Bayesian and frequentist inequality tests with ordinal data

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Outline

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Motivation: self-reported health status (SRHS)

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- "Our interest in health inequality stems from a more general interest in the distribution of welfare."
- SRHS is 1) "useful over the complete adult life cycle" and 2) strongly correlated with more objective measures (mortality, activities of daily living, etc.).
- Interested in "whether inequality in health status. . . increases with age" as well as "across socioeconomic groups."
- "Plausible that health shocks have both permanent and transitory components. . . the former implies that health status will be nonstationary. . . dispersion of health status will grow with age."

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- Goal: "to document the evolution of [SRHS] with age, looking at both cohort means and within-cohort dispersion."
- "Although some health shocks will have only temporary effects, others will leave a permanent residue, so that even if this residue is a small component of the original shock, the resulting health status will be non-stationary. . . . health of members of a cohort will disperse over time."

SRHS: empirical mean and variance?

From Deaton and Paxson (1998a), SRHS vs. age, males:

Mean   Variance

Data: Panel Study of Income Dynamics (PSID), 1984–92

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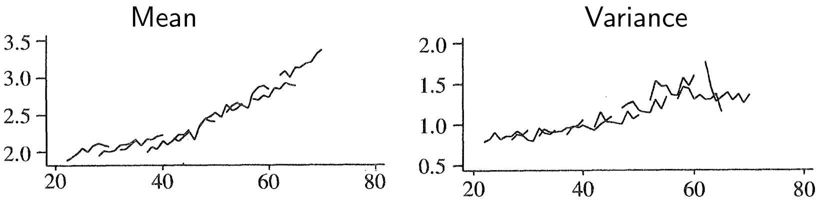

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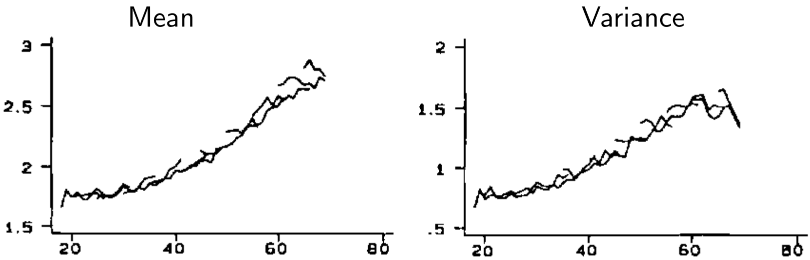

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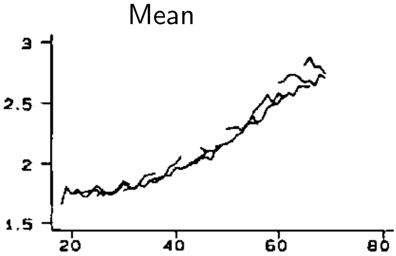

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- Dispersion increases with age?
- But: variance depends on cardinal values; SRHS is ordinal, "values" (1=excellent, ..., 5=poor) are arbitrary.

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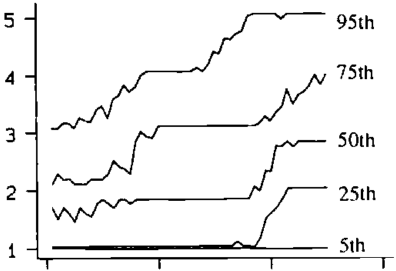

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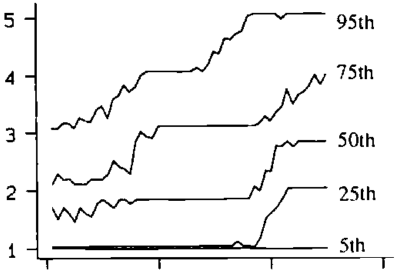

