Inference on Consensus Ranking of Distributions

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“Better”

Definitions and Consensus
“Better”? 

Two distributions (of earnings, productivity, …) 

Which is “better”?
“Better”? 

Two distributions (of earnings, productivity, ...) 

Which is “better”? 

► Would you prefer to (live there, buy this, use that, ...)
“Better”? 2 PDFs
“Better”? 

2 CDFs
Expected Utility

\[ I \text{ prefer } Y \succeq Z \iff E[u(Y)] \geq E[u(Z)] \text{ for my } u(\cdot) \]
Expected Utility

I prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for my $u(\cdot)$

You prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for your $u(\cdot)$
Expected Utility

I prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for my $u(\cdot)$

You prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for your $u(\cdot)$

All prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for all $u(\cdot)$
Expected Utility

I prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for my $u(\cdot)$

You prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for your $u(\cdot)$

All prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for all $u(\cdot)$

Most prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for $u \in \mathcal{D}$
Expected Utility

I prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for my $u(\cdot)$

You prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for your $u(\cdot)$

All prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for all $u(\cdot)$

Most prefer $Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)]$ for $u \in \mathcal{D}$

All: first-order stochastic dominance (SD$_1$)

Most: utility restricted stochastic dominance (SD$_D$)
Expected Utility

I prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for my $u(\cdot)$
You prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for your $u(\cdot)$
All prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for all $u(\cdot)$
Most prefer $Y \succeq Z \iff E[u(Y)] \geq E[u(Z)]$ for $u \in \mathcal{D}$

All: first-order stochastic dominance (SD$_1$)
Most: utility restricted stochastic dominance (SD$_\mathcal{D}$)
includes second-order SD (etc.)
Expected Utility

\[ \text{I prefer } Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)] \text{ for my } u(\cdot) \]

\[ \text{You prefer } Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)] \text{ for your } u(\cdot) \]

\[ \text{All prefer } Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)] \text{ for all } u(\cdot) \]

\[ \text{Most prefer } Y \succeq Z \iff \mathbb{E}[u(Y)] \geq \mathbb{E}[u(Z)] \text{ for } u \in \mathcal{D} \]

All: first-order stochastic dominance (SD\(_1\))

Most: utility restricted stochastic dominance (SD\(_{\mathcal{D}}\)) includes second-order SD (etc.)

“Deductive”: fix \( \mathcal{D} \), test SD\(_{\mathcal{D}}\)

“Inductive”: learn about the true \( \mathcal{D} \)
CDFs (Atkinson, 1987, §1)

- Poverty line: $v$
- Headcount poverty: $F_Y(v)$ and $F_Z(v)$
CDFs (Atkinson, 1987, §1)

Poverty line: $v$

Headcount poverty: $F_Y(v)$ and $F_Z(v)$

Me: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for my $v$

You: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for your $v$

All: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for all $v$

Most: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for $v \in \mathcal{V}$
CDFs (Atkinson, 1987, §1)

Poverty line: \( v \)

Headcount poverty: \( F_Y(v) \) and \( F_Z(v) \)

Me: \( Y \succeq Z \iff F_Y(v) \leq F_Z(v) \) for my \( v \)
You: \( Y \succeq Z \iff F_Y(v) \leq F_Z(v) \) for your \( v \)
All: \( Y \succeq Z \iff F_Y(v) \leq F_Z(v) \) for all \( v \)
Most: \( Y \succeq Z \iff F_Y(v) \leq F_Z(v) \) for \( v \in \mathcal{V} \)

All: first-order stochastic dominance (SD\(_1\))
Most: **CDF restricted stochastic dominance (SD\(_V\))**

Condition I of Atkinson (1987, p. 751)
CDFs (Atkinson, 1987, §1)

Poverty line: $v$

Headcount poverty: $F_Y(v)$ and $F_Z(v)$

Me: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for my $v$

You: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for your $v$

All: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for all $v$

Most: $Y \succeq Z \iff F_Y(v) \leq F_Z(v)$ for $v \in \mathcal{V}$

All: first-order stochastic dominance (SD$_1$)

Most: CDF restricted stochastic dominance (SD$_\mathcal{V}$)

Condition I of Atkinson (1987, p. 751)

Deductive (Davidson and Duclos, 2013)

