THE EVANS FUNCTION AND
THE WEYL-TITCHMARSH FUNCTION

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Abstract. We describe relations between the Evans function, a modern tool in the study of stability of traveling waves and other patterns for PDEs, and the classical Weyl-Titchmarsh function for singular Sturm-Liouville differential expressions and for matrix Hamiltonian systems. Also, we discuss a related issue of approximating eigenvalue problems on the whole line by that on finite segments.

1. Introduction

The primary goal of this semi-expository paper is to establish connections between the classical Weyl-Titchmarsh function for Hamiltonian ordinary differential equations, and the Evans function, a Wronskian-type determinant designed to detect point spectrum of ordinary differential operators arising in the study of stability of traveling waves and other patterns for partial differential equations. As a byproduct, we give an elementary explanation of the relations of the point spectrum and the Evans function for problems on the full line to that on finite segments.

In addition to the classical sources [9, 10] of the Weyl-Titchmarsh theory, we refer to recent work [7, 19, 20, 21, 27] containing illuminating surveys, and, especially, to [14]. There is of course big literature on approximating the spectra of differential operators on the line by the spectra of respective operators on finite segments, see, for instance, [4, 5, 22, 31, 33, 35, 39] and the literature cited therein. Finally, we refer to [3, 12, 13, 15, 25, 26, 30, 32, 36, 37] for the general discussion of the Evans function some relevant aspects of which will be briefly reviewed next.

Given a partial differential equation in one space dimension, say, a reaction diffusion equation on \((-\infty, +\infty)\), one is interested in studying stability of such special solutions as traveling waves. Passing to the moving coordinates and linearizing the partial differential equation about the wave, one considers an eigenvalue problem, \(Hu = zu\), for the linearized operator \(H\), an ordinary differential operator of some order. The instability of the wave is related to the presence of unstable eigenvalues, and thus the problem of locating the isolated eigenvalues is of importance. Rewriting the high order differential equation \(Hu = zu\) as a first order matrix differential equation, one is looking for the values of the spectral parameter \(z\) for which the latter equation has solutions exponentially decaying at both plus and minus infinity. These values are the zeros of the Evans function, defined as the determinant of the matrix whose columns are the solutions of the matrix differential equations.
that are exponentially decaying at $+\infty$ and at $-\infty$. For instance, in the particular case when $H$ is the Schrödinger operator with a summable real valued potential on the line, the Evans function is known to be equal, see [15, 26], to the Jost function, that is, to the rescaled Wronskian of the exponentially decaying at $+\infty$ and $-\infty$ Jost solutions of the Schrödinger equation, see, e.g., [6, Chapter XVII].

In summary, the main idea of constructing the Evans function is to match solutions with appropriate asymptotics at $+\infty$ and at $-\infty$. This idea works for many important situations, most notably, for quite general non-selfadjoint problems; however, the matching solutions are assumed to be exponentially decaying at the respective singular ends which are always assumed to be $+\infty$ and $-\infty$. Although the Weyl-Titchmarsh function is constructed only for selfadjoint problems, it is designed to produce solutions that are merely square summable at the respective singular ends (which, in addition, are not necessarily infinite). One of the many nice features of the Weyl-Titchmarsh function is that it is constructed by approximation, that is, using respective objects defined on regular (in particular, finite) segments. This is very much in concert with the approach of approximating the Evans function by respective objects on finite segments frequently used by numerical analysts working in stability analysis of traveling waves.

A typical result of the current paper, see, e.g., Corollary 2.18, is that the Evans function (which is matching solutions at two singular ends) is, essentially, equal to the difference of the two Weyl-Titchmarsh functions (corresponding to each of the singular ends). We do not really know of any previous literature relating the Evans and the Weyl-Titchmarsh functions. The relation between the Jost and the Weyl-Titchmarsh functions, however, should look familiar to experts in the Weyl-Titchmarsh theory although we were not able to pinpoint in the literature the exact facts that we want. Our methods are totally elementary, e.g., we even do not use connections of the Weyl-Titchmarsh functions and spectral measures.

The paper is organized as follows. In Section 2 we study general Sturm-Liouville differential operators on $(a, b) \subset \mathbb{R}$ where each end can be singular (finite or infinite, either in the limit point or in the limit circle case). After a brief review of the Weyl-Titchmarsh theory, we construct the solutions of the Sturm-Liouville equation whose Wronskian is equal to zero exactly at the points of the discrete spectrum of the differential operator. This could be viewed as constructing an analogue of the Evans function in our quite general situation. In Theorem 2.12 we offer formulas relating this Wronskian and the Weyl-Titchmarsh function for the values of the spectral parameter $z$ outside of the essential spectrum. (Formulas of this type are of course well known for non-real values of $z$; here, we just carefully recorded what happens on the real line, in particular, at the points of the discrete spectrum).

We then use the Weyl-Titchmarsh functions to describe the discrete spectrum, see Corollary 2.14. At the end of Section 2 we specialize to the situation of the Schrödinger operator on the line with a summable potential when the situation becomes more transparent.

In Section 3, still working on the model case of the Schrödinger equation, we discuss how to approximate the Jost (or the Evans) function, $J$, on the line by restricting the problem to finite intervals $[-L, L]$ and imposing some boundary conditions at $\pm L$. Of course, this topic belongs to a huge and well-studied area, but our contribution here is a formula, see Theorem 3.3, relating the asymptotic behavior for large $L$ of the Jost function $J_L$ on the finite segments to that of the
product of $\mathcal{J}$ and some quantities induced by the boundary conditions. We also offer an operator-theoretic interpretation of these quantities, which appears to be helpful in explaining some numerically important results in [4, 33].

In Section 4 we study the first order matrix self-adjoint Hamiltonian differential systems on $(a, b) \subset \mathbb{R}$ for which the matrix valued Weyl-Titchmarsh theory is available, see [7, 8, 23, 27, 28, 34]. Considering either the limit point or the limit circle case at either end, we describe square summable at each end solutions and match them by means of a matrix valued Wronskian. Our main result, Theorem 4.8, gives a formula relating the Evans function for the first order system and the Wronskians of the matrix valued Weyl-Titchmarsh functions. We conclude with a number of examples for which our results are applicable.

Notations. We denote by $[\alpha \, \beta]_T$ a row-vector or a rectangular block-matrix, so that $[\alpha \, \beta]_T \in \mathbb{C}^{j}$ is a column vector, where $T$ is transposition. We denote by $e_i = [\delta_{ij}]_{i=1}^n$, $i = 1, \ldots, n$, the standard unit column $(n \times 1)$ vectors in $\mathbb{C}^n$; the same symbols $e_i$, $i = 1, \ldots, 2n$, are used to denote the standard unit vectors in $\mathbb{C}^{2n}$. For an operator $T$, we denote by $\sigma(T)$ the spectrum, by $\sigma_d(T)$ the discrete spectrum (the set of isolated eigenvalues of finite algebraic multiplicity), and by $\sigma_{ess}(T) = \sigma(T) \setminus \sigma_d(T)$ the essential spectrum. Given a measurable and almost-everywhere positive function $p$, for $f, g$ and $pf', pg'$ absolutely continuous, we denote by $W_t(f, g) = f(t) pg'(t) - pf'(t)g(t)$ the value of the Wronskian $W(f, g)$ of the functions $f, g$ at the point $t$ (as a rule, we prefer to write $pf'(t)$ instead of $p(t)f'(t)$). Also, we will use notations $W_a(f, g) = \lim_{c \to a} W_c(f, g)$ with $a < c < 0$ and $W_b(f, g) = \lim_{d \to b} W_d(f, g)$ with $0 < d < b$. Given a measurable and almost-everywhere positive function $r$, we denote by $L^2(a, b)$ the Hilbert space with the scalar product $\langle f, g \rangle_{L^2(a, b)} = \int_a^b f(t) \overline{g(t)} r(t) dt$; here and below bar stands for complex conjugation.

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2. Singular Sturm-Liouville differential expressions

Let us consider the general Sturm-Liouville differential expression $\tau$,

$$\tau y(t) = \frac{1}{r(t)} \left( -(pg')'(t) + V(t) \right), \quad t \in (a, b), \quad -\infty \leq a < b \leq \infty, \quad (2.1)$$

and impose the following assumptions.

Hypothesis 2.1. Assume that $p, r, V$ are real-valued measurable functions on $(a, b)$ such that $p(t), r(t) > 0$ almost everywhere in $(a, b)$, and $1/p, r, V$ belong to $L^1_{loc}(a, b)$, that is, belong to $L^1(c, d)$ for any $a < c < d < b$.

First, we briefly review some basics of the Weyl-Titchmarsh theory [9, 10]. Without loss of generality we will assume that $0 \in (a, b)$. The point $b$ is called regular if $b$ is finite and assumptions in Hypothesis 2.1 hold on $(a, b)$, and singular otherwise; analogously for $a$. Following [9, Chapter IX], we fix an $\omega \in [0, \pi)$, and let $\theta(\cdot, z), \phi(\cdot, z)$ denote the solutions $(a, b)$ of the differential equation $\tau y = zy$, 

Then clearly \( \theta(\cdot, z) \), \( \phi(\cdot, z) \) are linearly independent solutions, and \( \theta, \theta', \phi, \phi' \) are entire functions of \( z \) and continuous in \( t, z \). Moreover, since \( W_0(\theta, \phi) = 1 \) one has \( W_t(\theta, \phi) = 1 \) for all \( t \). These solutions are real for real \( z \).

Every solution \( \chi \) of \( \tau y = zy \) except \( \phi \) is, up to a constant multiple, of the form
\[
\chi(\cdot, z) = \theta(\cdot, z) + m\phi(\cdot, z), \quad z \in \mathbb{C},
\]
for some \( m = m(z) \) which will depend on \( z \in \mathbb{C} \). Consider now a real boundary condition at some point \( d, 0 < d < b \),
\[
\cos \eta y(d) + \sin \eta y'(d) = 0, \quad 0 < \eta < \pi.
\]
We want to find \( m \) so that the solution \( \chi \) in (2.3) would satisfy (2.4). Clearly, \( m \) must satisfy:
\[
m = -\frac{\cot \eta \theta(d, z) + p\theta'(d, z)}{\cot \eta \phi(d, z) + p\phi'(d, z)}.
\]
As \( z, d, \eta \) vary, \( m \) becomes a function of these arguments, \( m = m(z, d, \eta) \), and since \( \theta, \theta', \phi, \phi' \) are entire in \( z \) it follows that \( m(\cdot, d, \eta) \) is meromorphic in \( z \) and real for real \( z \). For fixed \( (z, d) \), when \( \eta \) changes from 0 to \( \pi \), the complex numbers \( m(z, d, \eta) \) form a circle, denoted by \( C_d \), whose equation in the \( m \)-plane, by a direct verification, is given by the formula
\[
W_d(\chi, \overline{\chi}) = 0, \quad \text{where} \quad \chi(\cdot, z) = \theta(\cdot, z) + m\phi(\cdot, z), \quad m \in \mathbb{C}.
\]
The minimal operator $H_{\text{min}}$ in $L^2(a,b)$ associated with $\tau$ is defined as the closure of the operator $H'_{\text{min}}$ given as follows:

$$H'_{\text{min}}f = \tau f,$$

$$f \in \text{dom } H'_{\text{min}} = \{ g \in \text{dom } H_{\text{max}} \mid g \text{ has compact support in } (a,b) \}. \quad (2.8)$$

**Theorem 2.4.** ([29, Theorem 6.1], [39, Theorem 3.9]) Assume Hypothesis 2.1. Then the adjoint of $H_{\text{min}}$ defined in $L^2(a,b)$ is the maximal operator $H_{\text{max}}$:

$$H^*_{\text{min}}f = H_{\text{max}}f = \tau f,$$

$$f \in \text{dom } H^*_{\text{min}} = \{ g \in L^2(a,b) \mid g, pg' \in AC_{\text{loc}}(a,b); \tau g \in L^2(a,b) \}. \quad (2.9)$$

We will impose some boundary conditions in order to define self-adjoint extensions $H$ of the operator $H_{\text{min}}$ in $L^2(a,b)$, see [29, Section 6.3], [39, Section 4.5], and [38]. For this purpose, we define smooth functions $\rho_a$ and $\rho_b$ such that

$$\rho_a \equiv 1 \text{ near } a, \quad \rho_a \equiv 0 \text{ near } b, \quad (2.10)$$

$$\rho_b \equiv 1 \text{ near } b, \quad \rho_b \equiv 0 \text{ near } a. \quad (2.11)$$

Next, we fix $a \in \mathbb{C} \setminus \mathbb{R}$, then fix any $\hat{m}_a(z) \in C_a$ and $\hat{m}_b(z) \in C_b$ (in particular, $\hat{m}_a(z) = m_a(z)$ if $\tau$ is in $\text{hpc}$ at $a$ and $\hat{m}_b(z) = m_b(z)$ if $\tau$ is in $\text{hpc}$ at $b$), and define $u_a(\cdot, z) \in L^2(a,b)$ and $u_b(\cdot, z) \in L^2(a,b)$ by

$$u_a(t, z) = \rho_a(t)(\theta(t, z) + \hat{m}_a(z)\phi(t, z)), \quad (2.12)$$

$$u_b(t, z) = \rho_b(t)(\theta(t, z) + \hat{m}_b(z)\phi(t, z)).$$

Let us define in $L^2(a,b)$ the differential operator $H_1$ by

$$H_1 f = \tau f,$$

$$f \in \text{dom } H_1 = \text{dom } H'_{\text{min}} + \{ u_a(\cdot, z) \} + \{ u_b(\cdot, z) \}, \quad (2.13)$$

where $u_a(\cdot, z)$ and $u_b(\cdot, z)$ are given by (2.12). In other words, the domain of $H_1$ consists of compactly supported in $(a, b)$ functions from $\text{dom } H_{\text{max}}$ together with $u_a(\cdot, z), u_b(\cdot, z)$, and all linear combinations of finitely many such functions.

