Isospectral theory of Euler equations

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Abstract

Isospectral problem of both 2D and 3D Euler equations of inviscid fluids, is investigated. Connections with the Clay problem are described. Spectral theorem of the Lax pair is studied.

Keywords: Isospectral theory; Lax pair; Euler equation

1. Introduction

This note is a continuation of works [1,2]. It focuses upon the isospectral property of the Lax pairs of both 2D and 3D Euler equations. It provides efforts towards the connection between isospectral theory and the Clay problem on Navier–Stokes equations.

2. 2D Euler equation

The 2D Euler equation can be written in the vorticity form

$$\partial_t \Omega + \{\Psi, \Omega\} = 0,$$

where the bracket $\{\cdot, \cdot\}$ is defined as

$$\{ f, g \} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g).$$

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where $\Psi$ is the stream function given by

$$u = -\partial_y \Psi, \quad v = \partial_x \Psi,$$

where $u$ and $v$ are, respectively, the velocity components along $x$ and $y$ directions, and the relation between vorticity $\Omega$ and stream function $\Psi$ is

$$\Omega = \partial_x v - \partial_y u = \Delta \Psi.$$

**Theorem 2.1** (Li [1]). The Lax pair of the 2D Euler equation (2.1) is given as

$$\begin{cases}
L \phi = \lambda \phi, \\
\partial_t \phi + A \phi = 0,
\end{cases}$$

(2.2)

where

$$L \phi = \{\Omega, \phi\}, \quad A \phi = \{\Psi, \phi\},$$

and $\lambda$ is an imaginary constant, and $\phi$ is a complex-valued function.

### 2.1. Isospectral theory and conservation laws

Denote by $H^s$ the Sobolev space $H^s(R^2)$ or $H^s(T^2)$, and $\| \cdot \|_s$ the $H^s$ norm.

**Theorem 2.2.** Let $\Omega$ be a solution to the 2D Euler equation (2.1), $\phi$ be a solution to the Lax pair (2.2) at $(\Omega, \lambda)$; then

$$I = \frac{\|[\Omega, \phi]\|_s}{\|\phi\|_s}$$

is conservation law, i.e., $I$ is independent of $t$.

**Proof.** Take the $H^s$ norm on both side of the first equation in the Lax pair (2.2); then

$$I = \frac{\|[\Omega, \phi]\|_s}{\|\phi\|_s} = |\lambda|.$$

By the isospectral property of the Lax pair (2.2), $I$ is independent of $t$. $\Box$

An interesting idea is to try to use the conservation law $I$ to prove global well-posedness. In this 2D Euler case, the idea is not very important since the global well-posedness is already proved. The hope is to use this idea to prove the global well-posedness of 3D Euler equation.

**Lemma 2.1.** If $\phi$ solves the Lax pair (2.2), then $f(\phi)$ solves

$$\partial_t f(\phi) + \{\Psi, f(\phi)\} = 0$$

(2.3)

for any $f$ smooth enough.

**Proof.** The proof is completed by direct verification. $\Box$

**Lemma 2.2.** If $\{\phi_j\}_{j=1,2,...}$ is a complete base of $H^s$, where $\phi_j$’s are $f(\phi)$’s at different values of $\lambda$; then
\[ \Omega = \sum_{j=1}^{\infty} a_j \phi_j, \]

where \( a_j \)'s are complex constants.

**Proof.** Since Eq. (2.3) is a linear equation, the claim of Lemma 2.1 implies the current lemma. \( \square \)

### 2.2. The spectrum of \( L \)

Denote by \( f_\tau \) the flow generated by the vector field \((\Omega_y, -\Omega_x)\) on \( T^2 \).

**Theorem 2.3.** (1) If the vector field has at least two fixed points, then the essential spectrum of \( L \) in \( H^0(T^2) \) is the imaginary axis.

(2) Denote by \( \Lambda \) the largest Lyapunov exponent of the flow \( f_\tau \); then the essential spectrum of \( L \) in \( H^s(T^2) \) is the vertical band of width \( 2s\Lambda \), symmetric with respect to the imaginary axis.

For a proof of this theorem, see [3]. Here there can be point spectrum embedded in the essential spectrum.