Inductive (Goldman and Kaplan, 2018)
Brief Tangent: Economic Inequality

Literature on measuring inequality, comparing distributions

Similar issue (me/you/all/most), like

- $\epsilon$ of Atkinson (1970, p. 257)
- $\alpha$ of Cowell and Flachaire (2017, §4.3)
Another Tangent (sorry): Quantiles

Expected Quantile utility maximization

Another Tangent (sorry): Quantiles

Expected Quantile utility maximization


Equivariance: $Q_\tau(u(Y)) = u(Q_\tau(Y))$

$\Rightarrow Q_\tau(u(Y)) \geq Q_\tau(u(Z)) \iff Q_\tau(Y) \geq Q_\tau(Z)$

$\Rightarrow u(\cdot)$ irrelevant (!?)

All (SD $1\leq\tau\leq1$) $Q_\tau(Y) \geq Q_\tau(Z)$ for all $\tau \in (0, 1)$

Consensus: $Q_\tau(Y) \geq Q_\tau(Z)$ for $\tau \in [0, 1]$
Another Tangent (sorry): Quantiles

Expected Quantile utility maximization


Equivariance: $Q_{\tau}(u(Y)) = u(Q_{\tau}(Y))$

- $Q_{\tau}(u(Y)) \geq Q_{\tau}(u(Z)) \iff Q_{\tau}(Y) \geq Q_{\tau}(Z)$
- $u(\cdot)$ irrelevant (!?)

My $\tau$, your $\tau$, ...

All (SD$_1$): $Q_{\tau}(Y) \geq Q_{\tau}(Z)$ for all $\tau \in (0, 1)$

Consensus: $Q_{\tau}(Y) \geq Q_{\tau}(Z)$ for $\tau \in \mathcal{T}$
Inference

Learning from Data
Two features in common:

- Single $H_0$: all-or-nothing
- CDF-based
Literature: Testing

Two features in common:

▶ Single $H_0$: all-or-nothing
▶ CDF-based

$H_0: Y \ SD_1 \ Z$ (or $SD_2, \ldots$)

▶ 1-sided Kolmogorov–Smirnov
▶ Barrett and Donald (2003), many others
▶ Good for testing economic theory that implies $SD_1$
Two features in common:

- Single $H_0$: all-or-nothing
- CDF-based

$H_0: Y \text{ SD}_1 Z$ (or $\text{SD}_2, \ldots$)

- 1-sided Kolmogorov–Smirnov
- Barrett and Donald (2003), many others
- Good for testing economic theory that implies $\text{SD}_1$

$H_0: Y \text{ nonSD}_1 Z$ ($H_1: Y \text{ SD}_1 Z$)

- Want stronger evidence for $\text{SD}_1$ (analog: $H_0: \beta = 0$)
- Actually $\text{SD}_{[v_1,v_2]}$
New Perspectives

Single $H_0$ Multiple testing

- Goldman and Kaplan (2018)
- $H_{0v} : F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- Learn about $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$ ($Y$ SD $\mathcal{V}$ $Z$)
New Perspectives

Single $H_0$ Multiple testing

- Goldman and Kaplan (2018)
- $H_{0v}: F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- Learn about $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$ ($Y \text{ SD}_\mathcal{V} Z$)

CDF-based Utility-based

- Draft circulated for this talk
- $H_{0u}: E[u(Y)] \leq E[u(Z)]$ for each $u \in \mathcal{U}$
- Learn about $\mathcal{D} \equiv \{u : E[u(Y)] > E[u(Z)]\}$ ($Y \text{ SD}_\mathcal{D} Z$)
New Perspectives

Single $H_0$ Multiple testing

- Goldman and Kaplan (2018)
- $H_{0v}: F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- Learn about $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$ ($Y \text{ SD}_\mathcal{V} Z$)

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- Learn about $\mathcal{D} \equiv \{u : E[u(Y)] > E[u(Z)]\}$ ($Y \text{ SD}_\mathcal{D} Z$)

Quantile: learn about $\mathcal{T} \equiv \{\tau : Q_\tau(Y) > Q_\tau(Z)\}$
Examples (\texttt{distcomp} in Stata)
Examples (distcomp RDD)
Examples (\texttt{distcomp experiment})
Examples (CDF)

Empirical CDF

Monthly earnings (1980 USD)

GK reject
Non–urban
Urban
Examples (utility, CRRA $u(x - s)$)

inner 95% CS for higher expected utility (urban > non-urban)
CDF vs. Utility: Complementary Strengths

