Inference on Consensus Ranking of Distributions

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Econometrics Seminar
Fall 2020

“Better”

Definitions and Consensus

“Better”?

Two distributions (of earnings, productivity, . . .)

Which is “better”?

“Better”?

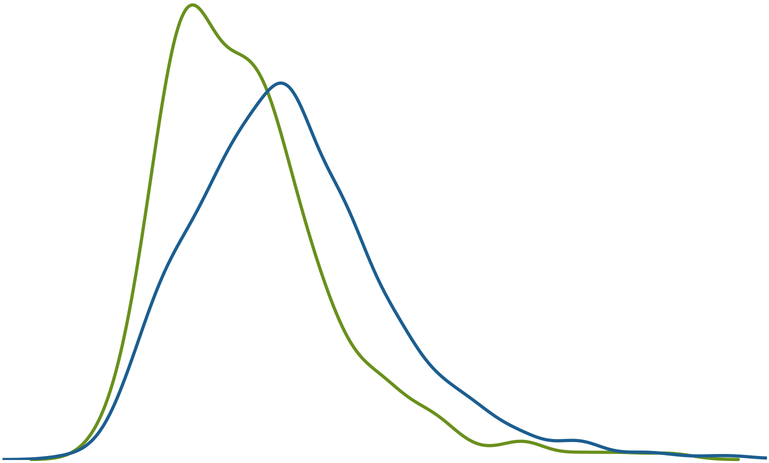
Two distributions (of earnings, productivity, ...)

Which is “better”?

- Would you prefer to (live there, buy this, use that, ...)

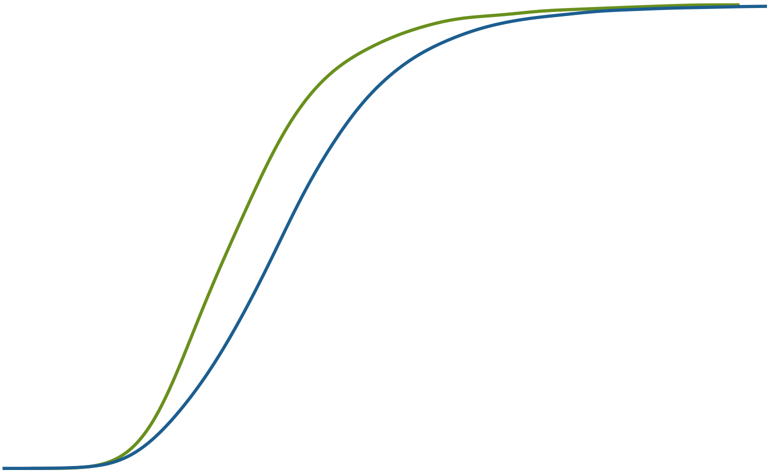
"Better"?

2 PDFs



“Better”?

2 CDFs



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Most: utility restricted stochastic dominance ($\mathrm{SD}_{\mathcal{D}}$)

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"Deductive": fix $\mathcal{D}$, test $\mathrm{SD}_{\mathcal{D}}$

"Inductive": learn about the true $\mathcal{D}$

CDFs (Atkinson, 1987, §1)

Poverty line: v

Headcount poverty: $F_Y(v)$ and $F_Z(v)$

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Condition I of Atkinson (1987, p. 751)

Deductive (Davidson and Duclos, 2013)
Inductive (Goldman and Kaplan, 2018)

Brief Tangent: Economic Inequality

Literature on measuring inequality, comparing distributions

Similar issue (me/you/all/most), like

- ϵ of Atkinson (1970, p. 257)
- α of Cowell and Flachaire (2017, §4.3)

Another Tangent (sorry): Quantiles

~~Expected~~ Quantile utility maximization

Manski (1988), Rostek (2010), de Castro and Galvao (2019), et al.

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My τ, your τ, ...

All (SD_1): $\mathrm{Q}_\tau(Y) \geq \mathrm{Q}_\tau(Z)$ for all $\tau \in (0,1)$

Consensus: $\mathrm{Q}_\tau(Y) \geq \mathrm{Q}_\tau(Z)$ for $\tau \in \mathcal{T}$

Inference

Learning from Data

Literature: Testing

Two features in common:

- Single H_0: all-or-nothing
- CDF-based

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$H_0\colon Y$ SD$_1$ Z (or SD$_2$, ...)