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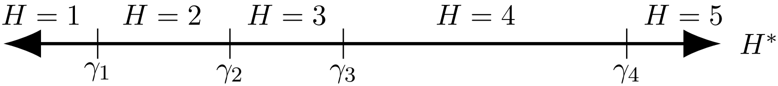
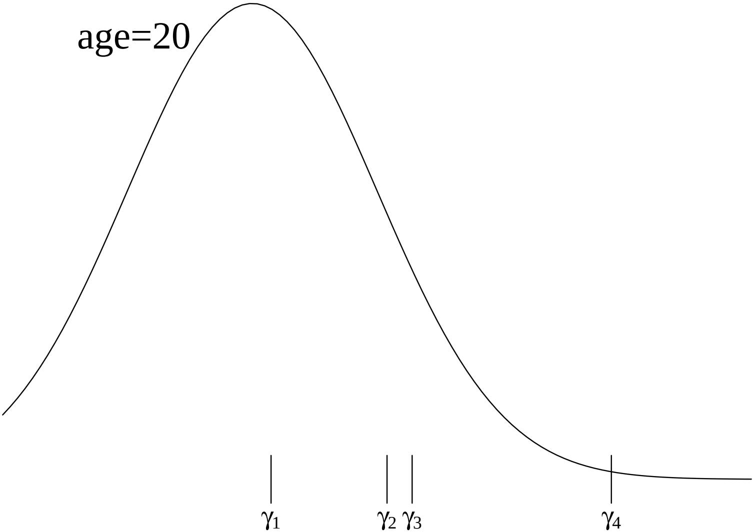
- Assume latent health H^*, SRHS H, fixed thresholds γ_j:



SRHS: simulated pure latent location shift



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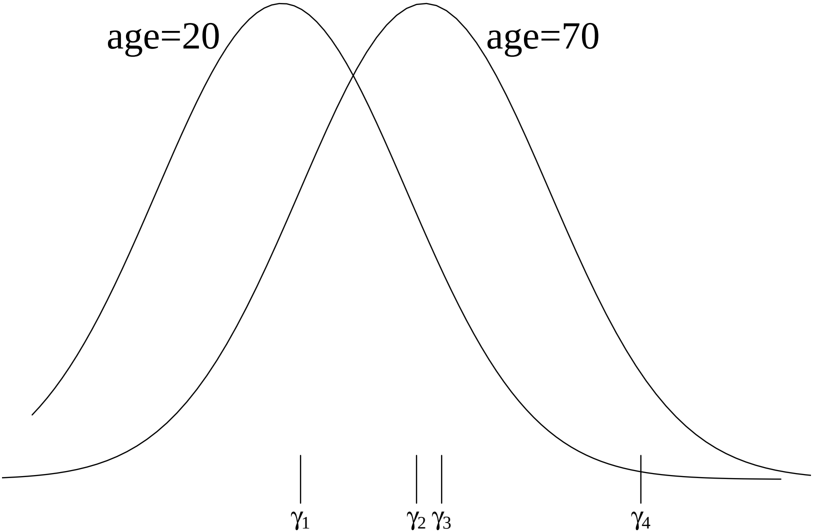
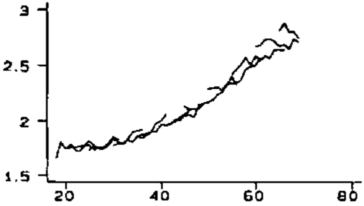
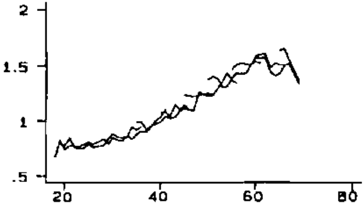