- Economic interpretation
- Top-coding; tail measurement error
- Unified framework for SD variants

Choice of $\mathcal{U}$:
- Economic restrictions
- Donsker: Cor. 3.1 of van der Vaart (1996)
- Computational limits; “sieve”? 
Multiple Testing Goal

Multiple testing procedure (MTP)

- Test $H_{0v}: F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$
Multiple Testing Goal

Multiple testing procedure (MTP)

- Test $H_{0v}: F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$

Familywise error rate (FWER)

- $\text{FWER} \equiv P(\text{reject any true } H_{0v})$
- “Strong control”: $\text{FWER} \leq \alpha$ regardless of $\mathcal{V}$
Multiple Testing Goal

Multiple testing procedure (MTP)

- Test $H_{0v}: F_Y(v) \geq F_Z(v)$ for each $v \in \mathbb{R}$
- $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$

Familywise error rate (FWER)

- FWER $\equiv P(\text{reject any true } H_{0v})$
- “Strong control”: FWER $\leq \alpha$ regardless of $\mathcal{V}$

Expected utility version

- Test $H_{0u}: E[u(Y)] \leq E[u(Z)]$ for each $u \in \mathcal{U}$
MTP vs. All-or-Nothing Test

If $H_0: Y SD_1 Z$ rejected:

- MTP shows where/why (which $v$ or $u$)
MTP vs. All-or-Nothing Test

If $H_0: Y \ SD_1 \ Z$ rejected:
- MTP shows where/why (which $v$ or $u$)

If $H_0: Y \ SD_1 \ Z$ not rejected:
- MTP shows evidence favoring $Y \ SD_1 \ Z$ vs. just uncertainty
- “Reject $H_0: Z \ SD_1 \ Y$” is a crude version of this idea
- Non-rejection may be type II error if small sample, etc.
“Outer Confidence Set” (CDF)

Usual

- Object of interest: $\mathbf{\theta} \in \mathbb{R}^k$
- Goal: $1 - \alpha \leq P(\mathbf{\theta} \in \hat{C})$
- Invert test: $\hat{C} = \{c \text{ : don’t reject } H_0 : \mathbf{\theta} = c\}$
Usual

- Object of interest: $\theta \in \mathbb{R}^k$
- Goal: $1 - \alpha \leq P(\theta \in \hat{C})$
- Invert test: $\hat{C} = \{c : \text{don’t reject } H_0: \theta = c\}$

Object of interest: $\mathcal{V} \equiv \{v : F_Y(v) \leq F_Z(v)\}$
“Outer Confidence Set” (CDF)

Usual

- Object of interest: $\theta \in \mathbb{R}^k$
- Goal: $1 - \alpha \leq P(\theta \in \hat{C})$
- Invert test: $\hat{C} = \{c : \text{don’t reject } H_0 : \theta = c\}$

Object of interest: $\mathcal{V} \equiv \{v : F_Y(v) \leq F_Z(v)\}$

“Outer CS”

- Goal: $1 - \alpha \leq P(\mathcal{V} \subseteq \hat{V})$
- Invert MTP: $\hat{V} = \{v : H_{0v} \text{ not rejected}\}$
  
  $H_{0v}: F_Y(v) \leq F_Z(v)$ \hspace{1em} ($H_{0v}: v \in \mathcal{V}$)

- $P(\mathcal{V} \subseteq \hat{V}) = P(\text{no true } H_{0v} \text{ rejected}) = 1 - \text{FWER} \geq 1 - \alpha$
“Inner Confidence Set” (CDF)

Object of interest: \( \mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\} \)

“Outer CS”: \( 1 - \alpha \leq P(\mathcal{V} \subseteq \hat{\mathcal{V}}) = P(\hat{\mathcal{V}}^c \subseteq \mathcal{V}^c) \)

“Inner CS”: \( 1 - \alpha \leq P(\hat{\mathcal{V}} \subseteq \mathcal{V}) \)
“Inner Confidence Set” (CDF)

Object of interest: $\mathcal{V} \equiv \{v : F_Y(v) < F_Z(v)\}$

“Outer CS”: $1 - \alpha \leq P(\mathcal{V} \subseteq \hat{\mathcal{V}}) = P(\hat{\mathcal{V}}^c \subseteq \mathcal{V}^c)$