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- Barrett and Donald (2003), many others
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$H_0\colon Y$ *non*SD$_1$ Z ($H_1\colon Y$ SD$_1$ Z)

- Davidson and Duclos (2013) (and Kaur, Prakasa Rao, and Singh, 1994, et al.)
- Want stronger evidence for SD$_1$ (analog: $H_0\colon \beta = 0$)
- Actually SD$_{[v_1,v_2]}$

New Perspectives

~~Single H_0~~ Multiple testing

- Goldman and Kaplan (2018)
- $H_{0v}\colon F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- Learn about $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$ (Y $\text{SD}_{\mathcal{V}}$ Z)

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~~CDF-based~~ Utility-based

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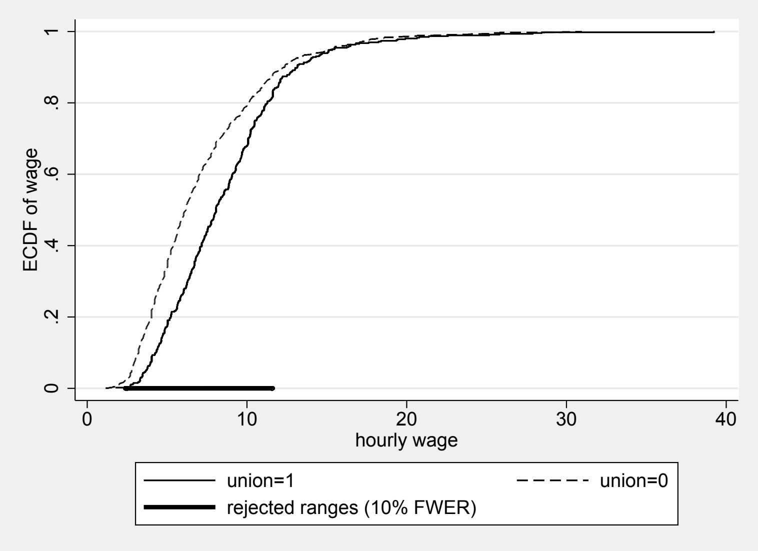
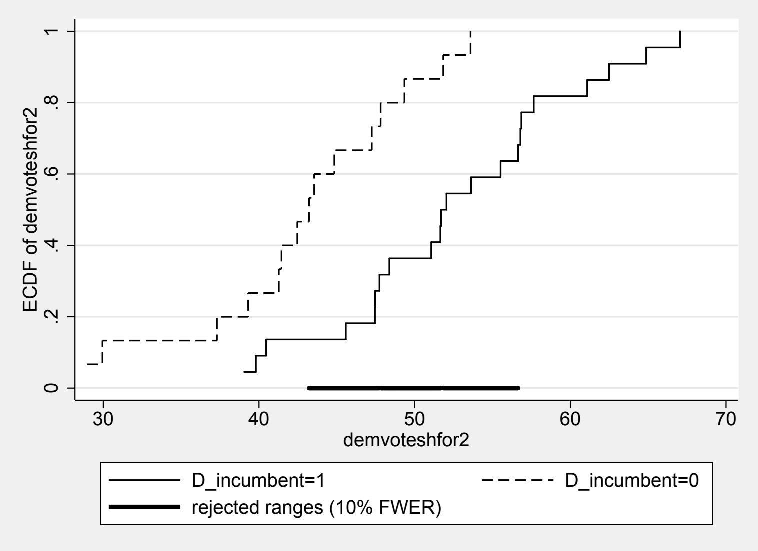
Quantile: learn about $\mathcal{T} \equiv \{\tau : \text{Q}_\tau(Y) > \text{Q}_\tau(Z)\}$

Examples (`distcomp` in Stata)



Examples (`distcomp` RDD)



