The Spectrum of a Linearized 2D Euler Operator

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We study the spectral properties of the linearized Euler operator obtained by linearizing the equations of incompressible two-dimensional fluid at a steady state with the vorticity that contains only two nonzero complex conjugate Fourier modes. We prove that the essential spectrum coincides with the imaginary axis, and give an estimate from above for the number of isolated nonimaginary eigenvalues. In addition, we prove that the spectral mapping theorem holds for the group generated by the linearized 2D Euler operator.

1. Introduction

Linear stability of fluids has long been an active area of research from Lord Rayleigh to G. I. Taylor and to C. C. Lin [1]. With the development of chaos in partial differential equations and new understanding of fluid turbulence, new interests are generated upon linearized fluid equations [2, 3]. A rather complete spectral theorem was obtained in [2]. The new result of the current article is an upper bound on the number of eigenvalues. This estimate guarantees that the number of eigenvalues are finite, in particular, there are only a finite number of unstable modes. The spectral theorem provides the foundation for any dynamical system study. Specifically, dynamical system studies start with linearized dynamics which involves the spectral theorem of the corresponding linear operator and the decomposition of the phase space into invariant linear
Invariant linear subspaces often result from spectral resolutions. The problem of spectral resolution is completely open. In fact, the central problem left for the linear 2D Euler operator is its spectral resolution. Once the spectral resolution is resolved, the next obstacle toward proving the existence of invariant manifolds is that the nonlinear term is non-Lipschitz. Existence of invariant manifolds has become a rather well-known open problem. There are many articles in this area; no one has directly targeted upon the open problem.

We consider 2D Euler equation under periodic boundary conditions. We apply Fourier transform and linearize the Euler equation about the simplest steady state that contains only two nonzero complex conjugate Fourier modes. Two issues are addressed. First, we reprove the spectral theorem in [2] using a different approach. Then we give an estimate from above for the number of the nonimaginary isolated eigenvalues in terms of the number of lattice points on a line segment inside a disc. Although the existence of nonimaginary eigenvalues is well-known, see, e.g., [4], we are not aware of any results in the literature that would give an estimate from above for the number of nonimaginary eigenvalues. The study in [2] indicates that this estimate is sharp. Second, we show that the spectral mapping theorem holds for the group generated by the linearized Euler operator \( L \), that is, we prove that \( \sigma(e^{tL}) = e^{t\sigma(L)}, \ t \neq 0 \), for the spectrum \( \sigma(\cdot) \). Note that the validity of this spectral mapping property for (nonanalytic) semigroups is a rather delicate issue. For instance, the spectral mapping property does not hold even for a group obtained by a first-order perturbation of a two-dimensional wave equation, see [5] and many other examples and general discussion of this phenomenon in, e.g., [6, 7].

Our strategy is to use some ideas from the theory of Jacobi matrices and operators on spaces with indefinite metrics as well as some general results from the theory of strongly continuous semigroups. It can be summarized as follows.

The steady state considered in this article gives a flow on the torus parallel to a vector \( p \in \mathbb{Z}^2 \). The commonly studied special case is the so-called cat’s eye flow. There are quite a lot physical applications of such flow, especially in the random convection area. The linearized Euler operator \( L \) is a difference operator acting on a space of sequences on \( \mathbb{Z}^2 \), see (4). To study the spectrum of \( L \) we at first “slice” the grid \( \mathbb{Z}^2 \) using subsets \( \Sigma_q, \ q \in \mathbb{Z}^2 \), of lines parallel to \( p \), see (5). This gives us a way to represent \( L \) as a direct sum of operators \( L_q \) acting on the space of sequences on \( \mathbb{Z} \). We show that \( \sigma(L) = \bigcup_q \sigma(L_q) \) and \( R(\lambda, L) = \bigoplus_q R(\lambda, L_q) \) for the resolvent operators. Using an appropriate rescaling and “symmetrization,” we replace the study of \( \sigma(L_q) \) by that of \( \sigma(B_q) \), where \( B_q \) is a certain two-diagonal infinite matrix. The essential spectrum \( \sigma_{\text{ess}}(B_q) \) is described using Weyl’s Theorem. We show that if \( \|q\| > \|p\| \) then \( B_q \) is a self-adjoint operator and, therefore, \( L_q \) does not have nonimaginary spectrum. If \( \|q\| \leq \|p\| \) then \( B_q \) is a finite rank perturbation of a self-adjoint operator. However, we give an appropriate choice of an indefinite metric that makes \( B_q \) a \( J \)-self-adjoint operator on a Pontryagin space with
finitely many positive squares. Standard facts about J-self-adjoint operators give the estimate for the cardinality of the nonimaginary point spectrum of \( L \) in terms of the number of points in \( \mathbb{Z}^2 \) located inside of the open disc with the radius \( \|p\| \). Finally, the proof of the spectral mapping theorem is based on a general Gearhart–Prüss theorem, see, e.g., [7] and an estimate for the norm of \( R(\lambda, L) \), \( \Re \lambda \neq 0 \).