“Inner CS”: $1 - \alpha \leq P(\hat{\mathcal{V}} \subseteq \mathcal{V})$

- “Invert” MTP: $\hat{\mathcal{V}} = \{v : H_{0v} \text{ rejected}\}$
  $H_{0v}: F_Y(v) \geq F_Z(v) \quad (H_{0v}: v \notin \mathcal{V})$
- $P(\hat{\mathcal{V}} \subseteq \mathcal{V}) = P(\text{only reject false } H_{0v}) = 1 - \text{FWER} \geq 1 - \alpha$
- Or: $\hat{\mathcal{V}}$ is a very conservative estimate of $\mathcal{V}$
“Inner Confidence Set” (CDF)

Object of interest: \( V \equiv \{ v : F_Y(v) < F_Z(v) \} \)

“Outer CS”: \( 1 - \alpha \leq P(V \subseteq \hat{V}) = P(\hat{V}^C \subseteq V^C) \)

“Inner CS”: \( 1 - \alpha \leq P(\hat{V} \subseteq V) \)
- “Invert” MTP: \( \hat{V} = \{ v : H_{0v} \text{ rejected} \} \)
  \( H_{0v}: F_Y(v) \geq F_Z(v) \) (\( H_{0v}: v \notin V \))
- \( P(\hat{V} \subseteq V) = P(\text{only reject false } H_{0v}) = 1 - \text{FWER} \geq 1 - \alpha \)
- Or: \( \hat{V} \) is a very conservative estimate of \( V \)

“2-sided CS”: \( 1 - \alpha \leq P(\hat{V}_1 \subseteq V \subseteq \hat{V}_2) \)
- Combine \( 1 - \alpha/2 \) inner & outer (Bonferroni)
Confidence Sets (Expected Utility)

Same arguments but with $\mathcal{D}$ instead of $\mathcal{V}$

$\mathcal{D} \equiv \{ u : E[u(Y)] > E[u(Z)] \}$
Confidence Sets (Expected Utility)

Same arguments but with $\mathcal{D}$ instead of $\mathcal{V}$

$\mathcal{D} \equiv \{ u : E[u(Y)] > E[u(Z)] \}$

Inner CS: $1 - \alpha \leq P(\hat{\mathcal{D}} \subseteq \mathcal{D})$

- “Invert” MTP: $\hat{\mathcal{D}} = \{ u : H_{0u} \text{ rejected} \}$
  - $H_{0u}: E[u(Y)] \leq E[u(Z)]$ \quad ($H_{0u}: u \notin \mathcal{D}$)
- $P(\hat{\mathcal{D}} \subseteq \mathcal{D}) = P(\text{only reject false } H_{0u}) = 1 - \text{FWER} \geq 1 - \alpha$

Outer CS: $1 - \alpha \leq P(\hat{\mathcal{D}} \supseteq \mathcal{D})$

- Invert MTP: $\hat{\mathcal{D}} = \{ u : H_{0u} \text{ not rejected} \}$
  - $H_{0u}: E[u(Y)] > E[u(Z)]$ \quad ($H_{0u}: u \in \mathcal{D}$)
- $P(\hat{\mathcal{D}} \supseteq \mathcal{D}) = P(\text{reject only false } H_{0u}) = 1 - \text{FWER} \geq 1 - \alpha$
Uniform confidence band for $\Delta(\cdot) \implies$ CS/MTP

$\Delta(v) \equiv F_Z(v) - F_Y(v) \quad \forall = \{v : \Delta(v) > 0\}$

$\Delta(u) \equiv E[u(Y)] - E[u(Z)] \quad \mathcal{D} = \{u : \Delta(u) > 0\}$

- Inner CS: values where lower band above zero
- Outer CS: values where upper band above zero
- MTP: equivalent to CS like before
Uniform confidence band for $\Delta(\cdot) \implies$ CS/MTP

$\Delta(v) \equiv F_Z(v) - F_Y(v) \quad V = \{v : \Delta(v) > 0\}$

$\Delta(u) \equiv E[u(Y)] - E[u(Z)] \quad D = \{u : \Delta(u) > 0\}$

- Inner CS: values where lower band above zero
- Outer CS: values where upper band above zero
- MTP: equivalent to CS like before