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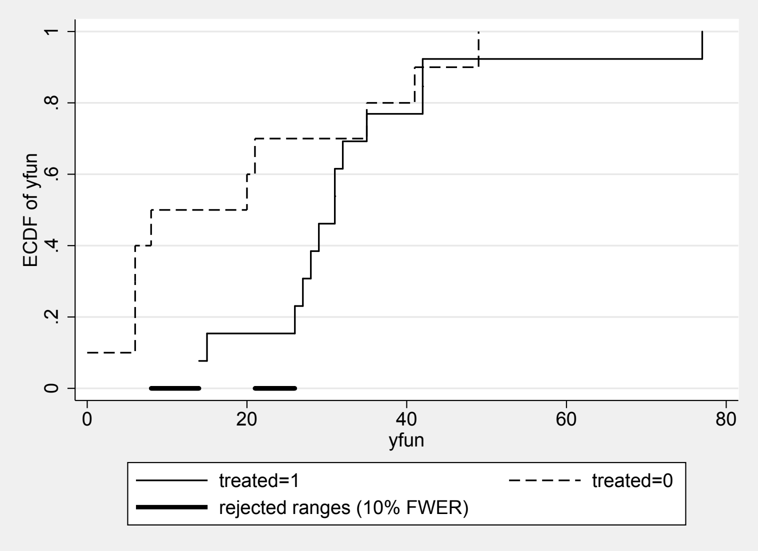
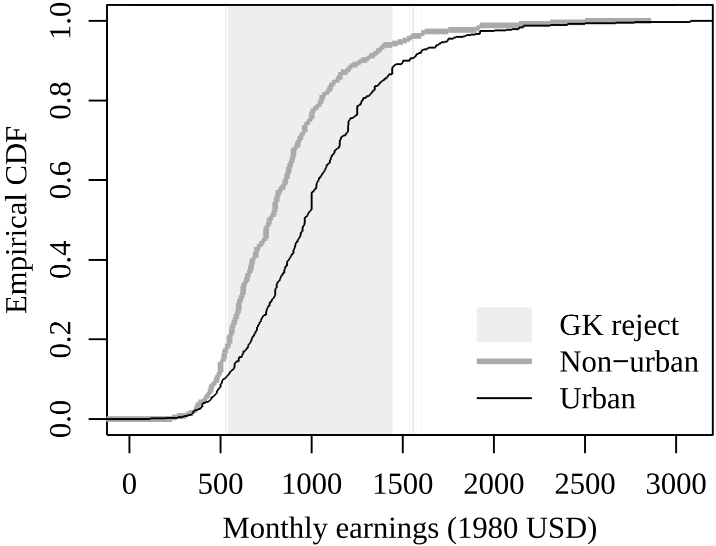
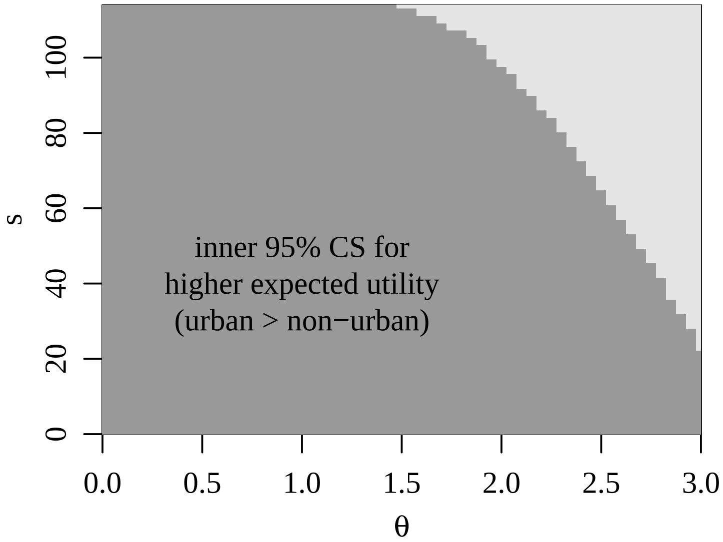


Examples (CDF)



Examples (utility, CRRA $u(x - s)$)



CDF vs. Utility: Complementary Strengths

Economic interpretation

Top-coding; tail measurement error

Unified framework for SD variants

Choice of $\mathcal{U}$:

- Economic restrictions
- Donsker: Cor. 3.1 of van der Vaart (1996)
- Computational limits; "sieve"?