SRHS: empirical vs. simulated pure latent location shift

Mean



Variance



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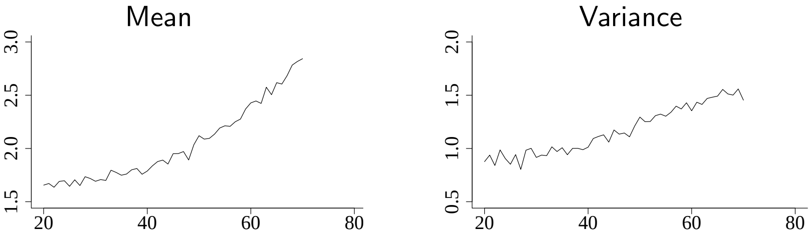
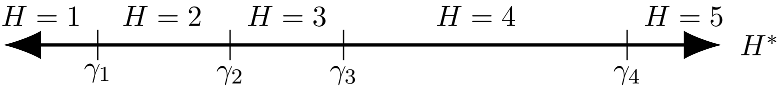


DGP: for ages $a = 20, \ldots, 70$, sample 1000 iid $\mathrm{N}(\mu_a, 1)$ each for increasing μ_a, convert to ordinal using fixed thresholds.

First-order stochastic dominance (SD1)



- Pure location shift of $H^* \implies$ SD1 in $H^* \implies$ SD1 in H.
- Proof: picture. $F(j) = F^*(\gamma_j)$, so $F_1^*(\gamma_j) \le F_2^*(\gamma_j) \iff F_1(j) \le F_2(j)$

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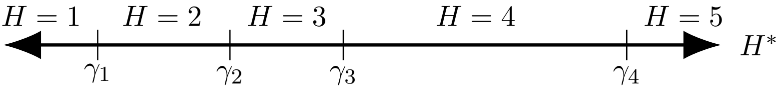

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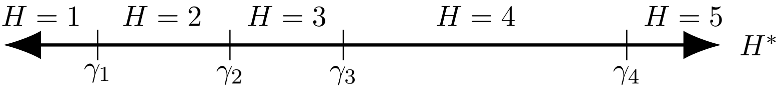

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- Latent SD1 can also suggest the other type of health inequality ("across socioeconomic groups").
- Rejecting ordinal SD1 $\implies$ rejecting latent SD1 and pure location shift.
- But ordinal SD1 does not imply latent SD1; latent SD1 is refutable but non-verifiable (without further assumptions).

Ordinal SD1 inference

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- Ordinal SD1: $H_2 \ \mathrm{SD}_1 \ H_1 \iff \theta_j \leq 0, \ j = 1, 2, 3, 4$.
- Can use recent moment inequality tests: Andrews and Barwick (2012), Romano, Shaikh, and Wolf (2014), McCloskey (2015), et al.
- Bayesian inference: Dirichlet–multinomial model.

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- Frequentist "too aggressive": Lindley's (1957) paradox and Berger and Sellke (1987), testing point (or small interval) hypothesis with prior $\mathrm{P}(H_0) = 1/2$; although Casella and Berger (1987b) disagree that $\mathrm{P}(H_0)$ is "objective."

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- Similar/same: Casella and Berger (1987a) one-sided hypothesis (scalar); Berger, Brown, and Wolpert (1994) conditional frequentist.

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- Allison and Foster (2004): "median-preserving spread" (MPS), called "the breakthrough in analyzing inequality with [SRHS] data" by Madden (2014, p. 206).
- SRHS-based inequality indexes (compute scalar summary of "inequality" based on ordinal probabilities): Abul Naga and Yalcin (2008), Reardon (2009), Silber and Yalonetzky (2011), Lazar and Silber (2013), Lv, Wang, and Xu (2015), and Yalonetzky (2016). Provides complete ordering of distributions, but many possible indexes and weighting parameters/functions, implicit assumptions.

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- Only Lazar and Silber (2013) mention statistical inference (w/o formal justification).

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- Meanwhile: we have these combinations of inequalities; does it matter (practically) if we use frequentist or Bayesian methods?

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- Limit experiment: ignore influence of prior (for now) (kind of).