**Theorem 2.5.** ([29, Theorem 6.5], [38, Theorem 5.8]) Assume Hypothesis 2.1. Let $H_1$ be as in (2.13). Then $H_1$ is an essential self-adjoint extension of the minimal operator $H_{\text{min}}$ given in (2.8), (2.9). Let $H$ be the closure of $H_1$. Then

$$H f = \tau f,$$

$$f \in \text{dom } H = \{ g \in L^2(a,b) \mid g, pg' \in AC_{\text{loc}}(a,b), \tau g \in L^2(a,b), \quad W_a(g, g_a(z)) = 0, W_b(g, g_b(z)) = 0 \}. \quad (2.14)$$

We will also need the operators $H_a$ in $L^2(a,0)$ and $H_b$ in $L^2(0,b)$ defined by

$$H_a f = \tau f,$$

$$f \in \text{dom } H_a = \{ g \in L^2(a,0) \mid g, pg' \in AC_{\text{loc}}(a,0), \tau g \in L^2(a,0), \quad W_a(g, g_a(z)) = 0, W_0(g, \phi) = \sin \omega g(0) - \cos \omega pg'(0) = 0 \}, \quad (2.15a)$$

$$H_b f = \tau f,$$

$$f \in \text{dom } H_b = \{ g \in L^2(0,b) \mid g, pg' \in AC_{\text{loc}}(0,b), \tau g \in L^2(0,b), \quad W_0(g, \phi) = \sin \omega g(0) - \cos \omega pg'(0) = 0, W_b(g, g_b(z)) = 0 \}. \quad (2.15c)$$
As shown in [29, page 25] or [27, Section VII.7], the boundary conditions at a and b in (2.14), (2.15c), (2.16c) are \( z \)-independent.

**Remark 2.6.** If \( \tau \) is in limit point case either at \( a \) or at \( b \) then the corresponding boundary condition at \( a \) or \( b \) in (2.14), (2.15c), (2.16c) may be omitted (that is, if \( \tau \) is in lpc at \( a \) (respectively, at \( b \)) then the condition \( W_a(g, \bar{u}(z)) = 0 \) (respectively, \( W_b(g, \bar{u}(z)) = 0 \)) is automatically satisfied for any \( g \in \text{dom} \, H_{\text{max}} \), see e.g. [38], [39, Theorem 5.7], [29, Theorem 6.5 (iv)]).

Since (see [10, Theorem XIII.7.4])

\[
\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_a) \cup \sigma_{\text{ess}}(H_b),
\]

the following lemma holds (see, e.g. [19, (A.25)]).

**Lemma 2.7.** Assume Hypothesis 2.1. Then the Weyl-Titchmarsh functions are meromorphic in \( z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H) \).

**Proof.** Let \( \hat{m}_a \) be the Weyl-Titchmarsh function (that is, a point on the circle \( C_a \) if \( \tau \) is in lcc at \( a \), or the point \( m_a \) if \( \tau \) is in lpc at \( a \)). Consider the operator \( H_a \) defined in \( L^2(a,0) \) by (2.15). Green’s function of \( H_a \) for \( z \in \mathbb{C} \setminus \sigma(H_a) \) is given by the formula

\[
G(t,s,z) = \begin{cases} 
\phi(t,z)\chi_a(s,z), & t \leq s, \\
\phi(s,z)\chi_a(t,z), & s \leq t,
\end{cases}
\]

where \( \chi_a(t,z) = \theta(t,z) + \hat{m}_a(z)\phi(t,z) \) (see [9, Chapter IX, (4.10)]). In particular,

\[
G(t,t,z) = \phi(t,z)(\theta(t,z) + \hat{m}_a(z)\phi(t,z)), \quad t \leq 0.
\]

Since Green’s function is meromorphic in \( z \) away from the essential spectrum of \( H_a \) for any fixed \( t, s \) (cf. [10, Section XIII.5, Theorem XIII.5.18, Corollary XIII.5.30]), \( \hat{m}_a \) is meromorphic for \( z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a) \) and, therefore, in \( \mathbb{C} \setminus \sigma_{\text{ess}}(H) \) due to (2.17). Analogously, one can prove that \( \hat{m}_b \) is meromorphic in \( \mathbb{C} \setminus \sigma_{\text{ess}}(H) \).

**Remark 2.8.** If \( \tau \) is in limit point case at \( b \), if \( m \) is any point on \( C_d \) and \( d \to b \), then \( m \to m_b \), where \( m_b \) is the limit point, and the relation \( m \to m_b \) holds independently of the choice of \( \eta \) in the boundary condition (2.4). In particular, it holds when \( \eta = 0 \), and thus the limit point is given by the formula

\[
m_b(z) = -\lim_{d \to b} \frac{\theta(d,z)}{\phi(d,z)} \quad \text{for} \ z \in \mathbb{C} \setminus \mathbb{R}.
\]

Similarly, if \( \tau \) is in limit point case at \( a \) then

\[
m_a(z) = -\lim_{c \to a} \frac{\theta(c,z)}{\phi(c,z)} \quad \text{for} \ z \in \mathbb{C} \setminus \mathbb{R}.
\]

The purpose of the next proposition is to establish the existence and uniqueness of the solutions \( f_a(\cdot, z) \) and \( f_b(\cdot, z) \) of the differential equation \( \tau y = zy \) that are square summable near \( a \) and \( b \), respectively, and satisfy the respective boundary conditions in (2.15c) and (2.16c), if any. For \( z \in \mathbb{C} \setminus \mathbb{R} \) this is contained in [10, Theorem XIII.2.32] or [29, Theorem 6.2]; we emphasize that \( z \) in Proposition 2.9 below can be real.

**Proposition 2.9.** Assume Hypothesis 2.1. If \( z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a) \) then the following assertions hold:
(i) If $\tau$ is in lpc at $a$ then there is a unique (up to a constant multiple) solution $f_a(\cdot, z)$ of $\tau y = zy$ that is square integrable near $a$; 
(ii) If $\tau$ is in lcc at $a$ then there is a unique (up to a constant multiple) solution $f_a(\cdot, z)$ of $\tau y = zy$ that is square integrable near $a$ and satisfy the boundary condition at $a$ in (2.15c).

Moreover, in either case, if $z \in \mathbb{C} \setminus \sigma(H_a)$ then (up to a constant multiple) $f_a(\cdot, z)$ is equal to $\chi_a(\cdot, z) = \theta(\cdot, z) + \hat{m}_a(z)\phi(\cdot, z)$, where $\hat{m}_a(z)$ is used in (2.12) and (2.15c) to define the operator $H_a$ if $\tau$ is in lcc at $a$, and $\hat{m}_a(z) = m_a(z)$ if $\tau$ is in lpc at $a$. Lastly, if $z \in \sigma_d(H_a)$ then $f_a(\cdot, z)$ is the nonzero eigenfunction of the operator $H_a - z$.

Analogous assertions hold for the point $b$.

Proof. First, assuming $z \in \mathbb{C} \setminus \sigma(H_a)$, we let $\mu$ denote the number of linearly independent solutions of $\tau y = zy$ that are square integrable near $a$ and satisfy the boundary condition at $a$ in (2.15c) provided $\tau$ is in lcc at $a$, and that are just square integrable near $a$ provided $\tau$ is in lpc at $a$. Also, we let $\nu$ denote the number of linearly independent solutions that are square integrable near 0 and satisfy the boundary condition at 0 in (2.15c). Then $\mu + \nu = 2$ by [10, Theorem XIII.3.11] or [38, Theorem 7.1]. Since 0 is a regular point, $\nu = 1$. Thus $\mu = 1$ proving (i) and (ii) for $z \in \mathbb{C} \setminus \sigma(H_a)$. Furthermore, if $\tau$ is in lpc at $a$ then the solution $\chi_a(\cdot, z) = \theta(\cdot, z) + \hat{m}_a(z)\phi(\cdot, z)$ is square integrable near $a$, and if $\tau$ is in lcc at $a$ then the solution $\chi_a(\cdot, z) = \theta(\cdot, z) + \hat{m}_a(z)\phi(\cdot, z)$ is square integrable near $a$ and satisfies the boundary condition at $a$ in (2.15c). Indeed, for nonreal $z$ this follows from the construction of the Weyl-Titchmarsh function in Theorem 2.2, cf. [10, Theorem XIII.2.32] or [29, Theorem 6.2]. For $z \in \mathbb{R} \setminus \sigma(H_a)$ this follows from the first line in formula (2.18) for Green’s function, and the fact that Green’s function is square integrable and $G(\cdot, s, z)$ (for a fixed $s$) satisfies the boundary conditions defining $H_a$, see, e.g., [10, Lemma XIII.3.4, Lemma XIII.3.7].

Next, we assume that $z \in \sigma_d(H_a)$, the discrete spectrum of $H_a$; in particular, $z \in \mathbb{R}$. If $\tau$ is in lpc at $a$, then the nonzero eigenfunction of $H_a$ corresponding to the given $z$ is the required solution $f_a(\cdot, z)$ since the number of linearly independent solutions that are square integrable near $a$ is at most one. It remains to consider the case when $z \in \sigma_d(H_a)$ and $\tau$ is in lcc at $a$. We know that the eigenfunction corresponding to the given $z$ is a solution of $\tau y = zy$ that is square integrable near $a$ and satisfy the boundary condition at $a$ in (2.15c). Seeking a contradiction, let us assume that there are two linearly independent solutions, $f_a^{(1)}(\cdot, z)$ and $f_a^{(2)}(\cdot, z)$, of $\tau y = zy$ that are square integrable near $a$ and satisfy the boundary condition at $a$ in (2.15c). According to [38, Theorem 5.8(v)], the domain of $H_a$ can be rewritten in the following way:

$$
\text{dom } H_a = \{g \in L^2(a, 0) | g, pg' \in AC_{loc}(a, 0), \tau g \in L^2(a, 0), W_a(v, g) = W_0(w, g) = 0\},$$

where $v$ and $w$ are nontrivial real solutions of $\tau y = zy$. Indeed, as required in [38, Theorem 5.8(iv)], $\tau$ is in lcc at both $a$ and 0, and the operator $H_a$ is defined using separated boundary conditions (2.15c). Let us introduce the functions $\tilde{f}_a^{(1)}$ and $\tilde{f}_a^{(2)}$ as follows:

$$
\tilde{f}_a^{(j)}(t, z) = \begin{cases} f_a^{(j)}(t, z) & \text{for } t \text{ close to } a, \\ 0 & \text{for } t \text{ close to } 0, \end{cases}
$$
and such that $\tilde{f}_a^{(j)}(\cdot, z) \in \text{dom } H_{a,\text{max}}$ (the latter inclusion is possible as described in [38, page 50]). Then $\tilde{f}_a^{(j)} \in \text{dom } H_{a}$. Consequently, $\tilde{f}_a^{(j)}(\cdot, z)$ and $\tilde{f}_a^{(2)}(\cdot, z)$ should satisfy the boundary condition at $a$ in (2.21). Therefore, $f_a^{(1)}(\cdot, z)$ and $f_a^{(2)}(\cdot, z)$ should satisfy the boundary condition at $a$ in (2.21) as well. Since $f_a^{(1)}(\cdot, z)$ and $v$ are solutions of $\tau y = zy$, they are linearly dependent. Analogously, $f_a^{(2)}(\cdot, z)$ and $v$ are linearly dependent. Thus, $f_a^{(1)}(\cdot, z)$ and $f_a^{(2)}(\cdot, z)$ are linearly dependent, a contradiction.

Remark 2.10. If $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a)$ and $f_a$ is the solution described in Proposition 2.9 then both Wronskians $W(\theta, f_a(\cdot, z))$ and $W(f_a(\cdot, z), \phi)$ are finite, and cannot be equal to 0 simultaneously as otherwise $f_a(0, z_0) = p f_a'(0, z_0) = 0$. Analogous facts hold for the point $b$.

Our next objective is to relate the Weyl-Titschmarsh $m$-function to the Wronskian determinant of the solutions $\theta, \phi, f_a$ and $f_b$. As the following lemma shows, this information can be used to characterize the discrete spectrum of the operators $H_a$, $H_b$ and $H$.

Lemma 2.11. Assume Hypothesis 2.1. If $z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a)$ then the following assertions are equivalent:

(i) $W(f_a(\cdot, z_0), \phi(\cdot, z_0)) = 0$;
(ii) $z_0 \in \sigma_d(H_a)$;
(iii) $z_0$ is a pole of $m_a(\cdot)$.

and if the equivalent assertions hold then $f_a(\cdot, z_0)$ is an eigenfunction of $H_a$. Analogous facts hold for the point $b$. Finally, if $z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(H)$ then the following assertions are equivalent:

(iv) $W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0$;
(v) $z_0 \in \sigma_d(H)$.

Proof. (i) $\iff$ (ii): By Proposition 2.9, $f_a$ is the unique solution of $\tau y = zy$ square summable at $a$ (when $\tau$ is in lpc at $a$) that satisfies the boundary condition at $a$ in (2.15c) (when $\tau$ is in lcc at $a$). Then (i) holds if and only if $f_a$ is proportional to $\phi$ if and only if $f_a$ satisfies both boundary conditions in (2.15c) (when $\tau$ is in lcc at $a$) if and only if $f_a$ is an eigenfunction of $H_a$.

(ii) $\iff$ (iii): Indeed, (ii) holds if and only if there exists at least one point $(t, s)$ for which Green’s function $G(t, s, \cdot)$ of $H_a$ has a pole at $z_0$ as a function of $z$, and this is the case if and only if $z_0$ is a pole of $m_a(\cdot)$ due to (2.18).

(iv) $\iff$ (v): Indeed, (iv) holds if and only if $f_a$ is proportional to $f_b$ if and only if $f_a$ satisfies Proposition 2.9 at both $a$ and $b$ if and only if $f_a$ is an eigenfunction of $H$.

We refer to [19, (A.36)] and [14, (5.8),(5.10)] for versions of the next theorem when $z \in \mathbb{C} \setminus \mathbb{R}$. Our main point, however, is that $z$ can be real in (2.23)-(2.25).

