On quasiperiodic boundary condition problem

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The paper raises the question of posing the quasiperiodic boundary condition in the Cauchy problem of partial differential equations. Using the one-dimensional cubic nonlinear Schrödinger as a simple example, we illustrated the various types of questions including global well-posedness, spectra of linear operators, and foliations. © 2005 American Institute of Physics. [DOI: 10.1063/1.1832754]

I. INTRODUCTION

The quasiperiodic boundary condition problem can be posed for a variety of partial differential equations (PDE) including, e.g., parabolic and hyperbolic equations. Questions that can be asked include local and global well-posedness, dynamics in phase spaces, and asymptotics, etc. Here we take a simple PDE—one-dimensional cubic nonlinear Schrödinger equation (NLS), to study its phase space foliations.

Typical fluid flows are defined on unbounded domain with nondecaying boundary conditions. For example, the Poiseuille flow or the boundary layer flow has nondecaying boundary conditions along the longitudinal direction. In fact, turbulence develops along this longitudinal direction. In many cases, turbulent fluid flows contain both temporal and spatial randomness. Temporal randomness is often caused by temporal chaotic motions. Spatial randomness is often caused by vortex (energy) cascade or inverse cascade. In such cases, periodic boundary conditions put too much constraint. Quasiperiodic or more general boundary conditions are more relevant.

The one-dimensional (1D) cubic NLS under periodic boundary conditions is well understood. It is globally well-posed. Under quasiperiodic boundary conditions, global well-posedness is not known. Under periodic boundary conditions, Stokes wave solution has a finite number of unstable eigenvalues. On the other hand, under quasiperiodic boundary conditions, it has infinitely many unstable eigenvalues dense on an interval. There is no spectral gap. But explicit expressions of the foliation in phase space can be obtained via a Darboux transformation.

II. FORMULATION OF THE PROBLEM

Consider the 1D cubic nonlinear Schrödinger equation

\[ iq_t = q_{xx} + 2|q|^2q, \]

where \( q \) is a complex-valued function of two real variables \( (t,x) \), \( i = \sqrt{-1} \). We pose a quasiperiodic boundary condition with two base frequencies \( \beta_1 \) and \( \beta_2 \); \( \beta_1 / \beta_2 \) is irrational. That is,

\[ q = q(t, \theta_1, \theta_2), \quad \theta_1 = \beta_1 x, \quad \theta_2 = \beta_2 x, \]

and \( q \) is periodic in both \( \theta_1 \) and \( \theta_2 \) with period \( 2\pi \). Thus
\[
q = q(t, \theta) = \sum_{k \in \mathbb{Z}^2} q_k(t) e^{ik \cdot \theta}, \quad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \quad k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.
\]

It seems that the more natural norm is
\[
\|q\|_{(2)}^2 = \sum_{k \in \mathbb{Z}^2} (1 + |k|^2) |q_k|^2,
\]
rather than
\[
\|q\|_{(1)}^2 = \sum_{k \in \mathbb{Z}^2} [1 + (k \cdot \beta)^2] |q_k|^2, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.
\]

In terms of Fourier transforms, (2.1) can be rewritten as
\[
\begin{align*}
\frac{idq_k}{dt} &= -\frac{\partial H}{\partial q_k}, \\
\frac{id\overline{q}_k}{dt} &= \frac{\partial H}{\partial \overline{q}_k},
\end{align*}
\]
where
\[
H = \sum_{k \in \mathbb{Z}^2} (k \cdot \beta)^2 |q_k|^2 - \sum_{k \in \mathbb{Z}^2} \left| \sum_{\tilde{k} \in \mathbb{Z}^2} q_{\tilde{k}} q_{k-k} \right|^2
\]
\[
= \lim_{a \to +\infty} \frac{1}{2a} \int_{-a}^{a} [q_k|^2 - |q|^4] dx.
\]

Using (2.4), the NLS (2.1) can be rewritten as
\[
\begin{align*}
iq_t &= -\frac{\delta H}{\delta q}, \\
i\overline{q}_t &= \frac{\delta H}{\delta \overline{q}}.
\end{align*}
\]

Obviously,
\[
I = \sum_{k \in \mathbb{Z}^2} |q_k|^2 = \lim_{a \to +\infty} \frac{1}{2a} \int_{-a}^{a} |q|^2 dx
\]
is an invariant.

