EXISTENCE OF CHAOS IN WEAKLY QUASILINEAR SYSTEMS

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Abstract. The aim of this article is twofold: (1). develop a strategy to prove
the existence of chaos in weakly quasilinear systems, (2). strengthen the ex-
isting results on chaos in partial differential equations. First, we study a sine-
Gordon equation containing weakly quasilinear terms, and existence of chaos
is proved. Then, we study a Ginzburg-Landau equation containing weakly
quasilinear terms, and existence of chaos is proved under generic conditions.
Finally, in the Appendix, we prove the existence of chaos in a reaction-diffusion
equation.

1. Introduction. Extensive studies have been conducted on chaos in ordinary
differential equations (ODE) [1], and intensive studies are focused upon Lorenz
equations [11]. Existence of chaos in Lorenz equations has not been proved by
hand. Recently, there have been a new trend of computer proofs [13]. In general,
for near integrable ODE with transversal homoclinics, existence of chaos can often
be proved by hand.

Chaos in partial differential equations (PDE) has long been an open area. During
the last decade, a standard program was established for proving the existence of
chaos in near integrable PDE [2]. Around transversal homoclinics, existence of
chaos can be proved by hand [4, 5, 6], while around non-transversal homoclinics,
existence of chaos can be proved by hand up to nasty generic conditions [7, 8, 3].
There are also attempts of computer proofs on the existence of chaos in PDE.

In contrast to ODE, PDE has a lot of novelties. For instance, boundary condi-
tion plays a major role on dynamics. So far, all the chaos in PDE is proved under
periodic boundary condition. In fact, odd or even constraint besides the periodic
boundary condition is crucial. Often one can prove the existence of chaos under
odd or even constraint, but not in general. Periodic boundary condition permits
a representation of solutions by Fourier series. The sequence of Fourier modes in
the Fourier series leads to a natural generalization of ODE to infinite dimensional
systems. The Fourier series also makes hand calculation possible. Under other
boundary conditions, hand calculation is often impossible. More importantly, the
dynamics is dramatically different. For example, for the whole space problem (i.e.
under decaying boundary conditions), existence of chaos has not been proved. Of
course, the PDE type is fundamental to dynamics. So far, the chaos proved is for
mixed hyperbolic and parabolic semilinear systems which are near hyperbolic semi-
linear integrable systems. For semilinear systems, existence of invariant manifolds

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is easy to obtain. For hyperbolic quasilinear systems, existence of invariant manifolds is a big open problem. Even for the simple integrable derivative nonlinear Schrödinger equation (DNLS)

\[ iq_t = q_{xx} - i |q|^2 q_x , \tag{1.1} \]

existence of invariant manifolds is open while its invariant subspaces are perfectly normal [10]. This difficulty is the major obstacle toward proving the existence of chaos when the DNLS is under perturbations. In this article, we study weak quasilinearity. That is, the unperturbed systems are still hyperbolic semilinear integrable systems, while in the perturbation we introduce quasilinearity and parabolicity. We show that in such cases one can prove the existence of chaos. The main idea is to use the compactness of the semigroup to recoup for the quasilinearity, together with the perturbative nature to establish the existence of chaos. First we will study a sine-Gordon equation, then we will study a Ginzburg-Landau equation. Finally, in the Appendix, we will illustrate how to prove the existence of chaos in a reaction-diffusion equation.

Chaos in PDE is a rich area, full of potential, with a crown goal of solving the problem of turbulence. Turbulence is governed by Navier-Stokes equations. Turbulence often happens when the Reynolds number is large. Formally setting the Reynolds number to infinity in the Navier-Stokes equations, one gets the Euler equations. Euler equations are a lot like hyperbolic quasilinear integrable systems. By posing periodic boundary condition to Navier-Stokes equations, one can study turbulence in the near Euler equations regime as a canonical problem.

The article is organized as follows: Section 2 deals with the well-posedness of the sine-Gordon problem. Section 3 is on the invariant manifolds of the sine-Gordon system. In section 4, we prove the existence of a pair of transversal homoclinic orbits and chaos for the sine-Gordon system. In section 5, we present results on the Ginzburg-Landau equation. Finally, in section 6 (Appendix), we prove the existence of chaos in a reaction-diffusion equation.

2. Well-posedness of the Sine-Gordon problem. We shall study a specific system as a representative of weakly quasilinear systems. The argument can be appropriately adjusted to other weakly quasilinear systems. Consider the perturbed sine-Gordon system (PSG)

\[ \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \sin u + \epsilon a \cos t (v_x)^2 \sin u \end{pmatrix}, \tag{2.1} \]

where

\[ L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v + \epsilon u_{xx} \\ c^2 u_{xx} + \epsilon v_{xx} \end{pmatrix}, \]

which is subject to periodic boundary condition and odd constraint

\[ w(t, x + 2\pi) = w(t, x) , \quad w(t, -x) = -w(t, x) , \quad (w = u, v) , \]

where \((u, v)\) are real-valued functions of two real variables \((t, x)\), \(c\) is a parameter, \(\frac{1}{2} < c < 1\), \(\epsilon\) is a small perturbation parameter, \(\epsilon \geq 0\), and \(a\) is a real parameter. The weakly quasilinear term is represented by \(v_x\). This term will always be controlled by \(\epsilon \partial_x^2\). Replacing \(v_x\) by \(u_x\) will make the problem easier and the control by \(\epsilon \partial_x^2\) unnecessary. The PSG (2.1) is a mixed hyperbolic and parabolic quasilinear problem with weak quasilinearity and parabolicity. Existence of chaos in such systems will be proved here for the first time, while the existing results on existence of chaos were
Lemma 2.1. If \((u, v)\) solves (2.1), so do \((-u, -v)\) and \((u(t, x + \pi), v(t, x + \pi))\).