Theorem 2.12. Assume Hypothesis 2.1 and suppose that $\theta$ and $\phi$ satisfy (2.2) for some $\omega \in [0, \pi)$. Let $f_a(\cdot, z)$ and $f_b(\cdot, z)$ denote the square integrable near $a$ and $b$ solutions of $\tau y = zy$ described in Proposition 2.9. Then the following formulas hold:

$$m_a(z) = \frac{W_0(\theta(\cdot, z), f_a(\cdot, z))}{W_0(f_a(\cdot, z), \phi(\cdot, z))}, \quad z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_a),$$

(2.23)
and, in particular, $\chi(z)$ at $z$ thus proving (2.24) for $\in \sigma_{ess}(H_b)$, (2.24)

$$m_b(z) = \frac{W_0(\theta(\cdot, z), f_b(\cdot, z))}{W_0(f_b(\cdot, z), \phi(\cdot, z))}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H_b),$$

$$m_b(z) - m_a(z) = \frac{W_0(f_a(\cdot, z), f_b(\cdot, z))}{W_0(f_b(\cdot, z), \phi(\cdot, z))}, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H).$$

(2.25)

Proof. Let us prove (2.24), the proof of (2.23) is analogous. By Proposition 2.9, $f_b(\cdot, z)$ and $\chi_b(\cdot, z) = \theta(\cdot, z) + m_b(z)\phi(\cdot, z)$ have to be proportional whenever $m_a(z)$ is finite, that is, for $z \in \mathbb{C} \setminus \sigma(H_b)$. Hence, there exists a $t$-independent constant $c_b(z)$ such that $f_b(t, z) = c_b(z)(\theta(t, z) + m_b(z)\phi(t, z))$. Differentiating, letting $t = 0$, and using (2.2) yields the system of equations for $m_b(z)$ and $c_b(z)$,

$$c_b(z) \sin \omega + c_b(z)m_b(z) \cos \omega = f_b(0, z),$$

$$-c_b(z) \cos \omega + c_b(z)m_b(z) \sin \omega = pf'_b(0, z),$$

whose solution is given by

$$m_b(z) = \frac{f_b(0, z) \cos \omega + pf'_b(0, z) \sin \omega}{f_b(0, z) \sin \omega - pf'_b(0, z) \cos \omega} \quad (2.26)$$

$$= \frac{W_0(\theta(\cdot, z), f_b(\cdot, z))}{W_0(f_b(\cdot, z), \phi(\cdot, z))},$$

thus proving (2.24) for $z \in \mathbb{C} \setminus \sigma(H_b)$. Further, if $z \in \sigma_d(H_b)$ then $m_b$ has a pole at $z$ by Lemma 2.11. Since $f_b(\cdot, z)$ is the eigenfunction of $H_b$ by Proposition 2.9, $W_0(f_b(\cdot, z), \phi(\cdot, z)) = 0$ by (2.16c). Also, $W_0(\theta(\cdot, z), f_b(\cdot, z)) \neq 0$ by Remark 2.10. This extends (2.24) for $z \in \mathbb{C} \setminus \sigma_{ess}(H_b)$. Using (2.26), its analogue for $a$, and (2.17), a short calculation yields (2.25).

Remark 2.13. Assume Hypothesis 2.1 and let $z \in \mathbb{C} \setminus (\sigma(H_a) \cup \sigma(H_b))$. If we choose $f_a(\cdot, z) = \chi_a(\cdot, z)$ and $f_b(\cdot, z) = \chi_b(\cdot, z)$ in Theorem 2.12, then (2.25) becomes

$$m_b(z) - m_a(z) = W(\chi_a(\cdot, z), \chi_b(\cdot, z)), \quad z \in \mathbb{C} \setminus (\sigma(H_a) \cup \sigma(H_b)).$$

(2.27)

This formula could be found in [19, A. 36)] for $a = -\infty, b = \infty$ and $z \in \mathbb{C} \setminus \mathbb{R}$. For $\omega = \pi/2$ formulas (2.23), (2.24), (2.25) become:

$$m_a(z) = \frac{pf'(0, z)}{f_a(0, z)}, \quad m_b(z) = \frac{pf'(0, z)}{f_b(0, z)}, \quad m_b(z) - m_a(z) = \frac{W(f_a(\cdot, z), f_b(\cdot, z))}{f_a(0, z)f_b(0, z)}.$$  

(2.28)

and, in particular, $m_b(z) = pf'(0, z), m_a(z) = pf'(0, z)$. Choosing $f_a = \chi_a, f_b = \chi_b$ shows that the Wronskians in (2.23), (2.24), (2.25) are not necessarily analytic in $z \in \mathbb{C} \setminus \sigma_{ess}(H_a), z \in \mathbb{C} \setminus \sigma_{ess}(H_b), z \in \mathbb{C} \setminus \sigma_{ess}(H)$.

As shown in Lemma 2.11, the Wronskian $W(f_a(\cdot, z_0), f_b(\cdot, z_0))$ plays the role of the Evans function. Usually, the Evans function is defined by means of exponentially decaying solutions, see e.g. [3, 15, 32, 36] and the literature therein. An important point of our analysis is that in the more general than in [32] settings of the current paper the Evans function is defined using square integrable solutions. Also, combining Lemma 2.11 and the next corollary, one can use the Weyl-Titchmarsh function to detect the discrete spectrum of $H$. 


Corollary 2.14. If the assumptions in Theorem 2.12 are satisfied, then for any $z_0 \in \mathbb{C} \setminus \sigma_{ess}(H)$ the Wronskian $W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0$ if and only if the following alternative holds: Either both $m_a(z_0)$ and $m_b(z_0)$ have poles at $z_0$, or they both have finite values at $z_0$ and $m_a(z_0) = m_b(z_0)$.

Proof. Since $m_a$ is meromorphic, (2.23) implies that $z_0$ is a pole of $m_a$ if and only if $W(f_a(\cdot, z_0), \phi) = 0$. Analogous assertion holds for $W(f_b(\cdot, z_0), \phi)$. Formula (2.25) yields:

$$W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = (m_b(z_0) - m_a(z_0))W(f_a(\cdot, z_0), \phi)W(f_b(\cdot, z_0), \phi)$$

$$= m_b(z_0)W(f_b(\cdot, z_0), \phi) \cdot W(f_a(\cdot, z_0), \phi) - m_a(z_0)W(f_a(\cdot, z_0), \phi) \cdot W(f_b(\cdot, z_0), \phi).$$

Thus, if both $W(f_a(\cdot, z_0), \phi)$ and $W(f_b(\cdot, z_0), \phi)$ are not zero then (2.29) implies that $W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0$ if and only if $m_b(z_0) = m_a(z_0)$. If one of the expressions $W(f_a(\cdot, z_0), \phi)$ or $W(f_b(\cdot, z_0), \phi)$ is not zero and another is zero, then one of the expressions in (2.30), (2.31) is zero and another is not; thus, $W(f_a(\cdot, z_0), f_b(\cdot, z_0))$ is not zero. Finally, if $W(f_a(\cdot, z_0), \phi) = W(f_b(\cdot, z_0), \phi) = 0$ then both (2.30), (2.31) are zero, and thus $W(f_a(\cdot, z_0), f_b(\cdot, z_0)) = 0$. $\square$

Let us now consider the Schrödinger equation on the line:

$$-y''(t) + V(t)y(t) = k^2 y(t), \quad -\infty < t < \infty. \quad (2.32)$$

We denote the spectral parameter by $k$ so that $z = k^2 \in \mathbb{C}$, and assume throughout that $\text{Im}(k) \geq 0$. The Schrödinger equation (2.32) is equivalent to the first order system

$$Y'(t) = (A(k) + R(t))Y(t), \quad A(k) = \begin{bmatrix} 0 & 1 \\ -k^2 & 0 \end{bmatrix}, \quad R(t) = \begin{bmatrix} 0 & 0 \\ V(t) & 0 \end{bmatrix}, \quad t \in \mathbb{R}, \quad (2.33)$$

where $Y(t) = [y(t) \quad y'(t)]^T$ is the column-vector. As soon as the values $y(t_0)$ and $y'(t_0)$ at any point $t_0$ are given, one can find the corresponding solution $y$ of (2.32) on $\mathbb{R}$ such that $y$, $y'$ are locally absolutely continuous functions by solving the corresponding Cauchy problem for (2.33).

Hypothesis 2.15. Assume that $a = -\infty$, $b = +\infty$, $p(t) = r(t) = 1$, $t \in \mathbb{R}$, and the potential in (2.32) is real valued and satisfies $V \in L^1(\mathbb{R})$.

As it is well known, see, e.g. [6], under Hypothesis 2.15 all solutions of the Schrödinger equation (2.32) can be obtained as linear combinations of the solutions $f_{\pm}(t, k)$ of (2.32) satisfying the asymptotic boundary conditions

$$\lim_{t \to \pm \infty} e^{\mpikt}f_{\pm}(t, k) = 1, \quad \text{Im}(k) > 0. \quad (2.34)$$

These solutions are called the Jost solutions; they are defined as solutions of the Volterra integral equations

$$f_{\pm}(t, k) = e^{\pmikt} - \int_t^{\pm \infty} \frac{\sin(k(t-s))}{k} V(s)f_{\pm}(s, k)ds, \quad \text{Im}(k) > 0, \quad t \in \mathbb{R}. \quad (2.35)$$
The Jost function, \( J = J(k) \), is defined by
\[
J(k) = \frac{1}{2ik} W(f_-(\cdot, k), f_+(\cdot, k)), \quad \text{Im}(k) > 0. \tag{2.36}
\]
The isolated zeros of the Jost function are the discrete eigenvalues of the self-adjoint operator \( H \) associated with (2.32), see, e.g., [6, Chapter XVII].

Assuming Hypothesis 2.15 and \( \text{Im}(k) > 0 \), the first order matrix system (2.33) has a solution \( Y_+(\cdot, k) \) exponentially decaying on \( \mathbb{R}_+ \) and a solution \( Y_-(\cdot, k) \) exponentially decaying on \( \mathbb{R}_- \); each of them is defined up to a constant multiple. The Evans function \( D(k) \) is then defined as \( D(k) = \det \begin{bmatrix} Y_+(0, k) & Y_-(0, k) \end{bmatrix} \), see, e.g., [32]. As shown in [15], the Jost function, \( J(k) \), for (2.32) is equal to the appropriately chosen Evans function, \( D(k) \), for (2.33). We will now show how to calculate the Evans and the Jost functions via the Weyl-Titchmarsh functions.

**Lemma 2.16.** [9, Problem 9.4] Assume Hypothesis 2.15. Then the equation \( \tau y = k^2 y \) is in the limit point case at both \(-\infty\) and \( \infty \).

**Proof.** As known from [9, Problem 9.4], for \( k > 0 \) the first equation in (2.35) has a bounded solution on the positive semi-axis and, hence, \( f_+(t, k) - e^{ikt} \to 0 \) as \( t \to \infty \). Therefore, \( f_+ \) is not square integrable at \( +\infty \) and thus according to Theorem 2.2 the equation \( \tau y = k^2 y \) is in the limit point case at \( +\infty \). Analogously one can show that the equation \( \tau y = k^2 y \) is in the limit point case at \(-\infty \). \( \square \)

**Lemma 2.17.** Assume Hypothesis 2.15. Then the Weyl-Titchmarsh functions \( m_{-\infty}(\cdot) \) and \( m_{\infty}(\cdot) \) are meromorphic in \( \mathbb{C}\setminus[0, \infty) \).

**Proof.** This follows from Lemma 2.7 and the fact that \( \sigma_{\text{ess}}(H) = [0, \infty) \), see, e.g., [38, Theorem 15.3]. \( \square \)

**Corollary 2.18.** Assume Hypothesis 2.15 and let \( \omega = \pi/2 \) in (2.2). Then \( z_0 \in \sigma_d(H) \) if and only if either \( m_\infty(z_0) = m_{-\infty}(z_0) \) or \( m_{-\infty} \) and \( m_\infty \) both have a simple pole at \( z_0 \). Moreover, the following formula holds:
\[
D(k) = J(k) = \frac{1}{2ik} (f_-(0, k)f_+(0, k))(m_\infty(k^2) - m_{-\infty}(k^2)), \quad \text{Im}(k) > 0. \tag{2.37}
\]

**Proof.** The Jost solutions \( f_- \) and \( f_+ \) are the solutions \( f_a, a = -\infty \), and \( f_b, b = +\infty \), described in Proposition 2.9. Thus, the required assertions follow from Lemma 2.11, Theorem 2.12, Corollary 2.14, (2.36), and equality \( D(k) = J(k) \) proved in [15, Theorems 5.4(i), 9.4]. \( \square \)

### 3. The reduced Jost function

In this section we derive an asymptotic formula relating the Jost function (2.36) for the Schrödinger equation (2.32) on the line, and the reduced Jost function for the Schrödinger equation on a large but finite segment. We assume throughout that Hypothesis 2.15 holds, and choose \( k \) with \( \text{Im}(k) > 0 \) so that \( e^{ikt} \to 0 \) as \( t \to \infty \). We recall that the Jost solutions \( f_{\pm}(\cdot, k) \) from (2.34), (2.35) are the solutions of the Schrödinger equation (2.32) that are asymptotic to the plane waves \( e^{\pm ikt} \) which are solutions of the Schrödinger equation with zero potential. The asymptotic properties of the Jost solutions are summarized in the following elementary lemma (for a related material see, e.g., [6, Sec. XVII.1.2]).
Lemma 3.1. Assume Hypothesis 2.15. Then, with \( \Im(k) > 0 \), the following asymptotic relations hold:

\[
\begin{align*}
\lim_{t \to \pm \infty} f_{\pm}(t,k)e^{\pm ikt} & = 1, & \quad \lim_{t \to \pm \infty} \frac{1}{\pm ik} f'_{\pm}(t,k)e^{\mp ikt} & = 1, \quad \text{(3.1)} \\
\lim_{t \to \pm \infty} f_{\pm}(t,k)e^{\pm ikt} & = \mathcal{J}(k), & \quad \lim_{t \to \pm \infty} \frac{1}{\mp ik} f'_{\pm}(t,k)e^{\mp ikt} & = \mathcal{J}(k). \quad \text{(3.2)}
\end{align*}
\]

Proof. Formulas (3.1) follow from (2.35). To show (3.2), we begin by recalling the following well-known property of the asymptotically constant coefficient ODE system (2.33) (see, e.g., [9, Problem III.29] or [11, Chapter 1]): There exist solutions \( Y_{\pm}(\cdot, k) \) and \( \tilde{Y}_{\pm}(\cdot, k) \) of (2.33) such that

\[
\lim_{t \to \pm \infty} Y_{\pm}(t,k)e^{\pm ikt} = [1 \pm ik]^T, \quad \lim_{t \to \pm \infty} \tilde{Y}_{\pm}(t,k)e^{\pm ikt} = [1 \mp ik]^T, \quad \text{(3.3)}
\]

where \( \pm ik \) are the eigenvalues and \([1 \pm ik]^T\) are the corresponding eigenvectors of the matrix \( A(k) \) defined in (2.33). By (3.1), we have \( Y_{\pm}(\cdot, k) = [f_{\pm}(\cdot, k) \ f'_\pm(\cdot, k)]^T \) with the Jost solutions \( f_{\pm}(\cdot, k) \) of (3.22). Denoting by \( f_{\pm}(\cdot, k) \) the solutions of (3.22) such that \( \tilde{Y}_{\pm}(\cdot, k) = [\tilde{f}_{\pm}(\cdot, k) \ \tilde{f}'_{\pm}(\cdot, k)]^T \), we derive from the second equation in (3.3) the following asymptotic formulas:

\[
\lim_{t \to \pm \infty} \tilde{f}_{\pm}(t,k)e^{\pm ikt} = 1, \quad \lim_{t \to \pm \infty} \frac{1}{\mp ik} \tilde{f}'_{\pm}(t,k)e^{\mp ikt} = 1. \quad \text{(3.4)}
\]

