Invariant Manifolds and Their Fibrations
for Perturbed Nonlinear Schrödinger
Equation

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1 Introduction

In this introductory chapter we want to give a brief survey of results on invariant manifolds and fibers in infinite dimensions as well as describe the aims and scopes of this monograph. Recent books by Wiggins [99] and Bronstein and Kopanskii [13] survey invariant manifold results in finite dimensions. Here we will be focussing solely on surveying results in the infinite dimensional setting.

1.1 Invariant Manifolds in Infinite Dimensions

The finite dimensional invariant manifold theory is not only very general, from the point of view of mathematical assumptions, but it is broadly applicable to a variety of specific systems arising in applications. Of course, this is true because the general mathematical assumptions are “reasonable” for many dynamical systems that arise in practice. For example, in the case of vector fields on $\mathbb{R}^n$, it is common for the evolution operators arising in applications to be smooth in both time and initial data. In contrast, the evolution operators for partial differential equations arising in applications are often at best continuous in time. More importantly, the nature of the evolution operators for partial differential equations differs greatly from one system to another. This fact makes a theory as general as in finite dimensional systems intractable.

From the literature one sees that there are two broad approaches to invariant manifold theory in infinite dimensions. One is to state a list of hypotheses satisfied by a class of infinite dimensional dynamical systems and then prove theorems for the class of dynamical systems satisfying such hypotheses. \(^1\) The other approach is to take a specific infinite dimensional dynamical system and prove the desired invariant manifold theorems specifically for the system (of course, one would first need some form of existence, uniqueness, and regularity theory for the specific dynamical system under investigation).

Many of the issues concerning invariant manifold theory in infinite dimensions are much the same as those in finite dimensions. The first approach described above is guided by the goal of “lifting” to infinite dimensions the

\(^1\)Here we will interpret the phrase *infinite dimensional dynamical system* broadly in that it applies to a semiflow, or flow on an appropriate infinite dimensional space. We also refer to evolutionary partial differential equations as infinite dimensional dynamical systems.
finite dimensional results. It is the second approach that often results in the discovery of phenomena that are truly infinite dimensional. Below we survey many of the known results related to invariant manifolds in infinite dimensions in the context of a number of specific issues that arise when seeking to develop and apply such results. Because the literature is so large we limit our survey to papers that prove new theorems for various classes of infinite dimensional dynamical systems, as well as a description of the areas of applications to which these theorems can be applied. As a result, we are leaving out a portion of the literature dealing with potential applications of these results.

Existence of Invariant Manifolds

As in the finite dimensional case, invariant manifold theory begins by assuming that some “basic” invariant manifold exists—the theory is then developed from this point onwards by building upon this basic invariant manifold. This may seem a bit strange, for one might think that invariant manifold theory is concerned with determining the existence of invariant manifolds. That is true, to a point. However, if one were to ask the general question:

*given a specific dynamical system (finite or infinite dimensional), what type of invariant manifolds does it possess?*

it probably could not be answered in any kind of generality. Thus, invariant manifold theory requires some type of “seed” in order to get started. One type of seed is knowledge of a particular orbit of the dynamical system. Such orbits can be viewed as invariant manifolds. Examples of orbits that can often be (relatively) easily found are:

1. Equilibrium points,
2. Periodic orbits,
3. Quasiperiodic or almost periodic orbits (invariant tori).

Starting from this point, one may then construct stable, unstable, and center manifolds associated with these basic invariant manifolds, as well as
fibrations of these manifolds, and also consider issues such as differentiability of the invariant manifolds and fibrations and their persistence under perturbation.

These particular invariant manifolds all share an important property. Namely, they all admit a global coordinate description in the sense that they are either graphs or parametrized curves. In this case the dynamical system is typically subjected to a “preparatory” coordinate transformation that serves to localize the dynamical system about the invariant manifold. This amounts to deriving a normal form in the neighborhood of an invariant manifold and it greatly facilitates the various estimates that are required in the analysis. Henry [45] has derived a normal form for a class of semilinear parabolic partial differential equations in the neighborhood of a compact, finite dimensional invariant manifold, with the important condition that the normal bundle of the invariant manifold is trivial. Bates et al. [5] consider the situation of a compact invariant manifold in Hilbert space that is not described as a graph and requires an atlas of local coordinate charts for their description. We will describe the results of Henry [45] and Bates et al. [5] in more detail shortly.

The other “seed” which may serve as a starting point for invariant manifold theory is if the dynamical system under consideration possesses some special structure so that the existence of an invariant manifold is “obvious”. For example, for the nonlinear Schrödinger equation it is clear from the structure of the equations that the solutions that are independent of the space variable form a two dimensional invariant plane in the infinite dimensional function space on which the nonlinear Schrödinger equation is defined. This trivial invariant plane will play a central role in the next chapter of this monograph.

**Behavior Near an Invariant Manifold–Stable, Unstable, and Center Manifolds**

A “stable manifold theorem” asserts that the set of points that approach an invariant manifold at an exponential rate as $t \to +\infty$ is an invariant manifold in its own right. The exponential rate of approach is inherited from the linearized dynamics as the stable manifold is constructed as a graph over the linearized stable *subspace* or *subbundle*. An “unstable manifold theorem” asserts similar behavior in the limit as $t \to -\infty$. Obviously, one may have problems with both of these concepts if the invariant manifold has a boundary.
The notion of a center manifold is more subtle. For equilibrium points and periodic orbits, a center manifold is an invariant manifold that is tangent to the linearized subspace corresponding to eigenvalues on the imaginary axis, and Floquet multipliers on the unit circle, respectively. In contrast to the situation with stable and unstable manifolds, the asymptotic behavior of orbits in the nonlinear center manifold may be very different from the asymptotic behavior of orbits in the linearized center subspaces, under the linearized dynamics.

Once the existence of stable, unstable, and center manifolds is established then questions related to persistence and differentiability arise quite naturally.

There are a number of papers that deal with stable, unstable, and center manifolds of equilibrium points of partial differential equations. In particular, we refer to Ball [3], Bates and Jones [4], Carr [15], Chow and Lu [18], da Prato and Lunardi [86], Henry [45], Keller [47], Mielke [76], Renardy [87], and Scarpellini [90], [91]. Below we summarize the types of equations and applications considered by these authors.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Type of Partial Differential Equation</th>
</tr>
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<tbody>
<tr>
<td>Ball [3]</td>
<td>semilinear hyperbolic equations</td>
</tr>
<tr>
<td>Bates and Jones [4]</td>
<td>semilinear hyperbolic and parabolic equations</td>
</tr>
<tr>
<td>Carr [15]</td>
<td>semilinear hyperbolic equations</td>
</tr>
<tr>
<td>Chow and Lu [18]</td>
<td>semilinear hyperbolic and parabolic equations</td>
</tr>
<tr>
<td>da Prato and Lunardi [86]</td>
<td>quasilinear parabolic equations</td>
</tr>
<tr>
<td>Henry [45]</td>
<td>semilinear parabolic equations</td>
</tr>
<tr>
<td>Keller [47]</td>
<td>semilinear hyperbolic equations</td>
</tr>
<tr>
<td>Renardy [87]</td>
<td>quasilinear hyperbolic equations</td>
</tr>
<tr>
<td>Scarpellini [90], [91]</td>
<td>semilinear hyperbolic equations</td>
</tr>
<tr>
<td>Mielke [76]</td>
<td>quasilinear hyperbolic equations</td>
</tr>
<tr>
<td>da Prato and Lunardi [86]</td>
<td>quasilinear parabolic equations</td>
</tr>
</tbody>
</table>

We remark that in dealing with center manifolds, the issue of whether or not it is infinite dimensional can pose some technical difficulties. The work of Bates and Jones [4] deals with the infinite dimensional case.

Many of the references above treat specific applications. We list them in the following table.
Regularity of Stable, Unstable, and Center Manifolds

Existence of stable, unstable, and center manifolds is typically proven by some form of contraction mapping argument. From this type of argument one obtains Lipschitz manifolds, even if the nonlinearity is differentiable. To prove the existence of smooth manifolds one formally differentiates the equation describing the manifold and, in this way, derives an equation which the derivative of the manifold must satisfy. Using contraction mapping arguments, one then proves that this equation has a solution. Finally, using the definition of Frechet derivative, one proves that the solution is indeed the derivative; thus, establishes the differentiability. One can repeat this argument and, thus, inductively prove the existence of more derivatives, provided the structure of the partial differential equation allows this. If one is interested in stability of the equilibrium solution, then Lipschitz is sufficient. However, for bifurcation theory one usually needs several derivatives.

We summarize the regularity of the manifolds in the references described above in the following table.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Application</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ball [3]</td>
<td>buckling beam</td>
</tr>
<tr>
<td>Bates and Jones [4]</td>
<td>nonlinear wave equations, nerve impulse equations</td>
</tr>
<tr>
<td>Carr [15]</td>
<td>semilinear wave equation</td>
</tr>
<tr>
<td>Chow and Lu [18]</td>
<td>reaction-diffusion equations, singularly perturbed wave equation</td>
</tr>
<tr>
<td>Henry [45]</td>
<td>combustion, Hodgkin-Huxley equations, plus many more</td>
</tr>
<tr>
<td>Keller [47]</td>
<td>nonlinear wave equations</td>
</tr>
<tr>
<td>Renardy [87]</td>
<td>Bénard problem for a viscoelastic fluid</td>
</tr>
<tr>
<td>Scarpellini [90], [91]</td>
<td>Sine-Gordon Equation</td>
</tr>
</tbody>
</table>
Fibrations of Stable and Unstable Manifolds — More Refined Behavior Near Invariant Manifolds

One may be interested in which orbits in the stable manifold approach the invariant manifold at a specified rate. Under certain conditions these orbits may lie on submanifolds of the stable manifold which are not invariant, but make up an invariant family of submanifolds that fiber or foliate the stable manifold. A similar situation may hold for the unstable manifold. Moreover, this fibration has the property that points in a fiber of the fibration asymptotically approach the trajectory in the invariant manifold that passes through the point of intersection of the fiber with the invariant manifold (the basepoint of the fiber). This is a generalization of the notion of asymptotic phase, that is familiar from studies of stability of periodic orbits, to arbitrary invariant manifolds. In recent years these foliations have seen many uses in applications in the finite dimensional setting, see Jones [46], and Wiggins [99]. Chow et al. [17] prove a fibration theorem in a very general setting. They do not consider any explicit examples in their paper, however they do remark that their results are related to inertial manifold type results. Li et al. [61] prove a fibration theorem for the nonlinear Schrödinger equation. Ruelle [88] constructs stable and unstable fibrations almost everywhere for a semi-flow in Hilbert space having a compact invariant set (he assumes that the linearized semi-flow is compact and injective, with dense range). His results were later extended by Mañé [80].

The Persistence and Differentiability of Invariant Manifolds Under Perturbation

The question of whether or not an invariant manifold persists under perturbation and, if so, if it maintains, loses, or gains differentiability is also important. In considering these issues it is important to characterize the stability of the unperturbed invariant manifold. This is where the notion of normal hyperbolicity arises. Roughly speaking, a manifold is normally hyperbolic if, under the dynamics linearized about the invariant manifold, the growth rate of vectors transverse to the manifold dominates the growth rate of vectors tangent to the manifold. For equilibrium points, these growth rates can be characterized in terms of eigenvalues associated with the linearization at the equilibria that are not on the imaginary axis, for periodic orbits these growth rates can be characterized in terms of the Floquet multipliers associated with the linearization about the periodic orbit that are
not on the unit circle, for invariant tori or more general invariant manifolds these growth rates can be characterized in terms of exponential dichotomies (see Coppel [21] or Sacker and Sell [89]) or by the notion of generalized Lyapunov type numbers (Fenichel [31]).

Characterizing growth rates in the fashion described above requires knowledge of the linearized dynamics near orbits on the invariant manifold as $t \to +\infty$ or $t \to -\infty$. Hence, if the invariant manifold has a boundary (which an equilibrium point, periodic orbit, or invariant torus does not have), then one must understand the nature of the dynamics at the boundary. Notions such as overflowing invariance or inflowing invariance were developed by Fenichel [31], [32], [33] to handle this. Invariant manifolds with boundary arise very often in applications, see Wiggins [98] for finite dimensional examples.

A question of obvious importance for applications is how does one compute whether or not an invariant manifold is normally hyperbolic? The answer is not satisfactory. For equilibria, the problem involves finding the spectrum of a linear operator. For invariant manifolds on which the dynamics is nontrivial the issues are more complicated.

However, one important class of dynamical systems which may have nontrivial invariant manifolds on which the dynamics is also nontrivial are integrable Hamiltonian systems, see Wiggins [98] for finite dimensional examples.

Bates et al. [5] have a very general result on the persistence of a compact, normally hyperbolic invariant manifold under a semiflow. They show that if both the unperturbed and perturbed semiflows are $C^1$, and the unperturbed invariant manifold is $C^2$, then the perturbed invariant manifold is $C^1$. They do not require the unperturbed invariant manifold to have a trivial normal bundle. Their work can be viewed as an infinite dimensional generalization of Fenichel’s [31] seminal finite dimensional work. Li et al. [61] prove a persistence theorem for normally hyperbolic invariant manifolds, as well as their stable and unstable manifolds and fibrations, for the nonlinear Schrödinger equation. The set-up is different than that of Bates et al. [5] in that Li et al. consider a noncompact invariant manifold invariant under a flow (rather than a semi-flow). Moreover, the unperturbed invariant manifold is globally represented as a graph. Their invariant manifold, along with its stable and unstable manifolds, are $C^k$, $k > 3$ and the fibers are $C^{k-2}$.

**Inertial Manifolds**
In recent years there has been much interest in a class of global, attracting, finite dimensional invariant manifolds in dissipative systems—*inertial manifolds*. These invariant manifolds are robust objects and contain the long term dynamical phenomena of a system. In this way, they provide a rigorous manner of reducing an infinite dimensional dynamical system to a finite dimensional dynamical system, when the questions of interest are concerned with asymptotic behavior. The following references give an account of this theory: Constantin et al. [20], Chow and Lu [18], Foias and Sell [35], Foias et al. [36], Hale [41], Mallet-Paret and Sell [66], Mora [78], Mora and Solà-Morales [79], Témam [94].

**Invariant Manifolds in Infinite Dimensional Hamiltonian Systems**

Invariant tori (quasiperiodic solutions) are a typical type of invariant manifold that arise in Hamiltonian systems. The KAM theorem is a well-known result describing persistence of invariant tori in perturbations of completely integrable systems in finite dimensions. In recent years there have been a number of generalizations of this result to various infinite dimensional settings, see Kuksin [56], [55], [54], [53], [52], Pöschel [84], [83], and Wayne [97], [95], [96], [12]. Nikolenko [81] considers the existence of asymptotically stable tori in a dissipative perturbation of an infinite dimensional Hamiltonian system (the Korteweg-de Vries equation). Mielke [75] gives a general treatment of invariant manifolds in a class of Hamiltonian partial differential equations.

**Normal Forms for Partial Differential Equations**

The classical Poincaré normal form theory is not unrelated to invariant manifold theory. Normal form theory is concerned with constructing coordinate changes that make the dynamical system as “simple as possible”.

“The simple” could mean linear, or it might mean eliminating terms whereby the invariant manifold structure of the resulting equation is transparent. Indeed, many invariant manifold theorems are stated in this way.

Normal forms for partial differential equations have been considered by Eckmann et al. [27], Nikolenko [82], and Shatah [92], Shatah and McKean [93]. A normal form calculation for the nonlinear Schrödinger equation is given in Li et al. [61], and indeed is central to their persistence theorems for invariant manifolds and fibrations.
Methods of Proof

As described in Wiggins [99], there are four approaches to the proof of invariant manifold results in finite dimensions.

- The Lyapunov- Perron Method
- Hadamard’s Method-The Graph Transform
- The Lie Transform or Deformation Method
- Irwin’s Method

In infinite dimensions the Liapunov-Perron method has been the most common approach. Bates and Jones [4] and Bates et al. [5] use the graph transform approach, and the results in this monograph are based entirely on the graph transform approach.

1.2 Aims and Scopes of This Monograph

This monograph is motivated by our work on the perturbatively damped and driven Nonlinear Schrödinger (NLS) equation described in [8], [9], [7], [14], [28], [29], [30], [42], [44], [43], [48], [49], [50], [51], [59], [64], [65], [63], [61], [70], [71], [73], [74], and [72]. In this monograph we prove the basic existence, persistence, and differentiability results for invariant manifolds and fibrations of the perturbatively, damped and driven NLS equation that we use in our analysis. Our technique is based on an infinite dimensional version of the graph transform. Our setting is somewhat general, however the generality is motivated by the structure of the perturbatively damped, driven NLS equation. In particular, the unperturbed NLS equation is completely integrable. Consequently, a great deal is known about the invariant manifold structure in the phase space. As a result, a number of “persistence of invariant manifold” questions arise naturally when one considers perturbations. Similar issues can be addressed in other completely integrable partial differential equations or soliton equations, and we now give a brief description on the scope of such problems.

1.2.1 Soliton Equations

Soliton equations have been a subject of much interest over the past thirty years since the discovery of the inverse scattering transformation (IST) for
the Korteweg-de Vries (KdV) equation [39]. The IST is a Cauchy problem solver. For (1+1)-dimensional (i.e. 1 temporal dimension and 1 spatial dimension) soliton equations, their IST solvers are parallel and can be viewed as nonlinear Fourier transforms [1]. For (1+2)-dimensional (i.e. 1 temporal dimension and 2 spatial dimensions) soliton equations, their IST solvers are not only very different from (1+1)-dimensional IST solvers, but also different among themselves. Here there are essentially two types: One is the “$\bar{\partial}$ approach”, the other is the “Riemann-Hilbert approach”. For a review, see for example [2],[37]. Aside from IST, other important tools of “soliton theory” are Backlund transformation theory [77][58][62], symplectic geometry [85],[63], algebraic geometry theory [26], modulation theory [34] [57], and long-time asymptotics theory [22],[23]. Interesting solutions of soliton equations include multi-solitons, breathers, kinks, anti-kinks, etc; which are of great physical importance.

Soliton equations are integrable Hamiltonian systems which can be classified into two groups (under periodic boundary conditions): Type 1. Equations whose phase space contains no hyperbolic structure, (i.e. all level sets are tori of elliptic stability type, referred to as “elliptic tori”), for example:

- Korteweg-de Vries equation [69],[68],
  \[ u_t + 6uu_x + u_{xxx} = 0, \]
  where \( u \) is a real-valued function of \((x,t)\).

- Defocusing nonlinear Schroedinger equation [40],
  \[ iq_t = q_{xx} - |q|^2q, \]
  where \( q \) is a complex-valued function of \((x,t)\).

- Sinh-Gordon equation [67],
  \[ u_{xt} = \sinh u, \]
  where \( u \) is a real-valued function of \((x,t)\).

- The modified KdV equation [25],
  \[ u_t - 6u^2u_x + u_{xxx} = 0, \]
  where \( u \) is a real-valued function of \((x,t)\).
Type 2. Equations whose phase space contains hyperbolic structures, for example:

- Focusing nonlinear Schroedinger equation, [63],

\[ iq_t = q_{xx} + |q|^2 q, \]

where \( q \) is a complex-valued function of \((x,t)\).

- Sine-Gordon equation, [29],

\[ u_{xt} = \sin u, \]

where \( u \) is a real-valued function of \((x,t)\).

- The modified KdV equation, [25],

\[ u_t + 6u^2u_x + u_{xxx} = 0, \]

where \( u \) is a real-valued function of \((x,t)\).

The hyperbolic structures of the type 2 soliton equations can be explicitly represented through Backlund-Darboux transformations, see [63] which establishes the connection between Backlund transformations and symplectic geometry. Moreover, Backlund-Darboux transformations also offer representations of fibers of the normally hyperbolic invariant manifolds. The phase block classification of the soliton equations can be studied through Morse-Bott-Fomenko theory, [63],[10],[38], which leads to a variety of interesting invariant submanifolds. The Morse-Bott-Fomenko theory is built upon invariants furnished through isospectral studies, [63]. The work [63] concentrates on the focusing nonlinear Schroedinger equation. However, it can be easily extended to other type 2 soliton equations.

There has been a variety of applications of soliton equations. For example, the KdV equation describes shallow water waves, ion plasma waves, waves in elastic media, etc. The nonlinear Schroedinger equation describes waves in nonlinear optics, vortex filament movement, etc. The sine-Gordon equation describes waves in Josephson junction, charge density waves in one dimensional metals, pseudo-sphere geometry, etc.
1.2.2 Perturbed Soliton Equations

Since soliton equations can have such a rich, and well-understood phase space structure, they are ideal “laboratories” for the development of geometric perturbation methods. According to the type of perturbations and the type of the soliton equations, the questions asked may be very different.

Hamiltonian Perturbations

Under Hamiltonian perturbations, one of the most commonly asked questions concerns the existence persistence of tori (or existence of periodic and/or quasiperiodic solutions) under perturbations. Under certain restrictions on the perturbations, the infinite dimensional KAM type methods described above may apply.

For Hamiltonian perturbations of type 2 soliton equations the persistence of normally hyperbolic invariant manifolds can be studied by the graph transform technique developed in this monograph. Generally, persistence of hyperbolic structures is not dependent on the Hamiltonian nature of the problem.

Dissipative Perturbations

For soliton equations under dissipative perturbations, inertial manifolds and attractors can be constructed, [19],[20],[6]. Inertial manifolds attract their neighborhoods in phase space at an exponential rate, while attractors are the limiting attracting sets. Their Hausdorff dimensions are all finite, and can be estimated. An inertial manifold can be viewed as a graph over a subspace spanned by a finite number of Fourier modes. The construction of an inertial manifold involves studies of spectral gaps and inequality estimates of various types. The construction of an attractor involves energy estimates. The existence of attractors does not require strongly dissipative perturbations, [6], while the existence of inertial manifolds usually requires strong dissipative perturbations in order to satisfy the spectral gap conditions [19].

For type 2 soliton equations under dissipative perturbations, persistence of normally hyperbolic invariant manifolds can be studied. This monograph illustrates such a study for a particular perturbed type 2 soliton equation—the perturbed nonlinear Schroedinger equation. Parallel arguments for other
type 2 soliton equations can be easily extended. Persistent hyperbolic foliations and fibrations are the basic tools for the study of chaos in perturbed soliton equations described in [61][60].
2 The Perturbed Nonlinear Schrödinger Equation

2.1 The Setting for the Perturbed Nonlinear Schrödinger Equation

Consider the perturbatively damped and driven nonlinear Schrödinger equation (PNLS):

\[ iq_t = q_{xx} + 2\left(\|q\|^2 - \omega^2\right)q + i\epsilon \left[-\alpha q + \hat{D}^2 q + \Gamma\right], \quad (2.1) \]

under even and periodic boundary condition

\[ q(-x) = q(x), \quad q(x+1) = q(x); \]

where \( \omega \in (\pi, 2\pi), \epsilon \in (-\epsilon_0, \epsilon_0) \) is the perturbation parameter, \( \alpha(>0) \) and \( \Gamma \) are real constants. The operator \( \hat{D}^2 \) is a regularized Laplacian, specifically given by

\[ \hat{D}^2 q = -\sum_{j=1}^{\infty} \beta_j k_j^2 \hat{q}_j \cos k_j x, \]

where \( \hat{q}_j \) is the Fourier transform of \( q \) and \( k_j \equiv 2\pi j \). The regularizing coefficient \( \beta_j \) is defined by

\[ \beta_j = \begin{cases} 
\beta & \text{for } j \leq N \\
\alpha_* k_j^{-2} & \text{for } j > N,
\end{cases} \]

where \( \alpha_* \) and \( \beta \) are positive constants and \( N \) is a large fixed positive integer. When \( \epsilon > 0 \), the terms \( -\epsilon\alpha q \) and \( \epsilon \hat{D}^2 q \) are perturbatively damping terms, the former is a linear damping, and the latter is a diffusion term; the term \( \epsilon \Gamma \) is a perturbatively driving term. As mentioned in the introduction, the system (2.1) has been extensively studied.

**Remark 2.1** The study in this book can be carried through for more general perturbations, and even for more general systems. The arguments will be parallel. We will restrict ourselves to the particular system (2.1) and give an intensive and detailed study.

We view the pde (2.1) as defining a flow on the following function space:

For any integer \( k (1 \leq k < \infty) \),

\[ S_k \equiv \left\{ \tilde{q} = \begin{pmatrix} q \\ r \end{pmatrix} \mid r = -\tilde{q}, \quad q(x+1) = q(x), \quad q(-x) = q(x), \forall x; \right\}, \quad q \in H_k \text{ (The Sobolev space of periodic functions)} \right\}. \]
The pde (2.1) is well posed in $S_k$ as the following two theorems state:

**Theorem 2.1 (Cauchy Problem)** For any $\vec{q}_0 \in S_k$, there exists a unique solution $\vec{q} \in C^0[(-\infty, \infty); S_k]$ to the perturbed NLS equation (2.1), such that $\vec{q} |_{t=t_0} = \vec{q}_0$.

Let $F_\epsilon^t$ denote the evolution operator of the perturbed NLS equation (2.1). Then we have

**Theorem 2.2 (Dependence on Data)** For any fixed $t \in (-\infty, \infty)$, and any fixed integers $n$ and $k$ ($1 \leq n, k < \infty$), $F_\epsilon^t$ is a $C^n$ diffeomorphism in $S_k$, and is also $C^n$ smooth in $\epsilon$.

The above two theorems are well-known. Local well-posedness is established following standard pde methods as described in [16]. A recent work concerning rough data is described in [11]. Global existence in $S_k$ follows by controlling for the PNLS flow (2.1) the constants of motion for the integrable ($\epsilon = 0$) system; in turn, these constants control the $H_k$ norm of the solution. Further details about the proofs of these two theorems may be found in the thesis [59].

### 2.2 Spatially Independent Solutions: An Invariant Plane

The plane of constants $\Pi (\equiv \{ \vec{q} \in S_k \mid \frac{d}{dx} \vec{q} \equiv 0 \})$ is an invariant plane for the PNLS equation. Phase plane methods are sufficient to describe motion on this plane, which is somewhat complicated for the perturbed PNLS system. On the other hand, the integrable NLS equation ($\epsilon = 0$), when restricted to this invariant plane, has very simple motion consisting of nested circles. Of these, the resonance circle $S_\omega$,

$$S_\omega \equiv \{ \vec{q} \in S_k \mid \vec{q} \in \Pi, \ |q| = \omega \},$$

consists in a circle of fixed points for the NLS equation.

The following simple linear stability analysis shows that, for $\pi < \omega < 2\pi$, at any point $\vec{q}_\gamma \equiv \omega e^{-i\gamma} \in S_\omega$, the stable space $E^s(\vec{q}_\gamma)$ is 1-dimensional, the unstable space $E^u(\vec{q}_\gamma)$ is 1-dimensional, and the center space $E^c(\vec{q}_\gamma)$ has codimension 2: Explicitly, we linearize the integrable NLS equation at $\vec{q}_\gamma \in S_\omega$:

$$q = (\omega + \delta \tilde{q}) \exp (-i\gamma), \quad \delta << 1:$$

\begin{equation}
\frac{i}{\partial \tau} \tilde{q} = \frac{\partial^2}{\partial x^2} \tilde{q} + 2\omega^2 (\tilde{q} + \overline{\tilde{q}}). \quad (2.2)
\end{equation}
The basic solutions of (2.2) have the following form:

\[ \tilde{q}_j^\pm = \left\{ A_j^\pm \exp(\Omega_j^\pm t) + B_j^\pm \exp(\overline{\Omega}_j^\pm t) \right\} \cos k_j x, \quad (2.3) \]

where

\[ \Omega_j^\pm = \pm k_j \sqrt{4\omega^2 - k_j^2}, \]
\[ k_j = 2j\pi, \quad j = 0, 1, 2, \ldots, \]
\[ B_j^\pm = \frac{2\omega^2}{k_j^2 - 2\omega^2 + i\Omega_j^\pm} \overline{A}_j^\pm. \]

For \( j = 0, 1; \) \( \Omega_j^\pm \) is real and satisfies \( 2\omega^2 = |k_j^2 - 2\omega^2 + i\Omega_j^\pm|. \) From this data, one can deduce instabilities as well as construct from these linearized solutions a basis of \( S_k. \)

Indeed, the following is a basis for \( S_k: \)

- **\( k_0 \) mode:**
  \[ e_0^+ = i \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e_0^- = \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \]

**Remark 2.2** \( \langle e_0^+, e_0^- \rangle = 0. \) (Here \( \langle, \rangle \) denotes the \( L^2 \) inner product.) If \( \tilde{q}(0) = \alpha_0^+ e_0^+ + \alpha_0^- e_0^-, \) for real numbers \( \alpha_0^+ \) and \( \alpha_0^-, \) then \( \tilde{q}(t) = (\alpha_+ - 4t\omega^2\alpha_) e_0^+ + \alpha_- e_0^- \).

- **\( k_1 \) mode:**
  \[ e_1^+ = \begin{pmatrix} \exp\{i\frac{\theta_1^+}{2}\} \\ -\exp\{-i\frac{\theta_1^+}{2}\} \end{pmatrix} \cos k_1 x, \quad e_1^- = \begin{pmatrix} \exp\{i\frac{\theta_1^-}{2}\} \\ -\exp\{-i\frac{\theta_1^-}{2}\} \end{pmatrix} \cos k_1 x, \]

where,

\[ \exp\{i\theta_1^\pm\} = \frac{2\omega^2}{k_1^2 - 2\omega^2 \pm i\Omega_1^\pm}, \]

i.e.

\[ \theta_1^\pm = \pm \arg\{k_1^2 - 2\omega^2 \pm i\Omega_1^\pm\}. \]

**Remark 2.3** \( \langle e_1^+, e_1^- \rangle \neq 0, \) in general. If \( \tilde{q}(0) = \alpha_1^+ e_1^+ + \alpha_1^- e_1^- \) for real numbers \( \alpha_1^\pm \), then \( \tilde{q}(t) = \alpha_1^+ e^{\Omega_1^+ t} e_1^+ + \alpha_1^- e^{\Omega_1^- t} e_1^- \).
\* \textbf{k}_j \text{ modes, for } j \geq 2:

\[ \Omega_j^\pm = \pm ik_j \sqrt{k_j^2 - 4\omega^2}, \]

Denote by

\[ \Omega_j \equiv k_j \sqrt{k_j^2 - 4\omega^2}. \]

Let

\[ r_j \equiv \frac{2\omega^2}{k_j^2 - 2\omega^2 + \Omega_j}. \]

\[ e_j^+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos k_j x, \quad e_j^- = i \frac{1 - r_j}{1 + r_j} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos k_j x. \]

**Remark 2.4** \( \langle e_j^+, e_j^- \rangle = 0 \); moreover, \( r_j \to 0 \), as \( j \to 0 \). If

\[ \tilde{q}(0) = \alpha_j^+ e_j^+ + \alpha_j^- e_j^- \]

or, in the more compact notation,

\[ \tilde{q}(0) = \begin{pmatrix} e_j^+ & e_j^- \end{pmatrix} \begin{pmatrix} \alpha_j^+ \\ \alpha_j^- \end{pmatrix}, \]

then

\[ \tilde{q}(t) = \begin{pmatrix} e_j^+ & e_j^- \end{pmatrix} \begin{pmatrix} \cos \Omega_j t & -\sin \Omega_j t \\ \sin \Omega_j t & \cos \Omega_j t \end{pmatrix} \begin{pmatrix} \alpha_j^+ \\ \alpha_j^- \end{pmatrix}. \]

In this notation, we have the following

**Proposition 2.1 (Tangent Spaces)** A basis for the tangent space at \( \tilde{q}_\gamma \in S_\omega \), \( TS_k(\tilde{q}_\gamma) = S_k \), is given by

\[ \{ e_j^\pm, j = 0, 1, \cdots \}. \]
Moreover, the linear spaces $E^s(\vec{q}_γ)$, $E^u(\vec{q}_γ)$, and $E^c(\vec{q}_γ)$ are given explicitly by

$$E^s(\vec{q}_γ) = \text{span} \{ e^-_1 \}$$
$$E^u(\vec{q}_γ) = \text{span} \{ e^+_1 \}$$
$$E^c(\vec{q}_γ) = \text{span} \{ e^\pm_j, j \neq 1 \}.$$ 

In addition, the center-stable and center-unstable spaces are given by

$$E^{cs}(\vec{q}_γ) = \text{span} \{ e^-_1; e^\pm_j, j \neq 1 \}$$
$$E^{cu}(\vec{q}_γ) = \text{span} \{ e^+_1; e^\pm_j, j \neq 1 \}.$$ 

### 2.3 Statement of the Persistence and Fiber Theorems

In this subsection, we will state the two main theorems on the pde (2.1) to be proved in this book. One is on persistent invariant manifolds, the other is on the fibration of the persistent invariant manifolds.

First we give the definitions of invariant, overflowing invariant, inflowing invariant, and locally invariant submanifolds.

**Definition 1** Let $M$ be a submanifold of $S_k$ with boundary $\partial M$, $\bar{M} = M \cup \partial M$; $F^t_\epsilon$ be the evolution operator of (2.1).

1. $M$ is said to be overflowing invariant under $F^t_\epsilon$, if for every $q \in \bar{M}$, $F^t_\epsilon(q) \in M$ for all $t \leq 0$; and for $q \in \partial M$, $F^t_\epsilon(q) \notin M$ for $t > 0$. 

2. $M$ is said to be inflowing invariant under $F^t_\epsilon$, if for every $q \in \bar{M}$, $F^t_\epsilon(q) \in M$ for all $t \geq 0$; and for $q \in \partial M$, $F^t_\epsilon(q) \notin M$ for $t < 0$. 

3. $M$ is said to be invariant under $F^t_\epsilon$, if for every $q \in \bar{M}$, $F^t_\epsilon(q) \in \bar{M}$ for all $t \in \mathbb{R}$. 

4. $M$ is locally invariant under $F^t_\epsilon$, if there exists a neighborhood $V$ of $M$ in $S_k$ such that for all $q \in M$, if $\tau \geq 0$ and $\bigcup_{t \in [0,\tau]} F^t_\epsilon(q) \subset V$, then $\bigcup_{t \in [0,\tau]} F^t_\epsilon(q) \subset M$; and if $\tau \leq 0$ and $\bigcup_{t \in [\tau,0]} F^t_\epsilon(q) \subset V$, then $\bigcup_{t \in [\tau,0]} F^t_\epsilon(q) \subset M$. (Intuitively speaking, the orbit $F^t_\epsilon(q)$ starting from any point $q$ in $M$, can leave $M$ in forward or backward time; nevertheless, it can only leave $M$ through the boundary $\partial M$.)

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Remark 2.5 Overflowing invariant submanifolds become inflowing invariant under time reversal and vice versa. $M$ can be an invariant manifold only if $F^s_\epsilon(q) \in \partial M$ for all $q \in \partial M$, or if $\partial M = \emptyset$.

**Theorem 2.3 (Persistent Invariant Manifold Theorem)** For any integers $k$ and $n$ ($1 \leq k, n < \infty$), there exist a positive constant $\epsilon_0$ and a neighborhood $U$ of the resonance circle $S_\omega$ in $S_k$, such that inside $U$, for any $\epsilon \in (-\epsilon_0, \epsilon_0)$, there exist $C^n$ locally invariant submanifolds $W^cu_\epsilon$ and $W^cs_\epsilon$ of codimension 1 and $W^n_\epsilon (= W^cu_\epsilon \cap W^cs_\epsilon)$ of codimension 2 under the evolution operator $F^i_\epsilon$ given by (2.1). When $\epsilon = 0$, $W^c_0$, $W^sc_0$, and $W^n_0$ are tangent to the center-unstable, center-stable, and center subspaces $\bigcup_{\gamma \in S_\omega} E^{cu}(\tilde{q}_\gamma)$ and $\bigcup_{\tilde{q}_\gamma \in S_\omega} E^{cs}(\tilde{q}_\gamma)$ of codimension 1, and $\bigcup_{\tilde{q}_\gamma \in S_\omega} E^{c}(\tilde{q}_\gamma)$ of codimension 2, along the resonance circle $S_\omega$. $W^c_\epsilon$, $W^sc_\epsilon$, and $W^n_\epsilon$ are smooth in $\epsilon$ for $\epsilon \in (-\epsilon_0, \epsilon_0)$.

**Remark 2.6** $W^c_\epsilon$, $W^sc_\epsilon$, and $W^n_\epsilon$ are called persistent center-unstable, center-stable, and center submanifolds near $S_\omega$.

**Theorem 2.4 (Fiber Theorem)** Inside the persistent center-unstable submanifold $W^c_\epsilon$ near $S_\omega$, there exists a family of $1$-dimensional $C^n$ smooth submanifolds (curves) $\{\tilde{F}^{(u,\epsilon)}(\tilde{q}) : \tilde{q} \in W^c_\epsilon\}$, called unstable fibers:

- $W^c_\epsilon$ can be represented as a union of these fibers,
  $$W^c_\epsilon = \bigcup_{\tilde{q} \in W^c_\epsilon} \tilde{F}^{(u,\epsilon)}(\tilde{q}).$$

- $\tilde{F}^{(u,\epsilon)}(\tilde{q})$ depends $C^{n-1}$ smoothly on both $\epsilon$ and $\tilde{q}$ for $\epsilon \in (-\epsilon_0, \epsilon_0)$ and $\tilde{q} \in W^c_\epsilon$, in the sense that $W$ defined by
  $$W = \left\{ (\tilde{q}_1, \tilde{q}, \epsilon) \mid \tilde{q}_1 \in \tilde{F}^{(u,\epsilon)}(\tilde{q}), \tilde{q} \in W^c_\epsilon, \epsilon \in (-\epsilon_0, \epsilon_0) \right\}$$
  is a $C^{n-1}$ smooth submanifold of $S_k \times S_k \times (-\epsilon_0, \epsilon_0)$.

- Each fiber $\tilde{F}^{(u,\epsilon)}(\tilde{q})$ intersects $W^c_\epsilon$ transversally at $\tilde{q}$, two fibers $\tilde{F}^{(u,\epsilon)}(\tilde{q}_1)$ and $\tilde{F}^{(u,\epsilon)}(\tilde{q}_2)$ are either disjoint or identical.

- The family of unstable fibers $\{\tilde{F}^{(u,\epsilon)}(\tilde{q}) : \tilde{q} \in W^c_\epsilon\}$ is negatively invariant, in the sense that the family of fibers commutes with the evolution operator $F^i_\epsilon$ in the following way:
  $$F^i_\epsilon(\tilde{F}^{(u,\epsilon)}(\tilde{q})) \subset \tilde{F}^{(u,\epsilon)}(F^i_\epsilon(\tilde{q}))$$
  for all $\tilde{q} \in W^c_\epsilon$ and all $t \leq 0$ such that $\bigcup_{t \in [t,0]} F^i_\epsilon(\tilde{q}) \subset W^c_\epsilon$. 

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There are positive constants $\kappa$ (e.g. $\kappa = 2\pi \sqrt{\omega^2 - \pi^2}$) and $C$ which are independent of $\epsilon$ such that if $\vec{q} \in W^c_\epsilon$ and $\vec{q}_1 \in \bar{F}^{(u,\epsilon)}(\vec{q})$, then

$$\left\| F^t_\epsilon(\vec{q}_1) - F^t_\epsilon(\vec{q}) \right\| \leq C e^{\kappa t} \left\| \vec{q}_1 - \vec{q} \right\|,$$

for all $t \leq 0$ such that $\bigcup_{\tau \in [t,0]} F^{\tau}_\epsilon(\vec{q}) \subset W^c_\epsilon$, where $\| \|$ denotes $H_k$ norm of periodic functions of period 1.

For any $\vec{q}, \vec{p} \in W^c_\epsilon$, $\vec{q} \neq \vec{p}$, any $\vec{q}_1 \in \bar{F}^{(u,\epsilon)}(\vec{q})$ and any $\vec{p}_1 \in \bar{F}^{(u,\epsilon)}(\vec{p})$; if $F^t_\epsilon(\vec{q}), F^t_\epsilon(\vec{p}) \in W^c_\epsilon$, $\forall t \in (-\infty, 0)$,

and

$$\left\| F^t_\epsilon(\vec{p}_1) - F^t_\epsilon(\vec{q}) \right\| \to 0, \text{ as } t \to -\infty;$$

then

$$\frac{1}{\epsilon^{\frac{1}{2}t}} \left( \frac{\left\| F^t_\epsilon(\vec{q}_1) - F^t_\epsilon(\vec{q}) \right\|}{\left\| F^t_\epsilon(\vec{p}_1) - F^t_\epsilon(\vec{q}) \right\|} \right) \to 0, \text{ as } t \to -\infty.$$

Similarly for $W^{cs}_\epsilon$.

### 2.4 Invariant Manifolds and Fibers for the Integrable NLS Equation

When $\epsilon = 0$, the pde (2.1) becomes the integrable nonlinear Schrödinger equation

$$i q_t = q_{xx} + 2 \left[ |q|^2 - \omega^2 \right] q, \quad (2.4)$$

under even and periodic boundary condition

$$q(-x) = q(x), \quad q(x + 1) = q(x).$$

The detailed study on the hyperbolic structure for the integrable NLS equation (2.4) is given in [63] [59]. Below we will give a brief account on that study.

#### 2.4.1 Integrable Preliminaries

The NLS equation (2.4) can be integrated with the Zakharov–Shabat linear system [100]:

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\[ \varphi_x = U^{(\lambda)} \varphi, \quad (2.5) \]
\[ \varphi_t = V^{(\lambda)} \varphi, \quad (2.6) \]

where

\[ U^{(\lambda)} = i\lambda \sigma_3 + i \begin{pmatrix} 0 & q \\ \bar{q} & 0 \end{pmatrix}; \quad (2.7) \]
\[ V^{(\lambda)} = 2i\lambda^2 \sigma_3 + i(\omega^2 - |q|^2) \sigma_3 + \begin{pmatrix} 0 & 2i\lambda q + q_x \\ 2i\lambda \bar{q} - \bar{q}_x & 0 \end{pmatrix}; \quad (2.8) \]

where \( \lambda \) is the spectral parameter, \( \sigma_3 \) denotes the third Pauli matrix \( \sigma_3 = \text{diag}(1, -1) \). Compatibility condition of (2.5,2.6) gives the NLS equation (2.4). Focusing attention upon the spatial part (2.5) of the linear system (2.5,2.6), let \( M = M(x; \lambda; \vec{q}) \) be the fundamental matrix solution of the ode (2.5); The Floquet discriminant:

\[ \Delta : C \times S_k \rightarrow C \]

is defined by

\[ \Delta(\lambda; \vec{q}) \equiv \text{trace}\{M(1; \lambda; \vec{q})\}. \]

For any \( \lambda \in C \), \( \Delta(\lambda; \vec{q}) \) is a constant of motion for the integrable NLS equation (2.4). \( \Delta(\lambda; \vec{q}) \) is an entire function in both \( \lambda \) and \( \vec{q} \). This leads to a countable number of functionally independent constants of motion. Next, we define some spectral points. Periodic and antiperiodic points \( \lambda^s \) are defined by

\[ \Delta(\lambda^s; \vec{q}) = \pm 2. \]

A critical point \( \lambda^c \) is defined by the condition

\[ \frac{\partial \Delta(\lambda^c; \vec{q})}{\partial \lambda} = 0. \]

A multiple point \( \lambda^m \) is a critical point which is also a periodic or antiperiodic point. The \textit{algebraic multiplicity} of \( \lambda^m \) is defined as the order of the zero
of $\Delta(\lambda) \pm 2$. Usually it is 2, but it can exceed 2; when it does equal 2, we call the multiple point a double point, and denote it by $\lambda^d$. The geometric multiplicity of $\lambda^m$ is defined as the dimension of the periodic (or antiperiodic) eigenspace of (2.5) at $\lambda^m$, and is either 1 or 2.

The following counting lemmas describe the number and locations of the spectral points.

**Lemma 2.1 (Counting Lemma for Critical Points)**  For $q \in H_1$, set $N = N(||q||_{H_1}) \in \mathbb{Z}^+$ by

$$N(||q||_{H_1}) = 2\left[||q||_2^2 \cosh ||q||_2 + 3||q||_{H_1} \sinh ||q||_2\right],$$

where $[x] \equiv \text{first integer greater than } x$. Consider

$$\Delta'(\lambda; q) = \frac{\partial}{\partial \lambda} \Delta (\lambda; q).$$

Then

- $\Delta'(\lambda; q)$ has exactly $2N + 1$ zeros (counted according to multiplicity) in the interior of the disc $D \equiv \{\lambda \in \mathbb{C}: |\lambda| < (2N + 1)\frac{\pi}{2}\}$.
- $\forall k \in \mathbb{Z}, |k| > N, \Delta'(\lambda, q)$ has exactly one zero in each disc $\{\lambda \in \mathbb{C}: |\lambda - k\pi| < \frac{\pi}{4}\}$.
- $\Delta'(\lambda; q)$ has no other zeros.
- For $|\lambda| > (2N + 1)\frac{\pi}{2}$, the zeros of $\Delta'$, $\{\lambda^c_j, |j| > N\}$, are all real, simple, and satisfy the asymptotics

$$\lambda^c_j = j\pi + o(1) \text{ as } |j| \to \infty.$$

**Lemma 2.2 (Counting Lemma for Periodic Points)**  For $q \in H_1$ and $N$ as in Counting Lemma 2.1, consider

$$\Delta_{\pm}(\lambda; q) \equiv \Delta(\lambda; q) \pm 2.$$

Then
• $\Delta_\pm(\lambda; q)$ has exactly $4N + 2$ zeros (counted according to multiplicity) in the interior of the disc $D \equiv \{ \lambda \in \mathbb{C} : |\lambda| < (2N + 1)\frac{\pi}{2} \}$.

• $\forall k \in \mathbb{Z}, |k| > N, \Delta_\pm(\lambda; q)$ has exactly two zeros in each disc $\{ \lambda \in \mathbb{C} : |\lambda - k\pi| < \frac{\pi}{2} \}$.

• $\Delta_\pm(\lambda; q)$ has no other zeros.

• For $|\lambda| > (2N + 1)\frac{\pi}{2}$, these zeros occur in conjugate pairs. If they coincide, they must be real. Conversely, if real, they must be double. They satisfy the asymptotics

$$\lambda_j = j\pi + o(1) \quad \text{as} \quad |j| \to \infty.$$ 

For the proof of the above two lemmas, see [59].

The important sequence of constants of motion $F_j$

$$F_j : \Omega \subset S_k \mapsto \mathbb{C}$$

is defined by

$$F_j(\bar{q}) = \Delta(\lambda_j^c(\bar{q}); \bar{q}).$$

Several properties of the functional $F_j$ must be emphasized: First, for $|j| > N$ (given in the counting lemmas), $F_j$ is real and $F_j \in [-2, +2]$; second, for complex critical points $|j| \leq N$, $F_j = F_j^R + iF_j^I$ is not necessarily real. The following theorem characterizes the critical structures of the functionals $F_j$.

**Theorem 2.5**

1. Except for the trivial case $q = 0$,

$$\frac{\delta F_j}{\delta \bar{q}}|_{\bar{q}_*} = 0 \iff \lambda_j^c(\bar{q}_*) \text{ is a multiple point with geometric multiplicity 2.}$$

2. If $\lambda_j^c(\bar{q}_*)$ is a real double point, then the Hessian of $F_j$ is positive (or negative) semi-definite.

3. If $\lambda_j^c(\bar{q}_*)$ is a complex double point, then the Hessian of $F_j$ is complex, both its real part and imaginary part are indefinite.
For the proof of the theorem, see [59]. Through a Morse type study based upon the above theorem, hyperbolic structures in the level sets of \( F_j \)'s can be identified. More specifically, if \( \lambda_j(q_*) \) is a complex double point, then the critical point \( q_* \) of \( F_j \) resides on a level set of \( F_j \) which is a normally hyperbolic invariant submanifold. In the subsequent subsections, we will give explicit expressions for such hyperbolic structures through Bäcklund-Darboux transformations.

### 2.4.2 Bäcklund-Darboux Transformations

Starting from a level set which is a normally hyperbolic invariant submanifold, its hyperbolic foliations can be constructed through Bäcklund-Darboux transformations.

Let \( \tilde{q}(x,t) \) be a solution to the NLS equation (2.4), and let \( \nu \) be a complex double point with geometric multiplicity 2. Denote two linearly independent solutions of the Zakharov-Shabat linear system (2.5,2.6) at \( \lambda = \nu \) by \((\tilde{\phi}^+, \tilde{\phi}^-)\). Thus, a general solution of the linear system at \((\tilde{q}, \nu)\) is given by

\[
\tilde{\phi}(x,t; \nu; c_+ , c_-) = c_+ \tilde{\phi}^+ + c_- \tilde{\phi}^-.
\]  

We use \( \tilde{\phi} \) to define a transformation matrix \( G \) by

\[
G = G(\lambda; \nu; \tilde{\phi}) \equiv N \begin{pmatrix} \lambda - \nu & 0 \\ 0 & \lambda - \bar{\nu} \end{pmatrix} N^{-1},
\]

where

\[
N \equiv \begin{bmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{bmatrix}
\]

(2.10)

Then we define \( Q \) and \( \Psi \) by

\[
Q(x,t) \equiv q(x,t) + 2(\nu - \bar{\nu}) \frac{\tilde{\phi}_1 \tilde{\phi}_2}{\phi_1 \bar{\phi}_1 + \phi_2 \bar{\phi}_2}
\]

(2.11)

and

\[
\tilde{\Psi}(x,t; \lambda) \equiv G(\lambda; \nu; \tilde{\phi}) \psi(x,t; \lambda)
\]

(2.12)

where \( \psi \) solves the linear system (2.5,2.6) at \((\tilde{q}, \lambda)\). Formulas (2.12) and (2.13) are the Bäcklund-Darboux transformations for the potential and eigenfunctions, respectively. We have the following theorem.
Theorem 2.6 Let \( \vec{q}(x,t) \) be a solution to the NLS equation (2.4), and let \( \nu \) be a complex double point with geometric multiplicity 2. Denote two linearly independent solutions of the Zakharov-Shabat linear system (2.5,2.6) at \( \lambda = \nu \) by \( (\vec{\phi}^+, \vec{\phi}^-) \), and define \( Q(x,t) \) and \( \vec{\Psi}(x,t;\lambda) \) by (2.12) and (2.13). Then

(i) \( Q(x,t) \) is also a solution to the NLS equation (2.4), and \( \vec{\Psi}(x,t;\lambda) \) solves the Zakharov-Shabat linear system (2.5,2.6) at \( (Q(x,t), \lambda) \).

(ii) The spectra of (2.5) at \( Q \) is the same with the spectra at \( q \).

2.4.3 Explicit Representations for Invariant Manifolds and Fibers

Consider the lowest dimensional level set: The circle \( S_c = \{ \vec{q} \in \Pi \mid |q| = c, \ c \in (\pi, 2\pi) \} \). Motion on this level set is a periodic orbit,

\[
q_\gamma = c \exp\{-i[2(c^2 - \omega^2)t + \gamma]\}. \tag{2.14}
\]

Applying the Bäcklund-Darboux transformation (2.12,2.13) to the solution (2.14), we have the expression for the two-dimensional level set \( W^s(S_c) = W^u(S_c) \) (called stable and unstable manifolds of the circle \( S_c \)):

\[
W^s(S_c) = W^u(S_c) = \bigcup_{\rho \in (-\infty, \infty), \gamma \in [0, 2\pi], \sigma = \pm} Q_H(x, 0; c, \gamma, \rho, \sigma), \tag{2.15}
\]

where

\[
Q_H = Q_H(x,t; c, \gamma, \rho, \pm) = q_\gamma \left[ 1 + \cos 2p - i \sin 2p \tanh \tau \over 1 \pm \sin p \sech \tau \cos(2\pi x) - 1 \right], \tag{2.16}
\]

where \( p = \arctan \left( \sqrt{c^2 - \pi^2} \over \pi \right), \ c \in (\pi, 2\pi), \ \tau = 4|\nu|\pi t - \rho (\pi + \nu = \ce^{ip}), \rho \) is called the Bäcklund parameter. Notice that as \( \tau \to \pm \infty, \)

\[
Q_H \to q_\gamma \ce^{\mp 2p}. \tag{2.17}
\]

That is, the orbit \( Q_H \) is asymptotic to \( q_\gamma \) up to phase shifts in both forward and backward time. For an illustration of \( W^s(S_c) = W^u(S_c) \), see Fig.2.1. \( Q_H \) coordinatizes \( W^s(S_c) = W^u(S_c) \) in the following way: \( \gamma \) coordinatizes the circle \( S_c, \ \pm \) selects one of the two symmetric “whiskers”, and \( \rho \) coordinatizes the “whiskers”. From the expression (2.16,2.17), we can read off the following expressions for stable and unstable fibers:
Figure 2.1: An illustration of the invariant submanifold $W^s(S_c) = W^u(S_c)$.

1. In $W^s(S_c)$, the stable fiber $\tilde{F}^s(q)$ with the base point $q = ce^{-i\gamma}$, has the expression

$$\tilde{F}^s(q) = qe^{i2p} \left[ 1 + \cos 2p - i \sin 2p \tanh \xi \left( 1 \pm \sin p \sech \xi \cos(2\pi x) \right) - 1 \right],$$

which is a curve parameterized by $\xi$, $\xi \in (-\infty, \infty)$; as $\xi \to \infty$, $\tilde{F}^s(q) \to q$.

2. In $W^u(S_c)$, the unstable fiber $\tilde{F}^u(q)$ with the base point $q = ce^{-i\gamma}$, has the expression

$$\tilde{F}^u(q) = qe^{-i2p} \left[ 1 + \cos 2p - i \sin 2p \tanh \xi \left( 1 \pm \sin p \sech \xi \cos(2\pi x) \right) - 1 \right],$$

which is a curve parameterized by $\xi$, $\xi \in (-\infty, \infty)$; as $\xi \to -\infty$, $\tilde{F}^u(q) \to q$.

For an geometric illustration of the fibers, see Fig.2.2. Fibers $\tilde{F}^s(q)$ and $\tilde{F}^u(q)$ in contrast to orbits $Q_H$, do not spiral around the circle $S_c$; therefore, they provide better coordinates for $W^s(S_c) = W^u(S_c)$:

$$W^s(S_c) = \bigcup_{\tilde{q} \in S_c} \tilde{F}^s(\tilde{q}), \quad W^u(S_c) = \bigcup_{\tilde{q} \in S_c} \tilde{F}^u(\tilde{q}).$$
2.5 Coordinates Centered on the Resonance Circle

In this section we center the equations in a manner which is natural to study behavior near the invariant plane Π. This centering notation takes some time to set up.

2.5.1 Definition of the $\tilde{H}$ Norms

It will be useful to use the basis $\{ e_j^\pm \}$ to define a norm on $S_k$ which is equivalent to the $H_k$ norm: Set

$$\vec{q} = \sum_{j=0}^{\infty} \{ (\alpha_j^+) e_j^+ + \alpha_j^- e_j^- \},$$

then define

$$\|\vec{q}\|_{\tilde{H}_1}^2 = \sum_{j=0}^{\infty} \{ (\alpha_j^+)^2 + (\alpha_j^-)^2 \} + \sum_{j=0}^{\infty} k_j^2 \{ (\alpha_j^+)^2 + (\alpha_j^-)^2 \},$$

with a similar definition for the $\tilde{H}_k$ norms, $k > 1$.

The $\tilde{H}_k$ is equivalent to the Sobolev $H_k$ norm; the advantage of the $\tilde{H}_k$ norm is that its temporal evolution under the linearization (2.2) of the integrable flow is easy to control.