Methods: decision-theoretic setup

- Test $H_0\colon \boldsymbol{\theta} \in \Theta_0$ vs. $H_a\colon \boldsymbol{\theta} \notin \Theta_0$.
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- This is the "Bayesian test" we consider.
- (Posterior is treated like p-value.)

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- If H_0 true: risk is rejection probability (RP) times $1-\alpha$ loss; RP bounded above by size.
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- Thus: size α gives max risk $\alpha(1-\alpha)$ in both cases.
- Even without unbiasedness, still approximately true (e.g., want size 0.052 instead of $\alpha = 0.05$).

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- $F(\cdot)$: continuous CDF, support $\mathbb{R}$, symmetry $F(-x) = 1 - F(x)$.
- F: properties satisfied by $\mathrm{N}(0, \sigma^2)$.

Assumptions

- Easier to have BvM with drifting centering value than drifting DGP:

$$\mathbf{X}_n = \sqrt{n}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_{0,n}) = \overbrace{\sqrt{n}(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})}^{\xrightarrow{d} \mathrm{N}(\mathbf{0}, \underline{\boldsymbol{\Sigma}})} + \overbrace{\sqrt{n}(\boldsymbol{\mu} - \boldsymbol{\mu}_{0,n})}^{\equiv \boldsymbol{\theta}_n \to \boldsymbol{\theta}}$$

$$\xrightarrow{d} \overbrace{\mathbf{X} \sim \mathrm{N}(\boldsymbol{\theta}, \underline{\boldsymbol{\Sigma}})}^{\text{limit experiment}}, \quad \underline{\boldsymbol{\Sigma}} \text{ known or } \hat{\underline{\boldsymbol{\Sigma}}} \xrightarrow{p} \underline{\boldsymbol{\Sigma}}.$$

- Posterior: $\boldsymbol{\theta}_n = \sqrt{n}(\boldsymbol{\mu} - \boldsymbol{\mu}_{0,n})$,

$$\boldsymbol{\theta}_n - \mathbf{X}_n = \sqrt{n}(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}) \xrightarrow{d} \mathrm{N}(\mathbf{0}, \underline{\boldsymbol{\Sigma}}).$$

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$$\theta \mid X \sim \mathrm{N}\left(\frac{\tau^2 X + m}{\tau^2 + 1}, \frac{\tau^2}{\tau^2 + 1}\right),$$
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- Parametric BvM: Theorem 10.1 in van der Vaart (1998, §10.2) and Theorems 20.1–3 in DasGupta (2008, §20.2).
- Semiparametric BvM: Shen (2002), Bickel and Kleijn (2012), Castillo and Rousseau (2015); Hahn (1997, Thm. G), Kwan (1999, Thm. 2), Kim (2002, Prop. 1), Lancaster (2003, Ex. 2), Schennach (2005, p. 36), Sims (2010, Sec. III.2), Norets (2015, Thm. 1).

Outline

Theorem: part (i), scalar case

- Let $X, \theta \in \mathbb{R}$, $X \sim \mathrm{N}(\theta, 1)$, $\theta \sim \mathrm{N}(X, 1)$, $H_0\colon \theta \leq 0$, $\alpha = 10\%$.

Theorem: part (i), scalar case

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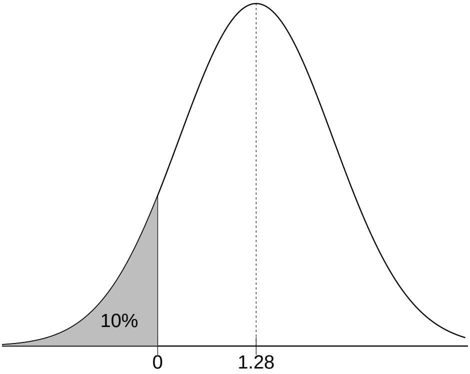

