Bridging steady states with renormalization group analysis

Yueheng Lan
Department of Physics
Tsinghua University

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Outline

Main contents

1 Introduction
   - Physics and heteroclinic connections
   - Renormalization group
   - The RG and differential equations

2 An extension of the RG analysis

3 Several examples
   - The Lotka-Volterra model of competition
   - The Gray-Scott model
   - The Kuramoto-Sivashinsky equation

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State and dynamics

\[ \dot{x}_1 = f_1(x_1, x_2, \cdots, x_n) \]
\[ \dot{x}_2 = f_2(x_1, x_2, \cdots, x_n) \]
\[ \cdots = \cdots \]
\[ \dot{x}_n = f_n(x_1, x_2, \cdots, x_n) \]

The phase space - a geometric representation

Vector field and trajectories

Invariant set and organization of trajectories
Connections

(a)

(b)

(c)

(d)
Pendulum orbits
State transition in chemical reactions

![3D diagram showing energy landscape with two coordinates](image-url)
Computation of heteroclinic connections

- Exact analytic solutions under certain conditions: nonlinear integrable systems, partially integrable systems.
- Asymptotic methods for analytic approximations: local stability analysis plus interpolation.
- Numerical methods: two-point boundary problem; shooting method; relaxation method.
- Challenge:
  1. Need to know orbit existence and both end points;
  2. Need to know the local behavior near two ends;
  3. Hard to represent the dynamics on the connection;
  4. Hard to derive analytic expressions.
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Renormalization group in physics

- RG investigates changes of physical laws at different scales.
- RG and scale invariance: a renormalizable system at one scale consists of self-similar copies of itself at a smaller scale, with convergent coupling parameters when scaled up.
- In statistical physics: block spin; In quantum physics: renormalization equation \( \frac{\partial g}{\partial \ln \mu} = \beta(g) \); In nonlinear dynamics: the universal route to chaos.

Block spin renormalization group for a spin system described by \( H(T, J):(T, J) \to (T', J') \to (T'', J'') \). Resummation...
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4 Summary
The Van der Pol equation is
\[
\frac{d^2y}{dt^2} + y = \epsilon \left[ \frac{dy}{dt} - \frac{1}{3} (dy/dt)^3 \right].
\]

A naive expansion
\[
y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots
\]
gives
\[
y(t) = R_0 \sin(t + \Theta_0) + \epsilon \left[ -\frac{R_0^3}{96} \cos(t + \Theta_0) + \frac{R_0^2}{2} \left( 1 - \frac{R_0^2}{4} \right) (t - t_0) \sin(t + \Theta_0) + \frac{R_0^3}{96} \cos 3(t + \Theta_0) \right] + O(\epsilon^2)
\]
where \( R_0, \Theta_0 \) are determined by the initial conditions.

The expansion breaks down when \( \epsilon(t - t_0) > 1 \). The arbitrary initial time \( t_0 \) may be treated as the ultraviolet cutoff in the usual field theory.
One example

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where \( R_0, \Theta_0 \) are determined by the initial conditions.

The expansion breaks down when \( \epsilon(t - t_0) > 1 \). The arbitrary initial time \( t_0 \) may be treated as the ultraviolet cutoff in the usual field theory.
• Split $t - t_0$ as $t - \tau + \tau - t_0$ and absorb the terms containing $\tau - t_0$ into the renormalized counterparts $R, \Theta$ of $R_0$ and $\Theta_0$.

• Assume $R_0(t_0) = Z_1(t_0, \tau)R(\tau), \Theta_0(t_0) = \Theta(\tau) + Z_2(t_0, \tau)$ where $Z_1 = 1 + \sum_1^\infty a_n \epsilon^n, Z_2 = \sum_1^\infty b_n \epsilon^n$. The choice $a_1 = -(1/2)(1 - R^2/4)(\tau - t_0), b_1 = 0$ removes the secular term to order $\epsilon$:

$$y(t) = [R + \epsilon \frac{R^2}{2}(-\frac{R^2}{4})(t - \tau)] \sin(t + \Theta) -$$

$$\epsilon \frac{R^3}{96} \cos(t + \Theta) + \epsilon \frac{R^3}{96} \cos 3(t + \Theta) + O(\epsilon^2),$$

where $R, \Theta$ are functions of $\tau$. 