From now on $\|\vec{q}\|$ stands for $\|\vec{q}\|_{\tilde{H}_k}$. 

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2.5.2 A Neighborhood of the Circle of Fixed Points

First, in order to study dynamics near the circle of fixed points we write

\[ q = \left[ \rho(t) + \delta f(x,t) \right] \exp i\theta(t), \quad 0 < \delta << 1, \quad (2.18) \]

where \( \rho \) is real and \( f \) has spatial mean 0. Defining the complex mean \( c \equiv \rho \exp i\theta \), the NLS equation takes the form

\[
\text{i} f_t = \left[ f_{xx} + 2 \rho^2 (f + \bar{f}) \right] \\
+ \delta \left[ 4 \rho (f \bar{f} - \langle f \bar{f} \rangle) + 2 \rho (f^2 - \langle f^2 \rangle) \right] \\
+ \delta^2 \left[ 2 (f \bar{f} - \langle f \bar{f} \rangle) - \left( \langle f^2 \rangle + \langle \bar{f}^2 \rangle + 4 \langle f \bar{f} \rangle \right) f \right] \\
- \epsilon \left[ \frac{\Gamma \sin \theta}{\rho} f + i(\alpha f - \hat{D}^2 f) \right] \\
- \frac{\delta^3}{\rho} (f \bar{f}(f + \bar{f})) \]

\[
\text{i} c_t = 2(\epsilon \bar{c} - \omega^2) c + i \epsilon (-\alpha c + \Gamma) \\
+ 2 \delta^2 \left( 2 (f \bar{f}) c + \langle f^2 \rangle \right) + 2 \delta^3 (f \bar{f} f) \exp i\theta.
\]

In these equations, \( \langle \cdots \rangle \) denotes spatial mean over one period. Equivalently, the second equation for the mean \( c \) can be written in polar coordinates \( c = \rho \exp i\theta \):

\[
\rho_t = \epsilon \left( -\alpha \rho + \Gamma \cos \theta \right) - i \delta^2 \left( \langle f^2 \rangle - \langle \bar{f}^2 \rangle \right) \rho \\
- i \delta^3 \left( \langle f \bar{f} (f - \bar{f}) \rangle \right)
\]

\[
\theta_t = -2(\rho^2 - \omega^2) - \frac{\Gamma}{\rho} \sin \theta \\
- \delta^2 \left( \langle f^2 \rangle + \langle \bar{f}^2 \rangle + 4 \langle f \bar{f} \rangle \right) \\
- \frac{\delta^3}{\rho} \left( \langle f \bar{f} (f + \bar{f}) \rangle \right)
\]
2.5.3 An Enlarged Phase Space

In this setting we have two small parameters – an amplitude parameter \( \delta \) and a perturbation parameter \( \epsilon \). While both are small, we will treat the two very differently. First, the perturbation parameter \( \epsilon \) will be treated as a variable by enlarging the system. In terms of the original variables \( q \), this enlargement yields the *enlarged system* (EPNLS):

\[
\begin{align*}
\dot{q} & = q_{xx} + 2[|q|^2 - \omega^2]q + i\epsilon \{-\alpha q + \hat{D}q + \Gamma\}, \\
\dot{\epsilon} & = 0.
\end{align*}
\]

(2.19)

The corresponding *enlarged function space* becomes \( \hat{S}_k \):

\[
\hat{S}_k = S_k \times (-\epsilon_0, \epsilon_0).
\]

(2.20)

We can define a family of invariant planes parametrized by \( \epsilon \), \( \hat{\Pi}_\epsilon \):

\[
\hat{\Pi}_\epsilon \equiv \{(\tilde{q}, \epsilon) \mid \tilde{q} \in S_k, \frac{d}{dx}\tilde{q} \equiv 0\}.
\]

It is easily seen that \( \hat{\Pi}_\epsilon \) is invariant under the EPNLS flow (2.19). Restricted to the invariant planes \( \hat{\Pi}_\epsilon \), the (2.19) flow becomes:

\[
\begin{align*}
\dot{q} & = 2[|q|^2 - \omega^2]q + i\epsilon \{-\alpha q + \Gamma\}, \\
\dot{\epsilon} & = 0.
\end{align*}
\]

In the enlarged function space, the resonance circle \( \hat{S}_\omega \) is defined as follows:

\[
\hat{S}_\omega \equiv \{(\tilde{q}, 0) \mid (\tilde{q}, 0) \in \hat{\Pi}_\epsilon, \ |	ilde{q}| = \omega\}.
\]

It is easily seen that \( \hat{S}_\omega \) is a set of equilibria of (2.19).

2.5.4 Scales through \( \delta \)

Next, with the amplitude parameter \( \delta \) we introduce three scales: \( O(\delta^0) \), \( O(\delta) \), and \( O(\delta^2) \). Along the resonance circle, the scale will be \( O(\delta^0) \); in the normal direction of \( S_\omega \) in \( S_k \), the \( H_k \) scale will be \( O(\delta) \); and along the \( \epsilon \) direction, the scale will be \( O(\delta^2) \). Explicitly, we have the following representation:

\[
\begin{align*}
q & = (\omega + \delta r)e^{i\theta} + \delta f e^{i\theta}, \\
\epsilon & = \delta^2 \epsilon_1, \\
< f > & = 0.
\end{align*}
\]

(2.21)
The coordinate transformation (2.21) maps \( \hat{\mathcal{S}}_k \) into a new phase space \( \tilde{\mathcal{S}}_k \):
\[
\tilde{\mathcal{S}}_k \equiv \left\{ (r, \theta, \tilde{f}, \epsilon_1) \mid (\tilde{q}, \epsilon) \text{ given by (2.21) belongs to } \hat{\mathcal{S}}_k \right\}
\]
coordinatized by
\[
Q \equiv (r, \theta, \tilde{f}, \epsilon_1),
\]
where \( \tilde{f} \equiv (f, -\bar{f})^T \), \( \tilde{q} \equiv (q, -\bar{q})^T \). We use \( \|Q\| \) to denote the norm inherited from the norm \( (\|q\|, |\epsilon|) \) through the transformation (2.21).

**Definition 2** Define the bounded region \( \tilde{D}_k \) in \( \tilde{\mathcal{S}}_k \) as follows:
\[
\tilde{D}_k \equiv \left\{ (r, \theta, \tilde{f}, \epsilon_1) \in \tilde{\mathcal{S}}_k \mid \left(r^2 + \|\tilde{f}\|_{H^k}^2\right)^{1/2} < 1, \ |\epsilon_1| < 1, \ \theta \in [0, 2\pi]\right\}
\]
Denote by \( \hat{D}_{k,\delta} \) the inverse image of \( \tilde{D}_k \) in \( \hat{\mathcal{S}}_k \) under the coordinate transformation (2.21).

**Remark 2.7** \( \tilde{D}_k \) is a bounded convex subset of \( \tilde{\mathcal{S}}_k \); moreover, it is independent of \( \delta \). On the other hand, \( \hat{D}_{k,\delta} \) depends on \( \delta \). For a fixed value of \( \delta \), \( \hat{D}_{k,\delta} \) is a solid torus.

### 2.5.5 The Equations in Their Final Setting

Substituting representation (2.21) into the enlarged perturbed NLS system (2.19) yields
\[
\begin{align*}
rt & = \delta \left[ -\epsilon_1 \left( \alpha(\omega + \delta r) - \Gamma \cos \theta \right) - i \left( \langle f^2 - \bar{f}^2 \rangle (\omega + \delta r) \right) - i \delta^2 \left( f \bar{f}(f - \bar{f}) \right) \right] \\
\theta_t & = -2\delta r(2\omega + \delta r) - \delta^2 \left[ \frac{\epsilon_1 \Gamma \sin \theta}{(\omega + \delta r)} + \langle f^2 + \bar{f}^2 + 4f \bar{f} \rangle \right] - \frac{\delta^3}{(\omega + \delta r)} \left[ f \bar{f}(f + \bar{f}) \right] \\
if_t & = f_{xx} + 2(\omega + \delta r)^2(f + \bar{f})
\end{align*}
\]
\[ + 2\delta \left[ (\omega + \delta r) \left( 2(f \bar{f} - \langle f \bar{f} \rangle) + (f^2 - \langle f^2 \rangle) \right) \right] \]
\[ + \delta^2 \left[ 2 (f \bar{f} f - \langle f \bar{f} f \rangle) - \left( \langle f^2 + \bar{f}^2 + 4f \bar{f} \rangle \right) f \right] \]
\[ - \epsilon_1 \left( \frac{\Gamma \sin \theta}{(\omega + \delta r)} f + i(\alpha f - \hat{D}^2 f) \right) \]
\[ - \frac{\delta^3}{(\omega + \delta r)} \left( f \bar{f}(f + \bar{f}) \right) f \]
\[ \epsilon_{1t} = 0. \tag{2.22} \]

The integrable linearized flow may be obtained by setting \( \delta = 0 \):
\[
\begin{align*}
    r_{t} &= 0, \\
    \theta_{t} &= 0, \\
    i f_{t} &= f_{xx} + 2\omega^2 (f + \bar{f}), \\
    \epsilon_{1t} &= 0. \tag{2.23}
\end{align*}
\]

**Remark 2.8** *In the equation (2.23), the \( f \) equation is the linearized equation (2.2) discussed in the last subsection, but the dynamics on \( \Pi \) is totally different from that in the last subsection, as a result of the new scaling. In fact, this scaling “slows down” motion on the annulus \( \Pi \cap \hat{D}_k, \delta \) to create an annulus of fixed points. Here, we want to emphasize that, from now on, we will view systems (2.22;2.23) as flows defined on \( \hat{D}_k \). Moreover, system (2.22) is a perturbation of system (2.23) with \( \delta \) as the perturbation parameter.*

Denote respectively by \( \tilde{F}^t \) and \( \tilde{F}_\delta^t \) the solution operators of systems (2.23) and (2.22) in \( \tilde{S}_k \).

**Proposition 2.2** *For any \( t \in (-\infty, \infty) \), there exist constants \( C = C(t) \) and \( d_1 = d_1(t) \), such that,
\[
\begin{align*}
    \left\| \tilde{F}_\delta^t(Q) - \tilde{F}^t(Q) \right\| &\leq C\delta, \\
    \left\| \nabla \tilde{F}_\delta^t(Q) - \nabla \tilde{F}^t(Q) \right\| &\leq C\delta,
\end{align*}
\]

for all \( Q \in \hat{D}_k \), and all \( \delta \in [0, d_1] \), where \( \nabla \) denotes the gradient.*
Proof: Notice that $F^t_\epsilon$ denotes the solution operator of the equation \( (2.1) \). By theorems (2.1;2.2),

\[
q(t) \equiv (\omega + \delta r(t))e^{i\theta(t)} + \delta f(x,t)e^{i\theta(t)} \\
= F^t_\epsilon(\omega e^{i\theta(0)}) + \frac{\partial}{\partial q} F^t_\epsilon(\omega e^{i\theta(0)}) \bullet \Delta q(0) \\
+ \frac{\partial}{\partial \epsilon} F^t_\epsilon(\omega e^{i\theta(0)}) \cdot \delta^2 \epsilon_1 + R,
\]

(2.24)

where

\[
\Delta q(0) = \delta \left[ r(0)e^{i\theta(0)} + f(x,0)e^{i\theta(0)} \right], \\
R = R(t;\omega e^{i\theta(0)};\Delta q(0);\delta^2 \epsilon_1);
\]

moreover, there exists a constant $C_1 = C_1(t;\omega e^{i\theta(0)})$, such that,

\[
\|R\| \leq C_1\|\Delta q(0)\|^2.
\]

Notice that,

\[
F^t_\epsilon(\omega e^{i\theta(0)}) = \omega e^{i\theta(0)}.
\]

Let

\[
\frac{\partial}{\partial q} F^t_\epsilon(\omega e^{i\theta(0)}) \bullet \Delta q(0) = \delta e^{i\theta(0)} q_1(t),
\]

then

\[
\tilde{q}_1(t) = r(0)e_0^{-} - 4t\omega^2 r(0)e_0^{+} \\
+ \alpha_1^+ e^{\Omega t} e_1^+ + \alpha_1^- e^{-\Omega t} e_1^- \\
+ \sum_{j=2}^{\infty} (e_j^+, e_j^-) \begin{pmatrix} \cos \Omega_j t & -\sin \Omega_j t \\ \sin \Omega_j t & \cos \Omega_j t \end{pmatrix} \begin{pmatrix} \alpha_j^+ \\ \alpha_j^- \end{pmatrix},
\]

where

\[
f(x,0) = \sum_{j=1}^{\infty} (\alpha_j^+ e_j^+ + \alpha_j^- e_j^-).
\]

Therefore, to order $O(\delta)$, Eq.\( (2.24) \) becomes

\[
(\omega + \delta r(t))e^{i\theta(t)} + \delta f(x,t)e^{i\theta(t)} = \omega e^{i\theta(0)} + \delta e^{i\theta(0)} q_1(t).
\]

(2.25)

Take spatial mean on both sides of \( (2.25) \), we have

\[
(\omega + \delta r(t))e^{i\theta(t)} = \omega e^{i\theta(0)} + \delta e^{i\theta(0)} (r(0) - i4t\omega^2 r(0)).
\]

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That is,
\[
\begin{align*}
\omega + \delta r(t) &= \sqrt{\omega^2 + 2\omega \delta r(0) + \delta^2 (r(0))^2 (1 + 16t^2 \omega^4)}, \\
\theta(t) &= \theta(0) + \theta_1(t);
\end{align*}
\]
where
\[
\theta_1(t) = \arg\{\omega + \delta r(0) - i4\delta t \omega^2 r(0)\}.
\]
Thus
\[
\begin{align*}
 r(t) &= r(0) + r_\delta(t), \\
\theta(t) &= \theta(0) + \theta_\delta(t);
\end{align*}
\]
where there exists a constant $C_2 = C_2(t)$, such that
\[
|r_\delta(t)| \leq C_2 \delta, \quad |	heta_\delta(t)| \leq C_2 \delta.
\]
The rest (mean zero) part of (2.25) leads to
\[
f(x,t)e^{i\theta(t)} = e^{i\theta(0)} \tilde{q}_2(t), \tag{2.26}
\]
where
\[
\tilde{q}_2(t) = \tilde{q}_1(t) - \langle \tilde{q}_1(t) \rangle.
\]
Eq.(2.26) can be rewritten as
\[
f(x,t) = \tilde{F}^t(f(x,0)) + f_\delta(x,t),
\]
where there exists a constant $C_3 = C_3(t)$, such that
\[
\|f_\delta(x,t)\| \leq C_3 \delta.
\]
Go back to Eq.(2.24), we see that, there exists a constant $C = C(t)$, such that, for all
\[
Q \equiv \left( r(0), \theta(0), \tilde{f}(x,0), \epsilon_1 \right) \in \tilde{D}_k,
\]
\[
\left\| \tilde{F}_k(Q) - \tilde{F}^t(Q) \right\| \leq C \delta.
\]
Similarly for $\nabla \tilde{F}_k^t$ and $\nabla \tilde{F}^t$. ♣
Corollary 1  For any \( t \in (-\infty, \infty) \), there exist constants \( \Lambda_1 = \Lambda_1(t) \) and \( d_2 = d_2(t) \), such that,

\[
\begin{align*}
\Lambda_1^{-1} &\leq \left\| \tilde{F}_\delta^t(Q) \right\| \leq \Lambda_1, \\
\Lambda_1^{-1} &\leq \left\| \tilde{F}^t(Q) \right\| \leq \Lambda_1, \\
\Lambda_1^{-1} &\leq \left\| \nabla \tilde{F}_\delta^t(Q) \right\| \leq \Lambda_1, \\
\Lambda_1^{-1} &\leq \left\| \nabla \tilde{F}^t(Q) \right\| \leq \Lambda_1,
\end{align*}
\]

for all \( Q \in D_k \), and all \( \delta \in [0, d_2] \).

Proof: The corresponding inequalities for \( \tilde{F}^t \) and \( \nabla \tilde{F}^t \) are obviously true from the calculation in previous sections. Then by the above proposition, the corresponding inequalities for \( \tilde{F}_\delta^t \) and \( \nabla \tilde{F}_\delta^t \) are also true. ♦

2.6 \((\delta = 0)\) Invariant Manifolds and the Introduction of a Bump Function

In this section, we consider the unperturbed \((\delta = 0)\) flow (2.23) and certain of its invariant manifolds. Then we introduce a bump function necessary for later studies.

2.6.1 \((\delta = 0)\) Invariant Manifolds

Consider the unperturbed \((\delta = 0)\) flow (2.23):

\[
\begin{align*}
\dot{r}_t &= 0, \\
\dot{\theta}_t &= 0, \\
if_t &= f_{xx} + 2\omega^2(f + \bar{f}), \\
\epsilon_{1t} &= 0,
\end{align*}
\]

(2.27)

and make a decomposition of \( f \):

\[
\begin{align*}
f &= f' + f'' \quad (2.28) \\
f' &\equiv a_1 \cos k_1 x, \quad (2.29)
\end{align*}
\]

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\[ f'' = \sum_{j=2}^{\infty} a_j \cos k_j x, \quad (2.30) \]

which, in vector notation takes the form

\[ \vec{f}' = \alpha_1^+ \vec{e}_1^+ + \alpha_1^- \vec{e}_1^- \].

(2.31)

Equation (2.27) for \( f \) becomes

\[
\begin{align*}
if'_t &= f''_{xx} + 2\omega^2 (f' + \vec{f}), \quad (2.32) \\
if''_t &= f'''_{xx} + 2\omega^2 (f'' + \vec{f}'). \\
\end{align*}
\]

(2.33)

Next in \( \tilde{D}_k \) we will define the center, center-stable, and center-unstable manifolds (subspaces) \( W_0, W_{0cs}^c, \) and \( W_{0cu}^c \) for the equilibrium set in \( \tilde{D}_k \) defined by

\[ \tilde{S}_\omega \equiv \{(r, \theta, \vec{f}, \vec{f}', \epsilon_1) \in \tilde{D}_k \mid \vec{f} = 0, \ r = 0, \ \epsilon_1 = 0 \} \]

of the unperturbed system (2.27). In fact, \( \tilde{S}_\omega \) is the image of \( \tilde{S}_\omega \) under the coordinate transformation (2.21).

**Definition 3**

\[
\begin{align*}
W_{0cs}^c &= \left\{ (r, \theta, \vec{f}, \vec{f}', \epsilon_1) \in \tilde{D}_k \mid \alpha_1^+ = 0 \right\}, \\
W_{0cu}^c &= \left\{ (r, \theta, \vec{f}, \vec{f}', \epsilon_1) \in \tilde{D}_k \mid \alpha_1^- = 0 \right\}, \\
W_0 &= W_{0cs}^c \cap W_{0cu}^c = \left\{ (r, \theta, \vec{f}, \vec{f}', \epsilon_1) \in \tilde{D}_k \mid \vec{f} = 0 \right\}.
\end{align*}
\]

From now on, we will concentrate our discussion on \( W_{0cs}^c \). The discussion on \( W_{0cs}^c \) goes similarly. For convenience of notation, we write \( M \) for \( W_{0cs}^c \):

\[ M \equiv W_{0cs}^c. \]
2.6.2 Tangent and Transversal Bundles of M

Next, we consider the tangent and transversal bundles of $M(\equiv W_0^{cu})$, denoted respectively by $TM$ and $N$. Since the linear flow $\nabla \tilde{F}^t(m)$ is independent of $m \in M$, these bundles are trivial. Moreover, since $\nabla \tilde{F}^t(m) = \tilde{F}^t$, we have the simple representations

$$TM(m) = \text{span}\{\hat{e}_1^+, \hat{e}_j^\pm, j = 0, 2, 3, \ldots; \hat{e}_\epsilon\}$$

$$N(m) = \text{span}\{\hat{e}^-\}$$

where

$$\hat{e}_0^+ = (1, 0, \bar{0}, 0) \in \tilde{S}_k,$$

$$\hat{e}_0^- = (0, 1, \bar{0}, 0) \in \tilde{S}_k,$$

$$\hat{e}_\epsilon = (0, 0, \bar{1}, 0) \in \tilde{S}_k,$$

$$\hat{e}_j^\pm = (0, 0, e_j^\pm, 0) \in \tilde{S}_k,$$

$$j = 1, 2, \ldots.$$

In addition, the temporal flows can be explicitly computed: Let $w_0 \in N(m), w_0 = \alpha_\epsilon^- \hat{e}_\epsilon^-; then$

$$w_{-t} = \nabla \tilde{F}^{-t}(m)w_0 = \tilde{F}^{-t}w_0 = e^{-\Omega_\epsilon^- t}w_0 = e^{\Omega_\epsilon^+ t}w_0.$$

Let $v_0 \in TM(m)$,

$$v_0 = \alpha_1^+ \hat{e}_1^+ + \sum_{j\neq 1} (\alpha_j^+ \hat{e}_j^+ + \alpha_j^- \hat{e}_j^-) + \epsilon_1 \hat{e}_\epsilon,$$

then,

$$v_{-t} = \tilde{F}^{-t}v_0 = \alpha_1^+ e^{-\Omega_1^+ t}\hat{e}_1^+ + \alpha_0^+ \hat{e}_0^+ + \alpha_0^- \hat{e}_0^- +$$

$$+ \sum_{j=2}^{\infty} \left( \begin{array}{c}
\hat{e}_j^+
\hat{e}_j^-
\end{array} \right) \left( \begin{array}{cc}
\cos \Omega_j t & - \sin \Omega_j t \\
\sin \Omega_j t & \cos \Omega_j t
\end{array} \right) \left( \begin{array}{c}
\alpha_j^+
\alpha_j^-
\end{array} \right) +$$

$$+ \epsilon_1 \hat{e}_\epsilon.$$

From these explicit formulas and definition (3), one sees that the manifold $M$ is overflowing invariant for the flow (2.27), but that it is not strictly overflowing invariant. At some points on the boundary $\partial M$, the unperturbed vector field is tangent to the boundary, while at other points it is pointing strictly outward. The points "of tangency" are too sensitive to perturbations. As is standard, we "bump the vector field" near the boundary of $M$ to make $M$ strictly overflowing (or inflowing) invariant.
2.6.3 Introduction of a Bump Function

To make the manifold $M$ strictly overflowing (or inflowing) invariant, we modify the flows defined by equations (2.23) and (2.22) in some neighborhood of the boundary of $M$. We emphasize that the modified flows will be identical with the original flows in the unmodified region of $M$.

Choose real numbers $a_1$, $a_2$, $a_3$, $a_4$, and $a_5$, satisfying

$$0 < a_5 < a_4 < a_3 < a_2 < a_1 = 1,$$

and choose a $C^\infty$ “bump” function $\psi : [0, 1] \to \mathbb{R}$, such that

- $\psi(a) = 0$, for $a \in [0, a_4]$,
- $\psi(a_3) > 0$,
- $\psi(a_2) < 0$,
- $\psi(a_1) > 0$.

See Fig.2.3. Let $R_c \equiv (r^2 + \|\tilde{f}^n\|_{H^2_{\tilde{h}}}^2)^{1/2}$. Then we modify the systems (2.23) and (2.22) into the following systems:

$$r_t = \delta_1 \psi(R_c) r,$$
\[ \theta_t = 0, \]
\[ i f'_t = f'_{xx} + 2\omega^2(f' + f'), \quad (2.36) \]
\[ i f''_t = f''_{xx} + 2\omega^2(f'' + f') + i\delta_1 \psi(R_c) f'' + i\delta \psi(R_c) r, \]
\[ \epsilon_{tt} = 0. \]

\[ r_t = \delta_1 \psi(R_c) r + \delta G_r, \]
\[ \theta_t = \delta G_\theta, \]
\[ i f'_t = f'_{xx} + 2\omega^2(f' + \bar{f}') + \delta G_{f'}, \quad (2.37) \]
\[ i f''_t = f''_{xx} + 2\omega^2(f'' + \bar{f}'') + i\delta_1 \psi(R_c) f'' + \delta G_{f''}, \]
\[ \epsilon_{tt} = 0. \]

If we drop the two terms
\[ \delta_1 \psi(R_c) r \quad \text{and} \quad i\delta_1 \psi(R_c) f'' \]
in the above systems (2.36;2.37), then they are identical to the systems (2.23;2.22). Thus,
\[ G_r, G_\theta, G_{f'}, \quad \text{and} \quad G_{f''} \]
represent the corresponding parts of the right hand side of (2.22). Here \( \delta_1 \)
is a small positive number. In fact, these are the final unperturbed and perturbed systems that we are going to study.

**Remark 2.9** We are not forced to modify the \( \epsilon_1 \) equation with a “bump function” because the \( \epsilon_1 \) equation is trivial; the later argument can go through without bumping the \( \epsilon_1 \) equation.

Denote the solution operators of (2.36) and (2.37) by
\[ F_{\delta_1} \quad \text{and} \quad F_{\delta_1}^{\delta}, \]
respectively. Following similar argument as for theorems (2.1; 2.2), proposition (2.2), and corollary (1), we have the following claims on systems (2.36;2.37):

**Theorem 2.7 (Cauchy Problem)** \( \forall Q_0 \in \tilde{S}_k \) and \( \forall t \in (-\infty, \infty), \exists \) a unique solution, \( Q(t, \cdot) \in \tilde{S}_k \), to (2.36) and (2.37), respectively; such that \( Q|_{t=t_0} = Q_0 \); moreover, \( Q(t, \cdot) \) depends continuously upon \( t \).
Theorem 2.8 (Dependence on Data) For any fixed $t \in (-\infty, \infty)$, any fixed $n \ (n < \infty)$, and any fixed $k \ (k < \infty)$, both $F^t_{\delta_1}$ and $F^t_{\delta, \delta_1}$ are $C^n$ diffeomorphisms in $\tilde{S}_k$.

Theorem 2.9 For any $t \in (-\infty, \infty)$, there exist constants $C_1 = C_1(t)$ and $d_3 = d_3(t)$, such that,
\[
\begin{align*}
\| F^t_{\delta_1}(Q) - \tilde{F}^t(Q) \| &\leq C_1 \delta_1, \\
\| F^t_{\delta, \delta_1}(Q) - \tilde{F}^t_\delta(Q) \| &\leq C_1 \delta_1, \\
\| \nabla F^t_{\delta_1}(Q) - \nabla \tilde{F}^t(Q) \| &\leq C_1 \delta_1, \\
\| \nabla F^t_{\delta, \delta_1}(Q) - \nabla \tilde{F}^t_\delta(Q) \| &\leq C_1 \delta_1,
\end{align*}
\]
for all $Q \in \tilde{D}_k$, all $\delta_1 \in [0, d_3]$, and all $\delta \in [0, d_3]$.

Theorem 2.10 For any $t \in (-\infty, \infty)$, there exist constants $C_2 = C_2(t)$ and $d_4 = d_4(t)$, such that,
\[
\begin{align*}
\| F^t_{\delta_1}(Q) - F^t_\delta(Q) \| &\leq C_2 \delta, \\
\| \nabla F^t_{\delta, \delta_1}(Q) - \nabla F^t_\delta(Q) \| &\leq C_2 \delta,
\end{align*}
\]
for all $Q \in \tilde{D}_k$, all $\delta \in [0, d_4]$, and all $\delta_1 \in [0, d_4]$.

Corollary 2 For any $t \in (-\infty, \infty)$, there exist constants $\Lambda = \Lambda(t)$ and $d_5 = d_5(t)$, such that,
\[
\begin{align*}
\Lambda^{-1} &\leq \| F^t_{\delta_1}(Q) \| \leq \Lambda, \\
\Lambda^{-1} &\leq \| \nabla F^t_{\delta_1}(Q) \| \leq \Lambda,
\end{align*}
\]
\[ \Lambda^{-1} \leq \left\| F^t_{\delta,\delta_1}(Q) \right\| \leq \Lambda, \]
\[ \Lambda^{-1} \leq \left\| \nabla F^t_{\delta,\delta_1}(Q) \right\| \leq \Lambda, \]

for all \( Q \in \tilde{D}_k \), all \( \delta_1 \in [0,d_5] \),
and all \( \delta \in [0,d_5] \).

**Remark 2.10** For any fixed \( \delta_1 \in (0,d_4] \), when \( \delta \) is sufficiently small, system (2.37) is a \( C^1 \) perturbation of system (2.36), in the sense described in Theorem (2.10), in \( \tilde{D}_k \); moreover, \( M \) is strictly overflowing invariant under (2.36).

**Theorem 2.11** For any \( t \in (-\infty, \infty) \), there exist constants \( \Lambda_* = \Lambda_*(t) \) and \( d_* = d_*(t) \), such that
\[ \left\| \nabla^2 F^t_{\delta,\delta_1}(Q) \right\| \leq \Lambda_* \]
\[ \left\| \nabla^3 F^t_{\delta,\delta_1}(Q) \right\| \leq \Lambda_*; \]

for all \( Q \in \tilde{D}_k \), all \( \delta_1 \in [0,d_*] \), and all \( \delta \in [0,d_*] \).

Proof: By theorem (2.8), \( F^t_{\delta,\delta_1} \) is a \( C^n \) diffeomorphism. In the original coordinate \((q,\epsilon) \) (2.21), for any fixed \( t \in (-\infty, \infty) \),
\[ \left\| \nabla^2 F^t_{\delta,\delta_1}(q = \omega e^{i\theta}, \epsilon = 0) \right\| \leq \frac{1}{2} \Lambda_*. \]

Since \( \left\| \nabla^2 F^t_{\delta,\delta_1}(q,\epsilon) \right\| \) is a continuous function of \((q,\epsilon)\), then when \( \delta \) is sufficiently small,
\[ \left\| \nabla^2 F^t_{\delta,\delta_1}(q = (\omega + \delta r)e^{i\theta} + \delta f e^{i\theta}, \epsilon = \delta^2 \epsilon_1) \right\| \leq \Lambda_*, \]
which is equivalent to the claim of the theorem. ♣

From now on, for simplicity of notations, we abbreviate \( F^t_{\delta_1} \) and \( F^t_{\delta,\delta_1} \) as
\[ F^t_0 \text{ and } F^t_{\delta} \]
respectively.
3 Persistent Invariant Manifolds

In this section, we establish the existence of certain infinite dimensional (codimension 1 and 2) invariant manifolds for the enlarged, perturbed system (EPNLS) (2.19), using a dichotomy of time scales; i.e., using normal hyperbolicity.

3.1 Statement of the Persistence Theorem and the Strategy of Proof

The following definitions describe the nature of the invariant manifolds which we will show to persist.

**Definition 4 (Locally Invariant Manifold)** Let $V$ be a submanifold in $\bar{S}_k$ with boundary $\partial V$, $F^t$ be a flow defined in $\bar{S}_k$. If $V$ has the following property: Whenever there exist a point $Q \in V$, and time $t$, such that

$$\bigcup_{\tau \in [0,t)} F^\tau(Q) \subset V, \ F^t(Q) \notin V; \ (t > 0)$$

or

$$\bigcup_{\tau \in (t,0]} F^\tau(Q) \subset V, \ F^t(Q) \notin V; \ (t < 0)$$

then

$$F^t(Q) \in \partial V.$$

Then, we call $V$ a locally invariant submanifold under $F^t$.

Notice that $\bar{S}_\omega$ is an equilibrium manifold for systems (2.36; 2.37;2.23;2.22), we define its center-unstable, center-stable, and center manifolds as follows:

**Definition 5 (Invariant Manifolds of $\bar{S}_\omega$)** We call $W^{cu}_\delta$ a center-unstable manifold of $\bar{S}_\omega$ under the flow $\bar{F}^t_\delta$ (2.22), if:

1. $\bar{S}_\omega \subset W^{cu}_\delta$;

2. $W^{cu}_\delta$ is locally invariant under the flow $\bar{F}^t_\delta$ (2.22);

3. As $\delta \to 0$, $W^{cu}_\delta$ coincides with $W^{cu}_0$.

Its center-stable manifold $W^{cs}_\delta$ is defined similarly. Its center manifold $W_\delta$ is defined as the intersection

$$W_\delta \equiv W^{cu}_\delta \cap W^{cs}_\delta.$$
Denote the center-unstable, center-stable, and center manifolds of \( \tilde{S}_\omega \) under the bumped flow (2.37) by

\[
W^{\text{cu}}_{\delta_1,\delta}, \ W^{\text{cs}}_{\delta_1,\delta}, \ \text{and} \ W_{\delta_1,\delta}
\]

respectively. Under the coordinate transformation (2.21), for any fixed \( \delta \), \( W^{\text{cu}}_{\delta}, W^{\text{cs}}_{\delta}, \) and \( W_{\delta} \) are transformed into center-unstable, center-stable, and center manifolds of \( \hat{S}_\omega \) in \( \hat{D}_{k,\delta} \) under the flow (2.19). Denote them respectively by

\[
W^{\text{cu}}, \ W^{\text{cs}}, \ \text{and} \ W.
\]

For any fixed \( \epsilon \), the center-unstable, center-stable, and center manifolds of \( S_\omega \) in \( S_k \) under (2.1) are defined by restriction of \( W^{\text{cu}}, W^{\text{cs}}, \) and \( W \) to the fixed value of \( \epsilon \),

\[
W^{\text{cu}}_\epsilon \equiv W^{\text{cu}} \mid_\epsilon, \ W^{\text{cs}}_\epsilon \equiv W^{\text{cs}} \mid_\epsilon, \ W_\epsilon \equiv W \mid_\epsilon.
\]

**Definition 6** Define the following subregions of \( \tilde{D}_k \):

\[
\tilde{D}^{(j)}_k \equiv \left\{ (r,\theta,\tilde{f},\epsilon_1) \in \tilde{D}_k \mid (r^2 + \|\tilde{f}\|^2_{\tilde{H}_k})^{1/2} < a_j \right\} \\
(j = 1, \ldots, 5)
\]

where \( a_j \)'s are given in the definition of the bump function \( \psi \).

**Proposition 3.1** There exists a positive constant \( \tilde{d}_1 \), for any \( \delta_1 \in (0,\tilde{d}_1] \), there exists a positive constant \( \tilde{d} = \tilde{d}(\delta_1) \), such that, for any \( \delta \in [0,\tilde{d}] \), there exist a \( C^n \) codimension 1 center-unstable, a \( C^n \) codimension 1 center-stable, and a \( C^n \) codimension 2 center manifolds

\[
W^{\text{cu}}_{\delta_1,\delta}, \ W^{\text{cs}}_{\delta_1,\delta}, \ \text{and} \ W_{\delta_1,\delta}
\]

of \( \tilde{S}_\omega \) in \( \tilde{D}^{(2)}_k \) under the bumped perturbed flow (2.37).

**Theorem 3.1** There exist a \( C^n \) codimension 1 center-unstable, a \( C^n \) codimension 1 center-stable, and a \( C^n \) codimension 2 center manifolds,

\[
W^{\text{cu}}_\delta \equiv \tilde{D}^{(5)}_k \cap W^{\text{cu}}_{\delta_1,\delta}, \\
W^{\text{cs}}_\delta \equiv \tilde{D}^{(5)}_k \cap W^{\text{cs}}_{\delta_1,\delta}, \\
W_\delta \equiv \tilde{D}^{(5)}_k \cap W_{\delta_1,\delta}
\]
of $\tilde{\mathcal{S}}_\omega$ in $\tilde{D}_k^{(5)}$ under the perturbed flow (2.22), where
\[
\delta \in [0, \tilde{d}], \quad \tilde{d} = \tilde{d}(\delta_1), \quad \delta_1 \in (0, \tilde{d}_1]
\]
are given in proposition (3.1).

Denote by $\hat{D}_k^{(j)} (j = 1, ..., 5)$ the image of $\tilde{D}_k^{(j)} (j = 1, ..., 5)$ in $\hat{S}_k$ under the coordinate transformation (2.21). Let
\[
\delta_\ast \equiv \tilde{d}(\delta_1 = \tilde{d}_1),
\]
then $\hat{D}_k^{(5)}_{\delta_\ast}$ is a fixed solid torus neighborhood of $\hat{S}_\omega$ in $\hat{S}_k$.

**Theorem 3.2** In the solid torus neighborhood $\hat{D}_k^{(5)}_{\delta_\ast}$ of $\hat{S}_\omega$ in $\hat{S}_k$, there exist a $C^n$ codimension 1 center-unstable, a $C^n$ codimension 1 center-stable, and a $C^n$ codimension 2 center manifolds,
\[
W^{cu}, \quad W^{cs}, \quad \text{and} \quad W
\]
of $\hat{S}_\omega$ under the perturbed flow (2.19).

**Theorem 3.3** For any $\epsilon \in (-\delta_2^2, \delta_2^2)$, there exist a $C^n$ codimension 1 center-unstable, a $C^n$ codimension 1 center-stable, and a $C^n$ codimension 2 center manifolds,
\[
W^{cu}_\epsilon, \quad W^{cs}_\epsilon, \quad \text{and} \quad W_\epsilon
\]
of $S_\omega$ under the perturbed flow (2.1), inside the solid torus neighborhood of $S_\omega$,
\[
\hat{D}_k^{(5)}_{\epsilon} \subset S_k.
\]
Moreover, $W^{cu}_\epsilon$, $W^{cs}_\epsilon$, and $W_\epsilon$ are $C^n$ in $\epsilon$ for $\epsilon \in (-\delta_2^2, \delta_2^2)$.

**Remark 3.1** Theorems (3.1;3.2;3.3) are corollaries of proposition (3.1). Inside the interior region $\hat{D}_k^{(5)}$ of $\hat{D}_k$, the bump function $\psi$ is identically equal to zero; therefore, the bumped flow (2.37) coincides with the unbumped flow (2.22) in $\hat{D}_k^{(5)}$. Thus theorem (3.1) follows immediately from proposition (3.1). If we fix the value of $\delta (= \delta_\ast)$ in theorem (3.1), and restate this theorem in $\hat{S}_k$, we get theorem (3.2). Theorem (3.3) follows immediately from theorem (3.2) through restriction.

The strategy of proof in next subsection can be summarized as follows:
1. Define graph transform on the section space of a transversal bundle over the unperturbed overflowing invariant manifold. Based upon certain rate inequalities, a contraction map argument is carried on the graph transform. The fixed point of the graph transform gives the persistent overflowing invariant manifold.

2. Prove the $C^n$ smoothness of the persistent invariant manifold: First formally differentiate the graph transform equation that the persistent invariant manifold satisfies; then, a contraction map argument leads to a solution; finally, by the definition of Frechet derivative and a weighted inequality argument, the $C^1$ smoothness is proved. Similarly for $C^n$ smoothness. The approach for proving smoothness here is used everywhere in this book, e.g. in proving smoothness of fibers and smoothness of fibers with respect to their base points in later subsections.

Next subsection is devoted to the proof of proposition (3.1).

As remarked above, we only need to prove proposition (3.1). Moreover, we will concentrate on proving the existence of $W^{cu}_{\delta_1,\delta}$ in $\tilde{D}_k^{(1)}$ ($\equiv \tilde{D}_k$), the existence of $W^{cs}_{\delta_1,\delta}$ in $\tilde{D}_k^{(2)}$ follows similarly through reversing the time ($t \rightarrow -t$).

### 3.2 Definition of the Graph Transform

Fix the manifold $M(\equiv W^{cu}_0)$. It is overflowing invariant under the “bumped linear flow” (2.36). We have defined the transversal bundle $N$ in section 4.2. Let $N_\kappa$ be the subset of $N$ such that $N_\kappa$ consists of pairs $(Q, w)$ with $\|w\| \leq \kappa$; i.e.,

$$N_\kappa \equiv \{ (Q, w) \mid Q \in M, (Q, w) \in N, \|w\| \leq \kappa. \}$$

(Here, and throughout this proof, $\|\cdot\|$ will always mean $\|\cdot\|_{\tilde{H}_k}$.)

Alternatively, we can view the transversal bundle as it is embedded in $\tilde{S}_k$:

$$em : N_\kappa \rightarrow \tilde{S}_k$$

$$em(Q, w) = Q + w.$$  

In either case, the “transversal fiber” $w$ can be realized explicitly through the basis element $\hat{e}_1^-$:

$$w = \{ \alpha \hat{e}_1^-, \alpha \in \mathbb{R} \},$$
where $\hat{e}_1$ is defined in section 4.2. This representation emphasizes that the fibers of the transversal bundle $N$ are one dimensional.

There exists a $\kappa_0$, such that for $0 < \kappa < \kappa_0$, $em$ maps $N_\kappa C^\infty$-diffeomorphically onto a neighborhood of $M$ in $\tilde{S}_k$. Let $\Sigma$ be the space of sections of $N_\kappa|_M$:

$$\Sigma = \{ \sigma : M \mapsto N_\kappa, \sigma(Q) = (Q, w_\sigma(Q)) \},$$

where, for each $\sigma$, the function $w_\sigma(Q)$ can be explicitly realized in terms of the basis element $\hat{e}_1$:

$$\sigma(Q) = \left(Q, \alpha_{\{\sigma\}}(Q)\hat{e}_1\right), \quad \alpha_{\{\sigma\}} : M \to \mathbb{R}.$$ 

Next, we define a Lipschitz semi-norm on $\Sigma$:

$$Lip(\sigma) \equiv \sup_{Q, Q' \in \bar{M}} \frac{\|w_\sigma(Q) - w_\sigma(Q')\|}{\|Q - Q'\|}.$$ 

Let $\Sigma_\zeta$ be a subset of $\Sigma$ defined by

$$\Sigma_\zeta \equiv \{ \sigma \in \Sigma \mid Lip(\sigma) \leq \zeta \}. \quad (3.1)$$

Next, let $M_1 \equiv F_{10}^{\delta}(M)$. By changing time scale, we can assume that $M_1$ is still overflowing invariant. For details see the following remark.

**Remark 3.2** $F_{10}^{\delta}(M)$ will be overflowing invariant for sufficiently small $t$, $t < t_0$ for example. In this case, the time $t_0 = t_0(\delta_1)$. Recall that with respect to $\delta$, $\delta_1$ is $O(1)$. Rescale $t$ so that $t_0$ is scaled into 1, then $F_{10}^{\delta}(M)$ is overflowing invariant, and $\delta$ is still a small parameter. So without loss of generality, we can assume $M_1$ is overflowing invariant.

We will need the following lemma in the proof of the next two propositions [24].

**Lemma 3.1 (Mean Value Theorem)** Let $E_1$, $E_2$ be two Banach spaces, $F$ be a continuous mapping from a neighborhood of a segment $l$ joining two points $q_0$, $q_0 + q_1$ of $E_1$, into $E_2$. If $F$ is differentiable at every point of $l$, then

$$\left\| F(q_0 + q_1) - F(q_0) \right\| \leq \|q_1\| \cdot \sup_{0 \leq \alpha \leq 1} \left\| \nabla F(q_0 + \alpha q_1) \right\|.$$

Denote $F_{10}^{\delta} \bullet em \bullet \sigma$ simply by $F_{10}^{\delta} \sigma$. The following lemma is an important prerequisite for the definition of the graph transform.
Lemma 3.2 Let $b: N_\kappa \mapsto M$ be the base projection; i.e., $b(Q,w) = Q$. For any finite $T > 0$, define $\Phi(Q) \equiv bF_T^T \sigma F_0^{-T}(Q)$. There exists a positive constant $d_6 = d_6(T)$, such that, when $\delta \in [0, d_6]$, for all $\sigma \in \Sigma_\zeta$ (defined in $N_\kappa$):

1. $\Phi(Q)$ is defined for all $Q \in M$,

2. $\bar{M} \subset \Phi(M_1)$,

3. Each point in $M$ is the $\Phi$ image of only one point in $M$.

Proof: When $\delta = 0$, by the discussion in sections 4.2 and 4.3, $\Phi |_{\delta=0} \equiv bF_0^T \sigma F_0^{-T} = \text{identity map.}$

Then,

$$\Phi(Q) = bF_T^T \sigma F_0^{-T}(Q) + (bF_T^T - bF_0^T)\sigma F_0^{-T}(Q) \equiv Q + \Phi_1(Q).$$

By lemma (3.1),

$$\|\Phi_1(Q_1) - \Phi_1(Q_2)\| \leq \|\sigma F_0^{-T}(Q_1) - \sigma F_0^{-T}(Q_2)\| \cdot \sup_{0 \leq \alpha \leq 1} \left\| \left( \nabla bF_\delta^T - \nabla bF_0^T \right) \left( \alpha \sigma F_0^{-T}(Q_1) + (1 - \alpha)\sigma F_0^{-T}(Q_2) \right) \right\|.$$ 

Then, by theorem (2.10), when $\delta \in [0, d_4]$,

$$\|\Phi_1(Q_1) - \Phi_1(Q_2)\| \leq C_2(T)\delta \|\sigma F_0^{-T}(Q_1) - \sigma F_0^{-T}(Q_2)\|.$$ 

By lemma (3.1) again,

$$\|\sigma F_0^{-T}(Q_1) - \sigma F_0^{-T}(Q_2)\| \leq \|b\sigma F_0^{-T}(Q_1) - b\sigma F_0^{-T}(Q_2)\| + \|h\sigma F_0^{-T}(Q_1) - h\sigma F_0^{-T}(Q_2)\| \leq (1 + \zeta)\|F_0^{-T}(Q_1) - F_0^{-T}(Q_2)\| \leq (1 + \zeta)\|Q_1 - Q_2\| \sup_{0 \leq \alpha \leq 1} \left\| \nabla F_0^{-T}(\alpha Q_1 + (1 - \alpha)Q_2) \right\|,$$
where \( h \) is the projection on fibers; i.e., \( h(Q, w) = w \). By corollary (2), when \( \delta \in [0, d_5] \),
\[
\sup_{0 \leq \alpha \leq 1} \left\| \nabla F_0^{-T}(\alpha Q_1 + (1 - \alpha)Q_2) \right\| \leq \Lambda.
\]
Finally, we have
\[
\left\| \Phi_1(Q_1) - \Phi_1(Q_2) \right\| \leq \delta C_2(1 + \zeta)\Lambda \| Q_1 - Q_2 \|.
\]
Thus, when \( \delta \) is small enough, \( \Phi \) is a perturbation of the identity map in the Lipschitz topology for map. Then the lemma follows immediately [31][24]. ♣

Next, we define the Graph Transform,
\[
G : \Sigma_\zeta \mapsto \Sigma
\]
\[
G\sigma(bF_\delta^T(\sigma(Q))) \equiv (bF_\delta^T(\sigma(Q)), hF_\delta^T(\sigma(Q))),
\]
where \( h \) is the projection on fibers; i.e., \( h(Q, w) = w \). See Figure *** for intuition.

**Lemma 3.3** There exists a constant \( d_7 = d_7(T) \), such that, for any \( \delta \in [0, d_7] \), the graph transform \( G \) defined in (3.2) is well-defined as a map \( G : \Sigma_\zeta \mapsto \Sigma \); i.e., \( G\sigma : M \mapsto N_\kappa \).

Proof: By lemma (3.2), for any \( Q \in M \), there exists a unique \( Q_1 \in M_1 \), such that,
\[
Q = bF_\delta^T\sigma F_0^{-T}(Q_1).
\]
Set
\[
Q_0 \equiv F_0^{-T}(Q_1),
\]
if \( T > 1 \), then
\[
Q_0 \in M.
\]
Moreover,
\[
Q = bF_\delta^T\sigma(Q_0).
\]
In order to prove \( G\sigma : M \mapsto N_\kappa \), we need to show that
\[
\left\| hF_\delta^T\sigma(Q_0) \right\| \leq \kappa.
\]
We can rewrite \( hF_\delta^T\sigma(Q_0) \) as:
\[
hF_\delta^T\sigma(Q_0) = hF_0^T\sigma(Q_0) + hF_\delta^T\sigma(Q_0) - hF_0^T\sigma(Q_0).
\]
50
Then,
\[ \left\| hF_\delta^T \sigma(Q_0) \right\| \leq \left\| hF_0^T \sigma(Q_0) \right\| + \left\| hF_\delta^T \sigma(Q_0) - hF_0^T \sigma(Q_0) \right\|. \]

By the calculation in sections 4.2 and 4.3, we see that when \( T \) is large enough (\( \geq \frac{1}{\Omega_1} \ln 2 \)),
\[ \left\| hF_0^T \sigma(Q_0) \right\| \leq \frac{1}{2} \kappa. \]

By theorem (2.10),
\[ \left\| hF_\delta^T \sigma(Q_0) - hF_0^T \sigma(Q_0) \right\| \leq C_2 \delta, \]
where \( C_2 = C_2(T) \). Let
\[ d_7 = \frac{\kappa}{2C_2}. \]
then
\[ \left\| hF_\delta^T \sigma(Q_0) - hF_0^T \sigma(Q_0) \right\| \leq \frac{1}{2} \kappa, \]
when \( \delta \in [0, d_7] \). Therefore,
\[ \left\| hF_\delta^T \sigma(Q_0) \right\| \leq \kappa. \]

The proof is completed. ♦

**Remark 3.3** Fixed points of the graph transform define invariant manifolds for the flow \( F_\delta^T \). To see this, consider a section \( \sigma \), which consists of the ordered pairs
\[ \sigma \equiv \left\{ \left( Q, w_\sigma(Q) \right) \forall Q \in \bar{M} \right\}; \]
and its graph transform \( G\sigma \)
\[ G\sigma \equiv \left\{ \left( Q', hF_\delta^T [Q + w_\sigma(Q)] \right) \forall Q' \in \bar{M} \right\}, \]
where \( Q \) and \( Q' \) are related by
\[ Q' = hF_\delta^T [Q + w_\sigma(Q)]. \]
If $\sigma$ is a fixed point of the graph transform $G$, these two sections are the same; that is,

$$Q' = bF^T_\delta[Q + w_\sigma(Q)]$$
$$w_\sigma(Q') = hF^T_\delta[Q + w_\sigma(Q)],$$

which, when embedded in the function space $\hat{S}_k$ yields

$$Q' + w_\sigma(Q') = F^T_\delta(Q + w_\sigma(Q)).$$

The latter certainly shows that the fixed point (section) $\sigma$ defines an invariant manifold for the flow $F^T_\delta$.

### 3.3 The Graph Transform as a $C^0$ Contraction

Here we are going to define a new norm for elements of $\Sigma$, the $C^0$ norm, and we will prove that the graph transform is a contraction map in this norm.

