important roles. To understand these, it will be useful to develop a formula for the gradient \( F'(q) \) in terms of Bloch functions.

**Remark.** The specific eigenfunctions (6.5) used in the example of a plane wave, independent of \( x \), were Bloch functions.

Let \( \bar{\psi}^{\pm}(x, \lambda) \) denote Bloch functions, that is, solutions of the Lax pair at \([q, \lambda]\). These functions are defined (up to normalization) by the transfer condition across one period:
\[
\bar{\psi}(x + 2\pi, \lambda) = \rho(\lambda) \bar{\psi}(x, \lambda).
\]
Here \( \rho(\lambda) \) denotes the Floquet multiplier, which is related to the Floquet discriminant by
\[
\rho(\lambda) = \frac{1}{2} \left[ \Delta(\lambda) + \sqrt{\Delta^2(\lambda) - 4} \right].
\]
On the Riemann surface for \((\lambda, \sqrt{\Delta^2(\lambda) - 4})\), the functions \( \rho \) and \( \bar{\psi} \) are well-defined, and \( \bar{\psi}^{\pm}(x, \lambda) \) denote the values of \( \bar{\psi} \) on the two sheets over \( \lambda \). At branch points (simple periodic or antiperiodic points), the two sheets touch
and the $\tilde{\psi}^\pm$ become linearly dependent. (This is compatible with the fact that at a simple eigenvalue, the eigenspace is one-dimensional.) At real multiple points, $\tilde{\psi}^\pm$ remain linearly independent, while at complex multiple points they may, but need not, become dependent. These two possibilities at the complex multiple points are a key to this non-self-adjoint spectral problem.

In any case, for fixed $\lambda$, these Bloch eigenfunctions can be represented explicitly in terms of the columns of the fundamental matrix $M(x; \lambda) = \text{column}\{Y^{(1)}(x; \lambda), Y^{(2)}(x; \lambda)\}$:

$$
\tilde{\psi}^\pm(x; \lambda) = \alpha^\pm \left\{ M_{21}(1; \lambda)Y^{(1)}(x; \lambda) + [M_{22}(1; \lambda) - \rho^\pm(\lambda)]Y^{(2)}(x; \lambda) \right\},
$$

where $\alpha^\pm$ denotes normalization constants.

The gradient of the Floquet discriminant (6.4) admits a beautiful representation in terms of these Bloch functions:

**COROLLARY 6.5** For $\lambda \neq a$ branch point (i.e., $\lambda \neq a$ a periodic or antiperiodic eigenvalue)

$$
\frac{\delta}{\delta \bar{q}} \Delta(\lambda; q, \bar{q}) = \frac{i}{W[\psi^+, \psi^-]} \begin{bmatrix}
\psi^+_2(x; \lambda)\psi^-_2(x; \lambda) \\
-\psi^+_1(x; \lambda)\psi^-_1(x; \lambda)
\end{bmatrix},
$$

where $\bar{q} = (q, \bar{q})^T$ and $W[\psi^+, \psi^-]$ denotes the Wronskian of $\psi^+$ and $\psi^-$. The representation is extended to the periodic or antiperiodic eigenvalues by continuity.

With this representation one has the following proposition:

**PROPOSITION 6.6**

$$
\text{grad } F(q, \bar{q}) = \frac{i}{W[\psi^+, \psi^-]} \begin{bmatrix}
\psi^+_2(x; \lambda)\psi^-_2(x; \lambda) \\
-\psi^+_1(x; \lambda)\psi^-_1(x; \lambda)
\end{bmatrix}.
$$

**Critical Points of $F$**

From this representation, one obtains the following theorem:

**THEOREM 6.7** The potential $q$ is a critical point of the functional $F$ if and only if $\lambda^c(q)$ is a multiple point with geometric multiplicity 2.
**Remark.** We note that if $\lambda^c$ is a real multiple point, its geometric multiplicity is always 2. Thus, a potential $q$ for which $\lambda^c(q)$ is a real double point is a critical point of $F$. On the other hand, if $\lambda^c(q)$ is a complex double point, the geometric multiplicity may be either 2 or 1, and $q$ may or may not be a critical point of $F$.

**Critical Tori**

To understand the critical points of $F$, we first note that, generically, the NLS level sets are tori $T^\infty = S \times S \times S \times \cdots$ of infinite dimension with the radius of the $j^{th}$ circle measured by $r_j = \Delta(\lambda^c_j) \mp 2$. For any $q_*$, where $\lambda^c_j(q_*)$ is a multiple point ($\Delta(\lambda^c_j) = \pm 2$), $r_j = 0$, and the $j^{th}$ circle is pinched off. In this case, the function $q_*$ resides on a torus which is singular in the sense that it has one dimension less than maximal because $r_j = 0$. If $\lambda^c_j = \lambda^c$, any $q_*$ on this singular torus is a critical point of $F$.

The existence of these critical tori is guaranteed by inverse spectral theory, which can be used to construct some of them in terms of finite genus theta functions. (It would be interesting to construct these critical tori directly from the variational problem $\delta F = 0$ without invoking inverse spectral theory except in a context that is natural to that particular variational problem. However, we have not done so. Rather, to construct critical tori, we have freely used results from the general theory of the inverse spectral transform.)

In reference [44] we have further studied these critical tori by calculating an explicit representation of the Hessian of $F$ at $q_*$. If the multiple point $\lambda^c(q_*)$ is real, the Hessian shows that $F$ is either a maximum or a minimum at $q*$; while if $\lambda^c(q_*)$ is a purely imaginary double point, $F$ has a saddle structure at $q_*$. In the latter case, if $F$ were the NLS Hamiltonian, $q_*$ would be an unstable fixed point; however, $F$ is in involution with the NLS Hamiltonian $H$. As such, $q_*$ lies on a singular torus that could be unstable (hyperbolic) under NLS dynamics. If the complex double point $\lambda^c(q_*) = \nu$ is indeed associated with an NLS instability, the Lax pair for the NLS flow can be used in a Bäcklund transformation $q_* \to q_h$ to provide a representation of the whiskers for the critical tori on which $q_*$ resides. In this manner one learns that

$$W^{cs}(q_*) = W^{cu}(q_*) = \{ q : F(q) = \pm 2 \},$$

depending upon whether $\lambda^c(q_*)$ is a periodic or antiperiodic eigenvalue.