Around \((u, v) = (0, 0)\), the PSG (2.1) can be rewritten as
\[
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} + \mathcal{N},
\]
where
\[
\mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v + cu_{xx} \\ e^2u_{xx} + u + \epsilon v_{xx} \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0 \\ \sin u - u + \epsilon a \cos t(v_x)^2 \sin u \end{pmatrix}.
\]
In fact
\[
\mathcal{L} = \epsilon \partial_x^2 + \mathcal{L}_0, \quad \text{and} \quad \mathcal{L}_0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ e^2u_{xx} + u \end{pmatrix}.
\]
For any
\[
\begin{pmatrix} u \\ v \end{pmatrix} = i \sum_{k \in \mathbb{Z}} \begin{pmatrix} u_k \\ v_k \end{pmatrix} e^{ikx},
\]
where \(u_k\) and \(v_k\) are real, and \(u_{-k} = -u_k, v_{-k} = -v_k\);
\[
e^{t\mathcal{L}} \begin{pmatrix} u \\ v \end{pmatrix} = - \left( u_1 + \frac{v_1}{\sqrt{1 - c^2}} \right) e^{(\sqrt{1 - c^2} - \epsilon)t} \sin x + \left( u_1 - \frac{v_1}{\sqrt{1 - c^2}} \right) e^{-(\sqrt{1 - c^2} + \epsilon)t} \sin x + i \sum_{|k| \geq 2} e^{-\epsilon k^2 t} \left( \frac{u_k \cos \sqrt{c^2k^2 - 1}t + \frac{v_k}{\sqrt{c^2k^2 - 1}} \sin \sqrt{c^2k^2 - 1}t}{-u_k \sqrt{c^2k^2 - 1} \sin \sqrt{c^2k^2 - 1}t + v_k \cos \sqrt{c^2k^2 - 1}t} \right) e^{ikx}.
\]
This representation clearly shows that one should pose the PSG (2.1) in the space \((u, v) \in H^{n+1} \times H^n (n \geq 1)\) where \(H^n\) is the Sobolev space and \(n\) is an integer. To understand \(\sin u\) for \(u \in H^n\) in an easy way, one can start with the Taylor series for any fixed \(x \in [0, 2\pi]\),
\[
\sin u(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m + 1)!} (u(x))^{2m+1}.
\]
When \(n \geq 1\), \(H^n\) is a Banach algebra, we have
\[
\|\sin u\|_n \leq \|u\|_n e^{|u|_n}, \quad \|\sin u_1 - \sin u_2\|_n \leq e\|u_1\|_n + \|u_2\|_n \|u_1 - u_2\|_n, \quad \|\sin u_1 - u_1\|_n \leq (\|u_1\|_n + \|u_2\|_n)^2 e\|u_1\|_n + \|u_2\|_n \|u_1 - u_2\|_n.
\]
In fact, \(\sin u\) is analytic in \(u\) in \(H^n (n \geq 1)\). \(\sin u\) is uniformly Lipschitz only in \(H^0\),
\[
\|\sin u_1 - \sin u_2\|_0 \leq \|u_1 - u_2\|_0.
\]
But \(\sin u\) is not differentiable in \(H^0\). By the method of variation of parameters, (2.2) can be written in the form
\[
\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{t\mathcal{L}} \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} + \int_0^t e^{(t-\tau)\mathcal{L}} \mathcal{N} d\tau.
\]
Theorem 2.2. For any \((u(0), v(0)) \in H^{n+1} \times H^n\) \((n \geq 1)\), there is a \(T = T(\|u(0)\|_{n+1}, \|v(0)\|_n) > 0\), such that (2.2) has a unique solution \((u(t), v(t)) \in C^0([0, T], H^{n+1} \times H^n)\) with the initial condition \((u(0), v(0))\). In particular, one can choose \(T\) to be independent of \(\epsilon\). Denote by \(F^t\) the evolution operator \((u(t), v(t)) = F^t(u(0), v(0))\). For any fixed \(t \in (0, T)\), \(F^t\) is a \(C^\infty\) map in a neighborhood of \((u(0), v(0))\) in \(H^{n+1} \times H^n\). For \((u(0), v(0)) \in H^{n+3} \times H^{n+2}\), there is a \(\bar{T} = T(\|u(0)\|_{n+3}, \|v(0)\|_{n+2}) > 0\), such that for any fixed \(t \in (0, \bar{T})\), \(F^t\) is \(C^1\) in \(\epsilon\) for \(\epsilon \in [0, \epsilon_0)\) for some \(\epsilon_0 > 0\) in the \(H^{n+1} \times H^n\) norm.

Proof. The most difficult part is on the regularity of \(F^t\) in \(\epsilon\) at \(\epsilon = 0\). Formally differentiating (2.5) with respect to \(\epsilon\) at \(\epsilon = 0\), one gets

\[
\begin{pmatrix}
  u_\epsilon(t) \\
  v_\epsilon(t)
\end{pmatrix} = \int_0^t e^{(t-\tau)L_0} \begin{pmatrix}
  0 \\
  (\cos u - 1)u_\epsilon
\end{pmatrix} d\tau + t\partial_2^2 e^{\epsilon L_0} \begin{pmatrix}
  u(0) \\
  v(0)
\end{pmatrix} + \int_0^t (t-\tau)\partial_2^2 e^{(t-\tau)L_0} \begin{pmatrix}
  0 \\
  \sin u - u
\end{pmatrix} d\tau + \int_0^t e^{(t-\tau)L_0} \begin{pmatrix}
  0 \\
  a \cos \tau (v_\epsilon)^2 \sin u
\end{pmatrix} d\tau. \tag{2.7}
\]