2. Linearization

Consider the 2D Euler equations in the vorticity form:

\[
\frac{\partial \Omega}{\partial t} = -u \frac{\partial \Omega}{\partial x} - v \frac{\partial \Omega}{\partial y}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1}
\]

Here \( \Omega = \text{curl} \ u = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) is the vorticity and \( u = (u, v) \) is the velocity. We assume that \( u = u(x, y) \) and \( v = v(x, y) \) are \( 2\pi \)-periodic, \( x, y \in [0, 2\pi] \), and have zero spatial means. If \( \psi \) is the stream function, then \( u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \), and \( \Omega = \Delta \psi \). If \( \Omega = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \omega_k e^{ik \cdot (x, y)} \), where \( \omega_{-k} = \omega_k^* \), \( k \in \mathbb{Z}^2 \setminus \{0\} \), then, after a short calculation, see [2, 3], we can rewrite (1) as follows:

\[
\frac{d\omega_k}{dt} = \sum_{q \in \mathbb{Z}^2 \setminus \{0\}} A(k - q, q)\omega_{k - q}\omega_q, \quad k \in \mathbb{Z}^2. \tag{2}
\]

Here \( A(p, q) \), \( p, q \in \mathbb{Z}^2 \), are defined by the formula

\[
A(p, q) = A(q, p) = \frac{1}{2} \left[ \left( \frac{1}{\|p\|^2} - \frac{1}{\|q\|^2} \right) \left| \begin{array}{cc} p_1 & q_1 \\ p_2 & q_2 \end{array} \right| \right],
\]

provided \( p \neq \pm q, \ p \neq 0, \ q \neq 0 \), and \( A(p, q) = A(q, p) = 0 \) otherwise. Here and below we denote \( p = (p_1, p_2), \ q = (q_1, q_2), \ k = (k_1, k_2) \); vertical bars denote the determinant, \( \|\cdot\| \) is the Euclidean norm.

Fix \( p \in \mathbb{Z}^2 \setminus \{0\} \) and \( \Gamma \in \mathbb{C} \), and consider the steady state \( \omega^0 = (\omega^0_k)_{k \in \mathbb{Z}^2} \) for (2) defined as follows:

\[
\omega^0_k = \begin{cases} \frac{1}{2} \Gamma, & k = p \\ 
\frac{1}{2} \Gamma^*, & k = -p \\
0, & k \neq \pm p
\end{cases} \tag{3}
\]

where bar denotes the complex conjugate.

If \( \Gamma = a + ib \) and \( p = (p_1, p_2) \) then the vorticity \( \Omega^0 \), corresponding to \( \omega^0 \), is given by the formula

\[
\Omega^0(x, y) = a \cos(p_1 x + p_2 y) - b \sin(p_1 x + p_2 y).
\]
The corresponding velocity field has the following components:

\[ u^0(x, y) = (-p_2a \sin p \cdot (x, y) - p_2b \cos p \cdot (x, y))\left\| p \right\|^2, \]
\[ v^0(x, y) = (p_1a \sin p \cdot (x, y) + p_1b \cos p \cdot (x, y))\left\| p \right\|^2. \]

The linearization of (2) around the steady state \( \omega^0 \) as in (3) gives the following operator \( L \):

\[ L : (\omega_k)_{k \in \mathbb{Z}^2} \mapsto (A(p, k - p)\Gamma \omega_{k-p} + A(-p, k + p)\tilde{\Gamma} \omega_{k+p})_{k \in \mathbb{Z}^2}. \quad (4) \]

The choice of the space on which we want to consider the operator \( L \) is related to the choice of the space for vorticity in (1). Thus, if \( \Omega \in H'(\mathbb{T}^2), r \geq 0 \), the Sobolev space, then \( L \) should be considered on the space \( \ell^2(\mathbb{Z}^2) \) of weighted \( \ell^2(\mathbb{Z}^2) \)-sequences with the weight \( (1 + \|k\|^2)^{r/2} \), \( k \in \mathbb{Z}^2 \). In what follows we consider the operator \( L \) on \( \ell^2(\mathbb{Z}^2) \), that is, for \( r = 0 \) and \( \Omega \in L^2(\mathbb{T}^2) \).