**Definition 7** The sections have the representation $\sigma(Q) = (Q, w_\sigma(Q))$ in terms of which the $C^0$ norm is defined as follows:

$$\|\sigma\|_{C^0} = \sup_{Q \in \bar{M}} \|w_\sigma(Q)\|.$$  

**Lemma 3.4** $\Sigma_\zeta$ is closed under the $C^0$ norm.

**Proof:** Suppose $\sigma_j$ is a Cauchy sequence (under the $C^0$ norm) in $\Sigma_\zeta$. Then for any $Q \in M$, $w_{\sigma_j}(Q)$ is a Cauchy sequence (under $\hat{H}_k$ norm). We know that $w_{\sigma_j}(Q)$ have the representation

$$w_{\sigma_j}(Q) = \alpha_{1{\{\sigma_j\}}} e^-_1;$$

thus, $\alpha_{1{\{\sigma_j\}}}$ is a real Cauchy sequence, which has a limit $\alpha^-_1$. Define a new section $\sigma$ by

$$\sigma(Q) \equiv (Q, w_\sigma(Q)),$$

where

$$w_\sigma(Q) = \alpha^-_1 e^-_1.$$

First, we want to show that $\sigma \in \Sigma$. Since,

$$(Q, w_{\sigma_j}(Q)) \in N_\kappa, \text{ for all } j;$$
i.e.,
\[ \| \alpha^{-1}_{1(j)} \hat{e}^{-1}_1 \| \leq \kappa, \text{ for all } j; \]
in another word,
\[ |\alpha^{-1}_{1(j)}| \leq \kappa \| \hat{e}^{-1}_1 \|^{-1}, \]
we have
\[ |\alpha^{-1}_1| \leq \kappa \| \hat{e}^{-1}_1 \|^{-1}; \]
i.e.,
\[ \| \alpha^{-1}_1 \hat{e}^{-1}_1 \| \leq \kappa. \]

Then,
\[
(Q, w_{\sigma}(Q)) \in N_\kappa.
\]
Thus, \( \sigma \in \Sigma. \) Then, we want to show that \( \sigma \in \Sigma_\zeta. \) We calculate \( \text{Lip}\{\sigma\}: \)

\[
\text{Lip}\{\sigma\} = \sup_{Q, Q' \in \bar{M}} \frac{\| w_{\sigma}(Q) - w_{\sigma}(Q') \|}{\| Q - Q' \|}
= \sup_{Q, Q' \in \bar{M}} \frac{|\alpha^{-1}_1(Q) - \alpha^{-1}_1(Q')| \| \hat{e}^{-1}_1 \|}{\| Q - Q' \|}
= \sup_{Q, Q' \in \bar{M}} \lim_{j \to \infty} \frac{|\alpha^{-1}_{1(\sigma_j)}(Q) - \alpha^{-1}_{1(\sigma_j)}(Q')| \| \hat{e}^{-1}_1 \|}{\| Q - Q' \|}.
\]

We know that
\[
\frac{|\alpha^{-1}_{1(\sigma_j)}(Q) - \alpha^{-1}_{1(\sigma_j)}(Q')| \| \hat{e}^{-1}_1 \|}{\| Q - Q' \|} \leq \zeta,
\]
so
\[
\lim_{j \to \infty} \frac{|\alpha^{-1}_{1(\sigma_j)}(Q) - \alpha^{-1}_{1(\sigma_j)}(Q')| \| \hat{e}^{-1}_1 \|}{\| Q - Q' \|} \leq \zeta,
\]
thus
\[
\sup_{Q, Q' \in \bar{M}} \lim_{j \to \infty} \frac{|\alpha^{-1}_{1(\sigma_j)}(Q) - \alpha^{-1}_{1(\sigma_j)}(Q')| \| \hat{e}^{-1}_1 \|}{\| Q - Q' \|} \leq \zeta;
\]
i.e. \( \sigma \in \Sigma_\zeta. \) \( \Sigma_\zeta \) is closed under the \( C^0 \) norm. ♠

**Proposition 3.2** The graph transform \( G \) defined above in (3.2) maps \( \Sigma_\zeta \) into \( \Sigma_\zeta. \)
Proof: Denote $F^t_z \cdot em \cdot (Q, w)$ by $F^t_z(Q, w)$, where $z = 0, \delta; \, Q \in M, w \in N(Q)$. From the explicit calculation in sections 4.2 and 4.3, we choose $T$ large enough; moreover, fixed from now on, such that, for some $\kappa_0 > 0$,

$$
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \nabla_w hF^T_0 (Q, w) \right\| < 1/4,
$$
(3.3)

$$
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \left\{ \nabla_Q bF^T_0 (Q, w) \right\}^{-1} \right\| \cdot
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \nabla_w hF^T_0 (Q, w) \right\| < 1/4,
$$
(3.4)

$$
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \left\{ \nabla_Q bF^T_0 (Q, w) \right\}^{-1} \right\| < \frac{1}{2} \Lambda_2,
$$
(3.5)

where $\Lambda_2 = \Lambda_2(T)$. Without loss of generality, we can take $\Lambda_2 = \Lambda$, where $\Lambda$ is given in corollary (2). By theorem (2.10), when $\delta$ is sufficiently small, we have

$$
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \nabla_w hF^T_\delta (Q, w) \right\| < 1/2,
$$
(3.6)

$$
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \left\{ \nabla_Q bF^T_\delta (Q, w) \right\}^{-1} \right\| \cdot
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \nabla_w hF^T_\delta (Q, w) \right\| < 1/2,
$$
(3.7)

$$
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \left\{ \nabla_Q bF^T_\delta (Q, w) \right\}^{-1} \right\| < \Lambda.
$$
(3.8)

Also from the explicit calculation in sections 4.2 and 4.3, we know that

$$
\nabla_Q hF^T_0 (Q, w) = 0, \forall (Q, w) \in N_{\kappa_0}.
$$
(3.9)

Thus, for any $\eta > 0$, there exists $\hat{\delta} > 0$, such that, when $\delta \in [0, \hat{\delta}]$;

$$
\sup_{(Q,w) \in N_{\kappa_0}} \left\| \nabla_Q hF^T_\delta (Q, w) \right\| < \eta.
$$
(3.10)

(Note that $\eta$ is independent of $\zeta$. Later we will first choose $\zeta$ small enough, then choose $\eta < \frac{\zeta}{8\Lambda}$.) Denote by

$$
\mathcal{A} = \sup_{(Q,w) \in N_{\kappa_0}} \left\| \nabla_w hF^T_\delta (Q, w) \right\|,
$$
(3.11)
\[ B = \sup_{(Q, w) \in \mathbb{N}_0} \left \| \nabla_Q h F_{\delta}^T (Q, w) \right \|, \quad (3.12) \]

\[ C = \sup_{(Q, w) \in \mathbb{N}_0} \left \| \left \{ \nabla_Q b F_{\delta}^T (Q, w) \right \}^{-1} \right \|. \quad (3.13) \]

Then, inequalities (3.6;3.7;3.8;3.10) can be rewritten as

\[ A < \frac{1}{2}, \quad (3.14) \]
\[ C A < \frac{1}{2}, \quad (3.15) \]
\[ C < \Lambda, \quad (3.16) \]
\[ B < \eta. \quad (3.17) \]

Set

\[ \hat{Q} \equiv b F_{\delta}^T (\sigma(Q)), \quad \hat{Q}' \equiv b F_{\delta}^T (\sigma(Q')). \]

Then the claim of the proposition amounts to

\[ \left \| h F_{\delta}^T (\sigma(Q)) - h F_{\delta}^T (\sigma(Q')) \right \| \leq \zeta \left \| \hat{Q} - \hat{Q}' \right \|, \]

for any \( \hat{Q}, \hat{Q}' \in M. \)

\[ \left \| h F_{\delta}^T (\sigma(Q)) - h F_{\delta}^T (\sigma(Q')) \right \| = \left \| h F_{\delta}^T (Q, h \sigma(Q)) - h F_{\delta}^T (Q', h \sigma(Q)) \right \|
\]
\[ + \left \| h F_{\delta}^T (Q', h \sigma(Q)) - h F_{\delta}^T (Q', h \sigma(Q')) \right \|
\]
\[ \leq \left \| h F_{\delta}^T (Q, h \sigma(Q)) - h F_{\delta}^T (Q', h \sigma(Q)) \right \|
\]
\[ + \left \| h F_{\delta}^T (Q', h \sigma(Q)) - h F_{\delta}^T (Q', h \sigma(Q')) \right \|. \]

By lemma (3.1),

\[ \left \| h F_{\delta}^T (Q, h \sigma(Q)) - h F_{\delta}^T (Q', h \sigma(Q)) \right \| \leq \| Q - Q' \| \cdot \sup_{0 \leq \alpha \leq 1} \left \| \nabla_Q h F_{\delta}^T (\alpha Q + (1 - \alpha)Q', h \sigma(Q)) \right \|
\]
\[ \leq \| Q - Q' \| B. \quad (3.18) \]
By lemma (3.1),
\[
\left\| hF^T_\delta (Q', h\sigma(Q)) - hF^T_\delta (Q', h\sigma(Q')) \right\| \leq \left\| h\sigma(Q) - h\sigma(Q') \right\| .
\]
\[
\cdot \sup_{0 \leq \alpha \leq 1} \left\| \nabla_w hF^T_\delta (Q', \alpha h\sigma(Q) + (1 - \alpha) h\sigma(Q')) \right\| \\
\leq \zeta \| Q - Q' \| A. 
\]  
(3.19)

Thus,
\[
\left\| hF^T_\delta (\sigma(Q)) - hF^T_\delta (\sigma(Q')) \right\| \leq (B + \zeta A) \| Q - Q' \|. 
\]  
(3.20)

We need a relation between \( \| Q - Q' \| \) and \( \| \hat{Q} - \hat{Q}' \| \):
\[
\| \hat{Q} - \hat{Q}' \| = \left\| bF^T_\delta (\sigma(Q)) - bF^T_\delta (\sigma(Q')) \right\| \\
= \left\| bF^T_\delta (Q, h\sigma(Q)) - bF^T_\delta (Q', h\sigma(Q)) \right\| \\
+ bF^T_\delta (Q', h\sigma(Q)) - bF^T_\delta (Q', h\sigma(Q')) \right\| \\
\geq \left\| bF^T_\delta (Q, h\sigma(Q)) - bF^T_\delta (Q', h\sigma(Q)) \right\| \\
- \left\| bF^T_\delta (Q', h\sigma(Q)) - bF^T_\delta (Q', h\sigma(Q')) \right\|. 
\]  
(3.21)

By lemma (3.1),
\[
\left\| bF^T_\delta (Q', h\sigma(Q)) - bF^T_\delta (Q', h\sigma(Q')) \right\| \leq \left\| h\sigma(Q) - h\sigma(Q') \right\| .
\]
\[
\cdot \sup_{0 \leq \alpha \leq 1} \left\| \nabla_w bF^T_\delta (Q', \alpha h\sigma(Q) + (1 - \alpha) h\sigma(Q')) \right\| .
\]

By corollary (2)
\[
\left\| \nabla_w bF^T_\delta (Q', \alpha h\sigma(Q) + (1 - \alpha) h\sigma(Q')) \right\| \leq \sup_{Q \in D_k} \left\| \nabla F^T_\delta (Q) \right\| \leq \Lambda, 
\]
where \( \Lambda = \Lambda(T) \). Thus,
\[
\left\| bF^T_\delta (Q', h\sigma(Q)) - bF^T_\delta (Q', h\sigma(Q')) \right\| \leq \zeta \Lambda \| Q - Q' \|. 
\]  
(3.22)
Next, we estimate $\left\| bF^T_\delta(Q, h\sigma(Q)) - bF^T_\delta(Q', h\sigma(Q)) \right\|$ in (3.21). Define the map

$$\varphi(Q_1) \equiv bF^T_\delta(F_0^{-T}(Q_1), h\sigma(Q)), \quad Q \text{ fixed.}$$

As discussed in lemma (3.2), if $\delta = 0$, $\varphi$ is the identity map. Then when $\delta$ is small enough, $\varphi$ is a perturbation of identity in the Lipschitz topology for map. Thus, $\varphi$ is $1 - 1$. We also know that $F_0^{-T}$ is $1 - 1$. Finally, the map

$$\hat{\varphi}(Q_2) \equiv bF^T_\delta(Q_2, h\sigma(Q)), \quad Q \text{ fixed;}$$

is $1 - 1$. We have

$$Q_2 = \hat{\varphi}^{-1}(bF^T_\delta(Q_2, h\sigma(Q))). \quad (3.23)$$

Differentiating (3.23) with respect to $Q_2$, we have

$$\text{Identity map} = \nabla \hat{\varphi}^{-1} \cdot \nabla_{Q_2} bF^T_\delta(Q_2, h\sigma(Q)),$$

then

$$\nabla \hat{\varphi}^{-1} = \left\{ \nabla_{Q_2} bF^T_\delta(Q_2, h\sigma(Q)) \right\}^{-1}.$$

By lemma (3.1),

$$\|Q - Q'\| \leq \left\| bF^T_\delta(Q, h\sigma(Q)) - bF^T_\delta(Q', h\sigma(Q)) \right\| \cdot \sup_{0 \leq \alpha \leq 1} \left\| \nabla \hat{\varphi}^{-1}(abF^T_\delta(Q, h\sigma(Q)) + (1 - \alpha)bF^T_\delta(Q', h\sigma(Q))) \right\|$$

$$\leq \left\| bF^T_\delta(Q, h\sigma(Q)) - bF^T_\delta(Q', h\sigma(Q)) \right\| \cdot C. \quad (3.24)$$

By (3.21;3.22;3.24),

$$\|\hat{Q} - \hat{Q}'\| \geq C^{-1}\|Q - Q'\| - \Lambda \zeta \|Q - Q'\|. \quad (3.25)$$

By (3.20;3.25)

$$\left\| hF^T_\delta(\sigma(Q)) - hF^T_\delta(\sigma(Q')) \right\| \leq \frac{B + \zeta A}{C - 1 - \Lambda \zeta} \|\hat{Q} - \hat{Q}'\|. \quad (3.26)$$

Since

$$\left\{ \nabla_{Q} bF^T_\delta(Q, w) \right\}^{-1} \left\{ \nabla_{Q} bF^T_\delta(Q, w) \right\} = \text{identity map},$$

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then,
\[ 1 \leq \left\| \nabla_{Q} bF_{\delta}^{T}(Q,w) \right\|^{-1} \left\| \nabla_{Q} bF_{\delta}^{T}(Q,w) \right\| \leq \Lambda C. \]
Thus,
\[ C \geq \Lambda^{-1}. \]
We know from (3.16)
\[ C < \Lambda. \]
Then,
\[ \frac{B + \zeta A}{C^{-1} - \Lambda \zeta} \leq \frac{C(B + \zeta A)}{1 - \Lambda^{2} \zeta} \leq \frac{\eta \Lambda + \frac{5}{7} \zeta}{1 - \Lambda^{2} \zeta}. \]
Choose \( \zeta \) small enough, such that,
\[ 1 - \Lambda^{2} \zeta > \frac{7}{8}, \]
then choose
\[ \eta < \frac{\zeta}{8 \Lambda}, \]
we have (3.26),
\[ \left\| hF_{\delta}^{T}(\sigma(Q)) - hF_{\delta}^{T}(\sigma(Q')) \right\| < \frac{5}{7} \zeta \| \hat{Q} - \hat{Q}' \|. \]
This completes the proof of the proposition.

**Proposition 3.3** \( G \) is a contraction map on \( \Sigma_{\zeta} \) under \( C^{0} \) norm.

Proof: The claim in the proposition amounts to
\[ \| G\sigma - G\sigma'\|_{C^{0}} < \nu \| \sigma - \sigma'\|_{C^{0}}, \quad 0 < \nu < 1, \]
for all \( \sigma, \sigma' \in \Sigma_{\zeta} \). Let,
\[ \hat{Q} = bF_{\delta}^{T}(\sigma(Q)) = bF_{\delta}^{T}(\sigma'(Q')), \]
then,
\[ \| hG\sigma(\hat{Q}) - hG\sigma'(\hat{Q}) \| = \| hF_{\delta}^{T}(\sigma(Q)) - hF_{\delta}^{T}(\sigma'(Q')) \| \]
\[ = \| hF_{\delta}^{T}(Q, h\sigma(Q)) - hF_{\delta}^{T}(Q', h\sigma(Q)) \| \]
Moreover, by (3.24),
\[
+ hF^T_\delta(Q', h\sigma(Q)) - hF^T_\delta(Q', h\sigma(Q'))
\]
\[
+ hF^T_\delta(Q', h\sigma(Q')) - hF^T_\delta(Q', h\sigma'(Q'))
\]
\[
\leq \left| hF^T_\delta(Q, h\sigma(Q)) - hF^T_\delta(Q', h\sigma(Q)) \right|
\]
\[
+ \left| hF^T_\delta(Q', h\sigma(Q)) - hF^T_\delta(Q', h\sigma(Q')) \right|
\]
\[
+ \left| hF^T_\delta(Q', h\sigma(Q')) - hF^T_\delta(Q', h\sigma'(Q')) \right|
\]
By (3.18;3.19),
\[
\left| hF^T_\delta(Q, h\sigma(Q)) - hF^T_\delta(Q', h\sigma(Q)) \right| \leq B\|Q - Q'\|
\]
\[
\left| hF^T_\delta(Q', h\sigma(Q)) - hF^T_\delta(Q', h\sigma(Q')) \right| \leq \zeta A\|Q - Q'\|
\]
Next, we estimate the term:
\[
\left| hF^T_\delta(Q', h\sigma(Q')) - hF^T_\delta(Q', h\sigma'(Q')) \right|
\]
\[
\leq \|h\sigma(Q') - h\sigma'(Q')\| \sup_{0 \leq \alpha \leq 1} \|\nabla_w hF^T_\delta(Q', \alpha h\sigma(Q') + (1 - \alpha)h\sigma'(Q'))\|
\]
\[
\leq \|h\sigma(Q') - h\sigma'(Q')\|A
\]
\[
\leq A\|\sigma - \sigma'\|_{C^0}.
\]
Thus,
\[
\left| hG\sigma(\hat{Q}) - hG\sigma'(\hat{Q}) \right| \leq (B + \zeta A)\|Q - Q'\| + A\|\sigma - \sigma'\|_{C^0}. \quad (3.29)
\]
We need a relation between \(\|Q - Q'\|\) and \(\|\sigma - \sigma'\|_{C^0}\). From (3.28), we see that
\[
\left| bF^T_\delta(Q, h\sigma(Q)) - bF^T_\delta(Q', h\sigma(Q)) \right| = \left| bF^T_\delta(Q', h\sigma'(Q')) - bF^T_\delta(Q', h\sigma(Q)) \right|
\]
Moreover, by (3.24),
\[
\left| bF^T_\delta(Q, h\sigma(Q)) - bF^T_\delta(Q', h\sigma(Q)) \right| \geq \zeta^{-1}\|Q - Q'\|. \quad (3.31)
\]
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Next, we estimate:

\[
\left\| bF_\delta^T(Q', h\sigma'(Q')) - bF_\delta^T(Q', h\sigma(Q)) \right\|
\leq \left\| bF_\delta^T(Q', h\sigma'(Q')) - bF_\delta^T(Q', h\sigma(Q)) \right\|
+ bF_\delta^T(Q', h\sigma(Q')) - bF_\delta^T(Q', h\sigma(Q))
\leq \left\| bF_\delta^T(Q', h\sigma'(Q')) - bF_\delta^T(Q', h\sigma(Q)) \right\|
+ \left\| bF_\delta^T(Q', h\sigma(Q')) - bF_\delta^T(Q', h\sigma(Q)) \right\|.
\]  \tag{3.32}

By (3.22),

\[
\left\| bF_\delta^T(Q', h\sigma(Q')) - bF_\delta^T(Q', h\sigma(Q)) \right\| \leq \zeta \Lambda \|Q - Q'\|. \tag{3.33}
\]

By lemma (3.1),

\[
\left\| bF_\delta^T(Q', h\sigma(Q')) - bF_\delta^T(Q', h\sigma(Q)) \right\|
\leq \|h\sigma'(Q') - h\sigma(Q')\| \sup_{0 \leq \alpha \leq 1} \left\| \nabla_w bF_\delta^T(Q', ah\sigma'(Q') + (1 - \alpha)h\sigma(Q')) \right\|.
\]

By corollary (2),

\[
\sup_{0 \leq \alpha \leq 1} \left\| \nabla_w bF_\delta^T(Q', ah\sigma'(Q') + (1 - \alpha)h\sigma(Q')) \right\| \leq \sup_{Q \in \mathcal{D}_k} \left\| \nabla F_\delta^T(Q) \right\| \leq \Lambda,
\]

then

\[
\left\| bF_\delta^T(Q', h\sigma'(Q')) - bF_\delta^T(Q', h\sigma(Q')) \right\| \leq \Lambda \|\sigma - \sigma'\|_{C^0}. \tag{3.34}
\]

Thus, by (3.32:3.33:3.34),

\[
\left\| bF_\delta^T(Q', h\sigma'(Q')) - bF_\delta^T(Q', h\sigma(Q)) \right\| \leq \Lambda \|\sigma - \sigma'\|_{C^0} + \zeta \Lambda \|Q - Q'\|. \tag{3.35}
\]

By (3.30:3.31:3.35), we have

\[
\|Q - Q'\| \leq C\Lambda \left( \|\sigma - \sigma'\|_{C^0} + \zeta \|Q - Q'\| \right),
\]
thus,
\[ \|Q - Q'\| \leq \frac{CA}{1 - \zeta CA} \|\sigma - \sigma'\|_{C^0}. \]  
\[ (3.36) \]

By (3.29;3.36),
\[ \left\| hG\sigma(\hat{Q}) - hG\sigma'(\hat{Q}) \right\| \leq \left( \frac{CA(B + \zeta A)}{1 - \zeta CA} + A \right) \|\sigma - \sigma'\|_{C^0}. \]

By (3.14;3.15;3.16;3.17),
\[ \left( \frac{CA(B + \zeta A)}{1 - \zeta CA} + A \right) < \frac{(\eta + \frac{1}{2}\zeta)\Lambda^2}{1 - \zeta \Lambda^2} + 1/2. \]

Choose \( \zeta \) small enough, and let \( \eta \) satisfy condition (3.27), such that,
\[ \frac{(\eta + \frac{1}{2}\zeta)\Lambda^2}{1 - \zeta \Lambda^2} < 1/4. \]

Thus,
\[ \left\| hG\sigma(\hat{Q}) - hG\sigma'(\hat{Q}) \right\| < \frac{3}{4} \|\sigma - \sigma'\|_{C^0}. \]  
\[ (3.37) \]

Take supremum on the left hand side of (3.37), we have
\[ \|G\sigma - G\sigma'\|_{C^0} \leq \frac{3}{4} \|\sigma - \sigma'\|_{C^0}. \]

This completes the proof of the proposition. ♣

3.4 The Existence of the Invariant Manifolds

The argument for existence is completed, once the following theorem is established:

**Theorem 3.4** There is a unique \( \sigma^* \in \Sigma \), such that, \( F_t^\Sigma(\text{graph } \sigma^*) \supset \text{graph } \sigma^* \) for all \( t > 0 \), i.e. \( \text{graph } \sigma^* \) is overflowing invariant under \( F_t^\Sigma \). Moreover, \( \sigma^* \in \Sigma_{\zeta} \).

Proof: \( \Sigma_{\zeta} \) is closed under \( C^0 \) norm, so \( G \) has a unique fixed point \( \sigma^* \) in \( \Sigma_{\zeta} \). Next, we show that \( \sigma^* \) is in fact the unique fixed point of \( G \) in \( \Sigma \). The proof of lemmas (3.2;3.3) can be immediately adjusted to show that these two lemmas also hold for \( \sigma \in \Sigma \). Moreover, in the proof of proposition (3.3) if only one of \( \sigma \) and \( \sigma' \) is Lipschitz and in \( \Sigma_{\zeta} \), the argument remains valid.
These facts show that $\sigma^*$ is in fact the unique fixed point of $G$ in $\Sigma$. Next, we show that graph $\sigma^*$ is an overflowing invariant manifold under $F_t^\delta$. For small $t > 0$,

$$F_t^\delta(\text{graph } \sigma^*) \cap N_\kappa$$

is the graph of an element $\sigma_t^* \in \Sigma_\zeta$. We know that

$$\text{graph } \sigma^* \subset F_t^T(\text{graph } \sigma^*),$$

then

$$F_t^\delta(\text{graph } \sigma^*) \subset F_t^\delta F_t^T(\text{graph } \sigma^*) = F_t^\delta T F_t^\delta(\text{graph } \sigma^*),$$

thus, $\sigma_t^*$ is also a fixed point of $G$. Therefore,

$$\sigma_t^* = \sigma^*;$$

i.e.,

$$\text{graph } \sigma^* \subset F_t^\delta(\text{graph } \sigma^*).$$

It is easily checked that, graph $\sigma^*$ is overflowing invariant under $F_t^\delta$ if and only if

$$\text{graph } \sigma^* \subset F_t^\delta(\text{graph } \sigma^*),$$

which completes the proof of this theorem. ♣

### 3.5 The Smoothness of the Invariant Manifolds

Let $\sigma$ denote the fixed point of the graph transform discussed in previous subsections. (We drop the star “*”. ) Graph $\sigma$ is the persistent overflowing invariant manifold. $\sigma$ satisfies the following functional equation:

$$\left\{ \begin{array}{l} \sigma(Q) = \left( bF_t^T(\sigma(Q')), hF_t^T(\sigma(Q')) \right), \\ Q = bF_t^T(\sigma(Q')); \end{array} \right.$$

i.e.

$$\left\{ \begin{array}{l} h\sigma(Q) = hF_t^\delta(\sigma(Q')), \\ Q = bF_t^\delta(\sigma(Q')). \end{array} \right. \quad (3.38)$$

Notice that,

$$\text{graph } \sigma \equiv \bigcup_{Q \in M} \left( Q + h\sigma(Q) \right) = \text{image } \left\{ (I + h\sigma)\big|_M \right\},$$

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where $I$ denotes the identity map. Since $M$ is $C^\infty$ smooth, the smoothness of graph $\sigma$ is characterized by the smoothness of the map $h\sigma$:

$$h\sigma : M \mapsto hN_\kappa,$$

where $M$ has codimension 1, and $hN_\kappa$ has dimension 1.

If $h\sigma \in C^1$, then

$$\nabla h\sigma \in C^0(M, L(TM, ThN_\kappa));$$

i.e., $\nabla h\sigma$ is a continuous map from $M$ to the linear space $L(TM, ThN_\kappa)$.

Denote by

$$\Sigma_1 \equiv \left\{ u \in C^0(M, L(TM, ThN_\kappa)) \right\}.$$

For any $u \in \Sigma_1$, define the norm:

$$\|u\| \equiv \sup_{Q \in M} \|u(Q)\|,$$  \hspace{1cm} (3.39)

where $\|u(Q)\|$ is the linear operator norm.

**Lemma 3.5** $\Sigma_1$ is a complete metric space, with respect to the norm defined above.

**Proof:** Let

$$\{u_j\}, \quad j = 1, 2, ..., \infty$$  \hspace{1cm} (3.40)

be a Cauchy sequence in $\Sigma_1$, then

$$\{u_j(Q)\}, \quad j = 1, 2, ..., \infty$$  \hspace{1cm} (3.41)

is a Cauchy sequence in $L(TQ^M, TQhN_\kappa)$. Thus, $\forall \xi \in TQ^M$,

$$\{u_j(Q) \cdot \xi\}, \quad j = 1, 2, ..., \infty$$  \hspace{1cm} (3.42)

is a Cauchy sequence in $TQhN_\kappa$. Since $TQhN_\kappa$ is a Hilbert space, the Cauchy sequence (3.42) has a limit in $TQhN_\kappa$:

$$w_{Q,\xi} \in TQhN_\kappa.$$

Define a map,

$$u_Q \ : \ TQ^M \mapsto TQhN_\kappa,$$

$$\forall \xi \in TQ^M, \quad u_Q(\xi) \equiv w_{Q,\xi}.$$
It follows immediately from the Cauchy sequence \((3.42)\) that

\[ u_Q \in L(T_Q M, T_Q h N_\kappa). \]

Next we show that

\[ u_j(Q) \to u_Q, \quad \text{as } j \to \infty, \]

in \(L(T_Q M, T_Q h N_\kappa)\). By \((3.41)\), for any \(\epsilon_s > 0\), there exists \(K\), such that

\[ \|u_j(Q) - u_k(Q)\| < \epsilon_s, \quad \forall j, k \geq K. \quad (3.43) \]

For any \(\xi \in T_Q M\), \(\|\xi\| = 1\), there exists \(K_\xi\), such that

\[ \|u_l(Q) \cdot \xi - u_Q(\xi)\| < \epsilon_s, \quad \forall l \geq K_\xi. \]

Without loss of generality, we can take \(K_\xi \geq K\). Then, for any \(j \geq K\), and any \(\xi \in T_Q M\), \(\|\xi\| = 1\),

\[
\begin{align*}
    \|u_j(Q) \cdot \xi - u_Q(\xi)\| &\leq \|u_j(Q) \cdot \xi - u_l(Q) \cdot \xi\| + \\
    \|u_l(Q) \cdot \xi - u_Q(\xi)\| &\leq \|u_l(Q) - u_l(Q)\| + \\
    \|u_l(Q) \cdot \xi - u_Q(\xi)\| &\leq 2\epsilon_s,
\end{align*}
\]

where \(l \geq K_\xi\). Then

\[
\sup_{\xi \in T_Q M, \|\xi\| = 1} \|u_j(Q) \cdot \xi - u_Q(\xi)\| \leq 2\epsilon_s;
\]

i.e.,

\[ \|u_j(Q) - u_Q\| \leq 2\epsilon_s. \]

Thus,

\[ u_j(Q) \to u_Q, \quad \text{as } j \to \infty \]

in \(L(T_Q M, T_Q h N_\kappa)\). Similar argument shows that

\[ u_j \to u, \quad \text{as } j \to \infty \]

in the norm \((3.39)\). This immediately implies that

\[ u \in C^0(M, L(TM, Th N_\kappa)). \]

Thus \(\Sigma_1\) is a complete metric space. This completes the proof of the lemma.

\[ \blacklozenge \]
Formally differentiating the functional equation (3.38) leads to

\[ u(Q) = Hu(Q); \]

in which,

\[ Hu(Q) \equiv [A + Bu(Q')] [C + Eu(Q')]^{-1}, \]

where \( Q = bF^T_\delta(\sigma(Q')) \); \( A, B, C, \) and \( E \) are linear operators defined as follows:

\[
\begin{align*}
A & \equiv \nabla Q' hF^T_\delta(\sigma(Q')), \\
B & \equiv \nabla w' hF^T_\delta(\sigma(Q')), \\
C & \equiv \nabla Q' bF^T_\delta(\sigma(Q')), \\
E & \equiv \nabla w' bF^T_\delta(\sigma(Q')).
\end{align*}
\]

**Definition 8** Define:

\[
\begin{align*}
u^0(Q) & \equiv 0, \\
u^{j+1}(Q) & = Hu^j(Q).
\end{align*}
\]

**Lemma 3.6** \( \|u^j\| < \zeta \), for all \( j \).

Proof: We will prove this lemma by induction: \( \|u^0\| = 0 < \zeta \), assume that \( \|u^j\| < \zeta \), and prove that \( \|u^{j+1}\| < \zeta \).

Notice that

\[ C + Eu^j(Q') = C(I + C^{-1}Eu^j(Q')), \]

then

\[ (C + Eu^j(Q'))^{-1} = [I + C^{-1}Eu^j(Q')]^{-1}C^{-1}. \]

We also know that

\[ \|C^{-1}Eu^j(Q')\| \leq \|C^{-1}\| \|E\| \|u^j(Q')\| \leq \Lambda^2\|u^j(Q')\| < \Lambda^2\zeta. \]

From the argument in the last subsection, we know that

\[ \zeta\Lambda^2 < 1/8. \]

Thus \( [1 + C^{-1}Eu^j(Q')]^{-1} \) is a bounded operator and equal to:

\[
I + \sum_{l=1}^{\infty} (-1)^l(C^{-1}Eu^j(Q'))^l;
\]
moreover,

\[ \|I + C^{-1}Ew^j(Q')\|^{-1} \leq \frac{1}{1 - \|C^{-1}Ew^j(Q')\|} < \frac{8}{7}. \]

\[ \|u^{j+1}(Q)\| = \|Hu^j(Q)\| = \|[A + Bu^j(Q')][C + Eu^j(Q')]^{-1}\| \]

\[ \leq \|[A + Bu^j(Q')]\| \|C^{-1}\| \|[I + C^{-1}Eu^j(Q')]^{-1}\| \]

\[ < \frac{8}{7}(\|A\| + \|B\| \|u^j(Q')\|)\|C^{-1}\|. \]

From the argument in the last subsection, we know that

\[ \|A\| \eta < \frac{\zeta}{8\Lambda}, \]

\[ \|B\| \|C^{-1}\| < 1/2; \]

thus

\[ \|u^{j+1}(Q)\| < \frac{8}{7}(\zeta/8 + \zeta/2) < \frac{5}{7}\zeta < \zeta. \]

Thus the proof of the lemma is completed. ♣

Lemma 3.7 \[ \|u^{j+1} - u^j\| < \frac{33}{29}\|u^j - u^{j-1}\|. \]

Proof:

\[ Hu^j - Hu^{j-1} = [A + Bu^j][C + Eu^j]^{-1} - [A + Bu^{j-1}][C + Eu^{j-1}]^{-1} \]

\[ = [A + Bu^j][C + Eu^j]^{-1} \left( [C + Eu^{j-1}] - [C + Eu^j] \right) [C + Eu^{j-1}]^{-1} + \]

\[ + \left( [A + Bu^j] - [A + Bu^{j-1}] \right) [C + Eu^{j-1}]^{-1} \]

\[ = [A + Bu^j][C + Eu^j]^{-1}[E(u^j - u^{j-1})][C + Eu^{j-1}]^{-1} + \]

\[ + [B(u^j - u^{j-1})][C + Eu^{j-1}]^{-1} \]

From the argument in the proof of the last lemma, we know that \[ \|[I + C^{-1}Eu^j]\|^{-1} < \frac{8}{7}, \]

for all \( j \). Thus

\[ \|Hu^j - Hu^{j-1}\| \leq \left( \frac{8}{7} \right)^2(\|A\| + \|B\|\|\zeta\|\|C^{-1}\|^2\|E\|\|u^j - u^{j-1}\| + \]

\[ + \frac{8}{7}\|B\|\|C^{-1}\|\|u^j - u^{j-1}\| \]

\[ \leq \left( \frac{8}{7}\right)^2(\eta\|C^{-1}\| + \|B\|\|C^{-1}\|\|\zeta\|\|C^{-1}\|^2\|E\|)\|u^j - u^{j-1}\| \]

\[ \leq \left( \frac{8}{7}\right)^2(\eta\Lambda + \frac{1}{2}\zeta\|C^{-1}\|^2\|E\|)\|u^j - u^{j-1}\|. \]
We know that $\eta < \frac{\zeta}{\Lambda}$ (cf: The proof of the last lemma), then
\[
\|Hu^j - Hu^{j-1}\| \leq \left[\frac{5}{8} \left(\frac{8}{7}\right)^2 \zeta \Lambda^2 + \frac{4}{7}\right] \|u^j - u^{j-1}\|.
\]

We also know that $\zeta \Lambda^2 < \frac{1}{8}$ (cf: The proof of the last lemma), then
\[
\|Hu^j - Hu^{j-1}\| < \frac{33}{49} \|u^j - u^{j-1}\|.
\]

The proof of the lemma is completed. ♣

Corollary 3 The sequence $u^j$ converges to a solution $u$ of the equation $u = Hu$. Moreover, $u \in \Sigma_1$. In particular, $u \in C^0(M, L(TM, ThN_\kappa))$, $\|u\| \leq \zeta$.

Theorem 3.5 For any $Q \in M$, $\nabla h \sigma(Q)$ exists and equals $u(Q)$. Therefore, $\nabla h \sigma \in C^0(M, L(TM, ThN_\kappa))$; i.e., $h \sigma \in C^1$.

Proof: For any $Q \in M$, define an increasing function:
\[
\Delta_Q : (0, 1) \mapsto R, \quad \Delta_Q(a) \equiv \sup_{Q' \in M, 0 < \|Q' - Q\| < a} \frac{\|h \sigma(Q') - h \sigma(Q) - u(Q) \cdot (Q' - Q)\|}{\|Q' - Q\|}.
\]

By corollary (3) and the property of $\sigma$, we know that, for any $Q \in M$, when $a$ is sufficiently small:
\[
\Delta_Q(a) \leq 2\zeta. \quad (3.44)
\]

For any $Q \in M$, since $\Delta_Q(a)$ is an increasing nonnegative function, the limit
\[
\lim_{a \to 0} \Delta_Q(a)
\]
exists. Denote this limit by $\Delta_Q(0)$. By (3.44),
\[
0 \leq \Delta_Q(0) \leq 2\zeta, \quad \forall Q \in M. \quad (3.45)
\]

For any $Q_1 \in M$, define the sequence:
\[
\{Q_j\}, \quad j = 1, 2, ..., \infty; \quad Q_j = bF_\delta^T(\sigma(Q_{j+1})).
\]
To prove the theorem, we need to show that:

\[ \Delta Q_1(0) = 0. \]

We will show that the inequality:

\[ \Delta Q_j(0) \leq \gamma \Delta Q_{j+1}(0), \quad \forall j = 1, 2, \ldots, \infty; (0 < \gamma < 1) \quad (3.46) \]

is valid. Then, by (3.45)

\[ \Delta Q_1(0) \leq 2 \zeta \gamma^m, \quad \forall m \in \mathbb{Z}^+. \]

Thus,

\[ \Delta Q_1(0) = 0. \]

Next, we prove the inequality (3.46). For any \( j \in \mathbb{Z}^+ \), there exists a small constant \( a_j \), such that, when \( \|Q_j' - Q_j\| < a_j \), all the Taylor expansions below are valid.

\[
Q_j' - Q_j = bF^T_\delta(\sigma(Q_j')) - bF^T_\delta(\sigma(Q_{j+1}))
= C(Q_{j+1}' - Q_{j+1}) + E(h\sigma(Q_{j+1}) - h\sigma(Q_{j+1})))
+ o(\|Q_{j+1}' - Q_{j+1}\|). \quad (3.47)
\]

\[
h\sigma(Q_j') - h\sigma(Q_j) = hF^T_\delta(\sigma(Q_j')) - hF^T_\delta(\sigma(Q_{j+1}))
= A(Q_{j+1}' - Q_{j+1}) + B(h\sigma(Q_{j+1}) - h\sigma(Q_{j+1})))
+ o(\|Q_{j+1}' - Q_{j+1}\|). \quad (3.48)
\]

We remark that here \( A, B, C, \) and \( E \) are defined with respect to \( \sigma(Q_{j+1}) \), instead of \( \sigma(Q_j') \) in their original definitions. Multiply both sides of (3.47) by \( C^{-1} \), we have

\[
Q_{j+1}' - Q_{j+1} = C^{-1}(Q_j' - Q_j) - C^{-1}E(h\sigma(Q_{j+1}) - h\sigma(Q_{j+1})))
+ o(\|Q_{j+1}' - Q_{j+1}\|).
\]

Then,

\[
\|Q_{j+1}' - Q_{j+1}\| \leq \|C^{-1}\| \|Q_j' - Q_j\| + 2\zeta \Lambda^2 \|Q_{j+1}' - Q_{j+1}\|,
\]

where we have chosen \( a_j \) small enough, so that

\[
\left\| o(\|Q_{j+1}' - Q_{j+1}\|) \right\| \leq \zeta \Lambda^2 \|Q_{j+1}' - Q_{j+1}\|.
\]

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Thus,
\[ \|Q_{j+1}' - Q_{j+1}\| \leq \frac{\|C^{-1}\|}{1 - 2\zeta \Lambda^2} \|Q'_j - Q_j\|. \] (3.49)

Eq. (3.47) can be rewritten as:
\[
Q'_j - Q_j = E \left( h\sigma(Q'_{j+1}) - h\sigma(Q_{j+1}) - u(Q_{j+1}) \cdot (Q'_{j+1} - Q_{j+1}) \right) \\
+ \left( C + Eu(Q_{j+1}) \right) \cdot (Q'_{j+1} - Q_{j+1}) \\
+ o(\|Q'_{j+1} - Q_{j+1}\|). \] (3.50)

By corollary (3),
\[
\begin{align*}
&h\sigma(Q'_j) - h\sigma(Q_j) - u(Q_j) \cdot (Q'_j - Q_j) \\
&= h\sigma(Q'_j) - h\sigma(Q_j) - Hu(Q_j) \cdot (Q'_j - Q_j) \\
&= h\sigma(Q'_j) - h\sigma(Q_j) - [A + Bu(Q_{j+1})][C + Eu(Q_{j+1})]^{-1} \\
&\quad \cdot (Q'_j - Q_j). \end{align*} \] (3.51)

Substitute (3.50) into (3.51), we have:
\[
\begin{align*}
&h\sigma(Q'_j) - h\sigma(Q_j) - u(Q_j) \cdot (Q'_j - Q_j) \\
&= h\sigma(Q'_j) - h\sigma(Q_j) - [A + Bu(Q_{j+1})][C + Eu(Q_{j+1})]^{-1} E \left( h\sigma(Q'_{j+1}) - h\sigma(Q_{j+1}) \\
&\quad - u(Q_{j+1}) \cdot (Q'_{j+1} - Q_{j+1}) \right) + o(\|Q'_{j+1} - Q_{j+1}\|). \end{align*} \] (3.52)

Substitute (3.48) into (3.52), we have:
\[
\begin{align*}
&h\sigma(Q'_j) - h\sigma(Q_j) - u(Q_j) \cdot (Q'_j - Q_j) \\
&= \left\{ B - [A + Bu(Q_{j+1})][C + Eu(Q_{j+1})]^{-1} E \right\} \left( h\sigma(Q'_{j+1}) - h\sigma(Q_{j+1}) \\
&\quad - u(Q_{j+1}) \cdot (Q'_{j+1} - Q_{j+1}) \right) + o(\|Q'_{j+1} - Q_{j+1}\|). \end{align*} \] (3.53)

By (3.49:3.53)
\[
\frac{\|h\sigma(Q'_j) - h\sigma(Q_j) - u(Q_j) \cdot (Q'_j - Q_j)\|}{\|Q'_j - Q_j\|} \]
\begin{align}
\leq & \left\Vert B - \left[ A + Bu(Q_{j+1}) \right]\left[ C + Eu(Q_{j+1}) \right]^{-1}E \right\Vert \cdot \frac{\|C^{-1}\|}{1 - 2\zeta A^2} \\
& \cdot \frac{\|h\sigma(Q'_{j+1}) - h\sigma(Q_{j+1}) - u(Q_{j+1}) \cdot (Q'_{j+1} - Q_{j+1})\|}{\|Q'_{j+1} - Q_{j+1}\|} \\
& + \frac{\|C^{-1}\|}{1 - 2\zeta A^2} \cdot \frac{\sigma(\|Q'_{j+1} - Q_{j+1}\|)}{\|Q'_{j+1} - Q_{j+1}\|}. \tag{3.54}
\end{align}

As mentioned before, we know that
\[
\sup_{Q' \in M} \|B\| \sup_{Q' \in M} \|C^{-1}\| < 1/2,
\]
\[
\sup_{Q' \in M} \|A\| < \frac{\zeta}{8\Lambda}.
\]
Moreover,
\[
\|u\| \leq \zeta.
\]
Therefore, when \(\zeta\) is sufficiently small, there exists a constant 0 < \(\gamma\) < 1, such that
\[
\left\Vert B - \left[ A + Bu(Q_{j+1}) \right]\left[ C + Eu(Q_{j+1}) \right]^{-1}E \right\Vert \cdot \frac{\|C^{-1}\|}{1 - 2\zeta A^2} \leq \gamma, \ \forall Q_{j+1} \in M.
\]
Denote by
\[
\lambda \equiv \frac{\|C^{-1}\|}{1 - 2\zeta A^2},
\]
then, by (3.49;3.55)
\[
\Delta Q_j(\|Q'_{j} - Q_{j}\|) \leq \gamma \Delta Q_{j+1}(\lambda\|Q'_{j} - Q_{j}\|) + r(\lambda\|Q'_{j} - Q_{j}\|), \tag{3.56}
\]
where
\[
r(\lambda\|Q'_{j} - Q_{j}\|) \to 0, \ \text{as} \ \|Q'_{j} - Q_{j}\| \to 0.
\]
Let \(\|Q'_{j} - Q_{j}\| \to 0\) in (3.56), we have
\[
\Delta Q_j(0) \leq \gamma \Delta Q_{j+1}(0),
\]
which is inequality (3.46). This completes the proof of the theorem. ♦

Additional smoothness follows similarly [31]. We summarize in the following:

**Theorem 3.6** \(h\sigma \in C^m\). Thus, graph \(\sigma\) defines a \(C^m\) manifold.
Proof: The proof is similar to that given above for $h\sigma \in C^1$. Therefore, here we only sketch the proof. If $h\sigma \in C^s$, then

$$\nabla^s h\sigma \in C^0(M, L^s(TM, ThN_\kappa)).$$

Denote by

$$\Sigma_s \equiv \left\{ u_s \in C^0(M, L^s(TM, ThN_\kappa)) \right\}.$$

For any $u_s \in \Sigma_s$, define the norm:

$$\|u_s\| \equiv \sup_{Q \in M} \|u_s(Q)\|,$$

where $\|u_s(Q)\|$ is the $s$–linear operator norm. Similar proof as for lemma (3.5) shows that $\Sigma_s$ is a complete metric space, with respect to the norm defined above. We know that $h\sigma \in C^1$,

$$\nabla h\sigma = u_1,$$

$$u_1(Q) = Hu_1(Q) \equiv [A + Bu_1(Q')][C + Eu_1(Q')]^{-1}. \quad (3.57)$$

Moreover,

$$\frac{dQ'}{dQ} = [C + Eu_1(Q')]^{-1}. \quad (3.58)$$

Assume $h\sigma \in C^{s-1}$, we want to show that $h\sigma \in C^s$, provided that $s \leq n$. Formally differentiating (3.57) $s-1$ times, and noticing the relation (3.58), we have

$$u_s(Q) = \left\{ Bu_s(Q') + [A + Bu_1(Q')][C + Eu_1(Q')]^{-1}Eu_s(Q') \right\} \cdot [C + Eu_1(Q')]^{-s} + \text{terms not involving } u_s. \quad (3.59)$$

Denote the right hand side of (3.59) by $H_s u_s(Q)$. By assumption,

$$h\sigma \in C^{s-1}, \quad \nabla^{s-1} h\sigma = u_{s-1}.$$

Therefore, $H_s$ is a linear map on $\Sigma_s$. We know that

$$\sup_{Q' \in M} \|A\| < \frac{\zeta}{8\Lambda}, \quad \|u_1\| \leq \zeta.$$

Thus, if

$$\|B\|\|C^{-1}\|^s < 1, \quad (3.60)$$
and $\zeta$ is sufficiently small, then $H_s$ is a contraction map. This is where the inequality (3.7) is used. By the inequality (3.7), if $s \leq n$, relation (3.60) holds, and $H_s$ is a contraction map. Therefore, there exists $u_s \in \Sigma_s$, such that

$$u_s = H_s u_s.$$ 

Similar argument as for theorem (3.5) shows that:

$$h \sigma \in C^s, \quad \nabla^s h \sigma = u_s.$$ 

This completes the proof of the theorem. ♣

### 3.6 Completion of the Proof of the Proposition

We have proved that there exists a $C^n$ codimension 1 overflowing invariant manifold

$$\text{graph } \sigma^*.$$ 

We need to check the following two conditions:

1. $\tilde{S}_\omega \subset \text{graph } \sigma^*$,
2. As $\delta \to 0$, $\text{graph } \sigma^*$ coincides with $M$.

Let $\sigma_0 = 0$ be the zero section, then

$$\sigma^* = \lim_{m \to \infty} G^m \sigma_0.$$ 

Since $\tilde{S}_\omega \subset M$, then

$$\tilde{S}_\omega \subset \text{graph } \sigma_0.$$ 

Notice that, $\tilde{S}_\omega$ is an equilibrium manifold under both $F^t_0$ and $F^t_\delta$ flows. Then from the definition of $G$ (3.2),

$$\tilde{S}_\omega \subset \text{graph } G \sigma_0.$$ 

Continuing such an argument, we know that

$$\tilde{S}_\omega \subset \text{graph } G^m \sigma_0, \quad \forall m.$$ 

Thus,

$$\tilde{S}_\omega \subset \text{graph } \sigma^*.$$
Next, we check the other condition. We know that
\[ \sigma^* \in \Sigma_\zeta; \]
moreover, graph \( \sigma^* \) is \( C^n \) smooth. Then,
\[ \left\| \nabla h\sigma^*(Q) \right\| \leq \zeta, \quad \forall Q \in M. \]
Since \( \eta \) depends on \( \delta \) by (3.10), then for any \( \zeta \) when \( \delta \) is sufficiently small, the relation (3.27)
\[ \eta < \frac{\zeta}{8\Lambda}, \]
is valid. The proof of proposition (3.2;3.3) can go through. Thus, as \( \delta \to 0 \),
\[ \sup_{Q \in M} \left\| \nabla h\sigma^*(Q) \right\| \to 0. \]
Together with condition 1, this shows that as \( \delta \to 0 \), graph \( \sigma^* \) coincides with \( M \).
Thus, we have completed the proof of the existence of the overflowing invariant center-unstable manifold \( W^{cu}_{\delta_1, \delta} \equiv \operatorname{graph} \sigma^* \) in \( \tilde{D}_k \). By changing \( t \) to \(-t\) and working inside the subset of \( \tilde{D}_k \):
\[ \left\{ (r, \theta, \vec{f}, \epsilon_1) \left| (r^2 + \| \vec{f}' \|_{\tilde{H}_k}^2)^{1/2} < a_2 \right. \right\}, \]
we have the existence of the inflowing invariant center-stable manifold \( W^{cs}_{\delta_1, \delta} \). Then,
\[ W_{\delta_1, \delta} \equiv W^{cs}_{\delta_1, \delta} \cap W^{cu}_{\delta_1, \delta} \]
is the center manifold.
4 Fibrations of the Persistent Invariant Manifolds

We continue to work in $\tilde{D}_k$ where the bumped perturbed flow (2.37) is defined. From proposition (3.1), we know the existence of the $C^n$ codimension 1 center-unstable manifold $W_{\delta_1,\delta}^\text{cu}$, the $C^n$ codimension 1 center-stable manifold $W_{\delta_1,\delta}^\text{cs}$, and the $C^n$ codimension 2 center manifold $W_{\delta_1,\delta}^\text{cs}$ under the bumped perturbed flow (2.37). More specifically, $W_{\delta_1,\delta}^\text{cu}$ exists in $\tilde{D}_k^{(1)}$; moreover, it is overflowing invariant, $W_{\delta_1,\delta}^\text{cs}$ exists in $\tilde{D}_k^{(2)}$; moreover, it is inflowing invariant, then $W_{\delta_1,\delta}^\text{cu} \cap W_{\delta_1,\delta}^\text{cs}$ exists in $\tilde{D}_k^{(2)}$, and it is inflowing invariant. Since the fibration theorem is concerned with the fiber representations of $W_{\delta_1,\delta}^\text{cu}$ and $W_{\delta_1,\delta}^\text{cs}$ with respect to $W_{\delta_1,\delta}$ as the base, we have to work in a region where $W_{\delta_1,\delta}$ exists. Therefore, we can only work inside $\tilde{D}_k^{(2)}$. We know that $W_{\delta_1,\delta}$ is inflowing invariant in $\tilde{D}_k^{(2)}$. Next, we will prove the lemma:

**Lemma 4.1** For any fixed $\delta_1$, if $\delta$ is sufficiently small, then $\tilde{D}_k^{(3)} \cap W_{\delta_1,\delta}^\text{cu}$ is overflowing invariant; consequently, $\tilde{D}_k^{(3)} \cap W_{\delta_1,\delta}$ is overflowing invariant.

**Proof:** For any fixed $\delta_1$, we know that $\tilde{D}_k^{(3)} \cap W_0^\text{cu}$ is overflowing invariant. Applying the same argument on the existence of $W_{\delta_1,\delta}^\text{cu}$ in $\tilde{D}_k$, to $\tilde{D}_k^{(3)} \cap W_0^\text{cu}$, we have the existence of an overflowing invariant manifold $\tilde{W}_{\delta_1,\delta}^\text{cu}$ when $\delta$ is sufficiently small. Moreover, $\tilde{W}_{\delta_1,\delta}^\text{cu}$ is the graph of the fixed point of the same graph transform as for $W_{\delta_1,\delta}^\text{cu}$ in $\tilde{D}_k$. By uniqueness of the fixed point, $\tilde{W}_{\delta_1,\delta}^\text{cu} \subset W_{\delta_1,\delta}^\text{cu}$, in fact,

$$\tilde{W}_{\delta_1,\delta}^\text{cu} \equiv \tilde{D}_k^{(3)} \cap W_{\delta_1,\delta}^\text{cu}.$$  

Thus, $\tilde{D}_k^{(3)} \cap W_{\delta_1,\delta}^\text{cu}$ is overflowing invariant. ♣

**Remark 4.1** The fibration of $W_{\delta_1,\delta}^\text{cu}$ will be proved in $\tilde{D}_k^{(3)}$. By reversing the time ($t \rightarrow -t$), the fibration of $W_{\delta_1,\delta}^\text{cs}$ in $\tilde{D}_k^{(2)}$ follows similarly. The fibration theorem is stated in $\tilde{D}_k^{(5)}$.

4.1 Statement of the Fiber Theorem and the Strategy of Proof

Now we state the unstable fiber theorem for $W_{\delta_1,\delta}^\text{cu}$ in $\tilde{D}_k^{(3)}$ to be proved in the subsequent sections.
Theorem 4.1 In a neighborhood of $W_{\delta_1,\delta}$ in $W_{\delta_1,\delta}^{cu}$, there exists a family of $C^n$ smooth curves $\{f^E(Q) : Q \in W_{\delta_1,\delta}\}$, called unstable fibers:

- $W_{\delta_1,\delta}^{cu}$ can be represented as a union of these fibers,
  $$W_{\delta_1,\delta}^{cu} = \bigcup_{Q \in W_{\delta_1,\delta}} f^E(Q).$$

- $f^E(Q)$ depends $C^{n-1}$ smoothly on $Q$, in the sense that $W$ defined by
  $$W = \{(Q_1,Q) \mid Q_1 \in f^E(Q), Q \in W_{\delta_1,\delta}\}$$
  is a $C^{n-1}$ smooth submanifold of $\tilde{S}_k \times \tilde{S}_k$.

- Each fiber $f^E(Q)$ intersects $W_{\delta_1,\delta}$ transversally at $Q$, two fibers $f^E(Q_1)$ and $f^E(Q_2)$ are either disjoint or identical.

- The family of unstable fibers $\{f^E(Q) : Q \in W_{\delta_1,\delta}\}$ is negatively invariant, in the sense that the family of fibers commutes with the evolution operator $F^t_\delta$ in the following way:
  $$F^t_\delta(f^E(Q)) \subset f^E(F^t_\delta(Q)),$$
  for all $Q \in W_{\delta_1,\delta}$ and all $t \leq 0$.

- There are positive constants $\kappa$ (e.g. $= 2\pi \sqrt{\omega^2 - \pi^2}$) and $C$ such that if $Q \in W_{\delta_1,\delta}$ and $Q_1 \in f^E(Q)$, then
  $$\|F^t_\delta(Q_1) - F^t_\delta(Q)\| \leq C e^{\kappa t}\|Q_1 - Q\|,$$
  for all $t \leq 0$.

- For any $Q, P \in W_{\delta_1,\delta}$, $Q \neq P$, any $Q_1 \in f^E(Q)$ and any $P_1 \in f^E(P)$; if
  $$\|F^t_\delta(P_1) - F^t_\delta(Q)\| \to 0, \text{ as } t \to -\infty;$$
  then
  $$\left(\frac{\|F^t_\delta(Q_1) - F^t_\delta(Q)\|}{\|F^t_\delta(P_1) - F^t_\delta(Q)\|}\right)^{1/2} e^{\kappa t} \to 0, \text{ as } t \to -\infty.$$
Rewrite the above theorem in original coordinates for (2.1), we have the fiber theorem (2.4) for the perturbed NLS equation (2.1).

The strategy of proof in subsequent subsections can be summarized as follows:

1. Establish certain rate lemmas needed for graph transform argument later on.

2. Prove the existence of a unique subbundle \( E \) of \( TW_{\delta_1, \delta}^{cu}\), which is overflowing invariant under \( TF_\delta \). \( E \) is a linear graph over the normal subbundle of \( W_{\delta_1, \delta} \) in \( TW_{\delta_1, \delta}^{cu}\).

3. Prove that \( E \) is a \( C^{n-1} \) smooth subbundle starting from the graph transform equation that \( E \) satisfies.

4. Prove the existence of unstable fibers as a graph over the subbundle \( E \).

5. Prove that for any fixed base point \( Q \), the fiber \( f^E(Q) \) is \( C^n \) smooth as a submanifold, starting from the graph transform equation that \( f^E(Q) \) satisfies.

6. Prove the metric characterization inequalities for fibers, starting from rate lemmas.

7. Prove that the fibers \( \{ f^E(Q) : Q \in W_{\delta_1, \delta} \} \) is \( C^{n-1} \) smooth in \( Q \) based upon the graph transform equation that \( f^E(Q) \) satisfies and the fact that \( E \) is a \( C^{n-1} \) smooth subbundle.

4.2 Rate Lemmas

For simplicity of notation, we denote

\[ W_{\delta_1, \delta}^{cu}, \quad W_{\delta_1, \delta}^{cs}, \quad \text{and} \quad W_{\delta_1, \delta}, \]

respectively by

\[ M^u, \quad M^s, \quad \text{and} \quad M. \]

Let \( TM^u|_M \) and \( TM^s|_M \) be respectively the tangent bundles of \( M^u \) and \( M^s \) restricted to \( M \). Let \( J^u \) and \( J^s \) be respectively the normal complements of \( TM \) in \( TM^u|_M \) and \( TM^s|_M \). Then, the tangent bundle of the whole phase space \( \tilde{S}_k \) restricted to \( M \), \( T\tilde{S}_k|_M \) splits as follows:

\[ T\tilde{S}_k|_M = J^u \oplus TM \oplus J^s. \]
Denote by \( \pi^u, \pi^M, \) and \( \pi^s \) the projections on \( J^u, TM, \) and \( J^s \) respectively. By definition of Frechet derivative, 
\[
\forall Q \in \tilde{S}_k, \nabla F_t^\delta(Q) \in L(T_Q\tilde{S}_k, T_{F_t^\delta(Q)}\tilde{S}_k).
\]
Let \( TF_t^\delta \) be the map induced by \( F_t^\delta \) on the tangent bundle of \( \tilde{S}_k, \)
\[
TF_t^\delta : T\tilde{S}_k \rightarrow T\tilde{S}_k, \quad \forall (Q, w) \in T\tilde{S}_k; \quad i.e., Q \in \tilde{S}_k, w \in T_Q\tilde{S}_k,
\]
\[
TF_t^\delta(Q, w) = \left(F_t^\delta(Q), \nabla F_t^\delta(Q) \cdot w\right).
\]
Working inside \( \tilde{D}_k^{(3)} \), we know that both \( TM^u|_M \) and \( TM \) are overflowing invariant under \( TF_t^\delta \), in the sense that
\[
TM^u|_M \subset TF_t^\delta(TM^u|_M),
\]
\[
TM \subset TF_t^\delta(TM).
\]
Recall the unperturbed center-unstable, center-stable, and center manifolds denoted respectively by:
\[
M^u_0, M^s_0, \) and \( M_0
\]
under the bumped unperturbed flow (2.36). We know that \( M^s \equiv \text{graph } \sigma(M^s_0); \)
that is, \( M^s \) has the representation:
\[
\hat{Q} = Q + h\sigma(Q),
\]
where \( Q \in M^s_0, \hat{Q} \in M^s, h \) is the projection on \( \hat{e}_1^+ \)-direction. Moreover,
\[
||h\sigma(Q)|| \leq \kappa, \quad \forall Q \in M^s_0,
\]
\[
||h\sigma(Q_1) - h\sigma(Q_2)|| \leq \zeta||Q_1 - Q_2||, \quad \forall Q_1, Q_2 \in M^s_0.
\]
Define the transversal bundle \( J_0 \) as follows:
\[
\forall Q \in M^s_0, J_0(Q) \equiv \text{span } \{\hat{e}_1^+\}.
\]
Next, we will show the following lemma:
Lemma 4.2 There exists a constant $\mu$; moreover, for any $t > 0$, there exist two constants $C_{s_j}(t)$ ($j = 1, 2$), for any $t < 0$, there exist two constants $C_{u_j}(t)$ ($j = 1, 2$), such that

1. For $t > 0$,
\[
\sup_{Q \in M} \|\nabla F^{s_j}(Q)\|_{TM} \leq \mu \sup_{Q' \in M^0_s} \|\nabla F^{s_j}(Q')\|_{TM^0_s} + \delta C_{s_1}(t) + \zeta C_{s_2}(t). \tag{4.1}
\]
\[
\sup_{Q \in M} \|\nabla F^{s_j}(Q)\|_{J} \leq \mu \sup_{Q' \in M^0_s} \|\nabla F^{s_j}(Q')\|_{TM^0_s} + \delta C_{s_1}(t) + \zeta C_{s_2}(t). \tag{4.2}
\]

2. For $t < 0$,
\[
\sup_{Q \in M} \|\nabla F^{s_j}(Q)\|_{TM} \leq \mu \sup_{Q' \in M^0_u} \|\nabla F^{s_j}(Q')\|_{TM^0_u} + \delta C_{u_1}(t) + \zeta C_{u_2}(t). \tag{4.3}
\]
\[
\sup_{Q \in M} \|\nabla F^{s_j}(Q)\|_{J} \leq \mu \sup_{Q' \in M^0_u} \|\nabla F^{s_j}(Q')\|_{J_0} + \delta C_{u_1}(t) + \zeta C_{u_2}(t). \tag{4.4}
\]

Where as $\delta \to 0$, $\zeta \to 0$.