For \((u(0), v(0)) \in H^{n+3} \times H^{n+2}\), \((n \geq 1)\); by a simple fixed point argument, this linear equation has a unique solution \((u_\epsilon(t), v_\epsilon(t)) \in C^0([0, \bar{T}], H^{n+1} \times H^n)\) where \(\bar{T} = T(\|u(0)\|_{n+3}, \|v(0)\|_{n+2}) > 0\) and \((u_\epsilon(0), v_\epsilon(0)) = (0, 0)\). Also both \((u(\epsilon, t), v(t, \epsilon))\) and \((u(0, t), v(t, 0))\) belong to \(C^0([0, \bar{T}], H^{n+3} \times H^{n+2})\). To prove that this formal derivative is indeed the derivative, one needs to show that

\[
\Delta = \sup_{t \in [0, \bar{T}]} \left\{ \|u(t, \epsilon) - u(t, 0) - \epsilon u_\epsilon(t)\|_{n+1} + \|v(t, \epsilon) - v(t, 0) - \epsilon v_\epsilon(t)\|_n \right\} \sim o(\epsilon). \tag{2.8}
\]

First one can show that \((u(t, \epsilon), v(t, \epsilon))\) is Lipschitz in \(\epsilon\) in \(H^{n+1} \times H^n\) directly from (2.5):

\[
\begin{pmatrix}
  u(t, \epsilon) - u(t, 0) \\
  v(t, \epsilon) - v(t, 0)
\end{pmatrix} = \hat{G} + \tilde{G},
\]

where

\[
\hat{G} = \begin{pmatrix}
  \hat{G}_1 \\
  \hat{G}_2
\end{pmatrix} = (e^{t\epsilon \mathcal{L}} - e^{t\epsilon L_0}) \begin{pmatrix}
  u(0) \\
  v(0)
\end{pmatrix},
\]

\[
\tilde{G} = \begin{pmatrix}
  \tilde{G}_1 \\
  \tilde{G}_2
\end{pmatrix} = \int_0^t \left[ e^{(t-\tau)L_0} \mathcal{N}(\tau, \epsilon) - e^{(t-\tau)L_0} \mathcal{N}(\tau, 0) \right] d\tau.
\]

Since \((u(0), v(0)) \in H^{n+3} \times H^{n+2}\),

\[
\|\hat{G}_1\|_{n+1} + \|\hat{G}_2\|_n \leq C\epsilon.
\]
The term $\tilde{G}$ can be splitted into $\tilde{G} = \tilde{G}^+ + \tilde{G}^-$ where
\[
\tilde{G}^+ = \int_0^t \left[ e^{(t-\tau)\mathcal{L}} \mathcal{N}(\tau, \epsilon) - e^{(t-\tau)\mathcal{L}_0} \mathcal{N}(\tau, \epsilon) \right] d\tau ,
\]
\[
\tilde{G}^- = \int_0^t e^{(t-\tau)\mathcal{L}_0} [\mathcal{N}(\tau, \epsilon) - \mathcal{N}(\tau, 0)] d\tau .
\]
Here $\tilde{G}^+$ can be further splitted into $\tilde{G}^+ = \tilde{G}^{(+, 1)} + \tilde{G}^{(+, 2)}$ where
\[
\tilde{G}^{(+, 1)} = \int_0^t \left[ e^{(t-\tau)\mathcal{L}} - e^{(t-\tau)\mathcal{L}_0} \right] \left( \begin{array}{c} 0 \\ \sin u(\tau, \epsilon) - u(\tau, \epsilon_0) \end{array} \right) d\tau ,
\]
\[
\tilde{G}^{(+, 2)} = \epsilon \int_0^t \left[ e^{(t-\tau)\mathcal{L}} - e^{(t-\tau)\mathcal{L}_0} \right] \left( \begin{array}{c} 0 \\ \cos \tau (v_x)^2 \sin u(\tau, \epsilon) \end{array} \right) d\tau .
\]
Since $(u(t, \epsilon), v(t, \epsilon)) \in H^{n+3} \times H^{n+2}$,
\[
\|\tilde{G}^{(+, 1)}\|_{n+1} + \|\tilde{G}^{(+, 2)}\|_n \leq C \epsilon . \tag{2.9}
\]
The argument goes as follows: Fix a $\epsilon_0$, then
\[
e^{(t-\tau)\mathcal{L}} \left( \begin{array}{c} 0 \\ \sin u(\tau, \epsilon_0) - u(\tau, \epsilon_0) \end{array} \right)
\]
is differentiable in $\epsilon$ everywhere for $\epsilon \geq 0$ in $H^{n+1} \times H^n$. By the mean value theorem, (2.9) is true for $\epsilon = \epsilon_0$ where $\epsilon_0$ is also arbitrary. We also obviously have
\[
\|\tilde{G}^{(+, 2)}\|_{n+1} + \|\tilde{G}^{(+, 2)}\|_n \leq C \epsilon .
\]
Similar arguments show that
\[
\|\tilde{G}^{(+, 1)}\|_{n+1} + \|\tilde{G}^{(+, 1)}\|_n \leq C_1 \tilde{T} \left( \|u(t, \epsilon) - u(t, 0)\|_{n+1} + \|v(t, \epsilon) - v(t, 0)\|_n \right) + C_2 \epsilon .
\]
When $\tilde{T}$ is small, we have
\[
\|u(t, \epsilon) - u(t, 0)\|_{n+1} + \|v(t, \epsilon) - v(t, 0)\|_n \leq C \epsilon . \tag{2.10}
\]
Starting from (2.10), one can go on to prove that the formal derivative is indeed the derivative.
\[
\left( \begin{array}{c} u(t, \epsilon) - u(t, 0) - \epsilon u_x(t) \\ v(t, \epsilon) - v(t, 0) - \epsilon v_x(t) \end{array} \right) = \tilde{E} + \tilde{E}^+ + E^- , \tag{2.11}
\]
where
\[
\tilde{E} = \left( e^{t\mathcal{L}} - e^{t\mathcal{L}_0} - \epsilon t \partial^2_x e^{(t-\tau)\mathcal{L}_0} \right) \begin{pmatrix} u(0) \\ v(0) \end{pmatrix},
\]