Denote by \( W \) the shift operator \( W : (\omega_k) \mapsto (\omega_{k-p})_{k \in \mathbb{Z}^2} \). This is a unitary operator on \( \ell^2(\mathbb{Z}^2) \). Also, consider the following operator \( D_p : (\omega_k) \mapsto (A(p, k)\Gamma \omega_k) \), and remark that \( D_{-p} = -D_p \). Thus, the linearized Euler operator \( L \) can be represented as follows:

\[ L = WD_p + W^*D_{-p}^* = WD_p + W^*D_p^*. \]

Here and below \(*\) denotes the adjoint operator, \( W^* : (\omega_k) \mapsto (\omega_{k+p}) \) and \( D_p^* : (\omega_k) \mapsto (A(p, k)\tilde{\Gamma} \omega_k) \).

**Remark 1**: The definition of \( A(p, q) \) implies that \( D_p = D_p^0 + D_p^1 \), where

\[ D_p^0 : (\omega_k) \mapsto \begin{pmatrix} \frac{1}{2\|p\|^2} & p_1 & k_1 \\ p_2 & k_2 & \Gamma \omega_k \end{pmatrix}. \]
\[ D_p^1 : (\omega_k) \mapsto \begin{pmatrix} -\frac{1}{2\|k\|^2} & p_1 & k_1 \\ p_2 & k_2 & \Gamma \omega_k \end{pmatrix}. \]

The operator \( D_p^1 \) is a compact operator on \( \ell^2(\mathbb{Z}^2) \), because the image sequence decays to zero as \( \|k\| \to \infty \). For a proof, see [2].

**Remark 2**: Let us define the operator \( L^0 = WD_p^0 - W^*D_p^{0*} \). Note that

\[ (WD_p^0 \omega)_k = 2^{-1\|p\|^{-2}} \begin{vmatrix} p_1 & k_1 - p_1 \\ p_2 & k_2 - p_2 \end{vmatrix} \Gamma \omega(k_1 - p_1, k_2 - p_2) \]

implies \( WD_p^0 \omega)_k = (D_p^0 W \omega)_k \). Therefore, \((L^0)^* = D_p^{0*} W^* - D_p^0 W = -L^0 \), and, as a result, \( \sigma(L^0) \subset i\mathbb{R} \). As we will see below, Weyl’s Theorem implies that \( \sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(L^0) \).
3. Decomposition and Symmetrization

In this section, we first perform a “decomposition” of the operator \( L \). The operator \( L \) is represented as a direct sum of certain operators \( L_q \) acting on the space \( \ell^2(\mathbb{Z}) \). Next, we perform a “symmetrization” of the operators \( L_q \). This procedure allows us to study, instead of the operators \( L_q \), certain operators \( B_q \) that are finite rank perturbations of self-adjoint operators.

For \( p \in \mathbb{Z}^2 \setminus \{0\} \) as above, and any \( q = (q_1, q_2) \in \mathbb{Z}^2 \) we denote

\[
\Sigma_q = \{q + np : n \in \mathbb{Z}\}.
\]

Note that \( \Sigma_q \) is a subset of the line \( \{q + tp : t \in \mathbb{R}\} \), but, generally, \( \Sigma_q \) does not contain all points with integer coordinates that belong to this line. For each \( q \in \mathbb{Z}^2 \) select \( \hat{q} = \hat{q}(q) \) such that \( \hat{q} \in \Sigma_q, \|\hat{q}\| = \inf\{\|q + np\| : n \in \mathbb{Z}\} \), and \( \hat{q} = q + \max\{n : \|q + np\| = \|q + np\|\}p \). The last condition just takes one of the two possible points \( q + np \in \Sigma_q \) such that \( \|q + np\| = \|\hat{q}\| \). Denote \( Q = \{\hat{q}(q) : q \in \mathbb{Z}^2\} \). Clearly, \( \Sigma_q = \Sigma_{\hat{q}(q)} \) and \( \cup_{q \in Q} \Sigma_q = \mathbb{Z}^2 \).

Fix \( q \in Q \) and let \( X_q = \{\{\omega_k\} \in \ell^2(\mathbb{Z}^2) : \omega_k = 0 \text{ for } k \notin \Sigma_q \} \). Note that \( \Sigma_q \cap \Sigma_{q'} = \emptyset \) provided \( q \neq q' \); therefore, \( \{X_q\}_{q \in Q} \) is a system of orthogonal subspaces in \( \ell^2(\mathbb{Z}^2) \) such that \( \oplus_{q \in Q} X_q = \ell^2(\mathbb{Z}^2) \).