Proof: First we prove inequalities (4.1:4.2). \(\forall Q \in M\), let \(Q_1 \in M^s\) be any point in a neighborhood of \(Q\), there exist \(Q', Q'_1 \in M^0_s\), such that
\[
Q = Q' + h\sigma(Q'), \tag{4.5}
Q_1 = Q'_1 + h\sigma(Q'_1).
\]
Then
\[
Q_1 - Q = Q'_1 - Q' + h\sigma(Q'_1) - h\sigma(Q').
\]
By this relation, we have the decomposition:
\[
\forall v \in (TM \oplus J^s)(Q); \quad v = v_0 + w_0, \tag{4.6}
\]
where
\[
v_0 \in TM^0_s(Q'), \quad w_0 \in J_0(Q');
\]
moreover,
\[
\|w_0\| \leq \zeta\|v_0\|. \tag{4.7}
\]
By theorem (2.10),

\[ F^t_\delta = F^t_0 + f^t, \]

where for any \( t \in (-\infty, \infty) \), and sufficiently small \( \delta \), there exists a constant \( C_2(t) \), such that

\[
\sup_{\hat{Q} \in \tilde{D}_k} \| f^t(\hat{Q}) \| \leq C_2(t)\delta, \\
\sup_{\hat{Q} \in \tilde{D}_k} \| \nabla f^t(\hat{Q}) \| \leq C_2(t)\delta. \tag{4.8}
\]

Thus,

\[ \nabla F^t_\delta(Q) \cdot v = \nabla F^t_0(Q) \cdot v + \nabla f^t(Q) \cdot v; \]

moreover, by (4.5)

\[ Q = Q' + h\sigma(Q'). \]

We know that (2.36),

\[ \nabla F^t_0(Q' + h\sigma(Q')) \cdot v = \nabla F^t_0(Q') \cdot v. \]

Therefore, by (4.6)

\[ \nabla F^t_0(Q) \cdot v = \nabla F^t_0(Q') \cdot v_0 + \nabla F^t_0(Q') \cdot w_0. \]

Finally,

\[ \nabla F^t_0(Q) \cdot v = \nabla F^t_0(Q') \cdot v_0 + \nabla F^t_0(Q') \cdot w_0 + \nabla f^t(Q) \cdot v. \]

By relations (4.7) and (4.8); i.e.,

\[ \|w_0\| \leq \zeta\|v\|, \quad \|\nabla f^t(Q)\| \leq C_2(t)\delta, \]

we see that inequalities (4.1;4.2) are valid. Notice that

\[ M = M^s \cap M^u, \]

if we apply the argument to \( M^u \) instead of \( M^s \), we get inequality (4.3). Next, we prove inequality (4.4). Let \( \hat{J}^u \) be the normal complement of \( TM \oplus J^s \) in \( T\bar{S}_k|M^s \):

\[ T\bar{S}_k|M = \hat{J}^u \oplus TM \oplus J^s, \tag{4.9} \]

with \( \hat{\pi}^u \) denoting projection on \( \hat{J}^u \). Let \( \hat{J}_0 \) be the normal complement of \( TM_0^s \) in \( T\tilde{S}_k|M_0^s \):

\[ T\tilde{S}_k|M_0^s = \hat{J}_0 \oplus TM_0^s \tag{4.10}. \]
Denote by $\pi_0$ and $\hat{\pi}_0$, the projections on $J_0$ and $\hat{J}_0$, respectively. Then,

$$\pi_0 \cdot \hat{\pi}_0 = \pi_0, \quad \hat{\pi}_0 \cdot \pi_0 = \hat{\pi}_0. \quad (4.11)$$

Notice that (4.10) is a trivial bundle splitting, $\forall Q \in M$, and $\forall v \in (TM \oplus J^s)(Q)$, $v$ can be decomposed as:

$$v = v_1 + w_1, \quad (4.12)$$

where

$$v_1 \in TM_0^s(Q'), \quad w_1 \equiv \hat{\pi}_0 v \in \hat{J}_0(Q').$$

Moreover, we have:

**Lemma 4.3** $\|w_1\| \leq 2\zeta\|v_1\|$.  

Proof: By (4.6),

$$\|v_0\| \leq \|v\| \leq (1 + \zeta)\|v_0\|. \quad (4.13)$$

By (4.11),

$$\|w_1\| = \|\hat{\pi}_0 v\| = \|\hat{\pi}_0 \cdot \pi_0 v\| \leq \|\pi_0 v\| = \|w_0\| \leq \zeta\|v_0\|. \quad (4.14)$$

By (4.12;4.13;4.14),

$$\|v_0\| \leq \|v\| \leq \|v_1\| + \|w_1\| \leq \|v_1\| + \zeta\|v_0\|,$$

then

$$\|v_0\| \leq \frac{1}{1 - 2\zeta}\|v_1\|. \quad (4.15)$$

By (4.14;4.15),

$$\|w_1\| \leq \frac{\zeta}{1 - 2\zeta}\|v_1\|,$$

then, if $\zeta$ is sufficiently small,

$$\|w_1\| \leq 2\zeta\|v_1\|.$$  

This completes the proof of lemma (4.3). ♦  

$\forall Q \in M$, and $\forall w \in J^v(Q)$, $w$ can be decomposed as:

$$w = v_2 + w_2, \quad (4.16)$$

where

$$v_2 \in TM_0^s(Q'), \quad w_2 \in \hat{J}_0(Q').$$

Moreover, we have:
Lemma 4.4 \[ \|v_2\| \leq 2\zeta \|w_2\|. \]

Proof: There exists \( \hat{v} \in (TM \oplus J^e)(Q) \), such that
\[ \hat{v} = v_2 + \hat{w}, \]
where
\[ \hat{w} \in \hat{J}_0(Q'), \quad \|\hat{w}\| \leq 2\zeta \|v_2\|. \]
The relation
\[ \langle \hat{v}, w \rangle = 0 \]
implies that
\[ \|v_2\|^2 = -\langle w_2, \hat{w} \rangle \leq \|\hat{w}\| \|w_2\| \leq 2\zeta \|v_2\| \|w_2\|. \]
Thus,
\[ \|v_2\| \leq 2\zeta \|w_2\|. \]
This completes the proof of the lemma (4.4).

Now we can show the fact,

Lemma 4.5
\[
\sup_{Q \in M} \|\bar{\pi}^u \nabla F^t_\delta(Q)\|_{J_u} \leq \mu \sup_{Q' \in M_0^u} \|\bar{\pi}_0 \nabla F^t_0(Q')\|_{J_0} + \delta C_{u1}(t) + \zeta C_{u2}(t).
\]

Proof: Let \( Q \in M, \ w \in \hat{J}^u(Q) \), then for \( t < 0 \),
\[ \nabla F^t_\delta(Q) \bullet w = \nabla F^t_0(Q) \bullet w + \nabla f^t(Q) \bullet w. \]
By (4.5) and property of \( F^t_0 \) (2.36),
\[ \nabla F^t_0(Q) \bullet w = \nabla F^t_0(Q') \bullet w. \]
By (4.16) and lemma (4.4),
\[ \nabla F^t_0(Q') \bullet w = \nabla F^t_0(Q') \bullet w_2 + \nabla F^t_0(Q') \bullet v_2, \]
where
\[ \|v_2\| \leq 2\zeta \|w_2\|. \] (4.17)
Then
\[
\hat{\pi}^u|_{F^t_\delta(Q)} \nabla F^t_0(Q) \cdot w = \hat{\pi}^u|_{F^t_\delta(Q)} \nabla F^t_0(Q') \cdot w_2 \\
+ \hat{\pi}^u|_{F^t_\delta(Q)} \left( \nabla F^t_0(Q') \cdot v_2 + \nabla f^t(Q) \cdot w \right). 
\]
(4.18)

We can decompose \( \nabla F^t_0(Q') \cdot w_2 \) as:
\[
\nabla F^t_0(Q') \cdot w_2 = v_3 + w_3 = v'_3 + w'_3,
\]
where
\[
\begin{align*}
v_3 &\in (TM \oplus J^s(F^t_0(Q))), \\
w_3 &\in \hat{\pi}^u|_{F^t_\delta(Q)} \nabla F^t_0(Q') \cdot w_2 \in \hat{J}^u(F^t_\delta(Q)), \\
v'_3 &\in TM^s_0(Q'), \\
w'_3 &\in \hat{\pi}_0 \nabla F^t_0(Q') \cdot w_2 \in \hat{J}_0(Q').
\end{align*}
\]
(4.19)

Notice that \( TM^s_0 \) and \( \hat{J}_0 \) are trivial bundles, by lemma (4.3),
\[
v_3 = \bar{v}_3 + \bar{w}_3,
\]
where
\[
\|\bar{w}_3\| \leq 2\zeta \|\bar{v}_3\|; 
\]
(4.21)
by lemma (4.4),
\[
w_3 = \hat{v}_3 + \hat{w}_3,
\]
where
\[
\|\hat{v}_3\| \leq 2\zeta \|\hat{w}_3\|. 
\]
(4.22)

Then,
\[
(\bar{v}_3 + \hat{v}_3) + (\bar{w}_3 + \hat{w}_3) = v'_3 + w'_3.
\]

Thus,
\[
w'_3 = \hat{w}_3 + \bar{w}_3, \quad v'_3 = \bar{v}_3 + \hat{v}_3.
\]

Therefore,
\[
w_3 = w'_3 + (\hat{v}_3 - \bar{v}_3). 
\]
(4.23)

By (4.21;4.22;4.23) and (4.19;4.20),
\[
\hat{\pi}^u|_{F^t_\delta(Q)} \nabla F^t_0(Q') \cdot w_2 = \hat{\pi}_0 \nabla F^t_0(Q') \cdot w_2 + \zeta G(t, Q), 
\]
(4.24)

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where \( G \) is a function of \( t \) and \( Q \). By (4.8; 4.17; 4.24), we have

\[
\| \hat{\pi}^u \nabla F_\delta^U(Q) |_{J_u} \| \leq \mu \| \hat{\pi}_0 \nabla F_\delta^U(Q') |_{J_0} \| + \delta C_{u1}(t) + \zeta C_{u2}(t),
\]

where \( \mu, C_{u1}(t) \), and \( C_{u2}(t) \) are sufficiently large. This completes the proof of lemma (4.5).

Since \( J_u \) and \( J_0 \) are both transversal bundles, there exists a constant \( \mu_1 \), such that

\[
\| \pi_u \|_Q \leq \mu_1, \quad \forall Q \in M,
\]

(4.25)

\[
\| \pi_0 \|_{Q'} \leq \mu_1, \quad \forall Q' \in M^s_0.
\]

(4.26)

Then, we have the lemma:

Lemma 4.6

\[
\mu_1^{-1} \| \pi^u \nabla F_\delta^U(Q) |_{J_u} \| \leq \| \hat{\pi}^u \nabla F_\delta^U(Q) |_{J_u} \| \leq \mu_1 \| \pi^u \nabla F_\delta^U(Q) |_{J_u} \|, \quad \forall Q \in M,
\]

(4.27)

\[
\mu_1^{-1} \| \pi_0 \nabla F_\delta^U(Q') |_{J_0} \| \leq \| \hat{\pi}_0 \nabla F_\delta^U(Q') |_{J_0} \| \leq \mu_1 \| \pi_0 \nabla F_\delta^U(Q') |_{J_0} \|, \quad \forall Q' \in M^s_0.
\]

(4.28)

Proof: First we prove inequality (4.27), then the other inequality (4.28) follows similarly. We know that

\[
\pi^u \cdot \hat{\pi}^u = \pi^u, \quad \hat{\pi}^u \cdot \pi^u = \hat{\pi}^u.
\]

(4.29)

For any \( Q \in M \), let

\[
w(0) \in J^u(Q), \quad w(t) = \pi^u \nabla F_\delta^U(Q) \cdot w(0).
\]

(4.30)

\( w(0) \) can be decomposed as:

\[
w(0) = v(0) + \hat{w}(0),
\]

(4.31)

where

\[
v(0) \in (TM + J^s)(Q), \quad \hat{w}(0) = \hat{\pi}^u |_Q w(0) \in \hat{J}^u(Q).
\]

Then, by (4.30; 4.31; 4.29)

\[
w(t) = \pi^u \nabla F_\delta^U(Q) \hat{w}(0) = \pi^u \cdot \hat{\pi}^u \nabla F_\delta^U(Q) \hat{w}(0).
\]

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Let 
\[ \hat{w}(t) \equiv \hat{\pi}^u \nabla F^t(Q) \hat{w}(0), \]
then
\[ w(t) = \pi^u \hat{w}(t). \] (4.32)
By (4.31),
\[ \hat{w}(0) = \hat{\pi}^u w(0). \] (4.33)
By (4.32;4.33),
\[ \| w(t) \| \leq \| \pi^u \| \| \hat{w}(t) \|, \quad \| \hat{w}(0) \| \leq \| \hat{\pi}^u \| \| w(0) \|. \]
By (4.25),
\[ \| w(t) \| \leq \mu_1 \| \hat{w}(t) \|, \quad \| \hat{w}(0) \| \leq \| w(0) \|. \]
Thus,
\[ \frac{\| w(t) \|}{\| w(0) \|} \leq \mu_1 \frac{\| \hat{w}(t) \|}{\| \hat{w}(0) \|}. \] (4.34)
Similarly, let \( \hat{w}_1(0) \in \tilde{J}^u(Q), \ Q \in M, \) and decompose \( \hat{w}_1(0) \) as:
\[ \hat{w}_1(0) = v_1(0) + w_1(0), \]
where
\[ v_1(0) \in (TM + J^s)(Q), \ w_1(0) = \pi^u \hat{w}_1(0) \in J^u(Q). \]
The same argument as above shows that:
\[ \frac{\| \hat{w}_1(t) \|}{\| \hat{w}_1(0) \|} \leq \mu_1 \frac{\| \hat{w}_1(t) \|}{\| \hat{w}_1(0) \|}. \] (4.35)
By (4.34;4.35), inequality (4.27) is valid. Similarly, for inequality (4.28).
This completes the proof of lemma (4.6). ♣

By lemmas (4.5;4.6), inequality (4.4) is valid. This completes the proof of the lemma (4.2). ♣ ♣

4.3 The Existence of an Invariant Subbundle \( E \)
Recall that both \( TM^u|_M \) and \( TM \) are overflowing invariant under \( TF^t_\delta \); moreover,
\[ TM^u|_M = J^u \oplus TM. \]
In general, \( J^u \) is not overflowing invariant under \( TF^t_\delta \). For later construction of fibers, we need to start from a subbundle of \( TM^u|_M \), which is invariant under \( TF^t_\delta \), and complementary to \( TM \). In fact, we have the fact:
Lemma 4.7 (Invariant Subbundle Lemma) There exists a unique subbundle $E$ of $TM^u|_M$, such that

1. $E$ is overflowing invariant under $TF^T_\delta$,
2. $E$ is transversally complementary to $TM$ in $TM^u|_M$, in the sense that, $TM^u|_M = E \oplus TM$;

moreover, if $\pi^E$ denotes the projection on $E$ in this splitting, there exists a constant $\mu_2$, such that

$$\|\pi^E_Q\| \leq \mu_2, \; \forall Q \in M.$$ 

Proof: Choose a large time $T > 0$ (fixed from now on), such that,

$$\mu^2 \sup_{Q \in M_3} \| \nabla F_0^{-T}(Q')|_{J_0} \| \sup_{Q \in M_3} \| \nabla F_0^T(Q')|_{TM_0} \| < 1/4.$$ 

By lemma (4.2), there exists a constant $d^u(T)$, such that, for any $\delta \in [0, d^u(T)]$,

$$\| \pi^u \nabla F^{-T}_\delta(Q)|_{J^u} \| \| \nabla F^T_\delta(F^{-T}_\delta(Q))|_{TM} \| < 1/2, \; \forall Q \in M. \; \; (4.36)$$

Let $\Sigma$ be a complete metric vector space which consists of families of linear maps:

$$\forall \sigma \in \Sigma, \; \sigma = \{\sigma_Q\}_{Q \in M}, \; \sigma_Q : J^u(Q) \mapsto TM(Q),$$

where $\sigma_Q$ is a linear transformation.

$$\forall \sigma^1, \sigma^2 \in \Sigma, \; \sigma^1 + \sigma^2 = \{\sigma^1_Q + \sigma^2_Q\}_{Q \in M},$$

$$\forall \alpha \in R, \; \forall \sigma \in \Sigma, \; \alpha \sigma = \{\alpha \sigma_Q\}_{Q \in M},$$

where $\sigma^1_Q + \sigma^2_Q$ and $\alpha \sigma_Q$ are defined as the usual addition and scalar multiplication of linear transformations.

$$\forall \sigma \in \Sigma, \; \|\sigma\| = \sup_{Q \in M} \|\sigma_Q\|,$$

where $\|\sigma_Q\|$ is defined as the usual norm of a linear transformation. For any $\sigma_1, \sigma_2 \in \Sigma$, the distance between them is defined as:

$$\text{distance } \{\sigma_1, \sigma_2\} \equiv \|\sigma_1 - \sigma_2\|.$$
(It is easily seen that \( \Sigma \) is a complete metric space under this metric.) If \( \sigma \) defines an overflowing invariant subbundle under \( T F^J_\delta \), then \( \sigma \) satisfies the equation:

\[
TF^J_\delta (F^{-T}_\delta (Q), \text{graph } \sigma_{F^{-T}_\delta (Q)}) = (Q, \text{graph } \sigma_Q), \quad \forall Q \in M.
\]

(4.37)

For any \( \xi \in J^n(F^{-T}_\delta (Q)) \), let:

\[
\xi_1 \equiv \pi^n \nabla F^T_\delta (F^{-T}_\delta (Q)) \bullet (\xi + \sigma_{F^{-T}_\delta (Q)}(\xi)),
\]

(4.38)

then Eq. (4.37) is equivalent to:

\[
\sigma_Q(\xi_1) = \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q)) \bullet (\xi + \sigma_{F^{-T}_\delta (Q)}(\xi)).
\]

(4.39)

Since \( TM \) is overflowing invariant under \( T F^J_\delta \), the term:

\[
\pi^n \nabla F^T_\delta (F^{-T}_\delta (Q)) \bullet \sigma_{F^{-T}_\delta (Q)}(\xi)
\]

in (4.38) vanishes. Then (4.38) becomes,

\[
\xi_1 = \pi^n \nabla F^T_\delta (F^{-T}_\delta (Q)) \bullet \xi
\]

(4.40)

\[
= \pi^n \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{J^n} \bullet \xi.
\]

(4.41)

Moreover, \( \pi^n \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{J^n} \) is invertible,

\[
\left[ \pi^n \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{J^n} \right]^{-1} = \pi^n \nabla F^T_\delta (Q) |_{J^n}.
\]

(4.42)

By (4.39;4.41), and dropping \( \xi \),

\[
\sigma_Q \bullet \pi^n \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{J^n} = \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{J^n}
\]

\[
+ \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{TM} \bullet \sigma_{F^{-T}_\delta (Q)}(\xi).
\]

(4.43)

By (4.42;4.43),

\[
\sigma_Q = \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{TM} \bullet \sigma_{F^{-T}_\delta (Q)} \bullet \pi^n \nabla F^T_\delta (Q) |_{J^n}
\]

\[
+ \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q)) |_{J^n} \bullet \pi^n \nabla F^T_\delta (Q) |_{J^n}.
\]

(4.44)

Therefore, searching for the invariant subbundle is equivalent to solving the linear functional equation (4.44). Define the graph transform:

\[
G : \Sigma \mapsto \Sigma,
\]

\[
\forall \sigma \in \Sigma, \quad G\sigma = \{(G\sigma)_Q \} \forall Q \in M.
\]

(4.45)
Next, we show that $G$ is a contraction map.

$$(G\sigma_1 - G\sigma_2)_Q = \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q))}|_{TM} \bullet (\sigma_1 - \sigma_2)_{F^{-T}_\delta (Q)} \bullet \pi^u \nabla F^{-T}_\delta (Q)|_{J^u}.$$ 

Then,

$$\| (G\sigma_1 - G\sigma_2)_Q \| \leq \| \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q))}|_{TM} \| \| \pi^u \nabla F^{-T}_\delta (Q)|_{J^u} \| \| (\sigma_1 - \sigma_2)_{F^{-T}_\delta (Q)} \|.$$ 

By (4.36),

$$\| (G\sigma_1 - G\sigma_2)_Q \| < \frac{1}{2} \| \sigma_1 - \sigma_2 \|, \quad \forall Q \in M.$$ 

Thus,

$$\| G\sigma_1 - G\sigma_2 \| \leq \frac{1}{2} \| \sigma_1 - \sigma_2 \|.$$ 

That is, $G$ is a contraction map. Since $\Sigma$ is a complete metric space, there exists a unique fixed point $\sigma^* \in \Sigma$ of $G$. Denote by $E$ the subbundle of $TM^u|_M$ defined by $\sigma^*$:

$$E(Q) \equiv \text{graph} \sigma^*_Q, \quad \forall Q \in M.$$ 

By (4.44;4.37), $E$ is overflowing invariant under $TF^T_\delta$,

$$E \subset TF^T_\delta (E).$$ 

(4.46)

For any $t > 0$, define

$$E_t \equiv TF^t_\delta (E).$$ 

Applying $TF^t_\delta$ on both sides of (4.46), we have

$$E_t \subset TF^t_\delta \bullet TF^T_\delta (E) = TF^T_\delta \bullet TF^t_\delta (E) = TF^T_\delta (E_t),$$ 

then $E_t$ corresponds to a fixed point of $G$. By uniqueness of the fixed point of $G$,

$$E_t = E; \quad \text{i.e., } E \subset TF^t_\delta (E).$$ 

Thus, $E$ is overflowing invariant under $TF^t_\delta$. This completes the proof of part 1 of the lemma. Next, we prove the other part. Since $\sigma^*$ solves (4.44),

$$\sup_{Q \in M} \| \sigma^*_Q \| \leq \frac{1}{2} \sup_{Q \in M} \| \sigma^*_Q \|$$

$$+ \sup_{Q \in M} \| \pi^M \nabla F^T_\delta (F^{-T}_\delta (Q))}|_{J^u} \| \sup_{Q \in M} \| \pi^u F^T_\delta (Q)|_{J^u} \|.$$ 

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Since both
\[ \sup_{Q \in M} \| \pi^M \nabla F^T_\delta (F_{\delta}^{-T}(Q)) \|_{J^u} \quad \text{and} \quad \sup_{Q \in M} \| \pi^u F^T_\delta (Q) \|_{J^u} \]
are bounded, there exists a constant \( \mu_2 \), such that
\[ \sup_{Q \in M} \| \sigma^*_Q \| \leq \mu_2. \quad (4.47) \]

This implies that (noticing that \( J^u \) is a normal complement of \( TM \) in \( TM^u|_M \)):
\[ \| \pi^E \| \leq \mu_2, \quad \forall Q \in M. \]

This completes the proof of the lemma. ♦

By lemma (4.7), we have a new splitting for \( T\tilde{S}_k|_M \):
\[ T\tilde{S}_k|_M = E \oplus TM \oplus J^s, \]
with the property that \( E \) is also an overflowing invariant subbundle under \( TF^\delta_\delta \). Denote respectively by
\[ \pi^E, \quad \pi^M, \quad \text{and} \quad \pi^s \]
the projections on
\[ E, \quad TM, \quad \text{and} \quad J^s, \]
in the above splitting. Next, we will show that lemma (4.2) is valid with \( J^u \) replaced by \( E \).

**Lemma 4.8** There exists a constant \( \mu_0 \); moreover, for any \( t > 0 \), there exist two constants \( C^+_j(t) \) (\( j = 1, 2 \)), for any \( t < 0 \), there exist two constants \( C^-_j(t) \) (\( j = 1, 2 \)), such that

1. For \( t > 0 \),
\[ \sup_{Q \in M} \| \nabla F^T_\delta(Q) \|_{TM} \leq \mu_0 \sup_{Q' \in M^0} \| \nabla F^T_0(Q') \|_{TM^0} \]
\[ + \delta C^+_1(t) + \zeta C^+_2(t). \quad (4.48) \]
\[ \sup_{Q \in M} \| \nabla F^T_\delta(Q) \|_{J^s} \leq \mu_0 \sup_{Q' \in M^0} \| \nabla F^T_0(Q') \|_{TM^0} \]
\[ + \delta C^-_1(t) + \zeta C^-_2(t). \quad (4.49) \]
2. For \( t < 0 \),

\[
\sup_{Q \in M} \| \nabla F_\delta(Q) \|_M \leq \mu_0 \sup_{Q' \in M_0} \| \nabla F_0(Q') \|_{TM_0^u} + \delta C_1(t) + \zeta C_2(t). \tag{4.50}
\]

\[
\sup_{Q \in M} \| \pi^E \nabla F_\delta(Q) \|_E \leq \mu_0 \sup_{Q' \in M_0} \| \nabla F_0(Q') \|_{J_0} + \delta C_1(t) + \zeta C_2(t). \tag{4.51}
\]

Where as \( \delta \to 0, \zeta \to 0 \).

Proof: Inequalities (4.48;4.49;4.50) follow directly from lemma (4.2). We need only to prove inequality (4.51). In fact, inequality (4.51) follows immediately from inequality (4.4) and the following inequality that we are going to show:

\[
\| \pi^E \nabla F_\delta(Q) \|_E \leq \mu_2 \| \pi^u \nabla F_\delta(Q) \|_{J^u}, \tag{4.52}
\]

where \( \mu_2 \) is given in lemma (4.7). The proof of this inequality is similar to the proof of lemma (4.6). We know that

\[
\pi^E \bullet \pi^u = \pi^E, \quad \pi^u \bullet \pi^E = \pi^u. \tag{4.53}
\]

For any \( Q \in M \), let \( w(0) \in E(Q) \),

\[
w(t) = \pi^E \nabla F_\delta(Q) \bullet w(0) = \nabla F_\delta(Q) \bullet w(0). \tag{4.54}
\]

\( w(0) \) can be decomposed as:

\[
w(0) = v(0) + \hat{w}(0), \tag{4.55}
\]

where

\[
v(0) \in TM(Q), \quad \hat{w}(0) \in J^u(Q).
\]

Then, by (4.54;4.55;4.53)

\[
w(t) = \pi^E \nabla F_\delta(Q) \hat{w}(0) = \pi^E \bullet \pi^u \nabla F_\delta(Q) \hat{w}(0).
\]

Let

\[
\hat{w}(t) \equiv \pi^u \nabla F_\delta(Q) \hat{w}(0),
\]

then

\[
w(t) = \pi^E \hat{w}(t). \tag{4.56}
\]
By (4.55),
\[ \hat{w}(0) = \pi^u w(0). \] (4.57)

By (4.56;4.57)
\[ \|w(t)\| \leq \|\pi^E\| \|\hat{w}(t)\|, \|\hat{w}(0)\| \leq \|\pi^u\| \|w(0)\|. \]

Inside the bundle $TM^u|_M$,
\[ \|\pi^u\| = 1, \|\pi^E\| \leq \mu_2; \]
in which the second inequality is given in lemma (4.7). Thus,
\[ \frac{\|w(t)\|}{\|w(0)\|} \leq \mu_2 \frac{\|\hat{w}(t)\|}{\|\hat{w}(0)\|}, \]
which implies that inequality (4.52) is valid. The proof of the lemma is completed.

\[ \square \]

### 4.4 Smoothness of the Invariant Subbundle $E$

In this subsection, we will prove that the invariant bundle $E$ is a $C^{n-1}$ bundle. The proof is similar to the proof of theorem (3.5) on the smoothness of the invariant manifolds, except that we need to use local coordinates.

Since $M$, $M^u$ and $M^s$ are $C^n$ manifolds, and $J^u$ and $J^s$ are the normal complements of $TM$ in $TM^u|_M$ and $TM^s|_M$ respectively, $J^u$ and $J^s$ are $C^{n-1}$ bundles. Making use of $J^u$ and $J^s$, we can define a $C^{n-1}$ local coordinate $(x, y, z)$ in the neighborhood of $M$; where $x \in V_k \subset \tilde{S}_k, \|x\| < d$ (for some constant $d$), $V_k$ is a codimension 2 subspace of $\tilde{S}_k$; $y \in R^1, z \in R^1, |y| < \eta_0, |z| < \eta_0$ (for some small constant \(\eta_0\)). $x$, $y$ and $z$ parametrize the codimension 2 submanifold $M$, the one-dimensional fiber of $J^u$ and the 1-dimensional fiber of $J^s$, respectively.

For example, we can take $x \in W_0$, $y \equiv \alpha^+_1$ and $z \equiv \alpha^-_1$; where $W_0$, $\alpha^+_1$ and $\alpha^-_1$ are defined in definition (3). In this frame, $M$ has the representation:

\[ \begin{cases} y = F(x, y), \\ z = G(x, y), \end{cases} \]

where $F$ and $G$ are $C^n$ smooth; moreover, their Lipschitz norms are small. Applying the implicit function theorem [24] to

\[ \begin{cases} y = F(x, G(x, y)), \\ z = G(x, F(x, y)), \end{cases} \]
we have the $C^n$ functions
\[
\begin{aligned}
y &= H_1(x), \\
z &= H_2(x).
\end{aligned}
\]
Then in the coordinate $(x, y, z)$; $M$ has the coordinate representation:
\[
(x, H_1(x), H_2(x))
\]
which defines the $C^n$ diffeomorphism $\tau$ from $x \in W_0$ to $Q \in M$. (There exists a constant $\nu_0$, such that $\|\nabla \tau\| < \nu_0$, $\|\nabla \tau^{-1}\| < \nu_0$. Without loss of generality, we can set $\nu_0$ equal to 1.) Then $TM$ is defined in terms of $x$.
\[
(TM^n|_M)
\]
is defined through
\[
(dx, dy, \nabla G(x, H_1(x)) \cdot (dx, dy)).
\]
$E(Q)$ is the normal complement of $T_QM$ in $T_QM^n|_M$. Thus $(x, y, z)$ is a $C^{n-1}$ local coordinate.

Notice that, for any $Q \in M$,
\[
E(Q) \equiv \text{graph } \sigma^*_Q = \text{image } \{I_Q + \sigma^*_Q\},
\]
where $I_Q$ is the identity map on $J^u(Q)$. Since $J^u(Q)$ is $C^{n-1}$ in $Q$, in order to prove that $E(Q)$ is $C^{n-1}$ in $Q$, we need to prove $\sigma^*_Q$ is $C^{n-1}$ in $Q$ in the local coordinate sense. In terms of the local coordinate $(x, y, z)$, $\sigma^*$ can be represented as follows: For any $x$ (corresponding to a point $Q \in M$), $\sigma^*_x$ is a linear map
\[
\sigma^*_x : R^1 \rightarrow V_k;
\]
\[
\forall y \in R^1, \|y\| < \eta_0; \; \sigma^*_x \cdot y \in V_k.
\]
In the local coordinate, $F^T_\delta$ has the representation:
\[
\tilde{x} = f(x, y, z); \; \tilde{y} = g(x, y, z); \; \tilde{z} = h(x, y, z);
\]
where $g(x, 0, 0) = h(x, 0, 0) = 0$. Then $\sigma^*_x$ satisfies the following linear equation (which is (4.44) rewritten in the local coordinate):
\[
\begin{aligned}
\sigma^*_{f(x,0,0)} &= \nabla_x f(x, 0, 0) \cdot \sigma^*_x \cdot [\nabla_y g(x, 0, 0)]^{-1} \\
&+ \nabla_y f(x, 0, 0) \cdot [\nabla_y g(x, 0, 0)]^{-1};
\end{aligned}
\]
in which,
\[
\nabla_x f(x, 0, 0), \; [\nabla_y g(x, 0, 0)]^{-1}, \; \text{and} \; \nabla_y f(x, 0, 0)
\]
are the respective, local coordinate representations of
\[
\nabla F^T_\delta(F^{-T}_\delta(Q))|_{TM}, \; \pi^u \nabla F^{-T}_\delta(Q)|_{J^u} \text{ and } \pi^M \nabla F^T_\delta(F^{-T}_\delta(Q))|_{J^u}.
\]
Definition 9 Define the Lipschitz norm of $\sigma^*_x$ in $x$ as follows:

$$Lip_x\{\sigma^*\} \equiv \sup_C \left\{ \left\| \sigma^*_{x_1} - \sigma^*_{x_2} \right\| \right\},$$

where $C$ stands for

$$C \equiv \{ x_1, x_2 \in V_k ; \|x_1\| < d, \|x_2\| < d \}.$$

Before we prove that $\sigma^*_x$ is $C^1$ in $x$, we first want to show that $\sigma^*_x$ is Lipschitz in $x$.

Lemma 4.9 There exists a constant $\chi$, such that

$$Lip_x\{\sigma^*\} \leq \chi.$$

Proof: By (4.58),

$$\sigma^*_{x_1} - \sigma^*_{x_2} = \nabla_x f(x'_1, 0, 0) \cdot \sigma^*_{x_1} \cdot [\nabla_y g(x'_1, 0, 0)]^{-1}$$

$$- \nabla_x f(x'_2, 0, 0) \cdot \sigma^*_{x_2} \cdot [\nabla_y g(x'_2, 0, 0)]^{-1}$$

$$+ \nabla_y f(x'_1, 0, 0) \cdot [\nabla_y g(x'_1, 0, 0)]^{-1}$$

$$- \nabla_y f(x'_2, 0, 0) \cdot [\nabla_y g(x'_2, 0, 0)]^{-1},$$

where $x_i = f(x'_i, 0, 0); \ i = 1, 2$. Then,

$$\left\| \sigma^*_{x_1} - \sigma^*_{x_2} \right\| \leq \left\| \nabla_x f(x'_1, 0, 0) - \nabla_x f(x'_2, 0, 0) \right\|$$

$$\left\| \sigma^*_{x_1} \right\| \left\| [\nabla_y g(x'_1, 0, 0)]^{-1} \right\|$$

$$+ \left\| \nabla_x f(x'_2, 0, 0) \right\| \left\| \sigma^*_{x_2} \right\| \left\| [\nabla_y g(x'_2, 0, 0)]^{-1} \right\|$$

$$+ \left\| \nabla_y f(x'_1, 0, 0) - \nabla_y f(x'_2, 0, 0) \right\|$$

$$\left\| [\nabla_y g(x'_1, 0, 0)]^{-1} \right\| + \left\| \nabla_y f(x'_2, 0, 0) \right\|$$

$$\left\| [\nabla_y g(x'_2, 0, 0)]^{-1} \right\|.$$ (4.61)

By lemma (3.1) and theorem (2.11),

$$\left\| \nabla_x f(x'_1, 0, 0) - \nabla_x f(x'_2, 0, 0) \right\| \leq \Lambda_s \left\| x'_1 - x'_2 \right\|,$$ (4.62)
\[
\| \nabla g(x_1', 0, 0)^{-1} - \nabla g(x_2', 0, 0)^{-1} \| \leq \Lambda_* \| x_1 - x_2 \|, \quad (4.63)
\]
\[
\| \nabla f(x_1', 0, 0) - \nabla f(x_2', 0, 0) \| \leq \Lambda_* \| x_1' - x_2' \|. \quad (4.64)
\]

Then by (4.61;4.62;4.63;4.64), corollary (2) and (4.47);
\[
\| \sigma_{x_1}^* - \sigma_{x_2}^* \| \leq \| \nabla x f(x_1', 0, 0) \| \| \nabla g(x_1', 0, 0)^{-1} \| \| \sigma_{x_1'}^* - \sigma_{x_2'}^* \|
\]
\[
+ \mu_2 \Lambda \Lambda_* \| x_1' - x_2' \| + \mu_2 \Lambda \Lambda_* \| x_1 - x_2 \|
\]
\[
+ \Lambda \Lambda_* \| x_1' - x_2' \| + \Lambda \Lambda_* \| x_1 - x_2 \|. \quad (4.65)
\]

Denote by
\[
\varphi(x) \equiv f(x, 0, 0), \ \forall x \in V_k, \| x \| < d.
\]
Then
\[
x_i' = \varphi^{-1}(x_i), \quad i = 1, 2.
\]

By lemma (3.1),
\[
\| x_1' - x_2' \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla x \varphi^{-1}(\alpha x_1 + (1 - \alpha)x_2) \| \| x_1 - x_2 \|. \quad (4.66)
\]

Notice that
\[
\varphi^{-1}(f(x, 0, 0)) = x,
\]
then
\[
\nabla \varphi^{-1}(x) = [\nabla x f(x, 0, 0)]^{-1}.
\]

Moreover,
\[
[\nabla x f(x, 0, 0)]^{-1}
\]
is the local coordinate representation of
\[
\nabla F_{\delta}^{-T}(Q)|_{TM}.
\]

By corollary (2) and (4.67),
\[
\| \nabla \varphi^{-1}(x) \| \leq \Lambda; \ \forall x \in V_k, \| x \| < d. \quad (4.70)
\]

By (4.66;4.70),
\[
\| x_1' - x_2' \| \leq \Lambda \| x_1 - x_2 \|. \quad (4.71)
\]

By (4.65;4.66;4.71),
\[
\| \sigma_{x_1}^* - \sigma_{x_2}^* \| \| x_1 - x_2 \| \leq \mu_2 \Lambda \Lambda_* \| x_1' - x_2' \| \| \sigma_{x_1'}^* - \sigma_{x_2'}^* \|
\]
\[
+ \mu_2 \Lambda \Lambda_* \| x_1' - x_2' \| + \mu_2 \Lambda \Lambda_* \| x_1 - x_2 \|. \quad (4.72)
\]

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By (2.3), choose "$T$" large enough, such that
\[
\mu^3 \sup_{Q' \in M_0^s} \| \nabla F_0^T(Q')|_{TM_0^s} \| \sup_{Q' \in M_0^s} \| \nabla F_0^{-T}(Q')|_{J_0} \|
\]
\[
\sup_{Q' \in M_0^s} \| \nabla F_0^{-T}(Q')|_{TM_0^s} \| \leq 1/4.
\]

Then by inequalities (4.1;4.4;4.3) in lemma (4.2), when $\delta$ is sufficiently small,
\[
\sup_{Q \in M_0^s} \| \nabla F_T(\delta)(Q')|_{TM_0^s} \| \sup_{Q \in M_0^s} \| \nabla F_0^{-T}(Q')|_{J_0} \|
\]
\[
\sup_{Q \in M_0^s} \| \nabla F_T(\delta)(Q')|_{TM_0^s} \| \leq 1/2.
\] (4.73)

By (4.59;4.60;4.67) and (4.73), relation (4.72) becomes
\[
\| \sigma_1^* - \sigma_2^* \| \leq \frac{1}{2} \| \sigma_1^* - \sigma_2^* \| \frac{1}{2} \| x_1 - x_2 \| + \frac{1}{2} \chi,
\] (4.74)

where
\[
\chi \equiv 2\Lambda\Lambda_{s}(\Lambda + 1)(\mu_2 + 1).
\]

From (4.74), we have
\[
\| \sigma_1^* - \sigma_2^* \| \leq \frac{1}{2} \text{Lip}_x \{ \sigma^* \} + \frac{1}{2} \chi.
\]

Taking supremum with respect to $x_1$ and $x_2$, we have
\[
\text{Lip}_x \{ \sigma^* \} \leq \frac{1}{2} \text{Lip}_x \{ \sigma^* \} + \frac{1}{2} \chi.
\]

That is,
\[
\text{Lip}_x \{ \sigma^* \} \leq \chi.
\]

This completes the proof of the lemma. ♣

Formally differentiating (4.58) with respect to $x_0 = f(x,0,0)$; we have:
\[
\left( \nabla_x \sigma^*_{x_0} \right) \cdot \Delta x_0 = \left( [H(\nabla_x \sigma^*)]_{x_0} \right) \cdot \Delta x_0,
\] (4.75)
\[
\left( [H(\nabla_x \sigma^*)]_{x_0} \right) \cdot \Delta x_0 \equiv \nabla_x f(x,0,0) \cdot \left( [\nabla_x \sigma^*_{x_0}] \cdot [\nabla_x f(x,0,0)]^{-1} \cdot \Delta x_0 \right)
\]
\[
\cdot [\nabla_y g(x,0,0)]^{-1} + K(\sigma^*,x);
\] (4.76)

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where $\Delta x_0 \in V_k$, $K(\sigma^*, x)$ does not contain $\nabla_x \sigma^*$:

$$
K(\sigma^*, x) \equiv \left( \nabla_x [\nabla_x f(x, 0, 0)] \cdot [\nabla_x f(x, 0, 0)]^{-1} \cdot \Delta x_0 \right)
$$

$$
\bullet [\nabla y g(x, 0, 0)]^{-1}
$$

$$
+ \left( \nabla_x [\nabla_y f(x, 0, 0)] \cdot [\nabla_x f(x, 0, 0)]^{-1} \cdot \Delta x_0 \right)
$$

$$
\bullet [\nabla y g(x, 0, 0)]^{-1} + \left[ \nabla_x f(x, 0, 0) \cdot [\nabla_y g(x, 0, 0)]^{-1} \cdot \Delta x_0 \right]
$$

$$
+ \nabla_y f(x, 0, 0) \cdot \left( \nabla x_0 [\nabla y g(x, 0, 0)]^{-1} \cdot \Delta x_0 \right).
$$

(4.77)

Let $B_k$ be the ball in $V_k$:

$$
B_k \equiv \{ x \in V_k \mid ||x|| < d. \}
$$

Then, if $\sigma^*_x$ is $C^1$ in $x$,

$$
\nabla_x \sigma^* \in C^0(B_k, L(V_k, L(R^1, V_k))).
$$

Denote by $\Sigma_1$ the space:

$$
\Sigma_1 \equiv C^0(B_k, L(V_k, L(R^1, V_k))).
$$

For any $u \in \Sigma_1$, define the norm:

$$
\| u \| \equiv \sup_{x \in B_k} ||u_x||,
$$

(4.78)

where $||u_x||$ is the linear operator norm.

**Lemma 4.10** $\Sigma_1$ is a complete metric space under the norm defined in (4.78).

Proof: Along the line laid down in the proof of lemma (3.5), this lemma follows immediately. ♣

For any $u \in \Sigma_1$, define $H(u)$ by replacing $\nabla_x \sigma^*$ by $u$ in (4.76):

$$
\left( [H(u)]_{x_0} \right) \cdot \Delta x_0 \equiv \nabla_x f(x, 0, 0) \cdot \left( u_x \cdot [\nabla_x f(x, 0, 0)]^{-1} \cdot \Delta x_0 \right)
$$

$$
\bullet [\nabla y g(x, 0, 0)]^{-1} + K(\sigma^*, x);
$$

(4.79)

where $x_0 = f(x, 0, 0)$, $\Delta x_0 \in V_k$; $K(\sigma^*, x)$ is given in (4.77).
Lemma 4.11 For any $u \in \Sigma_1$, $H(u) \in \Sigma_1$.

Proof: By definition, for any $u \in \Sigma_1$, it is obvious that

$$[H(u)]_{x_0} \in L(V_k, L(R^1, V_k)), \forall x_0 \in B_k.$$  

By lemma (4.9), $\sigma^*_x$ is Lipschitz in $x$. By assumption $u_x$ is $C^0$ in $x$. All other terms in (4.79) are also $C^0$ in $x$. Then,

$$H(u) \in C^0(B_k, L(V_k, L(R^1, V_k))).$$

Next, we want to show that

$$\|H(u)\| < \infty.$$  

By (4.47),

$$\|\sigma^*_x\| \leq \mu_2. \quad (4.80)$$

By assumption,

$$\|u\| < \infty. \quad (4.81)$$

By (4.79),

$$\|H(u)\| \leq \|\nabla f(x, 0, 0)^{-1}\|\|\nabla g(x, 0, 0)^{-1}\|$$

$$\left(\|\nabla f(x, 0, 0)\| u_x + \|\nabla f(x, 0, 0)\| \sigma^*_x\right)$$

$$\|\nabla [\nabla g(x, 0, 0)]\| \|\nabla g(x, 0, 0)^{-1}\|.$$  

(4.82)

By (4.80;4.81;4.82),

$$\|H(u)\| < \infty.$$  

This completes the proof of the lemma. ♣

Lemma 4.12 $H$ is a contraction map in $\Sigma_1$ under the norm defined in (4.78).

Proof: For any $u_i \in \Sigma_1, i = 1, 2$; by (4.79),

$$\left([H(u_1) - H(u_2)]_{x_0}\right) \cdot \Delta x_0 = \nabla f(x, 0, 0) \cdot [u_1 - u_2]_x \cdot [\nabla f(x, 0, 0)]^{-1} \cdot \Delta x_0 \cdot [\nabla g(x, 0, 0)]^{-1}. \quad (4.83)$$
Then,
\[
\|H(u_1) - H(u_2)\| \leq \sup_{x \in B_k} \|\nabla f(x, 0, 0)\| \sup_{x \in B_k} \|\nabla f(x, 0, 0)\|^{-1} \|
\]
\[
\sup_{x \in B_k} \|\nabla g(x, 0, 0)\|^{-1} \|
\]
\[
\left\|\left[\nabla x f(x, 0, 0)\right]^{-1}\right\| \|u_1 - u_2\|. \quad (4.84)
\]

By (2.3), we can choose "T" large enough, such that
\[
\mu^r \sup_{Q' \in M_0^s} \|\nabla F_0^{-T}(Q')\|_{M_0^s} \sup_{Q' \in M_0^s} \|\nabla F_0^T(Q')\|_{TM_0^s} \leq 1/4, \quad \forall 2 \leq r \leq n + 1. \quad (4.85)
\]

By (4.85) and lemma (4.2), when \(\delta\) is sufficiently small,
\[
\sup_{Q \in M} \|\nabla F_\delta^{-T}(Q)\|_{TM_0^s} \left( \sup_{Q \in M} \|\nabla F_\delta^{-T}(Q)\|_{TM_0^s} \right)^{-r} \leq 1/2, \quad \forall 0 \leq r \leq n - 1. \quad (4.86)
\]

By (4.86) and (4.59;4.60;4.68;4.69),
\[
\sup_{x \in B_k} \|\nabla g(x, 0, 0)\|^{-1} \sup_{x \in B_k} \|\nabla f(x, 0, 0)\| \left( \sup_{x \in B_k} \|\nabla f(x, 0, 0)\|^{-1}\right)^r \leq 1/2, \quad \forall 0 \leq r \leq n - 1. \quad (4.87)
\]

By (4.87;4.84),
\[
\|H(u_1) - H(u_2)\| \leq \frac{1}{2} \|u_1 - u_2\|. \quad (4.88)
\]

Thus \(H\) is a contraction in \(\Sigma_1\). This completes the proof of the lemma. ♠

**Corollary 4** \(H\) has a unique fixed point \(u^*\) in \(\Sigma_1\), \(H(u^*) = u^*\). In particular, \(u^* \in C^0(B_k, L(V_k, L(R^1, V_k)))\), \(\|u^*\| < \infty\).

**Proof:** This corollary follows immediately from lemmas (4.10;4.11;4.12). ♠

**Theorem 4.2** For any \(x \in B_k\), \(\nabla_x \sigma^*\) exists and equals \(u^*\). Therefore, \(\nabla_x \sigma^* \in C^0(B_k, L(V_k, L(R^1, V_k)))\), i.e. \(\sigma^*\) is \(C^1\) in \(x\).
Proof: For any \( x \in B_k \), define an increasing function:
\[
\Delta_x : (0, 1) \to \mathbb{R},
\]
\[
\Delta_x(a) \equiv \sup_{\tilde{x} \in B_k, 0 < \|\tilde{x} - x\| < a} \frac{\|\sigma^*_{\tilde{x}} - \sigma^*_{x} - u^*_x \cdot (\tilde{x} - x)\|}{\|\tilde{x} - x\|}.
\]
By lemma (4.9) and corollary (4), there exists a constant \( \chi_1 \), such that
\[
\Delta_x(a) \leq \chi_1, \quad \forall x \in B_k, \forall a \in (0, 1).
\]
For any \( x \in B_k \), since \( \Delta_x(a) \) is an increasing non-negative function, the limit
\[
\lim_{a \to 0} \Delta_x(a)
\]
exists. Denote this limit by \( \Delta_x(0) \). By (4.88),
\[
0 \leq \Delta_x(0) \leq \chi_1, \quad \forall x \in B_k.
\]
For any \( x_1 \in B_k \), define the sequence:
\[
\{x_j\}, \quad j = 1, 2, \ldots, \infty;
\]
\[
x_j = f(x_{j+1}, 0, 0).
\]
To prove the theorem, we need to show that:
\[
\Delta_{x_1}(0) = 0.
\]
We will show that the inequality:
\[
\Delta_{x_j}(0) \leq \gamma \Delta_{x_{j+1}}(0), \quad \forall j = 1, 2, \ldots, \infty; \quad (0 < \gamma < 1)
\]
is valid. Then by (4.89;4.90),
\[
\Delta_{x_1}(0) \leq \chi_1 \gamma^m, \quad \forall m \in \mathbb{Z}^+.
\]
Thus, \( \Delta_{x_1}(0) = 0 \).