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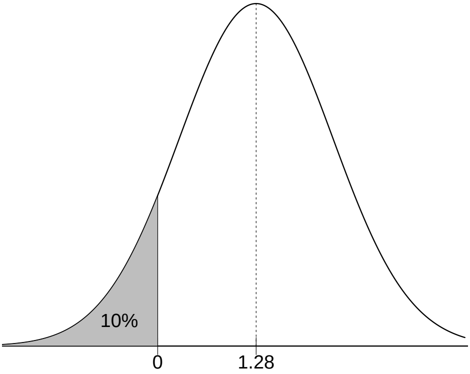

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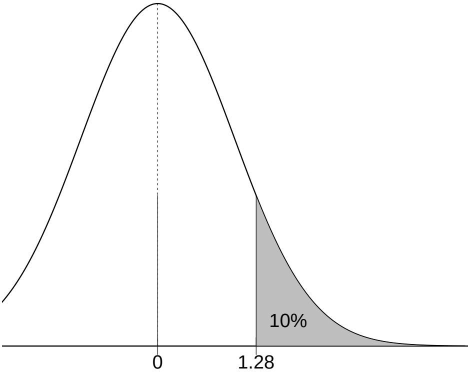

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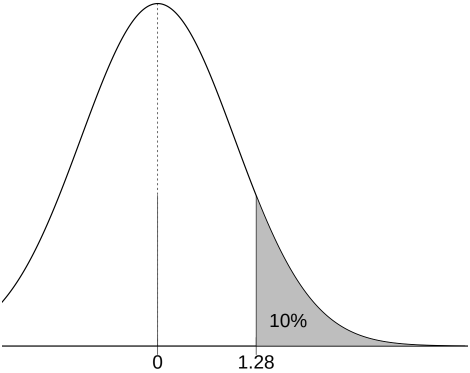
- $\mathrm{P}(H_0 \mid X = 1.28) = \alpha \implies$ reject iff $X \geq 1.28$.
- Size: $\sup_{\theta \leq 0} \mathrm{P}(\text{reject} \mid \theta) = \mathrm{P}(X \geq 1.28 \mid \theta = 0) = \alpha$.



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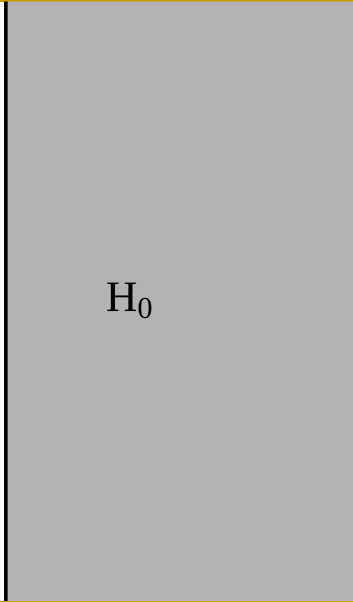
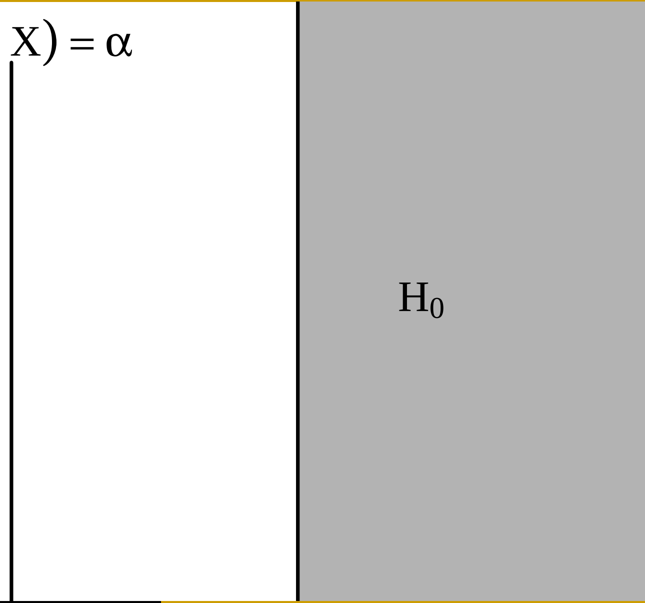
- Ex: $\mathbf{X} \sim \mathrm{N}(\boldsymbol{\theta}, \underline{\mathbf{V}})$, then for constant vector $\mathbf{c}$, $\mathbf{c}'\mathbf{X} \sim \mathrm{N}(\mathbf{c}'\boldsymbol{\theta}, \mathbf{c}'\underline{\mathbf{V}}\mathbf{c})$, scalar Gaussian.
- Ex: if $X(\cdot)$ is Gaussian process, then $X(r)$ is scalar Gaussian. So is $\phi(X(\cdot))$ if ϕ belongs to the dual of the Banach space of $X(\cdot)$.

