SIAM J. MATH. ANAL.
Vol. 42, No. 6, pp. 2434–2472
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# STABILITY OF TRAVELING WAVES FOR A CLASS OF REACTION-DIFFUSION SYSTEMS THAT ARISE IN CHEMICAL REACTION MODELS*

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**Abstract.** Stability results are proved for traveling waves in a class of reaction-diffusion systems that arise in chemical reaction models. The class includes systems in which there is no diffusion in some equations. A weight function that decays exponentially at one end is required to stabilize the essential spectrum. Perturbations of the wave in $H^1$ or $BUC$ that are small in both the weighted norm and the unweighted norm are shown to stay small in the unweighted norm and to decay exponentially to a shift of the traveling wave in the weighted norm. Perturbations that are in addition small in the $L^1$ norm decay algebraically to a shift of the wave in the $L^\infty$ norm. A decomposition of the variables that yields a triangular structure for the linearization at one end of the wave is exploited to prove the results. An application to exothermic-endothermic reactions is given.

**Key words.** traveling wave, nonlinear stability, reaction-diffusion system, degenerate diffusion matrix, weighted norm, combustion

**AMS subject classifications.** 35K57, 35C07, 35B40, 80A25

**DOI.** 10.1137/100786204

## 1. Introduction.

Consider a reaction-diffusion system

$$Y_t = DY_{xx} + R(Y), \tag{1.1}$$

where $Y \in \mathbb{R}^n$, $x \in \mathbb{R}$, $t \geq 0$, $D = \mathrm{diag}(d_1, \ldots, d_n)$ with all $d_i \geq 0$, and the function $R(Y)$ is smooth.

In applications modeled by (1.1), coherent structures of interest include traveling waves. These are solutions $Y_*(\xi)$, $\xi = x - ct$, of (1.1), where $c$ is the velocity of the wave. We are concerned with traveling waves that approach constant states as $\xi \to \pm\infty$:

$$\lim_{\xi\to-\infty} Y_*(\xi) = Y_-, \quad \lim_{\xi\to\infty} Y_*(\xi) = Y_+.$$

Such traveling waves are called *pulses* if $Y_- = Y_+$ and *fronts* if $Y_- \neq Y_+$. We must have $R(Y_-) = R(Y_+) = 0$. We shall always assume that traveling waves approach both end states at an exponential rate; i.e., there exist numbers $K > 0$ and $\omega_- < 0 < \omega_+$ such that for $\xi \leq 0$, $\|Y_*(\xi) - Y_-\| \leq Ke^{-\omega_-\xi}$, and for $\xi \geq 0$, $\|Y_*(\xi) - Y_+\| \leq Ke^{-\omega_+\xi}$.

There is an extensive literature on the stability of traveling waves in reaction-diffusion systems. We mention [15] and references therein.

Replacing the spatial variable $x$ by the moving variable $\xi$ in (1.1), we obtain

$$Y_t = DY_{\xi\xi} + cY_\xi + R(Y). \tag{1.2}$$

*Received by the editors February 17, 2010; accepted for publication (in revised form) July 23, 2010; published electronically October 5, 2010. This work was supported in part by the National Science Foundation under grants DMS-0754705, DMS-0708386, and DMS-0908009.
http://www.siam.org/journals/sima/42-6/78620.html

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The traveling wave $Y_*(\xi)$ is a stationary solution of (1.2). We shall say that the wave $Y_*$ is *stable* in the space $\mathcal{X}$ if a small perturbation of $Y_*$ of the form $Y = Y_* + \tilde{Y}$ with $\tilde{Y} \in \mathcal{X}$ decays to some shift of $Y_*$. (We shall use the word "stable" to mean what is more precisely termed asymptotically stable with asymptotic phase.) $Y_*$ is *exponentially stable* if the decay is exponential in time.

Information about the stability of the wave $Y_*$ is encoded in the spectrum of the operator obtained by linearizing (1.2) about $Y_*$,

$$\tilde{Y}_t = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + DR(Y_*)\tilde{Y} =: L\tilde{Y}. \tag{1.3}$$

Let $\mathcal{L} : \mathcal{X} \to \mathcal{X}$ be the operator on $\mathcal{X}$ given by $\tilde{Y} \to L\tilde{Y}$, with its natural domain. We shall say that the wave $Y_*$ is *spectrally stable* in the space $\mathcal{X}$ if the spectrum of $\mathcal{L}$ is contained in the half-plane $\operatorname{Re}\lambda < -\nu < 0$, with the exception of a simple eigenvalue 0. (A traveling wave has an eigenvalue 0, with eigenvector $Y_*'(\xi)$, in any space that contains $Y_*'$.) $Y_*$ is *linearly exponentially stable* in $\mathcal{X}$ if every solution of (1.3) decays exponentially to a multiple of $Y_*'$.

In [6] we studied a simple model for gasless combustion in a solid:

$$\partial_t y_1 = \partial_{xx} y_1 + y_2\rho(y_1), \tag{1.4}$$

$$\partial_t y_2 = -\beta y_2\rho(y_1), \tag{1.5}$$

with $\beta > 0$ and

$$\rho(y_1) = \begin{cases} e^{-\frac{1}{y_1}} & \text{if } y_1 > 0, \\ 0 & \text{if } y_1 \le 0. \end{cases} \tag{1.6}$$

In this system, $y_1$ is temperature, $y_2$ is concentration of unburned fuel, and $\rho$ is the unit reaction rate. The value $y_1 = 0$ represents a background temperature at which the reaction does not take place.

There is a number $c > 0$ for which (1.4)–(1.5) admits a traveling combustion front $(y_{1*}, y_{2*})(\xi)$, $\xi = x - ct$, such that $(y_{1*}, y_{2*})(-\infty) = (y_{1-}, 0)$ ($y_{1-} > 0$, the temperature of combustion, must be determined); $(y_{1*}, y_{2*})(\infty) = (0, 1)$ (the concentration of fuel in the medium is normalized to 1); and $(y_{1*}, y_{2*})(\xi)$ approaches the end states exponentially. If one attempts to prove stability of this traveling wave, one encounters three difficulties.

1. The traveling wave is not spectrally stable: the essential spectrum of the linearization of (1.4)–(1.5) at $(y_{1*}, y_{2*})(\xi)$ touches the imaginary axis. This can be cured by working in a weighted space with weight function $e^{\alpha\xi}$, $\alpha > 0$ small. In such a space, the traveling wave is spectrally stable. Such a space only includes functions that go to zero exponentially at the right. This is actually a natural restriction: the system (1.4)–(1.5) admits traveling waves other than $(y_{1*}, y_{2*})(\xi)$ that have the same end states but approach the right end state more slowly than exponentially. The wave $(y_{1*}, y_{2*})(\xi)$ would not be stable in a space that allowed such waves as small perturbations of it.

2. Because the reactant is a solid, there is no diffusion in (1.5); i.e., $d_2 = 0$. As a result the linearization of (1.4)–(1.5) at $(y_{1*}, y_{2*})(\xi)$ has a vertical line in its spectrum, so it is not a sectorial operator. Hence one cannot use standard theorems to conclude that in the weighted space, spectral stability implies linear exponential stability. In [6] we dealt with this issue with the aid of some special properties of (1.4)–(1.5). Later, in [5], we gave a more general result (see Theorem 1.1 below) that sometimes enables one to pass from spectral stability to linear exponential stability in the presence of vertical lines in the spectrum.

3. Unfortunately, in the weighted space, the nonlinear terms in (1.4)–(1.5) fail to yield a locally Lipschitz mapping, so the fact that the linearization has desirable properties is still not sufficient to prove stability of the traveling wave. Using an approach from [4], which treats combustion fronts for the system (1.4)–(1.5) with small diffusion added to the second equation (i.e., high Lewis number), we showed that perturbations of the traveling wave that are small in both the weighted norm and the unweighted norm decay exponentially to the traveling wave in the weighted norm, and, in fact, have additional nice behavior that yields a physically natural stability result.

It is the last paragraph that we will generalize in the present paper.

The following linear result will be key.

THEOREM 1.1. *Consider a linear PDE of the form*

$$\tilde{Y}_t = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + A(\xi)\tilde{Y} =: C\tilde{Y};$$

$D = \operatorname{diag}(d_1, \dots, d_n)$ *with all* $d_i \geq 0$*,* $A(\xi)$ *is smooth, and there exist matrices* $A_\pm$ *such that* $A(\xi) \to A_\pm$ *exponentially as* $\xi \to \pm\infty$*. Let* $\mathcal{E}_0$ *denote one of the standard Banach spaces* $L^1(\mathbb{R})$*,* $L^2(\mathbb{R})$*,* $H^1(\mathbb{R})$*, or* $BUC(\mathbb{R})$*, and let* $\mathcal{C}_0$ *denote the operator on* $\mathcal{E}_0^n$ *associated with* $C$*. Assume* (1) $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}_{\text{ess}}(\mathcal{C}_0) < 0\}$ *and* (2) $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ *is contained in the resolvent set of* $\mathcal{C}_0$*, except possibly for an eigenvalue* 0 *with generalized null space* $\mathcal{N}_0$*. Let* $\mathcal{P}_0$ *be the Riesz spectral projection for* $\mathcal{C}_0$ *whose kernel is equal to* $\mathcal{N}_0$*. (If* 0 *is not an eigenvalue, then* $\mathcal{P}_0$ *is the identity map.) Then there are positive numbers* $K$ *and* $\mu$ *such that* $\|e^{t\mathcal{C}_0\mathcal{P}_0}\|_{\mathcal{E}_0^n\to\mathcal{E}_0^n} \leq Ke^{-\mu t}$*.*

If all $d_i$'s are positive, then the operator associated with $C$ on each of these spaces is sectorial, and this result is contained in [8]. If some $d_i$'s are 0, and $\mathcal{E}_0$ is $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, it is proved in [5]. However, the proof in [5] also works for any $L^p(\mathbb{R})$, $1 \leq p < \infty$. The reason is that Palmer's theorem (see, e.g., [15]), which relates the Fredholm properties of first-order linear differential operators of the form $U \to \partial_\xi U - A(\xi)U(\xi)$ to the spectra of the constant-coefficient operators $U \to \partial_\xi U - A(\pm\infty)U(\xi)$, is true not only in the spaces used in [5] but also in any $L^p(\mathbb{R})$, $1 \leq p < \infty$; see [10].

Theorem 1.1 implies in particular that if the traveling wave $Y_*$ is spectrally stable in any of the spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, then it is linearly exponentially stable in that space.

For $\mathcal{E}_0$ equal to one of the spaces $H^1(\mathbb{R})$ or $BUC(\mathbb{R})$, which are suited to the study of nonlinear stability (because they are closed under multiplication), linearized exponential stability of the traveling wave $Y_*$ implies (nonlinear) stability; again, see [8] for the case in which all $d_i$'s are positive and [5] for the case in which some $d_i$'s are 0. On the other hand, the wave is not stable in $\mathcal{E}_0$ if there is spectrum in the half-plane $\operatorname{Re}\lambda > 0$; see [8, section 5.1], for the case in which all $d_i$'s are positive and [20] for the case in which some $d_i$'s are 0.

We remark that a weaker definition of spectral stability is sometimes used: in work on viscous conservation laws and related equations, a traveling wave is called *spectrally stable* in $\mathcal{X}$ if the spectrum of $\mathcal{L}$ is contained in $\{\lambda : \operatorname{Re}\lambda < 0\} \cup \{0\}$, and 0 is a simple eigenvalue of $\mathcal{L}$ [3]. If 0 is in the essential spectrum of $\mathcal{L}$, the simple eigenvalue condition means the following: the Evans function, an analytic function defined to the right of the essential spectrum of $\mathcal{L}$ whose zeros are eigenvalues of $\mathcal{L}$, can be analytically extended to a neighborhood of 0 and has a simple zero at 0. This weaker definition of spectral stability sometimes implies linear algebraic stability, which, in turn, sometimes implies (nonlinear) stability [25, 11, 12].

Let $\mathcal{E}_0$ be one of the spaces $H^1(\mathbb{R})$ or $BUC(\mathbb{R})$. Suppose that on $\mathcal{E}_0^n$, the linear operator associated with $L$ has essential spectrum in the half-plane $\operatorname{Re}\lambda \geq 0$. (Actually, the essential spectrum of the operator on either of these spaces is equal to its essential spectrum on $L^2(\mathbb{R})^n$.) As with the system (1.4)–(1.5), often one can introduce a weight function that shifts the essential spectrum to the left. We shall limit our attention to a class of weight functions of exponential type. Let $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$. We shall say that $\gamma_\alpha : \mathbb{R} \to \mathbb{R}$ is a *weight function of class* $\alpha$ if $\gamma_\alpha$ is $C^2$, $\gamma_\alpha(\xi) > 0$ for all $\xi$, $\gamma_\alpha(\xi) = e^{\alpha_-\xi}$ for large negative $\xi$, and $\gamma_\alpha(\xi) = e^{\alpha_+\xi}$ for large positive $\xi$.

Suppose that $\alpha_- \leq 0$ and $\alpha_+ \geq 0$ so that $\gamma_\alpha(\xi)$ is bounded below by a positive number. If, in the space with weight function $\gamma_\alpha(\xi)$ satisfying these conditions, the traveling wave is spectrally stable, then it is linearly exponentially stable and nonlinearly exponentially stable in the weighted space. The results in [8, 5] already mentioned imply this result; the proofs make essential use of the fact that in the weighted space the nonlinearity is locally Lipschitz [18]. If, for example, $\alpha_- = 0$ and $\alpha_+ > 0$, such a result shows that if a perturbation of the traveling wave is bounded as $\xi \to -\infty$ and decays like $e^{-\alpha_+\xi}$ as $\xi \to \infty$, then it decays in time, in the weighted norm, to some shift of the wave.

Our interest in this paper is in weight functions $\gamma_\alpha(\xi)$ with $\alpha_- > 0$ so that $\gamma_\alpha(\xi) \to 0$ as $\xi \to -\infty$. This is the type of weight function that was used in [6]. Such weight functions are also used in the study of *convective instability* [16]. Suppose that perturbations of a traveling wave with velocity $c$ do not decay in the sup norm but travel with velocity less than $c$. Then for the linearization of (1.2), if one uses a norm with weight function $\gamma_\alpha(\xi)$ with $\alpha_- > 0$, perturbations of the traveling wave may well decay.

In one sense there is no loss of generality in considering weight functions with $\alpha_- > 0$ rather than weight functions with $\alpha_+ < 0$. Since $D$ and $R$ are independent of $x$, we can always replace a traveling wave with velocity $c$ by one with velocity $-c$, in the process reversing $Y_-$ and $Y_+$.

However, in the examples with which we are familiar, for $c > 0$, the spectrum of the linearization of (1.2) at a zero of $R$ moves left when one uses a weight function $e^{\alpha_-\xi}$ with $\alpha_- > 0$ and vice versa. Since the weight function $e^{\alpha_-\xi}$ will be used to move the spectrum of the linearization of (1.2) at $Y_-$ to the left, in these examples we would need $c > 0$. Thus the traveling wave is moving to the right, and $Y_-$ is the state behind the front. However, the hypothesis $c > 0$ is not directly needed for any of our results, so we have not stated it. In Appendix B we give a sufficient condition for the spectrum of the linearization of (1.2) at a zero of $R$ to be moved to the left by a weight function $e^{\alpha_-\xi}$ with $\alpha_- > 0$.

Without loss of generality we shall take $Y_-$ to be 0.

Unfortunately, if one uses a weight function with $\alpha_- > 0$, then, as we mentioned for (1.4)–(1.5), in the weighted space, the nonlinear term typically is no longer a locally Lipschitz mapping. Making use of such a weight function to prove some sort of nonlinear stability of a traveling wave is therefore mathematically more difficult. Nevertheless, by using *both* such a weight function *and* the unweighted norm, one can sometimes obtain physically natural nonlinear stability results. This idea, which as we noted was used in [4, 6], goes back to [14]; see [6] for additional references. In the present paper we identify the key assumptions that make the nonlinear proofs in [4, 6] work, and we are thereby able to generalize the results of those papers.

We shall always assume that $0 < \alpha_- < -\omega_-$ and $0 \leq \alpha_+ < \omega_+$; $\omega_-$ and $\omega_+$ were defined at the start of this introduction. The condition $\alpha_+ < \omega_+$ ensures that $Y_*'$ is in the weighted space. The conditions $\alpha_- < -\omega_-$ and $0 \le \alpha_+$ ensure that $\gamma_\alpha^{-1}(\xi)Y_*(\xi)$ is bounded, which is required in section 8.

We remark that for a pulse, one could ensure that $\gamma_\alpha^{-1}(\xi)Y_*(\xi)$ is bounded by the weaker condition $-\omega_+ < \alpha_+$. However, since a pulse has $Y^- = Y^+ = 0$, and we are assuming that a weight function $e^{\alpha_-\xi}$ with $0 < \alpha_-$ is required to stabilize the linearization at $Y^-$, we need $0 < \alpha^+$ in order to stabilize the linearization at $Y_+$.

We assume that the traveling wave is spectrally stable in the weighted space. Since $\alpha_+ \ge 0$, this assumption is enough to prove stability at the right, where the weight function is bounded away from 0, but it is not enough to prove stability at the left. Because of this difficulty, we also assume a special form for the nonlinearity and some stability in the unweighted norm at the left state 0.

In appropriate variables $Y = (U, V)$, $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, we assume that for some constant $n_1 \times n_1$ matrix $A_1$, $R(U, 0) = (A_1U, 0)$. Then

$$R(U,V) = \begin{pmatrix} A_1U + \tilde{R}_1(U,V)V \\ \tilde{R}_2(U,V)V \end{pmatrix}, \tag{1.7}$$

where $\tilde{R}_1$ and $\tilde{R}_2$ are matrix-valued functions of size $n_1 \times n_2$ and $n_2 \times n_2$, respectively. This form with $A_1 = 0$ occurs in chemical reaction and combustion problems; see [6, 4] and section 9 for examples. In a combustion problem with $n-1$ reactants, suppose the left state of a combustion front with positive velocity has temperature $y_1 = y_{1-} > 0$ and reactant concentrations $(y_2, \dots, y_n) = (0, \dots, 0)$. In order to move the left state to the origin, let $u = y_1 - y_{1-}$ and let $(v_1, \dots, v_{n-1}) = (y_2, \dots, y_n)$. Since the reaction rate will be 0 when the reactant concentrations are all 0, the reaction term in the system of PDEs will take the form (1.7) with $n_1 = 1$, $n_2 = n - 1$, and $A_1 = 0$. We allow $A_1 \neq 0$ in (1.7) because our proofs work for this generalization, but we do not have an application in mind.

We write

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where each $D_i$ is a nonnegative diagonal matrix of size $n_i \times n_i$.

If we linearize (1.2) at $(0, 0)$, the constant-coefficient linear equation satisfied by $\tilde{V}_t$ depends only on $\tilde{V}$: $\tilde{V}_t = D_2V_{\xi\xi} + cV_\xi + \tilde{R}_2(0,0)\tilde{V} = L^{(2)}\tilde{V}$. We assume that in the unweighted norm the operator associated with $L^{(2)}$ has its spectrum in $\operatorname{Re}\lambda < -\rho < 0$ for some $\rho$.

In addition, we assume that when we linearize (1.2) at $(0, 0)$, the constant-coefficient linear equation satisfied by $\tilde{U}_t$ for $\tilde{V} = 0$—namely, $\tilde{U}_t = D_1\tilde{U}_{\xi\xi} + c\tilde{U}_\xi + A_1\tilde{U} = L^{(1)}\tilde{U}$—is such that in the unweighted norm the associated operator generates a bounded semigroup. This is the case when $A_1 = 0$; in Appendix A we give some other sufficient conditions for this assumption to hold.

With these assumptions we show that perturbations of the traveling wave that are initially small in both the unweighted and weighted norms stay small in the unweighted norm and decay exponentially in the weighted norm to some shift of the wave. In addition, the $V$-component of the perturbation decays exponentially in the unweighted norm.

Notice that in the unweighted norm the $U$-component of the perturbation may travel with velocity less than $c$ without decay. Our result therefore says that in the unweighted norm, any instability of the traveling wave is eventually concentrated in the $U$-component and is convected with velocity less than $c$.

We remark that in the case $\mathcal{E}_0 = BUC(\mathbb{R})$, as $\xi \to -\infty$, the allowed perturbations of the traveling wave need only be bounded.

The assumption that the operator associated with $L^{(1)}$ on the unweighted space generates a bounded semigroup implies that its spectrum is contained in the half-plane $\operatorname{Re}\lambda \leq 0$ but does not imply that its spectrum is contained in some half-plane $\operatorname{Re}\lambda < -\nu < 0$.

Suppose the linear equation $\tilde{U}_t = L^{(1)}\tilde{U}$ is parabolic; i.e., the corresponding $d_i$'s are all positive. If $A_1 = 0$, then on the space $(\mathcal{E}_0 \cap L^1(\mathbb{R}))^{n_1}$, the semigroup $S^{(1)}(t)$ generated by the operator associated with $L^{(1)}$ satisfies an algebraic decay estimate of the following type; see [9]. Let

$$h(t) = \min\left(1, t^{-\frac{1}{2}}\right), \quad t > 0. \tag{1.8}$$

Then there exists a constant $K > 0$ such that if $\tilde{U}^0 \in (\mathcal{E}_0 \cap L^1(\mathbb{R}))^{n_1}$, then

$$\|\tilde{U}(t)\|_{L^\infty} = \|S^{(1)}(t)\tilde{U}^0\|_{L^\infty} \leq Kh(t)\max\left(\|\tilde{U}^0\|_{L^\infty}, \|\tilde{U}^0\|_{L^1}\right). \tag{1.9}$$

Moreover, by Theorem 1.1, the hypotheses already given imply that on the space $L^1(\mathbb{R})^{n_2}$, the semigroup $S^{(2)}(t)$ generated by the operator associated with $L^{(2)}$ decays exponentially.

Under the additional assumption, which is automatically satisfied when $A_1 = 0$, that on the space $(\mathcal{E}_0 \cap L^1(\mathbb{R}))^{n_1}$, the semigroup $S^{(1)}(t)$ generated by the operator associated with $L^{(1)}$ satisfies the estimate (1.9), we show that for small perturbations of the traveling wave in $(\mathcal{E}_0 \cap L^1(\mathbb{R}))^{n_1}$, the $L^\infty$ norm of the $U$-component of the perturbation decays like $h(t)$ to the $U$-component of a shift of the traveling wave.

Our results have a natural interpretation in the case of combustion problems. Behind a combustion front moving to the right, temperature is high and there are no remaining reactants. If one makes a perturbation behind the front by adding reactants (the $v$-variables), they immediately burn because of the high temperature. On the other hand, if one makes a perturbation behind the front by adding heat (the $u$-variable), it simply diffuses. In a coordinate system moving at the velocity of the front, the perturbation is also convected to the left. In a weighted space with weight function that decays at the left, the perturbation will decay. In the unweighted space, it will remain bounded. If the perturbation is in $L^1$, then its $L^\infty$ norm will decay algebraically.

After giving some definitions in section 2, we list our assumptions and precisely state our results in section 3. In section 4 we convert (1.2) into a form more suitable for study. Our main nonlinear stability result is proved in section 5, and results that use the $L^1$ norm are proved in section 6. Estimates needed for the proofs are deferred to sections 7 and 8.

