Nonselfadjoint Operators, Infinite Determinants, and Some Applications

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Received September, 2005

Abstract. We study various spectral theoretic aspects of nonselfadjoint operators. Specifically, we consider a class of factorable nonselfadjoint perturbations of a given unperturbed nonselfadjoint operator and provide an in-depth study of a version of the Birman–Schwinger principle as well as local and global Weinstein–Aronszajn formulas. Our applications include a study of suitably symmetrized (modified) perturbation determinants of Schrödinger operators in dimensions $n = 1, 2, 3$ and their connection with Krein’s spectral shift function in two- and three-dimensional scattering theory. Moreover, we study an appropriate multi-dimensional analog of the celebrated formula by Jost and Pais that identifies Jost functions with suitable Fredholm (perturbation) determinants and hence reduces the latter to simple Wronski determinants.

1. INTRODUCTION

This paper was written in response to the increased demand of spectral theoretic aspects of nonselfadjoint operators in contemporary applied and mathematical physics. What we have in mind, in particular, concerns the following typical two scenarios: First, the construction of certain classes of solutions of a number of completely integrable hierarchies of evolution equations by means of the inverse scattering method, for instance, in the context of the focusing nonlinear Schrödinger equation in $(1 + 1)$-dimensions, naturally leads to nonselfadjoint Lax operators. Specifically, in the particular case of the focusing nonlinear Schrödinger equation, the corresponding Lax operator is a nonselfadjoint one-dimensional Dirac-type operator. Second, linearizations of nonlinear partial differential equations around steady state and solitary-type solutions, frequently, lead to a linear nonselfadjoint spectral problem. In the latter context, the use of the so-called Evans function (an analog of the one-dimensional Jost function for Schrödinger operators) in the course of a linear stability analysis has become a cornerstone of this circle of ideas. As shown in [15], the Evans function equals a (modified) Fredholm determinant associated with an underlying Birman–Schwinger-type operator. This observation naturally leads to the second main theme of this paper and a concrete application to nonselfadjoint operators, viz., a study of properly symmetrized (modified) perturbation determinants of nonselfadjoint Schrödinger operators in dimensions $n = 1, 2, 3$.

Next, we briefly summarize the content of each section. In Section 2, following the seminal work of Kato [24] (see also Konno and Kuroda [28] and Howland [20]), we consider a class of factorable nonselfadjoint perturbations, formally given by $B^*A$, of a given unperturbed nonselfadjoint operator $H_0$ in a Hilbert space $H$ and introduce a densely defined, closed linear operator $H$ in $H$ which represents an extension of $H_0 + B^*A$. Closely following Konno and Kuroda [28], we subsequently derive a general Birman–Schwinger principle for $H$ in Section 3. A variant of the essential spectrum of $H$ and a local Weinstein–Aronszajn formula is discussed in Section 4. The corresponding global Weinstein–Aronszajn formula in terms of modified Fredholm determinants associated with the Birman–Schwinger kernel of $H$ is the content of Section 5. Both, Sections 4 and 5 are modeled after an exemplary treatment of these topics by Howland [20] in the case where $H_0$ and $H$ are selfadjoint. In Section 6, we turn to concrete applications to properly symmetrized (modified)
perturbation determinants of nonselfadjoint Dirichlet- and Neumann-type Schrödinger operators in $L^2(\Omega; d^n x)$ with $\Omega = (0, \infty)$ in the case $n = 1$ and rather general open domains $\Omega \subseteq \mathbb{R}^n$ with a compact boundary in dimensions $n = 2, 3$. The corresponding potentials $V$ considered are of the form $V \in L^1((0, \infty)\; ; dx)$ for $n = 1$ and $V \in L^2(\Omega; d^n x)$ for $n = 2, 3$. Our principal result in this section concerns a reduction of the Fredholm determinant of the Birman–Schwinger kernel of $H$ in $L^2(\Omega; d^n x)$ to a Fredholm determinant associated with operators in $L^2(\partial \Omega; d^{n-1} \sigma)$. The latter should be viewed as a proper multi-dimensional extension of the celebrated result by Jost and Pais [23] concerning the equality of the Jost function (a Wronski determinant) and the associated Fredholm determinant of the underlying Birman–Schwinger kernel. In Section 7, we briefly discuss an application to scattering theory in dimensions $n = 2, 3$ and re-derive a formula for the Krein spectral shift function (related to the logarithm of the determinant of the scattering matrix) in terms of modified Fredholm determinants of the underlying Birman–Schwinger kernel. We present an alternative derivation of this formula originally due to Cheney [9] for $n = 2$ and Newton [38] for $n = 3$ (in the latter case, we obtain the result under weaker assumptions on the potential $V$ than in [38]). Finally, Appendix 8 summarizes results on Dirichlet and Neumann Laplacians in $L^2(\Omega; d^n x)$ for a general class of open domains $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, with compact boundary. We prove the equality of two natural definitions of Dirichlet and Neumann Laplacians for such domains and prove mapping properties between appropriate scales of Sobolev spaces. These results are crucial ingredients in our treatment of modified Fredholm determinants in Section 6, but they also seem to be of independent interest.

In this paper, we use the following notation. Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces, $(\cdot, \cdot)_{\mathcal{H}}$ and $(\cdot, \cdot)_{\mathcal{K}}$ the scalar products in $\mathcal{H}$ and $\mathcal{K}$ (linear in the second factor), and $I_{\mathcal{H}}$ and $I_{\mathcal{K}}$ the identity operators in $\mathcal{H}$ and $\mathcal{K}$, respectively. Next, let $T$ be a closed linear operator from $\text{dom}(T) \subseteq \mathcal{H}$ to $\text{ran}(T) \subseteq \mathcal{K}$, with $\text{dom}(T)$ and $\text{ran}(T)$ denoting the domain and range of $T$. The closure of a closable operator $S$ is denoted by $\overline{S}$. The kernel (null space) of $T$ is denoted by $\ker(T)$. The spectrum and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$. The Banach spaces of bounded and compact linear operators in $\mathcal{H}$ are denoted by $B(\mathcal{H})$ and $B_\infty(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $B_p(\mathcal{H})$, $p \in \mathbb{N}$. Analogous notation $B(\mathcal{H}_1, \mathcal{H}_2)$, $B_\infty(\mathcal{H}_1, \mathcal{H}_2)$, etc., will be used for bounded, compact, etc., operators between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. In addition, $\text{tr}(T)$ denotes the trace of a trace class operator $T \in B_1(\mathcal{H})$ and $\det_p(I_{\mathcal{H}} + S)$ represents the (modified) Fredholm determinant associated with an operator $S \in B_p(\mathcal{H})$, $p \in \mathbb{N}$ (for $p = 1$, we omit the subscript 1). Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$.

Finally, in Sections 6 and 7, we introduce various operators of multiplication, $M_f$, in $L^2(\Omega; d^n x)$ by elements $f \in L^1_{\text{loc}}(\Omega; d^n x)$, where $\Omega \subseteq \mathbb{R}^n$ is open and nonempty.

2. ABSTRACT PERTURBATION THEORY

In this section, following Kato [24], Konno and Kuroda [28], and Howland [20], we consider a class of factorable nonselfadjoint perturbations of a given unperturbed nonselfadjoint operator. For reasons of completeness, we will present proofs of many of the subsequent results even though most of them are only slight deviations from the original proofs in the selfadjoint context.

We start with our first set of hypotheses.

**Hypothesis 2.1.**

(i) Suppose that $H_0: \text{dom}(H_0) \to \mathcal{H}$, $\text{dom}(H_0) \subseteq \mathcal{H}$ is a densely defined, closed, linear operator in $\mathcal{H}$ with nonempty resolvent set,

$$\rho(H_0) \neq \emptyset,$$

(2.1)

A: $\text{dom}(A) \to K$, $\text{dom}(A) \subseteq \mathcal{H}$ a densely defined, closed, linear operator from $\mathcal{H}$ to $K$, and

B: $\text{dom}(B) \to K$, $\text{dom}(B) \subseteq \mathcal{H}$ a densely defined, closed, linear operator from $\mathcal{H}$ to $K$ such that

$$\text{dom}(A) \supseteq \text{dom}(H_0), \quad \text{dom}(B) \supseteq \text{dom}(H_0).$$

(2.2)

In the following, we denote

$$R_0(z) = (H_0 - zI_{\mathcal{H}})^{-1}, \quad z \in \rho(H_0).$$

(2.3)

(ii) For some (and hence for all) $z \in \rho(H_0)$, the operator $-AR_0(z)B^*$, defined on $\text{dom}(B^*)$, has a bounded extension in $K$, denoted by $K(z)$,

$$K(z) = -AR_0(z)B^* \in \mathcal{B}(K).$$

(2.4)

(iii) $1 \in \rho(K(z_0))$ for some $z_0 \in \rho(H_0).$

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If \( K(z_0) \in \mathcal{B}(\mathcal{K}) \) for some \( z_0 \in \rho(H_0) \), then \( K(z) \in \mathcal{B}(\mathcal{K}) \) for all \( z \in \rho(H_0) \) (as mentioned in Hypothesis 2.1 (ii)) is an immediate consequence of (2.2) and the resolvent equation for \( H_0 \).

We emphasize that, for the case in which \( H_0 \) is selfadjoint, the following results in Lemma 2.2, Theorem 2.3, and Remark 2.4 are due to Kato [24] (see also [20, 28]). The more general case we consider here requires only minor modifications. But for the convenience of the reader, we will sketch most of the proofs.

**Lemma 2.2.** Let \( z, z_1, z_2 \in \rho(H_0) \). Then Hypothesis 2.1 implies the following facts:

\[
AR_0(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad \overline{R_0(z)B^*} = [B(H_0^* - z)^{-1}]^* \in \mathcal{B}(\mathcal{K}, \mathcal{H}),
\]

\[
R_0(z_1)B^* - R_0(z_2)B^* = (z_1 - z_2)R_0(z_1)R_0(z_2)B^*
\]

\[
= (z_1 - z_2)R_0(z_2)R_0(z_1)B^*,
\]

\[
K(z) = -A[R_0(z)B^*], \quad K(\overline{z})^* = -B[R_0(\overline{z})^*A^*],
\]

\[
\text{ran}(R_0(z)B^*) \subseteq \text{dom}(A), \quad \text{ran}(R_0(\overline{z})^*A^*) \subseteq \text{dom}(B).
\]

**Proof.** Equations (2.5) follow from the relations in (2.2) and the Closed Graph Theorem. Equations (2.6) and (2.7) follow by combining (2.5) and the resolvent equation for \( H_0^* \). Next, let \( f \in \text{dom}(B^*), g \in \text{dom}(A^*) \), then

\[
(R_0(z)B^*f, A^*g)_{\mathcal{H}} = (R_0(z)B^*f, A^*g)_{\mathcal{H}} = (AR_0(z)B^*f, g)_{\mathcal{K}} = -(K(z)f, g)_{\mathcal{K}}.
\]

By continuity, this extends to all \( f \in \mathcal{K} \). Thus, \(-A[R_0(z)B^*]f \) exists and equals \( K(z)f \) for all \( f \in \mathcal{K} \). This proves the first assertions in (2.8) and (2.9). The remaining assertions in (2.8) and (2.9) are of course proved analogously. Multiplying (2.6) and (2.7) by \( A \) from the left and taking into account the first relation in (2.8), we prove (2.10) and (2.11).

Next, following Kato [24], one introduces

\[
R(z) = R_0(z) - R_0(z)B^*[I_K - K(z)]^{-1}AR_0(z), \quad z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(z))\}.
\]

**Theorem 2.3.** Assume Hypothesis 2.1 and suppose \( z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(z))\} \). Then, \( R(z) \) defined in (2.13) defines a densely defined, closed, linear operator \( H \) in \( \mathcal{H} \) by

\[
R(z) = (H - zI_{\mathcal{H}})^{-1}.
\]

Moreover,

\[
AR(z), BR(z)^* \in \mathcal{B}(\mathcal{H}, \mathcal{K})
\]

and

\[
R(z) = R_0(z) - R_0(z)B^*AR_0(z)
\]

\[
= R_0(z) - R_0(z)B^*AR_0(z).
\]

Finally, \( H \) is an extension of \( (H_0 + B^*A)|_{\text{dom}(H_0) \cap \text{dom}(B^*A)} \) (the latter intersection domain may consist of \{0\} only),

\[
H \supseteq (H_0 + B^*A)|_{\text{dom}(H_0) \cap \text{dom}(B^*A)}.
\]

**Proof.** Suppose \( z \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(z))\} \). Since, by (2.13),

\[
AR(z) = [I_K - K(z)]^{-1}AR_0(z),
\]

\[
BR(z)^* = [I_K - K(z)^*]^{-1}BR_0(z)^*,
\]

\[
R(z)f = 0 \text{ implies } AR(z)f = 0, \text{ and hence, by (2.19), } AR_0(z)f = 0. \text{ The latter implies } R_0(z)f = 0 \text{ by (2.13) and thus } f = 0. \text{ Consequently,}
\]

\[
\ker(R(z)) = \{0\}.
\]

Similarly, (2.20) implies

\[
\ker(R(z)^*) = \{0\} \text{ and hence } \overline{\text{ran}(R(z))} = \mathcal{H}.
\]

Next, combining (2.13), the resolvent equation for \( H_0 \), (2.6), (2.7), (2.10), and (2.11) proves the resolvent equation

\[
R(z_1) - R(z_2) = (z_1 - z_2)R(z_1)R(z_2), \quad z_1, z_2 \in \{\zeta \in \rho(H_0) \mid 1 \in \rho(K(z))\}.
\]
Thus, $R(z)$ is indeed the resolvent of a densely defined, closed, linear operator $H$ in $\mathcal{H}$ as claimed in connection with (2.14).

By (2.19) and (2.20), $AR(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $[BR(z)]^{*} = \overline{R(z)B^{*}} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, proving (2.15). A combination of (2.13), (2.19), and (2.20) then proves (2.16) and (2.17).

Finally, let $f \in \text{dom}(H_{0}) \cap \text{dom}(B^{*}A)$ and set $g = (H_{0} - zI_{\mathcal{H}})f$. Then $R_{0}(z)g = f$ and by (2.16), $R(z)g - f = -R(z)B^{*}Af$. Thus, $f \in \text{dom}(H)$ and $(H - zI_{\mathcal{H}})f = g + B^{*}Af = (H_{0} + B^{*}A - zI_{\mathcal{H}})f$, proving (2.18).

**Remark 2.4.** (i) Assume that $H_{0}$ is selfadjoint in $\mathcal{H}$. Then $H$ is also selfadjoint if

$$(Af, Bg)_{\mathcal{K}} = (Bf, Ag)_{\mathcal{K}}$$

for all $f, g \in \text{dom}(A) \cap \text{dom}(B)$. (2.24)

(ii) The formalism is symmetric with respect to $H_{0}$ and $H$ in the following sense: The densely defined operator $-AR(z)B^{*}$ has a bounded extension to all of $\mathcal{K}$ for all $z \in \{\xi \in \rho(H_{0}) \mid 1 \in \rho(K(\xi))\}$, in particular,

$$I_{\mathcal{K}} - AR(z)B^{*} = [I_{\mathcal{K}} - K(z)]^{-1}, \quad z \in \{\xi \in \rho(H_{0}) \mid 1 \in \rho(K(\xi))\}.$$  \hspace{1cm} (2.25)

Moreover,

$$R_{0}(z) = R(z) + R(z)B^{*}[I_{\mathcal{K}} - AR(z)B^{*}]^{-1}AR(z), \quad z \in \{\xi \in \rho(H_{0}) \mid 1 \in \rho(K(\xi))\}$$

and

$$H_{0} \supseteq (H - B^{*}A)|_{\text{dom}(H) \cap \text{dom}(B^{*}A)}.$$ \hspace{1cm} (2.27)

(iii) The basic hypotheses (2.2) which amount to

$$AR_{0}(z) \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad R_{0}(z)B^{*} = [B(H_{0}^{*} - z) - I_{\mathcal{K}}]^{*} \in \mathcal{B}(\mathcal{K}, \mathcal{H}), \quad z \in \rho(H_{0})$$

cf. (2.5) are more general than a quadratic form perturbation approach which would result in conditions of the form

$$AR_{0}(z)^{1/2} \in \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad R_{0}(z)^{1/2}B^{*} = [B(H_{0}^{*} - z)^{-1/2} - I_{\mathcal{K}}]^{*} \in \mathcal{B}(\mathcal{K}, \mathcal{H}), \quad z \in \rho(H_{0}),$$

or even an operator perturbation approach which would involve conditions of the form

$$[B^{*}A]R_{0}(z) \in \mathcal{B}(\mathcal{H}), \quad z \in \rho(H_{0}).$$ \hspace{1cm} (2.30)

3. A GENERAL BIRMAN–SCHWINGER PRINCIPLE

The principal result in this section represents an abstract version of (a variant of) the Birman–Schwinger principle due to Birman [4] and Schwinger [46] (cf. also [6, 13, 26, 27, 40, 42, 47, 51]).

We need to strengthen our hypotheses a bit and hence introduce the following assumption.

**Hypothesis 3.1.** In addition to Hypothesis 2.1, we assume the following condition:

(iv) $K(z) \in \mathcal{B}_{\infty}(\mathcal{K})$ for all $z \in \rho(H_{0})$.

Since, by (2.25),

$$-AR(z)B^{*} = [I_{\mathcal{K}} - K(z)]^{-1}K(z)$$

$$= -I_{\mathcal{K}} + [I_{\mathcal{K}} - K(z)]^{-1},$$ \hspace{1cm} (3.1)

Hypothesis 3.1 implies that $-AR(z)B^{*}$ extends to a compact operator in $\mathcal{K}$ as long as the right-hand side of (3.2) exists.

The following general result is due to Konno and Kuroda [28] for the case in which $H_{0}$ is selfadjoint. (The more general case presented here requires no modifications, but we present a proof for completeness.)