\section*{III. WELL-POSEDNESS}

Explicitly (2.2) can be written as
\[
\begin{align*}
iq_k &= -(k \cdot \beta)^2 q_k + 2 \sum_{\tilde{k},k \in \mathbb{Z}^2} \overline{q}_{\tilde{k}} \overline{q}_{\tilde{k} + k} q_{k-k}.
\end{align*}
\]

The method of variation of parameters leads to the integral equation
Notice that (3.1) bears more resemblance to two-dimensional (2D), rather than 1D, NLS under the periodic boundary condition. Local well-posedness can be easily established, since the nonlinear term is still locally Lipschitz.

**Theorem 3.1 (Local well-posedness):** For any \( q_0 \in H^s \), \( s \geq 2 \), there exists a unique solution \( q(t) \in C^0([0, \tau], H^s) \) where \( \tau = \tau(\|q_0\|_{L^2}) \), to the Cauchy problem of (3.1) with initial condition \( q(0) = q_0 \). For any fixed \( t \in [0, \tau] \), \( q(t) \) is \( C^\infty \) in \( q_0 \).

The interesting open problem is whether or not (3.1) has global well-posedness. On the one hand, it resembles 2D NLS under periodic boundary condition, therefore, it may not have global well-posedness. In fact, the first term in the Hamiltonian (2.3) is weaker than \( \sum_{k \in \mathbb{Z}} |\tilde{q}|^2 |\tilde{q}| \) of the 2D NLS periodic case. Thus the Hamiltonian cannot bound the \( H^s \) norm. On the other hand, it is still an integrable system, therefore, an infinite sequence of invariants is at one’s disposal.

**IV. THE SPECTRUM OF A LINEAR NLS OPERATOR**

Setting \( \partial_s = 0 \) in (2.1), one gets an ODE defined on the invariant complex plane

\[
iq = 2|q|^2 q
\]

with all periodic solutions (the so-called Stokes waves)

\[
q = ce^{-i(2c^2+\gamma)}.
\] (4.1)

Linearize the NLS in the manner

\[
q = (c + \hat{q})e^{-i(2c^2+\gamma)},
\]

one has

\[
i\hat{q} = \hat{q}_{xx} + 2c^2 (\hat{q} + \bar{\hat{q}}).
\]

Let

\[
\hat{q} = \sum_{k \in \mathbb{Z}^2} \hat{q}_k(t) e^{ik \cdot \theta},
\] (4.2)

one gets

\[
\frac{d}{dt} \left( \frac{\hat{q}_k}{\hat{q}_{-k}} \right) = \begin{pmatrix} 2c^2 - (k \cdot \beta)^2 & 2c^2 \\ -2c^2 & (k \cdot \beta)^2 - 2c^2 \end{pmatrix} \left( \frac{\hat{q}_k}{\hat{q}_{-k}} \right).
\] (4.3)

Let

\[
\left( \frac{\hat{q}_k}{\hat{q}_{-k}} \right) = e^{\lambda t} \begin{pmatrix} A \\ B \end{pmatrix},
\] (4.4)

where \( \lambda, A, \) and \( B \) are complex constants, then

\[
\lambda = \pm (k \cdot \beta) \sqrt{(2c^2 - (k \cdot \beta)^2)}.
\] (4.5)