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$$y(t) = [R + \epsilon \frac{R}{2}(-\frac{R^2}{4})(t - \tau)] \sin(t + \Theta) - \epsilon \frac{R^3}{96} \cos(t + \Theta) + \epsilon \frac{R^3}{96} \cos 3(t + \Theta) + O(\epsilon^2),$$

where $R, \Theta$ are functions of $\tau$. 

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Bridging steady states with RG analysis
The solution should not depend on $\tau$. Therefore 

$$(\partial y/\partial \tau)_t = 0:$$

$$
\frac{dR}{d\tau} = \epsilon \frac{R}{2} (1 - \frac{R^2}{4}) + O(\epsilon^2), \quad \frac{d\Theta}{d\tau} = O(\epsilon^2).
$$

The initial condition $R(0) = 2a$, $\Theta(0) = 0$ gives 

$$y(t) = R(t) \sin(t) + \frac{\epsilon}{96} R(t)^3 [\cos(3t) - \cos(t)] + O(\epsilon^2).$$
The solution should not depend on $\tau$. Therefore 
\[(\partial y/\partial \tau)_t = 0:\]

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Series expansion of a differential equation

- Suppose that we have a set of $n$-dimensional ODEs

$$\dot{x} = Lx + \epsilon N(x)$$

- We may make the expansion

$$x = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

which results in

$$u_0 = Lu_0$$
$$u_1 = Lu_1 + N(u_0)$$
$$u_2 = Lu_2 + N_2(u_0, u_1)$$
$$\ldots$$
Suppose that we have a set of \( n \)-dimensional ODEs

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We may make the expansion

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    u_0 &= Lu_0 \\
    u_1 &= Lu_1 + N(u_0) \\
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\end{align*}
\]

\[
\cdots
\]
This series of equations can be solved as

\[ u_0(t, t_0) = e^{L(t-t_0)}A(t_0) \]

\[ u_1(t, t_0) = e^{L(t-t_0)} \int_{t_0}^{t} e^{-L(\tau-t_0)} N(e^{L(t-t_0)}A) d\tau \]

\[ u_2(t, t_0) = e^{L(t-t_0)} \int_{t_0}^{t} e^{-L(\tau-t_0)} N_2(e^{L(t-t_0)}A, u_1(t, t_0)) d\tau. \]

The series expansion gives \( x = \tilde{x}(t; t_0, A(t_0)) \).

The RG equation is a set of equations for \( dA(t_0)/dt_0 \) derived from

\[ \frac{d\tilde{x}(t; t_0, A(t_0))}{dt_0} \bigg|_{t=t_0} = 0 \]
One simple example

- Consider the simple example

\[ \dot{x} = y, \quad \dot{y} = -x, \]

which can be solved exactly with

\[ x = R \sin(t - t_0 + \theta), \quad y = R \cos(t - t_0 + \theta), \]

where \( R = R(t_0), \theta = \theta(t_0) \) specify the initial condition.

- In phase space, orbits of the equation are circles with radius \( R \) and azimuth angle \( \theta \).

- The RG equation derived from

\[ \frac{\partial x(t; R(t_0), \theta(t_0), t_0)}{\partial t_0} |_{t=t_0} = 0, \quad \frac{\partial y(t; R(t_0), \theta(t_0), t_0)}{\partial t_0} |_{t=t_0} = 0. \]

is \( dR(t_0)/dt_0 = 0, d\theta(t_0)/dt_0 = 1 \) as expected.
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The RG analysis as a coordinate transformation

- Hamiltonian dynamics: action-angle variables. For a harmonic oscillator $H = 1/2p^2 + 1/2q^2 = I$.