**Theorem 3.2** [28]. Assume Hypothesis 3.1 and let $\lambda_{0} \in \rho(H_{0})$. Then

$$Hf = \lambda_{0}f, \quad 0 \neq f \in \text{dom}(H) \quad \text{implies} \quad K(\lambda_{0})g = g$$

where, for fixed $z_{0} \in \{\xi \in \rho(H_{0}) \mid 1 \in \rho(K(\xi))\}$, $z_{0} \neq \lambda_{0},$

$$0 \neq g = [I_{\mathcal{K}} - K(z_{0})]^{-1}AR_{0}(z_{0})f$$

$$= (\lambda_{0} - z_{0})^{-1}Af.$$ \hspace{1cm} (3.4)

Conversely,

$$K(\lambda_{0})g = g, \quad 0 \neq g \in \mathcal{K} \quad \text{implies} \quad Hf = \lambda_{0}f,$$ \hspace{1cm} (3.6)

where

$$0 \neq f = -R_{0}(\lambda_{0})B^{*}g \in \text{dom}(H).$$ \hspace{1cm} (3.7)

Moreover,

$$\dim(\ker(H - \lambda_{0}I_{\mathcal{H}})) = \dim(\ker(I_{\mathcal{K}} - K(\lambda_{0}))) < \infty.$$ \hspace{1cm} (3.8)

In particular, let $z \in \rho(H_{0})$, then

$$z \in \rho(H) \quad \text{if and only if} \quad 1 \in \rho(K(z)).$$ \hspace{1cm} (3.9)
Proof. \( Hf = \lambda_0 f, \ 0 \neq f \in \text{dom}(H), \) is equivalent to \( f = (\lambda_0 - z_0)R(z_0)f \) and applying (2.13), we obtain, after a simple rearrangement,
\[
(H_0 - \lambda_0 I_{\mathcal{H}})R_0(z_0)f = -(\lambda_0 - z_0)\overline{R_0(z_0)B^*[I_K - K(z_0)]^{-1}AR_0(z_0)f}.
\] (3.10)
Next, define \( g = [I_K - K(z_0)]^{-1}AR_0(z_0)f. \) Then \( g \neq 0 \) since otherwise
\[
(H_0 - \lambda_0 I_{\mathcal{H}})R_0(z_0)f = 0, \quad 0 \neq R_0(z_0)f \in \text{dom}(H_0),
\] and hence \( \lambda_0 \in \sigma(H_0), \)
would contradict our hypothesis \( \lambda_0 \in \rho(H_0). \) Applying \( [I_K - K(z_0)]^{-1}AR_0(\lambda_0) \) to (3.10) then yields
\[
[I_K - K(z_0)]^{-1}AR_0(\lambda_0)(H_0 - \lambda_0 I_{\mathcal{H}})R_0(z_0)f = [I_K - K(z_0)]^{-1}AR_0(z_0)f = g
\]
\[
= -(\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(\lambda_0)\overline{R_0(z_0)B^*[I_K - K(z_0)]^{-1}AR_0(z_0)f}
= -(\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(z_0)\overline{AR_0(z_0)B^*g}.
\] (3.12)
Thus, using (2.10), one infers
\[
g = -(\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(\lambda_0)\overline{AR_0(z_0)B^*g}
= [I_K - K(z_0)]^{-1}[K(\lambda_0) - K(z_0)]g = g - [I_K - K(z_0)]^{-1}[I_K - K(\lambda_0)]g
\] (3.13)
and hence \( K(\lambda_0)g = g, \) proving (3.3). Since \( f = (\lambda_0 - z_0)R(z_0)f, \) using (2.19), one computes
\[
Af = (\lambda_0 - z_0)AR(z_0)f = (\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(z_0)f = (\lambda_0 - z_0)g,
\] (3.14)
proving (3.5).

Conversely, suppose \( K(\lambda_0)g = g, \ 0 \neq g \in \mathcal{K} \) and define \( f = -\overline{R_0(\lambda_0)B^*g}. \) Then a simple computation using (2.10) shows
\[
g = g - [I_K - K(z_0)]^{-1}[I_K - K(\lambda_0)]g
= [I_K - K(z_0)]^{-1}[K(\lambda_0) - K(z_0)]g = (\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(z_0)f.
\] (3.15)
Thus, \( f \neq 0 \) since \( f = 0 \) would imply the contradiction \( g = 0. \) Next, inserting the definition of \( f \) into (3.15) yields
\[
g = (\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(z_0)f = - (\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(z_0)\overline{AR_0(z_0)B^*g}.
\] (3.16)
Applying \( \overline{R_0(z_0)B^*} \) to (3.16) and taking into account
\[
\overline{R_0(z_0)B^*}g = \overline{[R_0(\lambda_0) - (\lambda_0 - z_0)R_0(z_0)R_0(\lambda_0)]B^*g} = - f + (\lambda_0 - z_0)R_0(z_0)f,
\] (3.17)
a combination of (3.17) and (2.13) yields that
\[
f - (z_0 - \lambda_0)R_0(z_0)f = (\lambda_0 - z_0)\overline{R_0(z_0)B^*[I_K - K(z_0)]^{-1}AR_0(z_0)f}
= (\lambda_0 - z_0)[R_0(z_0) - R(z_0)].
\] (3.18)
The latter is equivalent to \( (\lambda_0 - z_0)(H - z_0 I_{\mathcal{H}})^{-1}f = f. \) Thus, \( f \in \text{dom}(H) \) and \( Hf = \lambda_0 f, \)
proving (3.6).

Since \( K(\lambda_0) \in \mathcal{B}_{\infty} (\mathcal{K}), \) the eigenspace of \( K(\lambda_0) \) corresponding to the eigenvalue 1 is finite-dimensional. The previous considerations established a one-to-one correspondence between the geometric eigenspace of \( K(\lambda_0) \) corresponding to the eigenvalue 1 and the geometric eigenspace of \( H \) corresponding to the eigenvalue \( \lambda_0. \) This proves (3.8).

Finally, (3.8), (2.13), and (2.25) prove (3.9).

Remark 3.3. It is possible to avoid the compactness assumption in Hypothesis 3.1 in Theorem 3.2 provided that (3.8) is replaced by the following statement:
the subspaces \( \ker(H - \lambda_0 I_{\mathcal{H}}) \) and \( \ker(I_K - K(\lambda_0)) \) are isomorphic. (3.19)
(Of course, (3.8) follows from (3.19) provided \( \ker(I_K - K(\lambda_0)) \) is finite-dimensional, which in turn follows from Hypothesis 3.1.) Indeed, by formula (2.19), we have \( AR(z_0) = [I_K - K(z_0)]^{-1}AR_0(z_0). \) By formula (3.4), if \( f \neq 0, \) then \( g = AR(z_0)f \neq 0, \) and thus the operator
\[
AR(z_0) = [I_K - K(z_0)]^{-1}AR_0(z_0): \ker(H - \lambda_0 I) \rightarrow \ker(K(\lambda_0) - I)
\] (3.20)
is injective. By formula (3.16), this operator is also surjective, since each \( g \in \ker(K(\lambda_0) - I) \) belongs to its range,
\[
g = (\lambda_0 - z_0)[I_K - K(z_0)]^{-1}AR_0(z_0)f = AR(z_0)f,
\] (3.21)
where \( f \in \ker(H - \lambda_0 I). \)
4. ESSENTIAL SPECTRA AND A LOCAL WEINSTEIN–ARONSZAJN FORMULA

In this section, we closely follow Howland [20] and prove a result which demonstrates the invariance of the essential spectrum. However, since we will extend Howland’s result to the nonselfadjoint case, this requires further explanation. Moreover, we will also re-derive Howland’s local Weinstein–Arenszaïjn formula.

**Definition 4.1.** Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose $\{L(z)\}_{z \in \Omega}$ is a family of compact operators in $K$, which is analytic on $\Omega$ except for isolated singularities. Following Howland, we call $\{L(z)\}_{z \in \Omega}$ completely meromorphic on $\Omega$ if $L$ is meromorphic on $\Omega$ and the principal part of $L$ at each of its poles is of finite rank.

We start with an auxiliary result due to Steinberg [55] with a modification by Howland [20].

**Lemma 4.2** [20, 55]. Let $\{L(z)\}_{z \in \Omega}$ be an analytic (resp., completely meromorphic) family in $K$ on an open connected set $\Omega \subseteq \mathbb{C}$. Then for each $z_0 \in \Omega$, there is a neighborhood $U(z_0)$ of $z_0$, and an analytic $B(K)$-valued function $M$ on $U(z_0)$, such that $M(z)^{-1} \in B(K)$ for all $z \in U(z_0)$ and

$$M(z)[I_K - L(z)] = I_K - F(z), \quad z \in U(z_0),$$

where $F$ is analytic (resp., meromorphic) on $U(z_0)$ with $F(z)$ of finite rank (except at poles) for all $z \in U(z_0)$.

The next auxiliary result is due to Ribaric and Vidav [45].

**Lemma 4.3** [45]. Let $\{L(z)\}_{z \in \Omega}$ be a completely meromorphic family in $K$ on an open connected set $\Omega \subseteq \mathbb{C}$ and suppose that $L(z)$ has finite rank for each $z \in \Omega$ (except at poles). Then the following assertions hold.

(i) $I_K - L(z)$ is not boundedly invertible for all $z \in \Omega$, or

(ii) $\{|I_K - L(z)|^{-1} - I_K\}_{z \in \Omega}$ is completely meromorphic on $\Omega$.

Further, we state the following result due to Howland [21].

**Lemma 4.4** [21]. Let $\{L(z)\}_{z \in \Omega}$ be an analytic (resp., meromorphic) family on an open set $\Omega \subseteq \mathbb{C}$ on $\Omega$ if

$$\sigma(\tilde{\lambda}(\cdot)) = \sigma(\tilde{\lambda}(\cdot)) = \sigma(\tilde{\lambda}(\cdot)) = \sigma(\tilde{\lambda}(\cdot)),$$

where $\tilde{\lambda}(\cdot)$ is a counterclockwise oriented circle centered at $\lambda_0$ with sufficiently small radius $\varepsilon > 0$ (excluding the rest of $\sigma(T)$). Then $m(z, T)$, $z \in \mathbb{C}$, is defined by

$$m(z, T) = \begin{cases} 
0 & \text{if } z \in \rho(T), \\
\dim(\text{ran}(P(z, T))) & \text{if } z \text{ is an isolated eigenvalue of } T \\
+\infty & \text{of finite algebraic multiplicity}, \\
& \text{otherwise}.
\end{cases}$$

We note that the dimension of the Riesz projection in (4.3) is finite if and only if $\lambda_0$ is an isolated eigenvalue of $T$ of finite algebraic multiplicity (cf. [25, p. 181]). In analogy to the selfadjoint case (but deviating from most definitions in the nonselfadjoint case, see [12, Sec. I.4, Chap. IX]), we now introduce the set

$$\sigma_{\varepsilon}(T) = \{\lambda \in \mathbb{C} \mid \lambda \in \sigma(T), \lambda \text{ is not an isolated eigenvalue of } T \text{ of finite algebraic multiplicity}\}.$$  

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Of course, $\tilde{\sigma}_e(T)$ coincides with the essential spectrum of $T$ if $T$ is selfadjoint in $\mathcal{H}$. In the nonselfadjoint case at hand, the set $\tilde{\sigma}_e(T)$ is most natural in our study of $H_0$ and $H$ as will subsequently be shown. It will also be convenient to introduce the complement of $\tilde{\sigma}_e(T)$ in $\mathbb{C}$,

$$\tilde{\Phi}(T) = \mathbb{C} \setminus \tilde{\sigma}_e(T) = \rho(T) \cup \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } T \text{ of finite algebraic multiplicity}\}. \quad (4.6)$$

If $T$ is selfadjoint in $\mathcal{H}$, then $\tilde{\Phi}(T)$ is the Fredholm domain of $T$.

If $\lambda_0 \in \mathbb{C}$ is an isolated eigenvalue of $T$ of finite algebraic multiplicity, then the singularity structure of the resolvent of $T$ is of the type

$$(T - zI_\mathcal{H})^{-1} = (\lambda_0 - z)^{-1}P(\lambda_0, T) + \sum_{k=1}^{\infty} (\lambda_0 - z)^{-k-1}(-1)^k D(\lambda_0, T)^k + \sum_{k=0}^{\infty} (\lambda_0 - z)^k (-1)^k S(\lambda_0, T)^{k+1} \quad (4.7)$$

for $z$ in a sufficiently small neighborhood of $\lambda_0$. Here

$$D(\lambda_0, T) = (T - \lambda_0 I_\mathcal{H}) P(\lambda_0, T) = \frac{1}{2\pi i} \oint_{C(\lambda_0; \varepsilon)} d\zeta (\lambda_0 - \zeta)(T - \zeta I_\mathcal{H})^{-1} \in \mathcal{B}(\mathcal{H}), \quad (4.8)$$

$$S(\lambda_0, T) = -\frac{1}{2\pi i} \oint_{C(\lambda_0; \varepsilon)} d\zeta (\lambda_0 - \zeta)^{-1}(T - \zeta I_\mathcal{H})^{-1} \in \mathcal{B}(\mathcal{H}), \quad (4.9)$$

and $D(\lambda_0, T)$ is nilpotent with its range contained in that of $P(\lambda_0, T)$,

$$D(\lambda_0, T) = P(\lambda_0, T) D(\lambda_0, T) = D(\lambda_0, T) P(\lambda_0, T). \quad (4.10)$$

Moreover,

$$S(\lambda_0, T) T \subset TS(\lambda_0, T), \quad (T - \lambda_0 I_\mathcal{H}) S(\lambda_0, T) = I_\mathcal{H} - P(\lambda_0, T),$$

$$S(\lambda_0, T) P(\lambda_0, T) = P(\lambda_0, T) S(\lambda_0, T) = 0. \quad (4.11)$$

Finally,

$$\mu(\lambda_0, T) \leq m(\lambda_0, T) = \dim(\text{ran}(P(\lambda_0, T))), \quad (4.12)$$

$$\text{tr}(P(\lambda_0, T)) = m(\lambda_0, T), \quad \text{tr}(D(\lambda_0, T)^k) = 0 \text{ for some } k \in \mathbb{N}. \quad (4.13)$$

Next, we need one more notation: let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $f : \Omega \to \mathbb{C} \cup \{\infty\}$ be meromorphic and not identically vanishing on $\Omega$. The multiplicity function $m(z; f)$, $z \in \Omega$, is then defined by

$$m(z; f) = \begin{cases} k & \text{if } z \text{ is a zero of } f \text{ of order } k, \\ -k & \text{if } z \text{ is a pole of order } k, \\ 0 & \text{otherwise.} \end{cases} \quad (4.14)$$

$$m(z; f) = \frac{1}{2\pi i} \oint_{C(z; \varepsilon)} d\zeta \frac{f'(\zeta)}{f(\zeta)}, \quad z \in \Omega, \quad (4.15)$$

for $\varepsilon > 0$ sufficiently small. If $f$ vanishes identically on $\Omega$, one defines

$$m(z; f) = +\infty, \quad z \in \Omega. \quad (4.16)$$

Here the circle $C(z; \varepsilon)$ is chosen so small that $C(z; \varepsilon)$ contains no other singularities or zeros of $f$ except, possibly, $z$.

The following result is due to Howland in the case where $H_0$ and $H$ are selfadjoint. We will closely follow his strategy of proof and present detailed arguments in the more general situation considered here.

**Theorem 4.5.** Assume Hypothesis 3.1. Then

$$\tilde{\sigma}_e(H) = \tilde{\sigma}_e(H_0). \quad (4.17)$$

In addition, let $\lambda_0 \in \mathbb{C} \setminus \tilde{\sigma}_e(H_0)$. Then there exists a neighborhood $U(\lambda_0)$ of $\lambda_0$ and a function $\Delta(\cdot)$ meromorphic on $U(\lambda_0)$, which does not vanish identically, such that the local Weinstein–Aronszajn formula

$$m(z, H) = m(z, H_0) + m(z; \Delta), \quad z \in U(\lambda_0), \quad (4.18)$$

holds.

**Proof.** By (2.10), $K(\cdot)$ is analytic on $\rho(H_0)$ and

$$K'(z) = -AR_0(z)[BR_0(z)^*]^*, \quad z \in \rho(H_0). \quad (4.19)$$
Let $z_0 \in \tilde{\Phi}(H_0)$, then by (4.7),
\begin{equation}
R_0(z) = (z_0 - z)^{-1} P_0 + \sum_{k=1}^{\mu_0} (z_0 - z)^{-k-1} (-1)^k D_k^0 + G_0(z),
\end{equation}
where $G_0(\cdot)$ is analytic in a neighborhood of $z_0$. Since
\begin{equation}
\text{ran}(D_0) \subseteq \text{ran}(R_0) \subseteq \text{dom}(H_0) \subseteq \text{dom}(A),
\end{equation}
$AR_0 B^\ast$, $AD_0 B^\ast$, and $AG_0(z)B^\ast$ have compact extensions from $\text{dom}(B^\ast)$ to $\mathcal{K}$, and the extensions of $AP_0 B^\ast$ and $AD_0 B^\ast$ are given by the finite-rank operators $AP_0 [BP_0]^{\ast}$ and $AP_0 D_0 P_0 B^\ast$, respectively. Moreover, it is easy to see that the extension of $AG_0(z)B^\ast$ is analytic near $z_0$. Consequently, $K(\cdot)$ is completely meromorphic on $\tilde{\Phi}(H_0)$.

Similarly, by (3.2) and Lemma 4.3, $-\overline{AR(z)B^\ast}$ is completely meromorphic on $\tilde{\Phi}(H_0)$. Moreover, by (3.2), any singularity $z_0$ of $-\overline{AR(z)B^\ast}$ is an isolated point of $\sigma(H)$. Since $R_0(z), AR_0(z),$ and $BR_0(z)$ all have finite-rank principal parts at their poles, (2.13) and (3.2) show that $R(\cdot)$ also has a finite-rank principal part at $z_0$. The latter implies that $z_0$ is an eigenvalue of $H$ of finite algebraic multiplicity. Thus, $\Phi(H_0) \subseteq \tilde{\Phi}(H)$. Since, by Remark 2.4 (ii), this formalism is symmetric with respect to $H_0$ and $H$, one also obtains $\Phi(H_0) \supseteq \tilde{\Phi}(H)$, and hence (4.17).