**Lemma 4.1:** The set \( \{k \cdot \beta\}_{k \in \mathbb{Z}^2} \) is dense in \( \mathbb{R} \).

**Proof:** This proof is furnished by Banks. For any real number \( z \), let \( [z] \) denote the greatest integer less than or equal to \( z \), and let \( \{z\} = z - [z] \) be the fractional part of \( z \); then \( 0 \leq \{z\} \leq 1 \). For any irrational number \( a \), it is known that the fractional parts \( \{na\}_{n \in \mathbb{Z}} \) are uniformly distributed over
the unit interval \([0,1]\). For any fixed \(b \in \mathbb{R}\), given any \(\varepsilon > 0\), let \(k_2\) be chosen such that
\[
|k_2(\beta_2/\beta_1) - \{b/\beta_1\}| < \varepsilon |\beta_1|,
\]
and choose \(k_1 = [b/\beta_1] - [k_2(\beta_2/\beta_1)]\), then
\[
|k_1 + k_2(\beta_2/\beta_1) - b/\beta_1| = |k_1 + [k_2(\beta_2/\beta_1)] - [b/\beta_1]| + |\{k_2(\beta_2/\beta_1) - \{b/\beta_1\}| \\
< \varepsilon |\beta_1|.
\]
Multiplying by \(\beta_1\), one obtains \(|k \cdot \beta| < \varepsilon\). This proves the lemma.

**Theorem 4.2**: The spectrum of the linear NLS operator in \(H_s\) is
\[
\sigma = \sigma_p \cup \sigma_c = [-2c^2, 2c^2] \cup i\mathbb{R}
\]
where \(\sigma_p\) is given by (4.5) and is everywhere dense in \(\sigma\).

**Proof**: The maximum of the function
\[
z^2((2c)^2 - z^2), \quad z \in \mathbb{R}
\]
is \(4c^2\). By Lemma 4.1 and the fact that the spectrum \(\sigma\) is a closed set, we have that
\[
[-2c^2, 2c^2] \cup i\mathbb{R} \subseteq \sigma.
\]
In terms of the Fourier transform (4.2), the linear NLS operator has the representation given by (4.3),
\[
L_k = -i \begin{pmatrix} 2c^2 - (k \cdot \beta)^2 & 2c^2 \\ -2c^2 & (k \cdot \beta)^2 - 2c^2 \end{pmatrix}.
\]
If \(\lambda \notin [-2c^2, 2c^2] \cup i\mathbb{R}\), then there is an absolute constant \(C\) such that
\[
\|L_k - \lambda \|^{-1} \leq C, \quad \forall k
\]
and this is true even for some \(k\), \((k \cdot \beta)^2\) might be equal to \((2c)^2\). Thus such \(\lambda\) belongs to the resolvent set, and
\[
\sigma = [-2c^2, 2c^2] \cup i\mathbb{R}.
\]
Let \(\lambda \in \sigma \setminus \sigma_p\) where \(\sigma_p\) is the point spectrum given by (4.5), then there is a sequence \(\lambda_j \in \sigma_p\) such that \(\lambda_j \to \lambda\), and
\[
\|L_k - \lambda \|^{-1} \geq 1/|\lambda_j - \lambda| \to +\infty;
\]
thus \(\lambda \in \sigma_c\) is the continuous spectrum. This proves the theorem.

**Remark 4.3**: For NLS under periodic boundary condition, the spectrum of the linear NLS operator consists of only discrete point spectrum given by
\[
\lambda = \pm k \beta \sqrt{(2c)^2 - (k\beta)^2},
\]
where \(k \in \mathbb{Z}\), and \(\beta\) is a positive constant. For any fixed \(c > 0\), there is a finite number of unstable modes. There are gaps among the unstable, center, and stable spectra. As shown above, under quasiperiodic boundary condition, the point spectrum is dense, and there is also a continuous spectrum. For any fixed \(c > 0\), there are infinitely many unstable modes. There is no gap among the unstable, center, and stable spectra.
V. FOLIATIONS

Although there is no spectral gap in this quasiperiodic setting, foliations can still be established via explicit expressions. The tool used is the so-called Darboux transformation. The NLS (2.1) has the Lax pair,

\[ \psi_x = U \psi, \quad \psi_t = V \psi, \]

where

\[ U = i \begin{pmatrix} \lambda & q \\ \bar{q} & -\lambda \end{pmatrix}, \]
\[ V = i \begin{pmatrix} 2\lambda^2 - |q|^2 & 2\lambda q - iq \bar{q} \\ 2\bar{q}^2 + i\bar{q} \bar{q} & -2\lambda^2 + |q|^2 \end{pmatrix}. \]