- In a general nonlinear dynamical system,
  \[ \dot{x} = f(x) \]

  which has the general solution $x(t) = \phi(t; A_0(t_0), t_0)$. The equation
  \[ \frac{\partial \phi(t; A_0(t_0), t_0)}{\partial t_0} \bigg|_{t=t_0} = 0 \]

  gives an equation for $dA_0(t_0)/dt_0$, which governs the evolution of the new coordinates $A_0$.

- The RG analysis is equivalent to a coordinate transformation in this sense, but often associated with approximations in the nonlinear case.
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Further development of the RG analysis

- It can be used for nonlinear partial equations and is able to derive the phase or amplitude equations.
- The invariance condition has been extended to the analysis of maps.
- It is also used to determine the center manifold near a bifurcation point.
- Problem: for the dynamics on a submanifold, the invariance equations contain less than $n$ unknowns!
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- Problem: for the dynamics on a submanifold, the $n$ invariance equations contain less than $n$ unknowns!?
Dynamics on a submanifold

- Without loss of generality, we will concentrate on the 1-d submanifold. Higher dimensional ones can be treated in a similar way.

- The initial vector $A$ should be taken as $A = (A_1, 0, 0, \cdots, 0)^t$.

- The $i$-th ($i \neq 1$) component of $u_1$ can be computed as

$$u_{1,i}(t, t_0) = e^{\lambda_i(t-t_0)} \int_{t_0}^{t} e^{-\lambda_i(\tau-t_0)} N(e^{L(t-t_0)} A) d\tau,$$

where $\int^t$ denotes integration without constant term.

- The first component

$$\frac{d\tilde{x}_1(t; t_0, A_1(t_0))}{dt_0} \bigg|_{t=t_0} = 0$$

is enough to derive the RG equation for $dA_1(t_0)/dt_0$, which also satisfies other component equations.
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- The first component
  
  $$\left. \frac{d\tilde{x}_1(t; t_0, A_1(t_0))}{dt_0} \right|_{t=t_0} = 0$$  

  is enough to derive the RG equation for $dA_1(t_0)/dt_0$, which also satisfies other component equations.
A proof by mathematical induction

- It is easy to write down an integral equation from its \( i \)-th \((i \neq 1)\) component

\[
x_i(t; t_0, A_1(t_0)) = \epsilon e^{\lambda_i(t-t_0)} \int_{t_0}^{t} e^{-\lambda_i(\tau-t_0)} N(x(t; t_0, A_1(t_0))) d\tau.
\]

- Take \( t_0 \)-derivatives on both sides and impose \( t \to t_0 \)

\[
\left. \frac{\partial x_i(t; t_0, A_1(t_0))}{\partial t_0} \right|_{t=t_0} = \epsilon \int_{t_0}^{t_0} e^{-\lambda_i(\tau-t_0)} \nabla N(x(t_0; t_0, A_1(t_0))) \cdot \left. \frac{\partial x(t; t_0, A_1(t_0))}{\partial t_0} \right|_{t=t_0} d\tau \sim O(\epsilon^{m+1}).
\]

- Our assertion is surely true for \( m = 0 \). By induction, it is true for all values of \( m \).
A proof by mathematical induction

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   - Renormalization group
   - The RG and differential equations

2 An extension of the RG analysis

3 Several examples
   - The Lotka-Volterra model of competition
   - The Gray-Scott model
   - The Kuramoto-Sivashinsky equation

4 Summary
The Lotka-Volterra model of competition is

\[ \dot{x} = x(3 - x - 2y) \]
\[ \dot{y} = y(2 - x - y) \]

The model describes the competition between the rabbits and the sheep fed on the grass of the same lawn.