To understand these singular tori more completely, one [44] must study the entire sequence of invariants

$$F_j(q) = \Delta(\lambda_j(q); q) \quad \forall j \in \mathbb{Z}.$$
However, for our purposes here, we don't need this degree of generality.

6.7 $F'(q_h)$

Let $q_*$ lie on a critical unstable torus with the instability associated with the purely imaginary double point $\nu$ and with whisker $q_h$ represented by the Bäcklund formulas (6.9). Here we seek a useful representation of the gradient $F'(q_h)$.

This representation can also be expressed rather explicitly using Bäcklund transformations. We begin from equation (6.16) for the grad $F$,

$$
\delta F \over \delta q = \lim_{\lambda \to \nu} i \frac{\sqrt{\Delta^2 - 4}}{W[\Psi^+, \Psi^-]} \begin{pmatrix} \psi_2^+ \psi_2^- \\ -\psi_1^+ \psi_1^- \end{pmatrix},
$$

where $\tilde{\Psi}^\pm(x, \lambda)$ are a Floquet basis at $(\tilde{q}_H, \nu)$. In [44] we compute this limit using the Bäcklund formulas. The result is

$$
\delta F \over \delta q = C_\nu \frac{c_+ c_- W[\psi^+, \psi^-]}{|\phi|^4} \begin{pmatrix} \bar{\phi}_1^2 \\ -\bar{\phi}_2^2 \end{pmatrix},
$$

where the constant $C_\nu$ is given by

$$
C_\nu \equiv i(\nu - \bar{\nu}) \sqrt{\Delta(\nu) \Delta''(\nu)}.
$$

**Remark.** Since

$$
\bar{\phi} = c_+ \tilde{\psi}^+ + c_- \tilde{\psi}^-,
$$

one sees explicitly from this formula that

$$
\delta F \over \delta q \bigg|_{(q_h, \nu)} \to 0 \quad \text{as} \quad c_+/c_- \to 0 \text{ or } \infty.
$$

Also, since the eigenfunctions $\tilde{\psi}^+$ and $\tilde{\psi}^-$ at the complex double point $\nu$ grow or decay exponentially,

$$
\tilde{\psi}^\pm \approx \exp(\pm \sigma_\nu t), \quad t \to \infty,
$$

the formula also shows explicitly that $\text{grad } F \big|_{(q_h, \nu)} \to 0$ as $t \to \infty$. The vector field $\text{grad } F$ must vanish because, in these limits, the point $\tilde{q}_H$ on the whisker tends to a critical function of $F$.

Once again, we will not need to work in such generality.
6.8 The Basic Example Revisited

In the case of \( q^* = c \exp\{ -i [2(c^2 - \omega^2)t - \gamma] \} = ce^{i\theta} \), explicit formulas can be used to produce the following representation of \( F'(q_h) \), which is valid when acting upon even functions of \( x \):

\[
\frac{\delta F}{\delta q} = a \frac{[\mp \sin p \cosh \tau \pm i \cos p \sinh \tau \cos x + 1]}{[1 \mp \sin p \sech \tau \cos x]^2} ce^{i\theta},
\]

where \( \tau = \sigma(t - t_0) \), \( \tan p = \sigma \), \( \sigma = \sqrt{4c^2 - 1} \), and \( a = 2\pi \sin^2 p \sech^2 \tau \).

From this representation, we see explicitly that

\[ F'(q_h(t)) \neq 0, \]

\[ \lim_{t \to \pm \infty} F'(q_h(t)) \to 0, \]

at an exponential (\( \exp(-\sigma|t|) \)) rate of approach. Thus, \( q_* \) is indeed critical, while the whisker \( q_h(t) \) is not.

7 Persistent Homoclinic Orbit (\( \epsilon \geq 0 \))

In this section we will combine the local analysis given so far with explicit global information from the unperturbed integrable system to establish the existence of a homoclinic orbit to the saddle point \( Q \) for the perturbed NLS equation,

\[
q_t = iH'(q) + \epsilon G(q),
\]

where \( H'(q) = -q_{xx} - 2(qq - \omega^2)q \) and \( G(q) = -\alpha q - \beta \tilde{B}q - 1 \) with \( \tilde{B} \) a bounded dissipative operator.

More specifically, we combine geometric singular perturbation with the construction of a Melnikov function to prove the existence of such orbits. The argument proceeds in two steps, which we shall call the “first measurement” and the “second measurement.” In the first measurement we will construct a distance function \( \Delta \) (not the Floquet discriminant of integrable theory) whose zeros correspond to orbits that do not lie in the invariant plane \( \Pi_c \) and are asymptotic to the saddle point \( Q \) in backward time and asymptotic to \( M_c \) in forward time. The second measurement consists of constructing a function \( d \) whose zeros correspond to one of these orbits intersecting a fiber whose base
point is in the stable manifold of $Q$. Therefore, from the definition of fibers, the simultaneous vanishing of $\Delta$ and $d$ ensures the existence of a homoclinic orbit to the point $Q$.