\[
\tilde{E} = \epsilon \left[ \int_0^t e^{(t-\tau)\mathcal{L}} \begin{pmatrix} 0 \\ \cos \tau(v_x)^2 \sin u \end{pmatrix} \right] d\tau
- \int_0^t e^{(t-\tau)\mathcal{L}_0} \begin{pmatrix} 0 \\ \cos \tau(v_x)^2 \sin u \end{pmatrix} \right] d\tau
\],

\[
E^+ = \int_0^t \left( e^{(t-\tau)\mathcal{L}} - e^{(t-\tau)\mathcal{L}_0} \right) \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] d\tau
- \epsilon \int_0^t (t-\tau) \partial^2_x e^{(t-\tau)\mathcal{L}_0} \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] d\tau,
\]

\[
E^- = \int_0^t e^{(t-\tau)\mathcal{L}_0} \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] d\tau
- \int_0^t e^{(t-\tau)\mathcal{L}_0} \begin{pmatrix} 0 \\ (\cos u - 1)u \end{pmatrix} \right] d\tau.
\]

Since \((u(0), v(0)) \in H^{n+3} \times H^{n+2},
\]

\[
\|\tilde{E}_1\|_{n+1} + \|\tilde{E}_2\|_n \sim o(\epsilon).
\]

The first integral in \(\tilde{E}\) is continuous in \(H^{n+1} \times H^n\), thus

\[
\|\tilde{E}_1\|_{n+1} + \|\tilde{E}_2\|_n \sim o(\epsilon).
\]

The term \(E^+\) can be splitted into \(E^+ = E^{(+,1)} + E^{(+,2)}\) where

\[
E^{(+,1)} = \int_0^t \left( e^{(t-\tau)\mathcal{L}} - e^{(t-\tau)\mathcal{L}_0} - \epsilon(t-\tau) \partial^2_x e^{(t-\tau)\mathcal{L}_0} \right) \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] d\tau,
\]

\[
E^{(+,2)} = \epsilon \int_0^t (t-\tau) \partial^2_x e^{(t-\tau)\mathcal{L}_0} \left[ \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] - \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] d\tau.
\]

To estimate \(E^{(+,1)}\), define

\[
f(\epsilon) = \left( e^{(t-\tau)\mathcal{L}} - e^{(t-\tau)\mathcal{L}_0} - \epsilon(t-\tau) \partial^2_x e^{(t-\tau)\mathcal{L}_0} \right) \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] \right)_{(\tau,\epsilon)}.
\]

Since \((u(\tau, \epsilon_0), v(\tau, \epsilon_0)) \in H^{n+3} \times H^{n+2}, f(\epsilon)\) is \(C^1\) in \(\epsilon\), thus

\[
\|f_1(\epsilon_0) - f_1(0)\|_{n+1} + \|f_2(\epsilon_0) - f_2(0)\|_n \leq \epsilon_0 (\|f'_1(\epsilon)\|_{n+1} + \|f'_2(\epsilon)\|_n),
\]

where \(\epsilon \in [0, \epsilon_0],
\]

\[
f'(\epsilon) = \left[ (t-\tau) \partial^2_x e^{(t-\tau)\mathcal{L}} - (t-\tau) \partial^2_x e^{(t-\tau)\mathcal{L}_0} \right] \begin{pmatrix} 0 \\ \sin u - u \end{pmatrix} \right] \right)_{(\tau,\epsilon_0)}.
\]

As \(\epsilon_0 \to 0,
\]

\[
\|f'_1(\epsilon)\|_{n+1} + \|f'_2(\epsilon)\|_n \to 0.
\]

Thus

\[
\|E^{(+,1)}\|_{n+1} + \|E^{(+,1)}\|_n \sim o(\epsilon).
\]
Obviously
\[ \|E_1^{(+,2)}\|_{n+1} + \|E_2^{(+,2)}\|_n \sim o(\epsilon) . \]
Finally, by the Taylor expansion of sin \( u \),
\[ \sin u(\tau, \epsilon) - u(\tau, \epsilon) = [\sin u(\tau, 0) - u(\tau, 0)] \]
\[ = [\cos u(\tau, 0) - 1] (u(\tau, \epsilon) - u(\tau, 0)) + o(\|u(\tau, \epsilon) - u(\tau, 0)\|_{n+1}) . \]
By (2.10),
\[ o(\|u(\tau, \epsilon) - u(\tau, 0)\|_{n+1}) \sim o(\epsilon) . \]
Thus
\[ \|E_1^-\|_{n+1} + \|E_2^-\|_n \leq C\tilde{T}\Delta + o(\epsilon) , \]
where \( \Delta \) is defined in (2.8). Then put together all the estimates above, we have
\[ \Delta \sim o(\epsilon) . \]
Thus the formal derivative is indeed the derivative. Notice that the above arguments do not need the inequality (2.6). When proving the existence of a solution to (2.5) and the differentiability of \( F^t \) in \( \epsilon \) at \( \epsilon > 0 \), one will need the inequality (2.6). But the arguments are easier due to the analyticity of \( e^{\epsilon \partial_x^2} \) when \( \epsilon > 0 \). In fact, for any \( (u(0), v(0)) \in H^{n+1} \times H^n \), \( F^t \) is \( C^1 \) in \( \epsilon \) when \( \epsilon > 0 \) in the same \( H^{n+1} \times H^n \) norm.

Remark 2.3. Differentiability of \( F^t \) in \( \epsilon \) at \( \epsilon = 0 \) is crucial. It serves as a bridge from the \( \epsilon = 0 \) integrable theory to the \( \epsilon > 0 \) chaos theory.