By letting \( t \to \pm \infty \), we note that (3.1), (3.4) also imply

\[
W(f_{\pm}(\cdot,k),\tilde{f}_{\pm}(\cdot,k)) = \mp 2ik, \quad \Im(k) > 0. \quad \text{(3.5)}
\]

Therefore, \( f_{\pm}(\cdot,k) \) and \( \tilde{f}_{\pm}(\cdot,k) \) are linearly independent solutions of (2.3), and thus there are some constants \( c_1, c_2 \) such that

\[
f_{\pm}(t,k) = c_1 f_{\pm}(t,k) + c_2 \tilde{f}_{\pm}(t,k) \quad \text{for all } t \in \mathbb{R}. \quad \text{(3.6)}
\]

Computing the Wronskian \( W(f_{\pm}(\cdot,k),f_{\pm}(\cdot,k)) \) in (3.6) and using (3.5), we see that \( c_2 = \mathcal{J}(k) \). Multiplying (3.6) by \( e^{ikt} \), letting \( t \to +\infty \), and using (3.1), (3.4) yields the first formula in (3.2), that is, \( \lim_{t \to +\infty} f_{\pm}(t,k)e^{\pm ikt} = \mathcal{J}(k) \). Differentiating (3.6), in the same manner one obtains the formula in (3.2) for the derivative of \( f_{\pm}(\cdot,k) \). Expressing \( f_{\pm}(\cdot,k) \) as a linear combination

\[
f_{\pm}(t,k) = c_1 f_{\pm}(t,k) + c_2 \tilde{f}_{\pm}(t,k) \quad \text{for all } t \in \mathbb{R}, \quad \text{(3.7)}
\]

computing the Wronskian \( W(f_{\pm}(\cdot,k),f_{\pm}(\cdot,k)) \) in (3.7) and using (3.5), we see that \( c_2 = \mathcal{J}(k) \). Multiplying (3.7) by \( e^{-ikt} \), letting \( t \to -\infty \), and using (3.1), (3.4) yields the formula \( \lim_{t \to -\infty} f_{\pm}(t,k)e^{-ikt} = \mathcal{J}(k) \) in (3.2). Differentiating (3.7), in the same manner one obtains the formula in (3.2) for the derivative of \( f_{\pm}(\cdot,k) \). \( \square \)

Let us introduce the operator \( Hf = -f'' + V(t)f \) on \( L^2(\mathbb{R}) \) with the domain

\[
\text{dom } H = \{ f \in L^2(\mathbb{R}) | f, f' \in AC_{\text{loc}}(\mathbb{R}), -f'' + Vf \in L^2(\mathbb{R}) \}, \quad \text{(3.8)}
\]

and remark that \( k^2 \in \sigma_d(H) \) if and only if \( k \in i\mathbb{R} \setminus \{0\} \) and \( \mathcal{J}(k) = 0 \), see, e.g., [6, Sec. XVII.13]. Indeed, if \( \mathcal{J}(k) = 0 \) then \( f_{\pm}(\cdot,k) \) is proportional to \( f_{\pm}(\cdot,k) \) and thus is an eigenfunction for \( H \) as it is exponentially decaying at both \(+\infty\) and \(-\infty\). Conversely, expressing an eigenfunction \( f \in L^2(\mathbb{R}) \) of \( H \) as a linear combination of \( f_{\pm}(\cdot,k) \) and \( f_{\pm}(\cdot,k) \), we see that the Jost solutions must be proportional yielding \( \mathcal{J}(k) = 0 \).
We will now construct a reduced Jost function, $J_L(k)$, that corresponds to the differential operator on the segment $[-L, L]$ with (large) positive $L$ whose discrete spectrum approximates the spectrum of $H$. Fix an $L > 0$ and two angles, $\omega_-$ and $\omega_+$, in $[0, \pi)$. We will consider the following boundary conditions at the points $-L$ and $+L$,

$$f(-L) \cos \omega_- + f'(-L) \sin \omega_- = 0, \quad (3.9)$$
$$f(+L) \cos \omega_+ + f'(+L) \sin \omega_+ = 0, \quad (3.10)$$

and define the operator $H_L f = -f'' + V(t)f$ on $L^2(-L, L)$ with the domain

$$\text{dom } H_L = \left\{ f \in L^2(-L, L) \bigg| f, f' \in AC_{\text{loc}}[-L, L], -f'' + Vf \in L^2(-L, L), \right. $$
$$\left. \text{ and both conditions (3.9), (3.10) hold } \right\}. $$

Solving the corresponding Cauchy problems for (2.33), we let $g_+ (\cdot, k)$ and $g_- (\cdot, k)$ denote the locally absolutely continuous together with their derivatives solutions of the Schrödinger equation (2.32) on $\mathbb{R}$ satisfying the following conditions at $\pm L$,

$$g_\pm (\pm L, k) = e^{ikL} \sin \omega_\pm, \quad g'_\pm (\pm L, k) = -e^{ikL} \cos \omega_\pm, \quad (3.12)$$

and define the reduced Jost function, $J_L = J_L(k)$, as follows:

$$J_L(k) = \frac{1}{2ik} W(g_-, g_+), \quad \text{Im}(k) > 0. \quad (3.13)$$

Clearly, $g_+(\cdot, k)$ satisfies (3.10) while $g_-(\cdot, k)$ satisfies (3.9). We claim that $k^2 \in \sigma_d(H_L)$ if and only if $J_L(k) = 0$. Indeed, if $J_L(k) = 0$ then $g_+(\cdot, k)$ is proportional to $g_-(\cdot, k)$ and thus is an eigenfunction for $H_L$ as it satisfies both conditions (3.9) and (3.10). Conversely, if $g$ is an eigenfunction for $H_L$ then it satisfies both conditions (3.9) and (3.10). Evaluating the Wronskian at $t = +L$, we notice that $W(g, g_+(\cdot, k)) = 0$ and hence $g$ is proportional to $g_+(\cdot, k)$. Evaluating the Wronskian at $t = -L$, we notice that $W(g, g_-(\cdot, k)) = 0$, and hence $g$ is proportional to $g_-(\cdot, k)$, proving the claim.

In addition, we will consider two more operators with zero potential, $H_{L,0}^+$ and $H_{L,0}^-$, such that $H_{L,0}^+ f = -f''$. The operator $H_{L,0}^+$ is defined in $L^2(-\infty, +L)$ with the domain

$$\text{dom } H_{L,0}^+ = \left\{ f \in L^2(-\infty, L) \bigg| f, f' \in AC_{\text{loc}}(-\infty, L), -f'' \in L^2(-\infty, L), \right. $$
$$\left. \text{ and condition (3.10) holds } \right\}. $$

The operator $H_{L,0}^-$ is defined in $L^2(-L, +\infty)$ with the domain

$$\text{dom } H_{L,0}^- = \left\{ f \in L^2(-L, +\infty) \bigg| f, f' \in AC_{\text{loc}}[-L, +\infty), -f'' \in L^2(-L, +\infty), \right. $$
$$\left. \text{ and condition (3.9) holds } \right\}. $$

Furthermore, we let $h_+ (\cdot, k)$ and $h_-(\cdot, k)$ denote the solutions of the differential equation $-h'' = k^2 h$ on $\mathbb{R}$ which are locally absolutely continuous together with their derivatives and satisfy the following conditions at $\pm L$,

$$h_\pm (\pm L, k) = e^{ikL} \sin \omega_\pm, \quad h'_\pm (\pm L, k) = -e^{ikL} \cos \omega_\pm, \quad (3.14)$$
and define the functions, $\mathcal{J}^{\pm}_{L,0} = \mathcal{J}^{\pm}_{L,0}(k)$, as follows:

$$\mathcal{J}^{+}_{L,0}(k) = \frac{1}{2ik} W(e^{-ikt}, h_+(\cdot, k)), \quad \mathcal{J}^{-}_{L,0}(k) = \frac{1}{2ik} W(h_-(\cdot, k), e^{ikt}), \quad \text{Im}(k) > 0.$$  \hfill (3.15)

Computing the Wronskian for $\mathcal{J}^{\pm}_{L,0}(k)$ at $\pm L$ yields

$$2ik \mathcal{J}^{+}_{L,0}(k) = -\det\begin{bmatrix} \sin \omega_+ & -ik \\ -\cos \omega_+ & 1 \end{bmatrix} = -\cos \omega_+ + ik \sin \omega_+, \quad (3.16)$$

$$2ik \mathcal{J}^{-}_{L,0}(k) = \det\begin{bmatrix} \sin \omega_- & 1 \\ -\cos \omega_- & ik \end{bmatrix} = \cos \omega_+ + ik \sin \omega_-. \quad (3.17)$$

in particular,

$$\mathcal{J}^{\pm}_{L,0}(k) \text{ does not depend on } L.$$

Clearly, $h_+(\cdot, k)$ satisfies (3.10) and $h_-(\cdot, k)$ satisfies (3.9) while $e^{ikt} \in L^2(-L, +\infty)$ and $e^{-ikt} \in L^2(\infty, +L)$ since $\text{Im}(k) > 0$.

**Remark 3.2.** We claim that $k^2 \in \sigma_d(H^{\pm}_{L,0})$ if and only if $\mathcal{J}^{\pm}_{L,0}(k) = 0$. Indeed, if, say, $\mathcal{J}^{+}_{L,0}(k) = 0$ then $h_+(\cdot, k)$ is proportional to $e^{-ikt}$ and thus is an eigenfunction for $H^{+}_{L,0}$ as it satisfies conditions (3.10) and $h_+(\cdot, k) \in L^2(-\infty, +L)$. Conversely, if $h$ is an eigenfunction for $H^{+}_{L,0}$ then it satisfies conditions (3.10) and $h \in L^2(\infty, +L)$. Evaluating the Wronskian at $t = +L$, we notice that $W(h, h_+(\cdot, k)) = 0$ and hence $h$ is proportional to $h_+(\cdot, k)$. Expressing $h \in L^2(-\infty, +L)$ as a linear combination of the plane waves $e^{-ikt}$ and $e^{ikt}$, we conclude that $h$ is proportional to $e^{-ikt}$, thus proving the claim for $H^{+}_{L,0}$. The argument for $H^{-}_{L,0}$ is analogous. \hfill \Box

We are ready to present the main result of this section (recall (3.18)).

**Theorem 3.3.** Assume that $V \in L^1(\mathbb{R}; \mathbb{R})$. Then, for the Jost functions $\mathcal{J}(k)$, $\mathcal{J}_{L}(k)$, $\mathcal{J}^{\pm}_{L,0}(k)$ defined in (2.36), (3.13), (3.15), we have:

$$\lim_{L \to \infty} \mathcal{J}_{L}(k) = \mathcal{J}(k) \mathcal{J}^{+}_{L,0}(k) \mathcal{J}^{-}_{L,0}(k), \quad \text{Im}(k) > 0.$$  \hfill (3.19)

**Proof.** Varying $k$ a little, if needed, with no loss of generality we may and will assume throughout the proof that $k^2$ is not an isolated eigenvalue of $H$, that is, that $\mathcal{J}(k) \neq 0$. Then the Jost solutions $f_+(\cdot, k)$ and $f_-(\cdot, k)$ are linearly independent and hence there exist constants $c_1^-, c_2^-$ and $c_1^+, c_2^+$ such that for all $t \in \mathbb{R}$ one has the following representations:

$$g_-(t, k) = c_1^- f_+(t, k) + c_2^- f_-(t, k),$$

$$g_+(t, k) = c_1^+ f_+(t, k) + c_2^+ f_-(t, k).$$  \hfill (3.20)

Computing the Wronskians, we find:

$$c_1^- = \frac{W(f_-(\cdot, k), g_-(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))}, \quad c_2^- = \frac{W(g_-(\cdot, k), f_+(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))},$$

$$c_1^+ = \frac{W(f_-(\cdot, k), g_+(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))}, \quad c_2^+ = \frac{W(g_+(\cdot, k), f_+(\cdot, k))}{W(f_-(\cdot, k), f_+(\cdot, k))}. \hfill (3.21)$$
A computation using (3.20), (3.21) reveals:

\[
W(g_1(-\cdot, k), g_2(\cdot, k)) = W(c_1 f_1(\cdot, k) + c_2 f_2(\cdot, k), c_1^+ f_1(\cdot, k) + c_2^+ f_2(\cdot, k)) = (c_1^+ c_2 - c_1 c_2^+) W(f_1(\cdot, k), f_2(\cdot, k)) = W_1 + W_2,
\]

where we temporarily introduced the following notations:

\[
W_1 = -W(f_-(\cdot, k), g_0(-\cdot, k)) W(g_1(\cdot, k), f_+(\cdot, k))/W(f_-(\cdot, k), f_+(\cdot, k)), \quad (3.23)
\]

\[
W_2 = W(g_0(\cdot, k), f_+(\cdot, k)) W(f_-(\cdot, k), g_0(\cdot, k))/W(f_-(\cdot, k), f_+(\cdot, k)). \quad (3.24)
\]