Note that \( J : X_q \hookrightarrow \ell^2(\mathbb{Z}) : (\omega_{k+np})_{n \in \mathbb{Z}} \mapsto (\omega_n)_{n \in \mathbb{Z}} \) is an isometric isomorphism. Also, the operator \( L \) leaves the subspace \( X_q \) invariant. Thus, \( L = \oplus_{q \in Q} L_q \), where \( L_q = VD - V^*D^* \), and we denote

\[
V : (\omega_n) \mapsto (\omega_{n-1}), \quad D = D(p, q) : (\omega_n) \mapsto (A(q + np, p) \Gamma \omega_n).
\]

**Remark 3:** If \( q = tp, t \in \mathbb{R} \), then \( D = 0 \) by the definition of \( A(p, q) \). Thus, in what follows we assume that \( q \neq tp \), that is, that \( \Sigma_q \) does not belong to the line \( \{tp : t \in \mathbb{R}\} \). In particular, \( q + np \neq 0 \) and \( q + np \neq \pm p \) for all \( n \in \mathbb{Z} \).

Our next step is to perform a symmetrization of the operator \( L_q \). Fix \( p \) as above, and \( q \in Q \) such that \( q \neq tp, t \in \mathbb{R} \). Denote

\[
\beta = -\frac{1}{2\|p\|^2} \begin{vmatrix} q_1 & p_1 \\ q_2 & p_2 \end{vmatrix} \Gamma, \quad \gamma_n = -\frac{\|p\|^2}{\|q + np\|^2}, \quad n \in \mathbb{Z}.
\]

Note that \( \beta \neq 0 \) due to \( q \neq tp \). By the definition of \( A(p, q) \) we have

\[
A(q + np, p)\Gamma = \frac{1}{2} \begin{vmatrix} 1 & 1 \\ \|q + np\|^2 & \|p\|^2 \end{vmatrix} \begin{vmatrix} q_1 & p_1 \\ q_2 & p_2 \end{vmatrix} \Gamma = \beta(1 + \gamma_n).
\]

Therefore,

\[
L_q = (V \beta - V^* \bar{\beta}) \text{diag}(1 + \gamma_n)_{n \in \mathbb{Z}}.
\]
Define $\alpha = i \beta^1/\bar{\beta}^1$, $|\alpha| = 1$, and set $S = \alpha V$. Note that
\[ V\beta - V^* \bar{\beta} = -i \bar{\beta}^1(S+S^*)\beta^1 = -i|\beta|(S+S^*). \]

Denote $M^0_q = S + S^*$, observe that $(M^0_q)^* = M^0_q$, and remark that
\[ M^0_q = \begin{bmatrix} \ddots & \alpha & 0 \\ 0 & 0 & \bar{\alpha} \\ \bar{\alpha} & 0 & \alpha \\ 0 & \bar{\alpha} & 0 \\ \ddots \end{bmatrix}, \quad |\alpha| = 1. \tag{8} \]

Let $M_q = M^0_q \text{diag}(1 + \gamma_n)_{n \in \mathbb{Z}}$, and note that $L_q = -i|\beta|M_q$. Thus,
\[ \sigma(L_q) = -i|\beta|\sigma(M_q) \quad \text{and} \quad L = \bigoplus_{q \in \mathbb{Q}} -i|\beta|M_q. \tag{9} \]

If $\lambda \notin \sigma(L)$, then $\lambda \notin \sigma(L_q)$ and we have
\[ R(\lambda, L) = \bigoplus_{q \in \mathbb{Q}} R(\lambda, -i|\beta|M_q) = \bigoplus_{q \in \mathbb{Q}} \frac{i}{|\beta|} R\left(\frac{\lambda}{i|\beta|}, M_q\right). \]

The operator $M_q$ is a multiplicative perturbation of a self-adjoint operator $M^0_q$. Instead of $M_q$ we consider a symmetric operator $B_q$ with the same spectrum. To do this, for $n \in \mathbb{Z}$ we define:
\[ \delta_n = |1 + \gamma_n|^1 \quad \text{if} \ 1 + \gamma_n \geq 0 \quad \text{and} \quad \delta_n = i|1 + \gamma_n|^1 \quad \text{if} \ 1 + \gamma_n \leq 0. \tag{10} \]

Note that $\delta_n \in \mathbb{R}$ if and only if $\|q + np\| \geq \|p\|$, and $\delta_n \in i\mathbb{R}$ if and only if $\|q + np\| < \|p\|$. Denote
\[ B_q = \text{diag}(\delta_n)M_q^0\text{diag}(\delta_n) = \begin{bmatrix} \ddots & 0 & \alpha \delta_{-1} \delta_0 & 0 \\ 0 & \bar{\alpha} \delta_{-1} \delta_0 & 0 & \alpha \delta_0 \delta_1 \\ \bar{\alpha} \delta_{-1} \delta_0 & 0 & \bar{\alpha} \delta_0 \delta_1 & 0 \\ \ddots \end{bmatrix}, \tag{11} \]

and note that
\[ M_q = M^0_q\text{diag}(\delta_n) \cdot \text{diag}(\delta_n), \quad B_q = \text{diag}(\delta_n)M_q^0\text{diag}(\delta_n). \tag{12} \]