Next, we prove the inequality (4.90). For any \( j \in \mathbb{Z}^+ \), there exists a small constant \( a_j \), such that, when \( \|\tilde{x}_j - x_j\| < a_j \), all the Taylor expansions below are valid. By corollary (4) and (4.58),
\[
\sigma^*_{\tilde{x}_j} - \sigma^*_{x_j} - u^*_x \cdot (\tilde{x}_j - x_j)
\]
\[
= \sigma^*_{\tilde{x}_j} - \sigma^*_{x_j} - [H(u^*)]_{x_j} \cdot (\tilde{x}_j - x_j)
\]
\[
\begin{align*}
\n & = \nabla_x f(\tilde{x}_{j+1}, 0, 0) \cdot \sigma^*_x \cdot [\nabla_y g(\tilde{x}_{j+1}, 0, 0)]^{-1} \\
& - \nabla_x f(x_{j+1}, 0, 0) \cdot \sigma^*_{x_{j+1}} \cdot [\nabla_y g(x_{j+1}, 0, 0)]^{-1} \\
& + \nabla_y f(\tilde{x}_{j+1}, 0, 0) \cdot [\nabla_y g(\tilde{x}_{j+1}, 0, 0)]^{-1} \\
& - \nabla_y f(x_{j+1}, 0, 0) \cdot [\nabla_y g(x_{j+1}, 0, 0)]^{-1} - [H(u^*)]_{x_j} \cdot (\tilde{x}_j - x_j) \\
& = \nabla_x f(x_{j+1}, 0, 0) \cdot (\sigma^*_{x_{j+1}} - \sigma^*_x) \cdot [\nabla_y g(x_{j+1}, 0, 0)]^{-1} \\
& + G_1 + G_2 + G_3 + G_4 - [H(u^*)]_{x_j} \cdot (\tilde{x}_j - x_j), \tag{4.91}
\end{align*}
\]

where
\[
G_1 \equiv \left( \nabla_x f(\tilde{x}_{j+1}, 0, 0) - \nabla_x f(x_{j+1}, 0, 0) \right) \cdot \sigma^*_{x_{j+1}} \cdot [\nabla_y g(\tilde{x}_{j+1}, 0, 0)]^{-1},
\]
\[
G_2 \equiv \nabla_x f(x_{j+1}, 0, 0) \cdot \sigma^*_{x_{j+1}} \left( [\nabla_y g(\tilde{x}_{j+1}, 0, 0)]^{-1} - [\nabla_y g(x_{j+1}, 0, 0)]^{-1} \right),
\]
\[
G_3 \equiv \left( \nabla_y f(\tilde{x}_{j+1}, 0, 0) - \nabla_y f(x_{j+1}, 0, 0) \right) \cdot [\nabla_y g(\tilde{x}_{j+1}, 0, 0)]^{-1},
\]
\[
G_4 \equiv \nabla_y f(x_{j+1}, 0, 0) \cdot \left( [\nabla_y g(\tilde{x}_{j+1}, 0, 0)]^{-1} - [\nabla_y g(x_{j+1}, 0, 0)]^{-1} \right).
\]

By lemma (4.9),
\[
\sigma^*_x = \sigma^*_{x_{j+1}} + O(\|\tilde{x}_{j+1} - x_{j+1}\|). \tag{4.92}
\]

By Taylor expansion and (4.92),
\[
G_1 = \left( \nabla_x [\nabla_x f(x_{j+1}, 0, 0)] \cdot (\tilde{x}_{j+1} - x_{j+1}) \right) \cdot \sigma^*_{x_{j+1}} \\
\cdot [\nabla_y g(x_{j+1}, 0, 0)]^{-1} + O(\|\tilde{x}_{j+1} - x_{j+1}\|^2); \tag{4.93}
\]
\[
G_2 = \nabla_x f(x_{j+1}, 0, 0) \cdot \sigma^*_{x_{j+1}} \left( \nabla_x [\nabla_y g(x_{j+1}, 0, 0)]^{-1} \right) \\
\cdot (\tilde{x}_j - x_j) + O(\|\tilde{x}_{j+1} - x_{j+1}\|^2); \tag{4.94}
\]
\[
G_3 = \left( \nabla_x [\nabla_y f(x_{j+1}, 0, 0)] \cdot (\tilde{x}_{j+1} - x_{j+1}) \right) \\
\cdot [\nabla_y g(x_{j+1}, 0, 0)]^{-1} + O(\|\tilde{x}_{j+1} - x_{j+1}\|^2); \tag{4.95}
\]
\[
G_4 = \nabla_y f(x_{j+1}, 0, 0) \cdot \left( \nabla_x [\nabla_y g(x_{j+1}, 0, 0)]^{-1} \right) \\
\cdot (\tilde{x}_j - x_j) + O(\|\tilde{x}_{j+1} - x_{j+1}\|^2). \tag{4.96}
\]
By (4.93;4.94;4.95;4.96) and (4.77),

\[
G_1 + G_2 + G_3 + G_4 - K(\sigma^*, x_{j+1}) = \nabla_x [\nabla_x f(x_{j+1}, 0, 0)] \bullet (\tilde{x}_{j+1} - x_{j+1})
\]

\[
- [\nabla_x f(x_{j+1}, 0, 0)]^{-1} \bullet (\tilde{x}_j - x_j)
\]

\[
\sigma^*_{x_{j+1}} \bullet [\nabla_y g(x_{j+1}, 0, 0)]^{-1}
\]

\[
+ [\nabla_x [\nabla_y f(x_{j+1}, 0, 0)] \bullet (\tilde{x}_{j+1} - x_{j+1} - [\nabla_x f(x_{j+1}, 0, 0)]^{-1}
\]

\[
\sigma^*_{x_{j+1}} \bullet [\nabla_y g(x_{j+1}, 0, 0)]^{-1} + O(\|\tilde{x}_{j+1} - x_{j+1}\|^2). \tag{4.97}
\]

Notice that

\[
\tilde{x}_{j+1} - x_{j+1} - [\nabla_x f(x_{j+1}, 0, 0)]^{-1} \bullet (\tilde{x}_j - x_j) = O(\|\tilde{x}_{j+1} - x_{j+1}\|^2). \tag{4.98}
\]

Then (4.97; 4.98) imply that

\[
G_1 + G_2 + G_3 + G_4 - K(\sigma^*, x_{j+1}) = O(\|\tilde{x}_{j+1} - x_{j+1}\|^2). \tag{4.99}
\]

By (4.91; 4.79; 4.99),

\[
\sigma^*_{\tilde{x}_j} - \sigma^*_{x_j} - u^*_{x_j} \bullet (\tilde{x}_j - x_j)
\]

\[
= \nabla_x f(x_{j+1}, 0, 0) \bullet \left[\sigma^*_{x_{j+1}} - \sigma^*_{x_{j+1}} - u^*_{x_{j+1}}
\right.
\]

\[
\left.\bullet (\nabla_x f(x_{j+1}, 0, 0)]^{-1} \bullet (\tilde{x}_j - x_j)\right]
\]

\[
\bullet [\nabla_y g(x_{j+1}, 0, 0)]^{-1} + O(\|\tilde{x}_{j+1} - x_{j+1}\|^2). \tag{4.100}
\]

By (4.98; 4.100),

\[
\sigma^*_{\tilde{x}_j} - \sigma^*_{x_j} - u^*_{x_j} \bullet (\tilde{x}_j - x_j)
\]

\[
= \nabla_x f(x_{j+1}, 0, 0) \left[\sigma^*_{x_{j+1}} - \sigma^*_{x_{j+1}} - u^*_{x_{j+1}} \bullet (\tilde{x}_{j+1} - x_{j+1})\right]
\]

\[
\bullet [\nabla_y g(x_{j+1}, 0, 0)]^{-1} + O(\|\tilde{x}_{j+1} - x_{j+1}\|^2). \tag{4.101}
\]

By (4.66; 4.67),

\[
\|\tilde{x}_{j+1} - x_{j+1}\| \leq \lambda_1 \|\tilde{x}_j - x_j\|, \tag{4.102}
\]

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where
\[ \lambda_1 \equiv \sup_{x \in B_k} \|[\nabla_x f(x, 0, 0)]^{-1}\|. \]

By (4.101;4.102),
\[
\frac{\|\sigma^*_{x_j} - \sigma^*_{\tilde{x}_j} + u^*_{x_j} \cdot (\tilde{x}_j - x_j)\|}{\|\tilde{x}_j - x_j\|} \\
\leq \sup_{x \in B_k} \|\nabla_x f(x, 0, 0)\| \sup_{x \in B_k} \|[\nabla_y g(x, 0, 0)]^{-1}\| \sup_{x \in B_k} \|[\nabla_x f(x, 0, 0)]^{-1}\|
\]
\[
\frac{\|\sigma^*_{x_{j+1}} - \sigma^*_{\tilde{x}_{j+1}} + u^*_{x_{j+1}} \cdot (\tilde{x}_{j+1} - x_{j+1})\|}{\|\tilde{x}_{j+1} - x_{j+1}\|} + O(\|\tilde{x}_{j+1} - x_{j+1}\|),
\]
(4.103)

By (4.103;4.87) and (4.102),
\[
\Delta_{x_j} (\|\tilde{x}_j - x_j\|) \leq \gamma \Delta_{x_{j+1}} (\lambda_1 \|\tilde{x}_j - x_j\|) + O(\|\tilde{x}_j - x_j\|), \quad \gamma = 1/2.
\]
(4.104)

Let \(\|\tilde{x}_j - x_j\| \to 0\) in (4.104), we have
\[
\Delta_{x_j} (0) \leq \gamma \Delta_{x_{j+1}} (0), \quad \gamma = 1/2.
\]

This completes the proof of the theorem. ✤

Additional smoothness follows similarly. We summarize in the following:

**Theorem 4.3** \(\sigma^*_x\) is \(C^{n-1}\) in \(x\). Thus, \(E\) is a \(C^{n-1}\) bundle.

**Proof:** The proof is similar to that given above for \(\sigma^*_x\) to be \(C^1\) in \(x\). Therefore, here we only sketch the proof. If \(\sigma^*_x\) is \(C^s\) in \(x\), then
\[
\nabla^s_x \sigma^* \in C^0(B_k, L^s(V_k, L(R^1, V_k))).
\]

Denote by
\[
\Sigma_s \equiv C^0(B_k, L^s(V_k, L(R^1, V_k))).
\]

For any \(u^{(s)} \in \Sigma_s\), define the norm:
\[
\|u^{(s)}\| \equiv \sup_{x \in B_k} \|u^{(s)}_x\|,
\]
where \(\|u^{(s)}_x\|\) is the s-linear operator norm. A similar proof for lemma (3.5) shows that \(\Sigma_s\) is a complete metric space under the norm defined above.

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Assuming $\sigma^*_s$ is $C^{s-1}$ in $x$, we want to show that $\sigma^*_s$ is $C^s$ in $x$, $2 \leq s \leq n - 1$. Formally differentiating (4.58) $s$ times with respect to $x$, we have

$$
\begin{align*}
\left( \nabla^s x \sigma^*_x \right) \bullet (\Delta x^1_0, \ldots, \Delta x^s_0) &= \left( [H(s)(\nabla^s x \sigma^*)]_{x_0} \right) \bullet (\Delta x^1_0, \ldots, \Delta x^s_0), \\
\end{align*}
$$

(4.105)

where

$$
\begin{align*}
\left( [H(s)(\nabla^s x \sigma^*)]_{x_0} \right) \bullet (\Delta x^1_0, \ldots, \Delta x^s_0)
&= \nabla_x f(x, 0, 0) \bullet \left( \left( \nabla^s x \sigma^*_x \right) \bullet \left( [\nabla_x f(x, 0, 0)]^{-1} \Delta x^1_0, \ldots, [\nabla_x f(x, 0, 0)]^{-1} \Delta x^s_0 \right) \right) \\
&\quad \bullet [\nabla_y g(x, 0, 0)]^{-1} + \text{terms not involving } \nabla^s x \sigma^*. (2 \leq s \leq n - 1) 
\end{align*}
$$

(4.106)

For any $u^{(s)} \in \Sigma_s$, define $H(s)(u^{(s)})$ by replacing $\nabla^s x \sigma^*$ with $u^{(s)}$ in (4.106). A similar proof for lemma (4.11) shows that

$$
H(s)(u^{(s)}) \in \Sigma_s, \quad (2 \leq s \leq n - 1).
$$

A similar proof for lemma (4.9) shows that $\nabla^{(s-1)} \sigma^*_x$ is Lipschitz in $x$. Then, by relation (4.87) and (4.106), a similar proof for lemma (4.12) shows that $H(s)$ is a contraction map in $\Sigma_s$. Thus, there exists a unique fixed point $u^{(s,s)}$ of $H(s)$ in $\Sigma_s$:

$$
u^{(s,s)} = H(s)(u^{(s,s)}), \quad (2 \leq s \leq n - 1).
$$

A similar argument for theorem (4.2) shows that:

$$
\nabla^s x \sigma^* \in \Sigma_1, \quad \nabla^s x \sigma^* = u^{(s,s)}, \quad (2 \leq s \leq n - 1).
$$

This completes the proof of the theorem.

**4.5 Existence of Fibers**

In this subsection, based upon the preliminary lemmas in last subsection, the existence of fibers is proved.

**Theorem 4.4 (Existence of Fibers)**  In a neighborhood of $M$ in $M^*$, there exists a family of 1-dimensional curves $f^E(Q)$, indexed by $Q \in M$, which are overflowing invariant under the flow $F^1_\delta$ in the sense that:

$$
f^E(Q) \subset F^1_\delta(f^E(F^t_\delta(Q))), \quad t \geq 0.
$$
Moreover, the curve $f^E(Q)$ is $C^n$ diffeomorphic to a segment of the straight line $E(Q)$, and is tangent to $E(Q)$ at $Q \in M$. $f^E(Q)$ intersects $M$ transversally at $Q$. Two fibers $f^E(Q_1)$ and $f^E(Q_2)$ are either disjoint or identical. ($f^E(Q)$ is called an unstable fiber.)

Proof: Denote by $\hat{N} \equiv TM \oplus J^s$, then,

$$T\tilde{S}_k|_M = E \oplus \hat{N}.$$ 

Define $E_\kappa$ and $\hat{N}_\kappa$ as follows:

$$E_\kappa(Q) \equiv \{ v \in E(Q) \mid \|v\| \leq \kappa \},$$ 

$$\hat{N}_\kappa(Q) \equiv \{ w \in \hat{N}(Q) \mid \|w\| \leq \kappa \}.$$ 

Let $\Sigma$ denote the set which consists of families of continuous maps of the form:

$$\forall \sigma \in \Sigma, \quad \sigma \equiv \{ \sigma_Q \}_{Q \in M},$$ 

$$\sigma_Q : E_\kappa(Q) \mapsto \hat{N}_\kappa(Q).$$ 

Define a Lipschitz norm:

$$\text{Lip} \{ \sigma \} \equiv \sup_C \left\{ \frac{\|\sigma_Q(x_1) - \sigma_Q(x_2)\|}{\|x_1 - x_2\|} \right\},$$ 

where $C$ stands for:

$$C \equiv \{ Q \in M; \ x_1, x_2 \in E_\kappa(Q); \ x_1 \neq x_2. \}$$ 

Define a subset of $\sigma$:

$$\Sigma_\zeta \equiv \left\{ \sigma \in \Sigma \mid \text{Lip} \{ \sigma \} \leq \zeta; \text{ moreover, } \sigma_Q(0) = 0, \forall Q \in M. \right\}$$ 

Define a distance in $\Sigma_\zeta$ as follows:

$$d(\sigma, \sigma') \equiv \sup_C \left\{ \frac{\|\sigma_Q(x) - \sigma'_Q(x)\|}{\|x\|} \right\},$$ 

where $C$ stands for:

$$C \equiv \{ Q \in M; \ x \in E_\kappa(Q); \ x \neq 0. \}$$
Remark 4.2 In the above definitions, we use the same notations "κ" and "ζ" as before. The context will always make the distinction clear.

Lemma 4.13 $d(\sigma, \sigma')$ exists for any $\sigma, \sigma' \in \Sigma_\zeta$.

Proof: By definition,
\[
\frac{\|\sigma_Q(x) - \sigma'_Q(x)\|/\|x\|}{\|x\|} \leq 2\zeta,
\]
for any $Q \in M$, $x \in E_\kappa(Q)$. Therefore, $d(\sigma, \sigma')$ exists. ⚫

Lemma 4.14 $\Sigma_\zeta$ is a complete metric space with respect to the distance $d(\sigma, \sigma')$ defined above.

Proof: Let
\[
\left\{ \sigma^{(j)}; \; j = 1, 2, ..., \infty \right\}
\]
be a Cauchy sequence in $\Sigma_\zeta$ with respect to the distance $d$; i.e., for any $\theta > 0$, there exists an integer $K$, such that
\[
d(\sigma^{(i)}, \sigma^{(j)}) < \theta; \; \forall i, j \geq K.
\]
Then for any fixed $Q \in M$, $x \in E_\kappa(Q)$, $x \neq 0$:
\[
\frac{\|\sigma_Q^{(i)}(x) - \sigma_Q^{(j)}(x)\|/\|x\|}{\|x\|} < \theta;
\]
i.e.,
\[
\frac{\|\sigma_Q^{(i)}(x) - \sigma_Q^{(j)}(x)\|}{\|x\|} < \theta\|x\|.
\]
Thus, for any fixed $Q \in M$, $x \in E_\kappa(Q)$, $x \neq 0$:
\[
\left\{ \sigma_Q^{(j)}(x); \; j = 1, 2, ..., \infty \right\}
\]
is a Cauchy sequence in $\hat{N}_\kappa(Q)$, which is a complete space. Therefore, there exists $y_{Q,x} \in \hat{N}_\kappa(Q)$, such that
\[
\sigma_Q^{(j)}(x) \to y_{Q,x}, \; \text{as} \; j \to \infty;
\]
in $\hat{N}_\kappa(Q)$. Define a family of maps $\sigma$:
\[
\sigma \equiv \{\sigma_Q\}_{Q \in M},
\]
$\sigma_Q : E_\kappa(Q) \mapsto \hat{N}_\kappa(Q)$,
$\forall x \in E_\kappa(Q)$, $\sigma_Q(x) = y_{Q,x}$.
Notice that,
\[ y_{Q,0} = 0, \quad \forall Q \in M. \]

Next, we will show that: \( \text{Lip} \{ \sigma \} \leq \zeta \), which implies that \( \sigma_Q \) is a continuous map, for any \( Q \in M \). More importantly, this implies that \( \sigma \in \Sigma_\zeta \). For any fixed \( Q \in M; x_1, x_2 \in E_\kappa(Q), x_1 \neq x_2 \):
\[
\frac{\| \sigma_Q^{(j)}(x_1) - \sigma_Q^{(j)}(x_2) \|}{\| x_1 - x_2 \|} \leq \zeta;
\]
let \( j \to \infty \), we have
\[
\frac{\| \sigma_Q(x_1) - \sigma_Q(x_2) \|}{\| x_1 - x_2 \|} \leq \zeta;
\]
then
\[
\sup_C \left\{ \frac{\| \sigma_Q(x_1) - \sigma_Q(x_2) \|}{\| x_1 - x_2 \|} \right\} \leq \zeta,
\]
where \( C \) stands for:
\[
C \equiv \{ Q \in M; x_1, x_2 \in E_\kappa(Q); x_1 \neq x_2. \}
\]
That is,
\[
\text{Lip} \{ \sigma \} \leq \zeta.
\]
Thus, \( \sigma \in \Sigma_\zeta \), \( \Sigma_\zeta \) is a complete metric space. This completes the proof of the lemma. ♣

Now we define a graph transform for elements of \( \Sigma_\zeta \). For any \( Q \in M \), let \( x \in E(Q), y \in \breve{N}(Q) \); then \((x, y)\) can be embedded in \( \breve{S}_k \) as a local coordinate in a neighborhood of \( Q \). Let \((\xi, \eta)\) be such a coordinate in a neighborhood of \( Q' = F_\delta^{-T}(Q) \). Then, in terms of such coordinates, the solution operator \( F_\delta^T \) has a representation:
\[
F_\delta^T : (\xi, \eta) \mapsto (x, y) = \left( f(\xi, \eta), g(\xi, \eta) \right).
\]
The graph transform \( G \) is defined as follows:
\[
\forall \sigma \in \Sigma_\zeta, \quad G\sigma \equiv \{(G\sigma)_Q \}_{Q \in M}, \\
(G\sigma)_Q(x) = g(\xi, \sigma_Q'(\xi)), \\
x = f(\xi, \sigma_Q'(\xi)), \\
Q' = F_\delta^{-T}(Q).
\]
The fixed point $\sigma^*$ (if it exists) of this graph transform satisfies the overflowing invariance condition:

$$\text{graph } \sigma^*_Q \subset F^T_\delta (\text{graph } \sigma^*_{Q'}) , \ Q' = F^{-T}_\delta (Q).$$

**Lemma 4.15** For any $\sigma \in \Sigma_{\zeta}$, $G\sigma \in \Sigma$; i.e., $G : \Sigma_{\zeta} \rightarrow \Sigma$.

Proof: In order to show that $G\sigma$ is well-defined as an element of $\Sigma$, first we need to show that $\mathcal{E}_\kappa(Q)$ is contained in the range of the map

$$\xi \mapsto f(\xi, \sigma_{Q'}(\xi)),$$

so that, every point $x$ in $\mathcal{E}_\kappa(Q)$ is assigned a value

$$(G\sigma)_Q(x) = g(\xi, \sigma_{Q'}(\xi));$$

moreover, each point in $\mathcal{E}_\kappa(Q)$ has a unique preimage point in $\mathcal{E}_\kappa(Q')$, so that, every point $x$ in $\mathcal{E}_\kappa(Q)$ is assigned a unique value. That is, the map

$$f(\bullet, \sigma_{Q'}(\bullet)) : U \subset \mathcal{E}_\kappa(Q') \mapsto \mathcal{E}_\kappa(Q),$$

is 1-1 and onto. (Where $U$ is the domain of definition.) Then we need to show that

$$\|g(\xi, \sigma_{Q'}(\xi))\| \leq \kappa, \ \forall \|x\| = \|f(\xi, \sigma_{Q'}(\xi))\| \leq \kappa.$$

Define the map:

$$\varphi(\xi) = [\nabla_\xi f(0,0)]^{-1}f(\xi, \sigma_{Q'}(\xi)).$$

Next we will show the $\varphi$ is a perturbation of identity in the Lipschitz topology for map. $\varphi$ can be rewritten as:

$$\varphi(\xi) = \xi + [\nabla_\xi f(0,0)]^{-1}\left( f(\xi, \sigma_{Q'}(\xi)) - \nabla_\xi f(0,0) \cdot \xi \right). \quad (4.108)$$

For any $\xi_0 \in \mathcal{E}_\kappa(Q')$, there is a small neighborhood $V_{\xi_0}$ of $\xi_0$ in $\mathcal{E}_\kappa(Q')$, such that, for any $\xi \in V_{\xi_0}$;

$$f(\xi, \sigma_{Q'}(\xi)) = f(\xi_0, \sigma_{Q'}(\xi_0)) + \nabla_\xi f(\xi_0, \sigma_{Q'}(\xi_0)) \cdot (\xi - \xi_0)$$

$$+ \nabla_\eta f(\xi_0, \sigma_{Q'}(\xi_0)) \cdot (\sigma_{Q'}(\xi) - \sigma_{Q'}(\xi_0))$$

$$+ o(\|\xi - \xi_0\|), \quad (4.109)$$

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where \( \| o(\| \xi - \xi_0 \|) \| \leq \zeta \| \xi - \xi_0 \|. \) By corollary (2),

\[
\| \nabla_\eta f(\xi_0, \sigma_{Q'}(\xi_0)) \bullet (\sigma_Q(\xi) - \sigma_{Q'}(\xi_0)) \| \leq \Lambda \| \sigma_Q(\xi) - \sigma_{Q'}(\xi_0) \| \leq \zeta \Lambda \| \xi - \xi_0 \|.
\]

By lemma (3.1) and theorem (2.11), for any \( \| \xi \| \leq \kappa \), \( \| \eta \| \leq \kappa \),

\[
\| \nabla_\xi f(\xi, \eta) - \nabla_\xi f(0,0) \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla \nabla_\xi f(\alpha(\xi, \eta)) \| \| (\xi, \eta) \| \leq 2 \kappa \Lambda_*.
\]

Thus, we can rewrite (4.109) as:

\[
f(\xi, \sigma_{Q'}(\xi)) - f(\xi_0, \sigma_{Q'}(\xi_0)) = \nabla_\xi f(0,0) \bullet (\xi - \xi_0) + G(\xi, \xi_0), \tag{4.110}
\]

where

\[
\| G(\xi, \xi_0) \| \leq (\kappa C_\kappa + \zeta C_\zeta) \| \xi - \xi_0 \|. \tag{4.111}
\]

in which \( C_\kappa = C_\kappa(T) \), and \( C_\zeta = C_\zeta(T) \) are constants. For any \( \xi^{(1)}, \xi^{(2)} \in E_\kappa(Q') \), denote by \( l_\xi \) the straight line segment joining \( \xi^{(1)} \) and \( \xi^{(2)} \) in \( E_\kappa(Q') \). Then, \( \{ V_{\xi_0} \}_{\xi_0 \in E_\kappa(Q')} \) is an open cover of \( l_\xi \). Since \( l_\xi \) is compact, we have a finite cover

\[
\{ V_{\xi_j} \}_{j=1,...,m}.
\]

Without loss of generality, we can take:

\[
\xi_1 = \xi^{(1)}, \quad \xi_m = \xi^{(2)}.
\]

Then,

\[
f(\xi^{(2)}, \sigma_{Q'}(\xi^{(2)})) - f(\xi^{(1)}, \sigma_{Q'}(\xi^{(1)})) = \sum_{j=1}^{m-1} [f(\xi_{j+1}, \sigma_{Q'}(\xi_{j+1})) - f(\xi_j, \sigma_{Q'}(\xi_j))].
\]

By (4.110),

\[
f(\xi^{(2)}, \sigma_{Q'}(\xi^{(2)})) - f(\xi^{(1)}, \sigma_{Q'}(\xi^{(1)})) = \nabla_\xi f(0,0) \bullet (\xi^{(2)} - \xi^{(1)}) + \sum_{j=1}^{m-1} G(\xi_{j+1}, \xi_j).
\]

Denote by

\[
G(\xi^{(2)}, \xi^{(1)}) = \sum_{j=1}^{m-1} G(\xi_{j+1}, \xi_j).
\]

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Notice that
\[ \| \xi^{(2)} - \xi^{(1)} \| = \sum_{j=1}^{m-1} \| \xi_{j+1} - \xi_j \|. \]

Then by (4.111),
\[ \| G(\xi^{(2)}, \xi^{(1)}) \| \leq (\kappa C_\kappa + \zeta C_\zeta) \| \xi^{(2)} - \xi^{(1)} \|. \] (4.112)

Finally,
\[ f(\xi^{(2)}, \sigma_Q'(\xi^{(2)})) - f(\xi^{(1)}, \sigma_Q'(\xi^{(1)})) = \nabla_\xi f(0, 0) \bullet (\xi^{(2)} - \xi^{(1)}) + G(\xi^{(2)}, \xi^{(1)}), \] (4.113)

where \( G(\xi^{(2)}, \xi^{(1)}) \) has the estimate (4.112). If we choose \( \xi^{(1)} = 0 \), and denote \( \xi^{(2)} \) by \( \xi \) in (4.113), then
\[ f(\xi, \sigma_Q'(\xi)) = \nabla_\xi f(0, 0) \bullet \xi + G(\xi, 0), \] (4.114)

where
\[ \| G(\xi, 0) \| \leq (\kappa C_\kappa + \zeta C_\zeta) \| \xi \|. \] (4.115)

Denote by
\[ H(\xi) \equiv [\nabla_\xi f(0, 0)]^{-1} \left( f(\xi, \sigma_Q'(\xi)) - \nabla_\xi f(0, 0) \bullet \xi \right). \]

By (4.114;4.115) and (4.113;4.112),
\[ \| H(\xi) \| \leq (\kappa C_\kappa + \zeta C_\zeta) \Lambda \| \xi \|, \quad \forall \xi \in E_\kappa(Q'); \] (4.116)
\[ \| H(\xi^{(2)}) - H(\xi^{(1)}) \| \leq (\kappa C_\kappa + \zeta C_\zeta) \Lambda \| \xi^{(2)} - \xi^{(1)} \|, \quad \forall \xi^{(1)}, \xi^{(2)} \in E_\kappa(Q'). \] (4.117)

Therefore, \( \varphi \) is a perturbation of identity in the Lipschitz topology for map. This implies that \( \varphi \) is an 1-1 map. Thus, \( f(\bullet, \sigma_Q'(\bullet)) \) is an 1-1 map.
\[ f(\xi, \sigma_Q'(\xi)) = [\nabla_\xi f(0, 0)] \left( \xi + H(\xi) \right). \] (4.118)

Notice that for any \( Q' \in M \),
\[ \nabla_\xi f(0, 0) \equiv \pi^E \nabla F_0^T(Q') |_E = \nabla F_0^T(Q') |_E. \]
Moreover,
\[
[\nabla_\xi f(0, 0)]^{-1} = \nabla F_\delta^{-T}(Q')|_E.
\]
By inequality (4.51) in lemma (4.8), when \(T\) is sufficiently large,
\[
\|[\nabla_\xi f(0, 0)]^{-1}\| \leq 1/4. \tag{4.119}
\]
Then by (4.118;4.119), when \(\kappa, \zeta\) are sufficiently small,
\[
\|\xi\| \leq \frac{1}{2}\|f(\xi, \sigma Q' (\xi))\| = \frac{1}{2}\|x\|. \tag{4.120}
\]
By (4.118;4.120), for any \(x \in E_\kappa(Q)\), there is a unique \(\xi \in E_\kappa(Q')\), such that
\[
x = f(\xi, \sigma Q'(\xi)).
\]
Next, we estimate \(\|g(\xi, \sigma Q'(\xi))\|\). By lemma (3.1),
\[
\|g(\xi, \sigma Q'(\xi))\| = \|g(\xi, \sigma Q'(\xi)) - g(0, \sigma Q'(\xi))
+ g(0, \sigma Q'(\xi)) - g(0, \sigma Q'(0))\|
\leq \sup_{0 \leq \alpha \leq 1} \|\nabla_\xi g(\alpha \xi, \sigma Q'(\xi))\|\|\xi\|
+ \sup_{0 \leq \alpha \leq 1} \|\nabla_\eta g(0, \alpha \sigma Q'(\xi))\|\|\sigma Q'(\xi) - \sigma Q'(0)\|, \tag{4.121}
\]
where \(\sigma Q'(0) = 0, g(0, 0) = 0\). Notice that
\[
\nabla_\xi g(0, 0) = 0.
\]
Moreover, by lemma (3.1) and theorem (2.11), for any \(\|\xi\| \leq \kappa, \|\eta\| \leq \kappa\),
\[
\|\nabla_\xi g(\xi, \eta) - \nabla_\xi g(0, 0)\| \leq \sup_{0 \leq \alpha \leq 1} \|\nabla \nabla_\xi g(\alpha \xi, \eta)\|\|\xi, \eta\| \leq 2\kappa \Lambda_\ast.
\]
Then,
\[
\sup_{\|\xi\| \leq \kappa, \|\eta\| \leq \kappa} \|\nabla_\xi g(\xi, \eta)\| \leq 2\kappa \Lambda_\ast. \tag{4.122}
\]
Thus, by (4.121;4.122),
\[
\|g(\xi, \sigma Q'(\xi))\| \leq (2\kappa \Lambda_\ast + \zeta \Lambda)\|\xi\|.
\]
By (4.120),
\[
\|g(\xi, \sigma Q'(\xi))\| \leq \frac{1}{2}(2\kappa \Lambda_\ast + \zeta \Lambda)\|x\|. \tag{4.123}
\]
Therefore, when \(\kappa\) and \(\zeta\) are sufficiently small, for any \(\|x\| \leq \kappa\),
\[
\|g(\xi, \sigma Q'(\xi))\| \leq \kappa.
\]
This completes the proof of the lemma. \(\clubsuit\)
Lemma 4.16 \( G : \Sigma_\zeta \mapsto \Sigma_\zeta \).

Proof: By lemma (4.15), we know that
\[
G : \Sigma_\zeta \mapsto \Sigma.
\]
Then, we need to prove that for any \( \sigma \in \Sigma_\zeta \),
\[
\text{Lip}\{G\sigma\} \leq \zeta; \quad (4.124)
\]
moreover,
\[
(G\sigma)Q(0) = 0, \quad \forall Q \in M. \quad (4.125)
\]
Since \( M \) is overflowing invariant under \( F^t \),
\[
f(0, \sigma Q' (0)) = f(0, 0) = 0.
\]
Moreover, by (4.118), \( f(\bullet, \sigma Q'(\bullet)) \) is an 1-1 map. Then \( x = f(\xi, \sigma Q'(\xi)) = 0 \) has a unique preimage \( \xi = 0 \). Since
\[
g(0, \sigma Q'(0)) = g(0, 0) = 0;
\]
then,
\[
(G\sigma)Q(0) = g(0, \sigma Q'(0)) = 0; \quad \forall Q \in M,
\]
which is equation (4.125). Next, we prove inequality (4.124). For any \( Q \in M; x_1, x_2 \in E_\kappa(Q); x_1 \neq x_2 \), let
\[
x_i = f(\xi_i, \sigma Q'(\xi_i)), \quad i = 1, 2; \quad (4.126)
\]
then
\[
(G\sigma)Q(x_i) = g(\xi_i, \sigma Q'(\xi_i)), \quad i = 1, 2.
\]
Thus,
\[
\|(G\sigma)Q(x_1) - (G\sigma)Q(x_2)\| = \|g(\xi_1, \sigma Q'(\xi_1)) - g(\xi_2, \sigma Q'(\xi_2))\|
\leq \|g(\xi_1, \sigma Q'(\xi_1)) - g(\xi_2, \sigma Q'(\xi_1))\| + \|g(\xi_2, \sigma Q'(\xi_1)) - g(\xi_2, \sigma Q'(\xi_2))\| \quad (4.127)
\]
Apply lemma (3.1) to (4.127), we have
\[
\|(G\sigma)Q(x_1) - (G\sigma)Q(x_2)\| \leq \sup_{0 \leq \alpha \leq 1} \|\nabla g(\alpha \xi_1 + (1-\alpha) \xi_2, \sigma Q'(\xi_1))\|\|\xi_1 - \xi_2\|.
\]

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+ \sup_{0 \leq \alpha \leq 1} \| \nabla_\eta g(\xi_2, \alpha \sigma Q(\xi_1) + (1 - \alpha) \sigma Q'(\xi_2)) \| \| \sigma Q'(\xi_1) - \sigma Q'(\xi_2) \|

\leq \left[ \sup_{0 \leq \alpha \leq 1} \| \nabla_\xi g(\alpha \xi_1 + (1 - \alpha) \xi_2, \sigma Q'(\xi_1)) \| + \zeta \sup_{0 \leq \alpha \leq 1} \| \nabla_\eta g(\xi_2, \alpha \sigma Q'(\xi_1) + (1 - \alpha) \sigma Q'(\xi_2)) \| \right] \| \xi_1 - \xi_2 \|. \tag{4.128}

By lemma (3.1) and theorem (2.11), for any \( \| \xi \| \leq \kappa, \| \eta \| \leq \kappa; \)

\[ \| \nabla_\xi g(\xi, \eta) - \nabla_\xi g(0, 0) \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_\xi g(\alpha(0, 0) + (1 - \alpha)(\xi, \eta)) \| \| (\xi, \eta) \| \leq 2\kappa \Lambda, \]

\[ \| \nabla_\eta g(\xi, \eta) - \nabla_\eta g(0, 0) \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_\eta g(\alpha(0, 0) + (1 - \alpha)(\xi, \eta)) \| \| (\xi, \eta) \| \leq 2\kappa \Lambda. \]

Since \( E \) is an overflowing invariant subbundle under \( TF^r \), then

\[ \nabla_\xi g(0, 0) = 0. \]

Thus,

\[ \| \nabla_\xi g(\xi, \eta) \| \leq 2\kappa \Lambda, \tag{4.129} \]

\[ \| \nabla_\eta g(\xi, \eta) \| \leq \| \nabla_\eta g(0, 0) \| + 2\kappa \Lambda. \tag{4.130} \]

By (4.128;4.129;4.130),

\[ \| (G\sigma)_Q(x_1) - (G\sigma)_Q(x_2) \| \leq 2\kappa \Lambda, \]

\[ + \zeta (\| \nabla_\eta g(0, 0) \| + 2\kappa \Lambda) \| \xi_1 - \xi_2 \|. \tag{4.131} \]

We need a relation between \( \| \xi_1 - \xi_2 \| \) and \( \| x_1 - x_2 \|. \) By (4.118),

\[ \xi = [\nabla_\xi f(0, 0)]^{-1} f(\xi, \sigma Q(\xi)) - H(\xi), \]

where \( H(\xi) \) has the estimates (4.116;4.117).

\[ \| \xi_1 - \xi_2 \| \leq \| [\nabla_\xi f(0, 0)]^{-1} f(\xi_1, \sigma Q'(\xi_1)) - f(\xi_2, \sigma Q'(\xi_2)) \| + \| H(\xi_1) - H(\xi_2) \|. \]

By (4.126) and (4.117),

\[ \| \xi_1 - \xi_2 \| \leq \| [\nabla_\xi f(0, 0)]^{-1} \| x_1 - x_2 \| + (\kappa C_\kappa + \zeta C_\zeta) \Lambda \| \xi_1 - \xi_2 \|. \]
Then,

$$\|\xi_1 - \xi_2\| \leq \frac{\|\nabla_\xi f(0,0)\|^{-1}}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} \|x_1 - x_2\|. \quad (4.132)$$

By (4.131;4.132),

$$\|(G\sigma)Q(x_1) - (G\sigma)Q(x_2)\| \leq \left[ 2\kappa \Lambda_* + \zeta(\|\nabla_\eta g(0,0)\| + 2\kappa \Lambda_*) \right] \cdot \frac{\|\nabla_\xi f(0,0)\|^{-1}}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} \|x_1 - x_2\|. \quad (4.133)$$

Next we will apply lemma (4.8) to (4.133). We can choose $T$ large enough, such that

$$\mu_0^2 \sup_{Q' \in M_0} \|\nabla F_0^{-T}(Q')|_{J_0}\| \sup_{Q' \in M_0} \|\nabla F_0^{-T}(Q')|_{TM_0}\| < 1/8.$$  

Then, by inequalities (4.48;4.49;4.51), when $\delta$ is sufficiently small (for this fixed $T$),

$$\sup_{Q \in M} \|\nabla F_\delta^{-T}(Q)|_{E}\| \left( \sup_{Q \in M} \|\nabla F_\delta^T(Q)|_{TM}\| + \sup_{Q \in M} \|\nabla F_\delta^T(Q)|_{J_\ast}\| \right) < 1/2. \quad (4.134)$$

Notice that

$$\nabla_\xi f(0,0)\|^{-1} \equiv \nabla F_\delta^{-T}(Q)|_{E},$$

$$\|\nabla_\eta g(0,0)\| \leq \sup_{Q \in M} \|\nabla F_\delta^T(Q)|_{TM}\| + \sup_{Q \in M} \|\nabla F_\delta^T(Q)|_{J_\ast}\|.$$  

Then, by (4.134)

$$\|\nabla_\xi f(0,0)\|^{-1} \|\nabla_\eta g(0,0)\| < 1/2. \quad (4.135)$$

Apply (4.135) to (4.133); when $\kappa, \zeta$ are sufficiently small,

$$\frac{\|\nabla_\eta g(0,0)\| + 2\kappa \Lambda_*}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} \frac{\|\nabla_\xi f(0,0)\|^{-1}}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} < 3/4. \quad (4.136)$$

Moreover, choose $\kappa$ small enough, such that

$$\frac{2\kappa \Lambda_* \|\nabla_\xi f(0,0)\|^{-1}}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} < \frac{1}{8\zeta}.$$  

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i.e.,

\[ \kappa < \frac{\zeta}{16} \cdot \frac{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)}{\Lambda \|\nabla \xi f(0, 0)\|^{-1}}. \]  

(4.137)

By (4.133;4.136;4.137),

\[ \|(G\sigma)Q(x_1) - (G\sigma)Q(x_2)\| \leq \frac{7}{8} \zeta \|x_1 - x_2\|. \]  

(4.138)

Since (4.138) is true for any \( Q \in M; x_1, x_2 \in E_\kappa(Q); x_1 \neq x_2; \) then

\[ \text{Lip}\{G\sigma\} \equiv \sup_{C} \left\{ \frac{\|(G\sigma)Q(x_1) - (G\sigma)Q(x_2)\|}{\|x_1 - x_2\|} \right\} \leq \zeta, \]

where \( C \) stands for

\[ C \equiv \{ Q \in M; x_1, x_2 \in E_\kappa(Q); x_1 \neq x_2. \} \]

This completes the proof of the lemma. ♣

**Lemma 4.17** \( G \) is a contraction on \( \Sigma_\zeta \) under the distance \( d(\sigma, \sigma') \).

Proof: For any \( \sigma^{(i)} \in \Sigma_\zeta, i = 1, 2; Q \in M, x \in E_\kappa(Q), x \neq 0; \) let

\[ x = f(\xi, \sigma^{(i)}_Q(\xi)); \ i = 1, 2; \]  

(4.139)

\[ Q' = F^{-T}_\delta (Q). \]

Then,

\[ (G\sigma^{(i)})_Q(x) = g(\xi, \sigma^{(i)}_Q(\xi)); \ i = 1, 2. \]

Thus,

\[ \|(G\sigma^{(1)})_Q(x) - (G\sigma^{(2)})_Q(x)\| \]

\[ = \|g(\xi_1, \sigma^{(1)}_Q(\xi_1)) - g(\xi_2, \sigma^{(2)}_Q(\xi_2))\| \]

\[ \leq \|g(\xi_1, \sigma^{(1)}_Q(\xi_1)) - g(\xi_2, \sigma^{(1)}_Q(\xi_2))\| \]

\[ + \|g(\xi_2, \sigma^{(1)}_Q(\xi_1)) - g(\xi_2, \sigma^{(1)}_Q(\xi_2))\| \]

\[ + \|g(\xi_2, \sigma^{(1)}_Q(\xi_2)) - g(\xi_2, \sigma^{(2)}_Q(\xi_2))\| \]

\[ \leq \sup_{0 \leq \alpha \leq 1} \|\nabla_\xi g(\alpha \xi_1 + (1 - \alpha)\xi_2, \sigma^{(1)}_Q(\xi_1))\| \|\xi_1 - \xi_2\| \]

\[ + \sup_{0 \leq \alpha \leq 1} \|\nabla_\eta g(\xi_2, \alpha \sigma^{(1)}_Q(\xi_1) + (1 - \alpha)\sigma^{(1)}_Q(\xi_2))\| \|\sigma^{(1)}_Q(\xi_1) - \sigma^{(1)}_Q(\xi_2)\| \]

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By corollary (2),
\[ \sup_{0 \leq \alpha \leq 1} \| \nabla g(\xi_2, \alpha \sigma^{(1)}_{\mathcal{Q'}}(\xi_2) + (1 - \alpha)\sigma^{(2)}_{\mathcal{Q'}}(\xi_2)) \| \leq \Lambda; \]
thus,
\[ \| f(\xi_2, \sigma^{(2)}_{\mathcal{Q'}}(\xi_2)) - f(\xi_2, \sigma^{(1)}_{\mathcal{Q'}}(\xi_2)) \| \leq \Lambda \| \sigma^{(1)}_{\mathcal{Q'}}(\xi_2) - \sigma^{(2)}_{\mathcal{Q'}}(\xi_2) \|. \]
In fact, we have a better estimate than (4.145). By lemma (3.1) and theorem (2.11), for any \( \|\xi\| \leq \kappa, \|\eta\| \leq \kappa; \)

\[
\| \nabla_\eta f(\xi, \eta) - \nabla_\eta f(0, 0) \| \\
\leq \sup_{0 \leq \alpha \leq 1} \| \nabla \nabla_\eta f(\alpha(0, 0) + (1 - \alpha)(\xi, \eta)) \| (\xi, \eta) \| \leq 2\kappa\Lambda_\ast.
\]

Moreover, since \( \hat{N} \) is a locally invariant subbundle under \( TF_3 \),

\[
\nabla_\eta f(0, 0) = 0.
\]

Thus,

\[
\| \nabla_\eta f(\xi, \eta) \| \leq 2\kappa\Lambda_\ast,
\]

which is a much better estimate than (4.145). By (4.142; 4.143; 4.146),

\[
\| \xi - \xi_2 \| \leq \frac{\Lambda \| \nabla_\xi f(0, 0) \|^{-1} \|}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} \| \sigma^{(1)}_{Q'}(\xi_2) - \sigma^{(2)}_{Q'}(\xi_2) \|. \quad (4.147)
\]

By (4.141; 4.147)

\[
\| (G\sigma^{(1)}_Q(x) - (G\sigma^{(2)}_Q(x)) \| \leq \lambda \| \sigma^{(1)}_{Q'}(\xi_2) - \sigma^{(2)}_{Q'}(\xi_2) \|, \quad (4.148)
\]

where

\[
\lambda \equiv \left[ 2\kappa\Lambda_\ast + \zeta(\| \nabla_\eta g(0, 0) \| + 2\kappa\Lambda_\ast) \right] \frac{\Lambda \| \nabla_\xi f(0, 0) \|^{-1} \|}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} \\
+ \| \nabla_\eta g(0, 0) \| + 2\kappa\Lambda_\ast. \quad (4.149)
\]

By (4.139),

\[
x = f(\xi_2, \sigma^{(2)}_{Q'}(\xi_2)).
\]

Then, by (4.118)

\[
\xi_2 = [\nabla_\xi f(0, 0)]^{-1} x - H(\xi_2),
\]

where \( H(\xi_2) \) has the estimate (4.116). Thus,

\[
\| \xi_2 \| \leq \|[\nabla_\xi f(0, 0)]^{-1} \| \| x \| + (\kappa C_\kappa + \zeta C_\zeta)\Lambda \| \xi_2 \|;
\]

i.e.,

\[
\| \xi_2 \| \leq \frac{\|[\nabla_\xi f(0, 0)]^{-1} \|}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} \| x \|. \quad (4.150)
\]

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By (4.148) and (4.150),

$$\| (G\sigma^{(1)})_Q(x) - (G\sigma^{(2)})_Q(x) \| / \| x \| \leq \frac{\lambda \| \nabla f(0,0)^{-1} \|}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} \| \sigma^{(1)}_Q(\xi_2) - \sigma^{(2)}_Q(\xi_2) \| / \| \xi_2 \|. \quad (4.151)$$

The largest term in (4.149) is $\| \nabla g(0,0) \|$. By (4.135),

$$\| \nabla f(0,0)^{-1} \| \| \nabla g(0,0) \| < 1/2.$$ 

Thus, when $\kappa$ and $\zeta$ are sufficiently small,

$$\frac{\lambda \| \nabla f(0,0)^{-1} \|}{1 - \Lambda(\kappa C_\kappa + \zeta C_\zeta)} < 3/4. \quad (4.152)$$

Therefore,

$$\| (G\sigma^{(1)})_Q(x) - (G\sigma^{(2)})_Q(x) \| / \| x \| < \frac{3}{4} \| \sigma^{(1)}_Q(\xi_2) - \sigma^{(2)}_Q(\xi_2) \| / \| \xi_2 \|. \quad (4.153)$$

Notice that the estimate (4.153) is valid for any $\sigma^{(i)} \in \Sigma_\zeta$, $i = 1, 2$; $Q \in M$, $x \in E_\kappa(Q)$, $x \neq 0$. First taking supremum with respect to

$$\{ Q' \in M, \xi_2 \in E_\kappa(Q'), \xi_2 \neq 0 \}$$

on the right hand side of (4.153), and then taking supremum with respect to

$$\{ Q \in M, x \in E_\kappa(Q), x \neq 0 \}$$

on the left hand side of (4.153), we have

$$d(G\sigma^{(1)}, G\sigma^{(2)}) \leq \frac{3}{4} d(\sigma^{(1)}, \sigma^{(2)});$$

i.e., $G$ is a contraction on $\Sigma_\zeta$. This completes the proof of the lemma. ♣

**Corollary 5** $G$ has a unique fixed point $\sigma^*$ in $\Sigma_\zeta$.

Proof: By lemma (4.14), $\Sigma_\zeta$ is a complete metric space with respect to the distance $d(\sigma, \sigma')$. By lemma (4.17), $G$ is a contraction on $\Sigma_\zeta$ under the distance $d(\sigma, \sigma')$. Thus, $G$ has a unique fixed point $\sigma^*$ in $\Sigma_\zeta$. ♣
Lemma 4.18 For any $Q \in M$, $\sigma^*_Q$ is a $C^m$ function.

Proof: Next subsection.

Lemma 4.19 Graph $\sigma^*_Q$ satisfies the overflowing invariance condition for long time $t \geq T$:

$$\text{Graph } \sigma^*_Q \subset F^T_\delta(\text{graph } \sigma^*_Q),$$

$$Q' = F^{-t}_\delta(Q).$$

Proof: By corollary (5) and the definition of the graph transform $G$:

$$\text{Graph } \sigma^*_Q \subset F^T_\delta(\text{graph } \sigma^*_Q), \quad Q_1 = F^{-T}_\delta(Q).$$

Moreover, if $\sigma$ satisfies (4.154) for any $Q \in M$, then $\sigma$ is a fixed point of $G$.

For any $t \geq T$, define:

$$\sigma^{(s,t)} = \{\sigma^{(s,t)}_Q\}_{Q \in M},$$

$$\sigma^{(s,t)}_Q : E_\kappa(Q) \mapsto N_\kappa(Q),$$

$$(4.155)$$

$$\sigma^{(s,t)}_Q(x) = g^t(\xi, \sigma^*_Q(\xi)), \quad x = f^t(\xi, \sigma^*_Q(\xi)), \quad Q' = F^{-t}_\delta(Q),$$

where,

$$F^t_\delta : (\xi, \eta) \mapsto (x, y) = (f^t(\xi, \eta), g^t(\xi, \eta)).$$

By lemma (4.15), the map (4.155) is well-defined. By lemma (4.16),

$$\sigma^{(s,t)} \in \Sigma_\zeta.$$ \hfill (4.156)

In fact,

$$\text{graph } \sigma^{(s,t)}_Q = F^T_\delta(\text{graph } \sigma^*_Q) \cap \left( E_\kappa(Q) \oplus N_\kappa(Q) \right),$$

$$Q' = F^{-t}_\delta(Q).$$

By (4.154),

$$F^T_\delta(\text{graph } \sigma^*_Q) \subset F^T_\delta F^{-T}_\delta(\text{graph } \sigma^*_Q), \quad Q'_1 = F^{-T}_\delta(Q').$$
That is, \( \sigma^{(s,t)} \) also satisfies (4.154), which, together with (4.156), implies that \( \sigma^{(s,t)} \) is also a fixed point of \( G \). By the uniqueness of the fixed point,

\[
\sigma^{(s,t)} = \sigma^*.
\]

Thus,

\[
\begin{align*}
\text{graph } \sigma_Q^* &\subset F^t_\delta(\text{graph } \sigma_{Q'}^*), \\
Q' &\equiv F^{-t}_\delta(Q).
\end{align*}
\]

This completes the proof of the lemma.

We know that \( \sigma^* \in \Sigma_{\zeta} \), \( \sigma^*_Q \) is defined in \( E_{\kappa}(Q) \); moreover, \( \kappa \) satisfies the constraint (4.137). Therefore, \( \sigma^* \) is labeled by \((\zeta, \kappa)\). In fact, we can denote \( \sigma^* \) in detail by \( \sigma^*_{(\zeta, \kappa)} \).

Lemma 4.20 For any \( Q \in M \), graph \( \sigma_Q^* \) is tangent to \( E(Q) \) at \( Q \).

Proof: For any fixed \( \delta \) which is sufficiently small, take \( \zeta \) arbitrarily small, and let \( \kappa \) satisfies the constraint (4.137); then the argument for lemmas (4.15; 4.16; 4.17) can be carried through. For any fixed \( \zeta_0 \) and \( \kappa_0 \) which satisfy the constraint (4.137), by the uniqueness of the fixed point of \( G \) in \( \Sigma_{\zeta_0} \) defined in \( E_{\kappa_0} \oplus \hat{N}_{\kappa_0} \):

\[
\text{graph } (\sigma^*_{(\zeta, \kappa)})_Q \subset \text{graph } (\sigma^*_{(\zeta_0, \kappa_0)})_Q,
\]

for any \( Q \in M \), \( \zeta \in (0, \zeta_0) \); \( \zeta \) and \( \kappa \) satisfy the constraint (4.137). Since we can take \( \zeta \) arbitrarily small; moreover, by lemma (4.18), \( (\sigma^*_{(\zeta_0, \kappa_0)})_Q \) is a differentiable function, we have

\[
\nabla_x (\sigma^*_{(\zeta_0, \kappa_0)})_Q(0) = 0;
\]

i.e., graph \( (\sigma^*_{(\zeta_0, \kappa_0)})_Q \) is tangent to \( E(Q) \) at \( Q \). This completes the proof of the lemma.

Lemma 4.21 When \( \zeta_1 \) and \( \kappa_1 \) are sufficiently small, graph \( (\sigma^*_{(\zeta_1, \kappa_1)})_Q \) satisfies the local invariance condition for short time \(-T < t < T\): For any \( Q \in M \), such that, \( F^t_-\delta(Q) \in M \); either

\[
\text{graph } (\sigma^*_{(\zeta_1, \kappa_1)})_Q \subset F^t_-\delta(\text{graph } (\sigma^*_{(\zeta_1, \kappa_1)})_Q'),
\]

or

\[
F^t_-\delta(\text{graph } (\sigma^*_{(\zeta_1, \kappa_1)})_Q) \subset \text{graph } (\sigma^*_{(\zeta_1, \kappa_1)})_Q;
\]

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where,

\[ Q' = F_{\delta}^{-t}(Q). \]

Moreover, for any \( Q \in M \), such that, \( F_{\delta}^{-t}(Q) \in M, t \in (-T, T); \)

\[ F_{\delta}^{t}(\text{graph}(\sigma_{(t, \kappa_{1})}^{*})Q') \subset \text{graph}(\sigma_{(t, \kappa_{2})}^{*})Q, \quad (4.157) \]

where \( \zeta_{2} = A\zeta_{1}, \kappa_{2} = B\kappa_{1}; A \) and \( B \) are constants.

Proof: For any \(-T < t < T\), define the family of continuous maps \( \sigma^{t} \): For any \( Q \in M \), such that, \( F_{\delta}^{-t}(Q) \in M; F_{\delta}^{t}(\text{graph}(\sigma_{(t, \kappa_{1})}^{*})F_{\delta}^{-t}(Q)) \) defines a continuous map,

\[ \sigma_{Q}^{t} : U(Q) \subset E(Q) \mapsto \hat{N}(Q), \]

where \( U(Q) \) is the domain of definition; for any \( x \in U(Q), \)

\[ x = f^{t}(\xi, \eta_{1}(\xi)), \quad \sigma_{Q}^{t}(x) = g^{t}(\xi, \eta_{1}(\xi)), \]

\[ \eta_{1}(\xi) = (\sigma_{(t, \kappa_{1})}^{*})_{\text{graph}}(\hat{N}(Q)), \quad Q' = F_{\delta}^{-t}(Q); \]

where \((\xi, \eta)\) is the coordinate for \( E(Q') \oplus \hat{N}(Q')\), \((x, y)\) is the coordinate for \( E(Q) \oplus N(Q)\), and \( F_{\delta}^{t} \) has the representation:

\[ F_{\delta}^{t} : (\xi, \eta) \mapsto (x, y) = (f^{t}(\xi, \eta), g^{t}(\xi, \eta)). \]

The same argument as for lemma (4.15) can be applied to the map:

\[ \xi \mapsto f^{t}(\xi, \eta_{1}(\xi)), \]

to show that it is an 1-1 map. In fact,

\[ f^{t}(\xi, \eta_{1}(\xi)) = (\nabla_{\xi}f^{t}(0, 0))[\xi + H^{t}(\xi)], \quad (4.158) \]

where

\[ \|H^{t}(\xi)\| \leq (\kappa_{1}\tilde{C}_{\kappa_{1}} + \zeta_{1}\tilde{C}_{\zeta_{1}})\tilde{\Lambda}\|\xi\|, \quad \forall \xi \in E_{\kappa_{1}}(Q'), \forall t \in (-T, T); \]

\[ \|H^{t}(\xi^{(2)}) - H^{t}(\xi^{(1)})\| \leq (\kappa_{1}\tilde{C}_{\kappa_{1}} + \zeta_{1}\tilde{C}_{\zeta_{1}})\tilde{\Lambda}\|\xi^{(2)} - \xi^{(1)}\|, \quad \forall \xi^{(1)}, \xi^{(2)} \in E_{\kappa_{1}}(Q'), \forall t \in (-T, T); \]

in which \( \tilde{\Lambda} = \tilde{\Lambda}(T), \tilde{C}_{\kappa_{1}} = \tilde{C}_{\kappa_{1}}(T), \tilde{C}_{\zeta_{1}} = \tilde{C}_{\zeta_{1}}(T) \). By (4.51), there exists a constant \( R_{1} = R_{1}(T) \), such that

\[ \sup_{t \in (-T,T); Q \in M} \|\nabla_{\xi}f^{t}(0, 0)\| = \sup_{t \in (-T,T); Q \in M} \|\nabla_{\xi}f^{t}(0, 0)^{-1}\| \leq R_{1}. \quad (4.159) \]
By (4.158;4.159),
\[
\| f^{t}(\xi, \eta_{1}(\xi)) \| \leq 2R_{1}\| \xi \|; \ \forall \xi \in E_{\kappa_{1}}(Q'), \ \forall t \in (-T, T).
\] (4.160)

Let \( \kappa_{2} = 2R_{1}\kappa_{1} \), then
\[
\| f^{t}(\xi, \eta_{1}(\xi)) \| \leq \kappa_{2}; \ \forall \xi \in E_{\kappa_{1}}(Q'), \ \forall t \in (-T, T).
\] (4.161)

The same argument as for lemma (4.15) shows that
\[
\| g^{t}(\xi, \eta_{1}(\xi)) \| \leq (2\kappa_{1}\tilde{\Lambda}_{\ast} + \zeta_{1}\tilde{\Lambda})\| \xi \|; \ \forall \xi \in E_{\kappa_{1}}(Q'), \ \forall t \in (-T, T);
\]
where \( \tilde{\Lambda}_{\ast} = \tilde{\Lambda}_{\ast}(T) \). When \( \kappa_{1} \) and \( \zeta_{1} \) are sufficiently small,
\[
\| g^{t}(\xi, \eta_{1}(\xi)) \| \leq \kappa_{2}; \ \forall \xi \in E_{\kappa_{1}}(Q'), \ \forall t \in (-T, T).
\] (4.162)

The same argument as for lemma (4.16) shows that (4.133):
\[
\| \sigma^{Q}(x_{1}) - \sigma^{Q}(x_{2}) \| \leq \left[ 2\kappa_{1}\tilde{\Lambda}_{\ast} + \zeta_{1}(\| \nabla_{\xi}g^{t}(0, 0) \| + 2\kappa_{1}\tilde{\Lambda}_{\ast}) \right] \cdot \frac{\| \nabla_{\xi}f^{t}(0, 0) \|}{1 - \tilde{\Lambda}(\kappa_{1}\tilde{C}_{\kappa_{1}} + \zeta_{1}\tilde{C}_{\zeta_{1}})} \| x_{1} - x_{2} \|; \ \forall t \in (-T, T), \ x_{1}, x_{2} \in U(Q).
\] (4.163)

By (4.48;4.49), there exists a constant \( R_{2} = R_{2}(T) \), such that
\[
\sup_{t \in (-T, T); Q \in M} \| \nabla_{\eta}g^{t}(0, 0) \| \leq R_{2}.
\] (4.164)

By (4.159;4.163;4.164), when \( \kappa_{1} \) and \( \zeta_{1} \) are sufficiently small,
\[
\| \sigma^{Q}(x_{1}) - \sigma^{Q}(x_{2}) \| \leq 2R_{1}R_{2}\kappa_{1}\| x_{1} - x_{2} \|, \ \forall t \in (-T, T); \ x_{1}, x_{2} \in U(Q).
\]

Let \( \zeta_{2} = 2R_{1}R_{2}\zeta_{1} \), then
\[
\| \sigma^{Q}(x_{1}) - \sigma^{Q}(x_{2}) \| \leq \zeta_{2}\| x_{1} - x_{2} \|, \ \forall t \in (-T, T); \ x_{1}, x_{2} \in U(Q).
\] (4.165)

When \( \zeta_{1} \) and \( \kappa_{1} \) are sufficiently small,
\[
\text{graph} \ (\sigma^{*}_{(\zeta_{1}, \kappa_{1})})_{Q_{1}} \subset F_{\delta}^{T}(\text{graph} \ (\sigma^{*}_{(\zeta_{1}, \kappa_{1})})_{Q_{1}}),
\]\[
\forall Q_{1} \in M, \ Q_{1}' = F_{\delta}^{T}(Q_{1}).
\] (4.166)
Then, for any $Q \in M$, such that, $F_{\delta}^{-t}(Q) \in M$;

$$F_{\delta}^{t}(\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q'}) \subset F_{\delta}^{t}F_{\delta}^{T}(\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q''})$$

$$= F_{\delta}^{T}F_{\delta}^{t}(\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q'})$$

$$Q' = F_{\delta}^{-t}(Q), \ Q'' = F_{\delta}^{-T}(Q').$$

Notice that, for any $Q \in M$, such that, $F_{\delta}^{-t}(Q) \in M$;

$$F_{\delta}^{t}(\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q'}) = \text{graph} \ \sigma_{Q'}^{t},$$

$$Q' = F_{\delta}^{-t}(Q).$$

Then, for any $Q \in M$, such that, $F_{\delta}^{-t}(Q) \in M$;

$$\text{graph} \ \sigma_{Q'}^{t} \subset F_{\delta}^{T}(\text{graph} \ \sigma_{Q_{0}}^{t}),$$

$$Q_{0} = F_{\delta}^{-T}(Q).$$

Consider the space $\Sigma_{\zeta_{2}}$ defined in $E_{\kappa_{2}} \oplus \tilde{N}_{\kappa_{2}}$, when $\zeta_{2}$ and $\kappa_{2}$ are sufficiently small, (Equivalently, when $\zeta_{1}$ and $\kappa_{1}$ are sufficiently small.) there exists a unique fixed point $\sigma_{(\zeta_{2},\kappa_{2})}^{*}$. By relations (4.161;4.162;4.165; 4.167), for any $Q \in M$, such that, $F_{\delta}^{-t}(Q) \in M$;

$$\text{graph} \ \sigma_{Q'}^{t} \subset (\sigma_{(\zeta_{2},\kappa_{2})}^{*})_{Q};$$

i.e.,

$$F_{\delta}^{t}(\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q'}) \subset (\sigma_{(\zeta_{2},\kappa_{2})}^{*})_{Q},$$

$$Q' = F_{\delta}^{-t}(Q).$$

Thus, relation (4.157) is proved. Without loss of generality, we assume that $\zeta_{2} \geq \zeta_{1}, \ \kappa_{2} \geq \kappa_{1}$. Then, by (4.166)

$$\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q} \subset (\sigma_{(\zeta_{2},\kappa_{2})}^{*})_{Q}, \ \forall Q \in M.$$  \hfill (4.169)

By (4.168;4.169), for any $t \in (-T, T)$, any $Q \in M$, such that $F_{\delta}^{-t}(Q) \in M$; either

$$\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q} \subset F_{\delta}^{t}(\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q'}),$$

or

$$F_{\delta}^{t}(\text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q'}) \subset \text{graph} \ (\sigma_{(\zeta_{1},\kappa_{1})}^{*})_{Q};$$

where $Q' = F_{\delta}^{-t}(Q)$. This completes the proof of the lemma. ♦
Definition 10 Define the 1-dimensional curve:

\[ f^E(Q) \equiv \bigcup_{t \in [0,T_Q)} F_{\delta}^{-t}(\text{graph } \sigma^*_{F_{\delta}^t(Q)}), \quad \forall Q \in M; \quad (4.170) \]

where \( T_Q = \min\{T, t_Q\} \), \( t_Q \) is the largest time \( t \), such that

\[ \bigcup_{\tau \in [0,t)} F_{\delta}^\tau(Q) \subset M. \]

Let \( \sigma^E \) be the family of continuous maps

\[ \sigma^E \equiv \{\sigma^E_Q\}_{Q \in M}, \]

\[ \sigma^E_Q : V_Q \subset E(Q) \mapsto \hat{N}(Q), \]

(Where \( V_Q \) is the domain of definition.) such that:

\[ f^E(Q) \equiv \text{graph } \sigma^E_Q, \quad \forall Q \in M. \]

Corollary 6 If we denote \( \sigma^* \) in detail by \( \sigma^*_\zeta, \kappa \), then when \( \zeta \) and \( \kappa \) are sufficiently small,

\[ f^E(Q) \equiv \text{graph } \sigma^E_Q \subset \text{graph } (\sigma^*_\zeta, \kappa)_Q, \quad \forall Q \in M; \quad (4.171) \]

where \( A \) and \( B \) are two constants.