Assertion (3.19) now follows from (3.12) and Lemma 3.1. Indeed, the expression

\[
W(f_-(\cdot, k), g_0(-\cdot, k)) = f_-(L, k) g_0(-L, k) - f'_-(L, k) g_0(-L, k)
\]

\[
= -f_-(L, k) e^{ikL} \cos \omega_+ - f'_-(L, k) e^{ikL} \sin \omega_+
\]

\[
= -\left(f_-(L, k) e^{ik(L - k)}\right) e^{2ikL} \cos \omega_+
\]

\[
- (ik) \left(\frac{1}{ik} e^{-ik(L-k)} f'_+(L, k)\right) \sin \omega_+
\]

tends to zero as \(L \to \infty\) due to (3.1) and \(\text{Im}(k) > 0\). This and a similar argument for \(W(g_1(\cdot, k), f_+(\cdot, k))\) yields \(W_1 \to 0\) as \(L \to \infty\), and hence it suffices to handle the term \(W_2\) in (3.22). If \(L \to \infty\) then, using (3.2), the expression

\[
W(g_0(\cdot, k), f_+(\cdot, k)) = g_0(-L, k) f_+(L, k) - g'_0(-L, k) f_+(L, k)
\]

\[
= e^{ikL} \sin \omega_+ f'_+(L, k) + e^{ikL} \cos \omega_+ f_+(L, k)
\]

\[
= \left(\frac{1}{ik} e^{ik(L-k)} f'_+(L, k)\right) (ik) \sin \omega_+ + \left(e^{-ik(L-k)} f_+(L, k)\right) \cos \omega_+
\]

tends to

\[
\mathcal{J}(k)(ik) \sin \omega_+ + \mathcal{J}(k) \cos \omega_+ = 2ik \mathcal{J}(k) \mathcal{J}^0_L(k),
\]

where in the last equality we used (3.17). Also, the expression

\[
W(f_-(\cdot, k), g_0(\cdot, k)) = f_-(L, k) g'_0(L, k) - f'_-(L, k) g_0(L, k)
\]

\[
= -f_-(L, k) e^{ikL} \cos \omega_+ - f'_-(L, k) e^{ikL} \sin \omega_+
\]

\[
= -\left(f_-(L, k) e^{ikL}\right) \cos \omega_+ + \left(\frac{1}{-ik} f'_-(L, k) e^{ikL}\right) (ik) \sin \omega_+
\]

tends to

\[-\mathcal{J}(k) \cos \omega_+ + \mathcal{J}(k)(ik) \sin \omega_+ = 2ik \mathcal{J}(k) \mathcal{J}^+_L(k),\]

where in the last equality we used (3.16). Combining this with (3.22) yields

\[
\lim_{L \to \infty} \mathcal{J}_L(k) = \lim_{L \to \infty} \frac{1}{2ik} W_2 = \lim_{L \to \infty} \frac{1}{2ik} W_1 = \lim_{L \to \infty} \frac{1}{2ik} W_2
\]

\[
= \frac{1}{2ik} \mathcal{J}_L(k) \mathcal{J}^+_L(k),
\]

as required. \(\square\)

We conclude this section with a short discussion of the relation between Theorem 3.3 and certain results in [33] and [4]. In particular, as a part of a general theory relating the spectra of first order differential operators on the line and on finite segments, [33, Theorem 3] relates the multiplicity of the eigenvalues of a general first order differential operator \(\mathcal{T}_L\) in the space \(L^2((-L, L) \cap \mathbb{C}^d)\) of \(d\)-dimensional vector
valued functions to that of the operator $T$ in $L^2(\mathbb{R}; \mathbb{C}^d)$, and to the multiplicity of zeros of certain functions, $D_{\pm}(z)$, of the spectral parameter $z$, determined by the boundary conditions at $\pm L$ used to define $T_{L}$. The discussion in [33] involves the Evans functions $D_{\infty}(z)$, defined for $T$, and $D_{L}(z)$, defined for $T_{L}$, such that the zeros of the Evans functions are the eigenvalues of the respective operators. One of the major conclusions in [33, Theorem 3], see also the preceding analysis in [4] and a more recent paper [31], is the following eigenvalue multiplicity result: Under certain natural assumptions, for $L$ sufficiently large, the algebraic multiplicity of an isolated eigenvalue of $T_{L}$ (that is, the order of a zero of $D_{L}$) is equal to the sum of the orders of zeros of the functions $D_{\infty}$, $D_{-}$, and $D_{+}$ in a small vicinity of the eigenvalue. We will now furnish the definitions of the functions $D_{\infty}(z)$, $D_{L}(z)$, $D_{\pm}(z)$ of the spectral parameter $z = k^2$ for the particular case of the first order system (2.33) corresponding to the Schrödinger equation (2.32) and relate them to the Jost functions $J(k)$, $J_{L}(k)$, $J_{L,0}^{\pm}(k)$ studied earlier in this section.

First, we recall that the Evans function $D_{\infty}(z)$ is the determinant of the $(2 \times 2)$ matrix whose first column is the initial data at $t = 0$ of the solution of (2.33) that exponentially decays to zero as $t \to +\infty$ and whose second column is the initial data at $t = 0$ of the solution of (2.33) that exponentially decays to zero as $t \to -\infty$. It is known that if these solutions are appropriately chosen then $D_{\infty}(z) = J(k)$, $z = k^2$, see [15, Section 9] and Corollary 2.18.

Next, we define $D_{L}(z)$ as follows. Let us fix two arbitrary unit vectors, denoted by $[\sin \omega_{-} - \cos \omega_{-}]^\top$ and $[\sin \omega_{+} - \cos \omega_{+}]^\top$, and let $Q_{\pm}$ denote the subspace spanned by the vector $[\sin \omega_{\pm} - \cos \omega_{\pm}]^\top$. The subspaces $Q_{-}$ and $Q_{+}$ determine the boundary conditions at $-L$ and $+L$ respectively in the sense that a solution $f$ of the Schrödinger equation (2.32) satisfies the boundary conditions (3.9), (3.10) if and only if the corresponding solution $Y(t) = [f(t) \ f'(t)]^\top$ of the first order system (2.33) satisfies the boundary conditions $Y(\pm L) \in Q_{\pm}$. Following [33], given two subspaces $F$ and $E$ of $\mathbb{C}^d$ with $\dim E + \dim F = d$, we denote by $E \wedge F$ the determinant of the $(d \times d)$ matrix whose columns are the bases vectors of $E$ and $F$ put consequently. We let $\varphi(t; s; z)$ denote the propagator of the system (2.33). With this notations, we define $D_{L}(z) = \varphi(0, -L; z)Q_{-} \wedge \varphi(0, L; z)Q_{+}$ (see, e.g., [33, (4.5)] where notation $D_{\text{rep}}$ was used instead of $D_{L}$). Of course, $D_{L}(z)$ depends, up to a constant factor, on the choice of the bases vectors in $Q_{\pm}$. In particular, if $g_{\pm}(\cdot; k)$ is the solutions of (2.32) that satisfy (3.12) then $[g_{\pm}(\pm L, k) \ g_{\pm}'(\pm L, k)]^\top \in Q_{\pm}$ and thus $D_{L}(z) = J_{L}(k)$ by (3.13).

Finally, we define $D_{\pm}(z)$. We recall that $\pm ik$ are the eigenvalues of the matrix $A(k)$ from (2.33) with the corresponding eigenvectors $[1 \pm ik]^\top$. Let us denote by $E_{\pm}^{\pm}(z)$ the linear subspace spanned by the vector $[1 + ik]^\top$, and by $E_{\pm}^{\mp}(z)$ the linear subspace spanned by the vector $[1 - ik]^\top$. Then $E_{\pm}^{\pm}(z)$ is the set of the initial data of the solutions of the constant coefficient differential equation $Y' = A(k)Y$ that exponentially grow to infinity as $t \to -\infty$ and $E_{\pm}^{\mp}(z)$ is the set of the initial data of the solutions of the constant coefficient differential equation $Y' = A(k)Y$ that exponentially grow to infinity as $t \to +\infty$. Using subspaces $Q_{\pm}$ introduced in the previous paragraph, following [33, (4.5)], we define $D_{-}(z) = Q_{-} \wedge E_{\pm}^{\pm}(z)$ and $D_{+}(z) = Q_{+} \wedge E_{\pm}^{\mp}(z)$. By (3.16), (3.17) we conclude that, up to a constant factor, $D_{\pm}(z)$ is equal to $J_{L,0}^{\pm}(k)$. Remark 3.2 provides an operator theoretical
interpretation of the quantities $D_{\pm}(z)$. Theorem 3.3 yields
\[
\lim_{L \to \infty} D_{L}(z) = D_{\infty}(z)D_{-}(z)D_{+}(z), \quad z \in \mathbb{C} \setminus [0, \infty),
\]  
which is consistent with the eigenvalue multiplicity results from [4, 33] mentioned above.

4. The Weyl-Titschmarsh $M$-function and Hamiltonian systems

Following [8, 23, 27, 28], we will briefly review the basics of the Weyl-Titschmarsh theory for the matrix valued Hamiltonian systems. We consider two singular endpoints problem for the $(2n \times 2n)$ Hamiltonian system
\[
JY'(t) = (zA(t) + B(t))Y(t), \quad t \in (a, b), \quad -\infty \leq a < b \leq \infty,
\]  
where $Y(t)$ is a $(2n \times 1)$-vector, $z$ is a complex parameter, and $J = \begin{bmatrix} 0 & -I_{n \times n} \\ I_{n \times n} & 0 \end{bmatrix}$. The coefficients $A, B$ satisfy the following assumptions.

**Hypothesis 4.1.** (i) $A(t)$ and $B(t)$ are $(2n \times 2n)$ Hermitian matrices for $t \in (a, b)$ whose entries are complex valued measurable and locally integrable functions on $(a, b)$;
(ii) $A(t) \geq 0$ a.e. in the sense of quadratic forms and satisfies Atkinson’s definiteness condition, i.e., if $Y$ is a nontrivial solution of (4.1) then
\[
\int_{c}^{d} Y(t)^{*}A(t)Y(t)dt > 0 \text{ for any } c, d \text{ such that } a < c < d < b.
\]

A solution of (4.1) is said to be $A$-square integrable if $\int_{a}^{b} Y^{*}(t)A(t)Y(t)dt < \infty$, and we denote this by $Y \in L_{A}^{2}(a, b)$. As already mentioned in (4.1), we allow the endpoints $a$ and $b$ to be finite or infinite, and, with no loss of generality, we assume that $0 \in (a, b)$. We will now discuss boundary conditions at $c, 0, d$ for any $c, d$ satisfying $a < c < 0 < d < b$.

Let us fix three $(n \times 2n)$ matrices, $[\alpha_{1} \alpha_{2}], [\gamma_{1} \gamma_{2}]$ and $[\beta_{1} \beta_{2}]$, where $\alpha_{1}, \alpha_{2}, \gamma_{1}, \gamma_{2}, \beta_{1}, \beta_{2}$ are $(n \times n)$ matrices satisfying the conditions
\[
\text{rank} [\alpha_{1} \alpha_{2}] = \text{rank} [\gamma_{1} \gamma_{2}] = \text{rank} [\beta_{1} \beta_{2}] = n,
\]
\[
\alpha_{1}\alpha_{2}^{*} - \alpha_{2}\alpha_{1}^{*} = 0_{n \times n}, \quad \gamma_{1}\gamma_{2}^{*} - \gamma_{2}\gamma_{1}^{*} = 0_{n \times n}, \quad \beta_{1}\beta_{2}^{*} - \beta_{2}\beta_{1}^{*} = 0_{n \times n}.
\]
If (4.2), (4.3) hold then, cf. [27, page 108], with no loss of generality we may and will assume that
\[
\alpha_{1}\alpha_{1}^{*} + \alpha_{2}\alpha_{2}^{*} = I, \quad \gamma_{1}\gamma_{1}^{*} + \gamma_{2}\gamma_{2}^{*} = I, \quad \beta_{1}\beta_{1}^{*} + \beta_{2}\beta_{2}^{*} = I.
\]

Let $Y(t, z)$ be the fundamental matrix solution of the differential equation (4.1) normalized such that
\[
Y(0, z) = \begin{bmatrix} \gamma_{1}^{*} & -\gamma_{2}^{*} \\ \gamma_{2}^{*} & \gamma_{1}^{*} \end{bmatrix}.
\]

Then $Y(t, z)$ satisfies the following conditions at $t = 0$:
\[
\begin{bmatrix} \gamma_{1} & \gamma_{2} \end{bmatrix} Y(0, z) = \begin{bmatrix} I_{n \times n} & 0_{n \times n} \end{bmatrix},
\]
\[
Y^{*}(0, z)Y(0, z) = I_{2n \times 2n} = Y(0, z)Y^{*}(0, z).
\]

We decompose $Y(\cdot, z)$ into $2n \times n$ blocks $\theta, \phi$, where $\theta$ and $\phi$ are further decomposed into $n \times n$ blocks as follows:
\[
Y(t, z) = \begin{bmatrix} \theta(t, z) & \phi(t, z) \\ \theta_{2}(t, z) & \phi_{2}(t, z) \end{bmatrix}.
\]
We note that $\theta, \phi$ are entire functions in $z$ and are continuous in $t, z$. Also, we let $\phi_1(t, z), \ldots, \phi_n(t, z)$ denote the columns of the $(2n \times n)$ matrix $\phi(t, z)$ and, for future use, derive from (4.5) the following identities:

\[
\phi(0, z) = J\theta(0, z), \ \theta(0, z) = -J\phi(0, z), \\
\phi^*(0, z) = -\theta^*(0, z)J, \ \theta^*(0, z) = \phi^*(0, z)J. \tag{4.8}
\]

We impose a regular, self-adjoint boundary condition at $c$ and $d$ for solutions of (4.1),

\[
[\alpha_1 \ \alpha_2] \ Y(c) = 0_{n \times 1}, \ [\beta_1 \ \beta_2] \ Y(d) = 0_{n \times 1}, \tag{4.9}
\]

For $z \in \mathbb{C} \setminus \mathbb{R}$, one attempts to satisfy the boundary condition (4.9) at $d$ for the solution $\chi_d(t, z) = \theta(t, z) + \phi(t, z)M_d(z)$ with some $(n \times n)$ matrix $M_d(z)$. Inserting $\chi_d$ into the equation $[\beta_1 \ \beta_2] \ Y(d) = 0$ shows that

\[
M_d(z) = -([\beta_1 \ \beta_2] \phi(d, z))^{-1} ([\beta_1 \ \beta_2] \theta(d, z)). \tag{4.10}
\]

A direct calculation shows that $M_d(z)$ satisfies the following circle equations:

\[
\pm\chi_d(d, z)^*(J/i)\chi_d(d, z) = 0, \text{ where } \chi_d(d, z) = \theta(d, z) + \phi(d, z)M_d(z). \tag{4.11}
\]

It can be shown, see [27, Section VII.3], that, as $d$ approaches $b$, $M_d(z)$ approaches one of the matrices of the form

\[
M_b(z) = C_b(z) + R_b(z)U_b(z)\overline{R}_b(z), \tag{4.12}
\]

where we define

\[
C_b(z) = -\lim_{d \to b} \left(2\Im(z) \int_0^d \phi^*A\phi dt\right)^{-1} \left(2\Im(z) \int_0^d \phi^*A\theta dt - iI_{n \times n}\right), \tag{4.13}
\]

\[
R_b(z) = \lim_{d \to b} \left(2\Im(z) \int_0^d \phi^*A\phi dt\right)^{-1/2}, \ z \in \mathbb{C} \setminus \mathbb{R},
\]

so that $\overline{R}_b(z) = R_b(\overline{z})$, and where $U_b(z)$ is a unitary matrix. It can be further shown, see [27, Section VII.4], that if $\chi_b(t, z) = \theta(t, z) + \phi(t, z)M_b(z)$, then

\[
\int_0^b \chi_b(t, z)A(t)\chi_b(t, z)dt \leq (M_b(z) - M^*_b(z))/(2i\Im(z)), \tag{4.14}
\]

and thus by Atkinson’s condition in Hypothesis 4.1 (ii) one has

\[
\Im(M_b(z))/\Im(z) = (M_b(z) - M^*_b(z))/(2i\Im(z)) > 0, \ z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.15}
\]