**Proposition 1.** The nonzero elements in $\sigma(M_q)$ and $\sigma(B_q)$ coincide.

**Proof:** This is a consequence of the following elementary fact: If $A$ and $B$ are bounded operators, then $\sigma(AB) \setminus \{0\} = \sigma(BA) \setminus \{0\}$. ■
Remark 4: Assume \( \|q\| \geq \|p\| \). By the choice of \( q \in Q \) we have \( \|q + np\| \geq \|q\| \). Therefore, \( \delta_n \in \mathbb{R} \) for all \( n \in \mathbb{Z} \) and \( B_q = B_q^* \). Hence, \( \sigma(B_q) \subset \mathbb{R} \) or \( \sigma(L_q) \subset i\mathbb{R} \). Because \( L_q \) is a bounded operator for each \( q \in Q \), we have that \( \cup_{\|q\| \leq \|p\|} \sigma(L_q) \) is a bounded set. Moreover, \( \cup_{q \in Q} \sigma(L_q) \setminus i\mathbb{R} \) is a bounded set.

Remark 5: Assume \( \|q\| < \|p\| \). Then \( \delta_n \delta_{n+1} \in i\mathbb{R} \) for exactly two values of \( n \in \mathbb{Z} \). Each of these values corresponds to a pair of two consecutive points in \( \Sigma_q \) such that one of the points lies inside and another outside of the disc with the radius \( \|p\| \). For all other values of \( n \in \mathbb{Z} \) we have \( \delta_n \delta_{n+1} \in \mathbb{R} \). Thus, \( B_q \) is a perturbation of a self-adjoint operator by a rank four skew-self-adjoint operator.

It is quite simple to describe the spectrum of the “constant coefficients” infinite matrix \( M_q^0 \). For \( q \in Q \) and \( \beta = \beta(q) \), defined in (6), let

\[
L_q^0 = V\beta - V^*\beta = -i|\beta| M_q^0,
\]

where \( M_q^0 \) is as in (8). Note that \( \|M_q^0\| \leq 2 \) and, thus, \( \sigma(M_q^0) \subset \{|z| \leq 2\} \) and \( \sigma(L_q^0) \subset \{|z| \leq |\beta|\} \).

If \( F : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}) \) is the Fourier transform, then

\[
F((\lambda + M_q^0)(\omega_n)) = (\alpha\bar{z} + \lambda + \bar{\alpha}z)F((\omega_n))(z), \quad |z| = 1,
\]

for each \( \lambda \in \mathbb{C} \), and \( \alpha = i\beta^1/\bar{\beta}^1 \). Therefore, \( -\lambda \not\in \sigma(M_q^0) \) if and only if the function \( (\lambda + M_q^0)(\cdot) \) defined by the formula \( (\lambda + M_q^0)(z) = \alpha\bar{z} + \lambda + \bar{\alpha}z \) is not equal to zero for all \( |z| = 1 \). If this is the case, then \( (\lambda + M_q^0)^{-1} = F^{-1}\frac{1}{(\lambda + M_q^0(\bar{z}))}F \).

Therefore, the following fact holds.

**Proposition 2.**

(a) \( \sigma(M_q^0) = \sigma_{ess}(M_q^0) = [-2, 2] \);

(b) \( \sigma(L_q^0) = \sigma_{ess}(L_q^0) = i[-2|\beta|, 2|\beta|] \).

We note that very interesting and deep results on the spectrum of the “variable coefficients” infinite matrices of the same type as \( M_q \) could be found in [8] and the literature on Jacobi matrices cited therein. However, the results that we need for the specific rate of decay of \( \gamma_n \) to zero for the problem in hand do not seem to be available.

4. Spectrum

In this section, we describe the spectrum of the linearized Euler operator \( L \). Let \( \sigma_p(\cdot) \) denote the point spectrum. For \( \lambda = a + i\tau, a \neq 0 \), and \( \beta = \beta(q) \) defined in (6) we denote \( z = \lambda/(i|\beta|) \).
Theorem 1.

(a) For each $q \in Q$ we have

$$\sigma_{\text{ess}}(L_q) = \sigma(V \beta - V^* \bar{\beta}) = i[-2|\beta|, 2|\beta|].$$

(b) $\sigma(L) = \bigcup_{q \in Q} \sigma(L_q)$;

(c) $\sigma_{\text{ess}}(L) = i\mathbb{R}$ and $\sigma_p(L) \setminus i\mathbb{R} = \bigcup_{\|q\| \leq \|p\|} (\sigma_p(L_q) \setminus i\mathbb{R})$ is a bounded set with accumulation points only on $i\mathbb{R}$.

