A correlation inequality for stable random vectors

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Abstract. Let $X_1,\ldots, X_n$ and $Y_1,\ldots, Y_n$ be jointly $q$-stable symmetric random variables, $0 < q \leq 2$, so that, for some $k \in \mathbb{N}$, $1 \leq k < n$, the vectors $(X_1,\ldots, X_k)$ and $(X_{k+1},\ldots, X_n)$ have the same distributions as $(Y_1,\ldots, Y_k)$ and $(Y_{k+1},\ldots, Y_n)$, respectively, but $Y_i$ and $Y_j$ are independent for every choice of $1 \leq i \leq k$, $k+1 \leq j \leq n$. Let $(\mathbb{R}^n, \cdot, \| \cdot \|)$ be an $n$-dimensional normed space such that $\|u \cdot v\| = \|(u, -v)\|$ for every $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$. We prove that, for every $p \in [n-3, n)$, $\mathbb{E}(\|X\|^{-p}) \geq \mathbb{E}(\|Y\|^{-p})$.

1. Introduction

Let $X_1,\ldots, X_n$ and $Y_1,\ldots, Y_n$ be jointly $q$-stable symmetric random variables, $0 < q \leq 2$, so that, for some $k \in \mathbb{N}$, $1 \leq k < n$, the vectors $(X_1,\ldots, X_k)$ and $(X_{k+1},\ldots, X_n)$ have the same distributions as $(Y_1,\ldots, Y_k)$ and $(Y_{k+1},\ldots, Y_n)$, respectively, but $Y_i$ and $Y_j$ are independent for every choice of $1 \leq i \leq k$, $k+1 \leq j \leq n$. Let $B = (\mathbb{R}^n, \cdot, \| \cdot \|)$ be an $n$-dimensional normed space.

The following result was established in [K, Th.4] and later proved by Houdré [H, Remark 2.4] by different methods:

Theorem A. If $0 < p < n$, $\|x\|^{-p}$ is a positive definite distribution and the norm satisfies a symmetry condition $\|(u, v)\| = \|(u, -v)\|$ for every $u \in \mathbb{R}^k$, $v \in \mathbb{R}^{n-k}$, then $\mathbb{E}(\|X\|^{-p}) \geq \mathbb{E}(\|Y\|^{-p})$.

Here we consider $\|x\|^{-p}$ as a tempered distribution. Recall that by L.Schwartz’s generalization of Bochner’s theorem (see [GV, p. 152]), a tempered distribution $f \in S'(\mathbb{R}^n)$ is positive definite if and only if its Fourier transform $\hat{f}$ is a positive distribution. The latter means that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function $\phi \in S(\mathbb{R}^n)$.

It was shown in [K, Corollary 2(ii)] that if $B$ is a subspace of $L_r$ with $0 < r \leq 2$ then $\|x\|^{-p}$ is positive definite for every $0 < p < n$. However, if $B = \ell_r^p$, $2 < r < \infty$, $n \geq 3$ then $\|x\|^{-p}$ is positive definite if and only if $p \in [n-3, n)$. In particular, for every $p \in [n-3, n)$, $n \geq 3$

$$
\mathbb{E}(\max_{i=1,\ldots,n} |X_i|^{-p}) \geq \mathbb{E}(\max_{i=1,\ldots,n} |Y_i|^{-p}).
$$

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In this article we show that the latter inequality is a part of a more general result:

**Theorem 1.** For every \( p \in [n - 3, n) \) and every \( n \)-dimensional normed space \( B = (\mathbb{R}^n, \| \cdot \|) \), \( n \geq 3 \), whose norm satisfies the symmetry condition \( \| (u,v) \| = \| (u,-v) \| \) for each \( u \in \mathbb{R}^n \), \( v \in \mathbb{R}^{n-k} \), we have \( \mathbb{E}(\| X \|^{-p}) \geq \mathbb{E}(\| Y \|^{-p}) \).

The proof is based on the fact that, for \( p \in [n - 3, n) \), the distribution \( \| x \|^{-p} \) is positive definite for every \( n \)-dimensional normed space \( B = (\mathbb{R}^n, \| \cdot \|) \). Theorem 1 follows immediately from this result and Theorem A.