Information vs. comprehension

- EU band more informative, CS/MTP easier to comprehend

Availability

- CDF diff: asymptotic band, but finite-sample CS/MTP

Refinements

- Stepdown, pre-test, …
CDF-based CS/MTP
Utility-based CS/MTP

inner 95% CS for higher expected utility (urban > non-urban)
Details

Theoretical & Otherwise
CDF: KS vs. Probability Integral Transform

Kolmogorov–Smirnov MTP/CS
- Reject $H_{0v}$ when $\hat{F}_Y(v) - \hat{F}_Z(v)$ exceeds KS critical value
- Prop. 3 of Goldman and Kaplan (2018)

KS: well-known low tail power
- `ks.test(c(1:15/21,10^6+1:5),punif)`
  - $D = 0.25$, p-value = 0.1376
CDF: KS vs. Probability Integral Transform

Kolmogorov–Smirnov MTP/CS
▶ Reject $H_{0v}$ when $\hat{F}_Y(v) - \hat{F}_Z(v)$ exceeds KS critical value
▶ Prop. 3 of Goldman and Kaplan (2018)

KS: well-known low tail power
▶ ks.test(c(1:15/21,10^6+1:5),punif)
  D = 0.25, p-value = 0.1376

If cts, $F_Y(Y_i) \sim \text{Unif}(0, 1)$
▶ Retain finite-sample properties
▶ Power more even than KS across distribution
▶ Goldman and Kaplan (2018): two-sample MTP, RDD, computation
CDF: KS vs. Probability Integral Transform
CDF: KS vs. Probability Integral Transform
CDF: KS vs. Probability Integral Transform

![Graph showing CDF comparison of Dirichlet, KS, and weighted KS methods.](image)
CDF: KS vs. Probability Integral Transform

![Graph showing CDF for KS and Dirichlet distributions.](image-url)
CDF: KS vs. Probability Integral Transform

Pointwise type I error, $nx=ny=80$, $Fx=Fy=\text{Unif}(0,1)$

Rejection probability

Dirichlet KS
Expected Utility: Asymptotics

CDF:

\[ \Delta(v) \equiv F_Z(v) - F_Y(v) = \mathbb{E}[1\{Z \leq v\}] - \mathbb{E}[1\{Y \leq v\}] \]

\[ \Delta(f) = \mathbb{E}[f(Z)] - \mathbb{E}[f(Y)], \quad f(x) = 1\{x \leq v\} \]

\( \{f_v(\cdot) : f_v(t) = 1\{t \leq v\}, v \in \mathbb{R}\} \) is Donsker

\[ \hat{\Delta}(\cdot) : \text{Gaussian limit} \]

EU:

\[ \Delta(u) \equiv \mathbb{E}[u(Y)] - \mathbb{E}[u(Z)] \]

\[ \hat{\Delta}(\cdot) : \text{Gaussian limit and bootstrap consistency if Donsker } \mathcal{U} \]
Expected Utility: Asymptotics

CDF:
- $\Delta(v) \equiv F_Z(v) - F_Y(v) = \mathbb{E}[\mathbb{1}\{Z \leq v\}] - \mathbb{E}[\mathbb{1}\{Y \leq v\}]$
- $\Delta(f) = \mathbb{E}[f(Z)] - \mathbb{E}[f(Y)], f(x) = \mathbb{1}\{x \leq v\}$
- $\{f_v(\cdot) : f_v(t) = \mathbb{1}\{t \leq v\}, v \in \mathbb{R}\}$ is Donsker
- $\hat{\Delta}(\cdot)$: Gaussian limit

EU:
- $\Delta(u) \equiv \mathbb{E}[u(Y)] - \mathbb{E}[u(Z)]$
- $\hat{\Delta}(\cdot)$: Gaussian limit and bootstrap consistency if Donsker $\mathcal{U}$

Cor. 3.1 of van der Vaart (1996): $\mathcal{U}$ Donsker if
- non-decreasing
- bounded from below (or above)
- $2 + \delta$ moments of envelope function
Expected Utility: MTP

\[ H_{0u}: \Delta(u) \equiv E[u(Y)] - E[u(Z)] \leq 0, \text{ each } u \in \mathcal{U} \]

Define pointwise t-statistics:

\[ \hat{T}_u = (\hat{\Delta}(u) - \Delta(u))/\hat{SE}_u \]
\[ \hat{T}^0_u = \hat{\Delta}(u)/\hat{SE}_u \]

Bootstrap cv: \(1 - \alpha\) quantile of \(\sup_{u \in \mathcal{U}} \hat{T}_u\)