Proof: Take \((\zeta, \kappa)\) as \((\zeta_1, \kappa_1)\) in lemma (4.21), then the corollary follows immediately from lemma (4.21) and definition (10). ♠

By lemma (4.19), we have:

Corollary 7 \( f^E(Q) \) has another equivalent representation:

\[ f^E(Q) = \bigcup_{t \in [0,t_Q)} F_{\delta}^{-t}(\text{graph } \sigma^*_{F_{\delta}^t(Q)}), \quad \forall Q \in M; \quad (4.172) \]

Proof: If \( t_Q \leq T \), representation (4.172) is identical with representation (4.170). If \( t_Q > T \), then \( T_Q = T \). Nevertheless, by lemma (4.19), when \( T \leq t < t_Q \):

\[ F_{\delta}^{-t}(\text{graph } \sigma^*_{F_{\delta}^t(Q)}) \subset \text{graph } \sigma^*_{F_{\delta}^t(Q)}. \]

Thus,

\[ \bigcup_{t \in [0,t_Q)} F_{\delta}^{-t}(\text{graph } \sigma^*_{F_{\delta}^t(Q)}) = \bigcup_{t \in [0,T_Q)} F_{\delta}^{-t}(\text{graph } \sigma^*_{F_{\delta}^t(Q)}). \]

This completes the proof of the corollary. ♠
Lemma 4.22 \( f^E(Q) \) satisfies the overflowing invariance condition:
\[
f^E(Q) \subset F_\delta^t(f^E(F_\delta^{-t}(Q))), \quad t \geq 0.
\]

Proof: By definition (10),
\[
t_{F_\delta^{-t}(Q)} = t_Q + t.
\]
Then, by (4.172),
\[
f^E(Q) = \bigcup_{\tau \in [0, t_Q)} F_\delta^{-\tau}(\text{graph } \sigma_{F_\delta}(Q)),
\]
\[
f^E(F_\delta^{-t}(Q)) = \bigcup_{\tau \in [0, t_Q+t)} F_\delta^{-\tau}(\text{graph } \sigma_{F_\delta}(F_\delta^{-t}(Q)))
\]
\[
= \bigcup_{\tau \in [-t, t_Q)} F_\delta^{-(t+\tau)}(\text{graph } \sigma_{F_\delta}(Q))
\]
\[
= F_\delta^{-(t)} \cdot \bigcup_{\tau \in [-t, t_Q)} F_\delta^{-\tau}(\text{graph } \sigma_{F_\delta}(Q)).
\]
Thus,
\[
F_\delta^t(f^E(F_\delta^{-t}(Q))) = \bigcup_{\tau \in [-t, t_Q]} F_\delta^{-\tau}(\text{graph } \sigma_{F_\delta}(Q)).
\]
By (4.173;4.174),
\[
f^E(Q) \subset F_\delta^t(f^E(F_\delta^{-t}(Q))), \quad t \geq 0.
\]
This completes the proof of the lemma.

The family \( \{f^E(Q)\}_{Q \in M} \) is the family of fibers stated in theorem (4.4).

By lemma (4.20), \( f^E(Q) \) intersects \( M \) transversally at \( M \). By corollary (5), two fibers \( f^E(Q_1), f^E(Q_2) \) are either disjoint or identical. Up to now, we have completed the proof of the theorem (4.4).

4.6 Smoothness of the Fiber \( f^E(Q) \) as a Submanifold

In this subsection, we study the smoothness of each fiber \( f^E(Q) \) as a curve.

By corollary (5) and the definition of the graph transform \( G (4.107) \): For any \( Q \in M \), any \( x \in E_\kappa(Q) \):
\[
\sigma_Q^*(x) = g(\xi, \sigma_Q^*(\xi)),
\]
\[
x = f(\xi, \sigma_Q^*(\xi)),
\]
\[
Q' = F_\delta^{-T}(Q).
\]
For any \( Q \in M \), if \( \sigma_Q^* \) is a \( C^1 \) function, then
\[
\nabla_x \sigma_Q^* \in C^0(E_\alpha(Q), L(E(Q), \hat{N}(Q))).
\]
Define the space:
\[
\Sigma_1 \equiv \left\{ u \mid u \equiv \{ u_Q \}_{Q \in M}, \ u_Q \in C^0(E_\alpha(Q), L(E(Q), \hat{N}(Q))) \right\}.
\]
For any \( u \in \Sigma_1 \), define the norm:
\[
\| u \| \equiv \sup_{Q \in M, x \in E_\alpha(Q)} \| u_Q(x) \|, \quad (4.176)
\]
where \( \| u_Q(x) \| \) is the linear operator norm.

**Lemma 4.23** \( \Sigma_1 \) is a complete metric space under the norm defined in (4.176).

Proof: Along the line laid down in the proof of the lemma (3.5), this lemma follows immediately. ♣

Formally differentiating (4.175) with respect to \( x \), we have:
\[
\nabla_x \sigma_Q^*(x) = \left[ R_1 + R_2 \cdot \nabla_\xi \sigma_Q^*(\xi) \right] \cdot \left[ R_3 + R_4 \cdot \nabla_\xi \sigma_Q^*(\xi) \right]^{-1}, \quad (4.177)
\]
where
\[
R_1 \equiv \nabla_\xi g(\xi, \sigma_Q^*(\xi)), \quad R_2 \equiv \nabla_\eta g(\xi, \sigma_Q^*(\xi));
\]
\[
R_3 \equiv \nabla_\xi f(\xi, \sigma_Q^*(\xi)), \quad R_4 \equiv \nabla_\eta f(\xi, \sigma_Q^*(\xi)).
\]

Define the map \( H \) in \( \Sigma_1 \): For any \( u \in \Sigma_1 \),
\[
[H(u)]_Q(x) \equiv \left[ R_1 + R_2 \cdot u_Q(\xi) \right] \cdot \left[ R_3 + R_4 \cdot u_Q(\xi) \right]^{-1}. \quad (4.178)
\]

**Definition 11** Let \( u^0 \) be the element of \( \Sigma_1 \), such that
\[
u_Q^0(x) = 0, \ \forall Q \in M, \ \forall x \in E_\alpha(Q);
\]
define inductively:
\[
u^{j+1} = H(u^j), \ j = 0, 1, 2, \ldots.
\]

**Lemma 4.24** \( \| u^j \| < \zeta \), for all \( j \).
Proof: We will prove this lemma by induction. We know that $\| u^0 \| < \zeta$. Assuming $\| u^j \| < \zeta$, we will show that $\| u^{j+1} \| < \zeta$. Notice that

$$
\left[ R_3 + R_4 \cdot u_{Q'}^j(\xi) \right]^{-1} = \left[ I + R_3^{-1} \cdot R_4 \cdot u_{Q'}^j(\xi) \right]^{-1} \cdot R_3^{-1}.
$$

(4.179)

Moreover,

$$
\| R_3^{-1} \cdot R_4 \cdot u_{Q'}^j(\xi) \| \leq \zeta \Lambda^2.
$$

(4.180)

By (4.179; 4.180),

$$
\| [R_3 + R_4 \cdot u_{Q'}^j(\xi)]^{-1} \| \leq \frac{\| R_3^{-1} \|}{1 - \zeta \Lambda^2}.
$$

(4.181)

By (4.178; 4.181),

$$
\| u_{Q'}^{j+1}(x) \| = \| [H(u^j)]_Q(x) \|
\leq \frac{\| R_3^{-1} \|}{1 - \zeta \Lambda^2} \left[ \| R_1 \| + \| R_2 \| \| u_{Q'}^j(\xi) \| \right].
$$

(4.182)

By (4.129; 4.130),

$$
\| R_1 \| \leq 2 \kappa \Lambda_\ast,
\| R_2 \| \leq \| \nabla \eta g(0,0) \| + 2 \kappa \Lambda_\ast.
$$

(4.183)

(4.184)

By lemma (3.1) and theorem (2.11),

$$
\nabla_{\xi}f(\xi, \sigma_{\ast}^Q(\xi)) = \nabla_{\xi}f(0,0) + r(\xi),
$$

where $\| r(\xi) \| \leq 2 \kappa \Lambda_\ast$. Then

$$
\left[ \nabla_{\xi}f(\xi, \sigma_{\ast}^Q(\xi)) \right]^{-1} = \left[ I + [\nabla_{\xi}f(0,0)]^{-1} \cdot r(\xi) \right]^{-1} \cdot [\nabla_{\xi}f(0,0)]^{-1},
$$

where $I$ denotes the identity map. Thus,

$$
\| R_3^{-1} \| \leq \frac{\| [\nabla_{\xi}f(0,0)]^{-1} \|}{1 - 2 \kappa \Lambda \Lambda_\ast}.
$$

(4.185)

By (4.182; 4.183; 4.184; 4.185),

$$
\| u_{Q'}^{j+1}(x) \| \leq \lambda_1 + \lambda_2 \zeta,
$$

(4.186)
where
\[
\lambda_1 \equiv \frac{2\kappa\Lambda_* \|
\nabla_\xi f(0,0)\|^{-1}}{(1 - \zeta\Lambda^2)(1 - 2\kappa\Lambda\Lambda_*)},
\]
\[
\lambda_2 \equiv \frac{\|
\nabla_\xi f(0,0)\|^{-1}(\|
\nabla_\eta g(0,0)\| + 2\kappa\Lambda_*)}{(1 - \zeta\Lambda^2)(1 - 2\kappa\Lambda\Lambda_*)}.
\]
By (4.135), when \(\zeta\) and \(\kappa\) are sufficiently small,
\[
\lambda_2 < \frac{3}{4}.
\]
Then choose \(\kappa\) small enough such that
\[
\lambda_1 < \frac{1}{8}\zeta.
\]
Thus,
\[
\| u_j^{j+1}(x) \| < \frac{7}{8}\zeta.
\]
The relation (4.187) is true for any \(Q \in M\) and any \(x \in E_\kappa(Q)\); then
\[
\| u_j^{j+1} \| \leq \frac{7}{8}\zeta.
\]
This completes the proof of the lemma. ♣

**Lemma 4.25** \(\| u_j^{j+1} - u_j^{j} \| < \gamma \| u_j^{j} - u_j^{j-1} \|, \text{ for all } j; 0 < \gamma < 1.\)

Proof: We know that
\[
\left[ H(u^j) \right]_Q(x) - \left[ H(u^{j-1}) \right]_Q(x) \\
= \left[ R_1 + R_2 \bullet u^j_Q(\xi) \right] \bullet \left[ R_3 + R_4 \bullet u^j_Q(\xi) \right]^{-1} \\
- \left[ R_1 + R_2 \bullet u^{j-1}_Q(\xi) \right] \bullet \left[ R_3 + R_4 \bullet u^{j-1}_Q(\xi) \right]^{-1} \\
= \left( \left[ R_1 + R_2 \bullet u^j_Q(\xi) \right] - \left[ R_1 + R_2 \bullet u^{j-1}_Q(\xi) \right] \right) \\
\bullet \left[ R_3 + R_4 \bullet u^j_Q(\xi) \right]^{-1} + \left[ R_1 + R_2 \bullet u^{j-1}_Q(\xi) \right] \\
\bullet \left\{ \left[ R_3 + R_4 \bullet u^j_Q(\xi) \right]^{-1} \bullet \left[ R_3 + R_4 \bullet u^{j-1}_Q(\xi) \right]^{-1} \right\} \\
- \left[ R_3 + R_4 \bullet u^j_Q(\xi) \right] \bullet \left[ R_3 + R_4 \bullet u^{j-1}_Q(\xi) \right]^{-1} \}
\[
\begin{align*}
= & \left\{ R_2 \bullet \left[ u_{Q'}(\xi) - u_{Q'}^{-1}(\xi) \right] \right\} \bullet \left[ R_3 + R_4 \bullet u_{Q'}(\xi) \right]^{-1} \\
+ & \left[ R_1 + R_2 \bullet u_{Q'}^{-1}(\xi) \right] \bullet \left[ R_3 + R_4 \bullet u_{Q'}(\xi) \right]^{-1} \\
- & \left\{ R_4 \bullet \left[ u_{Q'}^{i-1}(\xi) - u_{Q'}^{i}(\xi) \right] \right\} \bullet \left[ R_3 + R_4 \bullet u_{Q'}^{-1}(\xi) \right]^{-1}.
\end{align*}
\] (4.188)

By (4.181;4.183;4.184;4.185;4.188),
\[
\| \left[ H(u^j) \right]_{Q'}(x) - \left[ H(u^{j-1}) \right]_{Q'}(x) \| \leq \lambda_3 \| u_{Q'}^{i}(\xi) - u_{Q'}^{i-1}(\xi) \|,
\] (4.189)
where
\[
\lambda_3 \equiv \frac{\| \nabla g(0,0) \| + 2\kappa \Lambda^*_s}{(1 - \zeta \Lambda^2)(1 - 2\kappa \Lambda^*_s)} \left[ \frac{\| \nabla f(0,0) \|^{-1}}{(1 - \zeta \Lambda^2)(1 - 2\kappa \Lambda^*_s)} \right] R_3 \\
+ \left[ 2\kappa \Lambda^*_s + \zeta \| \nabla g(0,0) \| + 2\kappa \Lambda^*_s \right] \| R_4 \|
\] by (4.135), when \( \zeta \) and \( \kappa \) are sufficiently small,
\[
\lambda_3 < \frac{3}{4}.
\]

Taking supremum with respect to \( Q' \) and \( \xi \) in (4.189), we have
\[
\| u^{j+1} - u^j \| = \| H(u^j) - H(u^{j-1}) \| < \gamma \| u^j - u^{j-1} \|, \quad \gamma = \frac{4}{5}.
\]

This completes the proof of the lemma. ♣

**Corollary 8** The sequence \( \{ u^j \} \) converges to a solution \( u^* \) of the equation \( H(u) = u \). In particular, for any \( Q \in M \),
\[
u_{Q'}^* \in C^0(E_\kappa(Q), L(E(Q), \hat{N}(Q))); \quad \| u^* \| \leq \zeta.
\] (4.190)

**Theorem 4.5** For any \( Q \in M \), any \( x \in E_\kappa(Q) \); \( \nabla_x \sigma^*_Q(x) \) exists and equals \( u_{Q'}(x) \). Therefore, \( \nabla_x \sigma^*_Q \in C^0(E_\kappa(Q), L(E(Q), \hat{N}(Q))) \); moreover, \( \| \nabla_x \sigma^* \| \leq \zeta \).
Proof: For any \( Q \in M \), any \( x \in E_\kappa(Q) \); define an increasing non-negative function:
\[
\Delta_{(Q,x)} : (0,1) \rightarrow R,
\]
\[
\Delta_{(Q,x)}(a) \equiv \sup_{\bar{x} \in E_\kappa(Q), 0 < \|\bar{x} - x\| < a} \frac{\|\sigma^*_Q(\bar{x}) - \sigma^*_Q(x) - u^*_Q(x) \cdot (\bar{x} - x)\|}{\|\bar{x} - x\|}.
\]
By corollary (5) and (4.190), we know that for any \( Q \in M \), any \( x \in E_\kappa(Q) \), and any \( a \in (0,1) \):
\[
\Delta_{(Q,x)}(a) \leq 2\zeta.
\]
For any \( Q \in M \), and \( x \in E_\kappa(Q) \); since \( \Delta_{(Q,x)}(a) \) is an increasing non-negative function, the limit
\[
\lim_{a \to 0} \Delta_{(Q,x)}(a)
\]
exists. Denote this limit by \( \Delta_{(Q,x)}(0) \). By (4.191),
\[
0 \leq \Delta_{(Q,x)}(0) \leq 2\zeta; \forall Q \in M, \forall x \in E_\kappa(Q).
\]
For any \( Q_1 \in M \), any \( x_1 \in E_\kappa(Q_1) \), define the sequence:
\[
\{(Q_j,x_j)\}, \quad j = 1,2,\ldots;
Q_j = F_\delta^{-T}(Q_{j+1}),
\]
\[
x_j = f(x_{j+1},\sigma^*_{Q_{j+1}}(x_{j+1})).
\]
To prove the theorem, we need to show that
\[
\Delta_{(Q_1,x_1)}(0) = 0.
\]
We will show that the inequality
\[
\Delta_{(Q_j,x_j)}(0) \leq \gamma_1 \Delta_{(Q_{j+1},x_{j+1})}(0),
\]
\[
\forall j = 1,2,\ldots; (0 < \gamma_1 < 1)
\]
is valid. Then by (4.192; 4.193),
\[
\Delta_{(Q_1,x_1)}(0) \leq 2\zeta \gamma_1^m, \quad \forall m \in Z^+.
\]
Thus, \( \Delta_{(Q_1,x_1)}(0) = 0 \).
Next, we prove the inequality (4.193). For any \( j \in \mathbb{Z}^+ \), there exists a small constant \( a_j \) such that, when \( \| \tilde{x}_j - x_j \| < a_j \), all the Taylor expansions below are valid. By corollary (8) and (4.175),

\[
\Pi_j \equiv \sigma_{Q_j}^*(\tilde{x}_j) - \sigma_{Q_j}^*(x_j) - u_{Q_j}^*(x_j) \cdot (\tilde{x}_j - x_j)
\]

\[
= g(\tilde{x}_{j+1}, \sigma_{Q_{j+1}}^*(\tilde{x}_{j+1})) - g(x_{j+1}, \sigma_{Q_{j+1}}^*(x_{j+1}))
\]

\[- \left[ H(u^*)\right]_{Q_j}(x_j) \cdot (\tilde{x}_j - x_j), 
\]

(4.194)

where

\[
\tilde{x}_j = f(\tilde{x}_{j+1}, \sigma_{Q_{j+1}}^*(\tilde{x}_{j+1})),
\]

(4.195)

\[
x_j = f(x_{j+1}, \sigma_{Q_{j+1}}^*(x_{j+1})),
\]

(4.196)

\[
[H(u^*)]_{Q_j}(x_j) = \left[R_1^{(j)} + R_2^{(j)} \cdot u_{Q_{j+1}}^*(x_{j+1})\right]
\]

\[- \left[R_3^{(j)} + R_4^{(j)} \cdot u_{Q_{j+1}}^*(x_{j+1})\right]^{-1},
\]

\[
R_1^{(j)} \equiv \nabla_\xi g(x_{j+1}, \sigma_{Q_{j+1}}^*(x_{j+1})),
\]

\[
R_2^{(j)} \equiv \nabla_\eta g(x_{j+1}, \sigma_{Q_{j+1}}^*(x_{j+1})),
\]

\[
R_3^{(j)} \equiv \nabla_\xi f(x_{j+1}, \sigma_{Q_{j+1}}^*(x_{j+1})),
\]

\[
R_4^{(j)} \equiv \nabla_\eta f(x_{j+1}, \sigma_{Q_{j+1}}^*(x_{j+1})).
\]

From (4.194), by Taylor expansions, we have

\[
\Pi_j = R_1^{(j)} \cdot (\tilde{x}_{j+1} - x_{j+1}) + R_2^{(j)} \cdot \left[\sigma_{Q_{j+1}}^*(\tilde{x}_{j+1}) - \sigma_{Q_{j+1}}^*(x_{j+1})\right]
\]

\[- \left[H(u^*)\right]_{Q_j}(x_j) \cdot (\tilde{x}_j - x_j) + O(\| \tilde{x}_{j+1} - x_{j+1} \|^2). 
\]

(4.197)

By (4.195; 4.196),

\[
\tilde{x}_j - x_j = f(\tilde{x}_{j+1}, \sigma_{Q_{j+1}}^*(\tilde{x}_{j+1})) - f(x_{j+1}, \sigma_{Q_{j+1}}^*(x_{j+1}))
\]

\[
= R_3^{(j)} \cdot (\tilde{x}_{j+1} - x_{j+1}) + R_4^{(j)} \cdot \left[\sigma_{Q_{j+1}}^*(\tilde{x}_{j+1})\right]
\]

\[- \sigma_{Q_{j+1}}^*(x_{j+1}) + O(\| \tilde{x}_{j+1} - x_{j+1} \|^2)
\]

\[
= \left[R_3^{(j)} + R_4^{(j)} \cdot u_{Q_{j+1}}^*(x_{j+1})\right] \cdot (\tilde{x}_{j+1} - x_{j+1})
\]

\[
+ R_4^{(j)} \cdot \left[\sigma_{Q_{j+1}}^*(\tilde{x}_{j+1}) - \sigma_{Q_{j+1}}^*(x_{j+1}) - u_{Q_{j+1}}^*(x_{j+1})\right]
\]

\[- (\tilde{x}_{j+1} - x_{j+1}) + O(\| \tilde{x}_{j+1} - x_{j+1} \|^2). 
\]

(4.198)
By (4.194;4.198),
\[
[H(u^*)]_{Q_j}(x_j) \cdot (\bar{x}_j - x_j) \\
= \left[ R_1^{(j)} + R_2^{(j)} \cdot u^*_{Q_j+1}(x_{j+1}) \right] \cdot (\bar{x}_{j+1} - x_{j+1}) \\
+ [H(u^*)]_{Q_j}(x_j) \cdot R_4^{(j)} \cdot \Pi_{j+1} + O(\| \bar{x}_{j+1} - x_{j+1} \|^2),
\]
(4.199)
where
\[
\Pi_{j+1} = \sigma^*_{Q_j+1}(\bar{x}_{j+1}) - u^*_{Q_j+1}(x_{j+1}) \cdot (\bar{x}_{j+1} - x_{j+1}).
\]
By (4.197;4.199),
\[
\Pi_j = \left( R_2^{(j)} - [H(u^*)]_{Q_j}(x_j) \cdot R_4^{(j)} \right) \cdot \Pi_{j+1} \\
+ O(\| \bar{x}_{j+1} - x_{j+1} \|^2).
\]
(4.200)
By (4.195; 4.196) and (4.132),
\[
\| \bar{x}_{j+1} - x_{j+1} \| \leq \frac{\| [\nabla \xi f(0,0)]^{-1} \|}{1 - (\kappa C_{\kappa} + \zeta C_{\zeta})\Lambda} \| \bar{x}_j - x_j \|. \quad (4.201)
\]
By (4.200;4.201),
\[
\| \Pi_j \| \leq \lambda_1 \frac{\| \Pi_{j+1} \|}{\| \bar{x}_j - x_j \|} + O(\| \bar{x}_{j+1} - x_{j+1} \|),
\]
(4.202)
where
\[
\lambda_1 = \frac{\| [\nabla \xi f(0,0)]^{-1} \|}{1 - (\kappa C_{\kappa} + \zeta C_{\zeta})\Lambda} \| R_2^{(j)} - [H(u^*)]_{Q_j}(x_j) \cdot R_4^{(j)} \|.
\]
(4.203)
By (4.129;4.130),
\[
\| R_1^{(j)} \| \leq 2\kappa \Lambda_\kappa, \\
\| R_2^{(j)} \| \leq \| \nabla \eta g(0,0) \| + 2\kappa \Lambda_\kappa.
\]
(4.204)
\( (4.205) \)
We know that
\[
\left[ R_3^{(j)} + R_4^{(j)} \cdot u^*_{Q_j+1}(x_{j+1}) \right]^{-1} = \left[ I + [R_3^{(j)}]^{-1} \cdot R_4^{(j)} \cdot u^*_{Q_j+1}(x_{j+1}) \right]^{-1} \left[ R_3^{(j)} \right]^{-1},
\]
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where \( I \) denotes the identity map. By corollary (2) and (4.190),
\[
\left\| \left[ R_3^{(j)} + R_4^{(j)} \cdot u_{Q,j+1}^*(x_{j+1}) \right]^{-1} \right\| \leq \frac{\Lambda}{1 - \zeta \Lambda^2}.
\] (4.206)

By (4.204), corollary (2), (4.190) and (4.206),
\[
\left\| [H(u^*)]_{Q_j} \cdot R_4^{(j)} \right\| \leq (2\kappa \Lambda_\ast + \zeta \Lambda) \cdot \frac{\Lambda^2}{1 - \zeta \Lambda^2}.
\] (4.207)

By (4.203; 4.205; 4.207),
\[
\lambda_1 \leq \left\| \nabla f(0,0) \right\| - \frac{\kappa C_\kappa + \zeta C_\zeta}{\Lambda} \left\| \nabla g(0,0) \right\| + 2\kappa \Lambda_\ast
\] + \( (2\kappa \Lambda_\ast + \zeta \Lambda) \frac{\Lambda^2}{1 - \zeta \Lambda^2} \).\] (4.208)

By (4.135), when \( \kappa \) and \( \zeta \) are sufficiently small:
\[\lambda_1 \leq \frac{3}{4}.\]

Taking the limit \( \tilde{x}_j \to x_j \) in (4.202), we have
\[
\Delta(Q_j,x_j)(0) \leq \gamma_1 \Delta(Q_{j+1},x_{j+1})(0), \quad \gamma_1 = \frac{3}{4};
\]
which is inequality (4.193). This completes the proof of the theorem. ♦

Additional smoothness follows similarly. We summarize in the following:

**Theorem 4.6** For any \( Q \in M \), \( \sigma^*_Q(x) \) is \( C^m \) in \( x \).

Proof: The proof is similar to that given above for \( \sigma^*_Q(x) \) being \( C^1 \) in \( x \). Here we only sketch the proof. For any \( Q \in M \), if \( \sigma^*_Q(x) \) is \( C^s \) in \( x \), then
\[
\nabla^s x \sigma^*_Q \in C^0(\mathcal{E}_\kappa(Q), L^s(E(Q), \mathcal{N}(Q))).
\]

Define the space:
\[
\Sigma_s \equiv \left\{ u^{(s)} \mid u^{(s)} \equiv \{ u_Q^{(s)} \}_{Q \in M}, \quad u_Q^{(s)} \in C^0(\mathcal{E}_\kappa(Q), L^s(E(Q), \mathcal{N}(Q))) \right\}.
\]

For any \( u^{(s)} \in \Sigma_s \), define the norm:
\[
\left\| u^{(s)} \right\| \equiv \sup_{Q \in M, x \in \mathcal{E}_\kappa(Q)} \left\| u_Q^{(s)}(x) \right\|.
\]

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where \( \| u_s^{(s)}(x) \| \) is the s-linear operator norm. A similar proof as for lemma (3.5) shows that \( \Sigma_s \) is a complete metric space under the norm defined above. Formally differentiating (4.175) \( s \) times, we get the equation formally satisfied by \( \nabla^s \sigma^*_Q \):

\[
 u_s^{(s)}(x) = \left\{ R_2 \cdot u_Q^{(s)}(\xi) - \left[ R_1 + R_2 \cdot \nabla_x \sigma^*_Q(\xi) \right] \right. \\
 \left. \quad \cdot \left[ R_3 + R_4 \cdot \nabla_x \sigma^*_Q(\xi) \right]^{-1} \cdot R_4 \cdot u_Q^{(s)}(\xi) \right\} \cdot \left[ R_3 + R_4 \cdot \nabla_x \sigma^*_Q(\xi) \right]^{-s} \\
 + \text{terms not involving } u^{(s)}, (1 \leq s \leq n). 
\]

Denote the right hand side of (4.209) by \( [H_s(u_s^{(s)}))]_Q(x) \). Assume \( \sigma^*_Q \) is \( C^{s-1} \) in \( x \); moreover,

\[
 \nabla^s \sigma^* = u^{s-1};
\]

then prove \( \sigma^*_Q \) is \( C^s \) in \( x \). By (4.209), \( H_s \) is a linear map on \( \Sigma_s \). By (4.181;4.185),

\[
 \| \left[ R_3 + R_4 \cdot \nabla_x \sigma^*_Q(\xi) \right]^{-1} \| \leq \frac{\| [\nabla f \xi(0,0)]^{-1} \|}{(1 - 2k\Lambda \Lambda^{s})(1 - \zeta \Lambda^2)}. 
\]

By (4.183),

\[
 \| R_1 + R_2 \cdot \nabla_x \sigma^*_Q(\xi) \| \leq 2k\Lambda + \zeta \Lambda. 
\]

By (4.184),

\[
 \| R_2 \| \leq \| \nabla \eta g(0,0) \| + 2k\Lambda. 
\]

By lemma (4.8) and (4.210;4.211;4.212), we have that \( H_s \) is a contraction map for \( 1 \leq s \leq n \). Therefore, there exists \( u^{(s)}_s \in \Sigma_s \) such that

\[
 u^{(s)}_s = H_s(u^{(s)}_s). 
\]

A similar argument as for theorem (4.5) shows that \( \sigma^*_Q(x) \) is \( C^s \) in \( x \); moreover,

\[
 \nabla^s \sigma^* = u^{(s)}_s. 
\]

This completes the proof of the theorem. \( \clubsuit \)
4.7 Metric Characterization of the Fibers

In this subsection, we are going to study certain important metric characterization of the fibers.

Lemma 4.26 For any small \( \nu > 0 \), when \( \delta \) is sufficiently small, there exists a constant \( A_\nu \), such that:

- For any \( Q \in M \), any \( \hat{Q} \in f^E(Q) \), and any \( t \in (-\infty, 0] \);

\[
\| F^\delta_t(\hat{Q}) - F^\delta_t(Q) \| \leq A_\nu e^{\Omega_1^+(1-\nu)t} \| \hat{Q} - Q \|,
\]

where \( \Omega_1^+ = 4\pi \sqrt{\omega^2 - \pi^2} \) (2.3).

- For any \( Q_i \in M, i = 1, 2; Q_1 \neq Q_2 \); any \( \hat{Q}_i \in f^E(Q_i), i = 1, 2 \); if

\[
\| F^\delta_t(\hat{Q}_2) - F^\delta_t(Q_1) \| \to 0, \text{ as } t \to -\infty;
\]

then,

\[
\left( \frac{\| F^\delta_t(\hat{Q}_1) - F^\delta_t(Q_1) \|}{\| F^\delta_t(\hat{Q}_2) - F^\delta_t(Q_1) \|} \right) e^{\Omega_1^+(1-2\nu)t} \to 0, \text{ as } t \to -\infty.
\]

Proof: First, we prove inequality (4.213). By (2.3), \( \forall t \in (-\infty, \infty) \);

\[
\| \nabla F^\delta_t(Q) \|_{J_0} = e^{\Omega_1^+ t}, \forall Q \in M_0^+.
\]

(Notice that the bumping in (2.36) does not affect above claim (4.216).) By (4.51),

\[
\sup_{Q \in M} \| \nabla F^\delta_T(Q) \|_E \leq \mu_0 e^{-\Omega_1^+ T} + \delta C_1^-(-T) + \zeta C_2^+(-T).
\]

(Notice that “\( \zeta \)” here is defined in (3.2).) For any small \( \nu > 0 \), choose \( T \) large enough, such that

\[
4\mu_0 \leq e^{\Omega_1^+ \nu T}.
\]

Then, for this fixed “\( T \)”, when \( \delta \) and \( \zeta \) are sufficiently small in (4.217):

\[
\sup_{Q \in M} \| \nabla F^{-T}_\delta(Q) \|_E \leq 2\mu_0 e^{-\Omega_1^+ T}.
\]
By (4.218;4.219),
\[
\sup_{Q \in M} \|\nabla F_{\delta}^{-T}(Q)\|_{E} \leq \frac{1}{2}e^{-\Omega(1-\nu)T}. \tag{4.220}
\]

By corollary (6), for any \(Q \in M\), any \(\hat{Q} \in f^{E}(Q)\):
\[
\hat{Q} - Q = x + (\sigma^*_{(A\zeta,B\kappa)})Q(x), \tag{4.221}
\]
for some \(x \in E B_{\kappa}(Q)\);
\[
F_{\delta}^{-T}(\hat{Q}) - F_{\delta}^{-T}(Q) = \xi + (\sigma^*_{(A\zeta,B\kappa)})F_{\delta}^{-T}(Q)(\xi), \tag{4.222}
\]
for some \(\xi \in E B_{\kappa}(F_{\delta}^{-T}(Q))\).

By (4.107),
\[
x = f(\xi,(\sigma^*_{(A\zeta,B\kappa)})F_{\delta}^{-T}(Q)(\xi)). \tag{4.223}
\]

By (4.116;4.118),
\[
\|\xi\| \leq \frac{\|\nabla f(0,0)\|^{-1}}{1-(\kappa BC\kappa + \zeta AC\zeta)\Lambda}\|x\|. \tag{4.224}
\]

By (4.222),
\[
\|F_{\delta}^{-T}(\hat{Q}) - F_{\delta}^{-T}(Q)\| \leq (1 + A\zeta)\|\xi\|. \tag{4.225}
\]

By (4.221),
\[
\|x\| \leq \frac{1}{1 - A\zeta}\|\hat{Q} - Q\|. \tag{4.226}
\]

By (4.224;4.225;4.226),
\[
\|F_{\delta}^{-T}(\hat{Q}) - F_{\delta}^{-T}(Q)\| \leq \frac{1 + A\zeta}{(1 - A\zeta)(1 - (\kappa BC\kappa + \zeta AC\zeta)\Lambda)} \|\hat{Q} - Q\|. \tag{4.227}
\]

Choose \(\zeta\) and \(\kappa\) sufficiently small, such that
\[
\frac{1 + A\zeta}{(1 - A\zeta)(1 - (\kappa BC\kappa + \zeta AC\zeta)\Lambda)} \leq 2. \tag{4.228}
\]

Then,
\[
\|F_{\delta}^{-T}(\hat{Q}) - F_{\delta}^{-T}(Q)\| \leq 2\|\nabla f(0,0)\|^{-1}\|\hat{Q} - Q\|. \tag{4.229}
\]
Notice that,
\[ [\nabla_\xi f(0,0)]^{-1} \equiv \nabla F_\delta^{-T}(Q); \]
then, by (4.220;4.229),
\[ \| F_\delta^{-T}(\tilde{Q}) - F_\delta^{-T}(Q) \| \leq e^{-\Omega^+(1-\nu)T}\| \tilde{Q} - Q \|. \tag{4.230} \]
By corollary (2), for any \( t \in [-T,0]; \)
\[ \sup_{Q \in \tilde{D}_k} \| \nabla F_\delta^i(\tilde{Q}) \| \leq \Lambda(t). \]
Let
\[ \Lambda^{(-T,0)} \equiv \sup_{[-T,0]} \Lambda(t); \tag{4.231} \]
\( A_\nu \) be a constant, such that:
\[ \Lambda^{(-T,0)} \leq A_\nu e^{-\Omega^+(1-\nu)T}. \]
Then, for any \( t \in [-T,0], \)
\[ \sup_{\tilde{Q} \in \tilde{D}_k} \| \nabla F_\delta^i(\tilde{Q}) \| \leq A_\nu e^{\Omega^+(1-\nu)T}. \tag{4.232} \]
By lemma (3.1), for any \( \tilde{Q}_1, \tilde{Q}_2 \in \tilde{D}_k, \) any \( t \in [-T,0]; \)
\[ \| F_\delta^i(\tilde{Q}_1) - F_\delta^i(\tilde{Q}_2) \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla F_\delta^i(\alpha \tilde{Q}_1 + (1-\alpha)\tilde{Q}_2) \| \cdot \| \tilde{Q}_1 - \tilde{Q}_2 \|. \tag{4.233} \]
By (4.232;4.233), for any \( Q \in M, \) any \( \tilde{Q} \in f^E(Q), \) and any \( t \in [-T,0]; \)
\[ \| F_\delta^i(\tilde{Q}) - F_\delta^i(Q) \| \leq A_\nu e^{\Omega^+(1-\nu)T}\| \tilde{Q} - Q \|. \tag{4.234} \]
For any \( t \in (-\infty,0], \) let
\[ t = -jT + \tau, \ \tau \in (-T,0); \]
then, for any \( Q \in M, \) any \( \tilde{Q} \in f^E(Q); \)
\[ F_\delta^i(\tilde{Q}) - F_\delta^i(Q) = F_\delta^T(F_\delta^{-jT}(\tilde{Q}) - F_\delta^{-jT}(Q)), \]
\[ F_\delta^{-jT}(\tilde{Q}) - F_\delta^{-jT}(Q) = F_\delta^{-T}(F_\delta^{-j-1T}(\tilde{Q}) - F_\delta^{-j-1T}(Q)). \]
Thus, by (4.234;4.230),
\[
\|F_t^\delta(\hat{Q}) - F_t^\delta (Q)\| \leq A_\nu e^{\Omega_1^\nu(1-\nu)\tau}\|F_t^{-(j-1)T}(\hat{Q}) - F_t^{-(j-1)T}(Q)\|.
\]
\[
\|F_t^{-(j-1)T}(\hat{Q}) - F_t^{-(j-1)T}(Q)\| \leq e^{-\Omega_2^\nu(1-\nu)\tau}\|F_t^{-(j-1)T}(Q)\|.
\]
(4.235)

Iterating (4.235) on \( j \), we have
\[
\|F_t^\delta(\hat{Q}) - F_t^\delta (Q)\| \leq A_\nu e^{\Omega_1^\nu(1-\nu)\tau}\|\hat{Q} - Q\|.
\]
This completes the proof of the inequality (4.213). Next, we prove relation (4.215). By (4.213), as \( t \to -\infty \);
\[
\|F_t^\delta(\hat{Q}_2) - F_t^\delta (Q_2)\| \to 0.
\]
(4.236)

Then, relations (4.214;4.236) imply that:
\[
\|F_t^\delta(\hat{Q}_2) - F_t^\delta (Q_1)\| \to 0, \quad \text{as} \quad t \to -\infty.
\]
(4.237)

By (4.237), there exists \(-\infty < t_1 \leq 0\), such that, for any \( t \in (-\infty, t_1)\):
\[
\|F_t^\delta(\hat{Q}_2) - F_t^\delta (Q_1)\| \leq \kappa;
\]
(4.238)

moreover, in terms of the coordinate \((\xi,\eta)\) in the neighborhood of \(F_t^\delta(\hat{Q}_1)\), \(F_t^\delta(\hat{Q}_2)\) has the representation:
\[
F_t^\delta(\hat{Q}_2) = F_t^\delta(\hat{Q}_1) + (\xi_0,\eta_0); \quad \xi_0 \leq \frac{1}{10}\eta_0.
\]
(4.239)

For \( t \leq t_1 - T \), in terms of the coordinate \((x,y)\) in the neighborhood of \(F_t^{t+T}(\hat{Q}_1)\), \(F_t^{t+T}(\hat{Q}_2)\) has the representation:
\[
F_t^{t+T}(\hat{Q}_2) = F_t^{t+T}(\hat{Q}_1) + (x_0, y_0); \quad x_0 \leq \frac{1}{10}y_0,
\]
\[
x_0 = f(\xi_0,\eta_0), \quad y_0 = g(\xi_0,\eta_0);
\]
(4.240)

where \(F_t^T\) has the representation:
\[
F_t^T : (\xi,\eta) \mapsto (x,y) \equiv (f(\xi,\eta),g(\xi,\eta)).
\]

By (4.240),
\[
\|F_t^{t+T}(\hat{Q}_2) - F_t^{t+T}(\hat{Q}_1)\| \leq \frac{11}{10}\|y_0\| \leq \frac{11}{10}\left\{\|g(\xi_0,\eta_0) - g(\xi_0,0)\| + \|g(\xi_0,0) - g(0,0)\|\right\}.
\]
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Then, by lemma (3.1),
\[
\|F_{\delta}^{t+T}(Q_2) - F_{\delta}^{t+T}(Q_1)\| \leq \frac{11}{10} \left\{ \sup_{0 \leq \alpha \leq 1} \| \nabla_\eta g(\xi_0, \alpha \eta_0)\| \| \eta_0 \| \right. \\
\left. + \sup_{0 \leq \alpha \leq 1} \| \nabla_\xi g(\alpha \xi_0, 0)\| \| \xi_0 \| \right\}. \quad (4.241)
\]

By (4.239),
\[
\|\eta_0\| \leq \frac{10}{9} \|F_{\delta}^{t}(Q_2) - F_{\delta}^{t}(Q_1)\|. \quad (4.242)
\]

Thus, by (4.239;4.242;4.241),
\[
\|F_{\delta}^{t+T}(Q_2) - F_{\delta}^{t+T}(Q_1)\| \leq \frac{11}{9} \left\{ \sup_{0 \leq \alpha \leq 1} \| \nabla_\eta g(\xi_0, \alpha \eta_0)\| \right. \\
\left. + \frac{1}{10} \sup_{0 \leq \alpha \leq 1} \| \nabla_\xi g(\alpha \xi_0, 0)\| \right\} \|F_{\delta}^{t}(Q_2) - F_{\delta}^{t}(Q_1)\|. \quad (4.243)
\]

By (4.129;4.130),
\[
\sup_{0 \leq \alpha \leq 1} \| \nabla_\xi g(\alpha \xi_0, 0)\| \leq 2 \kappa \Lambda_\ast(T), \quad (4.244)
\]
\[
\sup_{0 \leq \alpha \leq 1} \| \nabla_\eta g(\xi_0, \alpha \eta_0)\| \leq \| \nabla_\eta g(0, 0)\| + 2 \kappa \Lambda_\ast(T). \quad (4.245)
\]

By (4.243;4.244;4.245),
\[
\|F_{\delta}^{t+T}(Q_2) - F_{\delta}^{t+T}(Q_1)\| \leq \frac{11}{9} \left\{ \| \nabla_\eta g(0, 0)\| \right. \\
\left. + \frac{11}{5} \kappa \Lambda_\ast(T) \right\} \|F_{\delta}^{t}(Q_2) - F_{\delta}^{t}(Q_1)\|. \quad (4.246)
\]

By definition,
\[
\| \nabla_\eta g(0, 0)\| \leq \sup_{Q \in M} \| \nabla F_{\delta}^{T}(Q)|_{TM}\| + \sup_{Q \in M} \| \nabla F_{\delta}^{T}(Q)|_{J^\ast}\|.
\]

By inequalities (4.48;4.49) in lemma (4.8),
\[
\| \nabla_\eta g(0, 0)\| \leq 2 \left\{ \mu_0 \sup_{Q' \in M_0} \| \nabla F_{\delta}^{T}(Q')|_{TM_0}\| + \delta C_1^+(T) + \zeta C_2^+(T) \right\}. \quad (4.247)
\]
(Notice that "ζ" here is defined in (3.1).) By (2.3) and theorem (2.9),
\[
\sup_{Q' \in M_0^t} \| \nabla F_0^T(Q') \|_{T M_0^t} \leq 1 + \delta_1 C_1(T). \tag{4.248}
\]
For any small \( \nu > 0 \), choose \( T \) large enough, such that
\[
32\mu_0 \leq e^{\frac{1}{2} \Omega_+^+ \nu T}. \tag{4.249}
\]
For such fixed \( T \), choose \( \delta_1 \) small enough, such that
\[
\delta_1 C_1(T) \leq 1. \tag{4.250}
\]
Then (4.248;4.249;4.250) imply that
\[
2\mu_0 \sup_{Q' \in M_0^t} \| \nabla F_0^T(Q') \|_{T M_0^t} \leq \frac{1}{8} e^{\frac{1}{2} \Omega_+^+ \nu T}. \tag{4.251}
\]
By (4.247;4.251),
\[
\| \nabla_0 g(0,0) \| \leq \frac{1}{8} e^{\frac{1}{2} \Omega_+^+ \nu T} + 2[\delta C_1^+ (T) + \zeta C_2^+ (T)]
\]
For the above fixed \( T \), choose \( \delta \) and \( \zeta \) small enough, such that
\[
2[\delta C_1^+ (T) + \zeta C_2^+ (T)] \leq \frac{1}{8} e^{\frac{1}{2} \Omega_+^+ \nu T}.
\]
Then,
\[
\| \nabla_0 g(0,0) \| \leq \frac{1}{4} e^{\frac{1}{2} \Omega_+^+ \nu T}. \tag{4.252}
\]
In relation (4.246), for the above fixed \( T \), choose \( \kappa \) small enough, such that
\[
\frac{11}{5} \kappa \Lambda_\kappa (T) \leq \frac{1}{4} e^{\frac{1}{2} \Omega_+^+ \nu T}. \tag{4.253}
\]
Then relations (4.246;4.252;4.253) imply that
\[
\| F^{t+T}_\delta (Q_2) - F^{t+T}_\delta (Q_1) \| \leq e^{\frac{1}{2} \Omega_+^+ \nu T} \| F^t_\delta (Q_2) - F^t_\delta (Q_1) \|. \tag{4.254}
\]
As in (4.231), let
\[
\Lambda^{(0,T)} \equiv \sup_{t \in [0,T]} \Lambda(t). \tag{4.255}
\]
As in (4.233), for any $\tilde{Q}_1, \tilde{Q}_2 \in \tilde{D}_k$, any $\tau \in [0, T]$:
\[
\|F_\delta^T(\tilde{Q}_1) - F_\delta^T(\tilde{Q}_2)\| \leq \Lambda(0, T)\|\tilde{Q}_1 - \tilde{Q}_2\|. \tag{4.256}
\]
For any $t \in (-\infty, t_1]$, let
\[
t - t_1 = -jT - \tau, \quad \tau \in [0, T]; \tag{4.257}
\]
then by (4.254),
\[
\|F_{\delta}^{t_{1}-jT}(Q_2) - F_{\delta}^{t_{1}-jT}(Q_1)\| \leq e^{\frac{1}{2}\Omega^\top_{j}e_{t_1}T}\|F_{\delta}^{t_{1}-jT}(Q_2) - F_{\delta}^{t_{1}-jT}(Q_1)\|, \tag{4.258}
\]
Iterating (4.257) on $l$, we have
\[
\|F_{\delta}^{t_{1}}(Q_2) - F_{\delta}^{t_{1}}(Q_1)\| \leq e^{\frac{1}{2}\Omega^\top_{j}e_{t_1}T}\|F_{\delta}^{t_{1}}(Q_2) - F_{\delta}^{t_{1}}(Q_1)\|. \tag{4.259}
\]
By (4.256),
\[
\|F_{\delta}^{t_{1}-jT}(Q_2) - F_{\delta}^{t_{1}-jT}(Q_1)\| \leq \Lambda(0, T)\|F_{\delta}^{t_{1}-jT}(Q_2) - F_{\delta}^{t_{1}-jT}(Q_1)\|, \tag{4.260}
\]
By (4.257:4.259:4.260),
\[
\|F_{\delta}^{t_{1}}(Q_2) - F_{\delta}^{t_{1}}(Q_1)\| \leq \Lambda(0, T)e^\frac{1}{2}\Omega^\top_{j}e_{t_1}T\|F_{\delta}^{t_{1}}(Q_2) - F_{\delta}^{t_{1}}(Q_1)\|. \tag{4.261}
\]
Thus,
\[
\|F_{\delta}^{t_{1}}(Q_2) - F_{\delta}^{t_{1}}(Q_1)\| \geq \lambda_1 e^\frac{1}{2}\Omega^\top_{j}e_{t_1}T, \tag{4.261}
\]
where
\[
\lambda_1 \equiv \left[\Lambda(0, T)\right]^{-1}e^{-\frac{1}{2}\Omega^\top_{j}e_{t_1}T}\|F_{\delta}^{t_{1}}(Q_2) - F_{\delta}^{t_{1}}(Q_1)\|.
\]
By (4.213) and (4.261), there exists $t_2, -\infty < t_2 \leq t_1$; such that, when $t \in (-\infty, t_2]$;
\[
\|F_{\delta}^{t}(Q_2) - F_{\delta}^{t}(Q_1)\| \leq \frac{1}{2}\|F_{\delta}^{t}(Q_2) - F_{\delta}^{t}(Q_1)\|. \tag{4.262}
\]
Then,
\[
\|F_{\delta}^{t}(\tilde{Q}_2) - F_{\delta}^{t}(\tilde{Q}_1)\| \geq \|F_{\delta}^{t}(Q_2) - F_{\delta}^{t}(Q_1)\| - \|F_{\delta}^{t}(\tilde{Q}_2) - F_{\delta}^{t}(\tilde{Q}_1)\|
\geq \frac{1}{2}\|F_{\delta}^{t}(Q_2) - F_{\delta}^{t}(Q_1)\|. \tag{4.263}
\]
By (4.263;4.261), when \( t \in (-\infty, t_2] \):
\[
\|F^t_\delta(\hat{Q}_2) - F^t_\delta(Q_1)\| \geq \frac{1}{2} \lambda_1 e^{\frac{1}{2} \Omega^t_1 \nu t}.
\] (4.264)

By (4.213),
\[
\|F^t_\delta(\hat{Q}_1) - F^t_\delta(Q_1)\| \leq A e^{\Omega^t_1 (1-\nu) t} \|\hat{Q}_1 - Q_1\|.
\] (4.265)

By (4.264;4.265), we have
\[
\frac{\{\|F^t_\delta(\hat{Q}_1) - F^t_\delta(Q_1)\|\}}{\|F^t_\delta(Q_1) - F^t_\delta(Q_2)\|} \xrightarrow{e^{\Omega^t_1 (1-2\nu) t}} 0,
\] as \( t \to -\infty \).

This completes the proof of the lemma.

**Lemma 4.27**  For any \( Q_i \in M, i = 1,2; \)
\[
f^E(Q_1) \cap f^E(Q_2) = \emptyset, \quad \text{unless } Q_1 = Q_2.
\]

**Proof:** Assume for some \( Q_i \in M, i = 1,2; \)
\[
f^E(Q_1) \cap f^E(Q_2) \neq \emptyset.
\]

Let
\[
\hat{Q} \in f^E(Q_1) \cap f^E(Q_2),
\] (4.266)

then by (4.213),
\[
\|F^t_\delta(\hat{Q}) - F^t_\delta(Q_1)\| \to 0, \quad i = 1,2; \quad \text{as } t \to -\infty.
\] (4.267)

Since
\[
\|F^t_\delta(Q_1) - F^t_\delta(Q_2)\| \leq \|F^t_\delta(Q_1) - F^t_\delta(\hat{Q})\| + \|F^t_\delta(\hat{Q}) - F^t_\delta(Q_2)\|,
\] (4.268)

we have
\[
\|F^t_\delta(Q_1) - F^t_\delta(Q_2)\| \to 0, \quad \text{as } t \to -\infty.
\] (4.269)

By (4.214;4.215) and (4.269;4.266),
\[
\|F^t_\delta(\hat{Q}) - F^t_\delta(Q_1)\| \bigg/ \| F^t_\delta(Q_2) - F^t_\delta(Q_1)\| \to 0, \quad \text{as } t \to -\infty; \tag{4.270}
\]
\[
\|F^t_\delta(\hat{Q}) - F^t_\delta(Q_2)\| \bigg/ \| F^t_\delta(Q_1) - F^t_\delta(Q_2)\| \to 0, \quad \text{as } t \to -\infty. \tag{4.271}
\]
By (4.268; 4.270; 4.271),

\[ 1 \leq \left( \| F^T_\delta(\hat{Q}) - F^T_\delta(Q_1) \| + \| F^T_\delta(\hat{Q}) - F^T_\delta(Q_2) \| \right) / \| F^T_\delta(Q_1) - F^T_\delta(Q_2) \| \]

\[ \to 0, \quad \text{as } t \to -\infty; \]

which is a contradiction. Thus, for any \( Q_i \in M, i = 1, 2; \)

\[ f^E(Q_1) \cap f^E(Q_2) = \emptyset, \quad \text{unless } Q_1 = Q_2. \]

This completes the proof of the lemma. ♣

4.8 Smoothness of the Fiber Family \( \{ f^E(Q) \}_{Q \in M} \) With Respect to Base Point \( Q \)

In this subsection, we study the smoothness of the fibers with respect to their base points.