**Theorem 5.1:** Let \( q(t,x) \) be a solution, and let \( \phi \) be an eigenfunction of the Lax pair at \( \lambda = \nu \) for any \( \nu \in \mathbb{C} \). Use \( \phi \) to define a matrix,

\[ G = \Gamma \begin{pmatrix} \lambda - \nu & 0 \\ 0 & \lambda - \bar{\nu} \end{pmatrix} \Gamma^{-1}, \]

where

\[ \Gamma = \begin{pmatrix} \phi_1 & -\phi_2 \\ \phi_2 & \phi_1 \end{pmatrix}. \]

We define \( Q \) and \( \Psi \) by

\[ Q = q + 2(\nu - \bar{\nu}) \frac{\phi_1 \bar{\phi}_2}{|\phi_1|^2 + |\phi_2|^2}, \quad \Psi = G \psi, \]

where \( \psi \) solves the Lax pair at \( (q, \lambda) \). Then \( \Psi \) solves the Lax pair at \( (Q, \lambda) \) and \( Q \) solves the NLS.

This is a well-known theorem in the integrable theory, see, e.g., Ref. 3. The transformation (5.1) is called a Darboux transformation.

For example, let

\[ q = ae^{i\theta(t)}, \quad \theta(t) = -[2a^2t + \gamma], \]

where \( a \) is the amplitude and \( \gamma \) is the phase. The eigenfunctions of the Lax pair are

\[ \phi^\pm = \begin{pmatrix} ae^{i\theta(t)/2} \\ (\pm \beta - \lambda)e^{-i\theta(t)/2} \end{pmatrix} e^{\pm 2\beta t \pm i\beta x}, \quad \lambda = \sqrt{\beta^2 - a^2}. \]

In order to have temporal growth, \( \beta^2 < a^2 \). For \( \beta = \beta_1, \lambda = \nu = i\sigma \), let

\[ \phi = c^+ \phi^+ + c^- \phi^-, \]

where \( c^\pm \) are two arbitrary complex constants. For \( \beta = \beta_2, \lambda = \nu = i\bar{\sigma} \), let

\[ \dot{\phi} = \dot{c}^+ \phi^+ + \dot{c}^- \phi^-, \]

where \( \dot{c}^\pm \) are two arbitrary complex constants. By iterating the Darboux transformation (5.1) at \( \nu \) and \( \nu \), one gets
\[
Q = q + 2(\nu - \nu') \frac{\phi_1 \phi_2}{|\phi_1|^2 + |\phi_2|^2} + 2(\hat{\nu} - \hat{\nu}') \frac{\Phi_1 \Phi_2}{|\Phi_1|^2 + |\Phi_2|^2},
\]

where
\[
\Phi_1 = \frac{1}{|\phi_1|^2 + |\phi_2|^2} \{[(\hat{\nu} - \nu)|\phi_1|^2 + (\hat{\nu} - \nu')|\phi_2|^2]\Phi_1 + (\hat{\nu} - \nu)\phi_1 \phi_2 \phi_2 \},
\]
\[
\Phi_2 = \frac{1}{|\phi_1|^2 + |\phi_2|^2} \{[(\nu - \nu')|\phi_1|^2 + (\nu - \nu')|\phi_2|^2]\Phi_1 + (\nu - \nu)\phi_1 \phi_2 \phi_2 \}.
\]