Multiple Testing Goal

Multiple testing procedure (MTP)

- Test $H_{0v}\colon F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
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Familywise error rate (FWER)

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- "Strong control": $\text{FWER} \leq \alpha$ regardless of $\mathcal{V}$

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Expected utility version

- Test $H_{0u}\colon \text{E}[u(Y)] \leq \text{E}[u(Z)]$ for each $u \in \mathcal{U}$

MTP vs. All-or-Nothing Test

If $H_0\colon Y\ \mathrm{SD}_1\ Z$ rejected:

- MTP shows where/why (which v or u)

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If $H_0\colon Y \text{ SD}_1 Z$ not rejected:

- MTP shows evidence favoring $Y \text{ SD}_1 Z$ vs. just uncertainty
- "Reject $H_0\colon Z \text{ SD}_1 Y$" is a crude version of this idea
- Non-rejection may be type II error if small sample, etc.

“Outer Confidence Set” (CDF)

Usual

- Object of interest: $\boldsymbol{\theta} \in \mathbb{R}^k$
- Goal: $1 - \alpha \leq \mathrm{P}(\boldsymbol{\theta} \in \hat{\mathcal{C}})$
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"Outer CS"

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“2-sided CS”: $1 - \alpha \leq \mathrm{P}(\hat{\mathcal{V}}_1 \subseteq \mathcal{V} \subseteq \hat{\mathcal{V}}_2)$

- Combine $1 - \alpha/2$ inner & outer (Bonferroni)

Confidence Sets (Expected Utility)

Same arguments but with $\mathcal{D}$ instead of $\mathcal{V}$

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CS/MTP from Uniform Confidence Band

Uniform confidence band for $\Delta(\cdot) \implies$ CS/MTP

- $\Delta(v) \equiv F_Z(v) - F_Y(v)$ $\quad \mathcal{V} = \{v : \Delta(v) > 0\}$
- $\Delta(u) \equiv \mathrm{E}[u(Y)] - \mathrm{E}[u(Z)]$ $\quad \mathcal{D} = \{u : \Delta(u) > 0\}$
- Inner CS: values where lower band above zero
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Information vs. comprehension