In section 9 we study a generalization from [22, 23, 24] of the model for gasless combustion with diffusive reactant that was studied in [4]. In [22, 23, 24] Simon et al. consider a model in which two chemical reactions occur at rates determined by temperature. One reaction is exothermic (produces heat); the other is endothermic (absorbs heat). Both reactants and heat can diffuse. In some parameter regimes the authors show numerically that traveling waves exist, that the zero eigenvalue of the linearization is simple, and that there are no other eigenvalues in the right half-plane. We show that these results together with our theorem imply the sort of nonlinear stability of the combustion front described above.

Our point in discussing the work of Simon et al. is not to "make it rigorous." Instead, our point is that a numerical study of the Evans function of the type done by

Simon et al., which takes considerable effort, can in some problems be coupled with rather routine checks of the remaining hypotheses of our theorems to produce quite detailed knowledge of the kind of nonlinear stability that the traveling wave enjoys.

**2. Spaces and operators.** Given $U \subset \mathbb{R}^l$, let $C^0(U)$ denote the space of bounded $C^0$ functions $m : U \to \mathbb{R}$ with the sup norm, which we denote $\| \cdot \|_{L^\infty}$. More generally, let $C^k(U)$ denote the space of $C^k$ functions $m : U \to \mathbb{R}$ such that $m$, $Dm$, …, $D^k m$ are all bounded continuous functions, with the following $C^k$ norm:

$$\|m\|_{C^k} = \|m\|_{L^\infty} + \|Dm\|_{L^\infty} + \cdots + \|D^k m\|_{L^\infty}.$$

Let $BUC(\mathbb{R})$ denote the closed subspace of $C^0(\mathbb{R})$ consisting of uniformly continuous functions. For $k \geq 1$, let $BUC^k(\mathbb{R})$ denote the closed subspace of $C^k(\mathbb{R})$ consisting of functions $m$ such that $m^{(k)} \in BUC(\mathbb{R})$.

Let $\mathcal{E}_0$ denote one of the standard Banach spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$. We denote the norm in $\mathcal{E}_0$ by $\| \, \|_0$. Recall the weight functions $\gamma_\alpha(\xi)$ defined in the introduction. For a fixed weight function $\gamma_\alpha$ of type $\alpha$, let $\mathcal{E}_\alpha = \{u : \gamma_\alpha(\xi)u(\xi) \in \mathcal{E}_0\}$, with norm $\|u\|_\alpha = \|\gamma_\alpha u\|_0$.

If $B$ is a system of $n$ differential expressions in $x$ or $\xi$, we shall denote by $\mathcal{B}_0 : \mathcal{E}_0^n \to \mathcal{E}_0^n$ and $\mathcal{B}_\alpha : \mathcal{E}_\alpha^n \to \mathcal{E}_\alpha^n$ the linear operators given by the formula $Y \to BY$, with their natural domains.

For example, consider the system of $n$ differential expressions $L$ given by (1.3). For $\mathcal{E}_0 = L^2(\mathbb{R})$, the domain of $\mathcal{L}_0$ is the set of $(y_1, \dots, y_n)$ in $\mathcal{E}_0^n$ such that $y_i \in H^2(\mathbb{R})$ if $d_i > 0$ and $y_i \in H^1(\mathbb{R})$ if $d_i = 0$. For $\alpha \in \mathbb{R}^2$, the domain of $\mathcal{L}_\alpha$ is the set of $(y_1, \dots, y_n)$ in $\mathcal{E}_\alpha^n$ such that $\gamma_\alpha(\xi)y_i(\xi) \in H^2(\mathbb{R})$ if $d_i > 0$ and $\gamma_\alpha(\xi)y_i(\xi) \in H^1(\mathbb{R})$ if $d_i = 0$. If $\mathcal{E}_0 = H^1(\mathbb{R})$, then $H^2(\mathbb{R})$ and $H^1(\mathbb{R})$ should be replaced by $H^3(\mathbb{R})$ and $H^2(\mathbb{R})$, respectively. If $\mathcal{E}_0 = BUC(\mathbb{R})$, then $H^2(\mathbb{R})$ and $H^1(\mathbb{R})$ should be replaced by $BUC^2(\mathbb{R})$ and $BUC^1(\mathbb{R})$, respectively. If $\mathcal{E}_0 = L^1(\mathbb{R})$, then $H^2(\mathbb{R}) = W_2^2(\mathbb{R})$ and $H^1 = W_1^2(\mathbb{R})$ should be replaced by the Sobolev spaces $W_2^1(\mathbb{R})$ and $W_1^1(\mathbb{R})$, respectively.

Let $\mathcal{X}$ be a Banach space, and let $\mathcal{B} : \mathcal{X} \to \mathcal{X}$ be a closed, densely defined linear operator. Its *resolvent set* $\rho(\mathcal{B})$ is the set of $\lambda \in \mathbb{C}$ such that $\mathcal{B} - \lambda I$ has a bounded inverse. The complement of $\rho(\mathcal{B})$ is the *spectrum* $\mathrm{Sp}(\mathcal{B})$. It is the union of the *discrete spectrum* $\mathrm{Sp}_{\mathrm{d}}(\mathcal{B})$, which is the set of isolated points in $\mathrm{Sp}(\mathcal{B})$ that are eigenvalues of $\mathcal{B}$ of finite algebraic multiplicity, and the *essential spectrum* $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{B})$, which is the rest.

**3. Assumptions and results.**

**3.1. The traveling wave and the linearized operator.** We consider the system (1.1).

*Hypothesis* 3.1. The function $R$ is $C^3$.

*Hypothesis* 3.2. The system (1.1) has a traveling wave solution $Y_*(\xi)$, $\xi = x - ct$, for which there exist numbers $K > 0$ and $\omega_- < 0 < \omega_+$ such that for $\xi \leq 0$, $\|Y_*(\xi)\| \leq Ke^{-\omega_- \xi}$, and for $\xi \geq 0$, $\|Y_*(\xi) - Y_+\| \leq Ke^{-\omega_+ \xi}$.

In other words, $Y_*(\xi) \to 0$ exponentially as $\xi \to -\infty$ and $Y_*(\xi) \to Y_+$ exponentially as $\xi \to \infty$.

Hypotheses 3.1 and 3.2 imply the following.

LEMMA 3.3. *There exists $K > 0$ such that the following is true. For $\xi \leq 0$, $\|Y_*^{(k)}(\xi)\| \leq Ke^{-\omega_- \xi}$ for $k = 1, 2, 3$, and for $\xi \geq 0$, $\|Y_*^{(k)}(\xi)\| \leq Ke^{-\omega_+ \xi}$ for $k =$ 1, 2, 3.*

Let $L^-$ and $L^+$ denote the constant-coefficient linear differential expressions obtained by linearizing the right-hand side of (1.2) at 0 and $Y_+$, respectively:

$$L^-\tilde{Y} = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + DR(0)\tilde{Y}, \quad L^+\tilde{Y} = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + DR(Y^+)\tilde{Y}. \tag{3.1}$$

To find $\mathrm{Sp}(\mathcal{L}_0^-)$ (respectively, $\mathrm{Sp}(\mathcal{L}_0^+)$) on $\mathcal{E}_0^n$ for $\mathcal{E}_0 = L^2(\mathbb{R})$, one uses the Fourier transform. The operator $\mathcal{L}_0^-$ (respectively, $\mathcal{L}_0^+$) on $L^2(\mathbb{R})^n$ is similar to the operator of multiplication on $L^2(\mathbb{R})^n$ by the matrix-valued function $M^-(\theta) = -\theta^2 D + i\theta cI + DR(0)$ (respectively, $M^+(\theta) = -\theta^2 D + i\theta cI + DR(Y_+)$). The spectrum of $\mathcal{L}_0^-$ (respectively, $\mathcal{L}_0^+$) on $\mathcal{E}_0^n$ for $\mathcal{E}_0 = L^2(\mathbb{R})$ is the closure of the union over $\theta \in \mathbb{R}$ of the spectra of the matrices $M^-(\theta)$ (respectively, $M^+(\theta)$). Hence the spectrum of $\mathcal{L}_0^-$ (respectively, $\mathcal{L}_0^+$) is equal to the set of $\lambda \in \mathbb{C}$ for which there exists $\theta \in \mathbb{R}$ such that $\det(-\theta^2 D + (i\theta c - \lambda)I + DR(0)) = 0$ (respectively, $\det(-\theta^2 D + (i\theta c - \lambda)I + DR(Y_+)) = 0$). It is a collection of curves of the form $\lambda = \lambda_k^-(\theta)$ (respectively, $\lambda = \lambda_k^+(\theta)$), where $\lambda_k^-(\theta)$ (respectively, $\lambda_k^+(\theta)$) are the eigenvalues of the matrices $M^-(\theta)$ (respectively, $M^+(\theta)$).

Actually, this calculation yields the spectrum of $\mathcal{L}_0^-$ (respectively, $\mathcal{L}_0^+$) on $\mathcal{E}_0^n$ for $\mathcal{E}_0$ equal to any of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$; see Lemma 2 in [8, Chapter 5, Appendix] and the proof of Lemma 3.11(1) below. It also yields important information about $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_0)$ for $\mathcal{E}_0$ equal to any of these spaces. We summarize as follows.

LEMMA 3.4.

(1) *The linear differential operators associated with $L^-$ (respectively, $L^+$) on $\mathcal{E}_0^n$ for $\mathcal{E}_0$ equal to any of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$ have the same spectra.*

(2) *If $\mathcal{E}_0$ is any of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, on $\mathcal{E}_0^n$ the right-hand boundary of $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_0)$ is exactly the right-hand boundary of the set $\mathrm{Sp}(\mathcal{L}_0^-) \cup \mathrm{Sp}(\mathcal{L}_0^+)$. Therefore the right-hand boundary of $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_0)$ is the same for $\mathcal{E}_0$ equal to any of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$.*

We will also need $\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_\alpha)$, which is most conveniently found as follows. The linear map $\mathcal{M} : \mathcal{E}_\alpha^n \to \mathcal{E}_0^n$ defined by $\mathcal{M}\tilde{Y} = \gamma_\alpha \tilde{Y}$ is an isomorphism. The linear map $\hat{\mathcal{L}}_0 = \mathcal{M}\mathcal{L}_\alpha\mathcal{M}^{-1}$ on $\mathcal{E}_0^n$ is therefore similar to $\mathcal{L}_\alpha$ and hence has the same spectrum. $\hat{\mathcal{L}}_0$ is given by the differential expression

$$\hat{L}W = \gamma_\alpha L \gamma_\alpha^{-1} W. \tag{3.2}$$

Setting $\xi = \pm\infty$ in (3.2) yields constant-coefficient linear differential expressions $\hat{L}^\pm$ given by

$$\hat{L}^\pm W = DW_{\xi\xi} + (c - 2\alpha_\pm)W_\xi + (\alpha_\pm^2 D - c\alpha_\pm I + DR(0))W,$$

with corresponding linear maps $\hat{\mathcal{L}}_0^\pm$ on $\mathcal{E}_0^n$. Via the Fourier transform, the operator $\hat{\mathcal{L}}_0^-$ (respectively, $\hat{\mathcal{L}}_0^+$) on $L^2(\mathbb{R})^n$ is similar to the operator of multiplication on $L^2(\mathbb{R})^n$ by the matrix-valued function $N^-(\theta) = -\theta^2 D + i\theta(c - 2\alpha_-)I + \alpha_-^2 D - c\alpha_- I + DR(0)$ (respectively, $N^+(\theta) = -\theta^2 D + i\theta(c - 2\alpha_+)I + \alpha_+^2 D - c\alpha_+ I + DR(Y_+)$). Hence the essential spectrum of $\hat{\mathcal{L}}_0^\pm$ on $L^2(\mathbb{R})^n$ equals that of multiplication by $N^\pm$ on $L^2(\mathbb{R})^n$.

We, of course, have the following analogue of Lemma 3.4.

LEMMA 3.5.

(1) *The linear differential operators associated with $\hat{L}^-$ (respectively, $\hat{L}^+$) on $\mathcal{E}_0^n$ for $\mathcal{E}_0$ equal to any of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$ have the same spectra.*

(2) *If* $\mathcal{E}_0$ *is any of* $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, *or* $BUC(\mathbb{R})$, *on* $\mathcal{E}_0^n$ *the right-hand boundary of* $\mathrm{Sp}_{\mathrm{ess}}(\hat{\mathcal{L}}_0) = \mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_\alpha)$ *is exactly the right-hand boundary of the set* $\mathrm{Sp}(\hat{\mathcal{L}}_0^-) \cup \mathrm{Sp}(\hat{\mathcal{L}}_0^+)$. *Therefore the right-hand boundary of* $\mathrm{Sp}_{\mathrm{ess}}(\hat{\mathcal{L}}_0) = \mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_\alpha)$ *is the same for* $\mathcal{E}_0$ *equal to any of* $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, *or* $BUC(\mathbb{R})$.

We are now ready to state the following.

*Hypothesis* 3.6. There exists $\alpha = (\alpha_-, \alpha_+) \in \mathbb{R}^2$ such that the following are true.

(1) $0 < \alpha_- < -\omega_-$.

(2) $0 \le \alpha_+ < \omega_+$.

(3) For the differential expression $L$ given by (1.3) and $\mathcal{E}_0 = L^2(\mathbb{R})$,

   (a) $\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_\alpha)\} < 0$,

   (b) the only element of $\mathrm{Sp}(\mathcal{L}_\alpha)$ in $\{\lambda : \mathrm{Re}\,\lambda \ge 0\}$ is a simple eigenvalue 0.

Hypothesis 3.2, Lemma 3.3, and Hypothesis 3.6(1) and (2) imply the following.

Lemma 3.7.

(1) $\gamma_\alpha^{-1} Y_* \in C^1(\mathbb{R})^n$.

(2) *As* $\xi \to \pm\infty$, $\gamma_\alpha(\xi) Y_*^{(k)}(\xi)$ *and* $\gamma_\alpha^{-1}(\xi) Y_*^{(k)}(\xi)$ *approach* 0 *exponentially for* $k = 1, 2, 3$.

Lemma 3.8.

(1) *Statements* (3a) *and* (3b) *of Hypothesis* 3.6 *are also true for* $\mathcal{E}_0 = L^1(\mathbb{R})$, $H^1(\mathbb{R})$, *and* $BUC(\mathbb{R})$.

(2) *For* $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, *or* $BUC(\mathbb{R})$, *the kernel of* $\mathcal{L}_\alpha$ *in* $\mathcal{E}_\alpha^n$ *is spanned by* $Y_*'$.

*Proof.* Lemma 3.5(2) implies that statement (3a) of Hypothesis 3.6 is also true for $\mathcal{E}_0 = L^1(\mathbb{R})$, $H^1(\mathbb{R})$, and $BUC(\mathbb{R})$.

We will now show that statement (3b) of Hypothesis 3.6 is also true for $\mathcal{E}_0 = L^1(\mathbb{R})$, $H^1(\mathbb{R})$, and $BUC(\mathbb{R})$, and at the same time we will show the second statement of the lemma. The eigenvalue equation $\lambda\tilde{Y} = L\tilde{Y}$ can be written as a first-order linear system of the form

$$Z_\xi = (B(\xi) + \lambda C)Z, \tag{3.3}$$

with $Z \in \mathbb{R}^{n+n_0}$; $n_0$ is the number of $d_i$'s in (1.1) that are positive. Statement (3a) of Hypothesis 3.6 and Palmer's theorem (see, e.g., [15] for $n = n_0$ and [5] for $n > n_0$) imply that there is a number $k$ such that for each $\lambda$ with $\mathrm{Re}\,\lambda \ge 0$, there is a $k$-dimensional space of solutions $E_-(\lambda)$ of (3.3) such that if $Z \in E_-(\lambda)$, then $e^{\alpha_-\xi} Z(\xi) \to 0$ exponentially as $\xi \to -\infty$; if $Z \notin E_-(\lambda)$ is any other solution of (3.3), then $e^{\alpha_-\xi} Z(\xi)$ grows exponentially as $\xi \to -\infty$. Similarly, there is an $(n-k)$-dimensional space of solutions $E_+(\lambda)$ of (3.3) such that if $Z \in E_+(\lambda)$, then $e^{\alpha_+\xi} Z(\xi) \to 0$ exponentially as $\xi \to \infty$; if $Z \notin E_+(\lambda)$ is any other solution of (3.3), then $e^{\alpha_+\xi} Z(\xi)$ grows exponentially as $\xi \to -\infty$. For $\mathcal{E}_0$ equal to any of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, $Z(\xi)$ is a solution of (3.3) that corresponds to an eigenfunction of $\mathcal{L}_\alpha$ if and only if $Z$ is a nonzero element of $E_-(\lambda) \cap E_+(\lambda)$. The result follows. ∎

**3.2. Product structure.** Let $Y = (U, V)$, $U \in \mathbb{R}^{n_1}$, $V \in \mathbb{R}^{n_2}$, and $n_1 + n_2 = n$. We write

$$R(Y) = \begin{pmatrix} R_1(U,V) \\ R_2(U,V) \end{pmatrix}, \quad R_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}, \quad i = 1,\ 2,$$

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad D_i = \mathrm{diag}(d_k^i), \quad d_k^i \ge 0, \quad i = 1,\ 2, \quad k = 1,\ 2,\ \ldots,\ n_i.$$

Equation (1.1) now reads

$$U_t = D_1 U_{xx} + R_1(U, V), \tag{3.4}$$

$$V_t = D_2 V_{xx} + R_2(U, V). \tag{3.5}$$

Equation (1.2) reads

$$U_t = D_1 U_{\xi\xi} + cU_\xi + R_1(U, V), \tag{3.6}$$

$$V_t = D_2 V_{\xi\xi} + cV_\xi + R_2(U, V). \tag{3.7}$$

We write $Y_*(\xi) = (U_*(\xi), V_*(\xi))$ and $Y_+ = (U_+, V_+)$.

Hypothesis 3.2 implies that $R(0, 0) = 0$. We assume in addition the following.

*Hypothesis* 3.9. There is an $n_1 \times n_1$ matrix $A_1$ such that $R(U, 0) = (A_1 U, 0)$.

As mentioned in the introduction, Hypothesis 3.9 implies that $R$ has the form (1.7). Hypothesis 3.9 is required to prove a key estimate, Lemma 8.3.

Let

$$L^{(1)} = D_1 \partial_{\xi\xi} + c\partial_\xi + D_U R_1(0, 0) = D_1 \partial_{\xi\xi} + c\partial_\xi + A_1, \tag{3.8}$$

$$L^{(2)} = D_2 \partial_{\xi\xi} + c\partial_\xi + D_V R_2(0, 0). \tag{3.9}$$

For $i = 1, 2$, $L^{(i)}$ is a constant-coefficient linear differential expression on $\mathbb{R}^{n_i}$. By Hypothesis 3.9,

$$L^- = \begin{pmatrix} L^{(1)} & D_V R_1(0,0) \\ 0 & L^{(2)} \end{pmatrix}. \tag{3.10}$$

For future reference, we note that from (1.3) and (3.1),

$$L\tilde{Y} = L^-\tilde{Y} + (DR(Y_*) - DR(0))\tilde{Y}, \tag{3.11}$$

and then from (3.10),

$$L\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} = \begin{pmatrix} L^{(1)} & D_V R_1(0,0) \\ 0 & L^{(2)} \end{pmatrix}\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix} + (DR(Y_*) - DR(0))\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}. \tag{3.12}$$

Our next hypothesis gives a degree of stability in the unweighted norm at the state $(0, 0)$ at one end of the traveling wave.

*Hypothesis* 3.10.

(1) For $\mathcal{E}_0 = L^2(\mathbb{R})$ or $BUC(\mathbb{R})$, the operator $\mathcal{L}_0^{(1)}$ on $\mathcal{E}_0^{n_1}$ generates a bounded semigroup.

(2) For $\mathcal{E}_0 = L^2(\mathbb{R})$, the operator $\mathcal{L}_0^{(2)}$ on $\mathcal{E}_0^{n_2}$ satisfies $\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_0^{(2)})\} < 0$.

If the matrix $A_1$ is dissipative (that is, if $\mathrm{Re}\langle A_1 U, U\rangle_{\mathbb{C}^{n_1}} \le 0$), then the operator $\mathcal{L}_0^{(1)}$ is dissipative on $L^2(\mathbb{R})^{n_1}$ and thus generates a contraction semigroup. In particular, if $A_1 = 0$, then Hypothesis 3.10(1) holds. In Appendix A we give another easily checked sufficient condition for Hypothesis 3.10(1) to hold in the case $\mathcal{E}_0 = L^2(\mathbb{R})$. Also, in Appendix A we give more sophisticated sufficient conditions for Hypothesis 3.10(1) to hold in the cases $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, or $BUC(\mathbb{R})$; they are based on general abstract conditions under which $C_0$-semigroups are bounded [7, 21].

Hypothesis 3.10 implies the following.

LEMMA 3.11.

(1) *For* $\mathcal{E}_0 = H^1(\mathbb{R})$, *it is again true that* $\mathcal{L}_0^{(1)}$ *generates a bounded semigroup.*

(2) *For* $\mathcal{E}_0 = L^1(\mathbb{R})$, $H^1(\mathbb{R})$, *or* $BUC(\mathbb{R})$, *it again true that* $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_0^{(2)})\} < 0$.

(3) *For* $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, *or* $BUC(\mathbb{R})$, *the following are true:*
   (a) $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_0^{(1)})\} \le 0$.
   (b) $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_0^{-})\} \le 0$.
   (c) *Choose* $\rho > 0$ *such that* $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_0^{(2)})\} < -\rho$. *Then there exists* $K > 0$ *such that for* $t \ge 0$, $\|e^{t\mathcal{L}_0^{(2)}}\|_{\mathcal{E}_0^{n_2} \to \mathcal{E}_0^{n_2}} \le K e^{-\rho t}$.

*Proof.* Statement (1) follows from Hypothesis 3.10(1) for $\mathcal{E}_0 = L^2(\mathbb{R})$. Indeed, we recall that the Fourier transform is an isomorphism of $H^1(\mathbb{R})$ onto $L^2_m(\mathbb{R})$, where the weight function is $m(\theta) = (1 + |\theta|)^{1/2}$, $\theta \in \mathbb{R}$. The operator of multiplication by the function $m(\theta)$ is an isomorphism of $L^2_m(\mathbb{R})$ onto $L^2(\mathbb{R})$. Under the Fourier transform followed by this isomorphism, the operator of differentiation on $H^1(\mathbb{R})$ is similar to the operator of multiplication by $i\theta$ on $L^2(\mathbb{R})$. The latter is in turn similar via the Fourier transform to the operator of differentiation on $L^2(\mathbb{R})$. It follows that operators on $H^1(\mathbb{R})^{n_1}$ and $L^2(\mathbb{R})^{n_1}$ associated with the same constant-coefficient differential expression are similar. Therefore the semigroups they generate are similar, so (1) is proved.

Statement (2) follows from Hypothesis 3.10(2) and the analogue of Lemma 3.4(1) for $L^{(1)}$ and $L^{(2)}$. Statement (3a) follows from Hypothesis 3.10(1). Statement (3b) follows from the analogous facts for $\mathcal{L}_0^{(1)}$ and $\mathcal{L}_0^{(2)}$, and (3c) follows from Theorem 1.1. ∎

**3.3. Nonlinear stability.** Let

$$\beta = (\min(0, \alpha_-), \max(0, \alpha_+)) = (0, \alpha_+). \tag{3.13}$$

Let $\gamma_\beta$ be a fixed weight function of class $\beta$ chosen so that for all $\xi$, $\max(1, \gamma_\alpha(\xi)) \le \gamma_\beta(\xi)$.