FWER = \(P(\text{reject any true}) \leq P(\sup_u \hat{T}_u > cv) \rightarrow \alpha\)
**Expected Utility: MTP**

\[ H_{0u} : \Delta(u) \equiv E[u(Y)] - E[u(Z)] \leq 0, \text{ each } u \in \mathcal{U} \]

Define pointwise \( t \)-statistics:
- \( \hat{T}_u = [\hat{\Delta}(u) - \Delta(u)]/\hat{SE}_u \)
- \( \hat{T}_u^0 = \hat{\Delta}(u)/\hat{SE}_u \)

Bootstrap cv: \( 1 - \alpha \) quantile of sup \( u \in \mathcal{U} \) \( \hat{T}_u \)

FWER = P(reject any true) \( \leq P(\text{sup}_u \hat{T}_u > cv) \rightarrow \alpha \)

Stepdown (Holm, 1979)
- \( \leq \) maybe very conservative if many \( \Delta(u) > 0 \)
- Re-compute bootstrap cv using only non-rejected \( u \)
- Iterate: bounded by oracle test using true \( \{ u : H_{0u} \text{ true} \} \)

Can also pre-test to remove \( \hat{\Delta}(u) \ll 0 \), etc.
Expected Utility: CS

Invert MTP to get CS
Simulation

Performance of New Methods
Setup

\[ Y_i \overset{iid}{\sim} \log N(0, 1) + 0.1, \ i = 1, \ldots, n \]

\[ Z_i \overset{iid}{\sim} \log N(\mu, \sigma) + 0.1, \ i = 1, \ldots, n \]

\( \mathcal{U} \): CRRA w/ risk aversion \( \theta \in [0, 3] \)

Band for \( \Delta(u) = E[u(Y)] - E[u(Z)] \)

\( \uparrow \) Equivalently: \( \Delta(\theta) \) on \( \theta \in [0, 3] \)

CSs for \( \mathcal{D} \equiv \{ u : E[u(Y)] > E[u(Z)] \} \)

\( \uparrow \) Equivalently: \( \mathcal{D} \) is subset of \( \theta \in [0, 3] \)
Results: \( n = 40 \)

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<th>( \sigma )</th>
<th>( \mu )</th>
<th>( { \theta : u_\theta \in D } )</th>
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<td>0.995</td>
<td>1.000</td>
</tr>
<tr>
<td>1.3</td>
<td>-0.3</td>
<td>[0.2, 3.0]</td>
<td>0.896</td>
<td>0.965</td>
<td>0.967</td>
<td>0.998</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0</td>
<td>[1.2, 3.0]</td>
<td>0.883</td>
<td>0.976</td>
<td>0.988</td>
<td>0.988</td>
</tr>
<tr>
<td>1.3</td>
<td>0.3</td>
<td>[2.5, 3.0]</td>
<td>0.861</td>
<td>0.962</td>
<td>0.994</td>
<td>0.968</td>
</tr>
</tbody>
</table>
Results: $n = 100$

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\mu$</th>
<th>${ \theta : u_\theta \in D }$</th>
<th>Coverage ($1 - \alpha = 0.9$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>-0.3</td>
<td>[0.0, 2.8]</td>
<td>0.907 0.968 0.975 0.993</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0</td>
<td>[0.0, 1.1]</td>
<td>0.897 0.977 0.993 0.984</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3</td>
<td>[ ]</td>
<td>0.908 0.999 0.999 1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.3</td>
<td>[0.0, 3.0]</td>
<td>0.934 1.000 1.000 1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>[ ]</td>
<td>0.929 0.965 0.965 1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3</td>
<td>[ ]</td>
<td>0.922 1.000 1.000 1.000</td>
</tr>
<tr>
<td>1.3</td>
<td>-0.3</td>
<td>[0.2, 3.0]</td>
<td>0.901 0.974 0.979 0.995</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0</td>
<td>[1.2, 3.0]</td>
<td>0.900 0.983 0.987 0.996</td>
</tr>
<tr>
<td>1.3</td>
<td>0.3</td>
<td>[2.5, 3.0]</td>
<td>0.887 0.964 0.992 0.972</td>
</tr>
</tbody>
</table>
Results: $n = 250$