3. Invariant manifolds. For quasilinear hyperbolic systems like the DNLS (1.1), existence of invariant manifolds is an open problem due to the quasilinearity. In fact, this is the main obstacle toward proving the existence of chaos in such systems under perturbations. The quasilinearity can often be controlled by sectorial operators. In our current system (2.2), the quasilinearity represented by the term containing \( v_x \) is weak and measured by \( \epsilon \). The sectorial operator is \( \epsilon \partial_x^2 \) which is also weak and measured by \( \epsilon \). In such a case, invariant manifold results can be obtained and take a novel form.

The \( C_0 \) semigroup generated by \( L \) has the representation (2.4). Notice that when \( \epsilon = 0 \), there are gaps among the unstable, stable and center eigenvalues. When \( \epsilon > 0 \), only the gap between the unstable and the rest eigenvalues is still \( O(1) \) as \( \epsilon \to 0^+ \). The gap between the stable and the center eigenvalues is smeared away. In fact, the center eigenvalues are all turned into stable eigenvalues, the largest real part of which is \(-4\epsilon\). Nevertheless, the following theorem on invariant manifolds and fibers can be established. The PSG (2.1) can be enlarged into an autonomous system by introducing an auxiliary variable \( \theta \) (\( \dot{\theta} = 1 \)). The fixed point \( (u, v) = (0, 0) \) is turned into a periodic orbit \( S \).

Theorem 3.1. For any \( n \geq 1 \), consider the phase space \( H^{n+1} \times H^n \times T_1 \), in an order \( O(1) \) neighborhood of \( S \), \( S \) has a 2-dimensional \( C^m \) \( (m \geq 3) \) center-unstable manifold \( W^{cu} \) and a 1-codimensional \( C^m \) center-stable manifold \( W^{cs} \). For some \( \epsilon_0 > 0 \), \( W^{cu} \) is \( C^1 \) in \( \epsilon \) for \( \epsilon \in [0, \epsilon_0] \). At \( (u, v) \in H^{n+3} \times H^{n+2} \), \( W^{cu} \) is \( C^1 \) in \( \epsilon \) for \( \epsilon \in [0, \epsilon_0] \). There is a \( C^m \) invariant family of 1-dimensional \( C^m \) unstable fibers \( \{ F^u(\theta) : \theta \in S \} \) such that
\[ W^{cu} = \bigcup_{\theta \in S} F^u(\theta) . \]
In an order $O(\sqrt{\epsilon})$ neighborhood of $S$, there is a $C^m$ invariant family of $2$-codimensional $C^m$ stable fibers $\{F^s(\theta) : \theta \in S\}$ such that $W^{cs}$ restricted to the $O(\sqrt{\epsilon})$ neighborhood of $S$ has the representation

$$W^{cs} = \bigcup_{\theta \in S} F^s(\theta).$$

There are positive constants $\kappa_u = \frac{3}{4}\sqrt{1-c^2}$, $\kappa_s = 3\epsilon$, and $C$ such that

$$\|F^s(q^-) - F^s(\theta)\| \leq Ce^{\kappa_u t}\|q^- - \theta\|, \forall t \in (-\infty, 0], \forall \theta \in S, \forall q^- \in F^u(\theta),$$

$$\|F^s(q^+) - F^s(\theta)\| \leq Ce^{-\kappa_s t}\|q^+ - \theta\|, \forall t \in [0, +\infty), \forall \theta \in S, \forall q^+ \in F^s(\theta),$$

where $F^s$ is the evolution operator of $(2.1)$.

Proof. In an order $O(1)$ [in $\epsilon$] neighborhood of $S$, the existence of a center-unstable manifold $W^{cu}$ is induced by the existence of unstable fibers $F^u(\theta)$ following from a proof as in [7]. The existence of a center-stable manifold $W^{cs}$ is proved directly as in [7]. Orbits inside $W^{cu}$ can have slow growth. In an order $O(\sqrt{\epsilon})$ neighborhood of $S$, one can prove the existence of stable fibers $F^s(\theta)$ by utilizing the small spectral gap between the largest stable eigenvalue $-4\epsilon$ and 0. Restricted to the order $O(\sqrt{\epsilon})$ neighborhood of $S$, any orbit inside $W^{cs}$ does not have small growth anymore and is characterized by the fiber decay property. The proof on the existence of stable fibers uses the fact that the unperturbed nonlinear term in $(2.2)$ is cubic, and the perturbed nonlinear term in $(2.2)$ contains an $\epsilon$ and can be controlled by $\epsilon \partial_x^2$ with inequality $(2.6)$. Below is a brief sketch of the proof on the case of the center-stable manifold $W^{cs}$. In the Fourier series $(2.3)$, define

$$\xi = -\left(u_1 + \frac{v_1}{\sqrt{1-c^2}}\right)\left(\frac{1}{\sqrt{1-c^2}}\right)\sin x, \quad \eta = \left(\frac{u}{v}\right) - \xi.$$

In terms of the new coordinates $(\xi, \eta)$, the system $(2.2)$ can be written as

$$\xi = \nu \xi + \mathcal{N}_1, \quad \eta = \hat{\mathcal{L}}\eta + \mathcal{N}_2,$$

where $\nu = \sqrt{1-c^2} - \epsilon$, $e^{\nu t}$ can be read from the representation $(2.4)$, and $\mathcal{N}_1$ and $\mathcal{N}_2$ are the projections of $\mathcal{N}$ on $\xi$ and $\eta$. Using the method of variation of parameters, we have

$$\begin{align*}
\xi(t) &= \xi(t_0)e^{\nu(t-t_0)} + \int_{t_0}^{t} e^{\nu(t-\tau)}\mathcal{N}_1(\tau)\,d\tau, \quad (3.1) \\
\eta(t) &= e^{(t-t_0)\hat{\mathcal{L}}}\eta(t_0) + \int_{t_0}^{t} e^{(t-t_\tau)}\hat{\mathcal{L}}\mathcal{N}_2(\tau)\,d\tau, \quad (3.2) \\
\theta(t) &= \theta(t_0) + (t-t_0). \quad (3.3)
\end{align*}$$