Vector field and trajectories

- four equilibria
  \[ P_1 = (0, 0), P_2 = (0, 2), P_3 = (1, 1), P_4 = (3, 0). \]
- Their approximation is
  \[ (1, 1), (2.908, -0.003), (-0.113, 2.105). \]
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Around the saddle $P_3$, we take a coordinate transformation

$$x = 1 - \sqrt{\frac{2}{3}} z + \sqrt{\frac{2}{3}} w, \quad y = 1 + \sqrt{\frac{1}{3}} z + \sqrt{\frac{1}{3}} w.$$  

Assume

$$z = \epsilon z_1 + \epsilon^2 z_2 + \epsilon^3 z_3 + O(\epsilon^4) \quad (2)$$
$$w = \epsilon w_1 + \epsilon^2 w_2 + \epsilon^3 w_3 + O(\epsilon^4), \quad (3)$$

we have

$$\mathcal{L} \circ z_1 = 0, \quad \mathcal{M} \circ w_1 = 0$$
$$\mathcal{L} \circ z_2 = F_2(z_1, w_1), \quad \mathcal{M} \circ w_2 = G_2(z_1, w_1)$$
$$\mathcal{L} \circ z_3 = F_3(z_1, w_1, z_2, w_2), \quad \mathcal{M} \circ w_3 = G_3(z_1, w_1, z_2, w_2),$$

where $\mathcal{L} \equiv 1 - \sqrt{2} + \frac{d}{dt}$, $\mathcal{M} \equiv 1 + \sqrt{2} + \frac{d}{dt}$ and $F_2, G_2, F_3, G_3$ are polynomial functions of their arguments.
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Series solution

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The general solution is

\[ z_1(t) = a(t_0)e^{(\sqrt{2}-1)(t-t_0)}, \quad w_1(t) = b(t_0)e^{-(1+\sqrt{2})(t-t_0)}. \]

Set \( b(t_0) = 0 \) and the solution is

\[
\begin{align*}
  z &= \epsilon a(t_0)e^{(\sqrt{2}-1)(t-t_0)} + \frac{\sqrt{3}\epsilon^2 a^2(t_0)}{6} (\sqrt{2} - 1) \\
  & \quad (e^{2(\sqrt{2}-1)(t-t_0)} - e^{(\sqrt{2}-1)(t-t_0)}) + O(\epsilon^3) \\
  w &= \frac{\sqrt{3}\epsilon^2 a^2(t_0)}{102} (1 + 3\sqrt{2}) e^{2(\sqrt{2}-1)(t-t_0)} + O(\epsilon^3).
\end{align*}
\]

From \( \partial z(t, t_0)/\partial t_0 = 0 \), we get

\[
\frac{da(t_0)}{dt_0} = a \left( \sqrt{2} - 1 - \frac{17\sqrt{3}(3 - 2\sqrt{2})}{102} \epsilon a \right).
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The Dependence on $\beta$

- The Gray-Scott model represents the cubic autocatalytic chemical reactions for two chemical species. The stationary patterns are described by

\[
\begin{align*}
  u'' &= uv^2 - \lambda(1 - v) \\
  \gamma v'' &= v - uv^2.
\end{align*}
\]

- The equation is invariant under $x \to -x$. Two heteroclinic orbits exist at $\gamma = 2/9$ and $\lambda = 9/2$, together with three equilibria $P_1 = (1, 0), P_2 = (1/3, 3), P_3 = (2/3, 3/2)$

- It can be converted to a 4-d dynamical system

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- The stability exponents of $P_1$ are $\pm 3\sqrt{2}/2$, both being doubly degenerate. We have to use two parameters $r_0(x_0), r_1$ to parametrize the initial position.

- We obtain

  \[ u = 1 + \epsilon\left(-\frac{\sqrt{2}r_0 f(x, x_0)}{3}\right) + \epsilon^2 \frac{4}{243}(f^2(x, x_0) - f(x, x_0))r_0^2 r_1^2 + \cdots \]

  \[ p = \epsilon f(x, x_0)r_0 - \epsilon^2 \frac{2\sqrt{2}}{81}(2f^2(x, x_0) - f(x, x_0))r_0^2 r_1^2 + \cdots \]

  \[ v = \epsilon\left(-\frac{\sqrt{2}r_0 r_1 f(x, x_0)}{3}\right) - \epsilon^2 \frac{2}{27}(f^2(x, x_0) - f(x, x_0))r_0^2 r_1^2 + \cdots \]