Next, by Lemma 4.2, let $U_0$ be a neighborhood of $\lambda_0$ such that
\begin{equation}
M(z)[I_K - K(z)] = I_K - F(z),
\end{equation}
with $M$ analytic and boundedly invertible on $U_0$ and some $F$ meromorphic and of finite rank on $U_0$. One defines
\begin{equation}
\Delta(z) = \det(I_K - F(z)), \quad z \in U_0.
\end{equation}
Since by Lemma 4.3, $[I_K - K(z)]^{-1}$ is meromorphic and $M(z)$ is boundedly invertible for all $z \in U_0$, $[I_K - F(z)]^{-1}$ is also meromorphic on $U_0$, and hence, $\Delta(\cdot)$ is not identically zero on $U_0$. By Lemma 4.4 (iii) and cyclicity of the trace (i.e., $\text{tr}(ST) = \text{tr}(TS)$ for $S$ and $T$ bounded operators such that $ST$ and $TS$ lie in the trace class, cf. [52, Corollary 3.8]),
\begin{equation}
\frac{\Delta'(z)}{\Delta(z)} = -\text{tr}([I_K - F(z)]^{-1} F'(z))
= \text{tr}([I_K - K(z)]^{-1} M(z)^{-1} M'(z) [I_K - K(z)] - [I_K - K(z)]^{-1} K'(z))
= \text{tr}(M(z)^{-1} M'(z) - K'(z) [I_K - K(z)]^{-1}).
\end{equation}
Let $z_0 \in U_0$ and $C(z_0; \varepsilon)$ be a clockwise oriented circle centered at $z_0$ of sufficiently small radius $\varepsilon$ (excluding all singularities of $[I_K - F(z)]^{-1}$, except possibly for $z_0$) contained in $U_0$. Then
\begin{equation}
m(z_0; \Delta) = \frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta \frac{\Delta'(\zeta)}{\Delta(\zeta)} = \frac{1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta \text{tr}(M(\zeta)^{-1} M'(\zeta) - K'(\zeta) [I_K - K(z)]^{-1}).
\end{equation}
Since $M$ is analytic and boundedly invertible on $U_0$, an interchange of the trace and the integral, using
\begin{equation}
\oint_{C(z_0; \varepsilon)} d\zeta M(\zeta)^{-1} M'(\zeta) = 0
\end{equation}
and (4.19), yields
\begin{equation}
m(z_0; \Delta) = \frac{1}{2\pi i} \text{tr} \left( \oint_{C(z_0; \varepsilon)} d\zeta A R_0(\zeta) [B R_0(\zeta)]^* [I_K - K(\zeta)]^{-1} \right)
= \frac{1}{2\pi i} \text{tr} \left( \oint_{C(z_0; \varepsilon)} d\zeta A R_0(\zeta) [B R(\zeta)]^* \right).
\end{equation}
Next, for $\varepsilon > 0$ sufficiently small, one infers from [25, p. 178] (cf. (4.13)) that
\begin{equation}
m(z_0, H) - m(z_0, H_0) = -\frac{1}{2\pi i} \text{tr} \left( \oint_{C(z_0; \varepsilon)} d\zeta [R(\zeta) - R_0(\zeta)] \right)
= \frac{1}{2\pi i} \text{tr} \left( \oint_{C(z_0; \varepsilon)} d\zeta [B R_0(\zeta)]^* [I_K - K(\zeta)]^{-1} A R_0(\zeta) \right)
= \frac{1}{2\pi i} \text{tr} \left( \oint_{C(z_0; \varepsilon)} d\zeta [B R(\zeta)]^* A R_0(\zeta) \right).
\end{equation}
At this point, we cannot simply change back the order of the trace and the integral and use the cyclicity of the trace to prove equality of (4.27) and (4.28) since now the integrand is not necessarily of trace class. But we can prove the equality of (4.27) and (4.28) directly as follows. Writing

\[ AR_0(z) = (z_0 - z)^{-1} P_0 + \sum_{k=1}^{\mu_0} (z_0 - z)^{-k-1} (-1)^k D_0^k \sum_{k=0}^{\infty} (z_0 - z)^k (-1)^k \tilde{S}_0^{k+1}, \]

(4.29)

\[ [BR(z)]^*(z_0 - z)^{-1} Q_0 + \sum_{k=1}^{\mu_0} (z_0 - z)^{-k-1} (-1)^k E_0^k + \sum_{k=0}^{\infty} (z_0 - z)^k (-1)^k \tilde{T}_0^{k+1} \]

(4.30)

(cf. (4.7)), we obtain

\[ \text{res}_{z=z_0}(AR_0(z)[BR(z)]^*) = \tilde{T}_0 \tilde{P}_0 + \tilde{S}_0 \tilde{Q}_0 + \sum_{k=1}^{\mu_0} \tilde{D}_0^k \tilde{T}_0^{k+1} + \sum_{k=1}^{\nu_0} \tilde{S}_0^{k+1} \tilde{E}_0^k, \]

(4.31)

\[ \text{res}_{z=z_0}([BR(z)]^*AR_0(z)) = \tilde{T}_0 \tilde{P}_0 + \tilde{Q}_0 \tilde{S}_0 + \sum_{k=1}^{\mu_0} \tilde{T}_0^{k+1} \tilde{D}_0^k + \sum_{k=1}^{\nu_0} \tilde{E}_0^k \tilde{T}_0^{k+1}. \]

(4.32)

Using the cyclicity of the trace and Cauchy’s theorem, we then prove that (4.27) and (4.28) are and hence (4.18).

**Remark 4.6.** Let \( H_0 \) be as in Hypothesis 2.1.

(i) Let \( V \in \mathcal{B}_{\infty}(H) \) and define \( H = H_0 + V, \text{dom}(H) = \text{dom}(H_0). \) Then (4.18) holds identifying \( A = V, B = I, K(z) = VR_0(z) \) in connection with (2.13).

(ii) Let \( V \) be of finite-rank and define \( H = H_0 + V, \text{dom}(H) = \text{dom}(H_0). \) Then (4.18) holds on \( \Phi(H_0) \) with \( \Delta(z) = \text{det}(I_K - K(z)), K(z) = VR_0(z), \) \( z \in \rho(H_0), \) and \( U(\lambda_0) = \Phi(H_0). \)

With the exception of the case discussed in Remark 4.6 (ii), Theorem 4.5 has the drawback that it yields a Weinstein–Aronszajn-type formula only locally on \( U(\lambda_0). \) However, by the same token, the great generality of this formalism, basically assuming only the compactness of \( K(z), \) must be emphasized. In the following section, we will present Howland’s global Aronszajn–Weinstein formula.

5. A GLOBAL WEINSTEIN–ARONSZAJN FORMULA

To this end, we introduce a new hypothesis on \( K. \)

**Hypothesis 5.1.** In addition to Hypothesis 3.1, we assume the following condition.

(v) For some \( p \in \mathbb{N}, \) we have \( K(z) \in \mathcal{B}_p(\mathcal{K}) \) for all \( z \in \rho(H_0). \)

We denote by \( \| \cdot \|_p \) the norm in \( \mathcal{B}_p(\mathcal{K}) \) and by \( \text{det}_p(\cdot) \) the regularized determinant of operators of the type \( I_K - L, L \in \mathcal{B}_p(\mathcal{K}) \) (cf. [16; 17; 18, Chaps. IX–XI; 19, Sec. 4.2; 50; 52, Chap. 9]).

We start by recalling the following result (cf. [19, pp. 162–163; 52, p. 107]).

**Lemma 5.2.** Let \( p \in \mathbb{N} \) and assume that \( \{L(z)\}_{z \in \Omega} \in \mathcal{B}_p(\mathcal{K}) \) is a family of \( \mathcal{B}_p(\mathcal{K}) \)-analytic operators on \( \Omega, \text{with } \Omega \subseteq \mathbb{C} \text{ open.} \) Let \( \{P_n\}_{n \in \mathbb{N}} \) be a sequence of orthogonal projections in \( \mathcal{K} \) converging strongly to \( I_K \) as \( n \to \infty. \) Then the following limits hold uniformly with respect to \( z \) as \( z \) varies in compact subsets of \( \Omega: \)

\[ \lim_{n \to \infty} \|P_n L(z) P_n - L(z)\|_p = 0, \]

(5.1)

\[ \lim_{n \to \infty} \text{det}_p(I_K - P_n L(z) P_n) = \text{det}_p(I_K - L(z)), \]

(5.2)

\[ \lim_{n \to \infty} \frac{d}{dz} \text{det}_p(I_K - P_n L(z) P_n) = \frac{d}{dz} \text{det}_p(I_K - L(z)). \]

(5.3)

Thus, whereas the situation for analytic \( \mathcal{B}_p(\mathcal{K}) \)-valued functions is very satisfactory, there is a problem with meromorphic (even completely meromorphic) \( \mathcal{B}_p(\mathcal{K}) \)-valued functions, as was pointed out by Howland. Indeed, suppose \( L(z), z \in \Omega, \) is meromorphic in \( \Omega \) and of finite rank. Then of course \( \text{det}(I_K - L(z)) \) is meromorphic in \( \Omega. \) However, the formula

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To sidestep this difficulty, Howland extends the definition of $m$ in (4.14), (4.15) to functions $f$ with isolated essential singularities as follows: suppose $f$ is meromorphic in $\Omega$ except at isolated essential singularities. Then we use (4.15) again to define
\[ m(z; f) = \frac{1}{2\pi i} \oint_{C(z)} \frac{f(\zeta)}{f'(\zeta)} \, d\zeta, \quad z \in \Omega, \]
where $\varepsilon > 0$ is chosen sufficiently small to exclude all singularities and zeros of $f$ except possibly $z$.

Given Lemma 5.2 and the extension of $m(\cdot; f)$ to meromorphic functions with isolated essential singularities, Howland [20] then proves the following fundamental result (the proof of which is independent of any selfadjointness hypotheses on $H_0$ and $H$ and hence omitted here).

**Lemma 5.3** [20]. Let $p \in \mathbb{N}$ and assume that \{L(z)\}$_{z \in \Omega}$ is a family of $B_p(K)$-valued completely meromorphic operators on $\Omega$, $\Omega \subseteq \mathbb{C}$ open. Let $M(z)$ be a boundedly invertible operator-valued analytic function on $\Omega$ such that
\[ M(z)[I_{\mathcal{K}} - L(z)] = I_{\mathcal{K}} - F(z), \quad z \in \Omega, \]
where $F(z)$ is meromorphic and of finite rank for all $z \in \Omega$. Define
\[ \Delta(z) = \text{det}(I_{\mathcal{K}} - F(z)), \quad z \in \Omega, \]
and
\[ \Delta_p(z) = \text{det}_p(I_{\mathcal{K}} - L(z)), \quad z \in \Omega. \]
Then
\[ m(z; \Delta) = m(z; \Delta_p), \quad z \in \Omega. \]

Combining Theorem 4.5 and Lemma 5.3 yields Howland’s global Weinstein–Aronszajn formula [20] extended to the nonselfadjoint case.

**Theorem 5.4.** Assume Hypothesis 5.1. Then the global Weinstein–Aronszajn formula
\[ m(z, H) = m(z, H_0) + m(z; \text{det}_p(I_{\mathcal{K}} - K(z))), \quad z \in \Phi(H_0), \]
holds.

**Remark 5.5.** Let $H_0$ be as in Hypothesis 2.1, fix $p \in \mathbb{N}$, and assume $VR_0(z) \in B_p(H)$. Define $H = H_0 + V$, $\text{dom}(H) = \text{dom}(H_0)$. Then (5.10) holds on $\Phi(H_0)$ with $K(z) = VR_0(z)$. In the special case $p = 1$, this was first obtained by Kuroda [32].

6. AN APPLICATION OF PERTURBATION DETERMINANTS TO SCHROEDINGER OPERATORS IN DIMENSION $n = 1, 2, 3$

In dimension one on the half-line $(0, \infty)$, the perturbation determinant associated with the Birman–Schwinger kernel corresponding to a Schrödinger operator with an integrable potential on $(0, \infty)$ is known to coincide with the corresponding Jost function and hence with a simple Wronski determinant (cf. Lemmas 6.2 and 6.3). This reduction of an infinite-dimensional determinant to a finite-dimensional one is quite remarkable and, in this section, we intend to give some ideas as to how this fact can be generalized to dimensions two and three.

We start with the one-dimensional situation on the half-line $\Omega = (0, \infty)$ and introduce the Dirichlet and Neumann Laplacians $H_{D,0}^0$ and $H_{N,0}^0$ in $L^2((0, \infty); dx)$ by
\[ H_{D,0}^0 f = -f'', \quad f \in \text{dom}(H_{D,0}^0) = \{ g \in L^2((0, \infty); dx) \mid g, g' \in AC([0, R]) \text{ for all } R > 0, \quad g(0) = 0, \quad g'' \in L^2((0, \infty); dx) \}, \]
\[ H_{N,0}^0 f = -f'', \quad f \in \text{dom}(H_{N,0}^0) = \{ g \in L^2((0, \infty); dx) \mid g, g' \in AC([0, R]) \text{ for all } R > 0, \quad g'(0) = 0, \quad g'' \in L^2((0, \infty); dx) \}. \]

Next, we make the following assumption on the potential $V$. 

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Hypothesis 6.1. Suppose $V \in L^1((0, \infty); dx)$.

Given Hypothesis 6.1, we introduce the perturbed operators $H^D_{0+}$ and $H^N_{0+}$ in $L^2((0, \infty); dx)$ by

$$H^D_{0+} f = -f'' + V f,$$

$$H^N_{0+} f = -f'' + V f,$$

$$f \in \text{dom}(H^D_{0+}) = \{ g \in L^2((0, \infty); dx) \mid g, g' \in AC([0, R]) \text{ for all } R > 0, \quad g(0) = 0, (-g'' + V g) \in L^2((0, \infty); dx) \},$$

$$f \in \text{dom}(H^N_{0+}) = \{ g \in L^2((0, \infty); dx) \mid g, g' \in AC([0, R]) \text{ for all } R > 0, \quad g'(0) = 0, (-g'' + V g) \in L^2((0, \infty); dx) \}.$$  

A fundamental system of solutions $\phi^D_+(z, \cdot)$, $\theta^D_+(z, \cdot)$, and the Jost solution $f_+(z, \cdot)$ of

$$-\psi''(z, x) + V \psi(z, x) = z \psi(z, x), \quad z \in \mathbb{C} \setminus \{0\}, \quad x \geq 0,$$  

are introduced by

$$\phi^D_+(z, x) = z^{-1/2} \sin(z^{1/2} x) + \int_0^x dx' g^{(0)}_+(z, x', x) V(x') \phi^D_+(z, x'),$$

$$\theta^D_+(z, x) = \cos(z^{1/2} x) + \int_0^x dx' g^{(0)}_+(z, x', x) V(x') \theta^D_+(z, x'),$$

$$f_+(z, x) = e^{i z^{1/2} x} - \int_x^\infty dx' g^{(0)}_+(z, x', x) V(x') f_+(z, x'),$$

$$\text{Im}(z^{1/2}) \geq 0, \quad z \in \mathbb{C} \setminus \{0\}, \quad x \geq 0,$$

where

$$g^{(0)}_+(z, x, x') = z^{-1/2} \sin(z^{1/2} x - x').$$

We introduce

$$u = \exp(i \text{arg}(V)) |V|^{1/2}, \quad v = |V|^{1/2}, \text{ so that } V = uv,$$

and denote by $I_+$ the identity operator in $L^2((0, \infty); dx)$. In addition, we let

$$W(f, g)(x) = f(x)g(x) - f'(x)g'(x), \quad x \geq 0,$$

denote the Wronskian of $f$ and $g$, where $f, g \in C^1((0, \infty))$. We also recall our convention to denote by $M_f$ the operator of multiplication in $L^2((0, \infty); dx)$ by an element $f \in L^1_{\text{loc}}((0, \infty); dx)$ (and similarly in the higher-dimensional context in the main part of this section).

The following is a modern formulation of a classical result by Jost and Pais [23].

Lemma 6.2 [14, Th. 4.3]. Assume Hypothesis 6.1 and $z \in \mathbb{C} \setminus [0, \infty]$ with $\text{Im}(z^{1/2}) > 0$. Then

$$M_u(H^D_{0+} - z I_+)^{-1} M_v \in \mathcal{B}_1(L^2((0, \infty); dx))$$

and

$$\det \left( I_+ + M_u(H^D_{0+} - z I_+)^{-1} M_v \right) = 1 + z^{-1/2} \int_0^\infty dx \sin(z^{1/2} x) V(x) f_+(z, x)$$

$$= W(f_+(z, \cdot), \phi^D_+(z, \cdot)) = f_+(z, 0).$$

Performing calculations similar to Section 4 in [14] for the pair of operators $H^N_{0+}$ and $H^N_+$, one also obtains the following result.

Lemma 6.3. Assume Hypothesis 6.1 and $z \in \mathbb{C} \setminus [0, \infty]$ with $\text{Im}(z^{1/2}) > 0$. In this case, we have

$$M_u(H^N_{0+} - z I_+)^{-1} M_v \in \mathcal{B}_1(L^2((0, \infty); dx))$$

and

$$\det \left( I_+ + M_u(H^N_{0+} - z I_+)^{-1} M_v \right) = 1 + iz^{-1/2} \int_0^\infty dx \cos(z^{1/2} x) V(x) f_+(z, x)$$

$$= -\frac{W(f_+(z, \cdot), \theta^D_+(z, \cdot))}{iz^{1/2}} = f'_+(z, 0) \frac{i z^{1/2}}{iz^{1/2}}.$$

We emphasize that (6.12) and (6.13) exhibit the remarkable fact that the Fredholm determinant associated with trace class operators in the infinite-dimensional space $L^2((0, \infty); dx)$ is reduced to a simple Wronski determinant of $\mathbb{C}$-valued distributional solutions of (6.5). This fact goes back
to Jost and Pais [23] (see also [14; 37; 39; 41, Sec. 12.1.2; 52, Prop. 5.7; 53], and the extensive literature cited in these references). The principal aim of this section is to explore the possibility to extend this fact to higher dimensions \( n = 2, 3 \). While a straightforward generalization of (6.12), (6.13) appears to be difficult, we will next derive a formula for the ratio of such determinants which permits a direct extension to the dimensions \( n = 2, 3 \).