7.1 The First Measurement

Recall that the invariant plane $\Pi_c \subset \mathcal{M} \cap \mathcal{M}_c$, and that $Q$ has a one-dimensional unstable manifold in $\Pi_c$ given by

$$q = (\omega^2 + \sqrt{\epsilon j_u(s)})^{1/2} e^{i\theta(s)}.$$ 

Let $q_b$ be a point on the above curve corresponding to $s = s_b$. The unperturbed flow has an orbit that passes through $q_b$ at $t = 0$,

$$q = r_b e^{-i(2(r_b^2 - \omega^2)t - \theta_b)},$$

and an orbit $q_h$ that is asymptotic to the above orbit as $t \to -\infty$,

$$q_h(t) = \left( \frac{\cos 2\pi - i \sin 2\pi \tanh \tau + \sin \pi \sech \tau \cos x}{1 - \sin \pi \sech \tau \cos x} \right) \times r_b \exp \left\{ -i \left[ 2(r_b^2 - \omega^2)t - \theta_b + 2\pi \right] \right\}$$

where

$$\tan p = \sqrt{4r_b^2 - 1},$$
$$\tau = (\tan p)(t + t_o),$$
$$r_b e^{i\theta_b} = (\omega^2 + \sqrt{\epsilon j_u(s_b)})^{1/2} e^{i\theta(s_b)}.$$ 

The asymptotic behavior of $q_h$ as $t \to \infty$ is given by

$$q_h(t) \to r_b e^{-i(2(r_b^2 - \omega^2)t - \theta_b + 4p)},$$

which implies that we have a phase shift of $-4p$,

$$e^{-4ip} = \left( 1 - i \sqrt{4r_b^2 - 1} \right)^4.$$ 

Choose $t_o$ so that the orbit is a distance of order $\delta$ from $q_b$ at $t = 0$,

$$\|q_h(0) - q_b\|_{H^1} = \frac{\delta}{4}.$$ (7.2)
Using the explicit expression of the unperturbed orbit $q_h$, we see that there exists a $T_\epsilon(\delta)$ such that $\text{dist}(q_h(t), S_\omega) \leq \delta/4$ for $t \geq T_\epsilon$, provided $\epsilon$ is small.

The point $q_b$ is the base point of fibers of length $\delta$. The $v$-coordinates of $q_b$ are $(0, 0, v_c = \eta_c)$, where $\eta_c = (\sqrt{\epsilon} j_u(s_b), \theta_u(s_b), v_o = 0)$, and the fibers through $q_b$ can be parametrized by $0 \leq \eta_u \leq \delta$ as follows:

$$
\begin{align*}
  v_c &= f^u(\eta_c, \eta_u; \epsilon) \\
  v_u &= \eta_u + h_u'(v_c; \epsilon) \\
  v_s &= h_s(v_u, v_c; \epsilon).
\end{align*}
$$

From equation (7.2), $q_h(0)$ belongs to an unperturbed fiber; that is, $\epsilon = 0$ with $\eta_u = \eta_u(0) \leq \delta/2$. Let $q_\delta$ be the point on the perturbed fiber corresponding to $\eta_u = \eta_u$ (see Figure 7.1). Since the functions $f$ and $h$ are $C^2$ with respect to $\epsilon$, we have

$$
\|q_\delta - q_h(0)\|_{H^1} \leq C\epsilon.
$$

Denote by $q_\epsilon(t)$ the solution of the perturbed equation with initial value $q_\epsilon(0) = q_\delta$. From the fiber construction of Section 5.2, we have that the orbit $q_\epsilon(t)$ is asymptotic to $Q$ as $t \to -\infty$, and by continuous dependence on the initial data we have that for any finite time $T$ and for $0 \leq t \leq T$,

$$
(7.3) \quad \|q_\epsilon(t) - q_h(t)\|_{H^1} \leq C(T)\epsilon.
$$

Define

$$
\begin{align*}
  q_0 &= q_h(T_\epsilon), \\
  q_\epsilon &= q_\epsilon(T_\epsilon).
\end{align*}
$$
The point $q_o$ belongs to $W_c^{cs}$ since as $t \to \infty$ we have $q_h \to \Pi_c$, and $q_o$ is at a distance less then $\delta/2$ from the plane $\Pi_c$:

$$q_o = r_0 e^{-i(2(r_0^2 - \omega^2)T_0 - \theta_0 + \phi)} + \delta q_o$$

where $\|\delta q_o\|_{H^1} \leq \delta/2$. By (7.3) we have

$$\|q_o - q_\ell\|_{H^1} \leq C(\delta) \epsilon < \delta/4$$

$$\text{dist}(q_\ell, S_\omega) \leq \delta/2$$

for small $\epsilon$.

The distance between $q_\ell$ and $W_c^{cs}$ will be measured along the normal to $W_c^{cs}$ at $q_o$. This is accomplished in the following manner: The manifold $W_c^{cs}$ is given as a graph

$$v_u = h_u(v_s, v_s; 0)$$

where the function $h_u$ has small derivative. This implies that the vector in the $v_u$-direction, $V = (1, 0, 0)$, is transversal to $W_c^{cs}$. This vector $V$ is the eigenfunction $e_u(x) = \frac{1}{2\sqrt{\pi}\omega}(1+i\sigma) \cos x$ given in (5.2). $W_c^{cs}$ is also characterized by $\{q \in H^1 : F(q) + 2 = 0\}$. Therefore the transversality of $e_u$ translates into

$$(7.4) \quad \langle F'(q_o), e_u \rangle \neq 0$$

where the above is a shorthand for the duality pairing

$$\langle \delta F/\delta q, \delta q \rangle + \langle \delta F/\delta \eta, \delta \eta \rangle.$$  

Since $h_u$ is $C^2$ in all of its arguments, then for every $q$ in an $\epsilon$-neighborhood of $q_o$, the straight line through $q$ in the direction of $e_u$ intersects $W_c^{cs}$ at a point of distance $\epsilon$ from $q$. Let $q_s$ be the intersection of the line through $q_\ell$ with the manifold $W_c^{cs}$. By (7.4), we can define

$$\Delta = \langle F'(q_o), q_\ell - q_s \rangle$$

as a measure of distance between $q_\ell$ and $q_s$ (see Figure 7.2).

To actually calculate $\Delta$ we define, for $t \leq 0$, the orbits

$$q_* (t) = q_h(t + T_*) ,$$

$$q_*(t) = q_\ell(t + T_*) .$$

For $t \geq 0$ we define $q_s(t)$ to be the solution of the cutoff flow with initial data $q_s$ and $q_*(t) = q_h(t + T_*)$. Note that since $q_h$ remains in a $\delta$-neighborhood of
Figure 7.2. Schematic diagram of the first measurement.