- EU band more informative, CS/MTP easier to comprehend

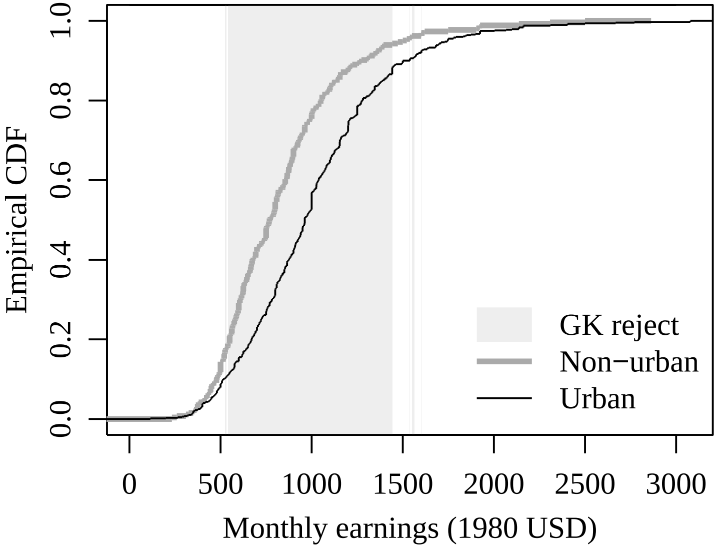
Availability

- CDF diff: asymptotic band, but finite-sample CS/MTP

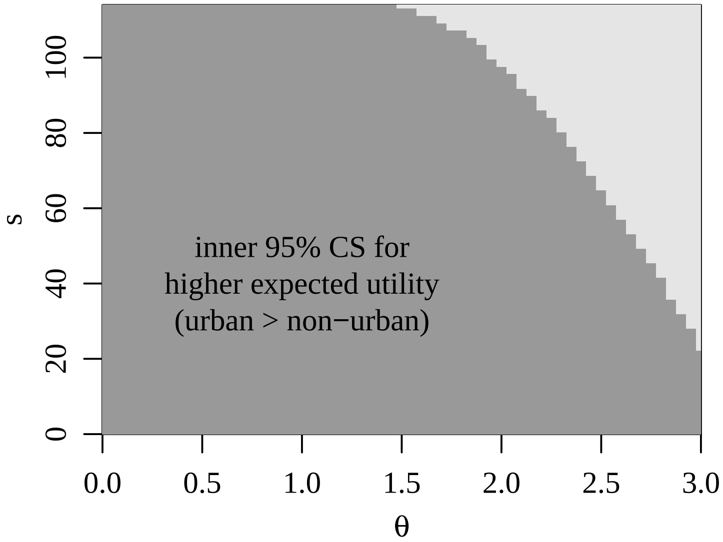
Refinements

- Stepdown, pre-test, …

CDF-based CS/MTP



Utility-based CS/MTP



Details

Theoretical & Otherwise

Skip

CDF: KS vs. Probability Integral Transform

Kolmogorov–Smirnov MTP/CS

- Reject H_{0v} when $\hat{F}_Y(v) - \hat{F}_Z(v)$ exceeds KS critical value
- Prop. 3 of Goldman and Kaplan (2018)

KS: well-known low tail power

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  ks.test(c(1:15/21,10^6+1:5),punif)
  D = 0.25, p-value = 0.1376
  ```
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If cts, $F_Y(Y_i) \sim \text{Unif}(0,1)$

- Retain finite-sample properties
- Power more even than KS across distribution
- Goldman and Kaplan (2018): two-sample MTP, RDD, computation
  ```

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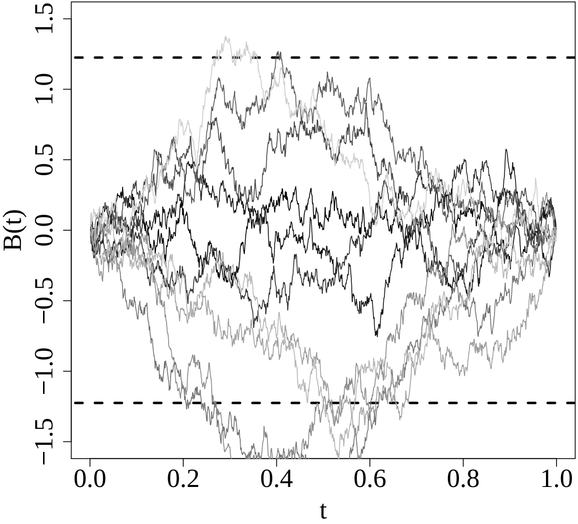

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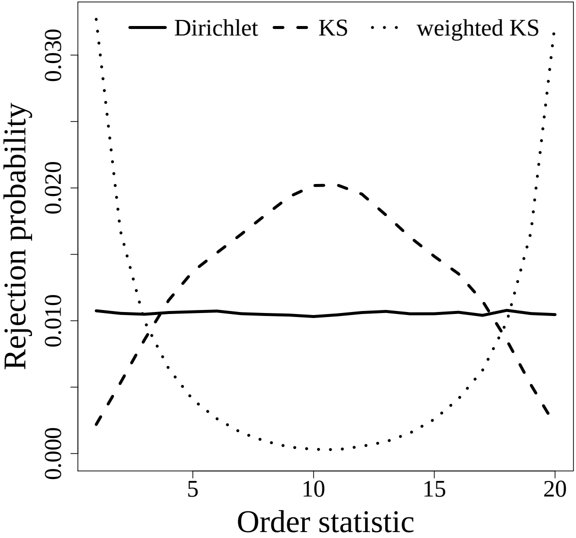