For $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, let $\mathcal{E}_\beta = \{u : \gamma_\beta(\xi)u(\xi) \in \mathcal{E}_0\}$, with norm $\|u\|_\beta = \|\gamma_\beta u\|_0$. We shall frequently use the facts in the following lemma without explicit mention.

LEMMA 3.12.

(1) *As vector spaces,* $\mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$.

(2) *On* $\mathcal{E}_\beta = \mathcal{E}_0 \cap \mathcal{E}_\alpha$, *the norms* $\|u\|_\beta$ *and* $\|u\| = \max(\|u\|_0, \|u\|_\alpha)$ *are equivalent.*

(3) $\mathcal{E}_\beta \hookrightarrow \mathcal{E}_\alpha$ *and* $\mathcal{E}_\beta \hookrightarrow \mathcal{E}_0$; *that is, if* $u \in \mathcal{E}_\beta$, *then*

$$\|u\|_0 \le \|u\|_\beta \quad \text{and} \quad \|u\|_\alpha \le \|u\|_\beta. \tag{3.14}$$

Since 0 is isolated in the spectrum of $\mathcal{L}_\alpha$ by Hypothesis 3.6(3) and Lemma 3.8, we can define the Riesz spectral projection $\mathcal{P}^c_\alpha$ of $\mathcal{E}^n_\alpha$ onto the one-dimensional space $N(\mathcal{L}_\alpha)$. $\mathcal{P}^c_\alpha$ commutes with $e^{t\mathcal{L}_\alpha}$ for all $t > 0$. Since $\mathcal{L}_\alpha$ is the Fredholm of index zero [5] and 0 is a simple eigenvalue of $\mathcal{L}_\alpha$, $\mathcal{E}^n_\alpha = \mathrm{R}(\mathcal{L}_\alpha) \oplus \mathrm{N}(\mathcal{L}_\alpha)$, and $\mathrm{N}(\mathcal{P}^c_\alpha) = \mathrm{R}(\mathcal{L}_\alpha)$. Since $\mathrm{R}(\mathcal{P}^c_\alpha) = \mathrm{N}(\mathcal{L}_\alpha)$ is spanned by $Y'_*$, we write $\mathcal{P}^c_\alpha Y = \pi_\alpha(Y)Y'_*$, where $\pi_\alpha : \mathcal{E}^n_\alpha \to \mathbb{R}$ is a bounded linear functional such that $\pi_\alpha(Y'_*) = 1$.

Let $\mathcal{P}^s_\alpha = I - \mathcal{P}^c_\alpha$. $\mathcal{P}^s_\alpha$ is projection onto $\mathrm{R}(\mathcal{L}_\alpha)$, with kernel $\mathrm{N}(\mathcal{L}_\alpha)$. It also commutes with $e^{t\mathcal{L}_\alpha}$ for all $t > 0$.

From Theorem 1.1 we have the following.

LEMMA 3.13. *Let* $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, *or* $BUC(\mathbb{R})$. *Choose* $\nu$, $0 < \nu < \rho$, *such that* $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_\alpha)$ *and* $\lambda \ne 0\} < -\nu$. *Then there exists* $K > 0$ *such that for* $t \ge 0$, $\|e^{t\mathcal{L}_\alpha \mathcal{P}^s_\alpha}\|_{\mathcal{E}^n_\alpha \to \mathcal{E}^n_\alpha} \le K e^{-\nu t}$.

Lemmas 3.3 and 3.7(2) imply that $Y_*' \in \mathcal{E}_\beta^n$. Therefore if $\tilde{Y} \in \mathcal{E}_\beta^n \subset \mathcal{E}_\alpha^n$, then $\mathcal{P}_\alpha^c \tilde{Y} \in \mathcal{E}_\beta^n$, and therefore $\mathcal{P}_\alpha^s \tilde{Y} = \tilde{Y} - \mathcal{P}_\alpha^c \tilde{Y} \in \mathcal{E}_\beta^n$. Hence we can define $\mathcal{P}_\beta^c$ and $\mathcal{P}_\beta^s$ to be operators from $\mathcal{E}_\beta^n$ to itself given by restricting $\mathcal{P}_\alpha^c$ and $\mathcal{P}_\alpha^s$, respectively, to $\mathcal{E}_\beta^n$. Since $\mathcal{E}_\beta \hookrightarrow \mathcal{E}_\alpha$, the one-dimensional operator $\mathcal{P}_\beta^c$ is a bounded operator on $\mathcal{E}_\beta^n$, so $\mathcal{P}_\beta^s = I - \mathcal{P}_\beta^c$ is also bounded. It is easy to see that $\mathcal{P}_\beta^c$ and $\mathcal{P}_\beta^s$ are projections and that the range of one is the kernel of the other. It follows that $\mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}})$ is a closed subspace of $\mathcal{E}_\beta^n$, and $\mathcal{E}_\beta^n = \mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) \oplus \mathrm{R}(\mathcal{P}_\beta^{\mathrm{c}})$. In particular, $\mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) = \mathrm{R}(\mathcal{L}_\alpha) \cap \mathcal{E}_\beta^{\mathrm{n}}$.

Given $Y^0 \in Y_* + \mathcal{E}_\beta^n$, let $Y(t) = Y(t, Y^0)$ be the solution of (1.2) in $Y_* + \mathcal{E}_\beta^n$ with $Y(0) = Y^0$, which we shall show exists. We shall show that there is a neighborhood $\mathcal{U}$ of $Y_*$ in $Y_* + \mathcal{E}_\beta^n$ such that if $Y^0 \in \mathcal{U}$, we can write

$$Y^0 = \tilde{Y}^0 + Y_*(\xi - q^0) \text{ with } (\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) \times \mathbb{R}. \tag{3.15}$$

Similarly, if $Y(t) \in \mathcal{U}$, we can write

$$Y(t) = \tilde{Y}(t) + Y_*(\xi - q(t)) \text{ with } (\tilde{Y}(t), q(t)) \in \mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) \times \mathbb{R}. \tag{3.16}$$

The following theorem gathers most of our nonlinear stability results. Let $\tilde{Y}(t) = (\tilde{U}(t), \tilde{V}(t))$.

Theorem 3.14. *Assume that Hypotheses* 3.1, 3.2, 3.6, 3.9, *and* 3.10 *hold. Choose* $\nu > 0$ *as in Lemma* 3.13. *Let* $\mathcal{E}_0 = H^1(\mathbb{R})$ *or* $BUC(\mathbb{R})$. *Then there is a constant* $C > 0$ *such that for each small* $\delta > 0$, *there exists* $\eta > 0$ *such that the following is true. Let* $Y^0 \in Y_* + \mathcal{E}_\beta^n$ *with* $\|Y^0 - Y_*\|_\beta \le \eta$, *and let* $(\tilde{Y}^0, q^0)$ *be given by* (3.15). *Let* $Y(t)$ *be the solution of* (1.2) *in* $Y_* + \mathcal{E}_\beta^n$ *with* $Y(0) = Y^0$. *Then for all* $t \ge 0$,

(1) $Y(t)$ *is defined;*
(2) $Y(t) \in \mathcal{U}$, *so we can define* $(\tilde{Y}(t), q(t))$ *by* (3.16)*;*
(3) $\|\tilde{Y}(t)\|_\beta + |q(t)| \le \delta$*;*
(4) $\|\tilde{Y}(t)\|_\alpha \le Ce^{-\nu t}\|\tilde{Y}^0\|_\alpha$*;*
(5) *there exists* $q^*$ *such that* $|q(t) - q^*| \le Ce^{-\nu t}\|\tilde{Y}^0\|_\alpha$*;*
(6) $\|\tilde{U}(t)\|_0 \le C\|\tilde{Y}^0\|_\beta$*;*
(7) $\|\tilde{V}(t)\|_0 \le Ce^{-\nu t}\|\tilde{Y}^0\|_\beta$.

Note that (4) and (5) imply easily that for a larger constant $\tilde{C}$, $\|Y(t) - Y_*(\xi - q^*)\|_\alpha \le \tilde{C}e^{-\nu t}\|\tilde{Y}^0\|_\alpha$.

**3.4. Algebraic decay.** Recall from (1.8) the function $h(t) = \min(1, t^{-\frac{1}{2}})$. For $\mathcal{E}_0 = L^2(\mathbb{R}), H^1(\mathbb{R})$, or $BUC(\mathbb{R})$, we consider the Banach space $\mathcal{E}_0 \cap L^1(\mathbb{R})$ with the norm

$$\|u\|_{\mathcal{E}_0 \cap L^1(\mathbb{R})} = \max\{\|u\|_{\mathcal{E}_0}, \|u\|_{L^1(\mathbb{R})}\}.$$

*Hypothesis* 3.15.

(1) The operator associated with $L^{(1)}$ on $L^1(\mathbb{R})^{n_1}$ generates a bounded semigroup.
(2) For $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$, the operator associated with $L^{(1)}$ on $(\mathcal{E}_0 \cap L^1(\mathbb{R}))^{n_1}$ generates a semigroup $S^{(1)}(t)$ that satisfies an estimate of the form (1.9).

We note that if $d_i > 0$ for $i = 1, \ldots, n-1$ and $A_1 = 0$, then Hypothesis 3.15 holds.

Theorem 3.16. *Assume that Hypotheses* 3.1, 3.2, 3.6, 3.9, 3.10, *and* 3.15 *hold. Let* $\mathcal{E}_0 = H^1(\mathbb{R})$ *or* $BUC(\mathbb{R})$. *Let* $Y^0 \in Y_* + (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$ *with* $\|Y^0 - Y_*\|_\beta$ *and*

*$\|Y^0 - Y_*\|_{L^1}$ sufficiently small, and let $(\tilde{Y}^0, q^0)$ be given by* (3.15)*. Let $Y(t)$ be the solution of* (1.2) *in $Y_* + \mathcal{E}_\beta^n$ with $Y(0) = Y^0$. Then for all $t \geq 0$, all conclusions of Theorem* 3.14 *hold, and in addition,*

(1) $Y(t) \in (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$;
(2) $\|\tilde{U}(t)\|_{L^1} \leq C \max(\|\tilde{U}^0\|_{L^1}, \|\tilde{Y}^0\|_\alpha)$;
(3) $\|\tilde{U}(t)\|_{L^\infty} \leq Ch(t) \max(\|\tilde{U}^0\|_{L^1}, \|\tilde{Y}^0\|_\beta)$;
(4) $\|\tilde{V}(t)\|_{L^1} \leq Ce^{-\nu t} \max(\|\tilde{V}^0\|_{L^1}, \|\tilde{Y}^0\|_\alpha)$.

**4. System to be studied.** Let $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$. We seek a solution to (3.6)–(3.7) in the form $Y(\xi, t) = Y_*(\xi - q(t)) + \tilde{Y}(\xi, t)$; i.e.,

$$\begin{pmatrix} U(\xi,t) \\ V(\xi,t) \end{pmatrix} = \begin{pmatrix} U_*(\xi - q(t)) \\ V_*(\xi - q(t)) \end{pmatrix} + \begin{pmatrix} \tilde{U}(\xi,t) \\ \tilde{V}(\xi,t) \end{pmatrix}, \tag{4.1}$$

with $\tilde{Y}(\xi, t)$ in $\mathcal{E}_\beta^n$ for each $t$. Let

$$Y_q = Y_*(\xi - q) = (U_*(\xi - q), V_*(\xi - q)) = (U_q, V_q).$$

With this notation, $\tilde{Y}$ satisfies

$$\tilde{Y}_t = D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + R(Y_q + \tilde{Y}) - R(Y_q) + Y_*'(\xi - q(t))q'(t). \tag{4.2}$$

Note that

$$R(Y + \tilde{Y}) - R(Y) - DR(Y)\tilde{Y} = \left( \int_0^1 DR(Y + t\tilde{Y}) - DR(Y)\, dt \right) \tilde{Y}.$$

We define

$$N(Y, \tilde{Y}) = \int_0^1 DR(Y + t\tilde{Y}) - DR(Y)\, dt, \tag{4.3}$$

as an $n \times n$ matrix-valued function of $(Y, \tilde{Y})$. Using (4.3), we rewrite (4.2) as

$$\begin{aligned} \tilde{Y}_t &= D\tilde{Y}_{\xi\xi} + c\tilde{Y}_\xi + DR(Y_*)\tilde{Y} + (DR(Y_q) - DR(Y_*))\tilde{Y} \\ &\quad + N(Y_q, \tilde{Y})\tilde{Y} + Y_*'(\xi - q(t))q'(t) \\ &= L\tilde{Y} + (DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y} + Y_*'(\xi - q(t))q'(t). \end{aligned} \tag{4.4}$$

Let us assume that $\tilde{Y}(\xi, t)$ is in $\mathrm{R}(\mathcal{L}_\alpha) \cap \mathcal{E}_\beta^n$ for every $t$. Applying $\mathcal{P}_\alpha^s$ and $\mathcal{P}_\alpha^c$ to (4.4) we obtain

$$\tilde{Y}_t = L\tilde{Y} + \mathcal{P}_\alpha^s\big((DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y} + Y_*'(\xi - q(t))q'(t)\big), \tag{4.5}$$

$$-q'(t)\mathcal{P}_\alpha^c Y_*'(\xi - q(t)) = \mathcal{P}_\alpha^c\big((DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y}\big). \tag{4.6}$$

From (4.6) we obtain

$$-q'(t)\pi_\alpha Y_*'(\xi - q(t)) = \pi_\alpha\big((DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y}\big) \tag{4.7}$$

LEMMA 4.1. *There is a number $\delta_1 > 0$ such that if $|q| \leq \delta_1$, then*

$$\frac{1}{2} \leq |\pi_\alpha Y_*'(\xi - q)| \leq \frac{3}{2}.$$

*Proof.* By Lemma 3.7(2), $\gamma_\alpha(\xi)Y_*''(\xi) \to 0$ exponentially as $\xi \to \pm\infty$. Therefore the mapping $q \to Y_*'(\xi - q)$ is continuous (in fact, differentiable) from $\mathbb{R}$ to $\mathcal{E}_\alpha$, and $\pi_\alpha(Y_*'(\xi)) = 1$. The lemma follows. $\square$

Assuming $|q| \le \delta_1$, we introduce the notation

$$G(\tilde{Y}, q) = (DR(Y_q) - DR(Y_*))\tilde{Y} + N(Y_q, \tilde{Y})\tilde{Y}, \tag{4.8}$$

$$\kappa(\tilde{Y}, q) = -(\pi_\alpha Y_*'(\xi - q))^{-1}\pi_\alpha G(\tilde{Y}, q). \tag{4.9}$$

We have

$$\frac{2}{3} \le |(\pi_\alpha Y_*'(\xi - q))^{-1}| \le 2. \tag{4.10}$$

Since $\kappa(\tilde{Y}, q)$ has been chosen to make

$$\mathcal{P}_\alpha^{c}\Big(G(\tilde{Y}, q) + \kappa(\tilde{Y}, q)Y_*'(\xi - q)\Big) = 0, \tag{4.11}$$

we may rewrite (4.5)–(4.6) as the following system on $(\mathrm{R}(\mathcal{L}_\alpha) \cap \mathcal{E}_\beta^{\mathrm{n}}) \times \mathbb{R}$:

$$\partial_t \tilde{Y} = L\tilde{Y} + G(\tilde{Y}, q) + \kappa(\tilde{Y}, q)Y_*'(\xi - q), \tag{4.12}$$

$$\dot{q} = \kappa(\tilde{Y}, q). \tag{4.13}$$

We recall from section 3 that $\mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) = \mathrm{R}(\mathcal{L}_\alpha) \cap \mathcal{E}_\beta^{\mathrm{n}}$.

## 5. Proof of nonlinear stability.

We continue to let $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$.

### 5.1. Existence of solutions and a priori bound for $\|\tilde{Y}(t)\|_\beta + |q(t)|$.

We shall study solutions of the system (4.12)–(4.13) on $\mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) \times \mathbb{R}$.

The operator $(\mathcal{L}_\beta, 0)$ generates a strongly continuous semigroup on $\mathcal{E}_\beta^n \times \mathbb{R}$. The nonlinearity is locally Lipschitz by Proposition 7.7, which will be proved in the following section. Therefore given initial data $(\tilde{Y}^0, q^0) \in \mathcal{E}_\beta^n \times \mathbb{R}$, the system (4.12)–(4.13) has a unique mild solution $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ with $(\tilde{Y}, q)(0, \tilde{Y}^0, q^0) = (\tilde{Y}^0, q^0)$. The solution is defined for $t$ in the maximal interval $0 \le t < t_{\max}(\tilde{Y}^0, q^0)$, where $0 < t_{\max}(\tilde{Y}^0, q^0) \le \infty$; see, e.g., [13, Theorem 6.1.4]. The set $\{(t, \tilde{Y}^0, q^0) \in \mathbb{R}_+ \times \mathcal{E}_\beta^n \times \mathbb{R} : 0 \le t < t_{\max}(\tilde{Y}^0, q^0)\}$ is open in $\mathbb{R}_+ \times \mathcal{E}_\beta^n \times \mathbb{R}$, and the map $(t, \tilde{Y}^0, q^0) \mapsto (\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ from this set to $\mathcal{E}_\beta^n \times \mathbb{R}$ is continuous; see, e.g., [19, Theorem 46.4].

Moreover, if $(\tilde{Y}, q) \in \mathcal{E}_\beta^n \times \mathbb{R}$, then we recall from (4.11) in section 4 that the right-hand side of (4.12) belongs to $\mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}})$, and $\mathcal{P}_\beta^s$ commutes with $\mathcal{L}_\beta$ and $e^{t\mathcal{L}_\beta}$. We may therefore consider (4.12)–(4.13) on $\mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) \times \mathbb{R}$. We conclude the following.

Proposition 5.1. *For each $\delta > 0$, if $0 < \gamma < \delta$, then there exists $T$, with $0 < T \le \infty$, such that the following is true: if $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) \times \mathbb{R}$ satisfies*

$$\|(\tilde{Y}^0, q^0)\|_{\mathcal{E}_\beta^n \times \mathbb{R}} = \|\tilde{Y}^0\|_\beta + |q^0| \le \gamma \tag{5.1}$$

*and $0 \le t < T$, then $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}_\beta^{\mathrm{s}}) \times \mathbb{R}$ is defined and satisfies*

$$\|\tilde{Y}(t, \tilde{Y}^0, q^0)\|_\beta + |q(t, \tilde{Y}^0, q^0)| \le \delta. \tag{5.2}$$

Let $T_{\max}(\delta, \gamma)$ denote the supremum of all $T$ such that (5.2) holds for all $0 \le t < T$ whenever (5.1) is satisfied.

**5.2. Decay of $\|\tilde{Y}(t)\|_\alpha$.** Let $\delta_1 < 1$ be chosen as in Lemma 4.1.

PROPOSITION 5.2. *Let $\nu > 0$ satisfy the hypothesis of Lemma 3.13. Then there exist $\delta_2$ in $(0, \delta_1)$, $C > 0$, and $K_\alpha > 0$ such that for every $\delta \in (0, \delta_2)$ and every $\gamma$ with $0 < \gamma < \delta$, the following is true. Let $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}_\beta) \times \mathbb{R}$ satisfy (5.1) so that $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ satisfies (5.2) for $0 \le t < T_{\max}(\delta, \gamma)$. Then*

$$(5.3)\quad \|\tilde{Y}(t)\|_\alpha \le K_\alpha e^{-\nu t}\|\tilde{Y}^0\|_\alpha \quad \textit{and} \quad |q(t) - q^0| \le C\|\tilde{Y}^0\|_\alpha \textit{ for } 0 \le t < T_{\max}(\delta, \gamma).$$

*Moreover, if $T_{\max}(\delta, \gamma) = \infty$, then there is $q^* \in \mathbb{R}$ such that*

$$|q(t) - q^*| \le Ce^{-\nu t}\|\tilde{Y}^0\|_\alpha \textit{ for all } t \ge 0. \tag{5.4}$$

*Proof.* Since $\tilde{Y}(t)$ is a mild solution of (4.12) in $\mathcal{E}^n_\beta$, it satisfies the integral equation

$$\tilde{Y}(t) = e^{t\mathcal{L}_\beta}\tilde{Y}^0 + \int_0^t e^{(t-s)\mathcal{L}_\beta}\Big(G(\tilde{Y}(s), q(s)) + \kappa(\tilde{Y}(s), q(s))Y_*'(\xi - q(s))\Big)\, ds. \tag{5.5}$$

Since $\tilde{Y}^0 \in \mathcal{E}^n_\beta$ by assumption and $G(\tilde{Y}(s), q(s)) + \kappa(\tilde{Y}(s), q(s))Y_*'(\xi - q(s))$ is in $\mathcal{E}^n_\beta$, we have $e^{t\mathcal{L}_\beta}\tilde{Y}^0 = e^{t\mathcal{L}_\alpha}\tilde{Y}^0$ and

$$\begin{aligned} &e^{(t-s)\mathcal{L}_\beta}\Big(G(\tilde{Y}(s), q(s)) + \kappa(\tilde{Y}(s), q(s))Y_*'(\xi - q(s))\Big) \\ &\qquad = e^{(t-s)\mathcal{L}_\alpha}\Big(G(\tilde{Y}(s), q(s)) + \kappa(\tilde{Y}(s), q(s))Y_*'(\xi - q(s))\Big). \end{aligned}$$

Therefore (5.5) holds with $\mathcal{L}_\beta$ replaced by $\mathcal{L}_\alpha$. In addition, $\tilde{Y}^0 \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}_\alpha)$, and we recall from section 4 (see (4.11)) that $G(\tilde{Y}(s), q(s)) + \kappa(\tilde{Y}(s), q(s))Y_*'(\xi - q(s))$ is in $\mathrm{R}(\mathcal{P}^{\mathrm{s}}_\alpha)$. Therefore (5.5) holds with $\mathcal{L}_\beta$ replaced by $\mathcal{L}_\alpha\mathcal{P}^s_\alpha$.

Choose $k > 1$ such that

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_\alpha) \text{ and } \lambda \ne 0\} < -\bar{\nu} := -k\nu.$$

By Lemma 3.13 there exists $K_\alpha > 0$ such that $\|e^{t\mathcal{L}_\alpha\mathcal{P}^s_\alpha}\| \le K_\alpha e^{-\bar{\nu}t}$. From Proposition 7.7(1), for $\|\tilde{Y}(s)\|_\beta + |q(s)| \le \delta$, with $\delta$ given by the a priori bound (5.2), there exists a constant $C_1$ such that

$$\begin{aligned} \|\tilde{Y}(t)\|_\alpha &\le K_\alpha e^{-\bar{\nu}t}\|\tilde{Y}^0\|_\alpha \\ &\quad + \int_0^t K_\alpha e^{-\bar{\nu}(t-s)}C_1\left(\|\tilde{Y}(s)\|_0 + |q(s)|\right)(1 + \|Y_*'(\xi - q(s))\,\|_\alpha)\|\tilde{Y}(s)\|_\alpha\, ds. \end{aligned}$$

Using the a priori bound (5.2) again, along with (3.14), one finds a constant $C_2$ so that

$$\|\tilde{Y}(t)\|_\alpha \le K_\alpha e^{-\bar{\nu}t}\|\tilde{Y}^0\|_\alpha + C_2\delta\int_0^t e^{-\bar{\nu}(t-s)}\|\tilde{Y}(s)\|_\alpha\, ds. \tag{5.6}$$

Choosing $\delta_2 < \min(\delta_1, (k-1)\frac{\nu}{C_2})$ and using Gronwall's inequality for the function $e^{\bar{\nu}t}\|\tilde{Y}(t)\|_\alpha$ (see, e.g., [8, Section 1.2.1]), we arrive at the first estimate in (5.3).