$S_\omega$ for $t \geq T_\omega$, $q_\omega$ is also a solution of the cutoff equation, and at $t = 0$ the initial data of all orbits, $q_\ell$, $q_s$, and $q_\omega$, are order $\epsilon$ apart. For $t \leq -T_\omega$ both orbits $q_u$ and $q_s$ remain in a $\delta$-neighborhood of $S_\omega$. Therefore by Gronwall's inequality applied to equation (5.8) and from (7.3), we have for $t < 0$

$$(7.5) \quad \| q_u - q_s \| H^1 \leq C(\delta) e^{-\delta t} \epsilon .$$

For $t \geq 0$ both orbits, $q_\omega$ and $q_s$, are solutions of the cutoff equations. Again by Gronwall's inequality we have

$$(7.6) \quad \| q_\omega(t) - q_s(t) \| H^1 \leq C e^{\delta t} \epsilon .$$

These orbits allow us to introduce the measurements

$\Delta^-(t) = \langle F'(q_\omega(t)), q_u(t) - q_\omega(t) \rangle, \quad t \leq 0,$

$\Delta^+(t) = \langle F'(q_\omega(t)), q_s(t) - q_\omega(t) \rangle, \quad t \geq 0,$

$\Delta = \Delta^-(0) - \Delta^+(0) .$

**Proposition 7.1** The distance $\Delta$ is given by

$$\Delta = \epsilon \int_{-\infty}^{\infty} \langle F'(q_\omega(t)), G(q_\omega(t)) \rangle \, dt + O(\epsilon^2) .$$

**Proof:** Note that $q_\omega$ is a smooth function in $(x, t)$, $F'(q_\omega)$ is smooth in $q_\omega$, and by Proposition 6.4 $F'(q_\omega)$ is a $C^1$ function in $t$ with values in $H^1$. Since $q_u$ and $q_s$ are $C^1$ functions in $t$ with values in $H^{-1}$, we conclude that $\Delta^-$ and $\Delta^+$ are $C^1$ functions of $t$.

We start by computing the time derivative of $\Delta^-(t)$ using the notation of equation (7.1):

$$\dot{\Delta}^- (t) = \langle F''(q_\omega) \dot{q}_\omega, q_u - q_\omega \rangle + \langle F'(q_\omega), \dot{q}_u - \dot{q}_s \rangle$$

$$= \langle F''(q_\omega) iH'(q_\omega), q_u - q_\omega \rangle$$

$$+ \langle F'(q_\omega), iH'(q_u) - iH'(q_\omega) + \epsilon G \rangle .$$

$$\Delta = \epsilon \int_{-\infty}^{\infty} \langle F'(q_\omega(t)), G(q_\omega(t)) \rangle \, dt + O(\epsilon^2) .$$

**Proof:** Note that $q_\omega$ is a smooth function in $(x, t)$, $F'(q_\omega)$ is smooth in $q_\omega$, and by Proposition 6.4 $F'(q_\omega)$ is a $C^1$ function in $t$ with values in $H^1$. Since $q_u$ and $q_s$ are $C^1$ functions in $t$ with values in $H^{-1}$, we conclude that $\Delta^-$ and $\Delta^+$ are $C^1$ functions of $t$.

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$$\dot{\Delta}^- (t) = \langle F''(q_\omega) \dot{q}_\omega, q_u - q_\omega \rangle + \langle F'(q_\omega), \dot{q}_u - \dot{q}_s \rangle$$

$$= \langle F''(q_\omega) iH'(q_\omega), q_u - q_\omega \rangle$$

$$+ \langle F'(q_\omega), iH'(q_u) - iH'(q_\omega) + \epsilon G \rangle .$$
By expanding the nonlinear part of $H'(q)$ around $q_*$ and using the fact that both orbits $q_u$ and $q_*$ are bounded for $t \leq 0$, we obtain

$$H'(q_u) - H'(q_*) = H''(q_*)(q_u - q_*) + R(q_u, q_*),$$

$$\|R\|_{H^1} \leq C\|q_u - q_*\|_{H^1}^2.$$  \hspace{1cm} (7.8)

Equation (7.7) can be written as

$$\dot{\Delta}^- = \langle F''(q_*) i H'(q_*) + F'(q_*) i H''(q_*)(q_u - q_*) \rangle + \epsilon \langle F'(q_*) G(q_u) + F'(q_*) i R \rangle.$$  

From [44] we have $\{F(q), H(q)\} = 0$, which implies that

$$\langle F''(q_*) i H'(q_*) + F'(q_*) i H''(q_*)(q_u - q_*) \rangle = 0,$$

and the equation for $\dot{\Delta}^-$ simplifies to

$$\dot{\Delta}^- = \epsilon \langle F'(q_*) G(q_u) + F'(q_*) i R \rangle.$$  

Finally, from (6.20), (7.5), and (7.8), we have for $t < 0$

$$\|F'(q_*)\|_{H^1} \leq Ce^{\sigma t},$$

$$\|q_u - q_*\|_{H^1} \leq Ce^{-\delta t},$$

$$\|R(t)\|_{H^1} \leq Ce^{-2\delta t}e^2,$$

with $2\delta < \sigma$ and $\Delta^-(t) \to 0$ as $t \to -\infty$. This implies

$$\Delta^-(0) = \epsilon \int_{-\infty}^0 \langle F'(q_*) G(q_u) \rangle + O(\epsilon^2).$$  \hspace{1cm} (7.9)