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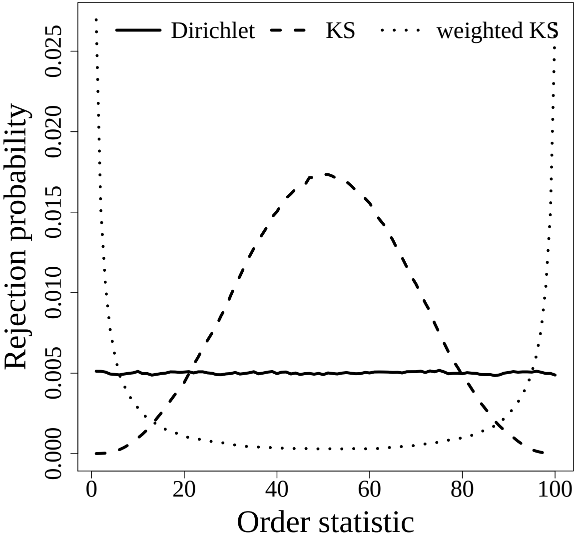
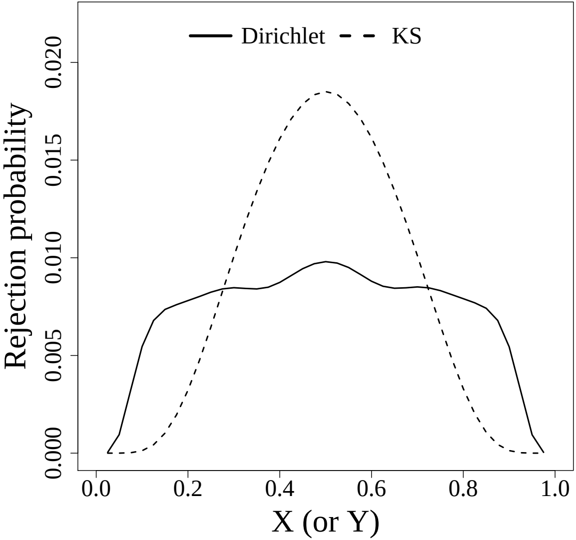
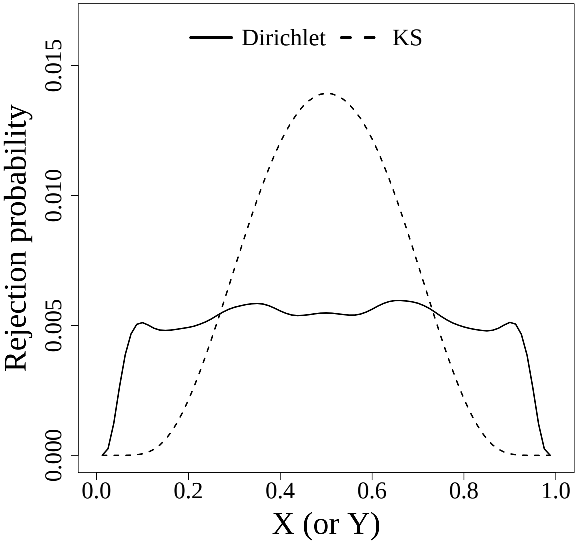


CDF: KS vs. Probability Integral Transform



CDF: KS vs. Probability Integral Transform



Expected Utility: Asymptotics

CDF:

- $\Delta(v) \equiv F_Z(v) - F_Y(v) = \mathrm{E}[\mathbb{1}\{Z \leq v\}] - \mathrm{E}[\mathbb{1}\{Y \leq v\}]$
- $\Delta(f) = \mathrm{E}[f(Z)] - \mathrm{E}[f(Y)]$, $f(x) = \mathbb{1}\{x \leq v\}$
- $\{f_v(\cdot) : f_v(t) = \mathbb{1}\{t \leq v\}, v \in \mathbb{R}\}$ is Donsker
- $\hat{\Delta}(\cdot)$: Gaussian limit

EU:

- $\Delta(u) \equiv \mathrm{E}[u(Y)] - \mathrm{E}[u(Z)]$
- $\hat{\Delta}(\cdot)$: Gaussian limit and bootstrap consistency if Donsker $\mathcal{U}$

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EU:

- $\Delta(u) \equiv \mathrm{E}[u(Y)] - \mathrm{E}[u(Z)]$
- $\hat{\Delta}(\cdot)$: Gaussian limit and bootstrap consistency if Donsker $\mathcal{U}$

Cor. 3.1 of van der Vaart (1996): $\mathcal{U}$ Donsker if

- non-decreasing
- bounded from below (or above)
- $2 + \delta$ moments of envelope function

Expected Utility: MTP

$H_{0u}\colon \Delta(u) \equiv \mathrm{E}[u(Y)] - \mathrm{E}[u(Z)] \leq 0$, each $u \in \mathcal{U}$