#### 2.3. Rossby wave

The Rossby wave equation is

\[ \partial_t \Omega + \{\Psi, \Omega\} + \beta \partial_x \Psi = 0, \]

where \( \Omega = \Omega(t, x, y) \) is the vorticity, \( \{\Psi, \Omega\} = \Psi_x \Omega_y - \Psi_y \Omega_x \), and \( \Psi = \Delta^{-1} \Omega \) is the stream function. Its Lax pair can be obtained by formally conducting the transformation, \( \Omega = \tilde{\Omega} + \beta y \), to the 2D Euler equation [1],

\[ \{\Omega, \varphi\} - \beta \partial_x \varphi = \lambda \varphi, \quad \partial_t \varphi + \{\Psi, \varphi\} = 0, \]

where \( \varphi \) is a complex-valued function, and \( \lambda \) is a complex parameter.

For the Rossby wave equation, parallel arguments to those in the previous subsections for 2D Euler equation can conducted.

### 3. 3D Euler equation

The 3D Euler equation can be written in vorticity form

\[ \partial_t \Omega + (u \cdot \nabla) \Omega - (\Omega \cdot \nabla) u = 0, \quad (3.1) \]

where \( u = (u_1, u_2, u_3) \) is the velocity, \( \Omega = (\Omega_1, \Omega_2, \Omega_3) \) is the vorticity, \( \nabla = (\partial_x, \partial_y, \partial_z) \), \( \Omega = \nabla \times u \), and \( \nabla \cdot u = 0 \). \( u \) can be represented by \( \Omega \) for example through Biot–Savart law.
Theorem 3.1 [4]. The Lax pair of the 3D Euler equation (3.1) is given as
\[
\begin{align*}
L\phi &= \lambda \phi, \\
\partial_t \phi + A\phi &= 0,
\end{align*}
\]
where
\[L\phi = (\Omega \cdot \nabla)\phi - (\phi \cdot \nabla)\Omega, \quad A\phi = (u \cdot \nabla)\phi - (\phi \cdot \nabla)u,
\]
\(\lambda\) is a complex constant, and \(\phi = (\phi_1, \phi_2, \phi_3)\) is a complex 3-vector valued function.

Theorem 3.2 [2]. Another Lax pair of the 3D Euler equation (3.1) is given as
\[
\begin{align*}
L\phi &= \lambda \phi, \\
\partial_t \phi + A\phi &= 0,
\end{align*}
\]
where
\[L\phi = (\Omega \cdot \nabla)\phi, \quad A\phi = (u \cdot \nabla)\phi,
\]
\(\lambda\) is a complex constant, and \(\phi\) is a complex scalar-valued function.

3.1. Isospectral theory and conservation laws

Denote by \(H^s\) the Sobolev space \(H^s(\mathbb{R}^3)\) or \(H^s(T^3)\), and \(\|\cdot\|_s\) the \(H^s\) norm.

Theorem 3.3. Let \(\Omega\) be a solution to the 3D Euler equation (3.1), \(\phi\) be a solution to the Lax pair (3.3) at \((\Omega, \lambda)\); then
\[
I = \frac{\| (\Omega \cdot \nabla)\phi \|_s}{\| \phi \|_s}
\]
is conservation law, i.e., \(I\) is independent of \(t\).

Proof. Take the \(H^s\) norm on both sides of the first equation in the Lax pair (3.2); then
\[
I = \frac{\| (\Omega \cdot \nabla)\phi \|_s}{\| \phi \|_s} = |\lambda|.
\]
By the isospectral property of the Lax pair (3.2), \(I\) is independent of \(t\). \(\square\)

Remark 3.1. Of course, the significance of the conservation laws comes from their potential in providing \textit{a priori} bounds and establishing the global well-posedness of 3D Euler equation, hence, of 3D Navier–Stokes equation (one of the Clay problems).

Lemma 3.1. If \(\{\varphi_j\}_{j=1,2,...}\) is a complete base of \(H^s\), where \(\varphi_j\)'s solve the Lax pair (3.2) at different values of \(\lambda\); then
\[
\Omega = \sum_{j=1}^{\infty} a_j \varphi_j,
\]
where \(a_j\)'s are complex constants.
**Proof.** The proof is completed by comparing the second equation in the Lax pair (3.2) and the 3D Euler equation. □

**References**