From Proposition 7.7(2), the a priori bound (5.2), and the first estimate in (5.3), we have

$$\begin{aligned} |\dot{q}(t)| = |\kappa(\tilde{Y}(t), q(t))| &\le C_1\left(|q(t)| + \|\tilde{Y}(t)\|_0\right)\|\tilde{Y}(t)\|_\alpha \le C_1\delta K_\alpha e^{-\nu t}\|\tilde{Y}^0\|_\alpha \\ &= Ce^{-\nu t}\|\tilde{Y}^0\|_\alpha, \end{aligned} \tag{5.7}$$

where $C = C_1 \delta K_\alpha$. Using (5.7) and

$$q(t) = q^0 + \int_0^t \dot{q}(s)\,ds, \quad 0 \le t < T_{\max}(\delta,\gamma), \tag{5.8}$$

we obtain the second estimate in (5.3):

$$|q(t) - q^0| \le \int_0^t |\dot{q}(s)|\,ds \le C\|\tilde{Y}^0\|_\alpha \int_0^t e^{-\nu t}\,ds \le \frac{C}{\nu}\|\tilde{Y}^0\|_\alpha. \tag{5.9}$$

Finally, if $T_{\max}(\delta,\gamma) = \infty$, then (5.7) implies that in (5.8), $\lim_{t\to\infty} q(t) = q^*$ exists. From (5.8) and (5.7) we have

$$|q^* - q(t)| \le \int_t^\infty |\dot{q}(s)|\,ds \le \frac{C}{\nu} e^{-\nu t}\|\tilde{Y}^0\|_\alpha. \qquad \square$$

**5.3. Bounds for $\|\tilde{Y}(t)\|_0$.**

PROPOSITION 5.3. *Let $\rho > 0$ satisfy the hypothesis of Lemma* 3.11. *Choose $\nu < \rho$ such that $\nu$ satisfies the hypothesis of Proposition* 5.2. *Let $\delta_2$ be given by Proposition* 5.2. *Then there exist $\delta_3$ in $(0,\delta_2)$ and $C > 0$ such that for every $\delta \in (0,\delta_3)$ and every $\gamma$ with $0 < \gamma < \delta$, the following is true: let $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}_\beta) \times \mathbb{R}$ satisfy* (5.1). *Then $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ satisfies* (5.2) *for $0 \le t < T_{\max}(\delta,\gamma)$, and the following estimates for $\tilde{Y}(t) = (\tilde{U}(t), \tilde{V}(t))$ hold for $0 \le t < T_{\max}(\delta,\gamma)$):*

$$\|\tilde{U}(t)\|_0 \le C\|\tilde{Y}^0\|_\beta, \tag{5.10}$$

$$\|\tilde{V}(t)\|_0 \le C\|\tilde{Y}^0\|_\beta e^{-\rho t}. \tag{5.11}$$

*Proof.* Using (3.12), we rewrite (4.12) as

$$\tilde{U}_t = L^{(1)}\tilde{U} + D_V R_1(0,0)\tilde{V} + H_1(\xi, \tilde{U}, \tilde{V}, q), \tag{5.12}$$

$$\tilde{V}_t = L^{(2)}\tilde{V} + H_2(\xi, \tilde{U}, \tilde{V}, q), \tag{5.13}$$

with

$$\begin{aligned} H_1(\xi,\tilde{U},\tilde{V},q) &= (DR_1(Y_*) - DR_1(0))\tilde{Y} + G_1(\tilde{Y},q) + \kappa(\tilde{Y},q)U_*'(\xi - q(t)) \\ &= (DR_1(Y_*) - DR_1(0))\tilde{Y} + (DR_1(Y_q) - DR_1(Y_*))\tilde{Y} + N_1(Y_q,\tilde{Y})\tilde{Y} \\ &\quad + \kappa(\tilde{Y},q)U_*'(\xi - q(t)), \end{aligned} \tag{5.14}$$

$$\begin{aligned} H_2(\xi,\tilde{U},\tilde{V},q) &= (DR_2(Y_*) - DR_2(0))\tilde{Y} + G_2(\tilde{Y},q) + \kappa(\tilde{Y},q)V_*'(\xi - q(t)) \\ &= (DR_2(Y_*) - DR_2(0))\tilde{Y} + (DR_2(Y_q) - DR_2(Y_*))\tilde{Y} + N_2(Y_q,\tilde{Y})\tilde{Y} \\ &\quad + \kappa(\tilde{Y},q)V_*'(\xi - q(t)). \end{aligned} \tag{5.15}$$

We consider the following nonautonomous linear system related to (5.12)–(5.13):

$$\hat{U}_t = L^{(1)}\hat{U} + D_V R_1(0,0)\tilde{V} + H_1(\xi, \tilde{U}(t), \tilde{V}(t), q(t)), \tag{5.16}$$

$$\hat{V}_t = L^{(2)}\hat{V} + H_2(\xi, \tilde{U}(t), \tilde{V}(t), q(t)), \tag{5.17}$$

where $(\tilde{U}, \tilde{V}, q)(t) = (\tilde{U}, \tilde{V}, q)(t, U^0, V^0, q^0)$. Since $(\tilde{U}, \tilde{V}, q)(t)$ is a fixed solution of (4.12)–(4.13) in $\mathcal{E}^n_\beta \times \mathbb{R}$, we can regard (5.16)–(5.17) as a nonautonomous linear system on $\mathcal{E}^n_0$. The solution with the value $(U^0, V^0)$ at $t = 0$ is, of course, $(\hat{U}, \hat{V})(t, U^0, V^0) = (\tilde{U}, \tilde{V})(t)$.

Let $(\tilde{Y}, q)$ lie in a bounded neighborhood $\mathcal{N}$ of $(0,0)$ in $\mathcal{E}^n_\beta \times \mathbb{R}$. Lemma 8.2, which will be proved in section 8, implies that in (5.14) and (5.15), $\| \|_0$ of the first two terms on the right is bounded by a constant times $\|\tilde{Y}\|_\alpha$. Proposition 7.7(2) implies that in each expression, $\| \|_0$ of the last term on the right is bounded by a constant times $\|\tilde{Y}\|_\alpha$. Finally, Lemma 8.3 implies that in each expression, $\| \|_0$ of the third term on the right is bounded by a constant times $\|\tilde{Y}\|_0 \|\tilde{V}\|_0$. Hence for each $\mathcal{N}$, there exists a constant $C_1 > 0$ such that for $(\tilde{Y}, q) \in \mathcal{N}$,

$$\|H_i(\xi, \tilde{U}, \tilde{V}, q)\|_0 \le C_1 \left( \|\tilde{Y}\|_0 \|\tilde{V}\|_0 + \|\tilde{Y}\|_\alpha \right), \quad i = 1, 2. \tag{5.18}$$

The solution of (5.17) in $\mathcal{E}^n_0$ is

$$\hat{V}(t) = \tilde{V}(t) = e^{t\mathcal{L}_0^{(2)}} \tilde{V}^0 + \int_0^t e^{(t-s)\mathcal{L}_0^{(2)}} H_2(\xi, \tilde{U}(s), \tilde{V}(s), q(s))\, ds. \tag{5.19}$$

Choose $k > 1$ such that

$$\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_0^{(2)})\} < -\bar{\rho} = -k\rho.$$

By Lemma 3.11 there exists $K_2 > 0$ such that $\|e^{t\mathcal{L}_0^{(2)}}\|_{\mathcal{E}_0^{n_2} \to \mathcal{E}_0^{n_2}} \le K_2 e^{-\bar{\rho}t}$. Using (5.18) we obtain the estimate

$$\|\tilde{V}(t)\|_0 \le K_2 e^{-\bar{\rho}t} \|\tilde{V}^0\|_0 + \int_0^t K_2 e^{-\bar{\rho}(t-s)} C_1 \left( \|\tilde{Y}(s)\|_0 \|\tilde{V}(s)\|_0 + \|\tilde{Y}(s)\|_\alpha \right) ds. \tag{5.20}$$

Let $0 < \gamma < \delta < \delta_2$, and let $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}_\beta) \times \mathbb{R}$ satisfy (5.1) so that $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ satisfies (5.2) for $0 \le t < T_{\max}(\delta, \gamma)$. Since $\nu < \rho < \bar{\rho}$, (5.20) and Proposition 5.2 yield

$$\begin{aligned}
\|\tilde{V}(t)\|_0 &\le K_2 e^{-\bar{\rho}t} \|\tilde{V}^0\|_0 + \int_0^t K_2 e^{-\bar{\rho}(t-s)} C_1 \left( \delta \|\tilde{V}(s)\|_0 + K_\alpha e^{-\nu s} \|\tilde{Y}^0\|_\alpha \right) ds \\
&\le K_2 e^{-\bar{\rho}t} \|\tilde{V}^0\|_0 + \frac{K_2 C_1 K_\alpha}{\bar{\rho} - \nu} e^{-\nu t} \|\tilde{Y}^0\|_\alpha + \int_0^t K_2 e^{-\bar{\rho}(t-s)} C_1 \delta \|\tilde{V}(s)\|_0\, ds \\
&\le C \|\tilde{Y}^0\|_\beta + \int_0^t K_2 C_1 \delta e^{-\bar{\rho}(t-s)} \|\tilde{V}(s)\|_0\, ds.
\end{aligned}$$

Using Gronwall's inequality for the function $e^{\bar{\rho}t}\|\tilde{V}(t)\|_0$, we obtain, for $0 \le t < T_{\max}(\delta, \gamma)$,

$$\|\tilde{V}(t)\|_0 \le C \|\tilde{Y}^0\|_\beta e^{(K_2 C_1 \delta - \bar{\rho})t}.$$

For $\delta_3 < \min(\delta_2, \frac{(k-1)\rho}{K_2 C_1})$, we have (5.11).

The solution of (5.16) in $\mathcal{E}^n_0$ is

$$\hat{U}(t) = \tilde{U}(t) = e^{\mathcal{L}_0^{(1)} t} \tilde{U}^0 + \int_0^t e^{\mathcal{L}_0^{(1)}(t-s)} \left( D_V R_1(0,0) \tilde{V}(s) + H_1(\xi, \tilde{U}(s), \tilde{V}(s), q(s)) \right) ds. \tag{5.21}$$

Using Hypothesis 3.10(1), (5.18), (5.11), and Proposition 5.2, we obtain the estimate

$$\begin{aligned}
\|\tilde{U}(t)\|_0 &\le K_1 \|\tilde{U}^0\|_0 + \int_0^t K_1 C_2 \left( \|\tilde{V}(s)\|_0 + \|\tilde{Y}(s)\|_0 \|\tilde{V}(s)\|_0 + \|\tilde{Y}(s)\|_\alpha \right) ds \\
&\le K_1 \|\tilde{Y}^0\|_\beta + \int_0^t K_1 C_2 \left( C(1+\delta) e^{-\rho s} \|\tilde{Y}^0\|_\beta + K_\alpha e^{-\nu s} \|\tilde{Y}^0\|_\beta \right) ds,
\end{aligned}$$

which implies (5.10). ∎

**5.4. Completion of proof of nonlinear stability.** Define a mapping $\mathcal{F}$ from $\mathrm{R}(\mathcal{P}^{\mathrm{s}}_\beta) \times \mathbb{R}$ to the affine space $\tilde{Y}_* + \mathcal{E}^n_\beta$ by

$$\mathcal{F}(\tilde{Y}, q) = Y = Y_*(\xi - q) + \tilde{Y} = Y_*(\xi) + (Y_*(\xi - q) - Y_*(\xi)) + \tilde{Y}(\xi).$$

Consider

$$Y_*(\xi - q) - Y_*(\xi) = -q \int_0^1 Y_*'(\xi - tq)dt \tag{5.22}$$

and

$$\begin{aligned} \gamma_\alpha(\xi)(Y_*(\xi - q) - Y_*(\xi)) &= -q\gamma_\alpha(\xi) \int_0^1 Y_*'(\xi - tq)dt \\ = -q \int_0^1 \frac{\gamma_\alpha(\xi)}{\gamma_\alpha(\xi - tq)} \gamma_\alpha(\xi - tq) Y_*'(\xi - tq)dt &= -q \int_0^1 \gamma_\alpha(tq)\gamma_\alpha(\xi - tq)Y_*'(\xi - tq)dt. \end{aligned} \tag{5.23}$$

By Lemmas 3.3 and 3.7, both (5.22) and (5.23) approach 0 exponentially as $\xi \to \pm\infty$. Therefore $Y_*(\xi - q) - Y_*(\xi)$ is in $\mathcal{E}^n_\beta$ as desired.

LEMMA 5.4. *$D\mathcal{F}(0,0)$ is an isomorphism, so $\mathcal{F}$ maps a neighborhood $\mathcal{V}$ of $(0,0)$ in $\mathrm{R}(\mathcal{P}^{\mathrm{s}}_\beta) \times \mathbb{R}$ diffeomorphically onto a neighborhood $\mathcal{U}$ of $Y_*$ in $Y_* + \mathcal{E}^n_\beta$.*

*Proof.* The mapping $q \to Y_*(\xi - q) - Y_*(\xi)$ is $C^1$ as a map from $\mathbb{R}$ to $\mathcal{E}^n_\beta$, so $\mathcal{F}$ is $C^1$. $R(\mathcal{P}^s_\beta)$ is a codimension-one subspace of $\mathcal{E}^2$, and $\frac{\partial \mathcal{F}}{\partial q}(0,0) = -Y_*'(\xi)$ is not in it. Therefore $D\mathcal{F}(0,0)$ is an isomorphism. The rest of the result is a consequence of the inverse function theorem. $\square$

Assume that $\mathcal{V}$ is chosen small enough so that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Lipschitz. Let $Q$ denote the Lipschitz constant of $\mathcal{F}^{-1}$.

We are now ready to prove Theorem 3.14.

*Proof of Theorem* 3.14. Let $\nu > 0$ satisfy the hypothesis of Theorem 3.14, and let $\rho > \nu$ satisfy the hypothesis of Proposition 5.3. Let $\delta_2$ be given by Proposition 5.2, and let $\delta_3$ be given by Proposition 5.3.

Choose $\delta_\mathcal{V}$, $0 < \delta_\mathcal{V} \le \delta_3$, such that (1) if $\|\tilde{Y}\|_\beta + |q| \le \delta_\mathcal{V}$, then $(\tilde{Y}, q) \in \mathcal{V}$, and (2) $\eta_\mathcal{U} = Q^{-1}\delta_\mathcal{V}$ is such that the closed ball of radius $\eta_\mathcal{U}$ about $Y_*$ in $Y_* + \mathcal{E}^n_\beta$ is contained in $\mathcal{U}$.

Given $Y^0 \in Y_* + \mathcal{E}^n_\beta$, let $Y(t) = Y(t, Y^0)$ be the solution of (1.2) in $Y_* + \mathcal{E}^n_\beta$ with $Y(0) = Y^0$. If $Y^0 \in \mathcal{U}$, we can use the decomposition (3.15); similarly, if $Y(t) \in \mathcal{U}$, we can use the decomposition (3.16).

To prove Theorem 3.14, we shall show that for each $\delta \in (0, \delta_\mathcal{V})$, there exists $\eta$ with $0 < \eta < \eta_\mathcal{U}$ with the properties given in the statement of the theorem.

Let $0 < \gamma_1 < \delta < \delta_\mathcal{V}$. Let $\gamma = C^{-1}\gamma_1$, where $C \ge 1$ is the largest of $K_\alpha$ in Proposition 5.2 and the constants $C$ appearing in Propositions 5.2 and 5.3. Let $\eta = Q^{-1}\gamma$.

Let $Y^0 \in Y_0^* + \mathcal{E}^n_\beta$ with $\|Y^0 - Y_*\| \le \eta$. Now $\eta = Q^{-1}\gamma \le Q^{-1}\gamma_1 < Q^{-1}\delta_\mathcal{V} = \eta_\mathcal{U}$, so $Y_0 \in \mathcal{U}$. Therefore there exists $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}^s_\beta) \times \mathbb{R}$ with $Y^0 = \tilde{Y}^0 + Y_*(\xi - q^0)$ and $\|\tilde{Y}_0\|_\beta + |q^0| \le Q\eta = \gamma < \delta$. By Proposition 5.1, $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ is defined for $0 \le t < T_{\max}(\delta, \gamma)$; by Propositions 5.1, 5.2, and 5.3, it satisfies (5.2), (5.3), (5.10), and (5.11).

We claim that $T_{\max}(\delta, \gamma) = \infty$. To see this, let $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}^s_\beta) \times \mathbb{R}$ with $\|\tilde{Y}_0\|_\beta + |q^0| \le \gamma$. For any $T$ in $(0, T_{\max}(\delta, \gamma))$, the inequalities (5.3), (5.10), and (5.11) yield

$$\|\tilde{Y}(T, \tilde{Y}^0, q^0)\|_\beta + |q(T, \tilde{Y}^0 q^0)| \le C(\|\tilde{Y}^0\|_\beta + |q^0|) \le C\gamma = \gamma_1. \tag{5.24}$$

Consider the solution with initial data $(\tilde{Y}^1, q^1) = (\tilde{Y}, q)(T, \tilde{Y}^0, q^0)$. Since $\|\tilde{Y}^1\|_\beta + |q^1| \le \gamma_1$, Proposition 5.1 applies to this solution. Therefore for all $t \in [0, T_{\max}(\delta, \gamma_1))$, we have

$$(5.25)\quad \|\tilde{Y}(t+T, \tilde{Y}^0, q^0)\|_\beta + |q(t+T, \tilde{Y}^0, q^0)| = \|\tilde{Y}(t, \tilde{Y}^1, q^1)\|_\beta + |q(t, \tilde{Y}^1, q^1)| \le \delta.$$

This shows that the a priori bound (5.2) for the solution with any initial data satisfying $\|\tilde{Y}^0\|_\beta + |q^0| \le \gamma$ holds for all $t \in [0, T + T_{\max}(\delta, \gamma_1))$. Therefore $T_{\max}(\delta, \gamma) \ge T + T_{\max}(\delta, \gamma_1)$, and thus, $T_{\max}(\delta, \gamma) \ge T_{\max}(\delta, \gamma) + T_{\max}(\delta, \gamma_1)$. Hence $T_{\max}(\delta, \gamma) = \infty$.

Hence for all $t \ge 0$, $\|\tilde{Y}(t))\|_\beta + |q(t)| \le \delta < \delta_{\mathcal{V}}$, so $(\tilde{Y}(t), q(t)) \in \mathcal{V}$ and $Y(t) = \tilde{Y}(t) + Y_*(\xi - q(t))$ is in $\mathcal{U}$; thus (1), (2), and (3) hold. Statement (4) is just (5.3); (5) is (5.4); (6) and (7) are (5.10) and (5.11), respectively. ∎

**6. Algebraic decay.** We continue to let $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$. In this section we shall study solutions of (1.2) of the form $Y = Y_* + \tilde{Y}$ with $\tilde{Y} \in (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$.

Since $(Y_*)' \in L^1(\mathbb{R})^n$ by Lemma 3.3 and $\mathcal{E}_\beta \cap L^1(\mathbb{R}) \hookrightarrow \mathcal{E}_\beta$, the one-dimensional operator $\mathcal{P}^c_\beta$ restricts to a bounded linear map of $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$ into itself. Therefore $\mathcal{P}^s_\beta = I - \mathcal{P}^c_\beta$ also restricts to a bounded linear map of $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$ into itself, which we denote $\mathcal{P}^s_1$. By analogy to what was done in subsection 5.1, instead of studying solutions of (1.2) of the form $Y = Y_* + \tilde{Y}$ with $\tilde{Y} \in (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$, we shall instead study the system (4.12)–(4.13) on $\mathrm{R}(\mathcal{P}^{\mathrm{s}}_1) \times \mathbb{R}$.

The operator $(\mathcal{L}_\beta, 0)$ restricted to $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ generates a strongly continuous semigroup on $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$. The nonlinearity is locally Lipschitz on on $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ by Proposition 7.7. Therefore given initial data $(\tilde{Y}^0, q^0) \in (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$, the system (4.12)–(4.13) has a unique mild solution $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ in $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ with $(\tilde{Y}, q)(0, \tilde{Y}^0, q^0) = (\tilde{Y}^0, q^0)$. The solution is defined for $t$ in the maximal interval $0 \le t < t_{\max}(\tilde{Y}^0, q^0)$, where $0 < t_{\max}(\tilde{Y}^0, q^0) \le \infty$. The set $\{(t, \tilde{Y}^0, q^0) \in \mathbb{R}_+ \times \mathcal{E}^n_\beta \times \mathbb{R} : 0 \le t < t_{\max}(\tilde{Y}^0, q^0)\}$ is open in $\mathbb{R}_+ \times (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$, and the map $(t, \tilde{Y}^0, q^0) \mapsto (\tilde{Y}, q)(t, \tilde{Y}^0, q^0)$ from this set to $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ is continuous. As in subsection 5.1, we conclude the following.

PROPOSITION 6.1. *For each $\delta > 0$, if $0 < \gamma < \delta$, then there exists $T$, with $0 < T \le \infty$, such that the following is true: if $(\tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}_1) \times \mathbb{R}$ satisfies*

$$(6.1)\qquad \|(\tilde{Y}^0, q^0)\|_{(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}} = \max\left(\|\tilde{Y}^0\|_\beta, \|\tilde{Y}\|_{L^1}\right) + |q^0| \le \gamma$$

*and $0 \le t < T$, then $(\tilde{Y}, q)(t, \tilde{Y}^0, q^0) \in \mathrm{R}(\mathcal{P}^{\mathrm{s}}_1) \times \mathbb{R}$ is defined and satisfies*

$$(6.2)\qquad \max\left(\|\tilde{Y}(t, \tilde{Y}^0, q^0)\|_\beta, \|\tilde{Y}(t, \tilde{Y}^0, q^0)\|_{L^1}\right) + |q(t, \tilde{Y}^0, q^0)| \le \delta.$$

We shall now prove Theorem 3.16, mimicking the proof of Theorem 3.14 in the previous section.