Define pointwise t-statistics:

- $\hat{T}_u = [\hat{\Delta}(u) - \Delta(u)]/\widehat{\mathrm{SE}}_u$
- $\hat{T}_u^0 = \hat{\Delta}(u)/\widehat{\mathrm{SE}}_u$

Bootstrap cv: $1 - \alpha$ quantile of $\sup_{u \in \mathcal{U}} \hat{T}_u$

FWER $= \mathrm{P}(\text{reject any true}) \leq \mathrm{P}(\sup_u \hat{T}_u > \text{cv}) \to \alpha$

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Stepdown (Holm, 1979)

- $\leq$ maybe very conservative if many $\Delta(u) > 0$
- Re-compute bootstrap cv using only non-rejected u
- Iterate: bounded by oracle test using true $\{u : H_{0u} \text{ true}\}$

Can also pre-test to remove $\hat{\Delta}(u) \ll 0$, etc.

Expected Utility: CS

Invert MTP to get CS

Simulation

Performance of New Methods

Skip

Setup

$Y_i \overset{iid}{\sim} \log \mathrm{N}(0, 1) + 0.1,\ i = 1, \ldots, n$

$Z_i \overset{iid}{\sim} \log \mathrm{N}(\mu, \sigma) + 0.1,\ i = 1, \ldots, n$

$\mathcal{U}$: CRRA w/ risk aversion $\theta \in [0, 3]$

Band for $\Delta(u) = \mathrm{E}[u(Y)] - \mathrm{E}[u(Z)]$

- Equivalently: $\Delta(\theta)$ on $\theta \in [0, 3]$

CSs for $\mathcal{D} \equiv \{u : \mathrm{E}[u(Y)] > \mathrm{E}[u(Z)]\}$

- Equivalently: $\mathcal{D}$ is subset of $\theta \in [0, 3]$

Results: $n = 40$

			Coverage ($1-\alpha=0.9$)			
σ	μ	$\{\theta : u_\theta \in \mathcal{D}\}$	band	2s CS	inner	outer
0.7	−0.3	[0.0, 2.8]	0.873	0.960	0.968	0.992
0.7	0.0	[0.0, 1.1]	0.865	0.972	0.990	0.982
0.7	0.3	[ ]	0.855	0.998	0.998	1.000
1.0	−0.3	[0.0, 3.0]	0.920	0.999	1.000	0.999
1.0	0.0	[ ]	0.938	0.972	0.972	1.000
1.0	0.3	[ ]	0.922	0.995	0.995	1.000
1.3	−0.3	[0.2, 3.0]	0.896	0.965	0.967	0.998
1.3	0.0	[1.2, 3.0]	0.883	0.976	0.988	0.988
1.3	0.3	[2.5, 3.0]	0.861	0.962	0.994	0.968

Results: $n = 100$

			Coverage ($1-\alpha=0.9$)			
σ	μ	$\{\theta : u_\theta \in \mathcal{D}\}$	band	2s CS	inner	outer
0.7	−0.3	[0.0, 2.8]	0.907	0.968	0.975	0.993
0.7	0.0	[0.0, 1.1]	0.897	0.977	0.993	0.984
0.7	0.3	[ ]	0.908	0.999	0.999	1.000
1.0	−0.3	[0.0, 3.0]	0.934	1.000	1.000	1.000
1.0	0.0	[ ]	0.929	0.965	0.965	1.000
1.0	0.3	[ ]	0.922	1.000	1.000	1.000
1.3	−0.3	[0.2, 3.0]	0.901	0.974	0.979	0.995
1.3	0.0	[1.2, 3.0]	0.900	0.983	0.987	0.996
1.3	0.3	[2.5, 3.0]	0.887	0.964	0.992	0.972

