Simple explicit formulae for finite time blow up solutions to the complex KdV equation

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Abstract

Simple explicit formulae for finite time blow up solutions to the complex KdV equation are obtained via a Darboux transformation. Diffusions induced by perturbations are calculated.

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1. The formulae

By viewing the variable \( u(t, x) \) in the KdV equation to be a complex-valued function of still two real variables \((t, x)\), one obtains the so-called complex KdV equation. Complex KdV generated some interests recently \([2,9,3,4]\). Unlike the original real KdV, complex KdV is an explosive equation that has abundant finite time blow up solutions \([2,9,1]\). Complex KdV also has applications \([5,7,8]\).

In this note, we will derive some simple explicit formulae for some finite time blow up solutions to the complex KdV equation. For example, the following

\[
 u(t, x) = i + \frac{8\exp[12(1 - t) + i(8t + 2x)]}{\exp[12(1 - t) + i(8t + 2x)] + 1^2}
\]

(1.1)

is a simple finite time blow up solution. When \( t = 0, u(0, x) \) is \( C^\infty \). Finite time blow up is developed when \( t = 1 \), with two singularities of \( u(1, x) \) at \( x = \frac{3\pi}{4} \) and \( x = \frac{5\pi}{4} \). When \( t \in [0, 1) \), \( u(t, x) \) is \( C^\infty \) in both \( x \) and \( t \). In fact, this solution represents a finite time blow up homoclinic orbit. As \( t \to \pm \infty \), \( u \to i \).

The complex KdV equation

\[
u_t = 6uu_x - u_{xxx}
\]

has a Lax pair

\[
 L\psi = \lambda \psi, \quad \psi_t = A\psi,
\]

where

\[
 L = - \partial_x^3 + u, \quad A = -4\partial_x^3 + 6u\partial_x + 3u_x.
\]

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$u$ and $\psi$ are complex-valued functions of two real variables $(t,x)$, and $\lambda$ is a complex spectral parameter.

Let $u$ be a solution to the complex KdV, and $\varphi$ be a solution to the Lax pair at $\lambda = \nu$ for some $\nu$, define

$$U = u - 2\partial_t^2 \ln \varphi, \quad \Psi = \psi_x - (\partial_x \ln \varphi)\psi,$$

does the Lax pair at $\lambda$ and an arbitrary $\lambda$. Then $U$ is also a solution to the complex KdV, and $\Psi$ solves the Lax pair at $\lambda$ and the same arbitrary $\lambda$. This is the so-called Darboux transformation.

Next we pose the periodic boundary condition

$$u(t,x + 2\pi) = u(t,x)$$

to the complex KdV. For any complex constant $a = a_c + i a_i$, $u(t,x) = a$ is a solution to the complex KdV. For any $k \in \mathbb{Z}$, the Lax pair has two linearly independent solutions at $\lambda = a$ and $\lambda = \nu = a + k^3$:

$$\psi_{\pm} = \exp\{\pm(\omega t + ikx)\},$$

where

$$\omega = 6iak + 4ik^3.$$

Let

$$\varphi = c_+\psi_+ + c_-\psi_-,$$

where $c_{\pm}$ are two arbitrary complex constants. Applying the Darboux transformation, one obtains

$$U = a + 2k^2 \left[ 1 - \frac{(\exp\{2\omega t + i2kx + \nu + i\gamma\} - 1)^2}{(\exp\{2\omega t + i2kx + \nu + i\gamma\} + 1)^2} \right],$$

where $c_+/c_- = \exp\{\nu + i\gamma\}$ and $(\nu, \gamma)$ are arbitrary real constants. Let $\omega = \omega_r + i\omega_i$, where

$$\omega_r = -6ak, \quad \omega_i = 6ak + 4k^3.$$

The singularities of the solution happens at

$$t = -\frac{\rho}{2\omega_r}, \quad x = \frac{1}{2k} \left[ \frac{\rho \omega_i}{\omega_r} - \gamma + (2n + 1)\pi \right], \quad \forall n \in \mathbb{Z}.$$

When $\omega_i \neq 0$, the solution represents a finite time blow up homoclinic orbit asymptotic to $u = a$. That is, as $t \to \pm\infty$, $U \to a$. When $\omega_i = 0$ and $\rho \neq 0$, the solution is a $C^\infty$ global complex-valued solution. When $\omega_i = 0$ and $\rho = 0$, the solution has singularities for any $t$. Finally, the choice $K = 1, a = i, \rho = 12$, and $\gamma = 0$ reduces the solution to (1.1).

Of course, one can iterate the Darboux transformation to get more and more solutions. Next we linearize the complex KdV around $u = a$, we get

$$u_t = \mathcal{L}u,$$

where

$$\mathcal{L}u = 6\omega_i u_x - u_{xxx}.$$

The spectrum of $\mathcal{L}$ consists of eigenvalues

$$\sigma = 6iak + ik^3, \quad k \in \mathbb{Z}.$$

When $\omega_i \neq 0$, $\mathcal{L}$ cannot generate a $C_0$ semigroup. In fact, for any $u(0,x) \in H^s$ (The Sobolev space on the periodic domain $[0,2\pi]$),

$$e^{\tau \mathcal{L}}u(0,x) \notin H^s, \quad \forall \tau \neq 0.$$

Locating the complex KdV around $u = a$ by setting $u = a + v$, one gets

$$v_t = \mathcal{L}v + 6\omega_i v_x.$$

The complex KdV in this case is not locally well-posed in $H^s$ for any $s$.