To prove the analogue of Proposition 5.3, the key estimate that we need is the following: given a bounded neighborhood $\mathcal{N}$ of $(0, 0)$ in $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$, there exists a constant $C_1 > 0$ such that for $(\tilde{Y}, q) \in \mathcal{N}$,

$$(6.3)\qquad \|H_i(\xi, \tilde{U}, \tilde{V}, q)\|_{L^1} \le C_1 \left(\|\tilde{Y}\|_0 \|\tilde{V}\|_{L^1} + \|\tilde{Y}\|_\alpha\right).$$

(Compare (5.18).) To justify (6.3), look at each term in (5.14) and (5.15). Since $U'_*$ and $V'_*$ are exponentially decaying, Proposition 7.7(2) implies that in (5.14) and (5.15), $\|\ \|_{L^1}$ of the last term on the right is bounded by a constant times $\|\tilde{Y}\|_\alpha$. Lemma 8.2

implies that in (5.14) and (5.15), $\| \, \|_{L^1}$ of the first two terms on the right is bounded by a constant times $\|\tilde{Y}\|_\alpha$. Finally, Lemma 8.3 implies that in each expression, $\| \, \|_{L^1}$ of the third term on the right is bounded by a constant times $\|\tilde{Y}\|_0(\|\tilde{V}\|_{L^1} + \|Y\|_\alpha)$. The estimate (6.4) follows.

From (5.19), (6.3) for $H_2$, and Lemma 3.11(3c) for the case $\mathcal{E}_0 = L^1(\mathbb{R})$, we obtain the following analogue of (5.20):

$$\|\tilde{V}(t)\|_{L^1} \le K_2 e^{-\tilde{\rho} t}\|\tilde{V}^0\|_{L^1} + \int_0^t K_2 e^{-\tilde{\rho}(t-s)} C_1 \left(\|\tilde{Y}(s)\|_0 \|\tilde{V}(s)\|_{L^1} + \|\tilde{Y}(s)\|_\alpha\right) ds. \tag{6.4}$$

Proceeding as in the proof of Proposition 5.3, we obtain the following analogue of (5.11):

$$\|\tilde{V}(t)\|_{L^1} \le C e^{-\rho t} \max\left(\|\tilde{V}^0\|_{L^1}, \|\tilde{Y}^0\|_\alpha\right). \tag{6.5}$$

From (5.21), Hypothesis 3.15(1), (6.3) for $H_1$, Theorem 3.16(4), and Proposition 5.2, we have

$$\begin{aligned} \|\tilde{U}(t)\|_{L^1} \le {} & K_1 \|\tilde{U}^0\|_{L^1} + \int_0^t K_1 C_2 \Big(\max\left(\|\tilde{V}(s)\|_0, \|\tilde{V}(s)\|_{L^1}\right) \\ & + \|\tilde{Y}(s)\|_0 \max\left(\|\tilde{V}(s)\|_0, \|\tilde{V}(s)\|_{L^1}\right) + \|\tilde{Y}(s)\|_\alpha\Big)\, ds \\ \le {} & K_1 \|\tilde{U}^0\|_{L^1} + \int_0^t K_1 C_2 \left(C(1+\delta) e^{-\rho s}\|\tilde{Y}^0\|_\alpha + K_\alpha e^{-\nu s}\|\tilde{Y}^0\|_\alpha\right) ds, \end{aligned} \tag{6.6}$$

which implies the following analogue of (5.10):

$$\|\tilde{U}(t)\|_{L^1} \le C \max\left(\|\tilde{U}^0\|_{L^1}, \|\tilde{Y}^0\|_\alpha\right). \tag{6.7}$$

The estimates (6.5) and (6.7) yield an analogue for Proposition 5.3. Then, arguing as in subsection 5.4, we use Propositions 6.1 and 5.2 and our analogue for Proposition 5.3 to show, under the assumptions of Theorem 3.16, that $Y(t)$ stays in $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$ for all $t \ge 0$, which is conclusion (1) of Theorem 3.16.

Conclusions (2) and (4) of the theorem are just (6.7) and (6.5), respectively.

Finally, we show conclusion (3) of Theorem 3.16. From (5.21), Hypothesis 3.15(2), (6.3) for $H_1$, Theorem 3.16(4), Theorem 3.14(4), and the fact that $0 < \nu < \rho$, we have

$$\begin{aligned} \|\tilde{U}(t)\|_{L^\infty} \le {} & K_1 h(t) \max\left(\|\tilde{U}^0\|_0, \|\tilde{U}^0\|_{L^1}\right) \\ & + \int_0^t K_1 h(t-s) C_3 \Big(\max\left(\|\tilde{V}(s)\|_0, \|\tilde{V}(s)\|_{L^1}\right) \\ & + \|\tilde{Y}(s)\|_0 \max\left(\|\tilde{V}(s)\|_0, \|\tilde{V}(s)\|_{L^1}\right) + \|\tilde{Y}(s)\|_\alpha\Big)\, ds \\ \le {} & K_1 h(t) \max\left(\|\tilde{U}^0\|_0, \|\tilde{U}^0\|_{L^1}\right) + \int_0^t K_1 h(t-s) C_3 (1+\delta) e^{-\nu s}\|\tilde{Y}^0\|_\alpha \, ds. \end{aligned} \tag{6.8}$$

We note that

$$\begin{aligned} \int_0^t h(t-s) e^{-\nu s}\, ds &= \int_0^{\frac{t}{2}} h(t-s) e^{-\nu s}\, ds + \int_{\frac{t}{2}}^t h(t-s) e^{-\nu s}\, ds \\ &\le \int_0^{\frac{t}{2}} h\left(\frac{t}{2}\right) e^{-\nu s}\, ds + \int_{\frac{t}{2}}^t e^{-\nu s}\, ds \le \frac{1}{\nu} h\left(\frac{t}{2}\right) + \frac{1}{\nu} e^{-\frac{\nu}{2} t}. \end{aligned} \tag{6.9}$$

Note that for $t \geq \max\left(2, \frac{1}{2\nu}\ln\nu\right)$,

$$h\left(\frac{t}{2}\right) + e^{-\frac{\nu}{2}t} \leq 2^{\frac{1}{2}}t^{-\frac{1}{2}} + t^{-\frac{1}{2}} = \left(2^{\frac{1}{2}}+1\right)h(t).$$

It follows easily that there is a constant $C_5$ such that for all $t \geq 0$,

$$\int_0^t h(t-s)e^{-\nu s}\,ds \leq C_5 h(t). \tag{6.10}$$

Theorem 3.16(3) follows from (6.8) and (6.10).

**7. Lipschitz properties of nonlinear operators.** Let $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$. We recall the properties of the weighted spaces $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ listed in Lemma 3.12.

PROPOSITION 7.1.

(1) *If $y \in \mathcal{E}_0$, then $y \in C^0(\mathbb{R})$, and there is a constant $C > 0$ such that $\|y\|_{L^\infty} \leq C\|y\|_0$.*

(2) *If $y$, $z \in \mathcal{E}_0$, then $yz \in \mathcal{E}_0$, and there is a constant $C > 0$ such that $\|yz\|_0 \leq C\|y\|_0\|z\|_0$.*

(3) *If $y$, $z \in \mathcal{E}_\beta$, then $yz \in \mathcal{E}_\alpha$, and there is a constant $C > 0$ such that $\|yz\|_\alpha \leq C\|y\|_0\|z\|_\alpha$.*

(4) *If $y$, $z \in \mathcal{E}_\beta$, then $yz \in \mathcal{E}_\beta$, and there is a constant $C > 0$ such that $\|yz\|_\beta \leq C\|y\|_\beta\|z\|_\beta$.*

(5) *If $y$, $z \in \mathcal{E}_\beta \cap L^1(\mathbb{R})$, then $yz \in \mathcal{E}_\beta \cap L^1(\mathbb{R})$, and there is a constant $C > 0$ such that $\|yz\|_{L^1} \leq C\|y\|_{L^1}\|z\|_{L^\infty} \leq C\|y\|_{L^1}\|z\|_\beta$.*

*Proof.* Statement (1) is obvious for $\mathcal{E}_0 = BUC(\mathbb{R})$ and well known for $\mathcal{E}_0 = H^1(\mathbb{R})$; the same is true for (2). To show (3), let $y$, $z \in \mathcal{E}_\beta$ and let $w = \gamma_\alpha z \in \mathcal{E}_0$ . Then, using (2),

$$\|yz\|_\alpha = \|\gamma_\alpha yz\|_0 = \|yw\|_0 \leq C\|y\|_0\|w\|_0 = C\|y\|_0\|z\|_\alpha.$$

To show (4), let $y$, $z \in \mathcal{E}_\beta$. Then by (2), $\|yz\|_0 \leq C\|y\|_0\|z\|_0 \leq C\|y\|_\beta\|z\|_\beta$, and by (3), $\|yz\|_\alpha \leq C\|y\|_0\|z\|_\alpha \leq C\|y\|_\beta\|z\|_\beta$. Therefore $yz \in \mathcal{E}_\beta$ and $\|yz\|_\beta \leq C\|y\|_\beta\|z\|_\beta$. Statement (5) follows from (4) and an obvious fact about the $L^1$ norm. ∎

PROPOSITION 7.2. *Let $m(\xi, q, y) \in C^2(\mathbb{R}^3)$. Consider the formula*

$$(q, y(\xi), z(\xi)) \mapsto m(\xi, q, y(\xi))z(\xi). \tag{7.1}$$

(1) *Formula (7.1) defines a mapping from $\mathbb{R}\times\mathcal{E}_0^2$ to $\mathcal{E}_0$ that is Lipschitz on any set of the form $\{(q, y, z) : |q|+\|y\|_0+\|z\|_0 \leq K\}$. If $m(\xi, 0, y)$ is identically $0$, then there is a constant $C$ such that on this set, $\|m(\xi, q, y(\xi))z(\xi)\|_0 \leq C|q|\|z\|_0$.*

(2) *Formula (7.1) defines a mapping from from $\mathbb{R}\times\mathcal{E}_\beta^2$ to $\mathcal{E}_\beta$ that is Lipschitz on any set of the form $\{(q, y, z) : |q| + \|y\|_\beta + \|z\|_\beta \leq K\}$. If $m(\xi, 0, y)$ is identically $0$, then there is a constant $C$ such that on this set, $\|m(\xi, q, y(\xi))z(\xi)\|_\alpha \leq C|q|\|z\|_\alpha$ and $\|m(\xi, q, y(\xi))z(\xi)\|_\beta \leq C|q|\|z\|_\beta$.*

(3) *Formula (7.1) defines a mapping from $\mathbb{R}\times(\mathcal{E}_\beta\cap L^1(\mathbb{R}))^2$ to $\mathcal{E}_\beta\cap L^1(\mathbb{R})$ that is Lipschitz on any set of the form $\{(q, y, z) : |q|+\|y\|_\beta+\|y\|_{L^1}+\|z\|_\beta+\|z\|_{L^1} \leq K\}$.*

*Proof.* We will only consider the case $\mathcal{E}_0 = H^1(\mathbb{R})$; the case $\mathcal{E}_0 = BUC(\mathbb{R})$ is easier. First we show that the mappings go into the correct spaces. We have

$$\|m(\xi, q, y)z\|_{L^k} \leq \|m\|_{L^\infty}\|z\|_{L^k}, \qquad k = 1, 2, \tag{7.2}$$

and

$$\text{(7.3)} \quad \begin{aligned}\|(mz)_\xi\|_{L^2} &\le \|m_\xi z\|_{L^2} + \|m_y y_\xi z\|_{L^2} + \|m z_\xi\|_{L^2} \\ &\le \|m\|_{C^1}\|z\|_{L^2} + \|m\|_{C^1}\|y_\xi\|_{L^2}\|z\|_{L^2} + \|m\|_{L^\infty}\|z_\xi\|_{L^2}.\end{aligned}$$

Therefore if $(q, y, z) \in \mathbb{R} \times H^1(\mathbb{R})^2$ (respectively, $\mathbb{R} \times (H^1(\mathbb{R}) \cap L^1(\mathbb{R}))^2$), then $m(\xi, q, y)z \in H^1(\mathbb{R})$ (respectively, $L^1(\mathbb{R})$). Next, we have

$$\text{(7.4)} \qquad \|\gamma_\alpha m(q, y, z)z\|_{L^2} \le \|m\|_{L^\infty}\|\gamma_\alpha z\|_{L^2}$$

and

$$\text{(7.5)} \quad \begin{aligned}\|\gamma_\alpha(mz)_\xi\|_{L^2} &\le \|\gamma_\alpha m_\xi z\|_{L^2} + \|\gamma_\alpha m_y y_\xi z\|_{L^2} + \|\gamma_\alpha m z_\xi\|_{L^2} \\ &\le \|m\|_{C^1}\|\gamma_\alpha z\|_{L^2} + \|m\|_{C^1}\|y_\xi\|_{L^2}\|\gamma_\alpha z\|_{L^2} + \|m\|_{L^\infty}\|\gamma_\alpha z_\xi\|_{L^2}.\end{aligned}$$

Therefore if $(q, y, z) \in \mathbb{R} \times H^1_\beta(\mathbb{R})^2$, then $m(\xi, q, y)z \in H^1_\alpha(\mathbb{R})$.

Now we show the Lipschitz properties.

First we consider variations in $q$. We have

$$m(\xi, q + \bar{q}, y(\xi))z(\xi) - m(\xi, q, y(\xi))z(\xi) = \int_0^1 m_q(\xi, q + t\bar{q}, y(\xi))\, dt\, \bar{q}z(\xi).$$

Therefore

$$\|m(\xi, q + \bar{q}, y)z - m(\xi, q, y)z\|_{L^k} \le \|m\|_{C^1}\|z\|_{L^k}|\bar{q}|, \quad k = 1,\, 2,$$

and

$$\|\gamma_\alpha(m(\xi, q + \bar{q}, y)z - m(\xi, q, y)z)\|_{L^2} \le \|m\|_{C^1}\|\gamma_\alpha z\|_{L^2}|\bar{q}|.$$

Also,

$$\begin{aligned}(m(\xi, q + \bar{q}, y(\xi))z(\xi) - m(\xi, q, y(\xi))z(\xi))_\xi &= \int_0^1 m_{q\xi}(\xi, q + t\bar{q}, y(\xi))\, dt\, \bar{q}z(\xi) \\ &+ \int_0^1 m_{qy}(\xi, q + t\bar{q}, y(\xi))\, dt\, \bar{q}y_\xi z(\xi) + \int_0^1 m_q(\xi, q + t\bar{q}, y(\xi))\, dt\, \bar{q}z_\xi.\end{aligned}$$

Therefore

$$\begin{aligned}&\|(m(\xi, q + \bar{q}, y(\xi))z(\xi) - m(\xi, q, y(\xi))z(\xi))_\xi\|_{L^2} \\ &\qquad \le (\|m\|_{C^2}\|z\|_{L^2} + \|m\|_{C^2}\|y_\xi\|_{L^2}\|z\|_{L^2} + \|m\|_{C^1}\|z_\xi\|_{L^2})|\bar{q}|\end{aligned}$$

and

$$\begin{aligned}&\|\gamma_\alpha(m(\xi, q + \bar{q}, y(\xi))z(\xi) - m(\xi, q, y(\xi))z(\xi))_\xi\|_{L^2} \\ &\qquad \le (\|m\|_{C^2}\|\gamma_\alpha z\|_{L^2} + \|m\|_{C^2}\|y_\xi\|_{L^2}\|\gamma_\alpha z\|_{L^2} + \|m\|_{C^1}\|\gamma_\alpha z_\xi\|_{L^2})|\bar{q}|.\end{aligned}$$

Next we consider variations in $y$. We have

$$m(\xi, q, y(\xi) + \bar{y}(\xi))z(\xi) - m(\xi, q, y(\xi))z(\xi) = \int_0^1 m_y(\xi, q, y(\xi) + t\bar{y}(\xi))\, dt\, \bar{y}(\xi)z(\xi).$$

Therefore

$$\|m(\xi,q,y+\bar{y})z-m(\xi,q,y)z\|_{L^k}\le\|m\|_{C^1}\|\bar{y}\|_{L^k}\|z\|_{L^\infty},\quad k=1,\,2,$$

and

$$\|\gamma_\alpha(m(\xi,q,y+\bar{y})z-m(\xi,q,y)z)\|_{L^2}\le\|m\|_{C^1}\|\bar{y}\|_{L^2}\|\gamma_\alpha z\|_{L^2}.$$

Also,

$$\begin{aligned}&(m(\xi,q,y(\xi)+\bar{y}(\xi))z(\xi)-m(\xi,q,y(\xi))z(\xi))_\xi=\\&\int_0^1 m_{y\xi}(\xi,q,y(\xi)+t\bar{y}(\xi))\,dt\,\bar{y}(\xi)z(\xi)+\int_0^1 m_{yy}(\xi,q,y(\xi)+t\bar{y}(\xi))(y_\xi+t\bar{y}_\xi)\,dt\,\bar{y}(\xi)z(\xi)\\&\qquad+\int_0^1 m_y(\xi,q,y(\xi)+t\bar{y}(\xi))\,dt\,\bar{y}_\xi z(\xi)+\int_0^1 m_y(\xi,q,y(\xi)+t\bar{y}(\xi))\,dt\,\bar{y}(\xi)z_\xi.\end{aligned}$$

Therefore

$$\begin{aligned}&\|(m(\xi,q,y(\xi)+\bar{y}(\xi))z(\xi)-m(\xi,q,y(\xi))z(\xi))_\xi\|_{L^2}\\&\qquad\le\|m\|_{C^2}\|\bar{y}\|_{L^2}\|z\|_{L^2}+\|m\|_{C^2}\|y_\xi\|_{L^2}C\|\bar{y}\|_{H^1}\|z\|_{H^1}\\&\quad+\frac{1}{2}\|m\|_{C^2}\|\bar{y}_\xi\|_{L^2}C\|\bar{y}\|_{H^1}\|z\|_{H^1}+\|m\|_{C^1}\|\bar{y}_\xi\|_{L^2}\|z\|_{L^2}+\|m\|_{C^1}\|\bar{y}\|_{L^2}\|z_\xi\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}&\|\gamma_\alpha(m(\xi,q,y(\xi)+\bar{y}(\xi))z(\xi)-m(\xi,q,y(\xi))z(\xi))_\xi\|_{L^2}\\&\qquad\le\|m\|_{C^2}\|\bar{y}\|_{L^2}\|\gamma_\alpha z\|_{L^2}+\|m\|_{C^2}\|y_\xi\|_{L^2}C\|\bar{y}\|_{H^1}\|\gamma_\alpha z\|_{H^1}\\&\qquad+\frac{1}{2}\|m\|_{C^2}\|\bar{y}_\xi\|_{L^2}C\|\bar{y}\|_{H^1}\|\gamma_\alpha z\|_{H^1}+\|m\|_{C^1}\|\bar{y}_\xi\|_{L^2}\|\gamma_\alpha z\|_{L^2}\\&\qquad\qquad+\|m\|_{C^1}\|\bar{y}\|_{L^2}\|\gamma_\alpha z_\xi\|_{L^2}.\end{aligned}$$

Finally, we consider variations in $z$. We have

$$m(\xi,q,y(\xi))(z(\xi)+\bar{z}(\xi))-m(\xi,q,y(\xi))z(\xi)=m(\xi,q,y(\xi))\bar{z}(\xi).$$

Estimates are left to the reader.

Using the separate Lipschitz estimates for variations in $q$, $y$, and $z$, one can easily show that the mappings are Lipschitz on the given sets.

To prove the estimates when $m(\xi,0,y)=0$, we note that this assumption implies that $\|m\|_{L^\infty}\le C|q|$ and $\|m\|_{C^1}\le C|q|$ on the given sets; then use (7.2)–(7.5). ◻

COROLLARY 7.3. *Let $m(\xi,q,z)\in C^2(\mathbb{R}^3)$. Then the formula*

$$(q,z(\xi))\mapsto m(\xi,q,z(\xi))z(\xi)$$

*defines mappings from $\mathbb{R}\times\mathcal{E}_0$ to $\mathcal{E}_0$, from $\mathbb{R}\times\mathcal{E}_\beta$ to $\mathcal{E}_\beta$, and from $\mathbb{R}\times(\mathcal{E}_\beta\cap L^1(\mathbb{R}))$ to $\mathcal{E}_\beta\cap L^1(\mathbb{R})$. The first is Lipschitz on any set of the form $\{(q,z):|q|+\|z\|_0\le K\}$; the second is Lipschitz on any set of the form $\{(q,z):|q|+\|z\|_\beta\le K\}$; the third is Lipschitz on any set of the form $\{(q,z):|q|+\|z\|_\beta+\|z\|_{L^1}\le K\}$.*

We remark that in both Proposition 7.2 and Corollary 7.3, it is enough to assume that $m\in C^2(U)$ for any set $U$ of the form $\{(\xi,q,y):|q|+|y|\le K\}$.

PROPOSITION 7.4.

(1) *The formula* $(\tilde{Y}(\xi), q) \mapsto (DR(Y_q) - DR(Y_*))\tilde{Y}$ *defines a mapping from* $\mathcal{E}_0^n \times \mathbb{R}$ *to* $\mathcal{E}_0^n$ *that is Lipschitz on any set of the form* $\{(\tilde{Y}, q) : \|\tilde{Y}\|_0 + |q| \leq K\}$. *On such a set there is a constant* $C$ *such that* $\|(DR(Y_q) - DR(Y_*))\tilde{Y}\|_0 \leq C|q|\|\tilde{Y}\|_0$.

(2) *The formula* $(\tilde{Y}(\xi), q) \mapsto (DR(Y_q) - DR(Y_*))\tilde{Y}$ *defines a mapping from* $\mathcal{E}_\beta^n \times \mathbb{R}$ *to* $\mathcal{E}_\beta^n$ *that is Lipschitz on any set of the form* $\{(\tilde{Y}, q) : \|\tilde{Y}\|_\beta + |q| \leq K\}$. *On such a set there is a constant* $C$ *such that* $\|(DR(Y_q) - DR(Y_*))\tilde{Y}\|_\alpha \leq C|q|\|\tilde{Y}\|_\alpha$ *and* $\|(DR(Y_q) - DR(Y_*))\tilde{Y}\|_\beta \leq C|q|\|\tilde{Y}\|_\beta$.

(3) *The formula* $(\tilde{Y}(\xi), q) \mapsto (DR(Y_q) - DR(Y_*))\tilde{Y}$ *defines a mapping from* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ *to* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$ *that is Lipschitz on any set of the form* $\{(\tilde{Y}, q) : \|\tilde{Y}\|_\beta + \|\tilde{Y}\|_{L^1} + |q| \leq K\}$.

*Proof.* Just apply Proposition 7.2 to each component of $(DR(Y_q) - DR(Y_*))\tilde{Y}$. (In this case the function $m$ depends only on $\xi$ and $p$. Note that it is important here that $R$ is $C^3$.) ∎

PROPOSITION 7.5.

(1) *The formula* $(\tilde{Y}, q) \mapsto N(Y_q, \tilde{Y})$ *defines a mapping from* $\mathcal{E}_0^n \times \mathbb{R}$ *to* $\mathcal{E}_0^{n^2}$ *that is Lipschitz and* $\mathcal{O}(\|\tilde{Y}\|_0)$ *on any bounded neighborhood of* $(0,0)$ *in* $\mathcal{E}_0^n \times \mathbb{R}$.

(2) *The formula* $(\tilde{Y}, q) \mapsto N(Y_q, \tilde{Y})\tilde{Y}$ *defines a mapping from* $\mathcal{E}_0^n \times \mathbb{R}$ *to* $\mathcal{E}_0^n$ *that is Lipschitz on any bounded neighborhood of* $(0,0)$ *in* $\mathcal{E}_0^n \times \mathbb{R}$.

*Proof.* (1) The Lipschitz property follows from Corollary 7.3. (Again, it is important here that $R$ is $C^3$.) The mapping is $\mathcal{O}(\|\tilde{Y}\|_0)$ on the given set because it is Lipschitz and $N(Y_q, 0) = 0$. (2) This follows from (1). ∎

PROPOSITION 7.6.