Results: $n = 250$

			Coverage ($1-\alpha=0.9$)			
σ	μ	$\{\theta : u_\theta \in \mathcal{D}\}$	band	2s CS	inner	outer
0.7	−0.3	[0.0, 2.8]	0.920	0.978	0.983	0.995
0.7	0.0	[0.0, 1.1]	0.912	0.981	0.995	0.986
0.7	0.3	[ ]	0.893	0.998	0.998	1.000
1.0	−0.3	[0.0, 3.0]	0.920	1.000	1.000	1.000
1.0	0.0	[ ]	0.937	0.968	0.968	1.000
1.0	0.3	[ ]	0.942	1.000	1.000	1.000
1.3	−0.3	[0.2, 3.0]	0.927	0.976	0.978	0.998
1.3	0.0	[1.2, 3.0]	0.902	0.979	0.988	0.991
1.3	0.3	[2.5, 3.0]	0.892	0.974	0.994	0.980

Bonus Material

Time Permitting

Quantile Utility Maximization

Maximize $\mathrm{Q}_\tau(u(X))$ instead of $\mathrm{E}[u(X)]$

- Manski (1988)
- Rostek (2010): axiomatization
- de Castro and Galvao (2019): dynamic

Quantile Utility Maximization

Maximize $\mathrm{Q}_\tau(u(X))$ instead of $\mathrm{E}[u(X)]$

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$\mathrm{Q}_\tau(u(X)) = u(\mathrm{Q}_\tau(X)) \implies u(\cdot)$ doesn't matter (!?)

Set of preferences $\iff$ set of τ

- Learn about $\mathcal{T} = \{\tau : \mathrm{Q}_\tau(Y) \geq \mathrm{Q}_\tau(Z)\}$

Uniform Confidence Band

Object of interest: $\Delta(u) = \mathrm{E}[u(Y)] - \mathrm{E}[u(Z)]$ over $u \in \mathcal{U}$

Goal: $1 - \alpha = \mathrm{P}\{\hat{b}_1(u) \le \Delta(u) \le \hat{b}_2(u) \text{ for all } u \in \mathcal{U}\}$

Asymptotically: $\sqrt{n}(\hat{\Delta}(\cdot) - \Delta(\cdot))$ Gaussian

Alg. 3 of Chernozhukov, Fernández-Val, and Melly (2013):

- Bootstrap to get std dev and t-stat for each u
- Bootstrap absolute sup t-stat to get critical value
- Band: $\hat{\Delta}(u) \pm \mathrm{cv}_{1-\alpha}\hat{\sigma}(u)/\sqrt{n}$

Test of Restricted (non)SD

H_0: $Z \text{ SD}_{\mathcal{D}}\, Y \iff \text{E}[u(Y)] - \text{E}[u(Z)] \leq 0$ for all $u \in \mathcal{D}$
H_1: $Z\ \mathit{non}\text{SD}_{\mathcal{D}}\, Y$

- Reject $\text{SD}_{\mathcal{D}} \implies$ reject SD
- Reject when sup t-stat exceeds bootstrap sup-t cv
- (Least favorable null: all zero)

Test of Restricted (non)SD

$H_0\colon Z \text{ SD}_{\mathcal{D}}\, Y \iff \text{E}[u(Y)] - \text{E}[u(Z)] \le 0 \text{ for all } u \in \mathcal{D}$
$H_1\colon Z\ non\text{SD}_{\mathcal{D}}\, Y$

- Reject $\text{SD}_{\mathcal{D}} \implies$ reject SD
- Reject when sup t-stat exceeds bootstrap sup-t cv
- (Least favorable null: all zero)

$H_0\colon Z\ non\text{SD}_{\mathcal{D}}\, Y$ vs. $H_1\colon Z \text{ SD}_{\mathcal{D}}\, Y$

- Utility version of Davidson and Duclos (2013)
- Reject when all t-stats below $\Phi^{-1}(\alpha) < 0$
- Least favorable null: $\text{E}[u^*(Y)] - \text{E}[u^*(Z)] \downarrow 0$ for single $u^* \in \mathcal{D}$; for $u \neq u^*$ $\text{E}[u(Y)] - \text{E}[u(Z)] \ll 0$

Epilogue

Past & Future

Conclusion

"Better": restricted stochastic dominance based on

- CDF
- Expected utility

Inference on set of:

- values with lower CDF
- utility functions with higher expected utility

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Future:

- non-iid, improve power, implement richer utility family
- economic inequality
- restricted stochastic monotonicity
- other ideas?

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Thank you / further questions & comments appreciated

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