The study below is along the line laid down in the subsection "Smoothness of the Invariant Bundle \( E \)". By theorem (4.3), \( E \) is a \( C^{n-1} \) bundle. Using \( E \) and \( J^S \), we set up a \( C^{n-1} \) local coordinate \((u, v, w)\) in the neighborhood of \( M \); where \( v \in V_k \subset \tilde{S}_k \), \( \| v \| < \hat{d} \) (for some constant \( \hat{d} \)), \( V_k \) is a codimension 2 subspace of \( \tilde{S}_k \); \( u \in R^1 \), \( w \in R^1 \), \( |u| < \hat{\eta}, |w| < \hat{\eta} \) (for some small constant \( \hat{\eta} \)). \( v \) parametrizes the codimension 2 submanifold \( M \); \( u \) and \( w \) parametrize the one-dimensional fiber of \( E \) and \( J^S \) respectively. Denote by \( B_k \) the ball in \( V_k \):

\[ B_k \equiv \{ v \in V_k \mid \| v \| < \hat{d} \}, \]

by \( R^1_\eta \) the segment in \( R^1 \):

\[ R^1_\eta \equiv \{ u \mid u \in R^1, |u| < \hat{\eta} \}. \]

In this local coordinate, \( F^T_\delta \) has the representation:

\[ \hat{u} = \hat{f}(u, v, w), \]
\[ \hat{v} = \hat{g}(u, v, w), \]
\[ \hat{w} = \hat{h}(u, v, w); \]

where \( \hat{f}(0, v, 0) = \hat{h}(0, v, 0) = 0. \) Since \( E \) is a \( C^{n-1} \) bundle, to prove the fiber \( f^E(Q) \) is \( C^{n-1} \) in its base point \( Q \), we need to prove that \( \sigma^*_Q \) is \( C^{n-1} \)
in \( Q \) in the local coordinate sense. In terms of the local coordinate \((u,v,w)\), \( \sigma_v^* \) can be represented as follows: For any \( v \in B_k \subset V_k \), \( \sigma_v^* \) is a Lipschitz map

\[
\sigma_v^* : R_\eta^1 \to V_k \times R^1;
\]

\[
\forall u \in R_\eta^1, \; \sigma_v^*(u) \in V_k \times R^1, \; \sigma_v^*(0) = 0, \; \text{Lip}\{\sigma_v^*\} \leq \zeta.
\]

By corollary (5) and the definition of the graph transform \( G \) (4.107): For any \( Q \in M \), any \( x \in E_\kappa(Q) \);

\[
\sigma_Q^*(x) = g(\xi, \sigma_Q^*(\xi)),
\]

\[
x = f(\xi, \sigma_Q^*(\xi)),
\]

\[
Q' = F^{-T}_{\delta}(Q).
\]

In the local coordinate \((u,v,w)\), Eq.(4.272) can be rewritten as:

\[
\sigma_v^*(u) = \left( \hat{g}(u', \sigma_v^*(u') + (v', 0)) - \hat{g}(0, v', 0), \; \hat{h}(u', \sigma_v^*(u') + (v', 0)) \right),
\]

\[
u = \hat{g}(0, v', 0).
\]

**Definition 12** Define the Lipschitz norm of \( \sigma_v^* \) in \( v \) as follows:

\[
\text{Lip}_v\{\sigma^*\} \equiv \sup_{v_1, v_2 \in B_k} \left\{ \frac{\| \sigma_{v_1}^* - \sigma_{v_2}^* \|}{\| v_1 - v_2 \|} \right\},
\]

where

\[
\| \sigma_{v_1}^* - \sigma_{v_2}^* \| = \sup_{u \in R_\eta^1} \left\{ \frac{\| \sigma_{v_1}^*(u) - \sigma_{v_2}^*(u) \|}{\| u \|} \right\}.
\]

Before we prove that \( \sigma_v^* \) is \( C^1 \) in \( v \), we first want to show that \( \sigma_v^* \) is Lipschitz in \( v \).

**Lemma 4.28** There exists a constant \( \chi \) such that

\[
\text{Lip}_v\{\sigma^*\} \leq \chi.
\]
Proof: By (4.273),
\[ \sigma^*_v(u) - \sigma^*_v(u) = (G_1, G_2), \]
(4.274)
\[ G_1 \equiv \hat{g}(u_1', \sigma^*_v(u_1') + (v_1', 0)) - \hat{g}(u_2', \sigma^*_v(u_2') + (v_2', 0)) \]
\[ - \hat{g}(0, v'_1, 0) + \hat{g}(0, v'_2, 0), \]
\[ G_2 \equiv \hat{h}(u_1', \sigma^*_v(u_1') + (v_1', 0)) - \hat{h}(u_2', \sigma^*_v(u_2') + (v_2', 0)); \]
de\[ \]
where
\[ v_i = \hat{g}(0, v'_i, 0), \ i = 1, 2; \]
\[ u = \hat{f}(u_1', \sigma^*_v(u_1') + (v_1', 0)) = \hat{f}(u_2', \sigma^*_v(u_2') + (v_2', 0)). \] (4.275)
Then
\[ || G_1 || \leq || \hat{g}(u_1', \sigma^*_v(u_1') + (v_1', 0)) - \hat{g}(u_2', \sigma^*_v(u_1') + (v_1', 0)) || \]
\[ + || \hat{g}(u_2', \sigma^*_v(u_1') + (v_1', 0)) - \hat{g}(u_2', \sigma^*_v(u_2') + (v_1', 0)) || \]
\[ + || \hat{g}(u_2', \sigma^*_v(u_2') + (v_1', 0)) - \hat{g}(u_2', \sigma^*_v(u_2') + (v_2', 0)) || \]
\[ - [\hat{g}(0, v'_1, 0) - \hat{g}(0, v'_2, 0)] ||. \] (4.276)
Define the function:
\[ \varphi(u, v, w) = \hat{g}(u, (v, w) + (v_1', 0)) - \hat{g}(u, (v, w) + (v_2', 0)). \] (4.277)
Apply lemma (3.1) to \( \varphi(u, v, w) \) in (4.277), we have
\[ || [\hat{g}(u_2', \sigma^*_v(u_2') + (v_1', 0)) - \hat{g}(u_2', \sigma^*_v(u_2') + (v_2', 0))] \]
\[ - [\hat{g}(0, v'_1, 0) - \hat{g}(0, v'_2, 0)] || \]
\[ \leq \sup_{0 \leq \alpha \leq 1} || \nabla_{(u', v, w)}[\hat{g}(\alpha u'_2, \alpha \sigma^*_v(u'_2) + (v'_1, 0)) \]
\[ - \hat{g}(\alpha u'_2, \alpha \sigma^*_v(u'_2) + (v'_2, 0))] \|| || (u'_2, \sigma^*_v(u'_2)) ||. \] (4.278)
By lemma (3.1) again,
\[ || \nabla_{(u', v, w)}[\hat{g}(\alpha u'_2, \alpha \sigma^*_v(u'_2) + (v'_1, 0)) \]
\[ - \hat{g}(\alpha u'_2, \alpha \sigma^*_v(u'_2) + (v'_2, 0))] || \]
\[ \leq \sup_{0 \leq \beta \leq 1} || \nabla_{v}[\nabla_{(u', v, w)}[\hat{g}(\alpha u'_2, \alpha \sigma^*_v(u'_2) \]
\[ + (\beta v'_1 + (1 - \beta)v'_2, 0))] \|| || v'_1 - v'_2 ||. \] (4.279)
Similarly,

\[
\| \hat{g}(u_2, \sigma_{v_2}^*(u_2) + (v_1', 0)) - \hat{g}(u_2, \sigma_{v_2}^*(u_2) + (v_2', 0)) \|
\]

- \[ \hat{g}(0, v_1', 0) - \hat{g}(0, v_2', 0) \| \]
\[
\leq \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_v \nabla_{(u, v, w)} \hat{g}(\alpha u_2', \alpha \sigma_{v_1}^*(u_2) + (\beta v_1' + (1 - \beta) v_2') \| \]
\[
\| v_1' - v_2' \| \| (u_2', \sigma_{v_2}^*(u_2)) \|. \tag{4.280}
\]

Then by (4.276), lemma (3.1), and (4.280);

\[
\| G_1 \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_u \hat{g}(\alpha u_1' + (1 - \alpha) u_2', \sigma_{v_1}^*(u_1') + (v_1', 0)) \| \| u_1' - u_2' \|
+ \sup_{0 \leq \alpha \leq 1} \| \nabla_v \nabla_{(u, w)} \hat{g}(u_2', \alpha \sigma_{v_1}^*(u_2') + (1 - \alpha) \sigma_{v_1}^*(u_2') + (v_1', 0)) \| \]
\[
\| \sigma_{v_1}^*(u_1') - \sigma_{v_1}^*(u_2') \|
+ \sup_{0 \leq \alpha \leq 1} \| \nabla_v \nabla_{(u, w)} \hat{g}(u_2', \alpha \sigma_{v_1}^*(u_2') + (1 - \alpha) \sigma_{v_2}^*(u_2') + (v_1', 0)) \| \]
\[
\| \sigma_{v_2}^*(u_2') - \sigma_{v_2}^*(u_2') \|
+ \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_v \nabla_{(u, v, w)} \hat{g}(\alpha u_2', \alpha \sigma_{v_2}^*(u_2') + (\beta v_1' + (1 - \beta) v_2', 0)) \| \]
\[
\| v_1' - v_2' \| \| (u_2', \sigma_{v_2}^*(u_2')) \|
\leq A_1 \| u_1' - u_2' \| + A_2 \| \sigma_{v_1}^*(u_2') - \sigma_{v_2}^*(u_2') \|
+ A_3 \| v_1' - v_2' \| \| u_2' \|, \tag{4.281}
\]

where

\[
A_1 \equiv \sup_{0 \leq \alpha \leq 1} \| \nabla_u \hat{g}(\alpha u_1' + (1 - \alpha) u_2', \sigma_{v_1}^*(u_1') + (v_1', 0)) \|
+ \zeta \sup_{0 \leq \alpha \leq 1} \| \nabla_v \nabla_{(u, w)} \hat{g}(u_2', \alpha \sigma_{v_1}^*(u_2') + (1 - \alpha) \sigma_{v_1}^*(u_2') + (v_1', 0)) \|
\]

\[
A_2 \equiv \sup_{0 \leq \alpha \leq 1} \| \nabla_v \nabla_{(u, w)} \hat{g}(u_2', \alpha \sigma_{v_1}^*(u_2') + (1 - \alpha) \sigma_{v_2}^*(u_2') + (v_1', 0)) \|
\]

\[
A_3 \equiv (1 + \zeta) \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_v \nabla_{(u, v, w)} \hat{g}(\alpha u_2', \alpha \sigma_{v_2}^*(u_2')
+ (\beta v_1' + (1 - \beta) v_2', 0)) \|. \tag{4.281}
\]

Similarly,

\[
\| G_2 \| \leq \| \hat{h}(u_1', \sigma_{v_1}^*(u_1') + (v_1', 0)) - \hat{h}(u_2', \sigma_{v_1}^*(u_1') + (v_1', 0)) \|
+ \| \hat{h}(u_2', \sigma_{v_1}^*(u_1') + (v_1', 0)) - \hat{h}(u_2', \sigma_{v_1}^*(u_2') + (v_1', 0)) \|
\]

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\[ \begin{align*}
+ \| \hat{h}(u_2', \sigma_{v_1}^*(u_2') + (v_1', 0)) - \hat{h}(u_2', \sigma_{v_1}^*(u_2') + (v_1', 0)) \| \\
+ \| \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (v_1', 0)) - \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (v_1', 0)) \| \quad & \text{for } 4.282
\end{align*} \]

By lemma (3.1),
\[ \| \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (v_1', 0)) - \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (v_1', 0)) \| \]
\[ \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_v \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (\alpha v_1')
+ (1 - \alpha) v_2', 0)) || v_1' - v_2' ||. \quad (4.283) \]

Notice that
\[ \nabla_v \hat{h}(0, \alpha v_1' + (1 - \alpha) v_2', 0) = 0, \]

then by lemma (3.1) again,
\[ \| \nabla_v \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (\alpha v_1' + (1 - \alpha) v_2', 0)) \|
\leq \sup_{0 \leq \alpha \leq 1} \| \nabla_v \hat{h}(\beta u_2', \beta \sigma_{v_2}^*(u_2')
+ (\alpha v_1' + (1 - \alpha) v_2', 0)) || (u_2', \sigma_{v_2}^*(u_2')) ||. \quad (4.284) \]

By (4.283; 4.284),
\[ \| \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (v_1', 0)) - \hat{h}(u_2', \sigma_{v_2}^*(u_2') + (v_1', 0)) \|
\leq \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_v \hat{h}(\beta u_2', \beta \sigma_{v_2}^*(u_2')
+ (\alpha v_1' + (1 - \alpha) v_2', 0)) || v_1' - v_2' || (u_2', \sigma_{v_2}^*(u_2')) ||. \quad (4.285) \]

By (4.282), lemma (3.1), and (4.285);
\[ \| G_2 \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_v \hat{h}(\alpha u_2' + (1 - \alpha) u_2', \sigma_{v_1}^*(u_1')
+ (v_1', 0)) || u_1' - u_2' || \\
+ \sup_{0 \leq \alpha \leq 1} \| \nabla_v \hat{h}(u_2', \alpha \sigma_{v_1}^*(u_1') + (1 - \alpha) \sigma_{v_1}^*(u_2')
+ (v_1', 0)) || \sigma_{v_1}^*(u_1') - \sigma_{v_1}^*(u_2') || \\
+ \sup_{0 \leq \alpha \leq 1} \| \nabla_v \hat{h}(u_2', \alpha \sigma_{v_1}^*(u_2') + (1 - \alpha) \sigma_{v_2}^*(u_2')
+ (v_1', 0)) || \sigma_{v_2}^*(u_2') - \sigma_{v_2}^*(u_2') || \\
+ \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_v \hat{h}(\beta u_2', \beta \sigma_{v_2}^*(u_2') + (\alpha v_1') \]

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\[ \begin{align*}
+ (1 - \alpha)v'_2, 0) \| \leq A_4 \| v'_1 - u'_2 \| + A_5 \| \sigma_{v'_2}^*(u'_2) - \sigma_{v'_1}^*(u'_1) \| \\
+ A_6 \| v'_1 - v'_2 \| \| u'_2 \|. 
\end{align*} \]

(4.286)

Where

\[ \begin{align*}
A_4 &= \sup_{0 \leq \alpha \leq 1} \| \nabla_u \hat{h}(\alpha u'_1 + (1 - \alpha)u'_2, \sigma_{v'_1}^*(u'_1) + (v'_1, 0)) \| \\
&+ \zeta \sup_{0 \leq \alpha \leq 1} \| \nabla_{(v,w)} \hat{h}(u'_2, \alpha \sigma_{v'_2}^*(u'_2) + (1 - \alpha)\sigma_{v'_1}^*(u'_1) + (v'_1, 0)) \|, \\
A_5 &= \sup_{0 \leq \alpha \leq 1} \| \nabla_{(v,w)} \hat{h}(u'_2, \alpha \sigma_{v'_2}^*(u'_2) + (1 - \alpha)\sigma_{v'_1}^*(u'_1) + (v'_1, 0)) \|, \\
A_6 &= (1 + \zeta) \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \nabla_u \hat{h}(\beta u'_2, \beta \sigma_{v'_2}^*(u'_2) + (\alpha v'_1 + (1 - \alpha)v'_2, 0)) \|. 
\end{align*} \]

By (4.275),

\[ \begin{align*}
\| \hat{f}(u'_1, \sigma_{v'_1}^*(u'_1) + (v'_1, 0)) - \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) \| \\
= \| \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) - \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) \|. 
\end{align*} \]

(4.287)

By (4.118:4.117),

\[ \| u'_1 - u'_2 \| \leq \frac{\| \nabla_u \hat{f}(0, v'_1, 0) \|^{-1}}{1 - (\kappa C_\kappa + \zeta C_\zeta)\Lambda} \| \hat{f}(u'_1, \sigma_{v'_1}^*(u'_1) + (v'_1, 0)) \\
- \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) \|. 
\]  

(4.288)

The right hand side of (4.287) has the estimate:

\[ \begin{align*}
\| \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) - \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_2, 0)) \| \\
\leq \| \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) - \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) \| \\
+ \| \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) - \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_2, 0)) \|. 
\]  

(4.289)

By lemma (3.1),

\[ \begin{align*}
\| \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_1, 0)) - \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (v'_2, 0)) \| \\
\leq \sup_{0 \leq \alpha \leq 1} \| \nabla_v \hat{f}(u'_2, \sigma_{v'_2}^*(u'_2) + (\alpha v'_1 \\
+ (1 - \alpha)v'_2, 0)) \| \| v'_1 - v'_2 \|. 
\]  

(4.290)
Notice that
\[ \nabla_{v} \hat{f}(0, \alpha v_1' + (1 - \alpha)v_2', 0) = 0, \]
then by lemma (3.1),
\[
\begin{align*}
&\| \nabla_{v} \hat{f}(u_2', \sigma_{\nu_2}^{*}(u_2') + (\alpha v_1' + (1 - \alpha)v_2', 0)) \| \\
&\leq \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_{(u,v,w)} \nabla_{v} \hat{f}(\beta u_2', \beta \sigma_{\nu_2}^{*}(u_2')) \\
&\quad + (\alpha v_1' + (1 - \alpha)v_2', 0)) \| \| (u_2', \sigma_{\nu_2}^{*}(u_2')) \|. \tag{4.291}
\end{align*}
\]
By (4.290; 4.291),
\[
\begin{align*}
&\| \hat{f}(u_2', \sigma_{\nu_2}^{*}(u_2') + (v_1', 0)) - \hat{f}(u_2', \sigma_{\nu_2}^{*}(u_2') + (v_2', 0)) \| \\
&\leq (1 + \zeta) \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_{(u,v,w)} \nabla_{v} \hat{f}(\beta u_2', \beta \sigma_{\nu_2}^{*}(u_2')) \\
&\quad + (\alpha v_1' + (1 - \alpha)v_2', 0)) \| \| v_1' - v_2' \| \| u_2' \|. \tag{4.292}
\end{align*}
\]
By (4.289; 4.292) and lemma (3.1) again,
\[
\begin{align*}
&\| \hat{f}(u_2', \sigma_{\nu_2}^{*}(u_2') + (v_1', 0)) - \hat{f}(u_2', \sigma_{\nu_2}^{*}(u_2') + (v_2', 0)) \| \\
&\leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \hat{f}(u_2', \alpha \sigma_{\nu_1}^{*}(u_2') + (1 - \alpha)\sigma_{\nu_2}^{*}(u_2') \\
&\quad + (v_1', 0)) \| \| \sigma_{\nu_1}^{*}(u_2') - \sigma_{\nu_2}^{*}(u_2') \| \\
&\quad + (1 + \zeta) \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_{(u,v,w)} \nabla_{v} \hat{f}(\beta u_2', \beta \sigma_{\nu_2}^{*}(u_2')) \\
&\quad + (\alpha v_1' + (1 - \alpha)v_2', 0)) \| \| v_1' - v_2' \| \| u_2' \|. \tag{4.293}
\end{align*}
\]
By (4.287;4.288;4.293),
\[
\| u_1' - u_2' \| \leq B_1 \| \sigma_{\nu_1}^{*}(u_2') - \sigma_{\nu_2}^{*}(u_2') \| + B_2 \| v_1' - v_2' \| \| u_2' \|. \tag{4.294}
\]
where
\[
B_1 = \frac{\| \nabla_{(u,v)} \hat{f}(0, v_1', 0) \|^{-1} \|}{1 - (\kappa C_{\zeta} + \zeta C_{\zeta})A} \left[ \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v)} \hat{f}(u_2', \alpha \sigma_{\nu_1}^{*}(u_2') \\
\quad + (1 - \alpha)\sigma_{\nu_2}^{*}(u_2') + (v_1', 0)) \| \right],
\]
\[
B_2 = \frac{\| \nabla_{(u,v)} \hat{f}(0, v_1', 0) \|^{-1} \|}{1 - (\kappa C_{\zeta} + \zeta C_{\zeta})A} \left[ (1 + \zeta) \sup_{0 \leq \alpha, \beta \leq 1} \| \nabla_{(u,v,w)} \nabla_{v} \hat{f}(\beta u_2', \beta \sigma_{\nu_2}^{*}(u_2') \\
\quad + (\alpha v_1' + (1 - \alpha)v_2', 0)) \| \right].
\]
We know that
\[ v_i = \hat{g}(0, v'_i, 0), \quad i = 1, 2. \] (4.295)

Define
\[ v = \phi(v') \equiv \hat{g}(0, v', 0), \quad \forall v' \in B_k. \] (4.296)

Then
\[ \nabla_v \phi^{-1}(v) = [\nabla_v \hat{g}(0, v', 0)]^{-1}. \] (4.297)

By (4.295;4.296),
\[ \| v'_1 - v'_2 \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_v \phi^{-1}(\alpha v_1 + (1 - \alpha)v_2) \|\| v_1 - v_2 \|. \] (4.298)

By (4.118;4.116) and (4.275),
\[ \| u'_2 \| \leq \| \nabla_u \hat{f}(0, v'_2, 0) \|^{-1} \| u \|. \] (4.299)

By (4.298;4.299),
\[ \| v'_1 - v'_2 \|\| u'_2 \| \leq B_3 \| v_1 - v_2 \|\| u \|; \] (4.300)

where,
\[ B_3 \equiv \| \nabla_u \hat{f}(0, v'_2, 0) \|^{-1} \| \nabla_v \phi^{-1}(\alpha v_1 + (1 - \alpha)v_2) \|. \]

By (4.281;4.294),
\[ \| G_1 \| \leq (A_1B_1 + A_2) \| \sigma^*_v(u'_1) - \sigma^*_v(u'_2) \| + (A_1B_2 + A_3) \| v'_1 - v'_2 \|\| u'_2 \|. \] (4.301)

By (4.300;4.301),
\[ \frac{\| G_1 \|}{\| v_1 - v_2 \|\| u \|} \leq \frac{B_3(A_1B_1 + A_2) \| \sigma^*_v(u'_1) - \sigma^*_v(u'_2) \|}{\| v'_1 - v'_2 \|\| u'_2 \|} + B_3(A_1B_2 + A_3). \] (4.302)

Similarly, by (4.286;4.294),
\[ \| G_2 \| \leq (A_4B_1 + A_5) \| \sigma^*_v(u'_1) - \sigma^*_v(u'_2) \| + (A_4B_2 + A_6) \| v'_1 - v'_2 \|\| u'_2 \|. \] (4.303)
By (4.300;4.303),
\[
\| G_2 \| \frac{\| v_1 - v_2 \| \| u \|}{\| v_1' - v_2' \| \| u_2' \|} \leq B_3 (A_4 B_1 + A_5) \frac{\| \sigma_{v_1'} (u_2') - \sigma_{v_2'} (u_2') \|}{\| v_1' - v_2' \| \| u_2' \|} + B_3 (A_4 B_2 + A_6).
\]  
(4.304)

By (4.274;4.302;4.304),
\[
\| \sigma^* (v_1) (u) - \sigma^* (v_2) (u) \| \| v_1 - v_2 \| \| u \| \leq B_3 (A_1 B_1 + A_4 B_1 + A_2 + A_5) \frac{\| \sigma^*_{v_1'} (u_2') - \sigma^*_{v_2'} (u_2') \|}{\| v_1' - v_2' \| \| u_2' \|} + B_3 (A_1 B_2 + A_4 B_2 + A_3 + A_6).
\]  
(4.305)

By lemma (3.1) and theorem (2.11):
\[
A_1 \leq \sup_{v_1' \in B_k} \| \nabla_u \hat{g}(0, v_1', 0) \| + 2\kappa \Lambda_*
\]
\[
+ \zeta \left( \sup_{v_1' \in B_k} \| \nabla_{(v,w)} \hat{g}(0, v_1', 0) \| + 2\kappa \Lambda_* \right);
\]

since
\[
\| \nabla_u \hat{g}(0, v_1', 0) \| = 0,
\]
\[
A_1 \leq 2\kappa \Lambda_* + \zeta \left( \sup_{v_1' \in B_k} \| \nabla_{(v,w)} \hat{g}(0, v_1', 0) \| + 2\kappa \Lambda_* \right). 
\]  
(4.306)

\[
A_2 \leq \sup_{v_1' \in B_k} \| \nabla_{(v,w)} \hat{g}(0, v_1', 0) \| + 2\kappa \Lambda_* .
\]  
(4.307)

\[
A_3 \leq 2\Lambda_* .
\]  
(4.308)

\[
A_4 \leq \sup_{v_1' \in B_k} \| \nabla_u \hat{h}(0, v_1', 0) \| + 2\kappa \Lambda_*
\]
\[
+ \zeta \left( \sup_{v_1' \in B_k} \| \nabla_{(v,w)} \hat{h}(0, v_1', 0) \| + 2\kappa \Lambda_* \right);
\]

since
\[
\| \nabla_u \hat{h}(0, v_1', 0) \| = 0,
\]
\[
A_4 \leq 2\kappa \Lambda_* + \zeta \left( \sup_{v_1' \in B_k} \| \nabla_{(v,w)} \hat{h}(0, v_1', 0) \| + 2\kappa \Lambda_* \right). 
\]  
(4.309)

\[
A_5 \leq \sup_{v_1' \in B_k} \| \nabla_{(v,w)} \hat{h}(0, v_1', 0) \| + 2\kappa \Lambda_* .
\]  
(4.310)

\[
A_6 \leq 2\Lambda_* .
\]  
(4.311)

\[
B_1 \leq 2 \| [\nabla_u \hat{f}(0, v_1', 0)]^{-1} \| \left( \sup_{v_1' \in B_k} \| \nabla_{(v,w)} \hat{f}(0, v_1', 0) \| + 2\kappa \Lambda_* \right);
\]

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since
\[ \| \nabla_{(v,w)} \hat{f}(0,v'_1,0) \| = 0, \]

\[ B_1 \leq 4 \kappa \| [\nabla_u \hat{f}(0,v'_1,0)]^{-1} \| \Lambda. \quad (4.312) \]
\[ B_2 \leq 4 \| [\nabla_u \hat{f}(0,v'_1,0)]^{-1} \| \Lambda. \quad (4.313) \]

By (4.297;4.300),
\[ B_3 \leq 2 \| [\nabla_u \hat{f}(0,v'_1,0)]^{-1} \| \sup_{v'_i \in B_k} \| [\nabla_v \hat{g}(0,v'_1,0)]^{-1} \|. \quad (4.314) \]

By (4.306;4.309;4.312;4.314), when \( \zeta \) and \( \kappa \) are sufficiently small,
\[ B_1 B_3 (A_1 + A_4) \leq \frac{1}{10}. \quad (4.315) \]

By (4.307;4.310;4.314),
\[ B_3 (A_2 + A_5) \leq 2 \| [\nabla_u \hat{f}(0,v'_1,0)]^{-1} \| \sup_{v'_i \in B_k} \| [\nabla_v \hat{g}(0,v'_1,0)]^{-1} \|
\left( \sup_{v'_i \in B_k} \| \nabla_{(v,w)} \hat{g}(0,v'_1,0) \| \right.
\left. + \sup_{v'_i \in B_k} \| \nabla_{(v,w)} \hat{h}(0,v'_1,0) \| + 4 \kappa A_s \right). \quad (4.316) \]

Notice that
\[ [\nabla_u \hat{f}(0,v'_1,0)]^{-1}, \quad [\nabla_v \hat{g}(0,v'_1,0)]^{-1}, \]
\[ \left( \nabla_{(v,w)} \hat{g}(0,v'_1,0), \nabla_{(v,w)} \hat{h}(0,v'_1,0) \right) \] (4.317)

are local coordinate representations of
\[ \nabla F_\delta^{-T}(Q)|_E, \nabla F_\delta^{-T}(Q)|_{TM}, \nabla F_\delta^{T}(Q)|_{TM \oplus J^s}. \quad (4.318) \]

As discussed above, by lemma (4.8), for sufficiently large \( T \), when \( \delta \) is sufficiently small; moreover, taking \( \kappa \) small enough, we have from (4.316):
\[ B_3 (A_2 + A_5) \leq \frac{2}{5}. \quad (4.319) \]
By (4.315;4.319;4.305), we have
\[ \| \sigma_{v_1}^*(u) - \sigma_{v_2}^*(u) \| \leq \frac{1}{2} \frac{\| \sigma_{v_1'}(u_2') - \sigma_{v_2'}(u_2') \|}{\| v_1' - v_2' \| \| u_2' \|} + \frac{1}{2} \chi, \] (4.320)

where
\[ \chi \equiv B_3(A_1B_2 + A_4B_2 + A_3 + A_6). \]

Thus,
\[ \text{Lip}_v \{ \sigma^* \} \leq \chi. \]

This completes the proof of the lemma. ♣

Before we prove that \( \sigma^* \) is \( C^1 \) in \( v \), we also want to prove the following lemma which will be needed later.

**Lemma 4.29** \( \nabla_u \sigma_v^* \) is \( C^0 \) in \( v \).

**Proof:** By theorem (4.5), \( \sigma_v^*(u) \) is \( C^1 \) in \( u \). In the local coordinate, \( \nabla_u \sigma_v^*(u) \) satisfies the equation:
\[ \nabla_u \sigma_v^*(u) = \left( G^{(1)}, G^{(2)} \right); \] (4.321)

\[ G^{(1)} \equiv \left[ \nabla_u \hat{g}(X) + \nabla_{(v,w)} \hat{g}(X) \cdot \nabla_u \sigma_v^*(u') \right] \]
\[ \cdot \left[ \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \cdot \nabla_u \sigma_v^*(u') \right]^{-1}, \]
\[ G^{(2)} \equiv \left[ \nabla_u \hat{h}(X) + \nabla_{(v,w)} \hat{h}(X) \cdot \nabla_u \sigma_v^*(u') \right] \]
\[ \cdot \left[ \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \cdot \nabla_u \sigma_v^*(u') \right]^{-1}; \]
\[ u = \hat{f}(u', \sigma_v^*(u') + (v', 0)), \]
\[ v = \hat{g}(0, v', 0), \]
\[ X = (u', \sigma_v^*(u') + (v', 0)). \]

For any \( v \in B_k \),
\[ \nabla_u \sigma_v^* \in C^0(R_1, L(R^1, V_k \times R^1)). \]

Then, for any \( v \in B_k \), define the norm:
\[ \| \nabla_u \sigma_v^* \| \equiv \sup_{u \in R_1} \| \nabla_u \sigma_v^*(u) \|. \]
where \( \| \nabla u \sigma^*_v(u) \| \) is the linear operator norm. For any \( v \in B_k \), define an increasing non-negative function:

\[
Z_v(a) : (0, 1) \rightarrow R,
Z_v(a) \equiv \sup_{0<\|\tilde{v} - v\|<a} \| \nabla_u \sigma^*_\tilde{v} - \nabla_u \sigma^*_v \| .
\]

By theorem (4.5), for any \( v \in B_k \) and any \( a \in (0, 1) \),

\[
Z_v(a) \leq 2\zeta. \tag{4.322}
\]

For any \( v \in B_k \), since \( Z_v(a) \) is an increasing non-negative function, the limit

\[
\lim_{a \rightarrow 0} Z_v(a)
\]

exists. Denote this limit by \( Z_v(0) \). By (4.322),

\[
0 \leq Z_v(0) \leq 2\zeta, \quad \forall v \in B_k. \tag{4.323}
\]

For any \( v_1 \in B_k \), define the sequence:

\[
\{v_j\}, \quad j = 1, 2, \ldots ;
\]

\[
v_j = \hat{g}(0, v_{j+1}, 0).
\]

To prove the lemma, we need to show that

\[
Z_{v_1}(0) = 0.
\]

We will show that the inequality

\[
Z_{v_j}(0) \leq \gamma Z_{v_{j+1}}(0), \quad \forall j = 1, 2, \ldots ; \quad (0 < \gamma < 1) \tag{4.324}
\]

is valid. Then by (4.323;4.324),

\[
Z_{v_1}(0) \leq 2\zeta \gamma^m, \quad \forall m \in Z^+.
\]

Thus

\[
Z_{v_1}(0) = 0.
\]

Next we prove the inequality (4.324). For any \( j \in Z^+ \), there exists a small constant \( a_j \) such that when \( \| \tilde{v}_j - v_j \|<a_j \), all the Taylor expansions below are valid. By (4.321),

\[
\nabla_u \sigma^*_\tilde{v}_j(u) - \nabla_u \sigma^*_v(u) = \begin{pmatrix} P^{(1)} \, P^{(2)} \end{pmatrix}; \tag{4.325}
\]

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\[ P^{(1)} \equiv [\nabla_u \tilde{g}(\tilde{X}) + \nabla_{(v,w)} \tilde{g}(\tilde{X}) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')] \]

\[ \bullet [\nabla_u \tilde{f}(\tilde{X}) + \nabla_{(v,w)} \tilde{f}(\tilde{X}) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')]^{-1} \]

\[ - [\nabla_u \bar{g}(X) + \nabla_{(v,w)} \bar{g}(X) \bullet \nabla_u \sigma^*_{\bar{v}_j+1}(u')] \]

\[ \bullet [\nabla_u \tilde{f}(X) + \nabla_{(v,w)} \tilde{f}(X) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(u')]^{-1} , \]

\[ P^{(2)} \equiv [\nabla_u \bar{h}(X) + \nabla_{(v,w)} \bar{h}(X) \bullet \nabla_u \sigma^*_{\bar{v}_j+1}(\bar{u}')] \]

\[ \bullet [\nabla_u \tilde{f}(X) + \nabla_{(v,w)} \tilde{f}(X) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')]^{-1} \]

\[ - [\nabla_u \bar{h}(X) + \nabla_{(v,w)} \bar{h}(X) \bullet \nabla_u \sigma^*_{\bar{v}_j+1}(u')] \]

\[ \bullet [\nabla_u \tilde{f}(X) + \nabla_{(v,w)} \tilde{f}(X) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(u')]^{-1} ; \]

\[ u = \tilde{f}(u', \sigma^*_{\tilde{v}_j+1}(u')) + (v_{j+1}, 0) \]

\[ = \tilde{f}(\tilde{u}', \sigma^*_{\tilde{v}_j+1}(\tilde{u}')) + (\tilde{v}_{j+1}, 0), \quad (4.326) \]

\[ v_j = \bar{g}(0, v_{j+1}, 0), \]

\[ \tilde{v}_j = \bar{g}(0, \tilde{v}_{j+1}, 0), \]

\[ X \equiv \tilde{f}(u', \sigma^*_{\tilde{v}_j+1}(u')) + (v_{j+1}, 0), \]

\[ \tilde{X} \equiv \tilde{f}(\tilde{u}', \sigma^*_{\tilde{v}_j+1}(\tilde{u}')) + (\tilde{v}_{j+1}, 0). \]

First we estimate \( P^{(1)} \). It can be rewritten as:

\[ P^{(1)} = \Gamma_1 A + B \Gamma_2 ; \quad (4.327) \]

where

\[ \Gamma_1 \equiv [\nabla_u \tilde{g}(\tilde{X}) + \nabla_{(v,w)} \tilde{g}(\tilde{X}) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')] \]

\[ - [\nabla_u \tilde{g}(X) + \nabla_{(v,w)} \tilde{g}(X) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')] , \]

\[ A \equiv [\nabla_u \tilde{f}(\tilde{X}) + \nabla_{(v,w)} \tilde{f}(\tilde{X}) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')]^{-1} , \]

\[ B \equiv [\nabla_u \bar{g}(X) + \nabla_{(v,w)} \bar{g}(X) \bullet \nabla_u \sigma^*_{\bar{v}_j+1}(u')] , \]

\[ \Gamma_2 \equiv [\nabla_u \tilde{f}(\tilde{X}) + \nabla_{(v,w)} \tilde{f}(\tilde{X}) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')]^{-1} \]

\[ - [\nabla_u \bar{h}(X) + \nabla_{(v,w)} \bar{h}(X) \bullet \nabla_u \sigma^*_{\bar{v}_j+1}(u')]^{-1} . \]

Moreover, \( \Gamma_2 \) can be rewritten as

\[ \Gamma_2 \equiv [\nabla_u \tilde{f}(\tilde{X}) + \nabla_{(v,w)} \tilde{f}(\tilde{X}) \bullet \nabla_u \sigma^*_{\tilde{v}_j+1}(\tilde{u}')]^{-1} \]
By (4.332),
\[
\left\{ \begin{array}{l}
\nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \bullet \nabla_u \sigma_{v_j+1}^s(u') \\
- \left[ \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \bullet \nabla_u \sigma_{v_j+1}^s(\tilde{u}') \right]
\end{array} \right\}
\]

By (4.329),
\[
\begin{align*}
\| \hat{f}(\tilde{u}', \sigma_{v_j+1}^s(\tilde{u}') + (v_{j+1}, 0)) - \hat{f}(u', \sigma_{v_j+1}^s(u') + (v_{j+1}, 0)) & = \| \hat{f}(\tilde{u}', \sigma_{v_j+1}^s(\tilde{u}') + (v_{j+1}, 0)) - \hat{f}(u', \sigma_{v_j+1}^s(\tilde{u}') + (\tilde{v}_{j+1}, 0)) \| \\
& \leq \Lambda \left[ \| \sigma_{v_j+1}^s(\tilde{u}') - \sigma_{v_j+1}^s(\tilde{u}') \| + \| v_{j+1} - \tilde{v}_{j+1} \| \right].
\end{align*}
\] (4.331)

By lemma (3.1),
\[
\| \hat{f}(\tilde{u}', \sigma_{v_j+1}^s(\tilde{u}') + (v_{j+1}, 0)) - \hat{f}(u', \sigma_{v_j+1}^s(u') + (\tilde{v}_{j+1}, 0)) \| \\
\leq \Lambda \left[ \| \sigma_{v_j+1}^s(\tilde{u}') - \sigma_{v_j+1}^s(\tilde{u}') \| + \| v_{j+1} - \tilde{v}_{j+1} \| \right].
\] (4.332)

Thus by (4.329:4.330:4.331:4.332), there exists a constant \( \chi_1 \) such that
\[
\| \tilde{u}' - u \| \leq \chi_1 \| v_{j+1} - \tilde{v}_{j+1} \|. 
\] (4.333)

We know that
\[
\| \tilde{X} - X \| \leq \| \tilde{u}' - u \| + \| \sigma_{v_j+1}^s(\tilde{u}') - \sigma_{v_j+1}^s(u') \| + \| \tilde{v}_{j+1} - v_{j+1} \| \\
\leq \| \tilde{u}' - u \| + \| \sigma_{v_j+1}^s(\tilde{u}') - \sigma_{v_j+1}^s(\tilde{u}') \| \\
+ \| \sigma_{v_j+1}^s(\tilde{u}') - \sigma_{v_j+1}^s(u') \| + \| \tilde{v}_{j+1} - v_{j+1} \|. 
\] (4.334)

By (4.332:4.333:4.334), there exists a constant \( \chi_2 \) such that
\[
\| \tilde{X} - X \| \leq \chi_2 \| \tilde{v}_{j+1} - v_{j+1} \|. 
\] (4.335)

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By lemma (3.1) and theorem (2.11),
\[
\| \Gamma_1 \| \leq \| \nabla_u \tilde{g}(\tilde{X}) - \nabla_u \tilde{g}(X) \| + \| \nabla_{(v,w)} \tilde{g}(\tilde{X}) \bullet \nabla_u \sigma_{\tilde{v}_{j+1}}^*(\tilde{u}') - \nabla_{(v,w)} \tilde{g}(X) \bullet \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \|
\]
\[
\leq \| \nabla_{(v,w)} \tilde{g}(X) \| \| \nabla_u \sigma_{\tilde{v}_{j+1}}^* (u') - \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \| + \| \nabla_{(v,w)} \tilde{g}(\tilde{X}) - \nabla_{(v,w)} \tilde{g}(X) \| \| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(\tilde{u}') - \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \|
\]
\[
+ \Lambda_s (1 + \zeta) \| \tilde{X} - X \|. \quad (4.336)
\]

By theorem (4.6) and lemma (3.1),
\[
\| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(\tilde{u}') - \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \| \leq \| \nabla_u \sigma_{\tilde{v}_{j+1}}^* \| \| \tilde{u}' - u' \|. \quad (4.337)
\]

By (4.333;4.335;4.336;4.337), there exists a constant \( \chi_3 \) such that
\[
\| \Gamma_1 \| \leq \| \nabla_{(v,w)} \tilde{g}(X) \| \| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') - \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \|
\]
\[
+ \chi_3 \| \tilde{v}_{j+1} - v_{j+1} \|. \quad (4.338)
\]

Similarly, by (4.328), there exists a constant \( \chi_4 \) such that
\[
\| \Gamma_2 \| \leq \left\{ \begin{array}{l}
\| \nabla_u \hat{f}(\tilde{X}) + \nabla_{(v,w)} \hat{f}(\tilde{X}) \bullet \nabla_u \sigma_{\hat{v}_{j+1}}^*(\hat{u}') \| \\
\| \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \bullet \nabla_u \sigma_{\hat{v}_{j+1}}^*(u') \| \\
\leq \chi_4 \| \hat{v}_{j+1} - v_{j+1} \|
\end{array} \right\}. \quad (4.339)
\]

Notice that
\[
\nabla_{(v,w)} \hat{f}(0, v_{j+1}, 0) = 0;
\]
by lemma (3.1) and theorem (2.11),
\[
\| \nabla_{(v,w)} \hat{f}(X) \| \leq \Lambda_s \| (u', \sigma_{v_{j+1}}^*(u')) \|
\]
\[
\leq (1 + \zeta) \Lambda_s \| u' \| \leq 2 \kappa \Lambda_s. \quad (4.340)
\]
Notice that

\[ A = \left[ I + [\nabla_u \hat{f}(\tilde{X})]^{-1} \bullet \nabla_{(v,w)} \hat{f}(\tilde{X}) \bullet \nabla_u \sigma_{\tilde{v}_{j+1}}^*(\tilde{u}') \right]^{-1} \bullet [\nabla_u \hat{f}(\tilde{X})]^{-1}, \]

where \( I \) denotes the identity map. Moreover,

\[ \| [\nabla_u \hat{f}(\tilde{X})]^{-1} \bullet \nabla_{(v,w)} \hat{f}(\tilde{X}) \bullet \nabla_u \sigma_{\tilde{v}_{j+1}}^*(\tilde{u}') \| \leq \Lambda^2 \zeta. \]

Thus, when \( \zeta \) is sufficiently small,

\[ \| I + [\nabla_u \hat{f}(\tilde{X})]^{-1} \bullet \nabla_{(v,w)} \hat{f}(\tilde{X}) \bullet \nabla_u \sigma_{\tilde{v}_{j+1}}^*(\tilde{u}') \| \leq \frac{1}{1 - \zeta \Lambda^2} \leq \frac{4}{3}. \]

By (4.341;4.343),

\[ \| A \| \leq \frac{4}{3} \| \nabla_u \hat{f}(\tilde{X}) \|^{-1}. \]

By (4.340;4.344;4.339),

\[ \| \Gamma_2 \| \leq \frac{16}{9} \| [\nabla_u \hat{f}(\tilde{X})]^{-1} \| \| \nabla_{(v,w)} \hat{f}(\tilde{X}) \| \left\{ 2\kappa \Lambda \| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') - \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \| + \chi_4 \| \tilde{v}_{j+1} - v_{j+1} \| \right\}. \]

By (4.338;4.344),

\[ \| A \| \| \Gamma_1 \| \leq \frac{4}{3} \| \nabla_u \hat{f}(\tilde{X}) \|^{-1} \| \nabla_{(v,w)} \hat{g}(X) \| \]

\[ \| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') - \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \| + \chi_3 \| \nabla_u \hat{f}(\tilde{X}) \|^{-1} \| \tilde{v}_{j+1} - v_{j+1} \|. \]

By (4.327;4.345;4.346), there exist constants \( \chi_5 \) and \( \chi_6 \) such that

\[ \| P^{(1)} \| \leq \left[ \frac{4}{3} \| \nabla_u \hat{f}(\tilde{X}) \|^{-1} \| \nabla_{(v,w)} \hat{g}(X) \| + \kappa \chi_5 \right] \]

\[ \| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') - \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') \| \]

\[ + \chi_6 \| \tilde{v}_{j+1} - v_{j+1} \|. \]
Similarly, there exist constants $\chi_7$ and $\chi_8$ such that
\[
\| P^{(2)} \| \leq \left[ \frac{4}{3} \| [\nabla_u \hat{f}(\tilde{X})]^{-1} \| \| \nabla_{(v,w)} \hat{h}(X) \| + \kappa \chi_7 \right] \\
\| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') - \nabla_u \sigma_{\tilde{v}_j}(u') \| \\
+ \chi_8 \| \tilde{v}_{j+1} - v_{j+1} \|. \tag{4.348}
\]
By (4.325;4.347;4.348),
\[
\| \nabla_u \sigma_{\tilde{v}_j}(u) - \nabla_u \sigma_{\tilde{v}_j}(u') \| \\
\leq \lambda_0 \| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') - \nabla_u \sigma_{\tilde{v}_j}(u') \| \\
+ (\chi_6 + \chi_8) \| \tilde{v}_{j+1} - v_{j+1} \|. \tag{4.349}
\]
where
\[
\lambda_0 = \frac{4}{3} \| [\nabla_u \hat{f}(\tilde{X})]^{-1} \| \left( \| \nabla_{(v,w)} \hat{g}(X) \| + \| \nabla_{(v,w)} \hat{h}(X) \| \right) + \kappa (\chi_5 + \chi_7).
\]
By lemma (3.1) there exists a constant $\chi_9$ such that
\[
\lambda_0 \leq \frac{4}{3} \| [\nabla_u \hat{f}(0, \tilde{v}_{j+1}, 0)]^{-1} \| \left( \| \nabla_{(v,w)} \hat{g}(0, v_{j+1}, 0) \| \\
+ \| \nabla_{(v,w)} \hat{h}(0, v_{j+1}, 0) \| \right) + \kappa \chi_9.
\]
By (4.317;4.318) and lemma (4.8), for a sufficiently large $T$, when $\delta$ is sufficiently small; moreover, taking $\kappa$ small enough, we have
\[
\lambda_0 \leq \frac{3}{4}. \tag{4.350}
\]
By (4.349;4.350),
\[
\| \nabla_u \sigma_{\tilde{v}_j}(u) - \nabla_u \sigma_{\tilde{v}_j}(u') \| \\
\leq \frac{3}{4} \| \nabla_u \sigma_{\tilde{v}_{j+1}}^*(u') - \nabla_u \sigma_{\tilde{v}_j}(u') \| \\
+ (\chi_6 + \chi_8) \| \tilde{v}_{j+1} - v_{j+1} \|. \tag{4.351}
\]
Taking supremum with respect to $u$ and $u'$ in (4.351), we have
\[
\| \nabla_u \sigma_{\tilde{v}_j} - \nabla_u \sigma_{\tilde{v}_j}^* \| \\
\leq \frac{3}{4} \| \nabla_u \sigma_{\tilde{v}_{j+1}}^* - \nabla_u \sigma_{\tilde{v}_j} \| \\
+ (\chi_6 + \chi_8) \| \tilde{v}_{j+1} - v_{j+1} \|. \tag{4.352}
\]
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Taking the limit \( \tilde{v}_j \to v_j \) in (4.352), we have

\[
Z_{v_j}(0) \leq \gamma Z_{v_{j+1}}(0), \quad \gamma = \frac{3}{4};
\]

which is inequality (4.324). This completes the proof of the lemma. ♦

Formally differentiating (4.273) with respect to \( v \), we have:

\[
[(\nabla_u \sigma^*_v) \bullet \Delta v](u) = \left( [H(\nabla_u \sigma^*_v)]_v \bullet \Delta v \right)(u); \quad (4.353)
\]

\[
\left( [H(\nabla_u \sigma^*_v)]_v \bullet \Delta v \right)(u) = \left( H^{(1)}, H^{(2)} \right);
\]

in which,

\[
H^{(1)} \equiv \nabla_{(v,w)} \tilde{g}(X) \bullet \left[ (\nabla_u \sigma^*_v \bullet \Delta v')(u') \right] - \left( \nabla_u \tilde{g}(X) + \nabla_{(v,w)} \tilde{g}(X) \bullet \nabla_u \sigma^*_v \bullet \Delta v'(u') \right) - \left[ \nabla_u f(X) + \nabla_{(v,w)} f(X) \bullet \nabla_u \sigma^*_v \bullet \Delta v'(u') \right]^{-1} \bullet \nabla_u f(X) \bullet \Delta v';
\]

\[
K^{(1)} \equiv \nabla_{(v,w)} \tilde{g}(X) \bullet \Delta v' - \nabla_{(v,w)} \tilde{g}(X_0) \bullet \Delta v';
\]

\[
H^{(2)} \equiv \nabla_{(v,w)} \tilde{h}(X) \bullet \left[ (\nabla_u \sigma^*_v \bullet \Delta v')(u') \right] - \left( \nabla_u \tilde{h}(X) + \nabla_{(v,w)} \tilde{h}(X) \bullet \nabla_u \sigma^*_v \bullet \Delta v'(u') \right) - \left[ \nabla_u f(X) + \nabla_{(v,w)} f(X) \bullet \nabla_u \sigma^*_v \bullet \Delta v'(u') \right]^{-1} \bullet \nabla_u f(X) \bullet \Delta v';
\]

\[
K^{(2)} \equiv \nabla \tilde{h}(X) \bullet \Delta v' - \left( \nabla \tilde{h}(X) + \nabla_{(v,w)} \tilde{h}(X) \bullet \nabla_u \sigma^*_v \bullet \Delta v'(u') \right) - \left[ \nabla_u f(X) + \nabla_{(v,w)} f(X) \bullet \nabla_u \sigma^*_v \bullet \Delta v'(u') \right]^{-1} \bullet \nabla_u f(X) \bullet \Delta v';
\]

where

\[
\Delta v' = [\nabla \tilde{g}(0, v', 0)]^{-1} \bullet \Delta v;
\]

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\[ X \equiv (u', \sigma^*_v(u') + (v', 0)), \]
\[ X_0 \equiv (0, v', 0), \]
\[ u = \hat{f}(u', \sigma^*_v(u') + (v', 0)), \]
\[ v = \hat{g}(0, v', 0). \]

If \( \sigma^*_v \) is \( C^1 \) in \( v \), then
\[ \nabla_v \sigma^* \in C^0(B_k, L(V_k, C^0(R^1, V_k \times R^1))); \]
moreover,
\[ (\nabla_v \sigma^*_v \cdot \Delta v)(0) = 0, \forall v \in B_k, \Delta v \in V_k. \]

Denote by \( \Sigma_1 \) the space:
\[ \Sigma_1 \equiv \left\{ \Psi \mid \Psi \in C^0(B_k, L(V_k, C^0(R^1, V_k \times R^1))); \right. \\
(\Psi_v \cdot \Delta v)(0) = 0, \forall v \in B_k, \Delta v \in V_k \right\}. \]

For any \( \Psi \in \Sigma_1 \), define the norm:
\[ \| \Psi \| \equiv \sup_{v \in B_k} \| \Psi_v \|, \quad (4.354) \]
where \( \| \Psi_v \| \) is the linear operator norm
\[ \| \Psi_v \| \equiv \sup_{\Delta v \in V_k} \frac{\| \Psi_v \cdot \Delta v \|}{\| \Delta v \|}, \]
\[ \| \Psi_v \cdot \Delta v \| \equiv \sup_{u \in R_k} \left\{ \| \Psi_v \cdot \Delta v(u) \| / \| u \| \right\}. \]

**Lemma 4.30** \( \Sigma_1 \) is a complete metric space under the norm defined in (4.354).

Proof: Along the line laid down in the proof of lemma (3.5), this lemma follows immediately. ♣

**Definition 13** For any \( \Psi \in \Sigma_1 \), define \( H(\Psi) \) by replacing \( \nabla_v \sigma^* \) by \( \Psi \) in (4.353).