We introduce the following Banach space: For $\sigma = \nu/100$ and $n \geq 1$, let

$$G = \left\{ g(t) = (\xi(t), \eta(t), \theta(t)) \mid t \in [0, +\infty), \ g(t) \text{ is continuous in } t \text{ in } H^{n+1} \times H^n \text{ norm}, \ ||g||_{\sigma,n} = \sup_{t \geq 0} e^{-\sigma t} \left[ ||\xi(t)|| + ||\theta(t)|| + ||\eta(t)||_{(n+1) \times n} \right] < \infty \right\}.$$
If $g(t) \in G$ is a solution to (3.1)-(3.3), by letting $t_0 \to +\infty$ in (3.1) and setting $t_0 = 0$ in (3.2)-(3.3), we have

$$
\begin{align*}
\xi(t) &= \int_{+\infty}^{t} e^{\nu(t-\tau)} N_1(\tau) d\tau, \\
\eta(t) &= e^{t\hat{L}} \eta(0) + \int_{0}^{t} e^{(t-\tau)\hat{L}} N_2(\tau) d\tau, \\
\theta(t) &= \theta(0) + t .
\end{align*}
$$

Viewing the right hand side as a map on the Banach space $G$, by a contraction map argument, there is a unique fixed point in $G$ which is the solution to the above integral equations, which leads to the center-stable manifold. For details, see [7]. In comparison with [7], the novel point is to control the quasilinear perturbed nonlinear term with the inequality (2.6). The difficult part is to prove the regularity of the center-stable manifold with respect to $\epsilon$ at $\epsilon = 0$. The proof of Theorem 2.2 and the arguments in [7] serve a good guide for the proof. The sizes of the neighborhoods of $S$ stated in the theorem are the results of the inequality (2.6) and the spectral gaps.

In summary, by virtue of the inequality (2.6) and the arguments in [7], the rest of the proof is quite straightforward.

4. Existence of a pair of transversal homoclinic orbits and chaos. When $\epsilon = 0$, the perturbed sine-Gordon system (2.1) reduces to the well-known sine-Gordon equation

$$
u_{tt} = c^2 u_{xx} + \sin u \tag{4.1}$$

for which there is a pair of homoclinic orbits [6] asymptotic to $(u, u_t) = (0, 0),

$$
u = \pm 4 \arctan \left[ \frac{\sqrt{1-c^2}}{c} \sech \tau \sin x \right], \quad \tau = \sqrt{1-c^2} (t-t_0) . \tag{4.2}$$

This expression was obtained via a Darboux transformation of (4.1). Notice that this pair of homoclinic orbits is not transversal. Under the temporally periodic perturbation in (2.1), the time introduces a new dimension. For some value of $t_0$, the above pair of homoclinic orbits can persist and the persistent pair of homoclinic orbits is transversal under the perturbed flow (2.1). Locating such a persistent pair of homoclinic orbits is our first goal. A simple but crucial fact to notice is that the homoclinic orbits (4.2) are classical solutions such that $u \in H^n$ for any $n \geq 1$. This fact together with the regularity of the evolution operator (Theorem 2.2) and the invariant manifold (Theorem 3.1) for (2.1), with respect to $\epsilon$ at $\epsilon = 0$ will bridge the unperturbed system (4.1) and the perturbed system (2.1). The persistent homoclinic orbits will be located through a Melnikov measurement and a tracking afterward. The Melnikov measurement involves a calculation of a Melnikov
By Theorem 3.1, every orbit in $W^{es}$ is constructed via integrable isospectral theory. The Melnikov integral built via the Melnikov vector $\nu$\[^6\]

\[
\int_{-\infty}^{\infty} \int_0^{2\pi} \left\{ \frac{\partial F_1}{\partial u} u_{xx} + \frac{\partial F_1}{\partial u_t} [u_{txx} + a \cos t(u_{tx})^2 \sin u] \right\} dx dt \quad \text{(4.5)}
\]

evaluated on (4.2)-(4.4). In fact, $\epsilon M$ is the leading order term of the distance between $W^{cu}$ and $W^{cs}$. The roots of $M$ and implicit function theorem imply the intersection of $W^{cu}$ and $W^{cs}$. It turns out that one can track orbits in $W^{cu} \cap W^{cs}$ to an order $O(\epsilon \ln \epsilon)$ neighborhood of $S$ in forward time. Once the orbits enter such a neighborhood, one can continue to track them via the stable fibers in Theorem 3.1. In fact, all such orbits will approach $S$ in forward time, and become homoclinic orbits asymptotic to $S$.

**Theorem 4.1** (Homoclinic orbit theorem). *For any $c \in (\frac{1}{2}, 1)$, there are an interval $I \subset \mathbb{R}^+$ and an $\epsilon_0 > 0$, such that for any $a \in I$ and $\epsilon \in (0, \epsilon_0)$, there is a pair of transversal homoclinic orbits $\ell_{\pm}$ asymptotic to the periodic orbit $S$.*