Finally we do some formal calculation on diffusions. There is an infinite sequence of invariants for (complex) KdV [6]:

$$I_n = \int_0^{2\pi} Q_n(u)dx, \quad n = 0, 1, 2, \ldots;$$
where
\[ Q_n(u) = [u^{(n)}]^2 + \alpha_1 u [u^{(n-1)}]^2 + \ldots \]
is a linear combination of monomials
\[ [u^{(m_1)}]_{m_1} \ldots [u^{(m_r)}]_{m_r} \]
of index
\[ \text{ind} = \sum_{i=1}^{r} m_i + \frac{1}{2} \sum_{r=1}^{c} m_rm_l \]
equal to \( n + 2 \). Of course, the mean
\[ M = \int_{0}^{2\pi} u \, dx \]
is also an invariant. We like to know the diffusions of these invariants under the perturbed complex KdV flow
\[ ut = 6uux - u_{xxx} + eF(u). \]
All the time derivatives of the invariants are evaluated along (1.1).

When \( F(u) = u_{xx} \), the real and imaginary parts of \( u \) receive equal decay from this perturbation:
\[ \frac{dM}{dt} = e \int_{0}^{2\pi} u \, dx = 0, \quad \text{except} \quad t = 1; \]
thus, \( \lim_{t \to 1} \frac{dM}{dt} = 0. \)

\[ \frac{dI_0}{dt} = 2e \int_{0}^{2\pi} |u|^2 \, dx \]
where \( \theta = 12(t - 1) + i(8t + 2x) \). Thus, \( \lim_{t \to 1} \frac{dI_0}{dt} = 0. \) In fact,
\[ \frac{dI_n}{dt} = 0, \quad \text{except} \quad t = 1; \]
for all \( n \). Thus, the perturbation \( F(u) = u_{xx} \) is a diffusionless perturbation.

When \( F(u) = \bar{u} \), the real part of \( u \) grows and the imaginary part decays from this perturbation:
\[ \frac{dM}{dt} = e \int_{0}^{2\pi} \bar{u} \, dx = -\epsilon \alpha 2\pi, \quad \text{except} \quad t = 1; \]
thus, \( \lim_{t \to 1} \frac{dM}{dt} = -\epsilon \alpha 2\pi. \)

\[ \frac{dI_0}{dt} = 2e \int_{0}^{2\pi} |u|^2 \, dx \]
where \( \theta_1 = 12(t - 1) \) and \( \theta_2 = 8t + 2x \). There is a \( \delta > 0 \) such that
\[ \cos \theta_2 < -1 + \frac{1}{2} \epsilon^2, \quad \zeta \in [-\delta, \delta], \quad \theta_2 = \pi + \zeta. \]

Let \( t = 1 - \beta \), then the integral is larger than
\[ \frac{1}{2} \int_{-\delta}^{\delta} \frac{1}{(1 + e^{2\beta} - 2e^{12\beta} + e^{12\beta} \zeta^2)^{\frac{3}{2}}} \, d\zeta \]
which is larger than \( \frac{1}{2} \epsilon^{-3} \) when \( \beta \) is sufficiently small. Letting \( \delta \to 0^+ \), one gets
\[ \lim_{t \to 1} \frac{dI_0}{dt} = +\infty. \]
Thus, the perturbation \( F(u) = \bar{u} \) is a super-diffusive perturbation.
Remark 0.1. One can also construct finite time blow up solutions for the whole line problem. By taking \( a = 0 \) and relaxing \( k \) to be complex in the formula of \( U \), one gets

\[
U = 2k^2 \left[ 1 - \frac{\exp\{i8k^3t + i2kx + \rho + i\gamma\} - 1}{\exp\{i8k^3t + i2kx + \rho + i\gamma\} + 1} \right] ^2.
\]

The plan is to choose \((k, \rho, \gamma)\) such that

\[
i8k^3t + i2kx + \rho + i\gamma \neq i\pi \text{ at } t = 0, \text{ and } = i\pi \text{ at some } t > 0.
\]

Let \( k = k_r + ik_i, k_r \neq 0 \) and \( k_i \neq 0 \); pick \( \rho \) and \( \gamma \) such that

\[
k_r\rho + k_i(\gamma - \pi) \neq 0,
\]

then at \( t = 0, U \) is \( C^\infty \) in \( x \) on the whole line, and decays exponentially at infinity. Its finite time blow up happens at

\[
t = \frac{k_r\rho + k_i(\gamma - \pi)}{16k_r k_i (k_r^2 + k_i^2)}, \quad x = \frac{k_r\rho (3k_r^2 - k_i^2) + k_i(\gamma - \pi)(k_r^2 - 3k_i^2)}{4k_r k_i (k_r^2 + k_i^2)}.
\]

A simple example of these finite time blow up solutions is

\[
U = \frac{16i \exp\{-2[(x - 8) + 8(t - 1)] + i2[(x - 8) - 8(t - 1)] + i\pi\} - 1}{\exp\{-2[(x - 8) + 8(t - 1)] + i2[(x - 8) - 8(t - 1)] + i\pi\} + 1}^2,
\]

for which the finite time blow up happens at

\[
t = 1, \quad x = 8.
\]

References