Theorem: part (ii,iii)



Theorem: part (ii,iii)

$$P(H_0 \mid X) = \alpha$$



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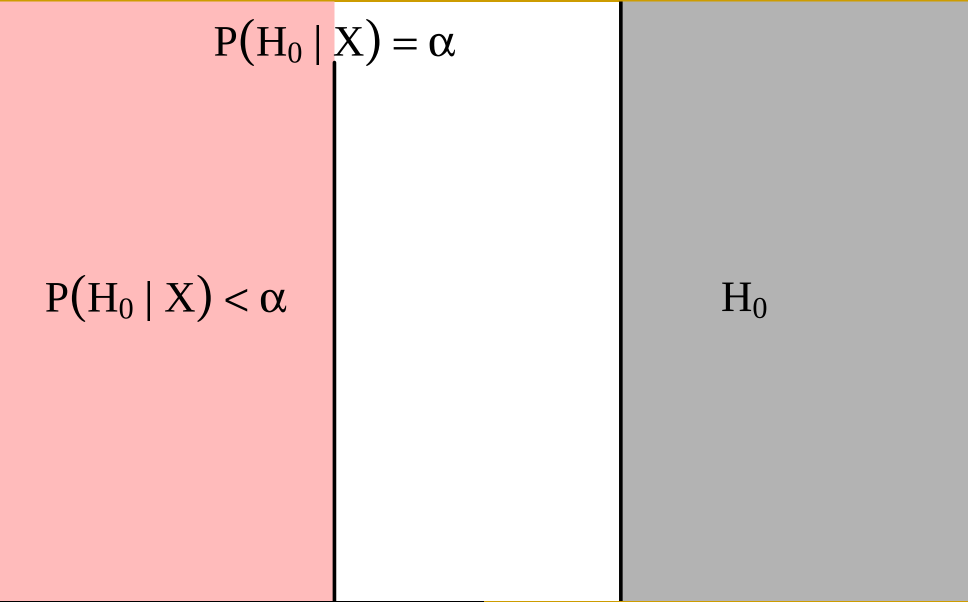