(1) *If* $\tilde{Y} \in \mathcal{E}_\beta^n$, *then* $N(Y_q, \tilde{Y})\tilde{Y} \in \mathcal{E}_\alpha^n$, *and on any bounded neighborhood of* $(0,0)$ *in* $\mathcal{E}_\beta^n \times \mathbb{R}$ *there is a constant* $C > 0$ *such that* $\|N(Y_q, \tilde{Y})\tilde{Y})\|_\alpha \leq C\|\tilde{Y}\|_0\|\tilde{Y}\|_\alpha$.

(2) *The formula* $(\tilde{Y}, q) \mapsto N(Y_q, \tilde{Y})$ *defines a mapping from* $\mathcal{E}_\beta^n \times \mathbb{R}$ *to* $\mathcal{E}_\beta^{n^2}$ *(respectively,* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ *to* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^{n^2}$*) that is Lipschitz on any bounded neighborhood of* $(0,0)$ *in* $\mathcal{E}_\beta^n \times \mathbb{R}$ *(respectively,* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$*).*

(3) *The formula* $(\tilde{Y}, q) \mapsto N(Y_q, \tilde{Y})\tilde{Y}$ *defines a mapping from* $\mathcal{E}_\beta^n \times \mathbb{R}$ *to* $\mathcal{E}_\beta^n$ *(respectively,* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ *to* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^{n^2}$*) that is Lipschitz on any bounded neighborhood of* $(0,0)$ *in* $\mathcal{E}_\beta^n \times \mathbb{R}$ *(respectively,* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$*).*

*Proof.* (1) Using Proposition 7.5(1),

$$\|N(Y_q, \tilde{Y})\tilde{Y}\|_\alpha = \|N(Y_q, \tilde{Y})\gamma_\alpha \tilde{Y}\|_0 \leq \|N(Y_q, \tilde{Y})\|_0 \|\gamma_\alpha \tilde{Y}\|_0 \leq C\|\tilde{Y}\|_0\|\tilde{Y}\|_\alpha;$$

(2) and (3) are proved like Proposition 7.5(1) and (2). ∎

PROPOSITION 7.7. *The formula* (4.8) *for* $G(\tilde{Y}, q)$ *defines mappings from* $\mathcal{E}_\beta^n \times \mathbb{R}$ *to* $\mathcal{E}_\beta^n$ *and from* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$ *to* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$. *The formula* (4.9) *for* $\kappa(\tilde{Y}, q)$ *defines a mapping from* $\mathcal{E}_\beta^n \times \mathbb{R}$ *to* $\mathbb{R}$. *Each of these mappings is Lipschitz on any bounded neighborhood of* $(0,0)$ *in its domain space. Moreover, there is a constant* $C$ *such that*

(1) $\|G(\tilde{Y}, q)\|_\alpha \leq C(\|\tilde{Y}\|_0 + |q|)\|\tilde{Y}\|_\alpha$;

(2) $|\kappa(\tilde{Y}, q)| \leq C(\|\tilde{Y}\|_0 + |q||)\|\tilde{Y}\|_\alpha$.

*Proof.* The Lipschitz statement follows from Proposition 7.4(2) and (3) and Proposition 7.6(3). The proof of (1) follows from (4.8) together with Proposition 7.4(2) and Proposition 7.6(1). For (2), note that

$$|\kappa(\tilde{Y}, q)| = |\pi Y_*'(\xi - q)|^{-1} |\pi G(\tilde{Y}, q)|.$$

By Lemma 4.1, $|\pi Y_*'(\xi - q)|^{-1} \leq 2$, and $|\pi G(\tilde{Y}, q)|$ is bounded by a constant times the bound on $\|G(\tilde{Y}, q)\|_\alpha$ given by (1). ∎

**8. Estimates for the nonlinear operator $N$.** We continue to let $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$.

LEMMA 8.1. *We have the following.*

(1) $(DR(Y_*) - DR(0))\gamma_\alpha^{-1}$ *is in* $C^1(\mathbb{R})^n$, $\mathcal{E}_0^n$, *and* $L^1(\mathbb{R})^n$.

(2) *For each* $q \in \mathbb{R}$, $\gamma_\alpha^{-1}(\xi)Y_*'(\xi - q)$ *is in* $C^1(\mathbb{R})^n$, $\mathcal{E}_0^n$, *and* $L^1(\mathbb{R})^n$.

*Proof.* Statement (1) follows from

$$(DR(Y_*) - DR(0))\gamma_\alpha^{-1} = \left(\int_0^1 D^2R(tY_*)\, dt\right) Y_*\gamma_\alpha^{-1}$$

and Lemma 3.7(1). To see (2), note that for $|\xi|$ large,

$$\gamma_\alpha^{-1}(\xi)Y_*'(\xi - q) = \frac{\gamma_\alpha(\xi - q)}{\gamma_\alpha(\xi)}\gamma_\alpha^{-1}(\xi - q)Y_*'(\xi - q) = \gamma_\alpha(-q)\gamma_\alpha^{-1}(\xi - q)Y_*'(\xi - q)$$

and use Lemma 3.7(2). ∎

LEMMA 8.2. *There is a constant* $K > 0$ *such that*

(1) $\|(DR(Y_*) - DR(0))\tilde{Y}\|_0 \leq K\|\tilde{Y}\|_\alpha$;

(2) $\|(DR(Y_*) - DR(0))\tilde{Y}\|_{L^1} \leq K\|\tilde{Y}\|_\alpha$;

(3) $\|(DR(Y_q) - DR(Y_*))\tilde{Y}\|_0 \leq K|q|\|\tilde{Y}\|_\alpha$;

(4) $\|(DR(Y_q) - DR(Y_*))\tilde{Y}\|_{L^1} \leq K|q|\|\tilde{Y}\|_\alpha$.

*Proof.* To see (1) and (2), write

$$(DR(Y_*) - DR(0))\tilde{Y} = (DR(Y_*) - DR(0))\gamma_\alpha^{-1}\gamma_\alpha\tilde{Y}.$$

By Lemma 8.1(1),

$$\|(DR(Y_*) - DR(0))\tilde{Y}\|_0 \leq \|(DR(Y_*) - DR(0))\gamma_\alpha^{-1}\|_0\|\gamma_\alpha\tilde{Y}\|_0 = K\|\tilde{Y}\|_\alpha,$$

and

$$\begin{aligned}\|(DR(Y_*) - DR(0))\tilde{Y}\|_{L^1} &\leq \|(DR(Y_*) - DR(0))\gamma_\alpha^{-1}\|_{L^1}\|\gamma_\alpha\tilde{Y}\|_{L^\infty} \\ &\leq \|(DR(Y_*) - DR(0))\gamma_\alpha^{-1}\|_{L^1}\|\gamma_\alpha\tilde{Y}\|_0 \leq K\|\tilde{Y}\|_\alpha.\end{aligned}$$

To see (3) and (4), write

$$\begin{aligned}(DR(Y_*(\xi - q)) - DR(Y_*(\xi)))\tilde{Y} &= -q\int_0^1 D^2R(Y_*(\xi - sq))Y_*'(\xi - sq)\tilde{Y}\, ds \\ &= -q\int_0^1 D^2R(Y_*(\xi - sq))\gamma_\alpha^{-1}Y_*'(\xi - sq)\gamma_\alpha\tilde{Y}\, ds.\end{aligned}$$

By Lemma 8.1(2), for each $s$, $\gamma_\alpha^{-1}(\xi)Y_*'(\xi - sq)$ is in $\mathcal{E}_0^n$ and $L^1(\mathbb{R})^n$. The remainder of the argument is similar to the proof of (1) and (2). ∎

LEMMA 8.3.

(1) *For each bounded neighborhood* $\mathcal{N}_0$ *of* $(0,0)$ *in* $\mathcal{E}_\beta^n \times \mathbb{R}$, *there exists a constant* $K > 0$ *such that for all* $(\tilde{Y}, p) \in \mathcal{N}_0$,

$$\|N(Y_q, \tilde{Y})\tilde{Y}\|_0 \leq K\|\tilde{Y}\|_0\left(\|\tilde{Y}\|_\alpha + \|\tilde{V}\|_0\right). \tag{8.1}$$

(2) *For each bounded neighborhood* $\mathcal{N}_1$ *of* $(0,0)$ *in* $(\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n \times \mathbb{R}$*, there exists a constant* $K > 0$ *such that for all* $(\tilde{Y}, q) \in \mathcal{N}_1$*,*

$$\|N(Y_q,\tilde{Y})\tilde{Y}\|_{L^1} \le K\|\tilde{Y}\|_0 \left(\|\tilde{Y}\|_\alpha + \|\tilde{V}\|_{L^1}\right). \tag{8.2}$$

*Proof.* Let

$$r(U,V) = \int_0^1 D_V R(U,tV)\,dt,$$

an $n \times n$ matrix that is a $C^2$ function of $(U,V)$. By Hypothesis 3.9,

$$R(U,V) = R(U,0) + R(U,V) - R(U,0) = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} U + r(U,V)V.$$

Therefore

$$\begin{aligned} D_U R(U,V)\tilde{U} &= \begin{pmatrix} A_1 \\ 0 \end{pmatrix} \tilde{U} + \left(D_U r(U,V)\tilde{U}\right) V, \\ D_V R(U,V)\tilde{V} &= \left(D_V r(U,V)\tilde{V}\right) V + r(U,V)\tilde{V}. \end{aligned}$$

Then

$$\begin{aligned} N(Y_q,\tilde{Y})\tilde{Y} &= \int_0^1 DR(Y_q + t\tilde{Y})\tilde{Y} - DR(Y_q)\tilde{Y}\,dt \\ &= \int_0^1 D_U R(Y_q + t\tilde{Y})\tilde{U} - D_U R(Y_q)\tilde{U}\,dt + \int_0^1 D_V R(Y_q + t\tilde{Y})\tilde{V} \\ &\quad - D_V R(Y_q)\tilde{V}\,dt \\ &= \int_0^1 \left(D_U r(Y_q + t\tilde{Y})\tilde{U}\right)(V_q + t\tilde{V}) - \left(D_U r(Y_q)\tilde{U}\right) V_q\,dt \\ &\quad + \int_0^1 \left(D_V r(Y_q + t\tilde{Y})\tilde{V}\right)(V_q + t\tilde{V}) - \left(D_V r(Y_q)\tilde{V}\right) V_q\,dt \\ &\quad + \int_0^1 \left(r(Y_q + t\tilde{Y}) - r(Y_q)\right)\tilde{V}\,dt \\ &= \int_0^1 \left(D_U r(Y_q + t\tilde{Y}) - D_U r(Y_q)\right)\tilde{U}V_q\,dt + \int_0^1 \left(D_U r(Y_q + t\tilde{Y})\tilde{U}\right) t\tilde{V}\,dt \\ &\quad + \int_0^1 \left(D_V r(Y_q + t\tilde{Y}) - D_V r(Y_q)\right)\tilde{V}V_q\,dt + \int_0^1 \left(D_V r(Y_q + t\tilde{Y})\tilde{V}\right) t\tilde{V}\,dt \\ &\quad + \int_0^1 \left(r(Y_q + t\tilde{Y}) - r(Y_q)\right)\tilde{V}\,dt. \end{aligned}$$

Thus $N(Y_q,\tilde{Y})\tilde{Y}$ is a sum of five integrals.

To estimate $\|N(Y_q,\tilde{Y})\tilde{Y}\|_0$, we note that if $(\tilde{Y},q) \in \mathcal{N}_0$, then in $\|\ \|_0$ the second through fifth integrals is each at most a constant times $\|\tilde{Y}\|_0\|\tilde{V}\|_0$. Similarly, to estimate $\|N(Y_q,\tilde{Y})\tilde{Y}\|_{L^1}$, we note that if $(\tilde{Y},q) \in \mathcal{N}_1$, then in $\|\ \|_{L^1}$ the second through fifth integrals is each at most a constant times $\|\tilde{Y}\|_0\|\tilde{V}\|_{L^1}$.

Finally, the first integral can be rewritten as

$$\int_0^1 \left(D_U r(Y_q + t\tilde{Y}) - D_U r(Y_q)\right)\tilde{U}V_q\,dt = \int_0^1 \left(D_U r(Y_q + t\tilde{Y}) - D_U r(Y_q)\right)\gamma_\alpha\tilde{U}\gamma_\alpha^{-1}V_q\,dt.$$

By Hypothesis 3.6(1), $\gamma_\alpha^{-1}V_q$ is in $C^1(\mathbb{R})^{n_1}$. Therefore in $\| \|_0$, the integral is at most $K\|\tilde{Y}\|_0\|\gamma_\alpha\tilde{U}\|_0 = K\|\tilde{Y}\|_0\|\tilde{U}\|_\alpha \leq K\|\tilde{Y}\|_0\|\tilde{Y}\|_\alpha$. Also, by Hypothesis 3.6(1), $\gamma_\alpha^{-1}V_q$ is in $L^1(\mathbb{R})^{n_1}$. Therefore in $\| \|_{L^1}$, the integral is at most

$$K\|\tilde{Y}\|_{L^\infty}\|\gamma_\alpha\tilde{U}\|_{L^\infty} \leq K\|\tilde{Y}\|_0\|\tilde{U}\|_\alpha \leq K\|\tilde{Y}\|_0\|\tilde{Y}\|_\alpha. \qquad \square$$

**9. Stability of traveling waves in an exothermic-endothermic reaction.** In [22, 23, 24], Simon et al. study the system

$$\partial_t z_1 = \partial_{xx} z_1 + z_2 f_2(z_1) - \sigma z_3 f_3(z_1), \tag{9.1}$$

$$\partial_t z_2 = d_2\partial_{xx} z_2 - z_2 f_2(z_1), \tag{9.2}$$

$$\partial_t z_3 = d_3\partial_{xx} z_3 - \tau z_3 f_3(z_1). \tag{9.3}$$

Here $z_1$ is temperature, $z_2$ is concentration of an exothermic reactant, and $z_3$ is concentration of an endothermic reactant. The parameters $d_2$, $d_3$, $\sigma$, and $\tau$ are positive, and there are positive constants $a_i$ and $b_i$ such that

$$f_i(u) = \begin{cases} a_i e^{-\frac{b_i}{u}} & \text{for } u > 0, \\ 0 & \text{for } u \leq 0. \end{cases}$$

We have changed the notation of Simon et al. a little to fit with ours. Simon et al. study existence of traveling waves for this system, and they study the discrete spectrum of the linearization at a traveling wave using the Evans function. We shall use our Theorem 3.14 to show what sort of stability is implied by their work.

The change of variables $\xi = x - ct$, $c > 0$, converts (9.1)–(9.3) to

$$\partial_t z_1 = \partial_{\xi\xi} z_1 + c\partial_\xi z_1 + z_2 f_2(z_1) - \sigma z_3 f_3(z_1), \tag{9.4}$$

$$\partial_t z_2 = d_2\partial_{\xi\xi} z_2 + c\partial_\xi z_2 - z_2 f_2(z_1), \tag{9.5}$$

$$\partial_t z_3 = d_3\partial_{\xi\xi} z_3 + c\partial_\xi z_3 - \tau z_3 f_3(z_1). \tag{9.6}$$

Let $Z_*(\xi)$ be a stationary solution of (9.4)–(9.6), i.e., a traveling wave solution of (9.1)–(9.3) with speed $c > 0$, with $Z_- = (z, 0, 0)$, $z > 0$, and $Z_+ = (0, 1, 1)$. It turns out that $z = 1 - \frac{\sigma}{\tau}$, so we must have $\sigma < \tau$. Simon et al. show numerically that in certain parameter regimes, such traveling waves exist for which both end states are approached at an exponential rate. (See [22, p. 544], for a discussion of why their numerical method should find traveling waves with this property.)

With $z = 1 - \frac{\sigma}{\tau}$, the change of variables $y_1 = z_1 - z$, $y_2 = z_2$, and $y_3 = z_3$ converts (9.4)–(9.6) to the system

$$\partial_t y_1 = \partial_{\xi\xi} y_1 + c\partial_\xi y_1 + y_2 f_2(z + y_1) - \sigma y_3 f_3(z + y_1), \tag{9.7}$$

$$\partial_t y_2 = d_2\partial_{\xi\xi} y_2 + c\partial_\xi y_2 - y_2 f_2(z + y_1), \tag{9.8}$$

$$\partial_t y_3 = d_3\partial_{\xi\xi} y_3 + c\partial_\xi y_3 - \tau y_3 f_3(z + y_1). \tag{9.9}$$

We write (9.7)–(9.9) as

$$\partial_t Y = D\partial_{\xi\xi} Y + c\partial_\xi Y + R(Y), \tag{9.10}$$

where

$$R(Y) = S(z + y_1, y_2, y_3) = (y_2 f_2(z + y_1) - \sigma y_3 f_3(z + y_1), -y_2 f_2(z + y_1), -\tau y_3 f_3(z + y_1)). \tag{9.11}$$

Let $Y_*(\xi) = (y_{1*}(\xi), y_{2*}(\xi), y_{3*}(\xi))$ be the stationary solution of (9.7) that corresponds to $Z_*(\xi)$ so that $Y_- = (0,0,0)$ (as required by our setup) and $Y_+ = (-z, 1, 1)$.

The linearization of (9.10) at $Y_*(\xi)$ is

$$\partial_t \tilde{Y} = L\tilde{Y} = D\partial_{\xi\xi}\tilde{Y} + c\partial_\xi \tilde{Y} + DR(Y_*(\xi))\tilde{Y}, \tag{9.12}$$

where

(9.13)

$$DR(Y_*(\xi))$$
$$= \begin{pmatrix} y_{2*}(\xi)f_2'(z+y_{1*}(\xi)) - \sigma y_{3*}(\xi)f_3'(z+y_{1*}(\xi)) & f_2(z+y_{1*}(\xi)) & -\sigma f_3(z+y_{1*}(\xi)) \\ -y_{2*}(\xi)f_2'(z+y_{1*}(\xi)) & -f_2(z+y_{1*}(\xi)) & 0 \\ -\tau y_{3*}(\xi)f_3'(z+y_{1*}(\xi)) & 0 & -\tau f_3(z+y_{1*}(\xi)) \end{pmatrix}.$$

THEOREM 9.1. *Suppose the constants $d_2$, $d_3$, $\sigma$, $\tau$, $a_i$, $b_i$, and $c > 0$ in* (9.10) *are chosen so that there is a stationary solution $Y_*(\xi)$ that approaches 0 exponentially as $\xi \to -\infty$ and approaches $Y_+ = (-z, 1, 1)$, $z = 1 - \frac{\sigma}{\tau} > 0$, exponentially as $\xi \to \infty$. Let $\alpha = (\alpha_-, \alpha_+)$ be as described in subsection* 9.4 *below; in particular $\alpha_- > 0$ and $\alpha_+ > 0$. Assume that Hypothesis* 3.6(3b) *holds. Let $\beta = (0, \alpha_+)$, and let $\mathcal{E}_0 = H^1(\mathbb{R})$ or $BUC(\mathbb{R})$. Suppose $Y^0 \in Y_* + \mathcal{E}_\beta^3$ with $\|Y^0 - Y_*\|_\beta$ small, and let $Y(t)$ be the solution of* (9.10) *in $Y_* + \mathcal{E}_\beta^3$ with $Y(0) = Y^0$. Then the following are true.*

(1) *$Y(t)$ is defined for all $t \geq 0$.*
(2) *$Y(t) = \tilde{Y}(t) + Y_*(\xi - q(t))$ with $\tilde{Y}(t)$ in a fixed subspace of $\mathcal{E}_\beta^3$ complementary to the span of $Y_*'$.*
(3) *$\|\tilde{Y}(t)\|_\beta + |q(t)|$ is small for all $t \geq 0$.*
(4) *$\|\tilde{Y}(t)\|_\alpha$ decays exponentially as $t \to \infty$.*
(5) *There exists $q^*$ such that $|q(t) - q^*|$ decays exponentially as $t \to \infty$.*
(6) *There is a constant $C$ independent of $Y^0$ such that $\|\tilde{y}_1(t)\|_0 \leq C\|\tilde{Y}^0\|_\beta$ for all $t \geq 0$.*
(7) *$\|(\tilde{y}_2, \tilde{y}_3)(t)\|_0$ decays exponentially as $t \to \infty$.*

*In addition, suppose $Y^0 \in Y_* + (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$ with $\|Y^0 - Y_*\|_\beta$ and $\|Y^0 - Y_*\|_{L^1}$ small. Then the following are true.*

(8) *$Y(t) \in (\mathcal{E}_\beta \cap L^1(\mathbb{R}))^n$ for all $t \geq 0$.*
(9) *$\|\tilde{y}_1(t)\|_{L^1}$ is small for all $t \geq 0$.*
(10) *$\|\tilde{y}_1(t)\|_{L^\infty}$ decays like $t^{-\frac{1}{2}}$ as $t \to \infty$.*
(11) *$\|(\tilde{y}_2, \tilde{y}_3)(t)\|_{L^1}$ decays exponentially as $t \to \infty$.*

This result follows from Theorems 3.14 and 3.16 by verifying their hypotheses. The steps are easy and are carried out below, except for the verification of Hypothesis 3.6(3b). This requires a numerical study of the Evans function, an analytic function whose zeros are eigenvalues of $\mathcal{L}_\alpha$. Such a study was carried out in [23, 24]. The point of the theorem, as discussed in the introduction, is that it shows the rather detailed information that such a study can yield about stability of the traveling wave.

**9.1. Traveling waves.** Let us briefly discuss the intuitive reason that traveling waves of (9.1)–(9.3) exist, which is related to a first integral that Simon et al. [22, 23, 24] don't mention. The traveling wave equation for (9.1)–(9.3), written as a first-order system, is

$$Z_\xi = V, \tag{9.14}$$

$$V_\xi = D^{-1}(-cV + S(Z)), \tag{9.15}$$

where $D = \mathrm{diag}(1, d_2, d_3)$ and

$$S(Z) = (z_2 f_2(z_1) - \sigma z_3 f_3(z_1), -z_2 f_2(z_1), -\tau z_3 f_3(z_1)).$$

Consider (9.4)–(9.6) with the left-hand side of each equation set to 0. After this substitution, if we add (9.4), (9.5), and $-\frac{\sigma}{\tau}$(9.6), we obtain

$$\partial_{\xi\xi} z_1 + c\partial_\xi z_1 + d_2 \partial_{\xi\xi} z_2 + c\partial_\xi z_2 - \frac{\sigma}{\tau} d_3 \partial_{\xi\xi} z_3 - \frac{\sigma}{\tau} c\partial_\xi z_3 = 0.$$

This expression can be integrated once to produce a function of $\xi$ that is constant along any traveling wave. Analogously, along any solution of (9.14)–(9.15) we have

$$v_1 + cz_1 + d_2 v_2 + cz_2 - \frac{\sigma}{\tau} d_3 v_3 - \frac{\sigma}{\tau} cz_3 = k.$$

For the solution that approaches $(z_1, z_2, z_3, v_1, v_2, v_3) = (0, 1, 1, 0, 0, 0)$ as $\xi \to \infty$, we must have $k = c(1 - \frac{\sigma}{\tau})$.