**Proof.** By Theorem 3.1, every orbit in $W^{cu}$ will approach $S$ in backward time, and is coordinated by the unstable fibers. Tracking $W^{cu}$ in finite forward time back to the $O(1) \left[ \ln \epsilon \right]$ neighborhood of Theorem 3.1. One can design a signed distance measurement between $W^{cu}$ and $W^{cs}$. For the details, see [7]. The signed distance

\[ d = \epsilon M + o(\epsilon) , \]

where $M$ is given in (4.5). Setting $M = 0$, one obtains

\[ \cos(t_0 + \gamma) = \frac{M_1}{a \sqrt{M_2^2 + M_3^2}} , \quad \text{(4.6)} \]

where

\[
\begin{align*}
M_1 &= -\int_{-\infty}^{+\infty} \int_0^{2\pi} \left\{ \frac{\partial F_1}{\partial u} u_{xx} + \frac{\partial F_1}{\partial u_t} u_{txxx} \right\} dx dt , \\
M_2 &= \int_{-\infty}^{+\infty} \int_0^{2\pi} \cos \frac{\tau}{\sqrt{1 - c^2}} \frac{\partial F_1}{\partial u_t} (u_{tx})^2 \sin u dx dt , \\
M_3 &= \int_{-\infty}^{+\infty} \int_0^{2\pi} \sin \frac{\tau}{\sqrt{1 - c^2}} \frac{\partial F_1}{\partial u_t} (u_{tx})^2 \sin u dx dt , \\
\cos \gamma &= \frac{M_2}{\sqrt{M_2^2 + M_3^2}} , \quad \sin \gamma = \frac{M_3}{\sqrt{M_2^2 + M_3^2}} .
\end{align*}
\]
Thus when $a$ is large enough, equation (4.6) will have solutions for $t_0$. Then by the implicit function theorem, $d$ will have roots nearby. Thus $W^{cu} \cap W^{cs} \neq \emptyset$. Let $\ell$ be an orbit in $W^{cu} \cap W^{cs}$, one can track $\ell$ in forward time to an order $O(\epsilon \ln \epsilon)$ neighborhood of $S$. In this neighborhood, the stable fibers in Theorem 3.1 exist, thus $\ell$ will approach $S$ in forward time and becomes a homoclinic orbit asymptotic to $S$. For the details on the estimates in the tracking, see [7]. Theorem 2.2 is needed here. By the symmetry in Lemma 2.1, there is a pair $\ell_\pm$ of such homoclinic orbits. The Melnikov integral also implies that $W^{cu}$ and $W^{cs}$ intersect transversally along these orbits. 

Let $P$ be the Poincaré period-$2\pi$ map

$$P : H^{n+1} \times H^n \to H^{n+1} \times H^n.$$ 

By Theorem 2.2, $P$ is a $C^\infty$ map that has no bounded inverse. Under $P$, $S$ turns into the saddle fixed point $(u,v) = (0,0)$, and $\ell_\pm$ turns into a pair of transversal discrete homoclinic orbits $h_\pm$ asymptotic to $(0,0)$. This is in the setting studied in [12]. One can easily generalize the argument in [4] [5] to this case too. By the shadowing lemma, the following chaos theorem is true [12].

**Theorem 4.2 (Chaos Theorem).** In a neighborhood of the pair of homoclinic orbits $h_\pm$, there is a Cantor set $\Xi$ of points which is invariant under the iterated Poincaré map $P^N$ for some $N$. The action of $P^N$ on $\Xi$ is topologically conjugate to the action of the Bernoulli shift on symbols.

In the product topology, the Bernoulli shift has sensitive dependence on initial condition. The conjugation of the Poincaré map $P^N$ to the Bernoulli shift, is taken as the definition of chaos here. For more details, see [2]. The key point in the proof of [12] for non-invertible map, is the claim 3 on pp.341 of [12]. When dealing with homoclinic tubes [5] under non-invertible map, neither the proof [5] nor [12] can be generalized. The splitting in claim 3 on pp.341 of [12] can still be done, but the bundles may not even be continuous.

5. **Ginzburg-Landau equation.** The arguments in the previous sections together with those of [7] [8] can be applied to the following weakly quasilinear Ginzburg-Landau equation

$$iq_t = (1 + i\epsilon)q_{xx} + (2|q|^2 - i\epsilon\gamma|q_x|^2)q - (2\omega^2 + i\epsilon\alpha)q + \epsilon \bar{\beta}q,$$

(5.1)

where $q$ is a complex-valued function of two variables $t$ and $x$, the parameters $\alpha$, $\beta$, $\gamma$ and $\omega$ are positive, $\omega \in (\frac{1}{2}, 1)$, and $\epsilon$ is the small perturbation parameter. We pose the periodic boundary condition and even constraint

$$q(t, x + 2\pi) = q(t, x), \quad q(t, -x) = q(t, x).$$

One can rewrite (5.1) in the perturbed nonlinear Schrödinger equation form

$$iq_t = q_{xx} + 2\left[|q|^2 - \omega^2\right]q + \epsilon \left[iq_{xx} - i\alpha q + \beta q - i\gamma|q_x|^2q\right].$$

As in previous sections, one can always control the nonlinear term $\epsilon|q_x|^2q$ by $\epsilon \tilde{\beta}^2$ with inequality (2.6). By going through the detailed arguments in [7] [8] [9], we obtain the following theorem.

**Theorem 5.1.** There exists a $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, there exists a codimension 1 surface in the space of $(\alpha, \beta, \gamma, \omega) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ where $\omega \in (\frac{1}{2}, 1)/S$, $S$ is a finite subset, and $\alpha < \beta$. For any $(\alpha, \beta, \gamma, \omega)$ on the codimension 1 surface, the Ginzburg-Landau equation (5.1) has a quartet of homoclinic
orbits consisting of two pairs of (non-transversal) homoclinic orbits, each of which is asymptotic to a saddle. Under similar generic assumptions as in [8], the Ginzburg-Landau equation (5.1) has chaos. That is, there exists a Cantor set \( \Lambda \) of points in the neighborhood of the quartet of homoclinic orbits. \( \Lambda \) is invariant under the Poincaré return map \( P \). \( P \) restricted to \( \Lambda \) is topologically conjugate to the Bernoulli shift on symbols.