Proof: (a) Split $L_q = L^0_q + L^\text{comp}_q$, where $L^0_q = V \beta - V^* \bar{\beta}$, and $L^\text{comp}_q = L^0_q \text{diag}(\gamma_n)_{n \in \mathbb{Z}}$ is a compact operator due to $\lim_{n \to \infty} |\gamma_n| = 0$. Apply Weyl’s theorem (see Lemma XIII.4.3 in [9]) for $A = L^0_q$ and $B = L_q$. Note that $\sigma(A) = i[-2|\beta|, 2|\beta|]$ has an empty interior in $\mathbb{C}$, and $\mathbb{C} \setminus \sigma(A)$ consists of only one component. Because $A - B$ is a compact operator, we have $\sigma_{\text{ess}}(L^0_q) = \sigma_{\text{ess}}(L_q)$. Using Proposition 2, we have (a).

(b) Split $L$ into the direct sum of two operators, $L^s$ and $L^b$, that correspond to “small” and “big” values of $\|q\|$, that is, write $L = L^s + L^b$, where $L^s = \bigoplus_{\|q\| \leq \|p\|} L_q$ and $L^b = \bigoplus_{\|q\| > \|p\|} L_q$. Because $L^s$ is a direct sum of finitely many operators, we have

$$\sigma(L) = \left( \bigcup_{\|q\| \leq \|p\|} \sigma(L_q) \right) \cup \sigma(L^b),$$

and we need to see only that $\sigma(L^b) \subset \bigcup_{\|q\| > \|p\|} \sigma(L_q)$. However, $|\beta| = |\beta(q)| \to \infty$ as $\|q\| \to \infty$. Using (a) in the theorem, we have

$$\bigcup_{\|q\| > \|p\|} \sigma(L_q) \supset \bigcup_{\|q\| > \|p\|} i[-2|\beta|, 2|\beta|] = i\mathbb{R}.$$ 

Thus, it suffices to show that $\sigma(L^b) \subset i\mathbb{R}$. Because

$$L^b = \bigoplus_{\|q\| > \|p\|} L_q = \bigoplus_{\|q\| > \|p\|} -i|\beta(q)|M_q,$$

it suffices to prove the following claim:

if $\lambda \notin i\mathbb{R}$, then $\sup_{\|q\| > \|p\|} \left\| \frac{1}{\beta(q)} (z + M_q)^{-1} \right\| < \infty$. (13)

Indeed, assume that (13) is proved. Then

$$\| (\lambda - L^b)^{-1} \| = \left\| \bigoplus_{\|q\| > \|p\|} \frac{1}{i|\beta(q)|} (z + M_q)^{-1} \right\| < \infty$$

and $\lambda \notin \sigma(L^b)$. 

To prove (13), fix $\lambda = a + i\tau$, $a \neq 0$. Note that for any $q \in Q$ we have (see (12))

$$z + M_q = z + M_q^0 + M_q^0 \text{diag}(\gamma_n)$$

$$= (z + M_q^0) \left[ I + (z + M_q^0)^{-1} M_q^0 \text{diag}(\gamma_n) \right].$$

(14)

Note that the resolvent $(z + M_q^0)^{-1}$ in this identity exists because $z \notin \mathbb{R}$ and $M_q^0$ is a self-adjoint operator; moreover,

$$\left\| (z + M_q^0)^{-1} \right\| = \frac{1}{|\text{Im } z|} = \frac{|\beta|}{|a|}.$$

(15)

**Proposition 3.** There exists a constant $c(p)$ such that for all $q \in Q$, $q \neq 0$, we have

$$|\beta(q)| \| \text{diag}(\gamma_n)_{n \in \mathbb{Z}} \| \leq c(p) / \| q \|.$$

**Proof:** Using (6), we have

$$|\beta| \| \text{diag}(\gamma_n)_{n \in \mathbb{Z}} \| = |\beta| \sup_{n \in \mathbb{Z}} |\gamma_n| \leq c \sup_{n \in \mathbb{Z}} \left| \begin{array}{cc} q_1 & p_1 \\ q_2 & p_2 \end{array} \right| \| q + np \|^{-2}$$

$$= c \sup_{n \in \mathbb{Z}} \left| \begin{array}{cc} q_1 + np_1 & p_1 \\ q_2 + np_2 & p_2 \end{array} \right| \| q + np \|^{-2}$$

$$= c \sup_{n \in \mathbb{Z}} \left| (q + np) \cdot p^\perp \| q + np \|^{-2} \right|,$$

where $p^\perp = (p_2, -p_1)$ is the $\mathbb{Z}^2$-vector, perpendicular to $p$. Using Cauchy–Schwartz inequality, we have that

$$|\beta| \| \text{diag}(\gamma_n)_{n \in \mathbb{Z}} \| \leq c \sup_{n \in \mathbb{Z}} \frac{\| q + np \| \| p^\perp \|}{\| q + np \|^2} \leq \frac{c_1}{\inf_{n \in \mathbb{Z}} \| q + np \|} = \frac{c_1}{\| q \|},$$

where the definition of $q \in Q$ has been used.