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\mu$</th>
<th>${\theta : u_\theta \in D}$</th>
<th>Coverage $(1 - \alpha = 0.9)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>-0.3</td>
<td>[0.0, 2.8]</td>
<td>0.920 0.978 0.983 0.995</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0</td>
<td>[0.0, 1.1]</td>
<td>0.912 0.981 0.995 0.986</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3</td>
<td>[   ]</td>
<td>0.893 0.998 0.998 1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.3</td>
<td>[0.0, 3.0]</td>
<td>0.920 1.000 1.000 1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0</td>
<td>[   ]</td>
<td>0.937 0.968 0.968 1.000</td>
</tr>
<tr>
<td>1.0</td>
<td>0.3</td>
<td>[   ]</td>
<td>0.942 1.000 1.000 1.000</td>
</tr>
<tr>
<td>1.3</td>
<td>-0.3</td>
<td>[0.2, 3.0]</td>
<td>0.927 0.976 0.978 0.998</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0</td>
<td>[1.2, 3.0]</td>
<td>0.902 0.979 0.988 0.991</td>
</tr>
<tr>
<td>1.3</td>
<td>0.3</td>
<td>[2.5, 3.0]</td>
<td>0.892 0.974 0.994 0.980</td>
</tr>
</tbody>
</table>
Bonus Material

Time Permitting
Quantile Utility Maximization

Maximize $Q_\tau(u(X))$ instead of $E[u(X)]$

- Manski (1988)
- Rostek (2010): axiomatization
- de Castro and Galvao (2019): dynamic
Quantile Utility Maximization

Maximize $Q_\tau(u(X))$ instead of $E[u(X)]$

- Manski (1988)
- Rostek (2010): axiomatization
- de Castro and Galvao (2019): dynamic

$$Q_\tau(u(X)) = u(Q_\tau(X)) \implies u(\cdot) \text{ doesn’t matter (!?)}$$

Set of preferences $\iff$ set of $\tau$

- Learn about $\mathcal{T} = \{\tau : Q_\tau(Y) \geq Q_\tau(Z)\}$
Object of interest: $\Delta(u) = E[u(Y)] - E[u(Z)]$ over $u \in \mathcal{U}$

Goal: $1 - \alpha = P\{\hat{b}_1(u) \leq \Delta(u) \leq \hat{b}_2(u) \text{ for all } u \in \mathcal{U}\}$

Asymptotically: $\sqrt{n}(\hat{\Delta}(\cdot) - \Delta(\cdot))$ Gaussian

Alg. 3 of Chernozhukov, Fernández-Val, and Melly (2013):

- Bootstrap to get std dev and $t$-stat for each $u$
- Bootstrap absolute sup $t$-stat to get critical value
- Band: $\hat{\Delta}(u) \pm cv_{1-\alpha} \hat{\sigma}(u) / \sqrt{n}$
Test of Restricted (non)SD

\[ H_0 : Z \ SD_D \ Y \iff E[u(Y)] - E[u(Z)] \leq 0 \text{ for all } u \in D \]
\[ H_1 : Z \ nonSD_D \ Y \]

- Reject SD_D \implies reject SD
- Reject when sup t-stat exceeds bootstrap sup-t cv
- (Least favorable null: all zero)
Test of Restricted (non)SD

\[ H_0: Z \text{ SD}_D Y \iff E[u(Y)] - E[u(Z)] \leq 0 \text{ for all } u \in D \]
\[ H_1: Z \text{ nonSD}_D Y \]

- Reject SD_D \implies reject SD
- Reject when sup t-stat exceeds bootstrap sup-t cv
- (Least favorable null: all zero)

\[ H_0: Z \text{ nonSD}_D Y \text{ vs. } H_1: Z \text{ SD}_D Y \]

- Utility version of Davidson and Duclos (2013)
- Reject when all t-stats below \( \Phi^{-1}(\alpha) < 0 \)
- Least favorable null: \( E[u^*(Y)] - E[u^*(Z)] \downarrow 0 \) for single \( u^* \in D \); for \( u \neq u^* \) \( E[u(Y)] - E[u(Z)] \ll 0 \)
Epilogue

Past & Future
Conclusion

“Better”: restricted stochastic dominance based on
- CDF
- Expected utility

Inference on set of:
- values with lower CDF
- utility functions with higher expected utility

Future:
- non-iid, improve power, implement richer utility family
- economic inequality
- restricted stochastic monotonicity
- other ideas?

Thank you / further questions & comments appreciated
Conclusion

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References I


References III