Similarly, one attempts to satisfy boundary conditions (4.9) at $c$ for the solution $\chi_c(t, z) = \theta(t, z) + \phi(t, z)M_c(z)$ with some $n \times n$ matrix $M_c(z)$. Inserting $\chi_c$ into the equation $[\alpha_1 \ \alpha_2] \ Y(c) = 0$ shows that

\[
M_c(z) = -([\alpha_1 \ \alpha_2] \phi(c, z))^{-1} ([\alpha_1 \ \alpha_2] \theta(c, z)). \tag{4.16}
\]

The circle equations, satisfied by $M_c(z)$, are as follows:

\[
\mp\chi_c(c, z)^*(J/i)\chi_c(c, z) = 0, \text{ where } \chi_c(c, z) = \theta(c, z) + \phi(c, z)M_c(z). \tag{4.17}
\]

As $c$ approaches $a$, $M_c(z)$ approaches

\[
M_a(z) = C_a(z) + R_a(z)U_a(z)\overline{R}_a(z), \tag{4.18}
\]
where

\[
 C_a(z) = -\lim_{\epsilon \to a} \left( 2 \Im(z) \int_{\epsilon}^0 \phi^* A \phi dt \right)^{-1} \left( 2 \Im(z) \int_{\epsilon}^0 \phi^* A \phi dt + iI \right),
\]

\[
 R_a(z) = \lim_{\epsilon \to a} \left( 2 \Im(z) \right)^{1/2} \left( 2 \Im(z) \int_{\epsilon}^0 \phi^* A \phi dt \right)^{-1/2}, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]

so that \( \mathcal{T}_a(z) = R_a(\tau) \), and where \( U_a(z) \) is a unitary matrix. If \( \chi_a(t, z) = \theta(t, z) + \phi(t, z) M_a(z) \), then

\[
 \int_a^b \chi_a^*(t) A(t) \chi_a(t) dt \leq (M_a^*(z) - M_a(z))/(2i \Im(z)),
\]

and by Atkinson’s condition in Hypothesis 4.1 (ii)

\[
 \Im(M_a(z))/\Im(z) = (M_a^*(z) - M_a(z))/(2i \Im(z)) < 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

The matrix valued functions \( M_a(z) \) and \( M_b(z) \) are called the Weyl-Titchmarsh \( M \)-functions.

We introduce the following spaces:

\[
 N(b, z) = \left\{ Y \in L^2_A(0, b) \mid JY'(t) = (zA(t) + B(t))Y(t) \text{ a.e. on } (0, b) \right\},
\]

\[
 N(a, z) = \left\{ Y \in L^2_A(a, 0) \mid JY'(t) = (zA(t) + B(t))Y(t) \text{ a.e. on } (a, 0) \right\}.
\]

The Hamiltonian system (4.1) is said to be (see [27, Page 88], [23, Page 274], [28]) in the limit point case at \( b \) (respectively, at \( a \)) whenever

\[
 \dim_C(N(b, z)) = n \quad \text{(respectively, } \dim_C(N(a, z)) = n) \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R},
\]

and in the limit circle case at \( b \) (respectively, at \( a \)) whenever

\[
 \dim_C(N(b, z)) = 2n \quad \text{(respectively, } \dim_C(N(a, z)) = 2n) \quad \text{for all } z \in \mathbb{C}.
\]

Let us define the maximal operator \( T_{\max} \) in \( L^2_A(a, b) \) associated with (4.1):

\[
 T_{\max} f = F,
\]

\[
 f \in \text{dom } T_{\max} = \left\{ g \in L^2_A(a, b) \mid g \in \text{AC}_{\text{loc}}((a, b); \mathbb{C}^{2n}) \text{ and there exists } F \in L^2_A(a, b) \text{ such that } Jg'(t) - B(t)g(t) = A(t)F(t) \text{ for a.e. } t \in (a, b) \right\}.
\]

The minimal operator \( T_{\min} \) in \( L^2_A(a, b) \) associated with (4.1) is defined as the closure of the operator \( T_{\min}^* \) given as follows:

\[
 T_{\min}^* f = F,
\]

\[
 f \in \text{dom } T_{\min} = \left\{ g \in \text{dom } T_{\max} \mid g \text{ has compact support in } (a, b) \right\}.
\]

We recall from [34, Theorem 4.1] that the deficiency indices of \( T_{\min} \) are given by

\[
 \dim \ker(T_{\max} - zI_{L^2_A}) = \dim_C(N(a, z)) + \dim_C(N(b, z)) - 2n, \quad z \in \mathbb{C} \setminus \mathbb{R}.
\]

Also, we recall from [27, Section VII.8] Green’s formula associated with (4.1): If \( F_1, F_2 \in \text{dom } T_{\max} \) then

\[
 \int_a^b \left( F_2^*(JJF_1^* - BF_1) - (JF_2^* - BF_2)F_1 \right) dt \]

\[
 = F_2^* JF_1 \bigg|_a^b := F_2^* (b) JF_1(b) - F_2^* (a) JF_1(a).
\]
The following assumption holds provided that the entries of $A$, $B$ are real valued. It is also holds for all examples discussed at the end of this section.

**Hypothesis 4.2.** In addition to Hypothesis 4.1 we assume that

$$\dim_{\mathbb{C}}(N(a,z)) = \dim_{\mathbb{C}}(N(a,\overline{z})),\quad \dim_{\mathbb{C}}(N(b,z)) = \dim_{\mathbb{C}}(N(b,\overline{z})), \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.29}$$

Next, we fix some $z \in \mathbb{C} \setminus \mathbb{R}$, and then fix any $M_a(z)$ and $M_b(z)$ in (4.18), (4.12). We introduce the operator $T$ on $L^2_{\mathbb{A}}(a,b)$ associated with (4.1) as follows:

$$Tf = F, \quad f \in \text{dom } T = \{g \in \text{dom } T_{\text{max}} \mid B_a(g) := \lim_{c \to a} \chi_a(c, \overline{z})^* Jg(c) = 0, \quad \gamma_1 \gamma_2 \ g(0) = 0\}. \tag{4.30}$$

**Theorem 4.3.** [27, Theorem VIII.3.4] Assume Hypothesis 4.2. The operator $T$ defined in (4.30) is self-adjoint.

Also, we introduce the self-adjoint operators $T_a$ in $L^2_{\mathbb{A}}(a,0)$ and $T_b$ in $L^2_{\mathbb{A}}(0,b)$ (see e.g., [27, Theorem VII.6.4]) by

$$T_a f = F, \quad f \in \text{dom } T_a = \{g \in \text{dom } T_{\text{max}} \mid B_a(g) := \lim_{c \to a} \chi_a(c, \overline{z})^* Jg(c) = 0, \quad \gamma_1 \gamma_2 \ g(0) = 0\}. \tag{4.32a}$$

$$T_b f = F, \quad f \in \text{dom } T_b = \{g \in \text{dom } T_{\text{max}} \mid \gamma_1 \gamma_2 \ g(0) = 0, \quad B_b(g) := \lim_{d \to b} \chi_b(d, \overline{z})^* Jg(d) = 0\}. \tag{4.33c}$$

As shown in [27, Section VII.7], the domain of the operators, defined via the boundary conditions at $a$ and $b$ in (4.30), (4.31), (4.32b), (4.33c), is $z$-independent.

**Remark 4.4.** Using Green’s formula (4.28), the deficiency index theorems in [38, Sections 4–6] can be proved for the closed symmetric operator $T_{\text{min}}$, cf. [34]. Also, as in Remark 2.6, if (4.1) is in lpc at $a$ (respectively, at $b$) then the boundary conditions in (4.30) and (4.32b) (respectively, in (4.31) and (4.33c)) can be omitted as they hold automatically for all $g \in \text{dom } T_{\text{max}}$, see, e.g., [27, Section VI.10].

Since (see [38, Theorem 11.5])

$$\sigma_{\text{ess}}(T) = \sigma_{\text{ess}}(T_a) \cup \sigma_{\text{ess}}(T_b), \tag{4.34}$$

the following lemma holds.

**Lemma 4.5.** Assume Hypothesis 4.2. Then the Weyl-Titchmarsh $M$-functions $M_a$ and $M_b$ are meromorphic in $\mathbb{C} \setminus \sigma_{\text{ess}}(T)$. 

**Proof.** Let $M_b$ be the matrix-valued Weyl-Titchmarsh function. Consider the operator $T_b$ defined in $L^2_{\mathbb{A}}(0,b)$ by (4.33). Green’s function of $T_b$ for $z \in \mathbb{C} \setminus \sigma(T_b)$ is given by the formula

$$G(t,s,z) = \begin{cases} 
\chi_b(t,z) \phi^*(s,\overline{z}), & s < t, \\
\phi(t,z) \chi_b^*(s,\overline{z}), & t < s,
\end{cases} \tag{4.35}$$
where \( \chi_h(s, z) = \theta(s, z) + \phi(s, z)M_h(z) \) (see [27, Theorem VII.6.3]). Denoting the columns of \( \phi \) by \( \phi_j \), for \( a < c < d < b \) and \( j = 1, \ldots, n \) we define

\[
\tilde{\phi}_j(t, z) = \begin{cases} 
\phi_j(t, z), & t \in (c, d), \\
0, & \text{otherwise},
\end{cases}
\]

and introduce the matrix valued function

\[
G(z) = \int_c^d \phi^*(t, z)A(t)\phi(t, z)dt.
\]

Clearly, \( G(\cdot) \) is analytic for \( z \in \mathbb{C} \). Due to Atkinson’s condition in Hypothesis 4.1 (ii), \( \langle \cdot, \cdot \rangle_{L^2(\mathbb{C}, d)} \) is an inner product in the vector space of the solutions of (4.1). Therefore, \( G(z) \) is the Gram matrix of the linearly independent solutions \( \phi_1, \ldots, \phi_n \). Consequently, \( G(z) \) is non-singular for any \( z \in \mathbb{C} \).

For \( z \in \rho(T_b) \) the resolvent operator \( (T_b - z)^{-1} \) satisfies:

\[
\langle (T_b - z)^{-1}\tilde{\phi}_j(\cdot, z), \tilde{\phi}_k(\cdot, z) \rangle_{L^2(A, b)} = \int_a^b \tilde{\phi}_j(t, z)A(t)\int_a^b \phi^*_j(t, z)A(t)\tilde{\phi}_k(s, z)dsdt
\]

\[
= \int_a^b \tilde{\phi}_j(t, z)A(t)\left(\phi(t, z)\int_t^b (\theta^*(s, z) + M^*(z)\phi^*(s, z))A(s)\tilde{\phi}_j(s, z)ds + (\theta(t, z) + \phi(t, z)M(z))\int_a^t \phi^*(s, z)A(s)\tilde{\phi}_j(s, z)ds\right)dt.
\]

Let \( R(z) \) denotes the \((n \times n)\) matrix whose entries \( R_{ij}(z) \) are defined as follows:

\[
R_{ij}(z) = \langle (T_b - z)^{-1}\tilde{\phi}_j(\cdot, z), \tilde{\phi}_k(\cdot, z) \rangle_{L^2(A, b)}
\]

\[
- \int_a^b \tilde{\phi}_j(t, z)A(t)\left(\phi(t, z)\int_t^b (\theta^*(s, z) + M^*(z)\phi^*(s, z))A(s)\tilde{\phi}_j(s, z)ds + \theta(t, z)\int_a^t \phi^*(s, z)A(s)\tilde{\phi}_j(s, z)ds\right)dt.
\]

Then (4.38) yields

\[
R_{ij}(z) = \left(\int_a^b \tilde{\phi}_j(t, z)A(t)\phi(t, z)dt\right) M_b(z) \left(\int_a^b \phi^*(s, z)A(s)\tilde{\phi}_j(s, z)ds\right)
\]

\[
= \left(c_i^2 \int_a^d \phi^*(t, z)A(t)dt\right) M_b(z) \left(\int_c^d \phi^*(s, z)A(s)\tilde{\phi}_j(s, z)ds \right)
\]

where \( c_i \) are the standard unit vectors in \( \mathbb{C}^n \). In other words,

\[
R(z) = G(z)M_b(z)G(z)^*,
\]

where \( G(z) \) is defined in (4.37). It is clear from (4.40) that \( M_b(\cdot) \) is meromorphic for \( z \in \mathbb{C} \setminus \sigma_{ess}(T_b) \) and, due to (4.34), for \( \mathbb{C} \setminus \sigma_{ess}(T) \) since \( R(z) \) is meromorphic and \( G(z) \) is analytic and nonsingular. Analogously, one can prove that \( M_a \) is meromorphic for \( z \in \mathbb{C} \setminus \sigma_{ess}(T) \).

Next, we prove an analog of Proposition 2.9. We stress again that \( z \) can be real in Proposition 4.6.

**Proposition 4.6.** Assume Hypothesis 4.1. If \( z \in \mathbb{C} \setminus \sigma_{ess}(T_a) \) then the following assertions hold:
(i) If the Hamiltonian system (4.1) is in lpc at a then there exist exactly $n$ linearly independent solutions of (4.1) that are $A$-square integrable near $a$ (that is, belonging to $L^2_A(a,0)$);

(ii) If the Hamiltonian system (4.1) is in lcc at $a$ then there exist exactly $n$ linearly independent solutions of (4.1) that are $A$-square integrable near $a$ and satisfy the boundary condition at $a$ in (4.32b).

Moreover, assuming that (4.1) is either in lpc or lcc at $a$, let $z \in \mathbb{C} \setminus \sigma(T_a)$ and let $F_a(\cdot, z)$ denote the $(2n \times n)$ matrix valued function whose columns are the solutions described in part (i) or (ii). Then $F_a(\cdot, z) = \chi_a(\cdot, z)C(z)$, where $\chi_a(\cdot, z) = \theta(\cdot, z) + \phi(\cdot, z)M_a(z)$, $C(z)$ is some constant matrix, and $M_a(z)$ is the Weyl-Titchmarsh $M$-function used in (4.32b) to define the operator $T_a$. Analogous assertions hold for the point $b$.