6. Appendix: existence of chaos in reaction-diffusion equations. In this appendix, we use a typical reaction-diffusion equation to illustrate how to prove the existence of chaos in such equations. This appendix is written for people who are used to ordinary differential equations (ODEs) and are willing to try partial differential equations (PDEs), especially for people who have studied traveling wave solutions of reaction-diffusion equations. Most of the studies so far on reaction-diffusion equations are focused upon traveling wave solutions, and the equations are reduced to ODEs. Here we shall study the full PDEs in the neighborhood of spatially-independent solutions. The mathematical machineries are easier that those in previous sections. The key component in the proof on the existence of chaos is the expression of the separatrix. Here the separatrix is spatially-independent and obtained trivially. On the other hand, the separatrices in the previous sections are nontrivial. We shall study a perturbation problem. The unperturbed system is a Klein-Gordon equation which is still a PDE. In the case that the unperturbed system is an ODE and the perturbed system is a PDE, proving the existence of chaos is open. The dispersive term in the Klein-Gordon equation helps to “neutralize” the spectrum. This is necessary in the proof of the existence of chaos. The diffusive terms are perturbative. That is, we are considering weak diffusions. The forcing term is perturbative too. In particular, the forcing term has an explicit spatial dependence which makes the spatially-independent subspace not invariant anymore. The persistent homoclinic orbits under perturbations will have spatial dependence. Solutions inside the chaos will have spatial dependence too.

We shall study the following specific system

\[
\begin{align*}
    u_t &= v + \epsilon u_{xx}, \\
    v_t &= 2u_{xx} + u - [1 + \epsilon f(t,x)]u^3 + \epsilon v_{xx},
\end{align*}
\]

where \( f(t, x) = (a + b \cos x) \cos t \), \((u, v)\) are real-valued functions of two real variables \((t, x)\), \(a\) and \(b\) are two real parameters, and \(\epsilon \geq 0\) is the small perturbation parameter. The proofs in this Appendix also works when adding quasilinear terms like \(\epsilon v^3\) and \(\epsilon u^3\) etc.. It is obvious that the system maintains a symmetry \((u, v) \to (-u, -v)\). We pose the periodic boundary condition

\[
w(t, x + 2\pi) = w(t, x), \quad (w = u, v).
\]

The system (6.1) can be written in a more convenient form

\[
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix} + \mathcal{N}, \quad (6.1)
\]

where

\[
L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v + \epsilon u_{xx} \\ 2u_{xx} + u + \epsilon v_{xx} \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} 0 \\ -(1 + \epsilon f(t,x))u^3 \end{pmatrix}.
\]
Using the Fourier series
\[
\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{k \in \mathbb{Z}} \left( u_k \ v_k \right) e^{ikx}
\]
where \( w_{-k} = \overline{w_k} \) \((w = u, v)\), then
\[
e^{tL} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{u_0 + v_0}{2} e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{u_0 - v_0}{2} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
\[
+ \sum_{k \neq 0} \left[ \frac{1}{2} \left( u_k + \frac{v_k}{i\sqrt{2k^2 - 1}} \right) e^{-\epsilon(k^2 t + i\sqrt{2k^2 - 1} t)} \begin{pmatrix} 1 \\ i\sqrt{2k^2 - 1} \end{pmatrix} \right.
\]
\[
\left. + \frac{1}{2} \left( u_k - \frac{v_k}{i\sqrt{2k^2 - 1}} \right) e^{-\epsilon(k^2 t - i\sqrt{2k^2 - 1} t)} \begin{pmatrix} 1 \\ -i\sqrt{2k^2 - 1} \end{pmatrix} \right] e^{ikx}. \quad (6.2)
\]
From the representation (6.2), we see that there is only one unstable mode. In such a case, we can prove the existence of chaos. Removing the \( 2u_{xx} \) term in (6.1), the \( \epsilon = 0 \) equations are ODEs, and the \( \epsilon \neq 0 \) equations are more common reaction-diffusion equations. In this case, there are infinitely many unstable modes (6.2), we do not know how to prove the existence of chaos. It is a very challenging problem.

When \( \epsilon = 0 \), (6.1) reduces to the Klein-Gordon equation
\[
u_{tt} = 2u_{xx} + u - u^3 \quad (6.3)
\]
for which there is a Hamiltonian
\[
H = \int_0^{2\pi} \left[ \frac{1}{2} v^2 + (u_x)^2 - \frac{1}{2} u^2 + \frac{1}{4} u^4 \right] dx, \quad (6.4)
\]
where \( v = u_t \). On the spatially-independent plane \( (\partial_x = 0) \), (6.3) has a pair of separatrices
\[
u = \pm \sqrt{2} \sech (t + t_0), \quad (6.5)
\]
\[
u = \mp \sqrt{2} \sech (t + t_0) \tanh(t + t_0). \quad (6.6)
\]
The Melnikov integral for (6.1) can be built with the Hamiltonian (6.4):
\[
M = \int_0^{+\infty} \int_{-\infty}^{+\infty} \left\{ \frac{\partial H}{\partial u} u_{xx} + \frac{\partial H}{\partial v} \left( v_{xx} - f(t, x) u^3 \right) \right\} dx dt
\]
evaluated along (6.5)-(6.6).
\[
M = 16a\pi \sin t_0 \int_0^{\infty} \sin \tau \sech^4 \tau \tanh \tau d\tau .
\]
The roots of \( M \) are \( t_0 = m\pi, \forall m \in \mathbb{Z} \). Theorems 2.2 and 3.1 for (2.1) hold for (6.1) too. The arguments in the previous sections for (2.1) apply to (6.1) too. We have the following theorem.

**Theorem 6.1.** There is a region \( D \subset \mathbb{R}^+ \times \mathbb{R}^+ \) and an \( \epsilon_0 > 0 \), such that for any \((a, b) \in D \) and \( \epsilon \in (0, \epsilon_0) \), there is a pair of transversal homoclinic orbits asymptotic to \((u, v) = (0, 0)\) under the Poincaré period-\(2\pi\) map. In a neighborhood of the pair of homoclinic orbits, there is a Cantor set of points which is invariant under some iterations of the Poincaré period map. Restricted to the Cantor set, the iterated Poincaré period map is topologically conjugate to the Bernoulli shift on symbols.
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