To obtain a similar expression for $\dot{\Delta}^+$, we repeat the same argument as above, keeping in mind that $q_s$ is a solution of the cutoff equation and that $q_*$ is a solution of the cutoff as well as the noncutoff equations. Thus expanding the cutoff equations around $q_*$ and using the fact that $\{F(q_*), H(q_*)\} = 0$, we obtain

$$\dot{\Delta}^+ = \langle F'(q_*) G_\delta(q_s) \rangle + \langle F'(q_*) i \tilde{R} \rangle$$

where $\tilde{R}$ is the remainder from expanding the cutoff flow $H'_\delta(q_s)$ around $q_*$,

$$\|\tilde{R}\|_{H^1} \leq C(\delta)\|q_s - q_*\|_{H^1}.$$
and $G_{\delta}(q_*)$ is the cutoff perturbation evaluated at $q_*$. Again from (6.20), (7.6), and (7.8), we have for $t \geq 0$
\begin{align*}
\|F'(q_*)\|_{H^1} &\leq C\epsilon^{-\alpha t}, \\
\|q_* - q_*\|_{H^1} &\leq C\epsilon^{\delta t}, \\
\|	ilde{R}(t)\|_{H^1} &\leq C\epsilon^{2\delta t}.
\end{align*}
Therefore the derivative of $\Delta^+$ can be written as
\begin{align*}
\Delta^+(0) &= -\epsilon \int_{0}^{\infty} \langle F'(q_*), G_{\delta}(q_*) \rangle + O(\epsilon^2) \\
&= -\epsilon \int_{0}^{\infty} \langle F'(q_*), G(q_*) \rangle + O(\epsilon^2),
\end{align*}
and the distance $\Delta$ has an expansion in $\epsilon$ given by
\begin{equation}
\Delta = \Delta^-(0) - \Delta^+(0) = \epsilon \int_{-\infty}^{\infty} \langle F'(q_*), G(q_*) \rangle + O(\epsilon^2).
\end{equation}

Since the base point of $q_*$ depends on $\epsilon$,
\begin{equation}
r_b e^{i\theta_b} = (\omega^2 + \sqrt{\epsilon} j_{\omega}(s_b))^{1/2} e^{i\theta(s_b)},
\end{equation}
we can simplify the expression for $\Delta$ further by using the homoclinic orbit $q_\omega(t)$ whose base point is $\omega e^{i\theta_b}$ and which is order $O(\sqrt{\epsilon})$ away from $q_*(t)$. Thus, using the explicit form of $G$ we obtain the following:

**Corollary 7.2** The distance $\Delta$ has an expansion in $\epsilon$ given by
\begin{equation}
\Delta = \epsilon M(\alpha, \beta, \theta_b) + O(\epsilon^{3/2})
\end{equation}

where
\begin{align*}
M(\alpha, \beta, \theta_b) &= \int_{-\infty}^{\infty} \langle F'(q_\omega(t)), G(q_\omega(t)) \rangle \, dt \\
&= -[\alpha M_\alpha + \beta M_\beta + M(\theta_b)]
\end{align*}

and
\begin{align*}
M_\alpha &= \int_{-\infty}^{\infty} \langle F'(q_\omega(t)), q_\omega(t) \rangle \, dt, \\
M_\beta &= \int_{-\infty}^{\infty} \langle F'(q_\omega(t)), \tilde{B} q_\omega(t) \rangle \, dt, \\
M(\theta_b) &= \int_{-\infty}^{\infty} \langle F'(q_\omega(t)), 1 \rangle \, dt.
\end{align*}
From the above corollary we conclude that to find a zero for $\Delta$ it is sufficient to find a nondegenerate zero of $M(\alpha, \beta, \theta_b)$ and then use the implicit function theorem.

The dependence of $M$ on $\theta_b$ can be computed by using the explicit formula for $F'$ given in (6.20),

$$\frac{\delta F}{\delta q} = 2\pi \sin^2 p \sech^2 \tau \left[ \left( -\sin p \cosh \tau + i \cos p \sinh \tau \right) \cos x + 1 \right] \frac{ce^{-i\theta}}{(1 - \sin p \sech \tau \cos x)^2},$$

to obtain

$$M(\theta_b) = \cos(\theta_b - 2p_0) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} dx \frac{4\pi \omega \sin^2 p_0 \sech \tau}{\sigma A^2}$$

$$\times \left( \sech \tau - \sin p_0 \cos x \right)$$

where $p_0 = \tan^{-1} \sqrt{\omega^2 - 1}$ and $A = (1 - \sin p_0 \sech \tau \cos x)$.

Assuming that $M_\alpha$ or $M_\beta$ is nonzero, then the function

$$-M(\alpha, \beta, \theta_b) = \alpha M_\alpha + \beta M_\beta + M_0 \cos(\theta_b - 2p_0)$$

has nondegenerate zero. This implies that we can choose our parameters so that $\Delta = 0$; that is,

$$q_\ell = q_s \in W^{cs}_\epsilon.$$

### 7.2 The Second Measurement

This measurement is performed to show that we can choose the parameter $\theta_b$ such that the point $q_\ell$ lies on a fiber whose base point, denoted by $q_{\ell,b}$, belongs to the stable manifold of $Q$. First, recall that the unperturbed orbit $q_*(t) = q_h(t + T_\ast)$ is asymptotic as $t \to \infty$ to the orbit

$$r_b e^{-i(2(r_b^2 - \omega^2)(t + T_\ast) - \theta_b + 4\pi)} ,$$

which implies that $q_o = q_h(T_\ast)$ belongs to an unperturbed fiber whose base point is

$$q_{o,b} = r_b e^{-i(2(r_b^2 - \omega^2)T_\ast - \theta_b + 4\pi)}.$$

The point $q_\ell \in W^{cs}_\epsilon$ belongs to a perturbed fiber whose base point $q_{\ell,b} \in M_\epsilon$ does not necessarily lie on the plane $\Pi_c$.