For this purpose, we introduce the boundary trace operators \( \gamma_D \) (Dirichlet trace) and \( \gamma_N \) (Neumann trace) which, in the current one-dimensional half-line situation, are just the functionals

\[
\begin{align*}
\gamma_D & : C^1([0, \infty)) \to \mathbb{C}, \\
& \quad g \mapsto g(0), \\
\gamma_N & : C^1([0, \infty)) \to \mathbb{C}, \\
& \quad h \mapsto -h'(0).
\end{align*}
\]

(6.14)

In addition, we denote by \( m_{0,+}, m_{N,+}^D, m_{N,+}^N, \) and \( m_N^N \) the Weyl–Titchmarsh \( m \)-functions corresponding to \( H_{0,+}^D, H_{0,+}^N, \) and \( H_N^+, \) respectively,

\[
\begin{align*}
m_{0,+}^D(z) &= iz^{1/2}, \\
m_{0,+}^N(z) &= -\frac{1}{m_{0,+}^D(z)} = iz^{-1/2}, \\
m_{+}^D(z) &= \frac{f_1'(z, 0)}{f_+(z, 0)}, \\
m_{+}^N(z) &= -\frac{1}{m_{+}^D(z)} = -\frac{f_+(z, 0)}{f_+(z, 0)}.
\end{align*}
\]

(6.15)

(6.16)

**Theorem 6.4.** Assume Hypothesis 6.1 and let \( z \in \mathbb{C} \setminus \sigma(H_+^D) \) with \( \text{Im}(z^{1/2}) > 0 \). Then

\[
\frac{\det (I_+ + M_u(h_{0,+}^N - zI_+)^{-1}M_v)}{\det (I_+ + M_u(h_{0,+}^D - zI_+)^{-1}M_v)} = \frac{W(f_+(z, 0), \phi_+^N(z))}{iz^{1/2}W(f_+(z, 0), \phi_+^D(z))} = \frac{f_1'(z, 0)}{m_{+}^N(z)} = \frac{m_{+}^D(z)}{m_{+}^N(z)} = \frac{m_{0,+}^N(z)}{m_{0,+}^D(z)}
\]

(6.17)

(6.18)

**Proof.** We start by noting that \( \sigma(H_{0,+}^D) = \sigma(H_{0,+}^N) = [0, \infty) \). Applying Lemmas 6.2 and 6.3 and equations (6.15) and (6.16), we obtain (6.17).

To verify the equality of (6.17) and (6.18) requires some preparations. First we recall that the Green’s functions (i.e., integral kernels) of the resolvents of \( H_{0,+}^D \) and \( H_{0,+}^N \) are given by

\[
(H_{0,+}^D - zI_+)^{-1}(x, x') = \begin{cases}
\sin(z^{1/2}x) e^{iz^{1/2}x'}, & 0 \leq x \leq x', \\
\sin(z^{1/2}x') e^{iz^{1/2}x}, & 0 \leq x' \leq x,
\end{cases}
\]

(6.19)

\[
(H_{0,+}^N - zI_+)^{-1}(x, x') = \begin{cases}
\cos(z^{1/2}x) e^{iz^{1/2}x'}, & 0 \leq x \leq x', \\
\cos(z^{1/2}x') e^{iz^{1/2}x}, & 0 \leq x' \leq x,
\end{cases}
\]

(6.20)

and hence Krein’s formula for the resolvent difference of \( H_{0,+}^D \) and \( H_{0,+}^N \) becomes

\[
(H_{0,+}^D - zI_+)^{-1} - (H_{0,+}^N - zI_+)^{-1} = -iz^{-1/2} \frac{\psi_0^+(z, \cdot)}{\psi_0^+(z, \cdot)} L_z((0, \infty); dx) \psi_0^+(z, \cdot),
\]

(6.21)

where we abbreviated

\[
\psi_0^+(z, x) = e^{iz^{1/2}x}, \quad \text{Im}(z^{1/2}) > 0, \quad x > 0.
\]

(6.22)

We also recall that

\[
(H_{+}^D - zI_+)^{-1}(x, x') = \begin{cases}
\phi_{+}^D(z, x) \psi_+(z, x'), & 0 \leq x \leq x', \\
\phi_{+}^D(z, x') \psi_+(z, x), & 0 \leq x' \leq x,
\end{cases}
\]

(6.23)

where

\[
\psi_+(z, x) = \theta_{+}^D(z, x) + m_{+}^D(z) \phi_{+}^D(z, x), \quad z \in \rho(H_{+}^D), \quad x > 0,
\]

(6.24)

and

\[
\psi_+(z, \cdot) = \frac{f_+(z, \cdot)}{f_+(z, 0)} \in L^2((0, \infty); dx), \quad z \in \rho(H_{+}^D).
\]

(6.25)

In fact, a standard iteration argument applied to (6.8) shows that

\[
|\psi_+(z, x)| \leq C(z) e^{-\text{Im}(z^{1/2})x}, \quad \text{Im}(z^{1/2}) > 0, \quad x > 0.
\]

(6.26)
In addition, we note that
\[ \gamma_N(H_{0,+}^D - zI_+)^{-1} g = -\int_0^\infty dx e^{iz^{1/2}x} g(x), \quad g \in L^2((0, \infty); dx), \]
(6.27)
\[ \gamma_N(H_{0,+}^D - zI_+)^{-1} g = -\int_0^\infty dx \psi_+(z, x) g(x), \quad g \in L^2((0, \infty); dx), \]
(6.28)
\[ \gamma_D(H_{0,+}^N - zI_+)^{-1} f = iz^{-1/2} \int_0^\infty dx e^{iz^{1/2}x} f(x), \quad f \in L^2((0, \infty); dx), \]
(6.29)
and hence
\[ \left[ \left[ \gamma_D(H_{0,+}^N - zI_+)^{-1} \right]^* c \right](\cdot) = icz^{-1/2}\psi_0(\cdot, \cdot), \quad c \in \mathbb{C}. \]
(6.30)
Then Krein’s formula (6.21) can be rewritten as
\[ (H_{0,+}^D - zI_+)^{-1} - (H_{0,+}^N - zI_+)^{-1} = \left[ \gamma_D(H_{0,+}^N - zI_+)^{-1} \right]^* \gamma_N(H_{0,+}^D - zI_+)^{-1}, \]
(6.31)
z ∈ \rho(H_{0,+}^D) ∩ \rho(H_{0,+}^N), \quad \text{Im}(z^{1/2}) > 0.

Finally, using the facts (cf. (6.8))
\[ f_+(z, 0) = 1 + z^{-1/2} \int_0^\infty dx \sin(z^{1/2}x)V(x)f_+(z, x), \]
(6.32)
\[ f'_+(z, 0) = iz^{1/2} - \int_0^\infty dx \cos(z^{1/2}x)V(x)f_+(z, x), \]
(6.33)
one computes (since v ∈ L^2(\mathbb{R}; dx) and ψ_+(z, ·) ∈ L^\infty(\mathbb{R}; dx))
\[ - \gamma_N(H_{0,+}^D - zI_+)^{-1} M_v \gamma_D(H_{0,+}^D - zI_+)^{-1} = iz^{-1/2}\gamma_N(H_{0,+}^D - zI_+)^{-1} M_u(\psi_0(\cdot, \cdot) \gamma_N(H_{0,+}^D - zI_+)^{-1} \gamma_D(H_{0,+}^N - zI_+)^{-1} \gamma_N(H_{0,+}^D - zI_+)^{-1} \gamma_D(H_{0,+}^N - zI_+)^{-1}
\]
(6.34)
At first sight, it may seem unusual to even attempt to prove (6.18) in the one-dimensional case since (6.17) already yields the reduction of a Fredholm determinant to a simple Wronskian determinant. However, we will see in Theorem 6.11 that it is precisely (6.18) that permits a straightforward extension to dimensions n = 2, 3.

**Remark 6.5.** As in Theorem 6.4, we assume Hypothesis 6.1 and suppose \( z \in \mathbb{C} \setminus \sigma(H_{0,+}^D) \). First we note that
\[ (H_{0,+}^D - zI_+)^{-1/2}(H_{0,+}^D - zI_+)(H_{0,+}^D - zI_+)^{-1/2} - I_+ \in B_1(L^2((0, \infty); dx)), \]
(6.35)
\[ (H_{0,+}^N - zI_+)^{-1/2}(H_{0,+}^N - zI_+)(H_{0,+}^N - zI_+)^{-1/2} - I_+ \in B_1(L^2((0, \infty); dx)). \]
(6.36)
Indeed, it follows from the proof of [14, Th. 4.2] (cf. also Lemma 6.8 below) that
\[ (H_{0,+}^D - zI_+)^{-1/2}M_v(H_{0,+}^D - zI_+)^{-1/2} \in B_2(L^2((0, \infty); dx)), \]
(6.37)
and hence
\[ (H_{0,+}^D - zI_+)^{-1/2}(H_{0,+}^D - zI_+)(H_{0,+}^D - zI_+)^{-1/2} - I_+ \]
(6.38)
\[ = (H_{0,+}^D - zI_+)^{-1/2}M_v(H_{0,+}^D - zI_+)^{-1/2} \in B_1(L^2((0, \infty); dx)). \]
(6.39)
This proves (6.35), and a similar argument yields (6.36). Using the cyclicity of det(·), one can then rewrite the left-hand side of (6.17) as follows:
\[ \frac{\det(I_+ + M_u(H_{0,+}^N - zI_+)^{-1}M_v)}{\det(I_+ + M_u(H_{0,+}^D - zI_+)^{-1}M_v)} = \frac{\det(I_+ + (H_{0,+}^N - zI_+)^{-1/2}M_v(H_{0,+}^N - zI_+)^{-1/2})}{\det(I_+ + (H_{0,+}^D - zI_+)^{-1/2}M_v(H_{0,+}^D - zI_+)^{-1/2})}
\]
\[ = \frac{\det((H_{0,+}^N - zI_+)^{-1/2}(H_{0,+}^N - zI_+)(H_{0,+}^N - zI_+)^{-1/2})}{\det((H_{0,+}^D - zI_+)^{-1/2}(H_{0,+}^D - zI_+)(H_{0,+}^D - zI_+)^{-1/2})}. \]
(6.40)
Equation (6.40) illustrates the kind of symmetrized perturbation determinants underlying Theorem 6.4.

Now we turn to dimensions $n = 2, 3$. As a general rule, we will have to replace Fredholm determinants by modified ones.

For the remainder of this section, we make the following assumptions on the domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, and the potential $V$.

**Hypothesis 6.6.** Let $n = 2, 3$.

(i) Assume that $\Omega \subset \mathbb{R}^n$ is an open nonempty domain of class $C^{1, \gamma}$ for some $(1/2) < \gamma < 1$ with a compact, nonempty boundary $\partial \Omega$. (For details we refer to Appendix A.)

(ii) Suppose that $V \in L^2(\Omega; d^n x)$.

First we introduce the boundary trace operator $\gamma_D^0$ (Dirichlet trace) by

$$
\gamma_D^0 : C(\overline{\Omega}) \to C(\partial \Omega), \quad \gamma_D^0 u = u|_{\partial \Omega}.
$$

(6.41)

Then there exists a bounded linear operator $\gamma_D$, $\gamma_D^1 : H^s(\Omega) \to H^{s-1/2}(\partial \Omega) \to L^2(\partial \Omega; d^{n-1} \sigma)$, $1/2 < s < 3/2,$

(6.42)

whose action is compatible with $\gamma_D^0$, i.e., the two Dirichlet trace operators coincide on the intersection of their domains. It is well-known (see, e.g., [33, Th. 3.38]), that $\gamma_D$ is bounded. Here $d^{n-1} \sigma$ denotes the surface measure on $\partial \Omega$ and we refer to Appendix A for our notation in connection with Sobolev spaces.

Next, let $I_{\partial \Omega}$ denote the identity operator in $L^2(\partial \Omega; d^{n-1} \sigma)$, and introduce the operator $\gamma_N$ (Neumann trace) by

$$
\gamma_N = \nu \cdot \gamma_D \nabla : H^{s+1}(\Omega) \to L^2(\partial \Omega; d^{n-1} \sigma), \quad 1/2 < s < 3/2,
$$

(6.43)

where $\nu$ denotes the outward pointing normal unit vector to $\partial \Omega$. It follows from (6.42) that $\gamma_N$ is also a bounded operator.

Given Hypothesis 6.6 (i), we introduce the Dirichlet and Neumann Laplacians $H_{0, \Omega}^D$ and $H_{0, \Omega}^N$ associated with the domain $\Omega$ as follows:

$$
H_{0, \Omega}^D = -\Delta, \quad \text{dom}(H_{0, \Omega}^D) = \{ u \in H^2(\Omega) \mid \gamma_D u = 0 \},
$$

(6.44)

$$
H_{0, \Omega}^N = -\Delta, \quad \text{dom}(H_{0, \Omega}^N) = \{ u \in H^2(\Omega) \mid \gamma_N u = 0 \}.
$$

(6.45)

In the following, we denote by $I_{\Omega}$ the identity operator in $L^2(\Omega; d^n x)$.

**Lemma 6.7.** Assume Hypothesis 6.6 (i). Then the operators $H_{0, \Omega}^D$ and $H_{0, \Omega}^N$ introduced in (6.44) and (6.45) are nonnegative and selfadjoint in $H = L^2(\Omega; d^n x)$ and the following mapping properties hold for all $q \in \mathbb{R}$ and $z \in \mathbb{C} \setminus [0, \infty)$:

$$
(H_{0, \Omega}^D - z I_{\Omega})^{-q}, (H_{0, \Omega}^N - z I_{\Omega})^{-q} \in \mathcal{B}(L^2(\Omega; d^n x), H^{2q}(\Omega)).
$$

(6.46)

The fractional powers in (6.46) (and in subsequent analogous cases such as in (6.54)) are defined via the functional calculus implied by the spectral theorem for selfadjoint operators. For the proof of Lemma 6.7, we refer to Lemmas A.1 and A.2 in Appendix A.

**Lemma 6.8.** Assume Hypothesis 6.6 (i) and let $(n/2)p < q \leq 1, p \geq 2, n = 2, 3, f \in L^p(\Omega; d^n x)$, and $z \in \mathbb{C} \setminus [0, \infty)$. Then

$$
M_f (H_{0, \Omega}^D - z I_{\Omega})^{-q} \in \mathcal{B}(L^2(\Omega; d^n x)),
$$

(6.47)

and for some $c > 0$ (independent of $z$ and $f$)

$$
\| M_f (H_{0, \Omega}^D - z I_{\Omega})^{-q} \|_{\mathcal{L}(L^p(\mathbb{R}^n; d^n x))} + \| M_f (H_{0, \Omega}^N - z I_{\Omega})^{-q} \|_{\mathcal{L}(L^p(\mathbb{R}^n; d^n x))} \leq c \| (|\cdot|^2 - z)^{-q} \|_{L^p(\mathbb{R}^n; d^n x)}.
$$

(6.48)

**Proof.** We start by noting that under the assumption that $\Omega$ is a Lipschitz domain, there is a bounded extension operator $\mathcal{E}$,

$$
\mathcal{E} \in \mathcal{B}(H^{2q}(\Omega), H^{2q}(\mathbb{R}^n)) \quad \text{such that} \quad (\mathcal{E} u)|_\Omega = u, \quad u \in H^{2q}(\Omega)
$$

(6.49)
(see, e.g., [33, Th. A.4]). Next, denote by $\mathcal{R}_\Omega$ the restriction operator

$$\mathcal{R}_\Omega: \left\{ \begin{array}{ll}
L^2(\mathbb{R}^n; d^n x) \to L^2(\Omega; d^n x), \\
u \mapsto u|_\Omega,
\end{array} \right.$$ (6.50)

and let $\tilde{f}$ denote the following extension of $f$,

$$\tilde{f}(x) = \left\{ \begin{array}{ll}
f(x), & x \in \Omega, \\
0, & x \in \mathbb{R}^n \setminus \Omega,
\end{array} \right. \tilde{f} \in L^p(\mathbb{R}^n; d^n x).$$ (6.51)

Then

$$M_j(H_{0,\Omega}^D - zI_\Omega)^{-q} = \mathcal{R}_\Omega M_j(H_0 - zI)^{-q}(H_0 - zI)^q \mathcal{E}(H_{0,\Omega}^D - zI_\Omega)^{-q},$$ (6.52)

where (for simplicity) $I$ denotes the identity operator in $L^2(\mathbb{R}^n; d^n x)$ and $H_0$ denotes the nonnegative selfadjoint operator

$$H_0 = -\Delta, \quad \text{dom}(H_0) = H^2(\mathbb{R}^n),$$ (6.53)

in $L^2(\mathbb{R}^n; d^n x)$. Utilizing the representation of $(H_0 - zI)^q$ as the operator of multiplication by $(|\xi|^2 - z)^q$ in the Fourier space $L^2(\mathbb{R}^n; d^n \xi)$, one obtains

$$(H_0 - zI)^q \in \mathcal{B}(L^{2q}(\mathbb{R}^n), L^2(\mathbb{R}^n; d^n x)),$$ (6.54)

which together with (6.46) and the mapping property of the extension operator $\mathcal{E}$ in (6.49) yields

$$(H_0 - zI)^q \mathcal{E}(H_{0,\Omega}^D - zI_\Omega)^{-q} \in \mathcal{B}(L^2(\Omega; d^n x), L^2(\mathbb{R}^n; d^n x)).$$ (6.55)

By [52, Th. 4.1] (or [43, Th. XI.20]), one also obtains

$$M_j(H_0 - zI)^{-q} \in \mathcal{B}(L^2(\mathbb{R}^n; d^n x))$$ (6.56)

and

$$\|M_j(H_0 - zI)^{-q}\|_{\mathcal{B}_p(L^2(\Omega; d^n x))} \leq c \|(|\xi|^2 - z)^{-q}\|_{L^p(\mathbb{R}^n; d^n x)} \|\tilde{f}\|_{L^p(\mathbb{R}^n; d^n x)}$$ (6.57)

Thus, the Dirichlet parts of (6.47) and (6.48) follow from (6.52), (6.55), (6.56), and (6.57).

Similar arguments prove the Neumann parts of (6.47) and (6.48).

**Lemma 6.9.** Assume Hypothesis 6.6 (i) and let $z \in (0, 1]$, $n = 2, 3$, and $z \in \mathbb{C}\setminus[0, \infty)$. Then

$$\gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-\frac{3+s}{2}}, \gamma_D(H_{0,\Omega}^N - zI_\Omega)^{-\frac{1+s}{2}} \in \mathcal{B}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1} \sigma)).$$ (6.58)

**Proof.** It follows from (6.46) that

$$(H_{0,\Omega}^D - zI_\Omega)^{-\frac{3+s}{2}} \in \mathcal{B}(L^2(\Omega; d^n x), H^{3+s}(\Omega)), \quad (6.59)$$

$$(H_{0,\Omega}^N - zI_\Omega)^{-\frac{1+s}{2}} \in \mathcal{B}(L^2(\Omega; d^n x), H^{1+s}(\Omega)), \quad (6.60)$$

and hence one infers the result from (6.42) and (6.43).