Explicitly, one has
\[
Q = \tilde{Q} + q \sin \theta_0 \frac{\Pi}{2} \Pi_1.
\]

where
\[
\tilde{Q} = q[1 + \sin \theta_0 \sech \tau \cos X]^{-1}[(\cos 2 \theta_0 - i \sin 2 \theta_0 \tanh \tau - \sin \theta_0 \sech \tau \cos X],
\]
\[
\Pi_1 = \left[ (\sin \theta_0)^2(1 + \sin \theta_0 \sech \tau \cos X)^2 + \frac{1}{8}(\sin 2 \theta_0)^2(\sech \tau)^2(1 - \cos 2X) \right]
\times (1 + \sin \theta_0 \sech \tau \cos \hat{X}) - \frac{1}{2} \sin 2 \theta_0 \sech \tau \sech \tau \sech (1 + \sin \theta_0 \sech \tau \cos X)
\times \sin X \sin \hat{X} + (\sin \theta_0)^2[1 + 2 \sin \theta_0 \sech \tau \cos X + (\cos X)^2 - (\cos \theta_0)^2](\sech \tau)^2]
\times (1 + \sin \theta_0 \sech \tau \cos \hat{X}) - 2 \sin \theta_0 \sin \theta_0 \cos \theta_0 \sech \tau \tanh \tau
+ (\sin \theta_0) \sech \tau \cos X)(\sin \theta_0 + \sech \tau \cos \hat{X})[1 + \sin \theta_0 \sech \tau \cos X),
\]
\[
\Pi_2 = \left[ -2(\sin \theta_0)^2(1 + \sin \theta_0 \sech \tau \cos X)^2 + \frac{1}{4}(\sin 2 \theta_0)^2(\sech \tau)^2(1 - \cos 2X) \right]
\times (\sin \theta_0 \sech \tau \cos X + \sin \theta_0 \sech \tau \cos X) + (\sin \theta_0 \sech \tau \cos X - \sin \theta_0 \sech \tau \cos X)
+ 2 \sin \theta_0 \sin \theta_0 \sech \tau \sech \tau \sech (2 \sin \theta_0 (1 + \sin \theta_0 \sech \tau \cos X)(1 + \sin \theta_0 \sech \tau \cos \hat{X})
\times (\sin \theta_0 \sech \tau \cos \hat{X} + \sin \theta_0 \sech \tau \cos \hat{X}) - 2 \sin \theta_0 \sin \theta_0 \sech \tau \sech \tau \sech \tau \sin \hat{X}],
\]
and
\[
\beta_1 + \nu = a e^{i \theta_0}, \quad \beta_2 + \hat{\nu} = a e^{i \hat{\theta}_0},
\]
\[
c^* c^- = e^{i \theta}, \quad \tilde{c}^* \tilde{c}^- = e^{i \tilde{\theta}},
\]
\[
\tau = 4 \sigma \beta_1 - \rho, \quad \hat{\tau} = 4 \hat{\sigma} \beta_2 - \hat{\rho}.
\]
\[ X = 2\beta_1 x + \vartheta - \vartheta_0 + \pi/2, \quad \dot{X} = 2\beta_2 x + \dot{\vartheta} - \dot{\vartheta}_0 + \pi/2. \]

The foliation here is with respect to the two linear unstable modes \((2\beta_1, 0)\) and \((0, 2\beta_2)\) in (4.5). The temporal growth condition \(\beta_1^2 < a^2\) or \(\beta_2^2 < a^2\) is in agreement with (4.5). Thus (5.2) represents a class of solutions with quasiperiodic boundary condition. For fixed \(a, \beta_1, \) and \(\beta_2, \) the parameters are \(\gamma, \rho, \dot{\gamma}, \vartheta,\) and \(\dot{\vartheta}.\) As \(t \to \pm \infty,\) e.g., \(\beta_1, \beta_2, \alpha, \) and \(\sigma\) are all positive,

\[ Q \to q e^{\pi i (\vartheta_0 + \dot{\vartheta}_0)}. \]

VI. CONCLUSION AND DISCUSSION

From the presentation in this paper, one can see that the first interesting question on such quasiperiodic boundary condition problem is the global well-posedness. In terms of Fourier transforms, one can see that the integrable NLS resembles the 2D more than the 1D periodic problem. I tend to believe that it may have finite-time blow-up solutions, which will be truly interesting. Also linearization in the quasiperiodic case often leads to a linear operator with continuous spectrum and with no spectral gap. Therefore, the phase space foliation is a challenging and interesting problem. In this paper, through Darboux transformation, such foliation can still be established.

2. V. Banks (private communication).