The distance between $q_{o,b}$ and $q_{\ell,b}$ is of order $\epsilon$, which can be proved as follows (see Figure 7.3): The point $q_o$ has $v$-coordinates

$$v_{o,c} = f^s(\eta_{o,s}, \eta_{o,c}; 0),$$

$$v_{o,s} = \eta_{o,s} + h_\epsilon^c(v_{o,c}; 0),$$

$$v_{o,u} = h_u(v_{o,s}, v_{o,c}; 0),$$
where $\eta_{0,s} \in [-\delta, \delta]$ and $\eta_{0,c} = (\sqrt{\epsilon} j \omega(s) \theta_b, 2(v^2 - \omega^2) T^* - \theta_b + 4p, 0)$. The base point $q_{0,b}$ corresponds to $\eta_{b} = 0$ in the above equations. The functions $f^s$ and $h^s_b$ are $C^1$ functions with well-defined inverses, and the points $q_0$ and $q_i$ are at a distance of order $O(\epsilon)$; therefore, by the inverse function theorem, the point $q_i$ has parameters $(\eta_{i,c}, \eta_{i,s})$, which differ from $(\eta_{0,c}, \eta_{0,s})$ by order $\epsilon$. The base point $q_{i,b}$ has $v$-coordinates

$$
\begin{align*}
v_{i,c} & = f^s(\eta_{i,c}, 0; \epsilon), \\
v_{i,s} & = h^s_b(v_{i,c}; \epsilon), \\
v_{i,a} & = h_{i}(v_{i,s}, v_{i,c}; \epsilon),
\end{align*}
$$

which differ from the $v$-coordinates of $q_{0,c}$ by order $\epsilon$. This difference implies that

$$
||q_{0,b} - q_{i,b}||_{H^1} = O(\epsilon).
$$

To construct a distance function from $q_{i,b} \in \mathcal{M}_i$ to $\mathcal{W}$, the stable manifold of $Q$, we recall that the curve $C^s_{\epsilon} \subset \mathcal{W} \cap \Pi_{\epsilon}$, expressed in the $y = (j, \theta)$-coordinates, is given by $y_*(s, \sqrt{\epsilon})$ for $s \in [0, s_0]$. In terms of $z$, where $z = (\omega^2 + \sqrt{\epsilon} j)^{1/2} \epsilon^{i\theta}(s)$, $C^s_{\epsilon}$ can be represented by

$$
z_*(s; \sqrt{\epsilon}) = (\omega^2 + \sqrt{\epsilon} j_0(s))^{1/2} \epsilon^{i\theta_0(s)} + O(\epsilon),
$$

where $(j_0(s), \theta_0(s))$ is the planar homoclinic orbit to the unperturbed $(j, \theta)$ equations. The manifolds $\mathcal{M}_i$ and $\mathcal{W}$ are given by the following:

$$
\begin{align*}
\mathcal{M}_i & = \{ v \in H^{1} : v_{a} = h^s_{i}(v_{c}; \epsilon), v_{s} = h^s_{b}(v_{c}; \epsilon) \}, \\
\mathcal{W} & = \{ v \in \mathcal{M}_i : r = g(s, v_{c}; \epsilon) \}.
\end{align*}
$$
where $s \in [0, s_0]$ and $\|v_0\|_{H^1} \in [0, \varepsilon^{3/4}]$. Here $r$ is the signed Euclidean distance on the plane $\Pi_c$ from a point $y$ to the curve $C^s_r$.

For every point $q \in \mathcal{M}$, that is a distance of order $O(\varepsilon)$ from $\Pi_c$ and a distance of order $O(\sqrt{\varepsilon})$ from $S_\omega$, we associate a point $\bar{q} \in \mathcal{W}$ as follows (see Figure 7.4):

- $q$ can be represented as $(z_q, v_{q, o})$ where $z_q = (\omega^2 + \sqrt{\varepsilon} j_q) e^{i \theta_q}$.
- In the $y = (j, \theta)$-coordinates, let $y_* (s_q) \in C^s_r$ be the point where the distance from $y_q = (j_q, \theta_q)$ to $C^s_r$ is achieved. By the implicit function theorem, such a point exists and is unique provided $(j_q, \theta_q)$ is in a fixed neighborhood $O$ of $C^s_r$.
- On the line joining $y_q$ to $y_* (s_q)$, let $y_{\bar{q}}$ to be the point that is at a distance $r_{\bar{q}} := g(s_q, v_{q, o}; \sqrt{\varepsilon})$ from $y_* (s_q)$. In the plane $\Pi_c$, this point has coordinates $z_{\bar{q}} = (\omega^2 + \sqrt{\varepsilon} j_{\bar{q}}) e^{i \theta_{\bar{q}}}$.
- Define $\bar{q} \in \mathcal{W}$ to be the point corresponding to $(z_{\bar{q}}, v_{q, o})$.

To measure the distance from $q$ to $\bar{q}$, we introduce the function

$$d(q) = E(y_q) - E(y_{\bar{q}}),$$

where $E$ is the Hamiltonian of the unperturbed ODE given in (3.12):

$$E(j, \theta) \equiv \frac{1}{2} j^2 - \omega (\sin \theta + \alpha \omega \theta).$$

**Proposition 7.3** The map $q \rightarrow \bar{q}$ has a fixed point if and only if $d(q) = 0$. Moreover, $d(q)$ has an expansion given by

$$d(q) = E(j_q, \theta_q) - E_0 + O(\sqrt{\varepsilon}).$$
PROOF: To show that the zeros of \( d \) correspond to fixed points of \( q \rightarrow \dot{q} \), we note that in a neighborhood of \( C^u_0 \), we have that the level curves of \( F \) intersect the normal to \( C^u_0 \) in exactly one point. (This can be shown by noting that in the neighborhood \( O \) the level curves of \( F \) intersect each normal to \( C^u_0 \) in exactly one point. By the implicit function theorem the same is true for \( C^u \) since the two curves are \( C^1 \) close.) Therefore on the line joining \( y_q \) to \( y_*(s_q) \), which is normal to \( C^u_0 \), we can use \( F \) as a measure of distance, and \( E(y_q) = E(y_\dot{q}) \) would imply \( y_q = y_\dot{q} \) or, equivalently, \( z_q = z_\dot{q} \). Since the \( v_0 \)-coordinates of \( q \) and \( \dot{q} \) are the same by definition, we conclude that the zeros of \( d(q) = E(y_q) - E(y_\dot{q}) \) correspond to fixed points of \( q \rightarrow \dot{q} \).

To expand \( d(q) \), we note that

\[
|y_q - y_*(s_q)| = |r_q| = O(\epsilon)
\]

since from equation (5.40) \( |r_q| = O(\epsilon) \). Moreover, \( y_*(s_q) \) is order \( \sqrt{\epsilon} \) away from \( C^u_0 \), the unperturbed stable manifold; therefore

\[
E(y_\dot{q}) = E(y_*(s_q)) + O(\epsilon) = E_0 + O(\sqrt{\epsilon})
\]

This completes the proof of the proposition.