**Corollary 6.10.** Let $f_1 \in L^{p_1}(\Omega; d^n x)$, $p_1 > 2n$, $f_2 \in L^{p_2}(\Omega; d^n x)$, $p_2 \geq 2$, $p_2 > 2n/3$, $n = 2, 3$, and $z \in \mathbb{C}\setminus[0, \infty)$. Then

$$\gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-1}M_{f_1} \in \mathcal{B}_{p_1}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1} \sigma)), \quad (6.61)$$

$$\gamma_D(H_{0,\Omega}^N - zI_\Omega)^{-1}M_{f_2} \in \mathcal{B}_{p_2}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1} \sigma)) \quad (6.62)$$

and for some $c_j(z) > 0$ (independent of $f_j$), $j = 1, 2$,

$$\|\gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-1}M_{f_1}\|_{\mathcal{B}_{p_1}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1} \sigma))} \leq c_1(z) \|f_1\|_{L^{p_1}(\Omega; d^n x)},$$ (6.63)

$$\|\gamma_D(H_{0,\Omega}^N - zI_\Omega)^{-1}M_{f_2}\|_{\mathcal{B}_{p_2}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1} \sigma))} \leq c_2(z) \|f_2\|_{L^{p_2}(\Omega; d^n x)}.$$(6.64)

**Proof.** Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ be such that $0 < \varepsilon_1 < 1 - (2n/p_1)$ and $0 < \varepsilon_2 < \min \{1, 3 - (2n/p_2)\}$. Then

$$\gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-1}M_{f_1} = \gamma_N(H_{0,\Omega}^D - zI_\Omega)^{-\frac{3+s}{2}}(H_{0,\Omega}^D - zI_\Omega)^{-\frac{1+s}{2}}M_{f_1},$$ (6.65)

$$\gamma_D(H_{0,\Omega}^N - zI_\Omega)^{-1}M_{f_2} = \gamma_D(H_{0,\Omega}^N - zI_\Omega)^{-\frac{1+s}{2}}(H_{0,\Omega}^N - zI_\Omega)^{-\frac{3+s}{2}}M_{f_2},$$ (6.66)

together with Lemmas 6.8 and 6.9 prove the corollary.
Next, we introduce the perturbed operators $H_D^0$ and $H_N^0$ in $L^2(\Omega; d^n x)$ as follows. We denote by $A = M_u$ and by $B = B^* = M_v$ the operators of multiplication by $u = \exp(i \arg(V)) |V|^{1/2}$ and $v = |V|^{1/2}$ in $L^2(\Omega; d^n x)$, respectively, so that $M_V = BA = M_u M_v$. Applying Lemma 6.8 to $f = u \in L^4(\Omega; d^n x)$ with $q = 1/2$ yields
\begin{align}
M_u(H_D^0 - zI_\Omega)^{-1/2}, (H_D^0 - zI_\Omega)^{-1/2}M_v &\in B_4(L^2(\Omega; d^n x)), \quad z \in \mathbb{C} \setminus [0, \infty), \quad (6.67) \\
M_u(H_N^0 - zI_\Omega)^{-1/2}, (H_N^0 - zI_\Omega)^{-1/2}M_v &\in B_4(L^2(\Omega; d^n x)), \quad z \in \mathbb{C} \setminus [0, \infty), \quad (6.68)
\end{align}
and hence, in particular,
\begin{align}
dom(A) &= \dom(B) \supseteq H^1(\Omega) \supseteq H^2(\Omega) \supseteq \dom(H_N^0), \quad (6.69) \\
dom(A) &= \dom(B) \supseteq H^1(\Omega) \supseteq \dom(H_D^0). \quad (6.70)
\end{align}
Thus, Hypothesis 2.1 (i) is satisfied for $H_D^0$ and $H_N^0$. Moreover, (6.67) and (6.68) imply
\begin{align}
M_u(H_D^0 - zI_\Omega)^{-1}M_v, M_u(H_N^0 - zI_\Omega)^{-1}M_v &\in B_2(L^2(\Omega; d^n x)), \quad z \in \mathbb{C} \setminus [0, \infty), \quad (6.71)
\end{align}
which verifies Hypothesis 2.1 (ii) for $H_D^0$ and $H_N^0$. One verifies Hypothesis 2.1 (iii) by utilizing (6.48) with sufficiently negative $z < 0$, so that the $B_4$-norms of the operators in (6.67) and (6.68) are less than 1, and hence, the Hilbert–Schmidt norms of the operators in (6.71) are less than 1. Thus, applying Theorem 2.3, one obtains the densely defined, closed operators $H_D^0$ and $H_N^0$ (which are extensions of $H_D^0 + M_V$ on $\dom(H_D^0) \cap \dom(M_V)$ and $H_N^0 + M_V$ on $\dom(H_N^0) \cap \dom(M_V)$, respectively).

We note in passing that (6.46)–(6.48), (6.58), (6.61)–(6.64), (6.67)–(6.71), etc., extend of course to all $z$ in the resolvent set of the corresponding operators $H_D^0$ and $H_N^0$.

The following result is a direct extension of the one-dimensional result in Theorem 6.4.

**Theorem 6.11.** Assume Hypothesis 6.6 and $z \in \mathbb{C} \setminus (\sigma(H_D^0) \cup \sigma(H_D^0) \cup \sigma(H_N^0))$. Then
\begin{align}
\gamma_N(H_D^0 - zI_\Omega)^{-1}M_V(H_D^0 - zI_\Omega)^{-1}M_V [\gamma_D(H_N^0 - zI_\Omega)^{-1}]^* &\in B_1(L^2(\partial \Omega; d^n - 1)) \quad (6.72) \\
\gamma_N(H_D^0 - zI_\Omega)^{-1}M_V [\gamma_D(H_N^0 - zI_\Omega)^{-1}]^* &\in B_1(L^2(\partial \Omega; d^n - 1)), \quad (6.73)
\end{align}
and
\begin{align}
\det_2 \left( I_\Omega + M_u(H_N^0 - zI_\Omega)^{-1}M_v \right) = \det_2 \left( I_\Omega - \gamma_N(H_D^0 - zI_\Omega)^{-1}M_V [\gamma_D(H_N^0 - zI_\Omega)^{-1}]^* \right) \\
\times \exp \left( \tr \left( \gamma_N(H_D^0 - zI_\Omega)^{-1}M_V(H_D^0 - zI_\Omega)^{-1}M_V [\gamma_D(H_N^0 - zI_\Omega)^{-1}]^* \right) \right). \quad (6.74)
\end{align}

**Proof.** From the outset, we note that the left-hand side of (6.74) is well-defined by (6.71). Let $z \in \mathbb{C} \setminus (\sigma(H_D^0) \cup \sigma(H_D^0) \cup \sigma(H_N^0))$ and
\begin{align}
u(x) &= \exp(i \arg(V(x))) |V(x)|^{1/2}, \quad v(x) = |V(x)|^{1/2}, \quad (6.75) \\
\tilde{u}(x) &= \exp(i \arg(V(x))) |V(x)|^{5/6}, \quad \tilde{v}(x) = |V(x)|^{1/6}. \quad (6.76)
\end{align}
Next, we introduce
\begin{align}
K_D(z) &= -M_u(H_D^0 - zI_\Omega)^{-1}M_v, \quad K_N(z) = -M_u(H_N^0 - zI_\Omega)^{-1}M_v \quad (6.77)
\end{align}
(cf. (2.4)) and utilize the following facts:
\begin{align}
[I_\Omega - K_D(z)]^{-1} &= I_\Omega + K_D(z)[I_\Omega - K_D(z)]^{-1}, \quad (6.78) \\
[I_\Omega - K_D(z)]^{-1} &\in B(L^2(\Omega; d^n x)), \quad (6.79)
\end{align}
and
\begin{align}
1 &= \det_2(I_\Omega) = \det_2 \left( [I_\Omega - K_D(z)](I - K_D(z))^{-1} \right) \quad (6.80) \\
&= \det_2 \left( I_\Omega - K_D(z) \right) \det_2 \left( [I_\Omega - K_D(z)]^{-1} \right) \exp \left( \tr \left( K_D(z)^2[I_\Omega - K_D(z)]^{-1} \right) \right).
\end{align}
Thus, one obtains
\[
\det_2 \left( [I_\Omega - K_N(z)][I_\Omega - K_D(z)]^{-1} \right) \\
= \frac{\det_2 (I_\Omega - K_N(z)) \det_2 ((I_\Omega - K_D(z))^{-1}) \exp \left( \operatorname{tr} \left( (K_N(z)K_D(z)[I_\Omega - K_D(z)]^{-1}) \right) \right)}{\det_2 (I_\Omega - K_D(z))} \exp \left( \operatorname{tr} \left( (K_N(z) - K_D(z))K_D(z)[I_\Omega - K_D(z)]^{-1}) \right) \right).
\] (6.81)

At this point, the left-hand side of (6.74) can be rewritten as
\[
\frac{\det_2 (I_\Omega + M_u(H^N_{0,\Omega} - zI_\Omega)^{-1}M_v)}{\det_2 (I_\Omega - K_N(z))} = \frac{\det_2 (I_\Omega - K_D(z))}{\det_2 (I_\Omega - K_D(z))} \exp \left( \operatorname{tr} \left( (K_D(z) - K_N(z))K_D(z)[I_\Omega - K_D(z)]^{-1}) \right) \right).
\] (6.82)

Next, temporarily suppose that \( V(x) \in L^2(\Omega; d^n x) \cap L^6(\Omega; d^n x) \). Using Lemma A.3 (an extension of a result of Nakamura [36, Lemma 6]) and Remark A.5, one finds
\[
K_D(z) - K_N(z) = -M_u[(H^D_{0,\Omega} - zI_\Omega)^{-1} - (H^N_{0,\Omega} - zI_\Omega)^{-1}]M_v \\
= -M_u[\gamma_D(H^N_{0,\Omega} - zI_\Omega)^{-1}]\gamma_N(D_{0,\Omega} - zI_\Omega)^{-1}M_v \\
= -[\gamma_D(H^N_{0,\Omega} - zI_\Omega)^{-1}]\gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v. \tag{6.83}
\]

Thus, inserting (6.83) into (6.82) yields
\[
\frac{\det_2 (I_\Omega + M_u(H^N_{0,\Omega} - zI_\Omega)^{-1}M_v)}{\det_2 (I_\Omega - K_N(z))} = \frac{\det_2 (I_\Omega - \left[ \gamma_D(H^N_{0,\Omega} - zI_\Omega)^{-1}\right] \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \left[ I_\Omega + M_u(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \right]^{-1} \right)}{\det_2 (I_\Omega - K_D(z))} \exp \left( \operatorname{tr} \left( \gamma_D(H^N_{0,\Omega} - zI_\Omega)^{-1} \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \right) \right).
\] (6.84)

Then, utilizing Corollary 6.10 with \( p_1 = 12 \) and \( p_2 = 12,5 \), one finds
\[
\frac{\gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v}{\gamma_D(H^N_{0,\Omega} - zI_\Omega)^{-1}M_v} = \mathcal{B}_{12}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1} \sigma)), \quad \gamma_D(H^N_{0,\Omega} - zI_\Omega)^{-1}M_v = \mathcal{B}_{12/5}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1} \sigma)), \tag{6.85}
\]

and hence the fact that
\[
\left[ I_\Omega + M_u(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \right]^{-1} \in \mathcal{B}(L^2(\Omega; d^n x)), \quad z \in \mathbb{C} \setminus (\sigma(H^D_{0,\Omega}) \cup \sigma(H^N_{0,\Omega})), \tag{6.87}
\]

one rearranges the terms in (6.84) as follows:
\[
\det_2 (I_\Omega + M_u(H^N_{0,\Omega} - zI_\Omega)^{-1}M_v) = \det_2 (I_\Omega - \gamma_D(H^N_{0,\Omega} - zI_\Omega)^{-1}M_v \left[ I_\Omega + M_u(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \right]^{-1} \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \right) \exp \left( \operatorname{tr} \left( \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1} \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \right) \right).
\] (6.88)

In the last equality, the following simple identities were employed:
\[
M_v[I_\Omega + M_u(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v]^{-1}M_u = M_v[I + M_u(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v]^{-1}M_u. \tag{6.89}
\]

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Vol. 12 No. 4 2005}
Utilizing (6.88) and the following analog of formula (2.20),

\[
(H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma \left( I_\sigma + M_\sigma (H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma \right)^{-1} = (H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma,
\]

one arrives at (6.74), subject to the extra assumption \( V \in L^2(\Omega; d^n x) \cap L^6(\Omega; d^n x) \).

Finally, assuming only \( V \in L^2(\Omega; d^n x) \) and utilizing Lemma 6.8 and Corollary 6.10 once again, one obtains

\[
M_\sigma(H_{0,\Omega}^D - zI_\Omega)^{-1/6} \in B_{12}(L^2(\Omega; d^n x)),
\]

\[
M_\sigma(H_{0,\Omega}^D - zI_\Omega)^{-5/6} \in B_{12}(L^2(\Omega; d^n x)),
\]

\[
\gamma(N(H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma \in B_{12}(L^2(\Omega; d^n x), L^2(\partial\Omega; d^n-1 x)),
\]

\[
\gamma(D(N(H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma \in B_{12/5}(L^2(\Omega; d^n x), \partial\Omega; d^n-1 x)),
\]

and hence

\[
M_\sigma(H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma \in B_2(L^2(\Omega; d^n x)).
\]

Relations (6.92)–(6.96) prove (6.72) and (6.73). Moreover, since

\[
[I_\sigma + M_\sigma(H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma]^{-1} \in B(L^2(\Omega; d^n x)), \quad z \in \mathbb{C} \setminus (\sigma(H_{0,\Omega}^D) \cup \sigma(H_{0,\Omega}^D))
\]

the left- and right-hand sides of (6.88), and hence of (6.74), are well-defined for \( V \in L^2(\Omega; d^n x) \).

Thus, using (6.48), (6.63), (6.64), the continuity of \( \text{det}_2(\cdot) \) with respect to the Hilbert–Schmidt norm \( \| \cdot \|_{B_2(L^2(\Omega; d^n x))} \), the continuity of \( \text{tr}(\cdot) \) with respect to the trace norm \( \| \cdot \|_{B_1(L^2(\Omega; d^n x))} \), and an approximation of \( V \in L^2(\Omega; d^n x) \) by a sequence of potentials \( v_k \in L^2(\Omega; d^n x) \cap L^6(\Omega; d^n x), k \in \mathbb{N} \), in the norm of \( L^2(\Omega; d^n x) \) as \( k \uparrow \infty \), then extends the result from \( V \in L^2(\Omega; d^n x) \cap L^6(\Omega; d^n x) \) to \( V \in L^2(\Omega; d^n x), n = 2, 3. \)

**Remark 6.12.** Thus, a comparison of Theorem 6.11 with the one-dimensional case in Theorem 6.4 shows that the reduction of Fredholm determinants associated with operators in \( L^2((0, \infty); dx) \) to simple Wronskian determinants, and hence to Jost functions as first observed by Jost and Pais \[23\], can be properly extended to higher dimensions and results in a reduction of appropriate ratios of Fredholm determinants associated with operators in \( L^2(\Omega; d^n x) \) to an appropriate Fredholm determinant associated with an operator in \( L^2(\partial\Omega; d^n-1 x) \).