To take advantage of these facts, we consider (9.14)–(9.15) on the invariant surface

$$v_1 = -cz_1 - d_2 v_2 - cz_2 + \frac{\sigma}{\tau} d_3 v_3 + \frac{\sigma}{\tau} cz_3 + c\left(1 - \frac{\sigma}{\tau}\right). \tag{9.16}$$

Using $(z_1, z_2, z_3, v_2, v_3)$ as variables, we obtain

$$\dot z_1 = -cz_1 - d_2 v_2 - cz_2 + \frac{\sigma}{\tau} d_3 v_3 + \frac{\sigma}{\tau} cz_3 + c\left(1 - \frac{\sigma}{\tau}\right), \tag{9.17}$$
$$\dot z_2 = v_2, \tag{9.18}$$
$$\dot z_3 = v_3, \tag{9.19}$$
$$\dot v_2 = d_2^{-1}(-cv_2 + z_2 f_2(z_1)), \tag{9.20}$$
$$\dot v_3 = d_3^{-1}(-cv_3 + \tau z_3 f_2(z_1)). \tag{9.21}$$

This system has equilibria at $(z_1, z_2, z_3, v_2, v_3) = (z, 0, 0, 0, 0)$ with $z = 1 - \frac{\sigma}{\tau}$, which corresponds to the equilibrium $(Z_-, 0)$ of (9.14)–(9.15), and $(0, z_2, z_3, 0, 0)$ with $z_2 - \frac{\sigma}{\tau} z_3 = 1 - \frac{\sigma}{\tau}$, which correspond to the line of equilibria $(0, z_2, z_3, 0, 0, 0)$ of (9.14)–(9.15). One of the equilibria on this line is $(0, 1, 1, 0, 0, 0) = (Z_+, 0)$.

The linearization of (9.17)–(9.21) at $(z, 0, 0, 0, 0)$ has two eigenvalues with positive real part and three with negative real part; at $(0, 1, 1, 0, 0)$ there are two eigenvalues with 0 real part and three with negative real part. We therefore expect that in the five-dimensional state space of (9.17)–(9.21), for isolated values of $c$ the two-dimensional unstable manifold of $(z, 0, 0, 0, 0)$ and the three-dimensional stable manifold of $(0, 1, 1, 0, 0)$ will intersect, producing a traveling wave that approaches both end states exponentially.

**9.2. Stability of end states in weighted spaces.** Let

$$\phi_2 = f_2(z) > 0, \quad \phi_3 = \tau f_3(z) > 0.$$

With this notation, the linearization of (9.10) at $Y_- = (0, 0, 0)$ is

$$\begin{pmatrix} \tilde y_{1t} \\ \tilde y_{2t} \\ \tilde y_{1t} \end{pmatrix} = L^- \begin{pmatrix} \tilde y_1 \\ \tilde y_2 \\ \tilde y_3 \end{pmatrix} = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi & 0 & 0 \\ 0 & d_2\partial_{\xi\xi} + c\partial_\xi - \phi_2 & 0 \\ 0 & 0 & d_3\partial_{\xi\xi} + c\partial_\xi - \phi_3 \end{pmatrix} \begin{pmatrix} \tilde y_1 \\ \tilde y_2 \\ \tilde y_3 \end{pmatrix}. \tag{9.22}$$

If $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ belongs to a weighted $L^2$ space with weight function $e^{\upsilon\xi}$, then $(\tilde{y}_1(\xi), \tilde{y}_2(\xi), \tilde{y}_3(\xi)) = e^{-\upsilon\xi}(\tilde{w}_1(\xi), \tilde{w}_2(\xi), \tilde{w}_3(\xi))$ with $(\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)$ in $L^2(\mathbb{R})^3$. Substituting into the formula for $L^-$ and multiplying by $e^{\upsilon\xi}$, we obtain the linear differential expression

$$\hat{L}^-\tilde{W} = (L^- - 2\upsilon\partial_\xi + \upsilon^2 D - c\upsilon I)\tilde{W}. \tag{9.23}$$

(Compare the discussion preceding Lemma 3.5.) Using the Fourier transform, we find that the spectrum of the operator associated with $\hat{L}^-$ on $L^2(\mathbb{R})^3$ is the union of the three curves $\lambda = -\theta^2 + (c-2\upsilon)i\theta + \upsilon^2 - c\upsilon$ and $\lambda = -d_k\theta^2 + (c-2\upsilon)i\theta + d_k\upsilon^2 - c\upsilon - \phi_k$, $k = 2, 3$, $\theta \in \mathbb{R}$. With a small abuse of notation, we use $\upsilon$ as shorthand for $(\upsilon, \upsilon)$. Then

$$\begin{aligned} \sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_\upsilon^-)\} &= \sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\hat{\mathcal{L}}_0^-)\} \\ &= \max\left(\upsilon^2 - c\upsilon, d_2\upsilon^2 - c\upsilon - \phi_2, d_3\upsilon^2 - c\upsilon - \phi_3\right), \end{aligned} \tag{9.24}$$

which is 0 for $\upsilon = 0$ but is negative for $\upsilon > 0$ sufficiently small.

Similarly, the linearization of (9.10) at $Y_+ = (-z, 1, 1)$ is

$$\begin{pmatrix} \tilde{y}_{1t} \\ \tilde{y}_{2t} \\ \tilde{y}_{1t} \end{pmatrix} = L^+ \begin{pmatrix} \tilde{u} \\ \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} \partial_{\xi\xi} + c\partial_\xi & 0 & 0 \\ 0 & d_2\partial_{\xi\xi} + c\partial_\xi & 0 \\ 0 & 0 & d_3\partial_{\xi\xi} + c\partial_\xi \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}. \tag{9.25}$$

Substituting $(\tilde{y}_1(\xi), \tilde{y}_2(\xi), \tilde{y}_3(\xi)) = e^{-\upsilon\xi}(\tilde{w}_1(\xi), \tilde{w}_2(\xi), \tilde{w}_3(\xi))$ into the formula for $L^+$ and multiplying by $e^{\upsilon\xi}$, we obtain the linear differential expression

$$\hat{L}^+\tilde{W} = (L^+ - 2\upsilon\partial_\xi + \upsilon^2 D - c\upsilon I)\tilde{W}. \tag{9.26}$$

We shall use the notation $d_1 = 1$ when it seems to result in simpler expressions. Using the Fourier transform, we find that the spectrum of the operator associated with $\hat{L}^+$ on $L^2(\mathbb{R})^3$ is the union of the three curves $\lambda = -d_k\theta^2 + (c - 2\upsilon)i\theta + d_k\upsilon^2 - c\upsilon$, $k = 1, 2, 3$, $\theta \in \mathbb{R}$. Then

$$\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_\upsilon^+)\} = \sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\hat{\mathcal{L}}_0^+)\} = \max_{k=1,2,3}\left(d_k\upsilon^2 - c\upsilon\right), \tag{9.27}$$

which again is 0 for $\upsilon = 0$ but is negative for $\upsilon > 0$ sufficiently small.

**9.3. Eigenvalue equation.** The eigenvalue equation for $L$ is $L\tilde{Y} = \lambda\tilde{Y}$, which we express as a first-order system:

$$\begin{pmatrix} \tilde{Y}_\xi \\ \tilde{Z}_\xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ D^{-1}(\lambda I - DR(Y_*(\xi)) & -cD^{-1} \end{pmatrix} \begin{pmatrix} \tilde{Y} \\ \tilde{Z} \end{pmatrix}. \tag{9.28}$$

As $\xi \to \pm\infty$, the linear system (9.28) approaches the constant-coefficient linear systems

$$\begin{pmatrix} \tilde{Y}_\xi \\ \tilde{Z}_\xi \end{pmatrix} = \begin{pmatrix} 0 & I \\ D^{-1}(\lambda I - DR(Y_\pm)) & -cD^{-1} \end{pmatrix} \begin{pmatrix} \tilde{Y} \\ \tilde{Z} \end{pmatrix}. \tag{9.29}$$

Eigenvalues $\mu$ and corresponding eigenvectors $(\tilde{Y}, \tilde{Z})$ of (9.29) satisfy the equations $\tilde{Z} = \mu\tilde{Y}$ and

$$\left(D^{-1}(\lambda I - DR(Y_\pm) - c\mu I) - \mu^2 I\right)\tilde{Y} = 0.$$

Therefore

$$\det\left(D^{-1}(\lambda I - DR(Y_\pm) - c\mu I) - \mu^2 I\right) = 0. \tag{9.30}$$

Now

$$DR(Y_-) = \begin{pmatrix} 0 & \phi_2 & -\sigma f_3(z) \\ 0 & -\phi_2 & 0 \\ 0 & 0 & -\phi_3 \end{pmatrix} \quad \text{and} \quad DR(Y_+) = 0.$$

Therefore at $Y_-$, (9.30) becomes

$$(\lambda - c\mu - \mu^2)(d_2^{-1}(\lambda + \phi_2 - c\mu) - \mu^2)(d_3^{-1}(\lambda + \phi_3 - c\mu) - \mu^2) = 0, \tag{9.31}$$

and at $Y_+$, (9.30) becomes

$$(\lambda - c\mu - \mu^2)(d_2^{-1}(\lambda - c\mu) - \mu^2)(d_3^{-1}(\lambda - c\mu) - \mu^2) = 0. \tag{9.32}$$

Hence at $Y_-$ the eigenvalues of (9.29) are

$$\mu_{-1\pm} = \frac{1}{2}\left(-c \pm \left(c^2 + 4\lambda\right)^{\frac{1}{2}}\right),$$
$$\mu_{-k\pm} = \frac{1}{2d_k}\left(-c \pm \left(c^2 + 4d_k\left(\lambda + \phi_k\right)\right)^{\frac{1}{2}}\right), \quad k = 1,\, 2.$$

(We shall always use $a^{\frac{1}{2}}$ to indicate a square root of $a$ with nonnegative real part.) At $Y_+$, the eigenvalues of (9.29) are

$$\mu_{+k\pm} = \frac{1}{2d_k}\left(-c \pm \left(c^2 + 4d_k\lambda\right)^{\frac{1}{2}}\right), \quad k = 1,\, 2,\, 3.$$

For $\lambda = 0$, note that at $Y_-$, all three $\mu_{-k-}$'s are negative, $\mu_{-1+} = 0$, and $\mu_{-2+}$ and $\mu_{-3+}$ are positive; at $Y_+$, all three $\mu_{+k-}$'s are negative, and all three $\mu_{+k+}$'s are 0. The six numbers $\mu_{-k-}$, $\mu_{-k+}$ (respectively, $\mu_{+k-}$, $\mu_{+k+}$) are also the eigenvalues of the linearization of (9.14)–(9.15) at the equilibrium $(Z_-, 0)$ (respectively, $(Z_+, 0)$).

If we drop one 0 from each of these lists of six eigenvalues, we obtain, respectively, the five eigenvalues of the linearization of (9.17)–(9.21) at $(z, 0, 0, 0, 0)$ and $(0, 1, 1, 0, 0)$. This justifies the assertions at the end of subsection 9.1.

We shall use the following elementary lemma.

Lemma 9.2. *Consider the quadratic equation $d_k\mu^2 + c\mu - b = 0$ with $d_k > 0$, $c > 0$, and $b \in \mathbb{C}$. The roots are*

$$\mu_\pm = \frac{1}{2d_k}\left(-c \pm \left(c^2 + 4d_k b\right)^{\frac{1}{2}}\right).$$

*Choose $\chi_- < \chi_+$ with $\chi_- < 0$. Then*

(1) *if $\operatorname{Re} b \geq \chi_-^2 + c\chi_-$, then $\operatorname{Re}\mu_- \leq \chi_-$;*

(2) *if $\operatorname{Re} b \geq \chi_+^2 + c\chi_+$, then $\operatorname{Re}\mu_+ \geq \chi_+$.*

Choose real numbers $\chi_- < \chi_+$ with $\chi_- < 0$. From Lemma 9.2 we have the following:

$$\operatorname{Re}\mu_{-1-} \leq \chi_- \text{ if } \operatorname{Re}\lambda \geq \chi_-^2 + c\chi_-, \tag{9.33}$$

$$\operatorname{Re}\mu_{-1+} \geq \chi_+ \text{ if } \operatorname{Re}\lambda \geq \chi_+^2 + c\chi_+, \tag{9.34}$$

$$\operatorname{Re}\mu_{-k-} \leq \chi_- \text{ if } \operatorname{Re}\lambda \geq d_k\chi_-^2 + c\chi_- - \phi_k, \quad k = 2,\, 3, \tag{9.35}$$

$$\operatorname{Re}\mu_{-k+} \geq \chi_+ \text{ if } \operatorname{Re}\lambda \geq d_k\chi_+^2 + c\chi_+ - \phi_k, \quad k = 2,\, 3, \tag{9.36}$$

$$\operatorname{Re}\mu_{+k-} \leq \chi_- \text{ if } \operatorname{Re}\lambda \geq d_k\chi_-^2 + c\chi_-, \quad k = 1,\, 2,\, 3, \tag{9.37}$$

$$\operatorname{Re}\mu_{+k+} \geq \chi_+ \text{ if } \operatorname{Re}\lambda \geq d_k\chi_+^2 + c\chi_+, \quad k = 1,\, 2,\, 3. \tag{9.38}$$

**9.4. Proof of Theorem 9.1.** To prove Theorem 9.1, we just need to verify the hypotheses of Theorems 3.14 and 3.16, other than Hypothesis 3.6(3b), which is assumed to hold.

The function $R$ defined by (9.11) is $C^\infty$, so Hypothesis 3.1 is satisfied.

Let let $-\omega_-$ denote the minimum of the two positive eigenvalues of the linearization of (9.14)–(9.15) at $(Z_-,0)$, and let $-\omega_+$ denote the maximum of the three negative eigenvalues of the linearization of (9.14)–(9.15) at $(Z_+,0)$. Then

$$\begin{aligned}
&\omega_- = -\min\left(\frac{1}{2d_2}\left(-c+\left(c^2+4d_2\phi_2\right)^{\frac12}\right), \frac{1}{2d_3}\left(-c+\left(c^2+4d_3\phi_3\right)^{\frac12}\right)\right) < 0,\\
&\omega_+ = \min\left(c, d_2^{-1}c, d_3^{-1}c\right) > 0.
\end{aligned}$$

With these values of $\omega_-$ and $\omega_+$, Hypothesis 3.2 is satisfied. (However, if the two positive eigenvalues of the linearization of (9.14)–(9.15) at $(Z_-,0)$ are equal, then $\omega_-$ should be increased slightly.)

Let $\alpha = (\alpha_-,\alpha_+)$, with $0 < \alpha_- < \min(c,-\omega_-)$ and $0 < \alpha_+ < \omega_+$, so that Hypothesis 3.6(1) and (2) are satisfied. Since $0 < \alpha_+ < \omega_+$, we see immediately that (9.27) with $\upsilon = \alpha_+$ is negative. Moreover,

$$0 < \alpha_- < \min\left(c, \frac{1}{2d_2}\left(c+\left(c^2+4d_2\phi_2\right)^{\frac12}\right), \frac{1}{2d_3}\left(c+\left(c^2+4d_3\phi_3\right)^{\frac12}\right)\right),$$

so (9.24) with $\upsilon = \alpha_-$ is also negative. Therefore Hypothesis 3.6(3a) is satisfied with

$$\begin{aligned}
\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}_{\mathrm{ess}}(\mathcal{L}_\alpha)\} = \max(&\alpha_-^2 - c\alpha_-, d_2\alpha_-^2 - c\alpha_- - \phi_2,\\
&d_3\alpha_-^2 - c\alpha_- - \phi_3, \alpha_+^2 - c\alpha_+, d_2\alpha_+^2 - c\alpha_+, d_3\alpha_+^2 - c\alpha_+).
\end{aligned} \tag{9.39}$$

We decompose $Y$-space as follows: $Y = (U,V)$ with $U = y_1$ and $V = (y_2,y_3)$. Since $R(y_1,0,0) = (0,0,0)$ from (9.11), Hypothesis 3.9 is satisfied with $A_1 = 0$. From (9.22) we have

$$L^{(1)} = \partial_{\xi\xi} + c\partial_\xi, \quad L^{(2)} = \begin{pmatrix} d_2\partial_{\xi\xi} + c\partial_\xi - \phi_2 & 0 \\ 0 & d_3\partial_{\xi\xi} + c\partial_\xi - \phi_3 \end{pmatrix}. \tag{9.40}$$

The semigroup on $L^2(\mathbb{R})$ or $BUC(\mathbb{R})$ generated by the operator associated with $L^{(1)}$ satisfies Hypotheses 3.10(1) and 3.15. The operator on $L^2(\mathbb{R})^2$ associated with $L^{(2)}$ has for its spectrum the union of the two curves $\lambda = -d_k\theta^2 + ci\theta - \phi_k$, $k = 2, 3$, $\theta \in \mathbb{R}$. Thus Hypothesis 3.10(2) is satisfied with

$$\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_0^{(2)})\} = \max(-\phi_2, -\phi_3). \tag{9.41}$$

**9.5. Discrete spectrum and the Evans function.** To verify Hypothesis 3.6(3b) we must consider eigenvalues $\lambda$ of $\mathcal{L}_\alpha$ on $L^2_\alpha(\mathbb{R})^3$. Their eigenfunctions are the $\tilde{Y}$-components of solutions $(\tilde{Y},\tilde{Z})$ of (9.28) such that $\gamma_\alpha(\xi)(\tilde{Y}(\xi),\tilde{Z}(\xi))$ decays exponentially as $\xi \to \pm\infty$.

Suppose we choose $\chi_- < \chi_+ < 0$ so that $\chi^2+c\chi$, $d_2\chi^2+c\chi-\phi_2$, and $d_3\chi^2+c\chi-\phi_3$, with $\chi = \chi_\pm$, are all negative, and let $-\nu_- < 0$ be the maximum of these six numbers. By (9.33)–(9.36), if $\operatorname{Re}\lambda \geq -\nu_-$, then for $k = 1, 2, 3$, $\operatorname{Re}\mu_{-k-} \leq \chi_-$ and $\operatorname{Re}\mu_{-k+} \geq \chi_+$.

In particular, let $\chi_+ = -\alpha_- < 0$. Then $\chi_+^2 + c\chi_+$, $d_2\chi_+^2 + c\chi_+ - \phi_2$, and $d_3\chi_+^2 + c\chi_+ - \phi_3$ are all negative. Choose $\chi_- < \chi_+$ such that $\chi_-^2 + c\chi_-$, $d_2\chi_-^2 + c\chi_- - \phi_2$, and

$d_3\chi_-^2 + c\chi_- - \phi_3$ are all negative, and define $-\nu_- < 0$ as in the previous paragraph. If $\mathrm{Re}\,\lambda \geq -\nu_-$, then for $k = 1, 2, 3$, $\mathrm{Re}\,\mu_{-k-} \leq \chi_-$ and $\mathrm{Re}\,\mu_{-k+} \geq \chi_+$.

Similarly, suppose we choose $\eta_- < \eta_+ < 0$ so that $d_k\eta^2 + c\eta$, $k = 1, 2, 3$, with $\eta = \eta_\pm$, are all negative, and let $-\nu_+ < 0$ be the maximum of these six numbers. By (9.37)–(9.38), if $\mathrm{Re}\,\lambda \geq -\nu_+$, then for $k = 1, 2, 3$, $\mathrm{Re}\,\mu_{+k-} \leq \eta_-$ and $\mathrm{Re}\,\mu_{-k+} \geq \eta_+$.

In particular, let $\eta_- = -\alpha_+ < 0$. Then $d_k\eta_-^2 + c\eta_-$, $k = 1, 2, 3$, are all negative. Choose $\eta_+$ such that $\eta_- < \eta_+ < 0$. Then $d_k\eta_+^2 + c\eta_+$, $k = 1, 2, 3$, are all negative. Define $-\nu_+ < 0$ as in the previous paragraph. If $\mathrm{Re}\,\lambda \geq -\nu_+$, then for $k = 1, 2, 3$, $\mathrm{Re}\,\mu_{+k-} \leq \eta_-$ and $\mathrm{Re}\,\mu_{-k+} \geq \eta_+$.

Choose $-\nu$ such that $\max(-\nu_-, -\nu_+) \leq -\nu < 0$, and let $\mathrm{Re}\,\lambda \geq -\nu$. Define $S_-(\lambda)$ to be, for the linear system (9.29) with $Y_-$, the three-dimensional sum of the eigenspaces for eigenvalues greater than $-\alpha_- < 0$; similarly, define $S_+(\lambda)$ to be, for the linear system (9.29) with $Y_+$, the three-dimensional sum of the eigenspaces for eigenvalues less than $-\alpha_+ < 0$. A solution of (9.28) lies in the space $L^2_\alpha(\mathbb{R})^6$ if and only if, when normalized, it approaches $S_-(\lambda)$ as $\xi \to -\infty$ and approaches $S_+(\lambda)$ as $\xi \to \infty$. Values of $\lambda$ for which such a solution exists are zeros of an analytic function $E(\lambda)$, the Evans function defined on $\mathrm{Re}\,\lambda \geq -\nu$.

A standard rescaling argument shows that given $\psi$, $0 < \psi < \pi$, there exists a number $R > 0$ such that all zeros of the Evans function in $\{\lambda = re^{i\theta} : r \geq 0 \text{ and } |\theta| \leq \psi\}$ have $r \leq R$. Therefore, when we study the Evans function on $\mathrm{Re}\,\lambda \geq -\nu$, there is a number $R > 0$ such that it suffices to study it on $\{\lambda : \mathrm{Re}\,\lambda \geq -\nu \text{ and } |\lambda| \leq R\}$.

Note that for $\mathrm{Re}\,\lambda \geq 0$, the space $S_-(\lambda)$ actually corresponds to eigenvalues with real part at most 0. Thus for $\mathrm{Re}\,\lambda \geq 0$, the Evans function actually detects eigenvalues with bounded eigenfunctions.

Simon et al. [22, 23, 24] show numerically that in a region of the form $\{\lambda : \mathrm{Re}\,\lambda \geq 0 \text{ and } |\lambda| \leq R\}$, the Evans function has no zeros except a simple 0 at the origin. Assuming this has been shown for $R$ sufficiently large, Hypothesis 3.6(3b) is verified.

Suppose (1) there continue to be no eigenvalues in $\{\lambda : \mathrm{Re}\,\lambda \geq -\nu \text{ and } |\lambda| \leq R\}$, and (2) $-\nu$ is greater than the maximum of (9.39) and (9.41). Then by Theorem 3.14, the number $-\nu$ can be used in the exponential rate conclusions of Theorem 9.1.

**Appendix A. Sufficient conditions for a bounded semigroup.** Hypotheses 3.10 and 3.15 require that certain semigroups be bounded. In this appendix we give some conditions that can be used to check this assumption.

Let $\mathcal{L}$ be the generator of a $C_0$-semigroup $\{e^{t\mathcal{L}}\}_{t\geq 0}$ on a Banach space $\mathcal{E}$. We recall that the semigroup is *bounded* if $\sup_{t\geq 0} \|e^{t\mathcal{L}}\| < \infty$. If the semigroup is bounded, then $\mathrm{Sp}(\mathcal{L}) \subset \{\lambda : \mathrm{Re}\,\lambda \leq 0\}$. Of course, this statement is not an equivalence, even for $2\times 2$ matrices.