**2. Proof of Theorem 1**

We use methods of convex geometry to prove positive definiteness of powers of the norm. Let \( K = \{ x \in \mathbb{R}^n : \| x \| \leq 1 \} \) be the unit ball of the space \( B \). For every unit vector \( \xi \in S^{n-1} \) the parallel section function \( A_\xi \) in the direction of \( \xi \) is defined as a function on \( \mathbb{R} \) so that for each \( t \in \mathbb{R} \), \( A_\xi(t) \) is the \( (n-1) \)-dimensional volume of the section of \( K \) by the hyperplane perpendicular to \( \xi \) and located at the distance \( t \) from the origin. We say that the space \( B \) is infinitely smooth if the restriction of the norm of \( B \) to the unit sphere \( S^{n-1} \) belongs to the space \( C^\infty(S^{n-1}) \) of infinitely differentiable functions on the sphere. If \( B \) is infinitely smooth then, for every \( \xi \in S^{n-1} \), \( A_\xi \) is an infinitely differentiable function in a neighborhood of zero. For \( \beta \in (-1, 0) \), the fractional derivative of order \( \beta \) of the function \( A_\xi \) at zero is defined by

\[
A_\xi^{(\beta)}(0) = \frac{1}{\Gamma(-\beta)} \int_0^\infty t^{-1-\beta} A_\xi(t) \, dt.
\]

If \( \beta \in (0, 2), \beta \neq 1 \) then

\[
A_\xi^{(\beta)}(0) = \frac{1}{\Gamma(-\beta)} \int_0^\infty t^{-1-\beta} (A_\xi(t) - A_\xi(0)) \, dt
\]

(note that \( A_\xi \) is an even function so its first derivative at zero is equal to zero; for more on fractional derivatives see, for example, [GKS, Section 3]).

Our main tool is the following theorem, which was proved in [GKS, Th.2] in a more general form (for every \( \beta \in \mathbb{C}, \Re(\beta) > -1, \beta \neq n-1 \)).

**Theorem B.** Let \( B \) be an infinitely smooth \( n \)-dimensional normed space, \( K \) is the unit ball of \( B \), \( \beta \in (-1, 2), \beta \) is not an integer. Then for every \( \xi \in S^{n-1} \)

\[
A_\xi^{(\beta)}(0) = \frac{\cos \frac{\beta \pi}{2}}{\pi (n - \beta - 1)} (\| x \|^{-n+\beta+1})^\wedge(\xi).
\]

Note that this result in its general form was used in [GKS] as one of the major ingredients of the solution to the Busemann-Petty problem on sections of convex bodies.

**Theorem 2.** Let \( B = (\mathbb{R}^n, \| \cdot \|) \) be an \( n \)-dimensional normed space. Then for every \( p \in [n - 3, n) \) the distribution \( \| x \|^{-p} \) is positive definite.
PROOF. First assume that \( B \) is infinitely smooth and \( p \) is not an integer. Put 
\[ \beta = n - p - 1 \in (-1, 2). \]
We are going to show that \((\|x\|^{-\beta + 1})^\wedge\) is a non-negative continuous function on \( S^{n-1} \). Since this function is also homogeneous of degree 
\( -\beta - 1 \) on \( \mathbb{R}^n \), \( n \geq 3 \), we deduce that it is non-negative and locally integrable on \( \mathbb{R}^n \). This would mean, in particular, that \((\|x\|^{-\beta + 1})^\wedge = (\|x\|^{-p})^\wedge\) is a positive distribution, and \( \|x\|^{-p} \) is positive definite.

Since the restriction of the norm to \( S^{n-1} \) is infinitely smooth and the volume of every section can be expressed in terms of the norm, it is easily seen that \( A_\xi^{(\beta)}(0) \) is a continuous function of \( \xi \in S^{n-1} \). By Theorem B, the restriction of the function \((\|x\|^{-\beta + 1})^\wedge\) to the sphere is continuous on \( S^{n-1} \).

Let us show that \((\|x\|^{-\beta + 1})^\wedge\) is a non-negative function. First let \( p \in (n - 1, n) \). Then \( \beta \in (-1, 0) \), so \( \Gamma(-\beta) > 0 \) and, by (1), \( A_\xi^{(\beta)}(0) > 0 \) for every \( \xi \in S^{n-1} \). Also \( \cos \frac{\beta \pi}{2} > 0 \), so (3) implies non-negativity.

If \( p \in (n - 2, n - 1) \) then \( \beta \in (0, 1) \), so \( \Gamma(-\beta) < 0 \). But, since the unit ball \( K \) of the space \( B \) is a convex body, the function \( A_\xi \) has maximum at zero (the central section has maximal volume among all sections perpendicular to \( \xi \); this follows for example from the Brunn-Minkowski theorem, see [S, Th. 6.1]). Therefore, the integral in (2) is less or equal to zero, and again \( A_\xi^{(\beta)}(0) \geq 0 \) for every \( \xi \in S^{n-1} \). Also \( \cos \frac{\beta \pi}{2} > 0 \), so the result follows from (3).