Proof. We let $\mu$ denote the number of linearly independent solutions of (4.1) that are $A$-square integrable near $a$ and satisfy the boundary conditions at $a$ in (4.32b) provided (4.1) is in lcc at $a$, and that are just $A$-square integrable near $a$ provided (4.1) is in lpc at $a$. Also, we let $\nu$ denote the number of linearly independent solutions that are $A$-square integrable near 0 and satisfy the boundary conditions at 0 in (4.32c). Assuming $z \in \mathbb{C} \setminus \sigma(T_a)$, we have $\mu + \nu = 2n$ by [38, Theorem 7.1]. Since 0 is a regular point, $\nu = n$ due to (4.32c) and (4.2). Thus $\mu = n$ proving (i) and (ii) for $z \in \mathbb{C} \setminus \sigma(T_a)$. Furthermore, if (4.1) is in lpc at $a$ then the columns of $\chi_a(\cdot, z) = \theta(\cdot, z) + \phi(\cdot, z)M_a(z)$ are the solutions of (4.1) that are $A$-square integrable near $a$, and if (4.1) is in lcc at $a$ then the columns of $\chi_a(\cdot, z) = \theta(\cdot, z) + \phi(\cdot, z)M_a(z)$ are the solutions of (4.1) that are $A$-square integrable near $a$ and satisfy the boundary conditions at $a$ in (4.32c). Indeed, this holds because $\chi_a(\cdot, z)$ enters the formula for Green’s function, cf. (4.35), and Green’s function is composed of the $A$-square integrable near $a$ solutions that satisfy the boundary conditions when appropriate (see, e.g., [38, Theorem 7.6]). This proves the representation $F_a(\cdot, z) = \chi_a(\cdot, z)C(z)$ for $z \in \mathbb{C} \setminus \sigma(T_a)$.

Next, we assume that $z \in \sigma_d(T_a)$, the discrete spectrum of $T_a$, and stress that $z$ is real. We claim that $\mu \geq n$ provided (4.1) is either in lpc or lcc at $a$. Starting the proof of the claim, let $F_a(\cdot, z)$ denote the matrix valued function whose columns are the solutions $Y_a^{(j)}(\cdot, z)$ of (4.1) with the properties described in assertions (i) or (ii). Seeking a contradiction, let us suppose that $\mu < n$. Since $\mu$ is the rank of $F_a(\cdot, z)$, by multiplying $F_a(\cdot, z)$ from the right by a constant $(n \times n)$ matrix, we may and will assume that the first $\mu$ columns of $F_a(\cdot, z)$ are linearly independent and the remaining columns are zero. Using $\mu = \text{rank } F_a(\cdot, z)$ again, let us suppose that the linearly independent rows of $F_a(\cdot, z)$ are located in the rows with the numbers $k_1, \ldots, k_\mu$, where $k_1 \geq \cdots \geq k_\mu$. Let us consider the $(n \times 2n)$ matrix $\tilde{\gamma} = [e_{k_1} \ldots e_{k_\mu} 0_{2n \times 1} \ldots 0_{2n \times 1}]^T$, where $e_t$ are the standard unit column vectors in $\mathbb{C}^{2n}$. If $j = 1, \ldots, \mu$, then the $j$-th row of the matrix $\tilde{\gamma}J$ is equal to $-e_{k_j+n}$ provided $k_j \leq n$ and is equal to $e_{k_j-n}$ provided $k_j > n$, and the remaining rows are zero. It follows that $\tilde{\gamma}J\tilde{\gamma}^* = 0$, and thus $\tilde{\gamma} = [\tilde{\gamma}_1 \tilde{\gamma}_2]$ satisfies (4.3). Since the first $\mu$ rows of the matrix $\tilde{\gamma}F_a(\cdot, z)$ are linearly independent, each of the first $\mu$ columns of this matrix is not zero. In other words, all columns of the product $[\tilde{\gamma}_1 \tilde{\gamma}_2] \cdot \begin{bmatrix} Y_a^{(1)}(\cdot, z) & \ldots & Y_a^{(\mu)}(\cdot, z) \end{bmatrix}$ are nonzero. We denote by $\tilde{T}_a$ the operator defined as in (4.32) but with $\gamma_1, \gamma_2$ replaced by $\tilde{\gamma}_1, \tilde{\gamma}_2$, and remark that the solutions $Y_a^{(1)}(\cdot, z), \ldots, Y_a^{(\mu)}(\cdot, z)$ of (4.1) do not satisfy the boundary condition
Therefore, \( z \) inclusion implies \( \mu \). Since \( \tilde{T}_a \) is yet another self-adjoint extension of the operator \( T_{\text{min}} \), one has \( \sigma_{\text{ess}}(\tilde{T}_a) = \sigma_{\text{ess}}(T_a) \), and thus \( z \notin \sigma_{\text{ess}}(\tilde{T}_a) \) since \( z \notin \sigma_{\text{ess}}(T_a) \) by the assumption in the proposition. Therefore, \( z \in \mathbb{C} \setminus \sigma(\tilde{T}_a) \). As we have seen in the previous paragraph, the latter inclusion implies \( \mu = n \), a contradiction which proves the claim.

It remains to prove that \( \mu \leq n \). Let us first consider the case when (4.1) is in lpc at \( a \) (still assuming \( z \in \sigma_d(T_a) \)). In this case the proof is analogous to the proof of [38, Theorem 4.8]. Indeed, since \( Y^j_a(\cdot, z) \) are solutions of (4.1) and \( z \) is real, we have \( Y^j_a(t, z)JY^j_a(t, z) = \text{const} \) for each \( i, j = 1, \ldots, \mu \). By Remark 4.4, these constants are in fact equal to zero since \( Y^j_a \in \text{dom} T_{\text{max}} \) and the boundary condition (4.30) and (4.32b) at \( a \) are satisfied automatically for all functions from \( \text{dom} T_{\text{max}} \) because (4.1) is in lpc at \( a \). So, \( Y^j_a(t, z)JY^j_a(t, z) = 0 \) for all \( i, j = 1, \ldots, \mu \) and \( t \in (a, 0) \). Fix any \( c \in (a, 0) \) and define the linear functionals on \( \mathbb{C}^m \) by \( \rho_i(c) = Y^i_a(c, z)Jc \), \( i = 1, \ldots, \mu \). They are linearly independent since \( Y^i_a(\cdot, z) \) are linearly independent. Then \( \cap_{i=1}^{\mu} \ker \rho_i \neq \ker \rho_j \) for each \( j = 1, \ldots, \mu \). This and \( \rho_j(Y^j_a(c, z)) = 0 \) for all \( i, j = 1, \ldots, \mu \) imply \( \mu \leq \dim \cap_{i=1}^{\mu} \ker \rho_i \leq 2n - \mu \), thus proving \( \mu \leq n \).

It remains to prove that \( \mu \leq n \) in the case when (4.1) is in lcc at \( a \) (still assuming \( z \in \sigma_d(T_a) \)). Let us assume that \( \mu > n \). Let \( Y^j_a \), \( j = 1, \ldots, \mu \), be the linearly independent solutions of (4.1) that are \( A \)-square integrable near \( a \) and satisfy the boundary conditions in (4.32b) at \( a \). Next, we introduce the functions \( \tilde{Y}^j_a \), \( j = 1, \ldots, \mu \), as follows:

\[
\tilde{Y}^j_a(t, z) = \begin{cases} Y^j_a(t, z) & \text{for } t \text{ close to } a, \\ 0 & \text{for } t \text{ close to } 0, \end{cases}
\]

and such that \( \tilde{Y}^j_a(\cdot, z) \in \text{dom} T_{a,\text{max}} \) (the latter inclusion is possible as described in [38, page 50]). Then \( \tilde{Y}^j_a \in \text{dom} T_a \). Moreover, \( \tilde{Y}^j_a \) are linearly independent modulo \( \text{dom} T_{a,\text{min}} \) (see, e.g., [38, Theorem 5.4(a)]). We also know that \( \nu = n \). Let \( Y^j_0 \), \( j = 1, \ldots, n \), be the linearly independent solutions of (4.1) that are \( A \)-square integrable near \( 0 \) and satisfy the boundary conditions in (4.32c) at \( 0 \). Now, we construct the functions \( \tilde{Y}^j_0 \), \( j = 1, \ldots, n \), as follows:

\[
\tilde{Y}^j_0(t, z) = \begin{cases} Y^j_0(t, z) & \text{for } t \text{ close to } 0, \\ 0 & \text{for } t \text{ close to } a, \end{cases}
\]

and such that \( \tilde{Y}^j_0 \in \text{dom} T_a \). Moreover, \( \tilde{Y}^j_0 \) are linearly independent modulo \( \text{dom} T_{a,\text{min}} \). Thus, there are \( \mu + n \) linearly independent modulo \( \text{dom} T_{a,\text{min}} \) elements from \( \text{dom} T_a \). This contradicts the fact that

\[
\dim \left( \text{dom} D(T_a) / \text{dom} D(T_{a,\text{min}}) \right) = 2n,
\]

which holds because the deficiency index of \( T_{a,\text{min}} \) is \( 2n \) since (4.1) is in lcc at \( a \).

Next, we describe the connections between the Weyl-Titchmarsh \( M \)-functions, the isolated eigenvalues of the respective differential operators, and the determinants of matrices composed of the solutions \( \theta, \phi, F_a \) and \( F_b \) here and below \( F_a(\cdot, z) \) and \( F_b(\cdot, z) \) are the matrix valued functions whose columns are the solutions of (4.1) with the properties described in assertions (i) or (ii) of Proposition 4.6.
Lemma 4.7. Assume Hypothesis 4.1. If \( z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(T_a) \) and (4.1) is either in lpc or lcc at \( a \) then the following assertions are equivalent:

(i) \( \det [F_a(t, z_0) \phi(t, z_0)] = 0 \) for some/all \( t \in (a, 0) \);

(ii) \( z_0 \in \sigma_d(T_a) \);

(iii) \( z_0 \) is a pole of \( M_a(\cdot) \).

Analogous facts hold for the point \( b \). Finally, if \( z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(T) \) and (4.1) is either in lpc or lcc at either \( a \) or \( b \) then the following assertions are equivalent:

(iv) \( \det [F_a(t, z_0) F_b(t, z_0)] = 0 \) for some/all \( t \in (a, b) \);

(v) \( z_0 \in \sigma_d(T) \).

Proof. (i) \( \Leftrightarrow \) (ii): By Proposition 4.6, the columns of \( F_a(\cdot, z_0) \) span the space of solutions of (4.1) that are \( A \)-square integrable at \( a \) (when (4.1) is in lpc at \( a \)) and that satisfy the boundary conditions at \( a \) in (2.15c) (when (4.1) is in lcc at \( a \)). Assertion (i) holds if and only if the columns of \( F_a(\cdot, z_0) \) and \( \phi(\cdot, z_0) \) are linearly dependent. Since \( \text{rank } F_a(\cdot, z_0) = \text{rank } \phi(\cdot, z_0) = n \), this holds if and only if there is a solution of (4.1) which is equal to a linear combination of the columns of \( F_a(\cdot, z_0) \) and is equal to a linear combination of the columns of \( \phi(\cdot, z_0) \). Thus, this solution satisfies both boundary conditions in (4.32b), (4.32c), yielding equivalence to (ii).

(ii) \( \Leftrightarrow \) (iii): By (4.40), \( z_0 \) is a pole of \( M_a(\cdot) \) if and only if \( z_0 \) is a pole of the resolvent \( (T_a - z)^{-1} \). Since \( z_0 \in \mathbb{C} \setminus \sigma_{\text{ess}}(T_a) \), the latter is equivalent to (ii).

(iii) \( \Leftrightarrow \) (v): The proof is similar to the proof of (i) \( \Leftrightarrow \) (ii). \( \square \)

Lemma 4.7 shows that the Evans function, \( D(z) \), for the Hamiltonian system (4.1) can be naturally defined in terms of \( F_a \) and \( F_b \) as follows:

\[
D(z) = \det [F_a(0, z) \quad F_b(0, z)], \quad z \in \mathbb{C} \setminus \sigma_{\text{ess}}(T),
\]

so that \( D(z_0) = 0 \) if and only if \( z_0 \in \sigma_d(T) \). We will now express the Evans function in terms of the Weyl-Titchmarsh \( M \)-functions. For this, we introduce an \((n \times n)\) matrix Wronskian, \( \mathcal{W}(F, G) \), by denoting

\[
\mathcal{W}(F, G) = F_1^T G_2 - F_2^T G_1 = -F^* J G
\]

for any \( 2n \times n \)-matrices \( F = [F_1 \quad F_2]^T \) and \( G = [G_1 \quad G_2]^T \), and remark that

\[
\mathcal{W}^*(F, G) = (-F^* J G)^* = -\mathcal{W}(G, F).
\]

Also, we recall that \( \psi(\cdot, z) \) is the fundamental matrix solution of (4.1) satisfying (4.5), (4.6).