In subsection A.1 we give a simple sufficient condition for our semigroups to be bounded that works when two matrices commute and $\mathcal{E}_0 = L^2(\mathbb{R})$. In subsection A.2 we give more sophisticated integral conditions based on an abstract theorem from [7, 21]. We give a necessary and sufficient integral condition for the case $\mathcal{E}_0 = L^2(\mathbb{R})$, and a sufficient integral condition that implies boundedness of the semigroup for all of the cases $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, and $BUC(\mathbb{R})$.

**A.1. A condition when two matrices commute.** For the case $\mathcal{E}_0 = L^2(\mathbb{R})$, we can relate Hypothesis 3.10(1) (respectively, (2)) to the matrix $D_UR_1(0,0)$ (respectively, $D_VR_2(0,0)$), provided the matrices $D_1$ and $D_UR_1(0,0)$ (respectively, $D_2$ and $D_VR_2(0,0)$) commute.

We recall that an eigenvalue of a matrix is called *semisimple* if its algebraic and geometric multiplicities coincide.

PROPOSITION A.1.

(1) *Suppose* (1) *the matrices* $D_1$ *and* $D_U R_1(0,0)$ *commute, and* (2) *for all eigenvalues* $\lambda$ *of the matrix* $D_U R_1(0,0)$*, we have* (a) $\operatorname{Re}\lambda \leq 0$*, and* (b) *if* $\operatorname{Re}\lambda = 0$*, then* $\lambda$ *is semisimple. Then Hypothesis* 3.10(1) *holds for* $\mathcal{E}_0 = L^2(\mathbb{R})$.

(2) *Suppose* (1) *the matrices* $D_2$ *and* $D_V R_2(0,0)$ *commute, and* (2) *for all eigenvalues* $\lambda$ *of the matrix* $D_V R_2(0,0)$*, we have* $\operatorname{Re}\lambda < 0$*. Then Hypothesis* 3.10(2) *holds for* $\mathcal{E}_0 = L^2(\mathbb{R})$.

*Proof.* Given a diagonal $m\times m$ matrix $D$ with nonnegative entries $d_j$ and an $m\times m$ matrix $A$ that commutes with $D$, for the differential operator $\mathcal{L} = D\partial_{\xi\xi}^2 + c\partial_\xi + A$ on $L^2(\mathbb{R})^m$, we claim that for $t \geq 0$,

$$\|e^{t\mathcal{L}}\|_{L^2(\mathbb{R})^n} = \|e^{tA}\|_{\mathbb{C}^n}. \tag{A.1}$$

Assuming the claim, we finish the proof of the proposition as follows. By (A.1) applied to the operator $\mathcal{L}_0^1$, Hypothesis 3.10(1) for $\mathcal{E}_0 = L^2(\mathbb{R})$ holds if and only if the matrix $D_U R_1(0,0)$ generates a bounded semigroup on $\mathbb{C}^{n_1}$. The first conclusion of the proposition then follows from [2, Corollary I.2.11]. By (A.1) applied to the operator $\mathcal{L}_0^2$, Hypothesis 3.10(2) for $\mathcal{E}_0 = L^2(\mathbb{R})$ holds if and only if the matrix $D_V R_2(0,0)$ generates an exponentially decaying semigroup on $\mathbb{C}^{n_1}$, yielding the second conclusion of the proposition.

To prove (A.1), we note that the semigroup generated by the operator $\mathcal{L}$ on $L^2(\mathbb{R})^n$ is similar via the Fourier transform to the semigroup generated on $L^2(\mathbb{R})^n$ by the operator $\mathcal{M}$ of multiplication by the matrix-valued function $M(\theta) = -\theta^2 D + i\theta c + A$, $\theta \in \mathbb{R}$. The norms of the respective semigroups are equal since the Fourier transform is an isometry on $L^2(\mathbb{R})$; cf., e.g., Theorem VI.5.12, Propositions I.4.11 and I.4.12, and Paragraph II.2.1 of [2]. It follows that

$$\begin{aligned}\|e^{t\mathcal{L}}\|_{L^2(\mathbb{R})^n} &= \|e^{t\mathcal{M}}\|_{L^2(\mathbb{R})^n} = \sup_{\theta\in\mathbb{R}} \|e^{tM(\theta)}\|_{\mathbb{C}^n} = \sup_{\theta\in\mathbb{R}} \|e^{t(-\theta^2 D + i\theta c + A)}\|_{\mathbb{C}^n} \\ &= \sup_{\eta\geq 0} \|e^{t(-\eta D + A)}\|_{\mathbb{C}^n},\end{aligned} \tag{A.2}$$

which yields the inequality ($\geq$) in (A.1). To prove the reverse inequality, for each $x \in \mathbb{C}^n$ with $\|x\|_{\mathbb{C}^n} = 1$, we denote $y(\eta, t, x) = e^{t(-\eta D + A)}x$. Since $A$ and $D$ commute, we have $\frac{d}{d\eta} e^{t(-\eta D + A)} = -tDe^{t(-\eta D + A)}$. We therefore calculate the following:

$$\begin{aligned}\frac{d}{d\eta}\big(\|y(\eta,t,x)\|^2\big) &= \frac{d}{d\eta}\langle e^{t(-\eta D+A)}x, e^{t(-\eta D+A)}x\rangle = -2t\langle Dy(\eta,t,x), y(\eta,t,x)\rangle \\ &= -2t \sum_{j:d_j\neq 0} d_j y_j(\eta,t,x)\overline{y_j(\eta,t,x)} \leq -2t \min_{j:d_j\neq 0} d_j \|y(\eta,t,x)\|_{\mathbb{C}^n}^2.\end{aligned}$$

We conclude easily that $\|y(\eta,t,x)\|_{\mathbb{C}^n}$ does not increase, so $\|y(\eta,t,x)\|_{\mathbb{C}^n} \leq \|y(0,t,x)\|_{\mathbb{C}^n}$ for all $\eta \geq 0$. Therefore

$$\begin{aligned}\|e^{t\mathcal{L}}\|_{L^2(\mathbb{R})^n} &= \sup_{\eta\geq 0}\|e^{t(-\eta D+A)}\|_{\mathbb{C}^n} = \sup_{\eta\geq 0}\sup_{\|x\|_{\mathbb{C}^n}=1}\|y(\eta,t,x)\|_{\mathbb{C}^n} \leq \sup_{\|x\|_{\mathbb{C}^n}=1}\|y(0,t,x)\|_{\mathbb{C}^n} \\ &= \|e^{tA}\|_{\mathbb{C}^n},\end{aligned}$$

which finishes the proof of (A.1). $\square$

**A.2. An integral condition.** We begin by recalling an abstract theorem from [7, 21]. We denote by $\langle f, g\rangle_{\mathcal{E},\mathcal{E}^*}$ the value of a functional $g \in \mathcal{E}^*$ on $f \in \mathcal{E}$.

THEOREM A.2. *Assume that the spectrum of the generator $\mathcal{L}$ of a $C_0$-semigroup $\{e^{t\mathcal{L}}\}_{t\geq 0}$ on $\mathcal{E}$ lies in $\{\lambda : \operatorname{Re}\lambda \leq 0\}$.*

(1) *Let $\mathcal{E}$ be a Hilbert space. Let $\mathcal{L}^*$ be the adjoint operator of $\mathcal{L}$. Then the semigroup is bounded if and only if*

$$\sup_{\omega>0} \omega \int_{-\infty}^{\infty} \left\|(\mathcal{L}-(\omega+i\tau)\mathcal{I})^{-1}f\right\|_{\mathcal{E}}^2 + \left\|(\mathcal{L}^*-(\omega+i\tau)\mathcal{I})^{-1}f\right\|_{\mathcal{E}}^2 d\tau < \infty \text{ for each } f\in\mathcal{E}. \tag{A.3}$$

(2) *Let $\mathcal{E}$ be a Banach space. Then the semigroup is bounded, provided*

$$\sup_{\omega>0} \omega \int_{-\infty}^{\infty} \left|\left\langle(\mathcal{L}-(\omega+i\tau)\mathcal{I})^{-2}f, g\right\rangle_{\mathcal{E},\mathcal{E}^*}\right| d\tau < \infty \text{ for each } f\in\mathcal{E},\ g\in\mathcal{E}^*. \tag{A.4}$$

This result is proved in [7, 21].

For the cases $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, or $BUC(\mathbb{R})$, we consider the operator $\mathcal{L}_0^{(1)}$ associated with $L^{(1)}$ defined by (3.8). Since the semigroup generated by $\mathcal{L}_0^{(1)}$ on $L^2(\mathbb{R})^{n_1}$ is bounded, $\operatorname{Sp}(\mathcal{L}_0^{(1)}) \subset \{\lambda : \operatorname{Re}\lambda \leq 0\}$, so the same is true of its Fourier transform. Therefore

$$\operatorname{Sp}(-D_1\theta^2 + ic\theta + D_U R_1(0,0)) \subset \{\lambda : \operatorname{Re}\lambda \leq 0\} \text{ for all } \theta\in\mathbb{R}. \tag{A.5}$$

Because of (A.5), we can define, for $(\theta,\omega,\tau)\in\mathbb{R}^3$ with $\omega>0$,

$$\begin{aligned} N(\theta,\omega,\tau) &= \big(-D_1\theta^2 + ic\theta + D_U R_1(0,0) - (\omega+i\tau)I\big)^{-1}, \\ m(\theta,\omega,\tau) &= \|N(\theta,\omega,\tau)\|_{\mathbb{C}^{n_1\times n_1}}. \end{aligned} \tag{A.6}$$

PROPOSITION A.3. *Assume* (A.5).

(1) *Suppose that $m(\cdot,\omega,\tau)\in L^\infty(\mathbb{R})$ for each $(\omega,\tau)\in\mathbb{R}^2$ with $\omega>0$, and*

$$\sup_{\omega>0} \omega \int_{-\infty}^{\infty} \|m(\cdot,\omega,\tau)\|_{L^\infty(\mathbb{R})}^2 d\tau < \infty. \tag{A.7}$$

*Then Hypothesis* 3.10(1) *holds for $\mathcal{E}_0 = L^2(\mathbb{R})$.*

(2) *Suppose that $m(\cdot,\omega,\tau)\in H^1(\mathbb{R})$ for each $(\omega,\tau)\in\mathbb{R}^2$ with $\omega>0$, and*

$$\sup_{\omega>0} \omega \int_{-\infty}^{\infty} \|m(\cdot,\omega,\tau)\|_{H^1(\mathbb{R})}^2 d\tau < \infty. \tag{A.8}$$

*Then Hypothesis* 3.10(1) *holds for both spaces, and Hypothesis* 3.15(1) *holds.*

*Proof.* First, we recall the definition and elementary properties of matrix-valued Fourier multipliers; see, e.g., [1] and the literature cited therein. Given an $L^\infty$ function $N:\mathbb{R}\to\mathbb{C}^{n_1\times n_1}$, we define an operator $T_N$ on the Schwartz space $\mathcal{S}(\mathbb{R})^{n_1}$ of smooth, rapidly decaying, vector-valued functions by $T_N h = \mathcal{F}^{-1}(N(\cdot)\mathcal{F}h)$, where $\mathcal{F}$ is the Fourier transform and $h\in\mathcal{S}(\mathbb{R})^{n_1}$. For $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, or $BUC(\mathbb{R})$, the function $N$ is called an $\mathcal{E}_0^{n_1}$*-Fourier multiplier* if the operator $T_N$ admits a bounded extension from $\mathcal{S}(\mathbb{R})^{n_1}$ to all of $\mathcal{E}_0^{n_1}$. Since $\mathcal{F}$ is an isometry on $L^2(\mathbb{R})^{n_1}$,

$$\begin{gathered} \text{every } L^\infty \text{function } N \text{ is an } L^2(\mathbb{R})^{n_1}\text{-Fourier multiplier,} \\ \text{and} \quad \|T_N\|_{L^2(\mathbb{R})^{n_1}\to L^2(\mathbb{R})^{n_1}} = \|N\|_{L^\infty}. \end{gathered} \tag{A.9}$$

For $\mathcal{E}_0 = L^1(\mathbb{R})$ or $BUC(\mathbb{R})$, various sufficient conditions are known for $N$ to be an $\mathcal{E}_0^{n_1}$-Fourier multiplier. We will use one of the simplest [1, Theorem 4.1]: let $m(\theta) = \|N(\theta)\|$. For $\mathcal{E}_0 = L^1(\mathbb{R})$, $L^2(\mathbb{R})$, or $BUC(\mathbb{R})$, if $m \in H^1(\mathbb{R})$, then $N$ is an $\mathcal{E}_0^{n_1}$-Fourier multiplier, and

$$\|T_N\|_{\mathcal{E}_0^{n_1}\to\mathcal{E}_0^{n_1}} \leq \|m\|_{H^1(\mathbb{R})}. \tag{A.10}$$

Next, let us consider the functions $N$ and $m$ defined by (A.6). In case (1) (respectively, (2)) of the proposition, we have $m(\cdot,\omega,\tau) \in L^\infty(\mathbb{R})$ (respectively, $m(\cdot,\omega,\tau) \in H^1(\mathbb{R}) \hookrightarrow C_0(\mathbb{R})$). In either case, $N$ is an $L^\infty$ function. For $h \in \mathcal{S}(\mathbb{R})^{n_1}$, we have $(\mathcal{L}_0^{(1)} - (\omega+i\tau)I)^{-1}h = T_N h$. For $(\omega,\tau) \in \mathbb{R}^2$ with $\omega > 0$, (A.9) implies that $T_N$ is bounded on $L^2(\mathbb{R})^{n_1}$. Therefore on $\mathcal{E}_0 = L^2(\mathbb{R})$ the operator $(\mathcal{L}_0^{(1)} - (\omega+i\tau)I)^{-1}$ is also bounded, and

$$(\mathcal{L}_0^{(1)} - (\omega+i\tau)I)^{-1} = \mathcal{F}^{-1}(N(\cdot)\mathcal{F}) \text{ on } L^2(\mathbb{R})^{n_1}. \tag{A.11}$$

Thus $\omega + i\tau \in \rho(\mathcal{L}_0^{(1)})$, so $\mathrm{Sp}(\mathcal{L}_0^{(1)}) \subset \{\lambda : \mathrm{Re}\,\lambda \leq 0\}$ for $\mathcal{E}_0 = L^2(\mathbb{R})$. By an analogue of Lemma 3.5(1) for $\mathcal{L}_0^{(1)}$, we have the same result for $\mathcal{E}_0 = L^1(\mathbb{R})$ and $BUC(\mathbb{R})$.

Suppose (A.7) holds. Then (A.11) yields, for each $f \in L^2(\mathbb{R})^{n_1}$,

$$\begin{aligned}
\sup_{\omega>0} \omega \int_{-\infty}^{\infty} \left\|(\mathcal{L}_0^{(1)} - (\omega+i\tau)I)^{-1}f\right\|^2_{L^2(\mathbb{R})^{n_1}} d\tau &= \sup_{\omega>0} \omega \int_{-\infty}^{\infty} \left\|\mathcal{F}^{-1}(N(\cdot)\mathcal{F}f)\right\|^2_{L^2(\mathbb{R})^{n_1}} d\tau \\
&= \sup_{\omega>0} \omega \int_{-\infty}^{\infty} \left\|N(\cdot)\mathcal{F}f\right\|^2_{L^2(\mathbb{R})^{n_1}} d\tau \\
&\leq \sup_{\omega>0} \omega \int_{-\infty}^{\infty} \|m(\cdot,\omega,\tau)\|_{L^\infty} d\tau \left\|\mathcal{F}f\right\|^2_{L^2(\mathbb{R})^{n_1}} < \infty.
\end{aligned}$$

This is half of what is needed to show that (A.3) holds. A similar argument yields the other half, and then Theorem A.2(1) gives the result.

Suppose (A.8) holds. Theorem A.2(2) implies, in particular, that the semigroup $\{e^{t\mathcal{L}}\}_{t\geq 0}$ is bounded on the Banach space $\mathcal{E}$, provided $\mathrm{Sp}(\mathcal{L}) \subset \{\lambda : \mathrm{Re}\,\lambda \leq 0\}$ and

$$\sup_{\omega>0} \omega \int_{-\infty}^{\infty} \left\|(\mathcal{L} - (\omega+i\tau)I)^{-1}\right\|^2_{\mathcal{E}\to\mathcal{E}} d\tau < \infty. \tag{A.12}$$

For $\mathcal{E}_0 = L^1(\mathbb{R}), L^2(\mathbb{R})$, and $BUC(\mathbb{R})$, we obtain, using (A.10),

$$\|(\mathcal{L}_0^{(1)} - (\omega+i\tau)I)^{-1}\|_{\mathcal{E}_0^{n_1}\to\mathcal{E}_0^{n_1}} = \|T_N\|_{\mathcal{E}_0^{n_1}\to\mathcal{E}_0^{n_1}} \leq \|m(\cdot,\omega,\tau)\|_{H^1(\mathbb{R})}. \tag{A.13}$$

Then (A.13) and (A.8) imply that (A.12) holds with $\mathcal{E} = \mathcal{E}_0$ and $\mathcal{L} = \mathcal{L}_0^{(1)}$, which yields the result. ∎

**Appendix B. Stabilizing weights.** We consider the linear PDE

$$Y_t = DY_{\xi\xi} + cY_\xi + AY, \tag{B.1}$$

with $Y \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $t \geq 0$, $D = \mathrm{diag}(d_1,\ldots,d_n)$ with all $d_i \geq 0$, and $A = (a_{kl})$ an $n \times n$ matrix.

In this appendix we will let $\alpha$ denote a real number, and we will use $L^2_\alpha(\mathbb{R})$ to denote a weighted $L^2$ space with weight function $e^{\alpha\xi}$ so that $Y \in L^2_\alpha(\mathbb{R})^n$ if and only if $e^{\alpha\xi}Y(\xi) \in L^2(\mathbb{R})^n$. Then

$$Y(\xi) = e^{-\alpha\xi}Z(\xi), \quad Z \in L^2(\mathbb{R})^n. \tag{B.2}$$

Substitute (B.2) into (B.1) and multiply by $e^{\alpha\xi}$:

$$Z_t = DZ_{\xi\xi} + (c - 2\alpha)Z_\xi + (\alpha^2 D - c\alpha + A)Z. \tag{B.3}$$

As explained in section 3, the operator on $L^2_\alpha(\mathbb{R})^n$ given by the right-hand side of (B.1) is similar to the operator on $L^2(\mathbb{R})^n$ given by the right-hand side of (B.3).

To find the spectrum of the constant-coefficient linear differential operator on the right-hand side of (B.3), we take the Fourier transform,

$$\hat{Z}_t = \left(-\theta^2 D + (c - 2\alpha)i\theta + \alpha^2 D - c\alpha + A\right)\hat{Z} = M(\theta, \alpha)\hat{Z}, \tag{B.4}$$

where

$$M(\theta, \alpha) = \operatorname{diag}\left(-\theta^2 d_k + (c - 2\alpha)i\theta + \alpha^2 d_k - c\alpha\right) + A.$$

Eigenvalues $\lambda$ of $M(\theta, \alpha)$ satisfy the equation $\det(M(\theta, \alpha) - \lambda I) = 0$ or

$$\det\Big(\operatorname{diag}\left(-\theta^2 d_k + (c - 2\alpha)i\theta + \alpha^2 d_k - c\alpha - \lambda\right) + A\Big) = 0. \tag{B.5}$$

For fixed $\alpha$, the spectrum of the operator on $L^2(\mathbb{R})^n$ given by the right-hand side of (B.3) is the closure of the set of $\lambda$ such that $\lambda$ satisfies (B.5) for some $\theta \in \mathbb{R}$.

Regard the left-hand side of (B.5) as a function of $(\theta, \alpha, \lambda)$ that we denote $f(\theta, \alpha, \lambda)$. Define $g : \mathbb{R}^{n^2} \to \mathbb{R}$ by $g(c_{kl}) = \det(c_{kl})$; the numbers $c_{kl}$ are the entries of an $n \times n$ matrix. Then $f$ can be regarded as the composite function,

$$f(\theta, \alpha, \lambda) = g\big(c_{kl}(\theta, \alpha, \lambda)\big),$$

where $c_{kl}(\theta, \alpha, \lambda)$ is the $kl$-entry of the matrix $M(\theta, \alpha) - \lambda I$. We have

$$c_{kl} = a_{kl} \text{ for } k \neq l, \quad c_{kk} = -\theta^2 d_k + (c - 2\alpha)i\theta + \alpha^2 d_k - c\alpha - \lambda + a_{kk}.$$

Suppose $f(\theta_0, 0, 0) = 0$; i.e., an eigenvalue of $M(\theta_0, 0) = \operatorname{diag}\left(-\theta_0^2 d_k + ic\theta_0\right) + A$ is 0. Hence in the unweighted space ($\alpha = 0$), the spectrum of the operator on the right-hand side of (B.3) includes 0, so the semigroup it generates is not exponentially stable. We want to move the spectrum to the left by introducing the weight $e^{\alpha\xi}$.

Suppose, in addition, that $\frac{\partial f}{\partial \lambda}(\theta_0, 0, 0) \neq 0$; i.e., 0 is a simple eigenvalue of $M(\theta_0, 0)$. Then by the implicit function theorem, the equation $f(\theta, \alpha, \lambda) = 0$ can be solved for $\lambda$ as a smooth function of $(\theta, \alpha)$ near $(\theta_0, 0, 0)$, so $\lambda(\theta_0, 0) = 0$.

Now $f(\theta, \alpha, \lambda(\theta, \alpha)) = g(c_{kl}(\theta, \alpha, \lambda(\theta, \alpha))) \equiv 0$ implies

$$\sum \frac{\partial g}{\partial c_{kl}} \left( \frac{\partial c_{kl}}{\partial \alpha} + \frac{\partial c_{kl}}{\partial \lambda} \frac{\partial \lambda}{\partial \alpha} \right) = 0.$$

This simplifies to

$$\sum \frac{\partial g}{\partial c_{kk}} \left( -2i\theta + 2\alpha d_k - c - \frac{\partial \lambda}{\partial \alpha} \right) = 0; \tag{B.6}$$

the partial derivatives of $g$ are evaluated at $M(\theta, \alpha) - \lambda(\theta, \alpha)I$.

If we substitute $(\theta, \alpha) = (\theta_0, 0)$ into (B.6), we must evaluate the partial derivatives of $g$ at $M(\theta_0, 0)$. Now $\frac{\partial g}{\partial c_{kk}}(M(\theta_0, 0)) = M_{kk}(\theta_0, 0)$, the $kk$-minor of $M(\theta_0, 0)$. Therefore

$$\left(\sum M_{kk}(\theta_0, 0)\right)\left(-2i\theta_0 - c - \frac{\partial \lambda}{\partial \alpha}(\theta_0, 0)\right) = 0.$$

It is not hard to check that the assumption that 0 a simple eigenvalue of $M(\theta_0, 0)$ implies that $\sum M_{kk}(\theta_0, 0) \neq 0$; actually, the latter is equivalent to $\frac{\partial f}{\partial \lambda}(\theta_0, 0, 0) \neq 0$. We therefore obtain

$$\frac{\partial \lambda}{\partial \alpha}(\theta_0, 0) = -2i\theta - c.$$

It follows that if $c > 0$, increasing $\alpha$ will move the spectrum to the left. On the other hand, if $c < 0$, increasing $\alpha$ will move the spectrum to the right.

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