We will use the function \( d \) to measure the distance from \( q_{\ell,b} \) to \( W \). From (7.12) we have \( q_{\ell,b} \) and \( q_{o,b} \) are order \( O(\epsilon) \) apart, which implies that

\[
d(q_{\ell,b}) = d(q_{o,b}) + O(\epsilon)
\]

Moreover, since the base point of \( q_{o,b} \) is

\[
r_0 \epsilon^{i(h_b - 4\pi)} = (\omega^2 + \sqrt{\epsilon} j_u(s_b))^{1/2} e^{i(h_b - 4\pi)}
\]

we have the following corollary:

**Corollary 7.4** The distance from \( q_{\ell,b} \) to \( W \) can be measured by

\[
d(q_{\ell,b}) = \omega [2 \sin 2p_0 \cos(\theta_b - 2p_0) + 4\omega p_0] + O(\sqrt{\epsilon})
\]

**Proof:** From the definition of \( d \) we have

\[
d(q_{o,b}) = E(j_u(s_b), \theta_b - 4p) - E_0,
\]

and since \( q_{o,b} \in C^u_0 \), which is an order-\( O(\sqrt{\epsilon}) \) perturbation of \( C^u_0 \), we also have

\[
E(j_u(s_b), \theta_b) - E_0 = O(\sqrt{\epsilon})
\]
This fact implies that
\[ d(q_{o,b}) = E(j_u(s_b), \theta_b - 4p) - E(j_u(s_b), \theta_b) + O(\sqrt{\epsilon}) \]
\[ = -\omega [\sin(\theta_b - 4p) - \sin \theta_b + 4\alpha \omega p] + O(\sqrt{\epsilon}) \]
\[ = \omega [2 \sin 2p \cos(\theta_b - 2p) + 4\alpha \omega p] + O(\sqrt{\epsilon}). \]

Finally, observing that
\[ p = \tan^{-1}(r'_b - 1)^{1/2} = \tan^{-1}(\omega^2 - 1)^{1/2} + O(\sqrt{\epsilon}) = p_0 + O(\sqrt{\epsilon}) \]
concludes the proof of the corollary.

Note that if we define
\[ (7.13) \quad \tilde{d}(\theta_b, \alpha) := 2 \sin 2p_0 \cos(\theta_b - 2p_0) + 4\alpha \omega p_0, \]
and if \( \tilde{d} \) has a nondegenerate zero, we again conclude that the parameters can be chosen so that \( d \) vanishes.

### 7.3 Existence of a Homoclinic Orbit

Fix \( \alpha \in (0, 1/\omega) \) and consider \( M \) and \( \tilde{d} \) as functions of \( \theta_b \) and \( \beta \),
\[ M(\alpha, \beta, \theta_b) = \alpha M_\alpha + \beta M_\beta + M_0 \cos(\theta_b - 2p_o), \]
\[ \dot{d}(\theta_b, \alpha) = 2 \sin 2p_0 \cos(\theta_b - 2p_0) + 4\alpha \omega p_0. \]

In order to prove the existence of a homoclinic orbit to \( Q \), it is sufficient to show that \( M \) and \( \tilde{d} \) vanish in a nondegenerate manner for some \( \beta > 0 \) and \( \theta_b \in (\theta_{\min}, \theta_0) \), which is the range of \( \theta \) for the unperturbed homoclinic orbit in the plane:
\[ \theta_0 = \tan^{-1} \left( \frac{\sqrt{1 - \alpha^2 \omega^2}}{\alpha \omega} \right) - \pi \]
\[ \sin(\theta_{\min}) + \alpha \omega \theta_{\min} = \sin \theta_0 + \alpha \omega \theta_0, \quad \theta_{\min} < \theta_0. \]

For \( \tilde{d} \) to vanish, the range of \( \alpha \) has to be restricted further to be in the interval
\[ (0, \alpha_*) := (0, 1/\omega) \cap (0, 2\omega p_o/\sin 2p_o). \]

Moreover, if \( M_\beta \) is not equal to 0, then we can solve \( \tilde{d} = 0 \) and \( M = 0 \) at the point
\[ \cos(\theta_b - 2p_o) = -\frac{2\alpha \omega p_o}{\sin 2p_o} \]
\[ \beta = -\alpha \left[ M_\alpha + \frac{2\omega p_o}{\sin 2p_o} M_0 \right] / M_\beta. \]
By the implicit function theorem, for $\epsilon$ small we can solve $d = \Delta = 0$ in a small neighborhood of the point given by equations (7.14) and (7.15).

**Theorem 7.5** Fix $\omega \in (1/\sqrt{2}, 1)$ and $\alpha \in (0, \alpha_\ast)$. If $\theta_0$ and $\beta$ given by equations (7.14) satisfy $\theta_0 \in (\theta_{\text{min}}, \theta_0)$ and $\beta > 0$, then for small $\epsilon$, equation (7.1) has a symmetric pair of homoclinic orbits.

**Proof:** By the implicit function theorem, for $\epsilon$ small we can solve $d = \Delta = 0$ in a small neighborhood of the point given by equations (7.14). Since $d = 0$, the point $q_t$ lies on a fiber whose base point $q_{t, b} \in W$, the stable manifold of $Q$. Therefore the orbit $q_s(t)$ for $t \geq 0$ (which solves the cutoff equations and passes through $q_t$ at $t = 0$) remains in a small neighborhood of $S_\omega$. This fact implies that $q_s(t)$ solves the original equation, and the orbit $q(t) = \begin{cases} q_u(t) & t \leq 0 \\ q_s(t) & t \geq 0 \end{cases}$ is homoclinic to $Q$. Since the unperturbed system has two homoclinic orbits $q^\pm_h$, this argument establishes the existence of a symmetric pair. 