**Remark 6.13.** As in Theorem 6.11, we assume Hypothesis 6.6 and suppose that we have \( z \in \mathbb{C} \setminus (\sigma(H_{0,\Omega}^D) \cup \sigma(H_{0,\Omega}^D) \cup \sigma(H_{0,\Omega}^D)) \). First we note that

\[
[ (H_{0,\Omega}^D - zI_\Omega)^{-1/2}(H_{0,\Omega}^D - zI_\Omega)(H_{0,\Omega}^D - zI_\Omega)^{-1/2} - I_\sigma ] \in B_2(L^2(\Omega; d^n x)),
\]

\[
[ (H_{0,\Omega}^N - zI_\Omega)^{-1/2}(H_{0,\Omega}^N - zI_\Omega)(H_{0,\Omega}^N - zI_\Omega)^{-1/2} - I_\sigma ] \in B_2(L^2(\Omega; d^n x)).
\]

Indeed, by (6.67) and (6.68), we obtain

\[
(H_{0,\Omega}^D - zI_\Omega)^{-1/2}(H_{0,\Omega}^D - zI_\Omega)(H_{0,\Omega}^D - zI_\Omega)^{-1/2} - I_\sigma
\]

\[
= (H_{0,\Omega}^D - zI_\Omega)^{-1/2} M_\sigma (H_{0,\Omega}^D - zI_\Omega)^{-1/2} \in B_2(L^2(\Omega; d^n x)),
\]

\[
(H_{0,\Omega}^N - zI_\Omega)^{-1/2}(H_{0,\Omega}^N - zI_\Omega)(H_{0,\Omega}^N - zI_\Omega)^{-1/2} - I_\sigma
\]

\[
= (H_{0,\Omega}^N - zI_\Omega)^{-1/2} M_\sigma (H_{0,\Omega}^N - zI_\Omega)^{-1/2} \in B_2(L^2(\Omega; d^n x)).
\]

Thus, using (6.67)–(6.71) and the cyclicity of \( \text{det}_2(\cdot) \), we rearrange the left-hand side of (6.74) as follows:

\[
\frac{\text{det}_2 \left( I_\sigma + M_\sigma (H_{0,\Omega}^N - zI_\Omega)^{-1} M_\sigma \right)}{\text{det}_2 \left( I_\sigma + M_\sigma (H_{0,\Omega}^D - zI_\Omega)^{-1} M_\sigma \right)} = \frac{\text{det}_2 \left( I_\sigma + (H_{0,\Omega}^N - zI_\Omega)^{-1/2} M_\sigma (H_{0,\Omega}^N - zI_\Omega)^{-1/2} \right)}{\text{det}_2 \left( I_\sigma + (H_{0,\Omega}^D - zI_\Omega)^{-1/2} M_\sigma (H_{0,\Omega}^D - zI_\Omega)^{-1/2} \right)}
\]

\[
= \frac{\text{det}_2 \left( (H_{0,\Omega}^N - zI_\Omega)^{-1/2}(H_{0,\Omega}^N - zI_\Omega)(H_{0,\Omega}^N - zI_\Omega)^{-1/2} \right)}{\text{det}_2 \left( (H_{0,\Omega}^D - zI_\Omega)^{-1/2}(H_{0,\Omega}^D - zI_\Omega)(H_{0,\Omega}^D - zI_\Omega)^{-1/2} \right)}.
\]

Again (6.102) illustrates that symmetrized perturbation determinants underly Theorem 6.11.
Remark 6.14. The following observation yields a simple application of formula (6.74). Since by Theorem 3.2, for any \( z \in \mathbb{C} \setminus (\sigma(H^D) \cup \sigma(H^N)) \), one has \( z \in \sigma(H^N) \) if and only if

\[
det_2 \left( I_\Omega + M_u(H^N_{0,\Omega} - zI_\Omega)^{-1}M_v \right) = 0,
\]

for all \( z \in \mathbb{C} \setminus (\sigma(H^D) \cup \sigma(H^N)) \), one has \( z \in \sigma(H^D) \)

if and only if

\[
det_2 \left( I_\Omega - \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_V \left[ \gamma_D(H^N_{0,\Omega} - \bar{z}I_\Omega)^{-1} \right]^* \right) = 0. \tag{6.103}
\]

One can also prove the following analog of (6.74):

\[
det_2 \left( I_\Omega + M_u(H^D_{0,\Omega} - zI_\Omega)^{-1}M_v \right) = det_2 \left( I_\Omega + \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_V \left[ \gamma_D(H^N_{0,\Omega} - \bar{z}I_\Omega)^{-1} \right]^* \right)
\times \exp \left( - \text{tr} \left( \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_V \left( \gamma_D(H^N_{0,\Omega} - \bar{z}I_\Omega)^{-1} \right)^* \right) \right). \tag{6.104}
\]

Then, proceeding as before, one obtains the following assertion:

for all \( z \in \mathbb{C} \setminus (\sigma(H^N) \cup \sigma(H^D_{0,\Omega}) \cup \sigma(H^N_{0,\Omega})) \), one has \( z \in \sigma(H^D) \)

if and only if

\[
det_2 \left( I_\Omega + \gamma_N(H^D_{0,\Omega} - zI_\Omega)^{-1}M_V \left[ \gamma_D(H^N_{0,\Omega} - \bar{z}I_\Omega)^{-1} \right]^* \right) = 0. \tag{6.105}
\]

7. AN APPLICATION TO SCATTERING THEORY

In this section, we relate Krein’s spectral shift function and hence the determinant of the scattering operator in connection with quantum mechanical scattering theory in dimensions \( n = 2, 3 \) with appropriately modified Fredholm determinants.

The results of this section are not new, they were first derived for \( n = 3 \) by Newton [38] and subsequently for \( n = 2 \) by Cheney [9]. However, since our method of proof nicely illustrates the use of infinite determinants in connection with scattering theory and is different from that in [38] and [9], and moreover, since our derivation in the case \( n = 3 \) is performed under slightly more general hypotheses than in [38], we think it worthwhile to include it at this point.

Hypothesis 7.1. Fix \( \delta > 0 \). Suppose \( V \in \mathcal{R}_{2,\delta} \) for \( n = 2 \) and \( V \in L^1(\mathbb{R}^2; d^3x) \cap \mathcal{R}_3 \) for \( n = 3 \), where

\[
\mathcal{R}_{2,\delta} = \left\{ V : \mathbb{R}^2 \to \mathbb{R} \text{ measurable} \mid V^{1+\delta}, (1 + |\cdot|^\delta)V \in L^1(\mathbb{R}^2; d^2x) \right\},
\]

\[
\mathcal{R}_3 = \left\{ V : \mathbb{R}^3 \to \mathbb{R} \text{ measurable} \mid \int_{\mathbb{R}^3} d^3x d^3x' |V(x)||V(x')||x - x'|^{-2} < \infty \right\}. \tag{7.2}
\]

Introduce \( H_0 \) as the following nonnegative selfadjoint operator in the Hilbert space \( L^2(\mathbb{R}^n; d^n x) \):

\[
H_0 = -\Delta, \quad \text{dom}(H_0) = H^2(\mathbb{R}^n), \quad n = 2, 3. \tag{7.3}
\]

Moreover, let \( A = M_u \) and \( B = B^* = M_v \) denote the operators of multiplication by \( u = \text{sign}(V) |V|^{1/2} \) and \( v = |V|^{1/2} \) in \( L^2(\mathbb{R}^n; d^n x) \), respectively, so that \( M_V = BA = M_u M_v \). Then, (cf. [48, Th. 1.21]) for \( n = 3 \) and [49] for \( n = 2 \)

\[
\text{dom}(A) = \text{dom}(B) \supseteq H^1(\mathbb{R}^n) \supseteq \text{dom}(H_0), \tag{7.4}
\]

and hence, Hypothesis 2.1 (i) is satisfied for \( H_0 \). It follows from Hypothesis 7.1 that

\[
M_u(H_0 - zI)^{-1}M_v \in B_2(L^2(\mathbb{R}^n; d^n x)), \quad z \in \mathbb{C} \setminus [0, \infty), \tag{7.5}
\]

where \( I \) now denotes the identity operator in \( L^2(\mathbb{R}^n; d^n x) \), and hence, Hypothesis 2.1 (ii) is satisfied. Taking \( z \in \mathbb{C} \setminus [0, \infty) \) with a sufficiently large absolute value, one also verifies Hypothesis 2.1 (iii).

Thus, applying Theorem 2.3 and Remark 2.4 (i), one obtains a selfadjoint operator \( H \) (which is an extension of \( H_0 + V \) to \( \text{dom}(H_0) \cap \text{dom}(V) \)).

Theorem 7.2. Assume Hypothesis 7.1 and let \( z \in \mathbb{C} \setminus (\sigma(H) \cup n = 2, 3. \) Then

\[
(H - zI)^{-1} - (H_0 - zI)^{-1} \in B_1(L^2(\mathbb{R}^n; d^n x)), \tag{7.6}
\]

and there is a unique real-valued spectral shift function

\[
\xi(\cdot, H, H_0) \in L^1(\mathbb{R}; (1 + \lambda^2)^{-1} d\lambda) \tag{7.7}
\]

such that \( \xi(\lambda, H, H_0) = 0 \) for \( \lambda < \inf(\sigma(H)) \), and

\[
\text{tr} \left( (H - zI)^{-1} - (H_0 - zI)^{-1} \right) = - \int_{\sigma(H)} \frac{d\lambda \xi(\lambda, H, H_0)}{(\lambda - z)^2}. \tag{7.8}
\]

We recall that \( \xi(\cdot, H, H_0) \) is called the spectral shift function for the pair of selfadjoint operators \((H, H_0)\). For background information on \( \xi(\cdot, H, H_0) \) and its connection with the scattering operator at fixed energy, we refer, for instance, to [3, Sec. 19.1; 5; 7; 62, Chap. 8].
Lemma 7.3. Assume Hypothesis 7.1 and let \( z \in \mathbb{C} \setminus \sigma(H) \) and \( n = 2, 3 \). Then

\[
M_u(H_0 - zI)^{-1}M_v \in B_2(L^2(\mathbb{R}^n; d^n x)),
\]

(7.9)

\[
(H_0 - zI)^{-1}M_V(H_0 - zI)^{-1} \in B_1(L^2(\mathbb{R}^n; d^n x)),
\]

(7.10)

and

\[
\frac{d}{dz} \ln \left( \det_2 \left( I + \frac{M_u}{M_v - zI} \right)^{-1} M_v \right) = - \text{tr} \left( (H_0 - zI)^{-1} - (H_0 - zI)^{-1} + (H_0 - zI)^{-1} M_V(H_0 - zI)^{-1} \right).
\]

(7.11)

The key ingredient in proving (7.6) is the fact that

\[
M_u(H_0 - zI)^{-1}, (H_0 - zI)^{-1}M_v \in B_2(L^2(\mathbb{R}^n; d^n x)), \quad z \in \mathbb{C} \setminus [0, \infty), \; n = 2, 3.
\]

(7.12)

This follows from either [52, Th. 4.1] (or [43, Th. XI.20]), or explicitly by an inspection of the corresponding integral kernels. For instance, the one for \( M_u(H_0 - zI)^{-1} \) reads:

\[
\left( M_u(H_0 - zI)^{-1}(x, x') \right) = \begin{cases} 
 u(x)(i/4)H_0^{(1)}(z^{1/2}|x - x'|), & x \neq x', \; x, x' \in \mathbb{R}^2, \\
[2\pi m]u(x)e^{iz^{1/2}|x - x'|/4\pi|x - x'|}, & x \neq x', \; x, x' \in \mathbb{R}^3, \\
\end{cases}
\]

\[
z \in \mathbb{C} \setminus [0, \infty), \; \text{Im}(z^{1/2}) > 0,
\]

(7.13)

where \( H_0^{(1)}(\cdot) \) denotes the Hankel function of order zero and first kind (see, e.g., [1, Sec. 9.1]). Hence, one only needs to apply equation (2.13) to conclude (7.6) and hence (7.10) (by factoring \( M_V = M_u M_v \)). (We note that (7.6) is proved in [43, Sec. XI.6] and [48, Th. II.37] for \( n = 3 \).) Relation (7.9) is then clear from \( V \in R_3 \) for \( n = 3 \) and follows from [49] for \( n = 2 \). Equation (7.11) is discussed in [8] for \( n = 2, 3 \). The trace formula (7.8) is a celebrated result of Krein [29, 30]; detailed accounts of it can be found in [3, Sec. 19.1.5; 7; 31; 62, Chap. 8].

Lemma 7.4. Assume Hypothesis 7.1. Then the following formula holds for a.e. \( \lambda \in \mathbb{R} \),

\[
2\pi i \xi(\lambda, H, H_0) = \ln \left( \frac{\det_2 \left( I + \frac{M_u}{M_v - (\lambda + i0)I} \right)^{-1} M_v}{\det_2 \left( I + \frac{M_u}{M_v - (\lambda - i0)I} \right)^{-1} M_v} \right) + \frac{i}{2\pi} \int_{\mathbb{R}^n} d^n x \; V(x) \times \begin{cases} 
\pi, & \lambda > 0, \; n = 2, \\
\lambda^{1/2}, & \lambda > 0, \; n = 3, \\
0, & \lambda \leq 0, \; n = 2, 3.
\end{cases}
\]

(7.14)

Proof. It follows from Theorem 7.2 and Lemma 7.3 that for \( z \in \mathbb{C} \setminus \sigma(H) \),

\[
\int_{\mathbb{R}} \frac{d\lambda \xi(\lambda, H, H_0)}{(\lambda - z)^2} = \frac{d}{dz} \ln \left( \frac{\det_2 \left( I + \frac{M_u}{M_v - zI} \right)^{-1} M_v}{\det_2 \left( I + \frac{M_u}{M_v - (\lambda - i0)I} \right)^{-1} M_v} \right) - \text{tr} \left( (H_0 - zI)^{-1} M_V(H_0 - zI)^{-1} \right).
\]

(7.15)

First, we rewrite the left-hand side of (7.15). Since \( \xi(\cdot, H, H_0) \in L^1(\mathbb{R}; d\lambda/(1 + \lambda^2)) \), we have the following formula:

\[
\int_{\mathbb{R}} \frac{d\lambda \xi(\lambda, H, H_0)}{(\lambda - z)^2} = \frac{d}{dz} \int_{\mathbb{R}} d\lambda \xi(\lambda, H, H_0) \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C} \setminus \sigma(H).
\]

(7.16)

Next, we compute the second term on the right-hand side of (7.15). By (7.12) and the cyclicity of the trace,

\[
\text{tr} \left( (H_0 - zI)^{-1} M_V(H_0 - zI)^{-1} \right) = \text{tr} \left( \frac{M_u}{M_v - zI} \right)^{-2} M_v), \quad z \in \mathbb{C} \setminus [0, \infty).
\]

(7.17)

Then \( M_u(H_0 - zI)^{-2} M_v = \frac{d}{dz}(H_0 - zI)^{-1} M_v \) has the integral kernel

\[
\left( M_u(H_0 - zI)^{-2} M_v \right)(x, x') = \begin{cases} 
\frac{u(x)^iH_0^{(1)}(z^{1/2}|x - x'|)|x - x'|}{8\pi z^{1/2}}, & x, x' \in \mathbb{R}^2, \\
\frac{2\pi m u(x)}{32\pi^{3/2}|x - x'|^{3/2}} v(x'), & x, x' \in \mathbb{R}^3, \\
x \neq x', \; z \in \mathbb{C} \setminus [0, \infty), \; \text{Im}(z^{1/2}) > 0,
\end{cases}
\]

(7.18)
and hence, utilizing [11, p. 1086], we compute, for \( z \in \mathbb{C} \setminus [0, \infty) \),
\[
\text{tr} \left( (H_0 - zI)^{-1} M V (H_0 - zI)^{-1} \right) = \frac{1}{4\pi} \int_{\mathbb{R}^n} d^n x \ V(x) \times \begin{cases} -z^{-1}, & n = 2 \\ i(2z^{1/2})^{-1}, & n = 3 \end{cases} \\
= \frac{1}{4\pi} \int_{\mathbb{R}^n} d^n x \ V(x) \times \begin{cases} -\ln(z), & n = 2 \\ iz^{1/2}, & n = 3 \end{cases}.
\]
(7.19)

Finally, using (7.15), (7.16), and (7.19), for \( z \in \mathbb{C} \setminus \sigma(H) \), we obtain
\[
\int_{\mathbb{R}} d\lambda \xi(\lambda, H, H_0) \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) + C \\
= \ln \left( \det (I + M_u (H_0 - zI)^{-1} M_v) \right) + \frac{1}{4\pi} \int_{\mathbb{R}^n} d^n x \ V(x) \times \begin{cases} -\ln(z), & n = 2 \\ iz^{1/2}, & n = 3 \end{cases},
\]
where \( C \in \mathbb{C} \) denotes an appropriate constant. To complete the proof, we digress for a moment and recall the Stieltjes inversion formula for Herglotz functions \( m \) (i.e., analytic maps \( m: \mathbb{C}_+ \rightarrow \mathbb{C}_+ \), where \( \mathbb{C}_+ \) denotes the open complex upper half-plane). Such functions \( m \) admit the Nevanlinna representation (Riesz–Herglotz representation, respectively),
\[
m(z) = c + dz + \int_{\mathbb{R}} d\omega(\lambda) \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+, \quad c = \Re[m(i)], \quad d = \lim_{\eta \to \infty} m(i\eta)/(i\eta) \geq 0,
\]
with a nonnegative measure \( d\omega \) on \( \mathbb{R} \) satisfying
\[
\int_{\mathbb{R}} d\omega(\lambda) < \infty.
\]
(7.21)
The absolutely continuous part \( d\omega_{ac} \) of \( d\omega \) with respect to Lebesgue measure \( d\lambda \) on \( \mathbb{R} \) is then known to be given by
\[
d\omega_{ac}(\lambda) = \pi^{-1} \Im[m(\lambda + i0)] d\lambda.
\]
(7.23)
In addition, we extend \( m \) to the open lower complex half-plane \( \mathbb{C}_- \) by
\[
m(z) = m(\overline{z}), \quad z \in \mathbb{C}_-.
\]
(7.24)
(We refer, e.g., to [2, Sec. 69] for details on (7.21)–(7.24).) Thus, in order to apply (7.21)–(7.24) to the computation of \( \xi(\cdot, H, H_0) = \xi_+ (\cdot, H, H_0) - \xi_-(\cdot, H, H_0) \) into its positive and negative parts \( \xi_{\pm}(\cdot, H, H_0) \geq 0 \) and separately consider the absolutely continuous measures \( \xi_{\pm}(\cdot, H, H_0)d\lambda \). Thus, letting \( z = \lambda \pm i\varepsilon \), taking the limit \( \varepsilon \downarrow 0 \) in (7.20), and subtracting the corresponding results, we obtain (7.14).

We conclude with the following result.

**Corollary 7.5.** Assume Hypothesis 7.1. Then, for a.e. \( \lambda > 0 \),
\[
det(S(\lambda)) = \frac{\det_2 (I + M_u (H_0 - (\lambda - i0)I)^{-1} M_v)}{\det_2 (I + M_u (H_0 - (\lambda + i0)I)^{-1} M_v)} \begin{cases} \exp \left( -\frac{i}{2} \int_{\mathbb{R}^n} d^n x \ V(x) \right), & n = 2 \\ \exp \left( -\frac{i\lambda^{1/2}}{2\pi} \int_{\mathbb{R}^n} d^n x \ V(x) \right), & n = 3 \end{cases},
\]
(7.25)

**Proof.** Hypothesis 7.1 implies that the scattering operator \( S(\lambda) \) at fixed energy \( \lambda > 0 \) in \( L^2(S^{n-1}; d^{n-1}\omega) \) satisfies
\[
[S(\lambda) - I] \in \mathcal{B}_1(L^2(S^{n-1}; d^{n-1}\omega)) \quad \text{for a.e. } \lambda > 0
\]
(7.26)
and
\[
det(S(\lambda)) = \exp(-2\pi i \xi(\lambda, H, H_0)) \quad \text{for a.e. } \lambda > 0
\]
(7.27)
(cf., e.g., [3, Secs. 19.1.4, 19.1.5; 5; 7; 62, Chap. 8]), where \( S^{n-1} \) denotes the unit sphere in \( \mathbb{R}^n \) and \( d^{n-1}\omega \) the corresponding surface measure on \( S^{n-1} \). Relation (7.25) then follows from Lemma 7.4 and (7.27).

We note again that Corollary 7.5 was derived earlier using different means by Cheney [9] for \( n = 2 \) and by Newton [38] for \( n = 3 \). (The stronger conditions \( V \in L^2(\mathbb{R}^3; dx^3) \) and the existence of \( a > 0 \) and \( 0 < C < \infty \) such that for all \( y \in \mathbb{R}^3 \), \( \int_{\mathbb{R}^3} d^3 x |V(x)|/(|x| + |y| + a)/(|x - y|)^2 \leq C \), are assumed in [38].)
APPENDIX A. PROPERTIES OF THE DIRICHLET AND NEUMANN LAPLACIANS

The purpose of this appendix is to derive some basic domain properties of Dirichlet and Neumann Laplacians on $C^{1,r}$-domains $\Omega \subset \mathbb{R}^n$ and to prove Lemma 6.7. Throughout this appendix, we assume $n \geq 2$, but we note that $n$ is restricted to $n = 2, 3$ in Sections 6 and 7.