If \( p \in (n - 3, n - 2) \) then \( \beta \in (1, 2) \), so \( \Gamma(-\beta) > 0 \). The integral in (2) is less or equal to zero for the same reason as in the case \( \beta \in (0, 1) \), so \( A_\xi^{(\beta)}(0) \leq 0 \) for every \( \xi \in S^{n-1} \). But now \( \cos \frac{\beta \pi}{2} < 0 \).

Now we have to free ourselves from the restrictions imposed in the beginning of the proof.

Suppose that \( B \) is not infinitely smooth. We can approximate the unit ball \( K \) of \( B \) in the Hausdorff metric by infinitely smooth convex bodies \( K_m \), \( m \in \mathbb{N} \) so that \( K_m \subset K \) for every \( m \). Let \( \| \cdot \|_m \) be the norm on \( \mathbb{R}^n \) with the unit ball \( K_m \). Since \( p < n \), the functions \( \|x\|_m^p \) are locally integrable on \( \mathbb{R}^n \). Hence, for every test function \( \phi \), the functions \( \|x\|_m^{-p} \widehat{\phi}(x) \) are integrable on \( \mathbb{R}^n \). Also these functions are majorated by an integrable function \( \|x\|^{-p} |\phi(x)| \). By definition of the Fourier transform of distributions and the dominated convergence theorem, for every non-negative test function \( \phi \) and every \( p \in [n - 3, n) \),

\[
\langle (\|x\|^{-p})^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|^{-p} \widehat{\phi}(x) \, dx = \\
\lim_{m \to -\infty} \int_{\mathbb{R}^n} \|x\|_m^{-p} \widehat{\phi}(x) \, dx = \lim_{m \to -\infty} \langle (\|x\|_m^{-p})^\wedge, \phi \rangle \geq 0
\]

because we have already proved that the distributions \( \|x\|_m^{-p} \) are positive definite.

Finally, let us show that the statement of Theorem 2 is true for \( p = n - 3, n - 2, n - 1 \). Suppose that \( 0 < p < n \) and \( p_i \) is a sequence of numbers that are not integers, belong to \( (n - 3, n - 2, n - 1) \), and \( \lim_{i \to \infty} p_i = p \). We can assume that there exists \( \epsilon > 0 \) so that \( 0 < p_i < p + \epsilon < n \) for every \( i \). Fix a non-negative test function \( \phi \). Then for every \( i \in \mathbb{N} \) we have \( \langle (\|x\|^{-p_i})^\wedge, \phi \rangle \geq 0 \). Define a function \( g \) on \( \mathbb{R}^n \) by \( g(x) = \|x\|^{-p_i} |\widehat{\phi}(x)| \) if \( |x| \leq 1 \), and \( g(x) = |\phi(x)| \) if \( |x| > 1 \). Since \( \|x\|^{-p} \) is a locally integrable function, the function \( g \) is integrable on \( \mathbb{R}^n \) and, for every \( i \in \mathbb{N}, x \in \mathbb{R}^n \), we have \( g(x) \geq \|x\|^{-p_i} |\widehat{\phi}(x)| \). By the dominated convergence
\[
\langle (\|x\|^{-p})^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|^{-p} \hat{\phi}(x) \, dx = \\
\lim_{i \to \infty} \int_{\mathbb{R}^n} \|x\|^{-p} \hat{\phi}(x) \, dx = \lim_{i \to \infty} \langle (\|x\|^{-p})^\wedge, \phi \rangle \geq 0,
\]
so \((\|x\|^{-p})^\wedge\) is a positive distribution, since we have already proved that \((\|x\|^{-p})^\wedge\) is positive for every \(i\).

\[\square\]

Theorem 1 immediately follows from Theorems A and 2. If \(n = 2\) the statement of Theorem 1 remains valid for \(p \in (0, 2)\), and the inequality for the expectations reverses if \(p \in (-\min(1, q), 0)\). To see that, note that every two-dimensional normed space embeds isometrically in \(L_1\), and use [K, Corollary 2(ii)] and [K, Proposition 1]. Also, note that if \(p \geq n\) in Theorem 1, then the function \(\|x\|^{-p}\) is not locally integrable, and the expectations do not exist.

References


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