Theorem 4.8. Assume Hypothesis 4.28 and that (4.1) is either in lpc or lcc at either \( a \) or \( b \). Let \( F_a(\cdot, z) \) and \( F_b(\cdot, z) \) be the matrix valued functions whose columns are the \( A \)-square integrable solutions of (4.1) described in assertions (i) or (ii) of Proposition 4.6. Then the following formulas hold:

\[
M_a(z) = \mathcal{W}(\theta(0, z), F_a(0, z)) \mathcal{W}^*(F_a(0, z), \phi(0, z))^{-1}, \quad z \in \mathbb{C} \setminus \sigma_{\text{ess}}(T_a),
\]

\[
M_b(z) = \mathcal{W}(\theta(0, z), F_b(0, z)) \mathcal{W}^*(F_b(0, z), \phi(0, z))^{-1}, \quad z \in \mathbb{C} \setminus \sigma_{\text{ess}}(T_b),
\]

\[
M_b(z) - M_a^*(z) = \left( \mathcal{W}(F_a(0, z), \phi(0, z)) \right)^{-1} \mathcal{W}(F_a(0, z), F_b(0, z))
\]

\[
\times \left( \mathcal{W}^*(F_b(0, z), \phi(0, z)) \right)^{-1}, \quad z \in \mathbb{C} \setminus \sigma_{\text{ess}}(T).
\]
Moreover, the Evans function $D(z)$ for (4.1) and the Weyl-Titchmarsh $M$-functions are related as follows:

$$D(z) = \det \mathcal{Y}(0, z) \det \mathcal{W}^*(F_a(0, z), \phi(0, z)) \det \mathcal{W}^*(F_b(0, z), \phi(0, z)) \times \det \left( M_b(z) - M_a(z) \right).$$  

(4.50)

**Proof.** Let us prove (4.48), the proof of (4.47) is analogous. By Proposition 4.6, $F_b(\cdot, z) = \chi_b(\cdot, z) C_b(z)$ for $z \in \mathbb{C} \setminus \sigma(T_b)$. Letting $t = 0$ and using (4.5) yield the system of equations for $M_b(z)$ and $C_b(z)$,

$$(\gamma_1^* - \gamma_2^* M_b(z)) C_b(z) = F_{b1}(0, z), \quad (\gamma_2^* + \gamma_1^* M_b(z)) C_b(z) = F_{b2}(0, z),$$  

(4.51)

where we subdivide $F_b(0, z) = [F_{b1}(0, z), F_{b2}(0, z)]^T$ into two $(n \times n)$ blocks. In other words, using (4.6), (4.7), (4.8), (4.45),

$$F_b(0, z) = \mathcal{Y}(0, z) \begin{pmatrix} C_b(z) \\ M_b(z) C_b(z) \end{pmatrix},$$

$$= \begin{pmatrix} \mathcal{Y}^*(0, z) F_b(0, z) \\ \theta^*(0, z) F_b(0, z) \end{pmatrix} = \begin{pmatrix} \phi^*(0, z) F_b(0, z) \\ \phi^*(0, z) F_b(0, z) \end{pmatrix} = \begin{pmatrix} -\mathcal{W}(\phi(0, z), F_b(0, z)) \\ \mathcal{W}(\theta(0, z), F_b(0, z)) \end{pmatrix},$$

(4.52)

By Lemma 4.7, $z \in \sigma_d(T_b)$ if and only if $M_b$ has a pole at $z$ or, equivalently, det $[F_b(0, z) \quad \phi(0, z)] = 0$. Since, cf. (4.6), (4.8),

$$\det \begin{pmatrix} \mathcal{Y}^*(0, z) [F_b(0, z) \quad \phi(0, z)] \end{pmatrix} = \det \begin{pmatrix} \mathcal{Y}^*(0, z) [F_b(0, z) \quad \phi(0, z)] \end{pmatrix} = \mathcal{Y}(0, z) [F_b(0, z) \quad \phi(0, z)] = \begin{pmatrix} \theta^*(0, z) F_b(0, z) \\ \phi^*(0, z) F_b(0, z) \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{W}(\phi(0, z), F_b(0, z)) \end{pmatrix} \begin{pmatrix} 0 \\ \mathcal{W}(\theta(0, z), F_b(0, z)) \end{pmatrix},$$

the matrix $\mathcal{W}^*(F_b(0, z), \phi(0, z))$ is singular if and only if $z \in \sigma_d(T)$. Now (4.52), (4.46) imply

$$M_b(z) = (\gamma_1 F_{b2}(0, z) - \gamma_2 F_{b1}(0, z))(\gamma_1 F_{b1}(0, z) + \gamma_2 F_{b2}(0, z))^{-1},$$

(4.53)

$$C_b(z) = \gamma_1 F_{b1}(0, z) + \gamma_2 F_{b2}(0, z) = \mathcal{W}^*(F_b(0, z), \phi(0, z)),$$

(4.54)

$$= -\mathcal{W}(\phi(0, z), F_b(0, z))$$

(4.55)

thus proving (4.48) for $z \in \mathbb{C} \setminus \sigma(T_b)$ and therefore for $z \in \mathbb{C} \setminus \sigma_{ess}(T_b)$ as both sides of (4.48) are not defined at $z \in \sigma_d(T_b)$ by Lemma 4.7.

Using (4.53), its analogue for $a$, and (4.3), a short calculation yields (4.49). Formula (4.50) follows from formulae (4.51), (4.54), from Theorem 4.8, and from the following computation:

$$\begin{pmatrix} F_a(0, z) & F_b(0, z) \end{pmatrix} = \mathcal{Y}(0, z) \begin{pmatrix} -\mathcal{W}(\phi(0, z), F_a(0, z)) & -\mathcal{W}(\phi(0, z), F_b(0, z)) \\ \mathcal{W}(\theta(0, z), F_a(0, z)) & \mathcal{W}(\theta(0, z), F_b(0, z)) \end{pmatrix} \begin{pmatrix} I \\ M_a(z) \end{pmatrix} - \mathcal{Y}(0, z) \begin{pmatrix} I \\ M_a(z) \end{pmatrix} \begin{pmatrix} I \gamma M_b(z) - M_a(z) \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}$$

(4.56)
We conclude the section with several examples of Hamiltonian systems where
the assumptions in Lemma 4.7 and Theorem 4.8 are satisfied.

**Example 1.** The Dirac-type system (see, e.g., [8]) is obtained by setting $A(t) = I_{2n \times 2n}$ in (4.1).

**Lemma 4.9.** [8, Lemma 2.15] The limit point case holds for Dirac-type systems at $a = -\infty$ and $b = +\infty$.

**Example 2.** The Schrödinger equation with matrix-valued potential,

$$-y''(t) + Q(t)y(t) = k^2y(t), \quad -\infty < t < \infty;$$

for which we impose the following assumption.

**Hypothesis 4.10.** Assume that $Q = Q^* \in L^1(\mathbb{R})^{n \times n}$.

Equation (4.56) may be re-written as the system

$$JY'(t) = (k^2A(t) + B(t))Y(t), \quad t \in (-\infty, +\infty),$$

where

$$Y(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} I_{n \times n} & 0 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} -Q(t) & 0 \\ 0 & I_{n \times n} \end{bmatrix}.$$  \hspace{1cm} (4.57)

As it is well known (see the proof of Theorem 1.4.1 [2]), equation (4.56) has a system of matrix valued solutions $f_+(\cdot, k)$ and $\tilde{f}_+(\cdot, k)$ satisfying the asymptotic boundary conditions

$$\lim_{t \to -\infty} e^{-ikt}f_+(t, k) = I_{n \times n}, \quad \lim_{t \to -\infty} e^{ikt}\tilde{f}_+(t, k) = I_{n \times n}, \quad \text{Im}(k) > 0.$$  \hspace{1cm} (4.59)

If $F_+(\cdot, k) = [f_+(\cdot, k) \quad f'_+(\cdot, k)]^\top$ and $\tilde{F}_+(\cdot, k) = [\tilde{f}_+(\cdot, k) \quad \tilde{f}'_+(\cdot, k)]^\top$, then $F(\cdot, k)$ and $\tilde{F}(\cdot, k)$ form a fundamental system of solutions of (4.57) satisfying asymptotic boundary conditions

$$\lim_{t \to -\infty} e^{-ikt}F_+(t, k) = \begin{bmatrix} I_{n \times n} & ikI_{n \times n} \end{bmatrix}^\top,$$

$$\lim_{t \to -\infty} e^{ikt}\tilde{F}_+(t, k) = \begin{bmatrix} I_{n \times n} & -ikI_{n \times n} \end{bmatrix}^\top, \quad \text{Im}(k) > 0.$$  \hspace{1cm} (4.60)

**Lemma 4.11.** Assume Hypothesis 4.10. Then equation (4.57) is in the limit point case at both $-\infty$ and $+\infty$.

**Proof.** Equation (4.57) is in the limit point case at $+\infty$ since the columns of $F_+(\cdot, k)$ are $A$-square integrable solutions of (4.57) while the columns of $\tilde{F}_+(\cdot, k)$ are not $A$-square integrable solutions of (4.57) which is clear from (4.60) and the formula

$$\int F^*(t, k)A(t)F(t, k)dt = \int f^*(t, k)f(t, k)dt,$$

which holds for any solution $F(\cdot, k) = [f(\cdot, k) \quad f'(\cdot, k)]^\top$ of (4.57). Similarly, it can be proved that (4.57) is in the limit point case at $-\infty$. \hfill \Box
The Zakharov-Shabat problem (see, e.g., [1]) is given by
\[ v' = \begin{bmatrix} -ik & r(t) \\ q(t) & ik \end{bmatrix} v, \quad -\infty < t < \infty, \]  
(4.62)
where we assume that the scalar functions \( r, q \in L^1(\mathbb{R}) \) satisfy \( r(t) = \frac{\bar{q}(t)}{q} \) for all \( t \in (-\infty, \infty) \). Let us make the change of variables
\[ u = \frac{1}{\sqrt{2}} \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix} v. \]  
(4.63)
Then (4.62) becomes
\[ Ju' = \left( kI_{2 \times 2} + \frac{1}{2} \begin{bmatrix} i(q-r) & r+q \\ r+q & i(r-q) \end{bmatrix} \right) u. \]  
(4.64)
Since \( r(t) = \frac{\bar{q}(t)}{q} \),
\[ Ju'(t) = (kI_{2 \times 2} + B(t))u(t), \quad \text{where} \quad B = B^* = \frac{1}{2} \begin{bmatrix} i(q-q^*) & q^*+q \\ q^*+q & i(q^*-q) \end{bmatrix}. \]  
(4.65)
Since \( q \in L^1(\mathbb{R}) \) and \( r(t) = \frac{\bar{q}(t)}{q} \), there exist four solutions \( f_\pm(\cdot, k), \tilde{f}_\pm(\cdot, k) \) of (4.62) (see [1, Chapter 1.3]) defined by the following boundary conditions:
\[ \lim_{t \to \infty} f_+(t, k)e^{-ikt} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^t, \quad \lim_{t \to -\infty} f_-(t, k)e^{ikt} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^t, \]  
(4.66)
\[ \lim_{t \to \infty} \tilde{f}_+(t, k)e^{ikt} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^t, \quad \lim_{t \to -\infty} \tilde{f}_-(t, k)e^{-ikt} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^t. \]  
(4.67)
Therefore, via (4.63) there exist four solutions \( F_\pm(k, k), \tilde{F}_\pm(k, k) \) of (4.65) defined by the following boundary conditions:
\[ \lim_{t \to \infty} F_+(t, k)e^{-ikt} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \end{bmatrix}^t, \quad \lim_{t \to -\infty} F_-(t, k)e^{ikt} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \end{bmatrix}^t, \]  
(4.68)
\[ \lim_{t \to \infty} \tilde{F}_+(t, k)e^{ikt} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \end{bmatrix}^t, \quad \lim_{t \to -\infty} \tilde{F}_-(t, k)e^{-ikt} = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \end{bmatrix}^t. \]  
(4.69)
It is clear from (4.68) and (4.69) that (4.65) is in lpc at both \(-\infty\) and \(+\infty\).

The nearly complex Ginzburg-Landau equation (see [24]) is
\[ u_t - \frac{1}{2} u_{xx} - u + |u|^2u = i\epsilon (\frac{1}{2} d_1 u_{xx} + d_2 u + d_3 |u|^2 u + d_4 |u|^4 u), \]  
(4.70)
where \( u = u(t, x), \ t \geq 0, x \in \mathbb{R}, \epsilon > 0 \) is small, and the other parameters are real and of \( O(1) \) as \( \epsilon \to 0 \). Assuming \( \epsilon = 0 \) and introducing \( v(t, x) = u(t, x) \), equation (4.70) can be rewritten as the system
\[ u_t - \frac{1}{2} u_{xx} - u + u^2 v = 0, \]  
(4.71)
\[ v_t - \frac{1}{2} v_{xx} - v + v^2 u = 0. \]

Linearizing (4.71) around the hole solution \( U(x) = \tanh x \) yields
\[ u_t - \frac{1}{2} u_{xx} - u + 2U^2(x)u + U^2(x)v = 0, \]  
\[ v_t - \frac{1}{2} v_{xx} - v + U^2(x)u + 2U^2(x)v = 0, \]  
(4.72)
for \( u = u(x), v = v(x) \), which induces the eigenvalue problem for \( z \in \mathbb{C} \setminus \mathbb{R}_+ \),

\[
\begin{align*}
- \frac{1}{2} u_{xx} - (1 - 2U^2(x))u + U^2(x)v &= z u, \\
- \frac{1}{2} v_{xx} - (1 - 2U^2(x))v + U^2(x)u &= z v.
\end{align*}
\] (4.73)

Letting \( Y(x) = \begin{bmatrix} u(x) & v(x) & u'(x) & v'(x) \end{bmatrix}^\top \), we rewrite (4.73) as the first order system

\[
\partial_x Y(x) = M(x, z) Y(x),
\] (4.74)

where

\[
M(x, z) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
2(-z - 1 + 2U^2(x)) & 2U^2(x) & 0 & 0 \\
2U^2(x) & 2(-z - 1 + 2U^2(x)) & 0 & 0
\end{pmatrix}.
\] (4.75)

Note that \( \lim_{x \to \pm \infty} M(x, z) = M_\infty(z) \) exponentially fast, where

\[
M_\infty(z) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
2(-z + 1) & 2 & 0 & 0 \\
2 & 2(-z + 1) & 0 & 0
\end{pmatrix}.
\] (4.76)

The eigenvalues of \( M(z) \) are \( \mu_{1,2} = \pm i\sqrt{2z} \), \( \mu_{3,4} = \pm i\sqrt{2z + 4} \) and thus nonimaginary if and only if \( z \in \mathbb{C} \setminus \mathbb{R}_+ \). Since \( M(\cdot, z) - M_\infty(z) \in L^1(\mathbb{R})^{4 \times 4} \), by the standard asymptotic theory (see, e.g., [11, Chapter I] or [9, Chapter 3, Problem 29]), there exist four solutions \( Y^{(j)}(x), j = 1, \ldots, 4 \), of (4.74) defined by the boundary conditions

\[
\lim_{x \to \pm \infty} Y^{(j)}(x, z)e^{-\mu_{j}x} = p_j, \quad \operatorname{Re}(\mu_j) \neq 0, \quad z \in \mathbb{C} \setminus \mathbb{R}_+,
\] (4.77)

where \( p_j \) is the eigenvector of \( M_\infty(z) \) corresponding to the eigenvalue \( \mu_j \). We can rewrite (4.74) as

\[
J(\partial_x Y)(x) = (zA + B(x))Y(x),
\] (4.78)

where

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
2(1 - 2U^2) & -2U^2 & 0 & 0 \\
-2U^2 & 2(1 - 2U^2) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad J = \begin{bmatrix}
0 & -I_2 \\
I_2 & 0
\end{bmatrix}.
\]

It is clear from (4.77) that (4.78) is in lpc at \( +\infty \). Similarly, it can be shown that (4.78) is in lpc at \( -\infty \).

**References**


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