In this paper, we use the following notation for the standard Sobolev Hilbert spaces ($s \in \mathbb{R}$),

$$H^s(\mathbb{R}^n) = \left\{ U \in \mathcal{S}(\mathbb{R}^n)^* \mid \|U\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} d^n x |\tilde{U}(\xi)|^2 \left(1 + |\xi|^2 \right) < \infty \right\},$$

(A.1)

$$H^s(\Omega) = \left\{ u \in C_0^\infty(\Omega) \mid u = 0 \text{ on } \partial \Omega \right\}$$

(A.2)

$$H^d_0(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in the norm of } H^d(\Omega).$$

(A.3)

Here $C_0^\infty(\Omega)^*$ denotes the usual set of distributions on $\Omega \subset \mathbb{R}^n$, $\Omega$ open and nonempty, $\mathcal{S}(\mathbb{R}^n)^*$ is the space of tempered distributions on $\mathbb{R}^n$, and $\tilde{U}$ denotes the Fourier transform of $U \in \mathcal{S}(\mathbb{R}^n)^*$.

It is then immediate that $H^{s_0}(\Omega) \hookrightarrow H^{s_1}(\Omega)$ whenever $-\infty < s_0 < s_1 < +\infty$.

(A.4)

continuously and densely.

Before we present a proof of Lemma 6.7, we recall the definition of a $C^{1,r}$-domain $\Omega \subset \mathbb{R}^n$, $\Omega$ open and nonempty, for convenience of the reader: let $\mathcal{N}$ be a space of real-valued functions in $\mathbb{R}^{n-1}$. We say that a bounded domain $\Omega \subset \mathbb{R}^n$ is of class $\mathcal{N}$ if there exists a finite open covering $\{O_j\}_{j \in N}$ of the boundary $\partial \Omega$ of $\Omega$ with the property that, for every $j \in \{1, \ldots, N\}$, $O_j \cap \Omega$ coincides with the portion of $O_j$ lying in the over-graph of a function $\varphi_j \in \mathcal{N}$ (considered in a new system of coordinates obtained from the original one via a rigid motion). Two special cases are going to play a particularly important role in the sequel. First, if $N$ is the space of Lipschitz functions satisfying a (global) Lipschitz condition in $\mathbb{R}^{n-1}$, we shall refer to $\Omega$ as being a Lipschitz domain; cf. [54, p. 189], where such domains are called "minimally smooth." Second, corresponding to the case when $N$ is the subspace of $\text{Lip}(\mathbb{R}^{n-1})$ consisting of functions whose first-order derivatives satisfy a (global) Hölder condition of order $r \in (0, 1)$, we shall say that $\Omega$ is of class $C^{1,r}$. The classical theorem of Rademacher on the almost everywhere differentiability of Lipschitz functions ensures that, for any Lipschitz domain $\Omega$, the surface measure $ds$ is well-defined on $\partial \Omega$ and that there exists an outward pointing normal vector $\nu$ at almost every point of $\partial \Omega$.

For a Lipschitz domain $\Omega \subset \mathbb{R}^n$, it is known that

$$(H^s(\Omega))^* = H^{-s}(\Omega), \quad -1/2 < s < 1/2.$$  

(A.5)

See [59] for this and other related properties.

Next, assume that $\Omega \subset \mathbb{R}^n$ is the domain lying above the graph of a function $\varphi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ of class $C^{1,r}$. Then for $0 \leq s < 1+r$, the Sobolev space $H^s(\partial \Omega)$ consists of functions $f \in L^2(\partial \Omega; d^{n-1}\sigma)$ such that $f(x', \varphi(x'))$, as function of $x' \in \mathbb{R}^{n-1}$, belongs to $H^s(\mathbb{R}^{n-1})$. This definition is easily adapted to the case in which $\Omega$ is a domain of class $C^{1,r}$ whose boundary is compact, by using a smooth partition of unity. Finally, we set $H^s(\partial \Omega) = (H^{-s}(\partial \Omega))^*$ for $-1-r < s < 0$. For additional background information in this context, we refer, for instance, to [33, Chap. 3; 61, Sec. I.4.2].

Assuming Hypothesis 6.6(i) (i.e., $\Omega$ is an open nonempty $C^{1,r}$-domain for some $(1/2) < r < 1$ with compact boundary $\partial \Omega$), we introduce the Dirichlet and Neumann Laplacians $\tilde{H}^D_{0,\Omega}$ and $\tilde{H}^N_{0,\Omega}$ associated with the domain $\Omega$ as the unique selfadjoint operators on $L^2(\Omega; d^n x)$ whose quadratic form equals $q(f, g) = \int_{\Omega} d^n x \nabla f \cdot \nabla g$ with the form domains $H^1_0(\Omega)$ and $H^1(\Omega)$, respectively. Then

$$\text{dom}(\tilde{H}^D_{0,\Omega}) = \{ u \in H^1_0(\Omega) \mid \exists f \in L^2(\Omega; d^n x) : q(u, v) = (f, v)_{L^2(\Omega; d^n x)} \forall v \in H^1_0(\Omega) \},$$

(A.6)

$$\text{dom}(\tilde{H}^N_{0,\Omega}) = \{ u \in H^1(\Omega) \mid \exists f \in L^2(\Omega; d^n x) : q(u, v) = (f, v)_{L^2(\Omega; d^n x)} \forall v \in H^1(\Omega) \},$$

(A.7)

with $(-, \cdot)_{L^2(\Omega; d^n x)}$ denoting the scalar product in $L^2(\Omega; d^n x)$. Equivalently, we introduce the densely defined closed linear operators

$$D = \nabla, \quad \text{dom}(D) = H^1_0(\Omega) \quad \text{and} \quad N = \nabla, \quad \text{dom}(N) = H^1(\Omega)$$

(A.8)

from $L^2(\Omega; d^n x)$ to $L^2(\Omega; d^n x)$ and note that

$$\tilde{H}^D_{0,\Omega} = D^* D \quad \text{and} \quad \tilde{H}^N_{0,\Omega} = N^* N.$$  

(A.9)

For details, we refer to [44, Secs. XIII.14, XIII.15]. Moreover, with $\text{div}(\cdot)$ denoting the divergence operator,

$$\text{dom}(D^*) = \{ w \in L^2(\Omega; d^n x) \mid \text{div}(w) \in L^2(\Omega; d^n x) \},$$

(A.10)
and hence
\[ \text{dom}(\tilde{H}_{0,\Omega}^D) = \{ u \in \text{dom}(D) \mid Du \in \text{dom}(D^*) \} = \{ u \in H_0^1(\Omega) \mid \Delta u \in L^2(\Omega; d^n x) \}. \] (A.11)

One can also define the map
\[ \{ w \in L^2(\Omega; d^n x)^n \mid \text{div}(w) \in (H^1(\Omega))^* \} \rightarrow H^{-1/2}(\partial \Omega) = (H^{1/2}(\partial \Omega))^*, \quad w \mapsto \nu \cdot w \] (A.12)
by setting
\[ \langle \nu \cdot w, \phi \rangle = \int_{\Omega} d^n x \, w(x) \cdot \nabla \Phi(x) + \langle \text{div}(w), \Phi \rangle \] (A.13)
whenever \( \phi \in H^{1/2}(\partial \Omega) \) and \( \Phi \in H^1(\Omega) \) is such that \( \gamma_D \Phi = \phi \). The last pairing in (A.13) is in the duality sense (which, in turn, is compatible with the (bilinear) distributional pairing). It should be remarked that the above definition is independent of the particular extension \( \Phi \in H^1(\Omega) \) of \( \phi \).

Indeed, by linearity, this comes down to proving that
\[ \langle \text{div}(w), \Phi \rangle = -\int_{\Omega} d^n x \, w(x) \cdot \nabla \Phi(x) \] (A.14)
if \( w \in L^2(\Omega; d^n x)^n \) has \( \text{div}(w) \in H^1(\Omega)^* \) and \( \Phi \in H^1(\Omega) \) has \( \gamma_D \Phi = 0 \). To see this, we rely on the existence of a sequence \( \Phi_j \in C_0^\infty(\Omega) \) such that \( \Phi_j \rightarrow \Phi, j \uparrow \infty, \) in \( H^1(\Omega) \). When \( \Omega \) is a bounded Lipschitz domain, this is well-known (see, e.g., [22, Remark 2.7] for a rather general result of this nature), and this result is easily extended to the case in which \( \Omega \) is an unbounded Lipschitz domain with a compact boundary. For if \( \xi \in C_0^\infty(B(0; 2)) \) is such that \( \xi = 1 \) on \( B(0; 1) \) and \( \xi_j(x) = \xi(x/j) \), \( j \in \mathbb{N} \) (here \( B(x_0; r_0) \) denotes the ball in \( \mathbb{R}^n \) of radius \( r_0 > 0 \) centered at \( x_0 \in \mathbb{R}^n \)), then \( \xi_j \Phi \rightarrow \Phi, j \uparrow \infty, \) in \( H^1(\Omega) \) and matters are reduced to approximating \( \xi_j \Phi \) in \( H^1(B(0; 2j) \cap \Omega) \) with test functions supported in \( B(0; 2j) \cap \Omega \), for each fixed \( j \in \mathbb{N} \). Since \( \gamma_D(\xi_j \Phi) = 0 \), the result for bounded Lipschitz domains applies.

Returning to the task of proving (A.14), it suffices to prove a similar identity with \( \Phi_j \) in place of \( \Phi \). This, in turn, follows from the definition of \( \text{div} \cdot (\cdot) \) in the sense of distributions and the fact that the duality between \((H^1(\Omega))^* \) and \( H^1(\Omega) \) is compatible with the duality between distributions and test functions.

Going further, we can introduce a (weak) Neumann trace operator \( \tilde{\gamma}_N \) as follows:
\[ \tilde{\gamma}_N : \{ u \in H^1(\Omega) \mid \Delta u \in (H^1(\Omega))^* \} \rightarrow H^{-1/2}(\partial \Omega), \quad \tilde{\gamma}_N u = \nu \cdot \nabla u, \] (A.15)
with the dot product understood in the sense of (A.12). We emphasize that the weak Neumann trace operator \( \tilde{\gamma}_N \) in (A.17) is an extension of the operator \( \gamma_N \) introduced in (6.43). Indeed, to see that \( \text{dom}(\gamma_N) \subset \text{dom}(\tilde{\gamma}_N) \), we note that if \( u \in H^{s+1}(\Omega) \) for some \( 1/2 < s < 3/2 \), then
\[ \Delta u \in H^{-1+s}(\Omega) = (H^{1-s}(\Omega))^* \hookrightarrow (H^1(\Omega))^*, \] by (A.5) and (A.4). With this in hand, it is then easy to show that \( \tilde{\gamma}_N \) in (A.17) and \( \gamma_N \) in (6.43) agree (on the smaller domain), as claimed.

We now return to the main discussion. From the above preamble, it follows that
\[ \text{dom}(N^*) = \{ w \in L^2(\Omega; d^n x)^n \mid \text{div}(w) \in L^2(\Omega; d^n x) \text{ and } \nu \cdot w = 0 \}, \] (A.16)
where the dot product operation is understood in the sense of (A.12). Consequenctly, with \( \tilde{H}_{0,\Omega}^N = N^* N \), we have
\[ \text{dom}(\tilde{H}_{0,\Omega}^N) = \{ u \in \text{dom}(N) \mid Nu \in \text{dom}(N^*) \} = \{ u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega; d^n x) \} \] (A.17)
\[ \text{dom}(\tilde{H}_{0,\Omega}^N) = \{ u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega; d^n x) \} \]
\[ \text{dom}(\tilde{H}_{0,\Omega}^N) \subseteq H^2(\Omega). \] (A.18)

Next, we shall prove that \( H^0_{0,\Omega} \equiv \tilde{H}_{0,\Omega}^D \) and \( H_0^N \equiv \tilde{H}_{0,\Omega}^N \), where \( H^0_{0,\Omega} \) and \( H_0^N \) denote the operators introduced in (6.44) and (6.45), respectively. Since it follows from the first Green’s formula (cf., e.g., [33, Th. 4.4]) that \( H^0_{0,\Omega} \subseteq \tilde{H}_{0,\Omega}^D \) and \( H_0^N \subseteq \tilde{H}_{0,\Omega}^N \), it remains to show that \( H^0_{0,\Omega} \supseteq \tilde{H}_{0,\Omega}^D \) and \( H_0^N \supseteq \tilde{H}_{0,\Omega}^N \). Moreover, it follows from comparing (6.44) with (A.11) and (6.45) with (A.17) that we only need to show that \( \text{dom}(\tilde{H}_{0,\Omega}^D), \text{dom}(\tilde{H}_{0,\Omega}^N) \subseteq H^2(\Omega) \).

**Lemma A.1.** Assume Hypothesis 6.6 (i). Then
\[ \text{dom}(H^0_{0,\Omega}) \subseteq H^2(\Omega), \quad \text{dom}(H_0^N) \subseteq H^2(\Omega). \] (A.18)
In particular,
\[ H^0_{0,\Omega} = \tilde{H}_{0,\Omega}^D, \quad H_0^N = \tilde{H}_{0,\Omega}^N. \] (A.19)
Indeed, with \( H \) consult, e.g., [34, 60] for jump relations in the context of Lipschitz domains), if

\[
\frac{\partial}{\partial \tau} \gamma_N u = 0.
\]

One convenient way to show that actually

\[
u \in H^2(\Omega)
\]

is to use layer potentials. Specifically, let \( E(x), x \in \mathbb{R}^n \backslash \{0\} \), be the fundamental solution of the Helmholtz operator \( \Delta - I_\Omega \) in \( \mathbb{R}^n \) and denote by \((\Delta - I_\Omega)^{-1}\) the operator of convolution with \( E \). Let us also define the associated single layer potential

\[
Sg(x) = \int_{\partial \Omega} d^{n-1} \sigma_y E(x-y)g(y), \quad x \in \Omega,
\]

where \( g \) is an arbitrary measurable function on \( \partial \Omega \). As is well-known (the interested reader may consult, e.g., [34, 60] for jump relations in the context of Lipschitz domains), if

\[
K^\# g(x) = \int_{\partial \Omega} d^{n-1} \sigma_y \partial_\nu_x E(x-y)g(y), \quad x \in \partial \Omega,
\]

stands for the so-called adjoint double layer on \( \partial \Omega \), the following jump formula holds:

\[
\gamma_N Sg = \frac{1}{2} I_\Omega \Omega + K^\# g.
\]

Now the solution \( u \) of (A.20) is given by

\[
u = (\Delta - I_\Omega)^{-1} \nu - Sg
\]

for a suitably chosen \( g \). In order to continue, we recall that the classical Calderón-Zygmund theory yields that, locally, \((\Delta - I_\Omega)^{-1}\) is smoothing of order 2 on the scale of Sobolev spaces, and since \( E \) has exponential decay at infinity, it follows that \((\Delta - I_\Omega)^{-1} \nu \in H^2(\Omega) \) whenever \( \nu \in L^2(\Omega; \mathbb{R}^n) \). We shall then require that

\[
\gamma_N Sg = \gamma_N (\Delta - I_\Omega)^{-1} \nu \text{ or } \left( \frac{1}{2} I_\Omega \Omega + K^\# \right) g = h = \gamma_N (\Delta - I_\Omega)^{-1} \nu \in H^{1/2}(\partial \Omega).
\]

Thus, formally, \( g = (\frac{1}{2} I_\Omega \Omega + K^\#)^{-1} h \) and (A.21) follows as soon as we prove that \( \frac{1}{2} I_\Omega \Omega + K^\# \) is invertible on \( H^{1/2}(\partial \Omega) \).

and that the operator

\[
S : H^{1/2}(\partial \Omega) \rightarrow H^2(\partial \Omega)
\]

is welldefined and bounded. The validity of (A.27) is essentially wellknown. See, for instance, [57, Prop. 4.5], which requires that \( \Omega \) be of class \( C^{1, r} \) for some \((1/2) < r < 1\). As for (A.28), we note, as a preliminary step, that

\[
S : H^{-s}(\partial \Omega) \rightarrow H^{-s+3/2}(\Omega)
\]

is welldefined and bounded for each \( s \in [0, 1] \), even when the boundary of \( \Omega \) is only Lipschitz. Indeed, with \( H^{-s+3/2}(\Omega) \) replaced by \( H^{-s+3/2}(\Omega \cap B) \) for a sufficiently large ball \( B \subset \mathbb{R}^n \), this is proved in [35] and the behavior at infinity is easily taken care of by employing the exponential decay of \( E \).

For a fixed arbitrary \( j \in \{1, \ldots, n\} \), next let us consider operator \( \partial_j S \) whose kernel is \( \partial_j E(x-y) = -\partial_y E(x-y) \). We write

\[
\partial_j S \nu = -D(\nu) + \sum_{k=1}^{n} \nu_k \partial_{\tau_{k,j}} h,
\]

where \( \partial_j \partial_{\tau_{k,j}} \nu = \nu_k \partial_j \nu - \nu_j \partial_k \nu \), \( j, k = 1, \ldots, n \), is the tangential derivative operator for which we have

\[
\int_{\partial \Omega} d^{n-1} \sigma \frac{\partial h_1}{\partial \tau_{j,k}} h_2 = -\int_{\partial \Omega} d^{n-1} \sigma h_1 \frac{\partial h_2}{\partial \tau_{j,k}}, \quad h_1, h_2 \in H^{1/2}(\partial \Omega). \]

It follows that

\[
\partial_j S \nu = -D(\nu) + \sum_{k=1}^{n} S \left( \frac{\partial (\nu_k h)}{\partial \tau_{k,j}} \right),
\]
where $\mathcal{D}$, the so-called double layer potential operator, is the integral operator with integral kernel $\partial_\nu E(x - y)$. Its mapping properties on the scale of Sobolev spaces have been analyzed in [35] and we note here that
\[ \mathcal{D} : H^s(\partial\Omega) \to H^{s+1/2}(\Omega), \quad 0 \leq s \leq 1, \] (A.33)
requires only that $\partial\Omega$ be Lipschitz.