**Remark.**

1. The simultaneous zeros of $d$ and $\Delta$, given by equations (7.14), provide qualitative information about the system. The first equation in the pair provides an approximate representation of the takeoff angle $\theta_0$ as a function of the parameters of the system. The second equation provides an approximate representation of the one constraint that the parameters must satisfy in order for a homoclinic orbit to persist.

2. It can be shown that if $\beta = 0$ and $\alpha > 0$, a simultaneous zero of the functions $d$ and $\Delta$, equations (7.14), does not exist. This shows the necessity for introducing additional freedom in the parameters of the system as is provided by the additional dissipation $\beta Bq$.

Examples of the dissipation operator $\widetilde{B}$ includes the discrete Laplacian,

\begin{equation}
\widetilde{B}q = q(x + h) - 2q(x) + q(x - h),
\end{equation}

for fixed $h > 0$, and a "smoothed Laplacian" defined through its symbol $b(k)$,

\begin{equation}
b(k) = \begin{cases} k^2 & k < \kappa \\ 0 & k \geq \kappa, \end{cases}
\end{equation}
where $\kappa$ is a large positive integer. We will describe this latter case in detail. Very similar arguments apply for the discrete Laplacian.

In the case of the smoothed Laplacian, equation (7.17), the dissipative terms $\alpha M_\alpha + \beta M_\beta$ of the Melnikov integral are given by the explicit formula

$$M_\alpha = \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} dx \frac{4\pi\omega^2 \sin^2 p_o \text{sech} \tau}{\sigma A^3} \times \left[ \text{sech} \tau + \sin p_o \tanh^2 \tau \cos x - \sin^2 p_o \text{sech} \tau (2 + \cos^2 x) + 2 \sin^3 p_o \text{sech}^3 \tau \cos x \right]$$

$$M_\beta = \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} dx \frac{4\pi\omega^2 \sin^2 p_o \text{sech} \tau}{\sigma A^5} \times \left[ \sin p_o \text{sech} \tau \cos x - \sin^2 p_o \text{sech}^2 \tau (1 + \sin^2 x) \right] \times \left[ 2 \text{sech} \tau - \sin p_o \text{sech}^2 \tau \cos x - \sin^2 p_o \text{sech} \tau + 2 \sin^3 p_o \text{sech}^2 \tau \cos x \right] + O(\sin^{\kappa-2} p_o),$$

where $p_o = \tan^{-1} \sqrt{\omega^2 - 1}$ and $A = 1 - \sin p_o \text{sech} \tau \cos x$, and where the $O(\sin^{\kappa-2} p_o)$ term in the $M_\beta$ equation is due to the fact that we used $-\partial_x^2$ instead of $\hat{B}$ in our computation.

With these formulas, one can return to equation (7.15) and write

$$\beta = \kappa(\omega) \alpha,$$

where

$$\kappa(\omega) \equiv - \left[ M_\alpha - \frac{2\omega p_o}{\sin 2p_o} M_0 \right] / M_\beta.$$
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Figure 7.5. \( \kappa \) as a function of \( \omega \).

expand \( M_\alpha \), \( M_\beta \), and \( M_0 \) in powers of \( p_0 \) (actually \( \sin p_0 \approx p_0 \)) to obtain

\[
M_\alpha = \frac{4\pi \omega^2 \sin^2 p_0}{\sigma} \int_{-\infty}^{\infty} \int_0^{2\pi} \left\{ \text{sech}^2 \tau + \sin^2 p_0 \text{sech}^2 \tau \times \left[ -2 + (6 \text{sech}^2 \tau + 3 \tanh^2 \tau - 1) \cos^2 x \right] \right\} + O(p_0^6)
\]

\[
M_\beta = \frac{4\pi \omega^2 \sin^4 p_0}{\sigma} \int_{-\infty}^{\infty} \int_0^{2\pi} \text{sech}^4 \tau (10 \cos^2 x - 4) + O(p_0^6 + \sin^{\kappa - 2} p_0)
\]

\[
M_0 = \frac{4\pi \omega \sin^2 p_0}{\sigma} \int_{-\infty}^{\infty} \int_0^{2\pi} \text{sech}^2 \tau + \sin^2 p_0 \text{sech}^2 \tau \times (3 \text{sech}^2 \tau - 2) \cos^2 x + O(p_0^6)
\]

Therefore if \( \omega \) is close to \( 1/\sqrt{2} \) and \( \kappa \) is large, we have \( M_\beta > 0 \) and

\[
M_\alpha - \frac{2\omega p_0}{\sin 2p_0} M_0 = M_\alpha - \omega M_0 - \frac{2}{3} \omega M_0 p_0^2 + O(p_0^4)
\]

\[
= -\frac{2}{3} \omega M_0 p_0^2 + O(p_0^4) < 0.
\]

This fact implies that \( \beta > 0 \). Now it is easy to check that this zero of \( \tilde{d} \) and \( M \) is nondegenerate, which implies by the implicit function theorem that for fixed \( \alpha \) and \( \omega \) we can solve for \( \theta_0 \) and \( \beta \) in terms of \( \epsilon \) to obtain \( d = \Delta = 0 \).
REMARK.

1. An identical analysis applies to other bounded approximations of the Laplacian, such as the finite difference operator (7.16).

2. Since the unperturbed problem has a symmetric pair of homoclinic orbits, we immediately conclude that the same is true for the perturbed problem. In addition, it is clear that equations (7.14) may have multiple solutions in the specified range. Indeed, if we plot the trajectory of \( q_{\alpha,b} \) as a function of \( \theta_b \), we obtain several intersections with the stable manifold \( C_0^\beta \) (these points are solutions of (7.14)), and the number of intersections depends on the value of \( \alpha \). Thus we may have several symmetric pairs of homoclinic orbits.

Acknowledgement. The authors would like to thank P. Bates and G. Haller for many useful discussions. David W. McLaughlin was funded in part by AFOSR-90-0161 and National Science Foundation Grant DMS 8922717 A01. Jalal Shatah was funded in part by National Science Foundation Grant DMS 9401558. S. Wiggins was funded in part by National Science Foundation Grant DMS 9403691.

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Received October 31, 1995.
Revised February 26, 1996.