For $\lambda = a + i\tau$ as above, and $c(p)$ from Proposition 3, fix $q_0 = q_0(a)$ such that $\| q_0 \| > \| p \|$ and if $\| q \| \geq \| q_0 \|$ then the inequality

$$\frac{2c(p)}{|a| \| q \|} \leq \frac{1}{2}$$

(16)

holds. Note that the set $Q_s := \{ q \in Q : \| q \| \in [\| p \|, \| q_0 \|] \}$ is finite, and let $Q_b := \{ q \in Q : \| q \| > \| q_0 \| \}$.

Using Proposition 3, and the inequality $\| M_q^0 \| \leq 2$, we have (thanks to (15)) that if $q \in Q_b$, then

$$\left\| (z + M_q^0)^{-1} M_q^0 \text{diag}(\gamma_n)_{n \in \mathbb{Z}} \right\| \leq \frac{2|\beta|}{|a|} \| \text{diag}(\gamma_n) \| \leq \frac{2c(p)}{|a| \| q \|} \leq \frac{1}{2}.$$
Thus, the operator \( I + (z + M_q^0) M_q^0 \text{diag}(\gamma_n) \) is invertible and, for \( q \in Q_b \),

\[
\left\| I + (z + M_q^0) M_q^0 \text{diag}(\gamma_n) \right\|^{-1} \leq 2. \tag{17}
\]

For each \( q \in Q_s \) we remark that \( z \not\in \sigma(B_q) \), see (12), and, hence, \( z \not\in \sigma(M_q) \). Because \( Q_s \) is a finite set, we have

\[
\sup_{q \in Q_s} \left\| \frac{1}{|\beta|} (z + M_q)^{-1} \right\| < \infty.
\]

**Remark 6:** Let \( L_s^b = \oplus_{q \in Q_b} L_q \). Then \( L_s^b \) is a bounded operator. Hence, if \( \lambda = a + i\tau, a \neq 0 \), then \( \| (\lambda - L_s^b)^{-1} \| = O(|\tau|^{-1}) \) as \( |\tau| \to \infty \).

To finish the proof of (13), we use (14)–(15), and (17):

\[
\sup_{q \in Q_b} \left\| |\beta|^{-1} (z + M_q)^{-1} \right\| \leq \sup_{q \in Q_s} \frac{2}{|\beta|} \left\| (z + M_q^0)^{-1} \right\|
\leq \sup_{q \in Q_s} 2 |\beta|^{-1} \cdot \frac{|\beta|}{|a|} = \frac{2}{|a|}. \tag{18}
\]

This proves (13) and (b) in the theorem.

(c) Because

\[
\sigma_{\text{ess}}(L) = \left( \bigcup_{||q|| \leq ||p||} \sigma_{\text{ess}}(L_q) \right) \bigcup \sigma_{\text{ess}}(L^b),
\]

the first statement follows from (a) in the theorem. The second statement follows from Remark 4 and (a) in the theorem.

Because \( L \) is the sum of a skew-adjoint operator \( L^0 \) and a compact operator, see Remarks 1 and 2, the operator \( L \) generates a strongly continuous group. The following spectral mapping theorem holds for the group \( \{e^{tL}\}_{t \geq 0} \). Its proof is similar to the proof of Theorem 1 in [10].

**Theorem 2.** If \( L \) is the linearized Euler operator, then

\[
\sigma(e^{tL}) = e^{t\sigma(L)}, \quad t \neq 0.
\]

**Proof:** Let \( L^b = \oplus_{q \in Q_b} L_q \). Inequality (18) shows that if \( \lambda = a + i\tau, a \neq 0 \), then \( \| (\lambda - L^b)^{-1} \| \leq 2/|a| \). Using Remark 6 we have that

\[
\| (\lambda - L)^{-1} \| = O(1) \quad \text{as} \quad |\tau| \to \infty. \tag{19}
\]

Now the assertion follows from the resolvent estimate (19) and the following general Gearhart–Prüss spectral mapping theorem: On a Hilbert space the spectrum \( \sigma(e^{itL}) \), \( t \neq 0 \), is the set of the points \( e^{it\mu} \) such that either \( \mu_n = \lambda + 2\pi n/t \) belongs to \( \sigma(L) \) for some \( n \in \mathbb{Z} \), or the sequence \( \| R(\mu_n, L) \|_{t \in \mathbb{Z}} \) is unbounded, see, e.g., [6]. Recall, that the spectral mapping property always
holds for the point spectrum. Due to (19) we conclude that $\sigma_{\text{ess}}(e^{itL})$ belongs to the unit circle.