Assuming that multiplication by (the components of) $\nu$ preserves the space $H^{1/2}(\partial\Omega)$ (which is the case if, e.g., $\Omega$ is of class $C^{1,r}$ for some $(1/2) < r < 1$), the desired conclusion about the operator (A.28) follows from (A.29), (A.32), and (A.33). This concludes the proof of the fact that $\text{dom}(H_{0,\Omega}^D) \subseteq H^2(\Omega)$.

To prove that $\text{dom}(H_{0,\Omega}^D) \subseteq H^2(\Omega)$, we proceed similarly, starting with the same representation (A.25). This time, the requirement on $g$ is that $Sg = h = \gamma_D(\Delta - I_\Omega)^{-1}f \in H^{3/2}(\partial\Omega)$, where $S = \gamma_D \circ \mathcal{S}$ is the trace of the single layer. Thus, in this scenario, it suffices to know that
\[ S : H^{1/2}(\partial\Omega) \to H^{3/2}(\partial\Omega) \] (A.34)
is an isomorphism. When $\partial\Omega$ is of class $C^\infty$, it has been proved in [57, Proposition 7.9] that $S : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega)$ is an isomorphism for each real number $s \in \mathbb{R}$ and, if $\Omega$ is of class $C^{1,r}$ with $(1/2) < r < 1$, the validity range of this result is limited to $-1 - r < s < r$, which covers (A.34). The latter fact follows from an inspection of Taylor’s original proof of Proposition 7.9 in [57]. Here we just note that the only significant difference is that $\partial\Omega$ is of class $C^{1,r}$ (instead of class $C^\infty$), then $S$ is a pseudodifferential operator whose symbol exhibits a limited amount of regularity in the space variable. Such classes of operators have been studied in, e.g., [34; 56, Chs. 1, 2].

We note that Lemma A.1 also follows from [10, Th. 8.2] in the case of $C^2$-domains $\Omega$ with compact boundary. This is proved in [10] by rather different methods and can be viewed as a generalization of the classical result for bounded $C^2$-domains.

**Lemma A.2.** Assume Hypothesis 6.6 (i) and let $q \in \mathbb{R}$. Then for each $z \in \mathcal{C} \setminus [0, \infty)$, one has
\[ (H_{0,\Omega}^D - zI_\Omega)^{-q}, (H_{0,\Omega}^N - zI_\Omega)^{-q} \in \mathcal{B}(L^2(\Omega; d^n x), H^{2q}(\Omega)). \] (A.35)

**Proof.** For notational convenience, we denote by $H_{0,\Omega}$ either one of the operators $H_{0,\Omega}^D$ or $H_{0,\Omega}^N$. The operator $H_{0,\Omega}$ is a semibounded selfadjoint operator in $L^2(\Omega; d^n x)$, and thus the resolvent set of $H_{0,\Omega}$ is linearly connected.

Step 1. We claim that it is enough to prove (A.35) for one point $z$ in the resolvent set of $H_{0,\Omega}$. Indeed, suppose that (A.35) holds, and $z'$ is any other point in the resolvent set of $H_{0,\Omega}$. Connecting $z$ and $z'$ by a curve in the resolvent set, and splitting this curve in small segments, without loss of generality we may assume that $z'$ is arbitrarily close to $z$ so that the operator $I_{\Omega} - (z' - z)(H_{0,\Omega} - zI_\Omega)^{-1}$ is invertible, and thus the operator $(I_{\Omega} - (z' - z)(H_{0,\Omega} - zI_\Omega)^{-1})^{-q}$ is a bounded operator on $L^2(\Omega; d^n x)$. Then (A.35) and the identity
\[ (H_{0,\Omega} - zI_\Omega)^{-q} = (H_{0,\Omega} - zI_\Omega)^{-q}(I_{\Omega} - (z' - z)(H_{0,\Omega} - zI_\Omega)^{-1})^{-q} \] (A.36)
implies (A.35) with $z$ replaced by $z'$, proving the claim.

Step 2. By [33, Th. B.8] (cf. also Th. 4.3.1.2 and Remark 4.3.1.2 in [58]), if $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain, $n \in \mathbb{N}$, and $s_0, s_1 \in \mathbb{R}$, then
\[ (H^{s_0}(\Omega), H^{s_1}(\Omega))_{\theta,2} = H^s(\Omega), \quad s = (1 - \theta)s_0 + \theta s_1, \quad 0 < \theta < 1. \] (A.37)

Here, for Banach spaces $\mathcal{X}_0$ and $\mathcal{X}_1$, we denote by $(\mathcal{X}_0, \mathcal{X}_1)_{\theta,p}$ the real interpolation space (obtained by the $K$-method), as discussed, for instance, in [33, Appendix B] and [58, Sec. 1.3]. Letting $s_0 = 0$, $s_1 = 2$, and $s = 2q$, one then infers
\[ (L^2(\Omega; d^n x), H^{2q}(\Omega))_{q,2} = H^{2q}(\Omega). \] (A.38)

Step 3. Using the claim in Step 1, we may assume without loss of generality that $H_{0,\Omega} - zI_\Omega$ is a strictly positive operator and thus the fractional power $(H_{0,\Omega} - zI_\Omega)^q$ can be defined via its spectral decomposition (see, e.g., [58, Sec. 1.18.10]). We remark that the operator $(H_{0,\Omega} - zI_\Omega)^q$ is an isomorphism between the Banach space $\text{dom}(H_{0,\Omega} - zI_\Omega)^q$, equipped with the graph-norm and the space $L^2(\Omega; d^n x)$, and thus
\[(H_{0,\Omega} - z I_\Omega)^{-q} \in \mathcal{B}(L^2(\Omega; d^n x), \text{dom}((H_{0,\Omega} - z I_\Omega)^q)). \tag{A.39}\]

By an abstract interpolation result for strictly positive, selfadjoint operators, see [58, Th. 1.18.10], for any \(\alpha, \beta \in \mathbb{C}\) with \(\text{Re} \alpha, \text{Re} \beta \geq 0\) and \(\theta \in (0, 1)\) we have

\[
\left(\text{dom}((H_{0,\Omega} - z I_\Omega)^{\alpha}), \text{dom}((H_{0,\Omega} - z I_\Omega)^{\beta})\right)_{\theta, 2} = \text{dom}((H_{0,\Omega} - z I_\Omega)^{\alpha(1-\theta)+\beta\theta}). \tag{A.40}\]

Applying this result with \(\alpha = 0\) and \(\beta = 1\), we infer

\[
\left(L^2(\Omega; d^n x), \text{dom}(H_{0,\Omega} - z I_\Omega)\right)_{\eta, 2} = \text{dom}((H_{0,\Omega} - z I_\Omega)^\eta). \tag{A.41}\]

Noting that \(\text{dom}(H_{0,\Omega}) = \text{dom}(H_{0,\Omega} - z I_\Omega)\), and using (A.38), (A.41), and Lemma A.1, we arrive at the continuous imbedding

\[
\text{dom}((H_{0,\Omega} - z I_\Omega)^\eta) \hookrightarrow H^{2q}(\Omega). \tag{A.42}\]

Thus, (A.35) is a consequence of (A.39) and (A.42).

Finally, we shall prove an extension of a result of Nakamura [36, Lemma 6] from a cube in \(\mathbb{R}^n\) to a Lipschitz domain \(\Omega\). This requires some preparations. First, we note that (A.15) and (A.13) yield the following Green formula

\[
(\bar{r}N u, \gamma_D \Phi) = (\nabla u, \nabla \Phi)_{L^2(\Omega; d^n x)} + \langle \Delta u, \Phi \rangle, \tag{A.43}\]

valid for any \(u \in H^1(\Omega)\) with \(\Delta u \in (H^1(\Omega))^*\) and any \(\Phi \in H^1(\Omega)\). The pairing on the left-hand side of (A.43) is between functionals in \((H^{1/2}(\partial \Omega))^*\) and elements in \(H^{1/2}(\partial \Omega)\), whereas the last pairing on the right-hand side is between functionals in \((H^1(\Omega))^*\) and elements in \(H^1(\Omega)\). For further use, we also note that the adjoint of (6.42) maps as follows:

\[
\gamma_D^* : (H^{s-1/2}(\partial \Omega))^* \to (H^s(\Omega))^*, \quad 1/2 < s < 3/2. \tag{A.44}\]

Next we observe that the operator \((\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1}, z \in \mathbb{C}\setminus\sigma(\bar{H}_{0,\Omega}^N)\), originally defined as

\[
(\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1} : L^2(\Omega; d^n x) \to L^2(\Omega; d^n x), \tag{A.45}\]

can be extended to a bounded operator mapping \((H^1(\Omega))^*\) into \(L^2(\Omega; d^n x)\). Specifically, since \((\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1} : L^2(\Omega; d^n x) \to \text{dom}(\bar{H}_{0,\Omega}^N)\) is bounded and since the inclusion \(\text{dom}(\bar{H}_{0,\Omega}^N) \hookrightarrow H^1(\Omega)\) is bounded, we can naturally view \((\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1}\) as an operator

\[
(\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1} : L^2(\Omega; d^n x) \to H^1(\Omega) \tag{A.46}\]

mapping in a linear, bounded fashion. Consequently, for its adjoint, we have

\[
((\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1})^* : (H^1(\Omega))^* \to L^2(\Omega; d^n x), \tag{A.47}\]

and it is easy to see that this latter operator extends the one in (A.45). Hence, there is no ambiguity in retaining the same symbol, i.e., \((\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1}\), both for the operator in (A.47) as well as for the operator in (A.45). Similar considerations and conventions apply to \((\bar{H}_{0,\Omega}^D - z I_\Omega)^{-1}\).

**Lemma A.3.** Let \(\Omega \subset \mathbb{R}^n, n \geq 2\), be a Lipschitz domain and let \(z \in \mathbb{C}\setminus(\sigma(\bar{H}_{0,\Omega}^D) \cup \sigma(\bar{H}_{0,\Omega}^N))\). Then, on \(L^2(\Omega; d^n x)\),

\[
(\bar{H}_{0,\Omega}^D - z I_\Omega)^{-1} - (\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1} = (\bar{H}_{0,\Omega}^D - z I_\Omega)^{-1}\gamma_D^*\gamma_N(\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1}, \tag{A.48}\]

where \(\gamma_D^*\) is an adjoint operator to \(\gamma_D\) in the sense of (A.44).

**Proof.** To set the stage, we note that the composition of operators appearing on the right-hand side of (A.48) is meaningful since

\[
(\bar{H}_{0,\Omega}^D - z I_\Omega)^{-1} : L^2(\Omega; d^n x) \to \text{dom}(\bar{H}_{0,\Omega}^D) \subset \{u \in H^1(\Omega) \mid \Delta u \in (H^1(\Omega))^*\}, \tag{A.49}\]

\[
\gamma_N : \{u \in H^1(\Omega) \mid \Delta u \in (H^1(\Omega))^*\} \to H^{-1/2}(\partial \Omega), \tag{A.50}\]

\[
\gamma_D^* : (H^{1/2}(\partial \Omega))^* \to (H^1(\Omega))^*, \tag{A.51}\]

\[
(\bar{H}_{0,\Omega}^N - z I_\Omega)^{-1} : (H^1(\Omega))^* \to L^2(\Omega; d^n x), \tag{A.52}\]

with the convention made just before the statement of the lemma used in the last line. Next, let \(\phi_1, \psi_1 \in L^2(\Omega; d^n x)\) be arbitrary and define
\[ \phi = (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\phi_1 \in \text{dom}(\tilde{H}_{0,\Omega}^N) \subset H^1(\Omega), \]
\[ \psi = (\tilde{H}_{0,\Omega}^D - zI_\Omega)^{-1}\psi_1 \in \text{dom}(\tilde{H}_{0,\Omega}^D) \subset H^1(\Omega). \]

It therefore suffices to show that the following identity holds:
\[ (\phi_1, (\tilde{H}_{0,\Omega}^D - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega; d^n x)} - (\phi_1, (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega; d^n x)} = (\phi_1, (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\gamma_N^* \tilde{\gamma}_N (\tilde{H}_{0,\Omega}^D - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega; d^n x)}. \]

We note that, according to (A.53), we have
\[ (\phi_1, (\tilde{H}_{0,\Omega}^D - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega; d^n x)} = ((\tilde{H}_{0,\Omega}^N - zI_\Omega)\phi, \psi)_{L^2(\Omega; d^n x)}, \]
\[ (\phi_1, (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega; d^n x)} = (((\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1})^* \phi, \psi_1)_{L^2(\Omega; d^n x)} \]
\[ = (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\phi_1, \psi_1)_{L^2(\Omega; d^n x)} = (\phi_1, (\tilde{H}_{0,\Omega}^D - zI_\Omega)\psi)_{L^2(\Omega; d^n x)}, \]

and, keeping in mind the convention adopted prior to the statement of the lemma,
\[ (\phi_1, (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\gamma_d^* \tilde{\gamma}_N (\tilde{H}_{0,\Omega}^D - zI_\Omega)^{-1}\psi_1)_{L^2(\Omega; d^n x)} = (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\gamma_d^* \tilde{\gamma}_N (\tilde{H}_{0,\Omega}^D - zI_\Omega)^{-1}\psi_1 \]
\[ = (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\phi_1, \psi_1)_{L^2(\Omega; d^n x)} = (\psi_1, (\tilde{H}_{0,\Omega}^D - zI_\Omega)\phi)_{L^2(\Omega; d^n x)} = (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\phi_1, \psi_1)_{L^2(\Omega; d^n x)}. \]

where \((\cdot, \cdot)\) stands for pairings between Sobolev spaces \((\Omega)\) and \(\partial\Omega\) and their duals. Thus, matters have been reduced to proving that
\[ ((\tilde{H}_{0,\Omega}^N - zI_\Omega)\phi, \psi)_{L^2(\Omega; d^n x)} - (\phi, (\tilde{H}_{0,\Omega}^D - zI_\Omega)\psi)_{L^2(\Omega; d^n x)} = (\tilde{H}_{0,\Omega}^N - zI_\Omega)^{-1}\phi_1, \psi_1)_{L^2(\Omega; d^n x)}. \]

Using (A.43) for the left-hand side of (A.58), we obtain
\[ ((\tilde{H}_{0,\Omega}^N - zI_\Omega)\phi, \psi)_{L^2(\Omega; d^n x)} - (\phi, (\tilde{H}_{0,\Omega}^D - zI_\Omega)\psi)_{L^2(\Omega; d^n x)} = -(\Delta \phi, \psi)_{L^2(\Omega; d^n x)} + (\phi, \Delta \psi)_{L^2(\Omega; d^n x)} \]
\[ = (\nabla \phi, \nabla \psi)_{L^2(\Omega; d^n x)} - \langle \tilde{\gamma}_N \phi, \gamma_D \psi \rangle - (\nabla \psi, \nabla \psi)_{L^2(\Omega; d^n x)} + \langle \psi_1, \tilde{\gamma}_N \psi \rangle \]
\[ = -\langle \nabla \phi, \nabla \psi \rangle + \langle \gamma_D \phi, \tilde{\gamma}_N \psi \rangle. \]

Observing that \(\tilde{\gamma}_N \phi = 0\) since \(\phi \in \text{dom}(H_{0,\Omega}^N\), we obtain (A.58).

**Remark A.4.** While it is tempting to view \(\gamma_D\) as an unbounded but densely defined operator on \(L^2(\Omega; d^n x)\) whose domain contains the space \(C_0^\infty(\Omega)\), one should note that in this case its adjoint \(\gamma_d^*\) is not densely defined; indeed, the adjoint \(\gamma_d^*\) of \(\gamma_D\) would have to be an unbounded operator from \(L^2(\partial\Omega; d^{n-1} x)\) to \(L^2(\Omega; d^n x)\) such that
\[ (\gamma_d f, g)_{L^2(\partial\Omega; d^{n-1} x)} = (f, \gamma_d^* g)_{L^2(\Omega; d^n x)} \quad \text{for all} \quad f, g \in \text{dom}(\gamma_D^*). \]

In particular, choosing \(f \in C_0^\infty(\Omega)\), in which case \(\gamma_D f = 0\), one concludes that \((f, \gamma_d^* g)_{L^2(\Omega; d^n x)} = 0 \quad \text{for all} \quad f \in C_0^\infty(\Omega)\). Thus, one obtains \(\gamma_d^* g = 0 \quad \text{for all} \quad g \in \text{dom}(\gamma_D^*)\). Since we obviously have \(\gamma_D \neq 0\), relation (A.60) implies \(\text{dom}(\gamma_d^*) = \{0\} \) and hence \(\gamma_D^*\) is not a closable linear operator in \(L^2(\Omega; d^n x)\).

**Remark A.5.** In the case of a domain of \(\Omega\) of class \(C^{1, r}\), \(1/2 < r < 1\), the operators \(\tilde{H}_{0,\Omega}^N\) and \(\tilde{H}_{0,\Omega}^D\) coincide with the operators \(H_{0,\Omega}^N\) and \(H_{0,\Omega}^D\), respectively, and hence one can use the operators \(H_{0,\Omega}^D\) and \(H_{0,\Omega}^N\) in Lemma A.3. Moreover, since \(\text{dom}(H_{0,\Omega}^D) \subseteq H^2(\Omega)\), one can also replace \(\tilde{\gamma}_N\) by \(\gamma_N\) (cf. (6.43)) in Lemma A.3. In particular,
\[ (H_{0,\Omega}^D - zI_\Omega)^{-1} - (H_{0,\Omega}^N - zI_\Omega)^{-1} = \left[ \gamma_D (H_{0,\Omega}^N - zI_\Omega)^{-1} \right]^* \gamma_N (H_{0,\Omega}^D - zI_\Omega)^{-1}. \]

**ACKNOWLEDGMENTS**

We are indebted to Konstantin Makarov, Alexander Pushnitski, Roland Schnaubelt, and Rico Zacher for very helpful discussions.

Fritz Gesztesy and Yuri Latushkin gratefully acknowledge a research leave for the academic year 2005/06 granted by the Research Council and the Office of Research of the University of Missouri–Columbia. Moreover, Yuri Latushkin gratefully acknowledges support by the Research Board of the University of Missouri.
REFERENCES

41. R. G. Newton, Scattering Theory of Waves and Particles, 2nd ed. (Dover, New York, 2002).