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The RG equation

- The RG equation for \( r_0(x_0) \) is given by setting
  \[
  \partial u(x, x_0) / \partial x_0 = 0 \quad \text{followed by} \quad x \to x_0
  \]
  \[
  \frac{dr_0(x_0)}{dx_0} = -\frac{3r_0}{\sqrt{2}} + \frac{2}{27} \epsilon r_0^2 r_1^2 + \frac{r_0}{21870} (-45 \sqrt{2} r_1^2 (9 + 2r_1) \epsilon^2 r_0^2 + 8 r_1^3 (9 + 2r_1) \epsilon^3 r_0^3) + \cdots.
  \]

- By setting \( r_1 = -9/2 \), we have
  \[
  \frac{dr_0(x_0)}{dx_0} = -\frac{3}{\sqrt{2}} r_0 + \frac{3}{2} r_0^2.
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  which has the solution
  \[
  r_0(x_0) = \frac{\sqrt{2}}{2} (1 - \tanh \frac{3x_0}{2\sqrt{2}}).
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  \[ r_0(x_0) = \frac{\sqrt{2}}{2} (1 - \tanh \frac{3x_0}{2\sqrt{2}}) . \]
The exact analytic solution of the original equation is thus

\[ u(x) = 1 - \frac{\sqrt{2}}{3} r_0(x) = \frac{1}{3} \left( 2 + \tanh \frac{3x}{2\sqrt{2}} \right) \]

\[ v(x) = \frac{3}{\sqrt{2}} r_0(x) = \frac{3}{2} \left( 1 - \tanh \frac{3x}{2\sqrt{2}} \right). \]
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The Kuramoto-Sivashinsky equation

- The Kuramoto-Sivashinsky equation is an important physics model

\[ u_t = (u^2)_x - u_{xx} - \nu u_{xxxx} , \]

where \( \nu > 0 \) is the hyper-viscosity parameter.

- With periodic boundary condition on \([0, 2\pi]\), we may expand

\[ u(t, x) = i \sum_{k=-\infty}^{\infty} a_k e^{ikx} . \]

- For the antisymmetric solution \( u(t, -x) = -u(t, x) \), \( a_k \) is real and \( a_{-k} = -a_k \). The equation becomes

\[ \dot{a}_k = (k^2 - \nu k^4) a_k - k \sum_{m=-\infty}^{\infty} a_m a_{k-m} . \]
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Perturbation analysis

- Assume

\[ a_k = \epsilon a_{k,1} + \epsilon^2 a_{k,2} + \epsilon^3 a_{k,3} + \cdots. \]

- For the $1-d$ unstable manifold of the origin at $\nu < 1$, we may get

\[ a_{1,1}(t, t_0) = r(t_0)e^{(1-\nu)(t-t_0)}, a_{k,1} = 0 \text{ for } k > 1. \]

where $r(t_0)$ is the renormalization parameter.

- The RG equation for $r(t_0)$ is obtained from

\[ \frac{dr_0}{dt_0} \bigg|_{t=t_0} = 0: \]

\[ \frac{dr_0}{dt_0} = (1 - \nu)r_0 + \frac{2r_0^3}{1 - 7\nu} - \frac{6r_0^5}{(1 - 7\nu)^2(-1 + 13\nu)} + \cdots. \]
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The time evolution on the connection

\[ \nu = 0.5 \]
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\[ \nu = 0.3 \]
The manifold and physical observable

[Image: Two graphs showing a 3D plot and a 2D plot with annotations]

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An extension of the RG method has been proposed and was successfully used for the determination of heteroclinic orbits in the phase space.

The method was applied to three typical physical systems: the Lotka-Volterra model of competition, the Gray-Scott model and the Kuramoto-Sivashinsky equation.

There seems no obstacles to generalize the current technique to the treatment of dynamics on invariant submanifolds of dimension higher than one.

Problems and challenges:
1. How to adapt the current scheme to the oscillatory case is an interesting problem.
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