5. $J$-Theory

In this section, we obtain an estimate from above for the number of nonimaginary isolated eigenvalues of the operator $L$. Recall that $\sigma_p(L_q) \setminus i\mathbb{R}$ is empty as soon as $\|q\| \geq \|p\|$. Let $\varkappa$ denote the number of points $q \in \mathbb{Z}^2$ that belongs to the open disk of radius $\|p\|$, and such that $q \neq tp$. Because for such $q$ we have $(\pm q_1)^2 + (\pm q_2)^2 < \|p\|^2$, we conclude that $\varkappa$ is even.

**Theorem 3.** The number of nonimaginary eigenvalues of $L$ (counting the multiplicities) does not exceed $2\varkappa$.

**Proof:** Since $L_q = 0$ for $q = tp$, only those $L_q$ for which $\|q\| < \|p\|$ and $q \neq tp$ will contribute to the nonimaginary point spectrum of $L$. For each such $q$ let $n'_q$ denote the smallest and $n''_q$ the largest integer such that $\delta_n \in i\mathbb{R}$ for $n = n'_q, \ldots, n''_q$, see (10) for the definition of $\delta_n$. Let $J_q = \text{diag}(jn)_{n \in \mathbb{Z}}$, where $jn = 1$ if $n = n'_q, \ldots, n''_q$ and $jn = -1$ otherwise. Note that $J_q = J_q^{-1} = J_q^*$ and

$$J_q \text{diag}(\delta_n) = \text{diag}(\delta_n)J_q = (\text{diag}(\delta_n))^*.$$ 

Using (11), we have that $J_q B_q J_q = B_q^*$, that is, that the operator $B_q$ is $J_q$-self-adjoint. Let $\langle \omega, \omega' \rangle$ be the standard scalar product in $\ell^2(\mathbb{Z})$. Note that the formula $[\omega, \omega'] = \langle J_q \omega, \omega' \rangle$ defines an indefinite metric on $\ell^2(\mathbb{Z})$. Thus, $\ell^2(\mathbb{Z})$ is a Pontryagin space with $n''_q - n'_q$ positive squares. By a standard result of the theory of $J$-self-adjoint operators on Pontryagin spaces (see, e.g., [11, Cor. II.3.15]), we have that the number of nonreal eigenvalues of $B_q$ does not exceed $2(n''_q - n'_q)$. By (12), (9), and $\varkappa = \sum_q (n''_q - n'_q)$ we have the result. \blacksquare

**Proposition 4.** The nonimaginary eigenvalues of $L$ are symmetric about the coordinate axes.

**Proof:** Rewrite (7) as $L_q = N \text{diag}(1 + \gamma_n)$, where $N = V \beta - V^* \beta = -N^*$. Using the argument in the proof of Proposition 1, we have

$$\sigma(L_q^*) \setminus \{0\} = \sigma\left(\text{diag}(1 + \gamma_n)N^*\right) \setminus \{0\} = -\sigma\left(\text{diag}(1 + \gamma_n)N\right) \setminus \{0\},$$

$$-\sigma(N\text{diag}(1 + \gamma_n)) \setminus \{0\} = -\sigma(L_q) \setminus \{0\}.$$ 

Thus, $\sigma(L_q) \setminus \{0\} = \overline{\sigma(L_q)} \setminus \{0\} = -\sigma(L_q) \setminus \{0\}$, and the nonimaginary eigenvalues of $L$ are symmetric about $i\mathbb{R}$. Next, define $\hat{J}$ on $\ell^2(\mathbb{Z})$ by $\hat{J} : (\omega_q) \mapsto ((-1)^n \omega_n)$ and note that $V \hat{J} = -\hat{J} V$ and $\hat{J}V^* = -V^* \hat{J}$. Thus,

$$\hat{J} L_q \hat{J} = \hat{J} \left(V \text{diag}(1 + \gamma_n) \beta - V^* \text{diag}(1 + \gamma_n) \beta\right) \hat{J} = -L_q,$$

and $\sigma(L_q) = \sigma(\hat{J} L_q \hat{J}) = -\sigma(L_q)$. \blacksquare
Note that the change of variables $\xi = p_1x + p_2y$, $\eta = -p_2x + p_1y$ in (1) converts $\Omega^0$ to the vorticity $\tilde{\Omega}^0 = \tilde{\Omega}^0(\xi)$, that is, to a parallel shear flow. However, this does not simplify our analysis, because the new flow loses $2\pi$-periodicity, and the results from, say, [4] cannot be applied directly.

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References


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