**Lemma 4.31** For any \( \Psi \in \Sigma_1 \), \( H(\Psi) \in \Sigma_1 \).
Proof: By definition, for any $\Psi \in \Sigma_1$,

$$[H(\Psi)]_v \in L(V_k, C^0(R^1_\eta, V_k \times R^1)), \ \forall v \in B_k.$$  

By lemma (4.28), $\sigma^*_v$ is Lipschitz in $v$. By assumption, $\Psi_v$ is $C^0$ in $v$. By lemma (4.29), $\nabla u \sigma^*_v$ is $C^0$ in $v$. All other terms in (4.353) are also $C^0$ in $v$. Then

$$H(\Psi) \in C^0(B_k, L(V_k, C^0(R^1_\eta, V_k \times R^1))).$$

Next, we want to show that

$$\| H(\Psi) \| < \infty.$$  

First we estimate $\| K^{(1)} \|$: By lemma (3.1) and theorem (2.11),

$$\| \nabla_v \hat{g}(X) \bullet \Delta v' - \nabla_v \hat{g}(X_0) \bullet \Delta v' \| \leq \| \nabla_v \hat{g}(X) - \nabla_v \hat{g}(X_0) \| \| \Delta v' \|$$

$$\leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \nabla_v \hat{g}(\alpha u', \alpha \sigma^*_v(u') + (v', 0)) \| \| (u', \sigma^*_v(u')) \| \| \Delta v' \|$$

$$\leq \Lambda^* (1 + \zeta) \| u' \| \| \Delta v' \| . \quad (4.355)$$

Since $\nabla_v \hat{f}(X_0) = 0$; then,

$$\| \nabla_v \hat{f}(X) \bullet \Delta v' \| \leq \| \nabla_v \hat{f}(X) - \nabla_v \hat{f}(X_0) \| \| \Delta v' \|$$

$$\leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \nabla_v \hat{f}(\alpha u', \alpha \sigma^*_v(u') + (v', 0)) \| \| (u', \sigma^*_v(u')) \| \| \Delta v' \|$$

$$\leq \Lambda^* (1 + \zeta) \| u' \| \| \Delta v' \| . \quad (4.356)$$

We know that

$$\Delta v' = [\nabla_v \hat{g}(0, v', 0)]^{-1} \bullet \Delta v. \quad (4.357)$$

By (4.118:4.116),

$$\| u' \| \leq \frac{\| \nabla_u \hat{f}(0, v', 0) \|^{-1}}{1 - (\kappa C_\kappa + \zeta C_\zeta) \Lambda} \| u \|. \quad (4.358)$$

Then, by (4.355:4.356:4.357:4.358), we have:

$$\sup_{v \in B_k, \Delta v \in V_k, u \in R^1_\eta} \left\{ \frac{\| K^{(1)} \|}{\| \Delta v \| \| u \|} \right\} < \infty. \quad (4.359)$$

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By assumption, \( \| \Psi \| < \infty. \) \( (4.360) \)

By (4.359;4.360),

\[
\sup_{v \in B_k, \Delta v \in V_k, u \in R_\eta} \left\{ \frac{\| H^{(1)} \|}{\| \Delta v \| \| u \|} \right\} < \infty.
\]

Similarly,

\[
\sup_{v \in B_k, \Delta v \in V_k, u \in R_\eta} \left\{ \frac{\| H^{(2)} \|}{\| \Delta v \| \| u \|} \right\} < \infty.
\]

Thus,

\[ \| H(\Psi) \| < \infty. \]

This completes the proof of the lemma. ♣

**Lemma 4.32** \( H \) is a contraction map in \( \Sigma_1 \) under the norm defined in (4.354).

Proof: By (4.353), for any \( \Psi_i \in \Sigma_1, i = 1, 2; \)

\[
\left( [H(\Psi_1) - H(\Psi_2)]_u \cdot \Delta v \right)(u) \equiv \left( \Delta H^{(1)}, \Delta H^{(2)} \right);
\]

\[
(4.361)
\]

in which,

\[
\Delta H^{(1)} \equiv \nabla_{(v,w)} \hat{g}(X) \cdot \left( [(\Psi_1 - \Psi_2)_{v'} \cdot \Delta v'](u') \right) - \left[ \nabla_u \hat{g}(X) + \nabla_{(v,w)} \hat{g}(X) \cdot \nabla_u^* \sigma_{v'}(u') \right] - \left[ \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \cdot \nabla_u^* \sigma_{v'}(u') \right]^{-1} \left[ \nabla_{(v,w)} \hat{f}(X) \cdot \left( [(\Psi_1 - \Psi_2)_{v'} \cdot \Delta v'](u') \right) \right],
\]

\[
\Delta H^{(2)} \equiv \nabla_{(v,w)} \hat{h}(X) \cdot \left( [(\Psi_1 - \Psi_2)_{v'} \cdot \Delta v'](u') \right) - \left[ \nabla_u \hat{h}(X) + \nabla_{(v,w)} \hat{h}(X) \cdot \nabla_u^* \sigma_{v'}(u') \right] - \left[ \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \cdot \nabla_u^* \sigma_{v'}(u') \right]^{-1} \left[ \nabla_{(v,w)} \hat{f}(X) \cdot \left( [(\Psi_1 - \Psi_2)_{v'} \cdot \Delta v'](u') \right) \right].
\]
Thus, by (4.357;4.358),

$$
\frac{\| (H(\Psi_1) - H(\Psi_2))_v \cdot \Delta v(u) \|}{\| \Delta v \| \| u \|} \leq \lambda \frac{\| (\Psi_1 - \Psi_2)_v \cdot \Delta v'(u') \|}{\| \Delta v' \| \| u' \|},
$$

(4.362)

where

$$
\lambda \equiv P_0(P_1 + P_2 + P_3 + P_4),
$$

$$
P_0 = \frac{\| \nabla_u \hat{f}(0, v', 0) \|^{-1} \| \nabla_v \hat{g}(0, v', 0) \|^{-1}}{1 - (\kappa C_\kappa + \zeta C_\zeta) \Lambda},
$$

$$
P_1 = \| \nabla_{(v,w)} \hat{g}(X) \|,
$$

$$
P_2 = \| \nabla_u \hat{g}(X) + \nabla_{(v,w)} \hat{g}(X) \cdot \nabla_u \sigma^*_v(u') \|
$$

$$
\| \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \cdot \nabla_u \sigma^*_v(u') \|^{-1} \| \nabla_{(v,w)} \hat{f}(X) \|,
$$

$$
P_3 = \| \nabla_{(v,w)} \hat{h}(X) \|,
$$

$$
P_4 = \| \nabla_u \hat{h}(X) + \nabla_{(v,w)} \hat{h}(X) \cdot \nabla_u \sigma^*_v(u') \|
$$

$$
\| \nabla_u \hat{f}(X) + \nabla_{(v,w)} \hat{f}(X) \cdot \nabla_u \sigma^*_v(u') \|^{-1} \| \nabla_{(v,w)} \hat{f}(X) \|. \tag{4.365}
$$

Notice that \( \nabla_{(v,w)} \hat{f}(X_0) = 0 \) ; then by lemma (3.1) and theorem (2.11),

$$
\| \nabla_{(v,w)} \hat{f}(X) \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \nabla_{(v,w)} \hat{f}(\alpha u', \alpha \sigma^*_v(u')
$$

$$
\quad + (v', 0)) \| \| (u', \sigma^*_v(u')) \| \leq \Lambda_* (1 + \zeta) \| u' \| \leq \hat{\eta} \Lambda_* (1 + \zeta), \tag{4.363}
$$

where \( \hat{\eta} \) can be taken to equal \( \kappa \). Notice that by (4.190),

$$
\| \nabla_u \sigma^*_v(u') \| \leq \zeta. \tag{4.364}
$$

By (4.363;4.364),

$$
P_2 \leq 4\kappa \Lambda^2, \quad P_4 \leq 4\kappa \Lambda^2. \tag{4.365}
$$

By lemma (3.1) and theorem (2.11),

$$
P_1 \leq \| \nabla_{(v,w)} \hat{g}(X_0) \| + 2\kappa \Lambda_* \tag{4.366}
$$

$$
P_3 \leq \| \nabla_{(v,w)} \hat{g}(X_0) \| + 2\kappa \Lambda_* \tag{4.367}
$$

By (4.317;4.318) and lemma (4.8), for sufficiently large \( T \), when \( \delta \) is sufficiently small, and taking \( \kappa \) small enough, we have from (4.365;4.366;4.367) that

$$
\lambda \equiv P_0(P_1 + P_2 + P_3 + P_4) \leq \frac{3}{4}. \tag{4.369}
$$
Then (4.362) implies that
\[ \| H(\Psi_1) - H(\Psi_2) \| \leq \frac{3}{4} \| \Psi_1 - \Psi_2 \|. \]
This completes the proof of the lemma. ♣

**Corollary 9** \( H \) has a unique fixed point \( \Psi^* \in \Sigma_1 \), \( H(\Psi^*) = \Psi^* \). In particular, \( \| \Psi^* \| < \infty \).

**Proof:** This corollary follows immediately from lemmas (4.30;4.31;4.32). ♣

**Theorem 4.7** For any \( v \in B_k \), \( \nabla_v \sigma^* \) exists and equals \( \Psi^* \). Therefore, \( \nabla_v \sigma^* \in C^0(B_k,L(V_k,C^0(R^1_{\eta},V_k \times R^1))) \), i.e. \( \sigma^* \) is \( C^1 \) in \( v \).

**Proof:** For any \( v \in B_k \), define an increasing nonnegative function:
\[
\Delta_v : (0,1) \rightarrow R,
\]
\[
\Delta_v(a) \equiv \sup_{\tilde{v} \in B_k, 0 < \| \tilde{v} - v \| < a} \frac{\| \sigma^*_v - \sigma^*_v - \Psi^*_v \cdot (\tilde{v} - v) \|}{\| \tilde{v} - v \|},
\]
where
\[
\| \sigma^*_v - \sigma^*_v - \Psi^*_v \cdot (\tilde{v} - v) \| \equiv \sup_{u \in R^1_{\eta}} \frac{\| \sigma^*_v(u) - \sigma^*_v(u) - [\Psi^*_v \cdot (\tilde{v} - v)](u) \|}{\| u \|}.
\]

By lemma (4.28) and corollary (9), there exists a constant \( \chi_1 \), such that
\[
\nabla_v(a) \leq \chi_1, \forall v \in B_k, \forall a \in (0,1).
\] (4.368)

For any \( v \in B_k \), since \( \Delta_v(a) \) is an increasing nonnegative function, the limit
\[
\lim_{a \rightarrow 0} \Delta_v(a)
\]
exists. Denote this limit by \( \Delta_v(0) \). By (4.368),
\[
0 \leq \Delta_v(0) \leq \chi_1, \forall v \in B_k.
\] (4.369)

For any \( v_1 \in B_k \), define the sequence:
\[
\{v_j\}, ~ j = 1, 2, \ldots ;
\]
\[
v_j = \hat{g}(0,v_{j+1},0).
\]

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To prove the theorem, we need to show that:

$$\Delta v_1(0) = 0.$$  

We will show that the inequality

$$\Delta v_j(0) \leq \gamma \Delta v_{j+1}(0), \quad \forall j = 1, 2, \ldots; \quad (0 < \gamma < 1) \quad (4.370)$$

is valid. Then by (4.369;4.370),

$$\Delta v_1(0) \leq \chi_1 \gamma^m, \quad \forall m \in \mathbb{Z}^+.$$  

Thus

$$\Delta v_1(0) = 0.$$  

Next we prove the inequality (4.370). For any \( j \in \mathbb{Z}^+ \), there exists a small constant \( a_j \), such that, when \( \| \tilde{v}_j - v_j \| < a_j \), all the Taylor expansions below are valid. By corollary (9) and (4.273), for any \( u \in \mathbb{R} \eta \):

$$\sigma_{v_j}(u) - \sigma_{v_j}(u) - \left[ \Psi_{v_j} \bullet (\tilde{v}_j - v_j) \right](u) = (I_1, I_2) - \left[ (H(\Psi^{*}))_{v_j} \bullet (\tilde{v}_j - v_j) \right](u), \quad (4.371)$$

where

$$I_1 = \hat{g}(u_1, \sigma_{v_{j+1}}(\tilde{u}_1)) - \hat{g}(u_1, \sigma_{v_{j+1}}(u_1)) + \left[ \hat{g}(0, \tilde{v}_{j+1}, 0) - \hat{g}(0, v_{j+1}, 0) \right],$$

$$I_2 = \hat{h}(u_1, \sigma_{v_{j+1}}(\tilde{u}_1)) - \hat{h}(u_1, \sigma_{v_{j+1}}(u_1)) + \left[ \hat{h}(0, v_{j+1}, 0) \right],$$

$$v_j = \hat{g}(0, v_{j+1}, 0),$$

$$\tilde{v}_j = \hat{g}(0, \tilde{v}_{j+1}, 0),$$

$$u = \hat{f}(u_1, \sigma_{v_{j+1}}(u_1)) + (v_{j+1}, 0) = \hat{f}(u_1, \sigma_{v_{j+1}}(\tilde{u}_1)) + (v_{j+1}, 0).$$  

By lemma (4.28),

$$\| \sigma_{v_j}(u') - \sigma_{v'}(u') \| \leq \chi \| v - v' \| \quad \forall v, v' \in \mathbb{R}_1^k, \quad \forall u \in \mathbb{R}_1.$$  

(4.373)

By (4.275;4.372) and (4.294), there exist constants \( b_1 \) and \( b_2 \) such that

$$\| \tilde{u}_1 - u_1 \| \leq b_1 \| \sigma_{\tilde{v}_{j+1}}(u_1) - \sigma_{\tilde{v}_{j+1}}(u_1) \|
+ b_2 \| \tilde{v}_{j+1} - v_{j+1} \| \leq a_1.$$  

(4.374)
By (4.373:4.374), there exists a constant $b_3$ such that
\[
\| \tilde{u}_1 - u_1 \| \leq b_3 \| \tilde{v}_{j+1} - v_{j+1} \| \| u_1 \| .
\] (4.375)

Notice that
\[
\sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_{v_{j+1}}(u_1) = \left[ \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_{v_{j+1}}(\tilde{u}_1) \right] \\
+ \left[ \sigma^*_{v_{j+1}}(\tilde{u}_1) - \sigma^*_{v_{j+1}}(u_1) \right] = \left[ \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_{v_{j+1}}(\tilde{u}_1) \right] \\
+ \nabla_v \sigma^*_{v_{j+1}}(u_1) \bullet (\tilde{u}_1 - u_1) + O(\| \tilde{u}_1 - u_1 \|^2). (4.376)
\]

By (4.373:4.375:4.376), there exists a constant $b_4$ such that
\[
\| \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_{v_{j+1}}(u_1) \| \leq b_4 \| \tilde{v}_{j+1} - v_{j+1} \| \| u_1 \| .
\] (4.377)

By (4.372),
\[
\left[ \hat{f}(\tilde{u}_1, \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) + (\tilde{v}_{j+1}, 0)) - \hat{f}(\tilde{u}_1, \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1)) \right] \\
+ \left[ \hat{f}(\tilde{u}_1, \sigma^*_{v_{j+1}}(\tilde{u}_1) + (v_{j+1}, 0)) \\
- \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)) \right] = 0. (4.378)
\]

In order to get a relation between $(\tilde{u}_1 - u_1)$ and $(\sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_{v_{j+1}}(u_1))$ from (4.378), we first estimate the quantity:
\[
\Pi \equiv \hat{f}(\tilde{u}_1, \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) + (\tilde{v}_{j+1}, 0)) - \hat{f}(\tilde{u}_1, \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1)) \\
- \nabla_v \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)) \bullet (\tilde{v}_{j+1} - v_{j+1}) \\
= \left[ \hat{f}(\tilde{u}_1, \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) + (\tilde{v}_{j+1}, 0)) - \nabla_v \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)) \bullet \tilde{v}_{j+1} \right] \\
- \left[ \hat{f}(\tilde{u}_1, \sigma^*_{v_{j+1}}(\tilde{u}_1) + (v_{j+1}, 0)) - \nabla_v \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)) \bullet v_{j+1} \right] \\
\equiv \Theta(\tilde{v}_{j+1}) - \Theta(v_{j+1}), (4.379)
\]

where
\[
\Theta(x) \equiv \hat{f}(\tilde{u}_1, \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) + (x, 0)) \\
- \nabla_v \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)) \bullet x; \ \forall x \in B_k.
\]
Apply lemma (3.1) to (4.379), so that we have

\[
\| \Pi \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_v \Theta(\alpha v_{j+1} + (1 - \alpha) \tilde{v}_{j+1}) \| \| \tilde{v}_{j+1} - v_{j+1} \|
\]

\[
= \sup_{0 \leq \alpha \leq 1} \| \nabla_v \hat{f}(\tilde{u}_1, \sigma_{v_{j+1}}^*(\tilde{u}_1) + (\alpha v_{j+1} + (1 - \alpha) \tilde{v}_{j+1}, 0)) \|
\]

\[
- \nabla_v \hat{f}(u_1 \sigma_{v_{j+1}}^*(u_1) + (v_{j+1}, 0)) \| \| \tilde{v}_{j+1} - v_{j+1} \|. \quad (4.380)
\]

Moreover,

\[
\rho \equiv \| \nabla_v \hat{f}(\tilde{u}_1, \sigma_{v_{j+1}}^*(\tilde{u}_1) + (\alpha v_{j+1} + (1 - \alpha) \tilde{v}_{j+1}, 0)) \|
\]

\[
- \nabla_v \hat{f}(u_1 \sigma_{v_{j+1}}^*(u_1) + (v_{j+1}, 0)) \|
\]

\[
\leq \| \nabla_v \hat{f}(\tilde{u}_1, \sigma_{v_{j+1}}^*(\tilde{u}_1) + (\alpha v_{j+1} + (1 - \alpha) \tilde{v}_{j+1}, 0)) \|
\]

\[
- \nabla_v \hat{f}(u_1 \sigma_{v_{j+1}}^*(u_1) + (\alpha v_{j+1} + (1 - \alpha) \tilde{v}_{j+1}, 0)) \|
\]

\[
+ \| \nabla_v \hat{f}(u_1 \sigma_{v_{j+1}}^*(u_1) + (\alpha v_{j+1} + (1 - \alpha) \tilde{v}_{j+1}, 0)) \|
\]

\[
- \nabla_v \hat{f}(u_1 \sigma_{v_{j+1}}^*(u_1) + (v_{j+1}, 0)) \| \| \tilde{v}_{j+1} - v_{j+1} \|. \quad (4.381)
\]

By (4.375:4.377:4.381) and theorem (2.11),

\[
\rho \leq \Lambda_*(b_3 + b_4) \| \tilde{v}_{j+1} - v_{j+1} \| \| u_1 \|
\]

\[
+ (1 - \alpha) \sup_{0 \leq \beta \leq 1} \| \nabla_v \nabla_v \hat{f}(u_1, \sigma_{v_{j+1}}^*(u_1) \|
\]

\[
+ (x_{(\alpha, \beta), 0}) \| \| \tilde{v}_{j+1} - v_{j+1} \|. \quad (4.382)
\]

where

\[
x_{(\alpha, \beta), 0} \equiv \beta [\alpha v_{j+1} + (1 - \alpha) \tilde{v}_{j+1}] + (1 - \beta)v_{j+1}.
\]

Notice that,

\[
\nabla_v \nabla_v \hat{f}(0, v, 0) = 0, \quad \forall v \in B_k;
\]

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By Taylor expansion,
\[
\rho_1 \equiv \| \nabla_v \nabla_v \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (x(\alpha, \beta), 0)) \| \\
\leq \sup_{0 \leq \alpha_1 \leq 1} \| \nabla_{(u,v,u)} \nabla_v \hat{f}(\alpha_1 u_1, \alpha_1 \sigma^*_{v_{j+1}}(u_1)) \\
+ \langle x(\alpha, \beta), 0 \rangle \| (u_1, \sigma^*_{v_{j+1}}(u_1)) \|.
\]
(4.383)

By theorem (2.11),
\[
\rho_1 \leq \Lambda_1 (1 + \zeta) \| u_1 \|.
\]
(4.384)

By (4.382:4.384),
\[
\rho \leq \Lambda_1 [b_3 + b_4 + (1 - \alpha)(1 + \zeta)] \| \tilde{v}_{j+1} - v_{j+1} \| \| u_1 \|.
\]
(4.385)

By (4.380:4.385),
\[
\| \Pi \| \leq \Lambda_1 [b_3 + b_4 + 2] \| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|.
\]
(4.386)

By (4.379:4.386),
\[
\hat{f}(\hat{u}_1, \sigma^*_{v_{j+1}}(\hat{u}_1) + (\tilde{v}_{j+1}, 0)) - \hat{f}(\hat{u}_1, \sigma^*_{v_{j+1}}(\hat{u}_1) + (v_{j+1}, 0)) \\
= \nabla_v \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)) \cdot (\tilde{v}_{j+1} - v_{j+1}) \\
+ O(\| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|).
\]
(4.387)

By Taylor expansion,
\[
\Pi_1 \equiv \hat{f}(\hat{u}_1, \sigma^*_{v_{j+1}}(\hat{u}_1) + (v_{j+1}, 0)) - \hat{f}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)) \\
= \nabla_u \hat{f}(X_1) \cdot (\tilde{u}_1 - u_1) + \nabla_{(v,w)} \hat{f}(X_1) \cdot \left[ \sigma^*_{v_{j+1}}(\hat{u}_1) - \sigma^*_{v_{j+1}}(u_1) \right] \\
+ O(\| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|^2),
\]
(4.388)

where \( X_1 \equiv (u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0). \) By (4.376) and (4.388),
\[
\Pi_1 = \left[ \nabla_u \hat{f}(X_1) + \nabla_{(v,w)} \hat{f}(X_1) \cdot \nabla_u \sigma^*_{v_{j+1}}(u_1) \right] \cdot (\tilde{u}_1 - u_1) \\
+ \nabla_{(v,w)} \hat{f}(X_1) \cdot \left[ \sigma^*_{v_{j+1}}(\tilde{u}_1) - \sigma^*_{v_{j+1}}(\hat{u}_1) \right] \\
+ O(\| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|^2).
\]
(4.389)

\[
\tilde{u}_1 - u_1 = - \left[ \nabla_u \hat{f}(X_1) + \nabla_{(v,w)} \hat{f}(X_1) \cdot \nabla_u \sigma^*_{v_{j+1}}(u_1) \right]^{-1}
\]

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\[ \nabla_{(v,w)} \hat{f}(X_1) \bullet \left[ \sigma_{u_j+1}(\tilde{u}_1) - \sigma_{v_j+1}(\tilde{u}_1) \right] \]

\[ + \ \nabla_v \hat{f}(X_1) \bullet (\tilde{v}_{j+1} - v_{j+1}) \]

\[ + \ O(\| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|). \]

(4.390)

Next, we estimate \( I_1 \) defined in (4.371).

\[ I_1 = I_{11} + I_{12}, \]

(4.391)

\[ I_{11} \equiv \hat{g}(\tilde{u}_1, \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) + (\tilde{v}_{j+1}, 0)) - \hat{g}(u_1, \sigma^*_{v_{j+1}}(u_1) + (v_{j+1}, 0)), \]

\[ I_{12} \equiv \left[ \hat{g}(u_1, \sigma^*_{v_{j+1}}(u_1) + (\tilde{v}_{j+1}, 0)) - \hat{g}(u_1, \sigma^*_{\tilde{v}_{j+1}}(u_1) + (v_{j+1}, 0)) \right] \]

\[ - \left[ \hat{g}(0, \tilde{v}_{j+1}, 0) - \hat{g}(0, v_{j+1}, 0) \right]. \]

We first estimate:

\[ \hat{\hat{I}}_{12} = \hat{I}_{12} - \left[ \nabla_v \hat{g}(X_1) \bullet [\nabla_v \hat{g}(X_{10})]^{-1} \bullet (\tilde{v}_j - v_j) \right] \]

\[ - \ \nabla_v \hat{g}(X_{10}) \bullet [\nabla_v \hat{g}(X_{10})]^{-1} \bullet (\tilde{v}_j - v_j) \]  

(4.392)

where

\[ X_1 \equiv (u_1, \sigma^*_{\tilde{v}_{j+1}}(u_1) + (v_{j+1}, 0)), \]

\[ X_{10} \equiv (0, v_{j+1}, 0). \]

Define

\[ \varphi_1(u', v', w') \equiv \hat{g}(u', v' + \tilde{v}_{j+1}, w') - \hat{g}(u', v' + v_{j+1}, w') \]

\[ - \left[ \nabla_v \hat{g}(u', v' + v_{j+1}, w') \bullet [\nabla_v \hat{g}(X_{10})]^{-1} \bullet (\tilde{v}_j - v_j) \right]. \]

Then,

\[ \hat{I}_{12} = \varphi_1(u_1, \sigma^*_{v_{j+1}}(u_1)) - \varphi_1(0, 0, 0). \]

Thus,

\[ \| \hat{I}_{12} \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \varphi_1(\alpha u_1, \alpha \sigma^*_{v_{j+1}}(u_1)) \| \| (u_1, \sigma^*_{v_{j+1}}(u_1)) \| \]

\[ \equiv \sup_{0 \leq \alpha \leq 1} \rho^{(1)}(1 + \zeta) \| u_1 \|, \]

(4.393)
where
\[ \rho_{\alpha}^{(1)} \equiv \| \nabla_{(u,v,w)} \hat{g}(z_\alpha + (0, \tilde{v}_{j+1}, 0)) - \nabla_{(u,v,w)} \hat{g}(z_\alpha + (0, v_{j+1}, 0)) \]
\[ - \left[ \nabla_{(u,v,w)} \nabla_v \hat{g}(z_\alpha + (0, v_{j+1}, 0)) \right] \cdot (\tilde{v}_{j+1} - v_{j+1}) \]
\[ \cdot \left[ \nabla_v \hat{g}(X_{10}) \right]^{-1} \cdot (\tilde{v}_j - v_j) \|, \]
where \( z_\alpha \equiv (\alpha u_1, \alpha \sigma_{v_{j+1}}^*(u_1)) \). Notice that
\[ \tilde{v}_j - v_j = \left[ \nabla_v \hat{g}(X_{10}) \right] \cdot (\tilde{v}_{j+1} - v_{j+1}) + O(\| \tilde{v}_{j+1} - v_{j+1} \|^2). \]

Then,
\[ \rho_{\alpha}^{(1)} \leq \| \nabla_{(u,v,w)} \hat{g}(z_\alpha + (0, \tilde{v}_{j+1}, 0)) - \nabla_{(u,v,w)} \hat{g}(z_\alpha + (0, v_{j+1}, 0)) \]
\[ - \left[ \nabla_{(u,v,w)} \nabla_v \hat{g}(z_\alpha + (0, v_{j+1}, 0)) \right] \cdot (\tilde{v}_{j+1} - v_{j+1}) \|
\[ + O(\| \tilde{v}_{j+1} - v_{j+1} \|^2). \] (4.394)

Notice that
\[ \nabla_{(u,v,w)} \hat{g}(z_\alpha + (0, \tilde{v}_{j+1}, 0)) - \nabla_{(u,v,w)} \hat{g}(z_\alpha + (0, v_{j+1}, 0)) \]
\[ = \left[ \nabla_{(u,v,w)} \nabla_v \hat{g}(z_\alpha + (0, v_{j+1}, 0)) \right] \cdot (\tilde{v}_{j+1} - v_{j+1}) \] \[ + O(\| \tilde{v}_{j+1} - v_{j+1} \|^2). \] (4.395)

Then (4.394; 4.395) imply that there exists a constant \( b_5 \) such that
\[ \rho_{\alpha}^{(1)} \leq b_5 \| \tilde{v}_{j+1} - v_{j+1} \|^2. \] (4.396)

By (4.393; 4.396),
\[ \| I_{12} \| \leq 2b_5 \| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|. \] (4.397)

By (4.392; 4.397),
\[ I_{12} = \left[ \nabla_v \hat{g}(X_{1}) - \nabla_v \hat{g}(X_{10}) \right] \cdot \left[ \nabla_v \hat{g}(X_{10}) \right]^{-1} \cdot (\tilde{v}_j - v_j) + O(\| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|). \] (4.398)

Next we estimate \( I_{11} \) (4.391). By Taylor expansion,
\[ I_{11} = \nabla_a \hat{g}(X_2) \cdot (\tilde{u}_1 - u_1) + \nabla_{(v,w)} \hat{g}(X_2) \cdot \left( \sigma_{v_{j+1}}^*(\tilde{u}_1) - \sigma_{v_{j+1}}^*(u_1) \right) + O(\| \tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|^2), \] (4.399)
where $X_2 \equiv (u_1, \sigma^*_v(u_1) + (\tilde{v}_{j+1}, 0))$. By (4.376),

\[
I_{11} = \left[ \nabla_u \hat{g}(X_2) + \nabla_{(v,w)} \hat{g}(X_2) \cdot \nabla_u \sigma^*_v(u_1) \right] \cdot (\tilde{u}_1 - u_1) + \nabla_{(v,w)} \hat{g}(X_2) \cdot \left[ \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_v(u_1) \right] + O(||\tilde{v}_{j+1} - v_{j+1}||^2 ||u_1||^2). \tag{4.400}
\]

Notice that

\[
X_2 = X_1 + (0, \tilde{v}_{j+1} - v_{j+1}, 0), \tag{4.401}
\]

by (4.373;4.375),

\[
I_{11} = \left[ \nabla_u \hat{g}(X_1) + \nabla_{(v,w)} \hat{g}(X_1) \cdot \nabla_u \sigma^*_v(u_1) \right] \cdot (\tilde{u}_1 - u_1) + \nabla_{(v,w)} \hat{g}(X_1) \cdot \left[ \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_v(u_1) \right] + O(||\tilde{v}_{j+1} - v_{j+1}||^2 ||u_1||). \tag{4.402}
\]

By (4.391;4.398;4.402),

\[
I_1 = \nabla_{(v,w)} \hat{g}(X_1) \cdot \left[ \sigma^*_{\tilde{v}_{j+1}}(\tilde{u}_1) - \sigma^*_v(u_1) \right] + \nabla_{(v,w)} \hat{g}(X_1) \cdot \nabla_u \sigma^*_v(u_1) \cdot (\tilde{u}_1 - u_1) + [\nabla_v \hat{g}(X_1) - \nabla_{(v,w)} \hat{g}(X_{10})] \cdot [\nabla_u \hat{g}(X_{10})]^{-1} \cdot (\tilde{v}_j - v_j) + O(||\tilde{v}_{j+1} - v_{j+1}||^2 ||u_1||). \tag{4.403}
\]

Notice that

\[
[H(\Psi^*)]_{v_j} \cdot (\tilde{v}_j - v_j) (u) = \left( H^{(1)}(\Psi^*), H^{(2)}(\Psi^*) \right); \tag{4.404}
\]

in which:

\[
H^{(1)}(\Psi^*) \equiv \nabla_{(v,w)} \hat{g}(X_1) \cdot \left[ (\Psi^*_{\tilde{v}_{j+1}} \cdot \Delta v_{j+1})(u_1) \right] - \left[ \nabla_u \hat{g}(X_1) + \nabla_{(v,w)} \hat{g}(X_1) \cdot \nabla_u \sigma^*_v(u_1) \right] \cdot \left[ \nabla_u \hat{f}(X_1) + \nabla_{(v,w)} \hat{f}(X_1) \cdot \nabla_u \sigma^*_v(u_1) \right]^{-1} \cdot \left( \nabla_{(v,w)} \hat{f}(X_1) \cdot (\Psi^*_{\tilde{v}_{j+1}} \cdot \Delta v_{j+1})(u_1) \right) + K^{(1)}(\Psi^*);
\]

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\[
K^{(1)}(\Psi^*) = \left( \nabla_v \tilde{g}(X_1) - \nabla_v \tilde{g}(X_{10}) \right) \cdot \Delta v_{j+1}
- \left[ \nabla_u \tilde{g}(X_1) + \nabla_{(v,w)} \tilde{g}(X_1) \cdot \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]
- \left[ \nabla_u \tilde{f}(X_1) + \nabla_{(v,w)} \tilde{f}(X_1) \cdot \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]^{-1}
- \left[ \nabla_v \tilde{f}(X_1) \cdot \Delta v_{j+1} \right] ;
\]

\[
H^{(2)}(\Psi^*) = \nabla_{(v,w)} \tilde{h}(X_1) \cdot \left[ (\Psi_{v_{j+1}}^* \cdot \Delta v_{j+1})(u_1) \right]
- \left[ \nabla_u \tilde{h}(X_1) + \nabla_{(v,w)} \tilde{h}(X_1) \cdot \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]
- \left[ \nabla_u \tilde{f}(X_1) + \nabla_{(v,w)} \tilde{f}(X_1) \cdot \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]^{-1}
+ \left( \nabla_{(v,w)} \tilde{f}(X_1) \cdot \left[ (\Psi_{v_{j+1}}^* \cdot \Delta v_{j+1})(u_1) \right] \right)
+ K^{(2)}(\Psi^*);
\]

\[
K^{(2)}(\Psi^*) = \nabla_v \tilde{h}(X_1) \cdot \Delta v_{j+1} - \left[ \nabla_u \tilde{h}(X_1) \right]
+ \nabla_{(v,w)} \tilde{h}(X_1) \cdot \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]
- \left[ \nabla_u \tilde{f}(X_1) + \nabla_{(v,w)} \tilde{f}(X_1) \cdot \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]^{-1}
+ \left[ \nabla_v \tilde{f}(X_1) \cdot \Delta v_{j+1} \right] ;
\]

where
\[
\Delta v_{j+1} = \left[ \nabla_v \tilde{g}(X_{10}) \right]^{-1} \cdot (\tilde{v}_j - v_j),
X_1 = (u_1, \sigma_{v_{j+1}}^*(u_1) + (v_{j+1}, 0)),
X_{10} = (0, v_{j+1}, 0).
\]

Moreover,
\[
\Delta v_{j+1} = \tilde{v}_{j+1} - v_{j+1} + O(\| \tilde{v}_{j+1} - v_{j+1} \|^2).
\]

We know that
\[
\nabla_v \tilde{f}(X_{10}) = 0;
\]

then by lemma (3.1),
\[
\| \nabla_v \tilde{f}(X_1) \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \nabla_v \tilde{f}(\alpha u_1, \alpha \sigma_{v_{j+1}}^*(u_1) + (v_{j+1}, 0)) \| \cdot \| (u_1, \sigma_{v_{j+1}}^*(u_1)) \| .
\]

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By theorem (2.11),
\[ ||\nabla \hat{f}(X_1) || \leq (1 + \zeta)\Lambda \| u_1 \|. \] (4.406)

By (4.405, 4.406),
\[ \nabla \hat{f}(X_1) \bullet (\tilde{v}_{j+1} - v_{j+1}) = \nabla \hat{f}(X_1) \bullet \Delta v_{j+1} + O(||\tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \||). \] (4.407)

By (4.390, 4.407),
\[ \tilde{u}_1 - u_1 = - \left[ \nabla_u \hat{f}(X_1) + \nabla_{(v,w)} \hat{f}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]^{-1} \]
\[ \cdot \left( \nabla_{(v,w)} \hat{f}(X_1) \bullet \left[ \sigma_{v_{j+1}}^*(\tilde{u}_1) - \sigma_{v_{j+1}}^*(\tilde{u}_1) \right] \right) \]
\[ + \nabla \hat{f}(X_1) \bullet \Delta v_{j+1} + O(||\tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \||). \] (4.408)

By (4.403, 4.404, 4.408),
\[ I_1 - K^{(1)}(\Psi^*) = \nabla_{(v,w)} \tilde{g}(X_1) \bullet \left[ \sigma_{v_{j+1}}^*(\tilde{u}_1) - \sigma_{v_{j+1}}^*(\tilde{u}_1) \right] \]
\[ - \left[ \nabla_u \hat{g}(X_1) + \nabla_{(v,w)} \hat{g}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^*(u_1) \right] \]
\[ \cdot \left( \nabla_{(v,w)} \hat{g}(X_1) \bullet \left[ \sigma_{v_{j+1}}^*(\tilde{u}_1) - \sigma_{v_{j+1}}^*(\tilde{u}_1) \right] \right) \]
\[ + O(||\tilde{v}_{j+1} - v_{j+1} \|^2 \| u_1 \||). \] (4.409)

Notice that
\[ ||[\sigma_{v_{j+1}}^*(\tilde{u}_1) - \sigma_{v_{j+1}}^*(\tilde{u}_1)] - [\sigma_{v_{j+1}}^*(u_1) - \sigma_{v_{j+1}}^*(u_1)] || \]
\[ \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_u \sigma_{v_{j+1}}^*(\alpha u_1 + (1 - \alpha)\tilde{u}_1) - \nabla_u \sigma_{v_{j+1}}^*(\alpha u_1 + (1 - \alpha)\tilde{u}_1) \| \| \tilde{u}_1 - u_1 \|. \] (4.410)

By lemma (4.29), \( \nabla_u \sigma_v^* \) is \( C^0 \) in \( v \), we have
\[ \sup_{u \in \mathbb{R}^l_u} \| \nabla_u \sigma_{v_{j+1}}^*(u) - \nabla_u \sigma_{v_{j+1}}^*(u) \| \rightarrow 0, \text{ as } \tilde{v}_{j+1} \rightarrow v_{j+1}. \] (4.411)
By (4.375;4.410;4.411),
\[
\begin{align*}
\sigma^* v_j + 1 (u_1) - \sigma^* v_j + 1 (u_1) &= O(r(\tilde{v}_j + 1, v_j + 1) \parallel \tilde{v}_j + 1 - v_j + 1 \parallel u_1),
\end{align*}
\]
where
\[
r(\tilde{v}_j + 1, v_j + 1) \to 0, \text{ as } \parallel \tilde{v}_j + 1 - v_j + 1 \parallel \to 0.
\]

By (4.404;4.405;4.409;4.412),
\[
I_1 - H^{(1)}(\Psi^*) = \lambda^{(1)} \left\{ \sigma^* v_j + 1 (u_1) - \sigma^* v_j + 1 (u_1) \right\} \\
- \left[ \Psi^* v_j + 1 \cdot (\tilde{v}_j + 1 - v_j + 1) \right] (u_1) \\
+ O(r^{(1)}(\tilde{v}_j + 1, v_j + 1) \parallel \tilde{v}_j + 1 - v_j + 1 \parallel u_1),
\]
where
\[
\lambda^{(1)} = \nabla_{(v,w)} \hat{g}(X_1) - \left[ \nabla_u \hat{g}(X_1) + \nabla_{(v,w)} \hat{g}(X_1) \\
\cdot \nabla_u \sigma^* v_j + 1 (u_1) \right] \cdot \nabla_u \hat{f}(X_1) + \nabla_{(v,w)} \hat{f}(X_1) \\
\cdot \nabla_u \sigma^* v_j + 1 (u_1)^{-1} \cdot \nabla_{(v,w)} \hat{f}(X_1); \quad (4.414)
\]
moreover,
\[
r^{(1)}(\tilde{v}_j + 1, v_j + 1) \to 0, \text{ as } \parallel \tilde{v}_j + 1 - v_j + 1 \parallel \to 0.
\]
Next we estimate \( I_2 \) (4.371).
\[
I_2 = I_{21} + I_{22},
\]
\[
I_{21} = \hat{h}(\tilde{u}_1, \sigma^* v_j + 1 (u_1) + (\tilde{v}_j + 1, 0) - \hat{h}(u_1, \sigma^* v_j + 1 (u_1) + (\tilde{v}_j + 1, 0)),
\]
\[
I_{22} = \hat{h}(u_1, \sigma^* v_j + 1 (u_1) + (\tilde{v}_j + 1, 0) - \hat{h}(u_1, \sigma^* v_j + 1 (u_1) + (v_j + 1, 0)).
\]

We first estimate
\[
\hat{I}_{22} \equiv I_{22} - \nabla_v \hat{h}(X_1) \cdot \Delta v_j + 1.
\]

Define
\[
\varphi_2(u', v', w') = \hat{h}(u', v' + \tilde{v}_j + 1, w') - \hat{h}(u', v' + v_j + 1, w') \\
- \left[ \nabla_v \hat{h}(u', v' + v_j + 1, w') \right] \cdot \Delta v_j + 1.
\]

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Then, 
\[ \varphi_2(0, 0, 0) = 0. \]

Thus,
\[ \hat{I}_{22} = \varphi_2(u_1, \sigma_{v_{j+1}}^*(u_1)) - \varphi_2(0, 0, 0). \]

\[ \| \hat{I}_{22} \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \varphi_2(\alpha u_1, \alpha \sigma_{v_{j+1}}^*(u_1)) \| \| (u_1, \sigma_{v_{j+1}}^*(u_1)) \| \]
\[ \equiv \sup_{0 \leq \alpha \leq 1} \rho_{\alpha}^{(2)} (1 + \zeta) \| u_1 \|, \quad (4.417) \]

where
\[ \rho_{\alpha}^{(2)} = \| \nabla_{(u,v,w)} \hat{h}(z_\alpha + (0, \bar{v}_{j+1}, 0)) \]
\[ - \nabla_{(u,v,w)} \hat{h}(z_\alpha + (0, v_{j+1}, 0)) \]
\[ - \left[ \nabla_{(u,v,w)} \nabla_v \hat{h}(z_\alpha + (0, v_{j+1}, 0)) \right] \cdot \Delta v_{j+1} \|, \]

where 
\[ z_\alpha \equiv (\alpha u_1, \alpha \sigma_{v_{j+1}}^*(u_1)). \]

By (4.405), then
\[ \rho_{\alpha}^{(2)} \leq \| \nabla_{(u,v,w)} \hat{h}(z_\alpha + (0, \bar{v}_{j+1}, 0)) \]
\[ - \nabla_{(u,v,w)} \hat{h}(z_\alpha + (0, v_{j+1}, 0)) \]
\[ - \left[ \nabla_{(u,v,w)} \nabla_v \hat{h}(z_\alpha + (0, v_{j+1}, 0)) \right] \cdot (\bar{v}_{j+1} - v_{j+1}) \| + O(\| \bar{v}_{j+1} - v_{j+1} \|^2). \quad (4.418) \]

Notice that
\[ \nabla_{(u,v,w)} \hat{h}(z_\alpha + (0, \bar{v}_{j+1}, 0)) - \nabla_{(u,v,w)} \hat{h}(z_\alpha + (0, v_{j+1}, 0)) \]
\[ = \left[ \nabla_{(u,v,w)} \nabla_v \hat{h}(z_\alpha + (0, v_{j+1}, 0)) \right] \cdot (\bar{v}_{j+1} - v_{j+1}) \]
\[ + O(\| \bar{v}_{j+1} - v_{j+1} \|^2). \quad (4.419) \]

Then (4.418;4.419) imply that there exists a constant \( b_6 \) such that
\[ \rho_{\alpha}^{(2)} \leq b_6 \| \bar{v}_{j+1} - v_{j+1} \|^2. \quad (4.420) \]

By (4.417;4.420),
\[ \| \hat{I}_{22} \| \leq 2b_6 \| \bar{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|. \quad (4.421) \]
By (4.416:4.421),
\[ I_{22} = \nabla_v \dot{h}(X_1) \bullet \Delta v_{j+1} + O(\| \dot{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|). \] (4.422)

Next we estimate \( I_{21} \) (4.415). By Taylor expansion,
\[
I_{21} = \nabla_u \dot{h}(X_2) \bullet (\ddot{u}_1 - u_1) + \nabla_{(v,w)} \dot{h}(X_2)
\]
\[ \bullet \left( \sigma_{\ddot{v}_{j+1}}^\ast (\ddot{u}_1) - \sigma_{v_{j+1}}^\ast (u_1) \right) + O(\| \ddot{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|^2), \] (4.423)

where \( X_2 \equiv (u_1, \sigma_{v_{j+1}}^\ast (u_1) + (\ddot{v}_{j+1}, 0)) \). By (4.376),
\[
I_{21} = \left[ \nabla_u \dot{h}(X_2) + \nabla_{(v,w)} \dot{h}(X_2) \bullet \nabla_u \sigma_{v_{j+1}}^\ast (u_1) \right]
\]
\[ \bullet (\ddot{u}_1 - u_1) + \nabla_{(v,w)} \dot{h}(X_2) \bullet \left[ \sigma_{\ddot{v}_{j+1}}^\ast (\ddot{u}_1) - \sigma_{v_{j+1}}^\ast (\ddot{u}_1) \right] + O(\| \ddot{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|^2). \] (4.424)

By (4.373:4.375:4.401),
\[
I_{21} = \left[ \nabla_u \dot{h}(X_1) + \nabla_{(v,w)} \dot{h}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^\ast (u_1) \right]
\]
\[ \bullet (\ddot{u}_1 - u_1) + \nabla_{(v,w)} \dot{h}(X_1) \bullet \left[ \sigma_{\ddot{v}_{j+1}}^\ast (\ddot{u}_1) - \sigma_{v_{j+1}}^\ast (\ddot{u}_1) \right] + O(\| \ddot{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|). \] (4.425)

By (4.415:4.422:4.425),
\[
I_2 = \left[ \nabla_u \dot{h}(X_1) + \nabla_{(v,w)} \dot{h}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^\ast (u_1) \right]
\]
\[ \bullet (\ddot{u}_1 - u_1) + \nabla_{(v,w)} \dot{h}(X_1) \bullet \left[ \sigma_{\ddot{v}_{j+1}}^\ast (\ddot{u}_1) - \sigma_{v_{j+1}}^\ast (\ddot{u}_1) \right] + \nabla_u \dot{h}(X_1) \bullet \Delta v_{j+1} + O(\| \ddot{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|). \] (4.426)

By (4.404:4.408:4.426),
\[
I_2 - K^{(2)}(\Psi^\ast) = \left[ \nabla_{(v,w)} \dot{h}(X_1) \bullet \left[ \sigma_{\ddot{v}_{j+1}}^\ast (\ddot{u}_1) - \sigma_{v_{j+1}}^\ast (\ddot{u}_1) \right] \right]
\]
\[ - \left[ \nabla_u \dot{h}(X_1) + \nabla_{(v,w)} \dot{h}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^\ast (u_1) \right]
\]
\[ \bullet \left[ \nabla_u \tilde{f}(X_1) + \nabla_{(v,w)} \tilde{f}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^\ast (u_1) \right]^{-1}
\]
\[ \bullet \left( \nabla_{(v,w)} \tilde{f}(X_1) \bullet \left[ \sigma_{\ddot{v}_{j+1}}^\ast (\ddot{u}_1) - \sigma_{v_{j+1}}^\ast (\ddot{u}_1) \right] \right) \]
\[ + O(\| \ddot{v}_{j+1} - v_{j+1} \|^2 \| u_1 \|). \] (4.427)
By (4.404;4.405;4.412;4.427),
\[
I_2 - H^{(2)}(\Psi^*) = \lambda^{(2)} \left\{ \sigma_{\tilde{v}_{j+1}}^*(u_1) - \sigma_{v_{j+1}}^*(u_1) \right. \\
- \left[ \Psi_{v_{j+1}}^* \bullet (\tilde{v}_{j+1} - v_{j+1}) \right] (u_1) \right\} \\
+ O(r^{(2)}(\tilde{v}_{j+1}, v_{j+1}) \parallel \tilde{v}_{j+1} - v_{j+1} \parallel \parallel u_1 \parallel),
\]
where
\[
\lambda^{(2)} = \nabla_{(v,w)} \hat{h}(X_1) - \left[ \nabla_u \hat{h}(X_1) + \nabla_{(v,w)} \hat{h}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^*(u_1) \right] \\
\cdot \left[ \nabla_u \hat{f}(X_1) + \nabla_{(v,w)} \hat{f}(X_1) \bullet \nabla_u \sigma_{v_{j+1}}^*(u_1) \right]^{-1} \\
\cdot \nabla_{(v,w)} \hat{f}(X_1);
\]
moreover,
\[
r^{(2)}(\tilde{v}_{j+1}, v_{j+1}) \to 0, \quad \text{as} \parallel \tilde{v}_{j+1} - v_{j+1} \parallel \to 0.
\]
By (4.371;4.404;4.413;4.428),
\[
\parallel \sigma_{\tilde{v}_{j}}^*(u) - \sigma_{v_{j}}^*(u) - \left[ \Psi_{v_{j}}^* \bullet (\tilde{v}_{j} - v_{j}) \right] (u) \parallel \\
\leq \left( \parallel \lambda^{(1)} \parallel + \parallel \lambda^{(2)} \parallel \right) \left\{ \parallel \sigma_{\tilde{v}_{j+1}}^*(u_1) - \sigma_{v_{j+1}}^*(u_1) \parallel \\
- \left[ \Psi_{v_{j+1}}^* \bullet (\tilde{v}_{j+1} - v_{j+1}) \right] (u_1) \right\} \\
+ O(r^{(3)}(\tilde{v}_{j+1}, v_{j+1}) \parallel \tilde{v}_{j+1} - v_{j+1} \parallel \parallel u_1 \parallel),
\]
where \(\lambda^{(1)}, \lambda^{(2)}\) are given in (4.414;4.429),
\[
r^{(3)}(\tilde{v}_{j+1}, v_{j+1}) \to 0, \quad \text{as} \parallel \tilde{v}_{j+1} - v_{j+1} \parallel \to 0.
\]
By (4.372) and (4.118;4.116),
\[
\parallel u_1 \parallel \leq \frac{\parallel [\nabla_u \hat{f}(0,v_{j+1},0)]^{-1} \parallel}{1 - (\kappa C' + \zeta C' \zeta)\Lambda} \parallel u \parallel.
\]
By (4.297;4.298),
\[
\parallel \tilde{v}_{j+1} - v_{j+1} \parallel \leq \sup_{v' \in B_k} \parallel [\nabla_v \hat{g}(0,v',0)]^{-1} \parallel \parallel \tilde{v}_j - v_j \parallel.
\]
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Thus, when $\kappa$ and $\zeta$ are sufficiently small,
\[
\| u_1 \| \| \tilde{v}_{j+1} - v_{j+1} \| \leq \tau \| u \| \| \tilde{v}_j - v_j \|, 
\]  
(4.433)
where
\[
\tau = \frac{5}{4} \| \nabla_u f(0, v_{j+1}, 0)^{-1} \| \sup_{v' \in B_k} \| \nabla_v \hat{g}(0, v', 0)^{-1} \|. 
\]  
(4.434)
By (4.430;4.433),
\[
\| \sigma^*_{\tilde{v}_j}(u) - \sigma^*_{v_j}(u) - [\Psi^*_{v_j} \bullet (\tilde{v}_j - v_j)](u) \| \leq \rho \| u \| \| \tilde{v}_j - v_j \| 
\]  
(4.435)
where
\[
\rho \equiv \tau(\| \chi(1) \| + \| \chi(2) \|), 
\]  
\[
r^{(4)}(\tilde{v}_{j+1}, v_{j+1}) \rightarrow 0, \text{ as } \| \tilde{v}_{j+1} - v_{j+1} \| \rightarrow 0. 
\]  
Next we estimate $\rho$. First we estimate $\| \chi(1) \|$ and $\| \chi(2) \|$ : Since $E$ is an invariant bundle,
\[
\nabla_u \hat{g}(X_1) = \nabla_u \hat{h}(X_1) = 0; 
\]  
then by lemma (3.1),
\[
\| \nabla_u \hat{g}(X_1) \| \leq \sup_{0 \leq \alpha \leq 1} \| \nabla_{(u,v,w)} \nabla_u \hat{g}(\alpha u_1, \alpha \sigma^*_{v_{j+1}}(u_1) \\
+ (v_{j+1}, 0)) \| \| (u_1, \sigma^*_{v_{j+1}}(u_1)) \| 
\]  
(4.436)
similarly,
\[
\| \nabla_u \hat{h}(X_1) \| \leq (1 + \zeta)\Lambda_* \| u_1 \| \leq (1 + \zeta)\Lambda_* \kappa; 
\]  
(4.437)
By (4.190),
\[
\| \nabla_u \sigma^*_{v_{j+1}}(u_1) \| \leq \zeta. 
\]  
(4.438)
By (4.436;4.437;4.438), there exist constants $b_7$ and $b_8$ such that
\[
\| \chi(1) \| \leq \| \nabla_{(v,w)} \hat{g}(X_1) \| + b_7 \kappa + b_8 \zeta, 
\]  
(4.439)
\[
\| \chi(2) \| \leq \| \nabla_{(v,w)} \hat{h}(X_1) \| + b_7 \kappa + b_8 \zeta. 
\]  
(4.440)
Thus,

\[ \rho \leq \frac{5}{4} \left\| \left[ \nabla_u \hat{f}(X_{10}) \right]^{-1} \sup_{v' \in B_k} \left\| \left[ \nabla_v \hat{g}(0, v', 0) \right]^{-1} \right\| \right. 
\left. \cdot \left( \left\| \nabla_{(v,w)} \hat{g}(X_1) \right\| + \left\| \nabla_{(v,w)} \hat{h}(X_1) \right\| 
+ 2(b_7 \kappa + b_8 \zeta) \right) \right) \right]. \tag{4.441} \]

By (4.317;4.318) and lemma (4.8), for sufficiently large \( T \), when \( \delta, \kappa \) and \( \zeta \) are sufficiently small, we have from (4.441):

\[ \rho \leq \frac{4}{5}. \tag{4.442} \]

Taking supremum with respect to \( u \) and \( u_1 \), and taking limit \( \| \tilde{v}_j - v_j \| \to 0 \) in (4.435), we have:

\[ \Delta v_j(0) \leq \frac{4}{5} \Delta v_{j+1}(0), \]

which is inequality (4.370) with \( \gamma = 4/5 \). This completes the proof of the theorem. ♣

Additional smoothness follows similarly. We summarize in the following:

**Theorem 4.8** \( \sigma^*_v \) is \( C^{a-1} \) smooth in \( v \). Thus, the family of fibers \( \{ f^E(Q) \}_{Q \in M} \) is \( C^{a-1} \) in \( Q \).

Proof: The proof is similar to that given above for \( \sigma^*_v \) being \( C^1 \) in \( v \). Here, we only sketch the proof. If \( \sigma^*_v \) is \( C^1 \) in \( v \), then

\[ \nabla_v \sigma^*_v \in C^0(B_k, L^s(V_k, C^0(R^1, V_k \times R^1))); \]

Moreover,

\[ \left[ \nabla_v \sigma^*_v \cdot (\Delta v^1, \ldots, \Delta v^s) \right](0) = 0, \ \forall v \in B_k, \ \forall \Delta v^j \in V_k, (1 \leq j \leq s). \]

Denote by \( \Sigma_s \) the space:

\[ \Sigma_s = \left\{ \psi(s) \mid \psi(s) \in C^0(B_k, L^s(V_k, C^0(R^1, V_k \times R^1))); \right. 
\left. \left[ \psi_v(s) \cdot (\Delta v^1, \ldots, \Delta v^s) \right](0) = 0, \ \forall v \in B_k, 
\forall \Delta v^j \in V_k, (j = 1, \ldots, s). \right\} \]
For any $\Psi^{(s)} \in \Sigma_s$, define the norm:

$$
\| \Psi^{(s)} \| \equiv \sup_{v \in B_k} \| \Psi^{(s)} v \|,
$$

where $\| \Psi^{(s)} v \|$ is the s-linear operator norm,

$$
\| \Psi^{(s)} v \| \equiv \sup_{\Delta v \in V_k} \| \Psi^{(s)} v \| \Delta v^1 \cdots \Delta v^s \|
$$

and

$$
\| \Psi^{(s)} v \| \equiv \sup_{u \in R_k^1} \left\{ \| \Psi^{(s)} v \| (\Delta v^1, \ldots, \Delta v^s) \| u \| \right\}.
$$

A similar proof as for lemma (4.30) shows that $\Sigma_s$ is a complete metric space under the norm defined above. Formally differentiating (4.273) $s$ times with respect to $v$, we have

$$
\left[ \nabla^s v^* \cdot (\Delta v^1, \ldots, \Delta v^s) \right] (u) = \left( H_s^{(1)}(u), H_s^{(2)}(u) \right),
$$

where

$$
H_s^{(i)} \equiv \Gamma^{(i)} \left[ \nabla^s v^* \cdot (\Delta v^1, \ldots, \Delta v^s) \right] (\tilde{u}) + K_s^{(i)}, \quad (i = 1, 2);
$$

and

$$
\Gamma^{(1)} \equiv \nabla (v, w) g(X) - \left[ \nabla u g(X) + \nabla (v, w) g(X) \cdot \nabla u \sigma^*(\tilde{u}) \right]
$$

and

$$
\Gamma^{(2)} \equiv \nabla (v, w) h(X) - \left[ \nabla u h(X) + \nabla (v, w) h(X) \cdot \nabla u \sigma^*(\tilde{u}) \right]
$$

and

$$
\Delta \tilde{v}^j = \left[ \nabla u g(0, \tilde{v}, 0) \right]^{-1} \cdot \Delta v^j, \quad j = 1, \ldots, s;
$$

and

$$
X \equiv (\tilde{u}, \sigma^*(\tilde{u}) + (\tilde{v}, 0)),
$$

and

$$
u = \tilde{f}(\tilde{u}, \sigma^*(\tilde{u}) + (\tilde{v}, 0)),
$$

and

$$
v = \tilde{g}(0, \tilde{v}, 0).
$$

$K_s^{(i)} (i = 1, 2)$ contains terms not involving $\nabla^s v^*$. Nevertheless, $K_s^{(i)} (i = 1, 2)$ contains terms involving

$$
\nabla^s u \nabla^s v^* \sigma^*, \quad (0 < s' \leq s)
$$

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\[ \nabla^{s_1} \nabla^{s_2} \sigma^*, \quad (s_1 + s_2 < s; s_1, s_2 \geq 0). \]

Additional smoothness is proved inductively. For example, since \( \sigma^*_v(u) \) is \( C^1 \) in both \( u \) and \( v \), starting from the equation (4.321) which involves \( \nabla_u \sigma^* \) only, we can prove that \( \nabla_u \sigma^* \) is \( C^1 \) in \( v \), along the line laid down for the proof that \( \sigma^* \) is \( C^1 \) in \( v \). That is, \( \nabla_v \nabla_u \sigma^*_v \) exists and is \( C^0 \) in \( v \). We already know from theorem (4.6) that \( \sigma^*_v(u) \) is \( C^2 \) in \( u \), then we can prove \( \sigma^*_v \) is \( C^2 \) in \( v \) through the equation (4.443). ♣
References


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