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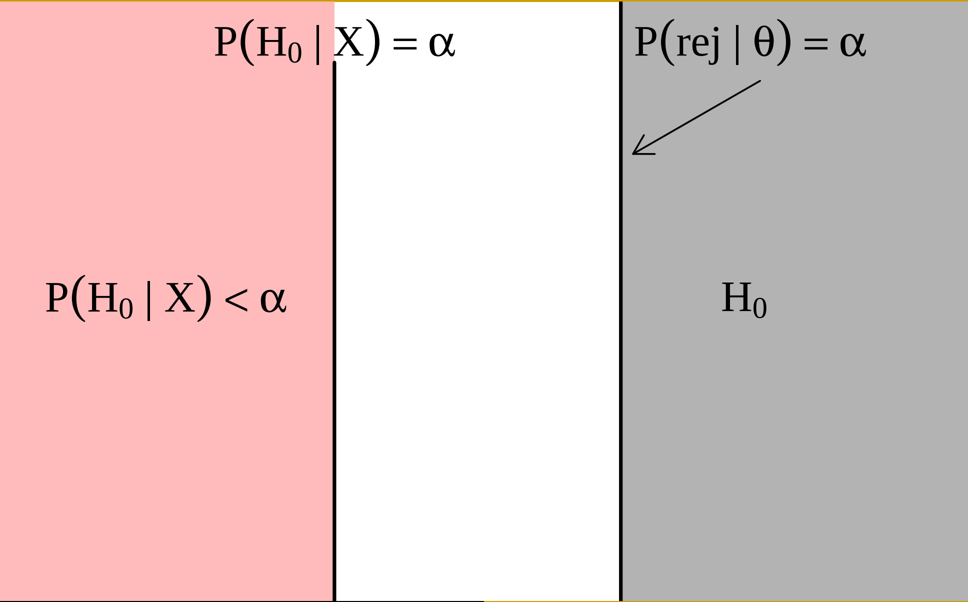

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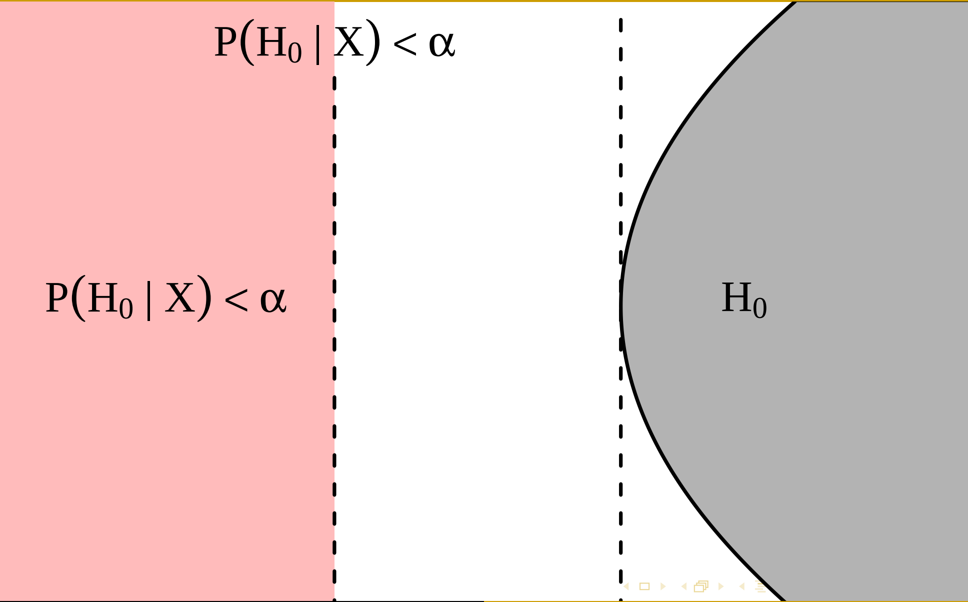

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- What if $d > 2$ dimensions, or infinite?
- $\mathbb{R}^d$: same argument applies if carved away part has positive Lebesgue measure and distribution has support on $\mathbb{R}^d$.
- Infinite: basically, check if there is a finite-dimensional test of a necessary (not sufficient) condition of the infinite-dimensional H_0.
- $A \implies B$ means "reject B" $\implies$ "reject A." So, $\mathrm{P}(\mathrm{rej}\ B \mid \theta(\cdot)) < \mathrm{P}(\mathrm{rej}\ A \mid \theta(\cdot))$, and $\mathrm{P}(\mathrm{rej}\ B \mid \theta(\cdot)) > \alpha \implies \mathrm{P}(\mathrm{rej}\ A \mid \theta(\cdot)) > \alpha$.
- Ex: $H_0 \colon \theta(\cdot) \le 0(\cdot) \implies (\theta(r_1), \theta(r_2)) \le (0, 0)$.

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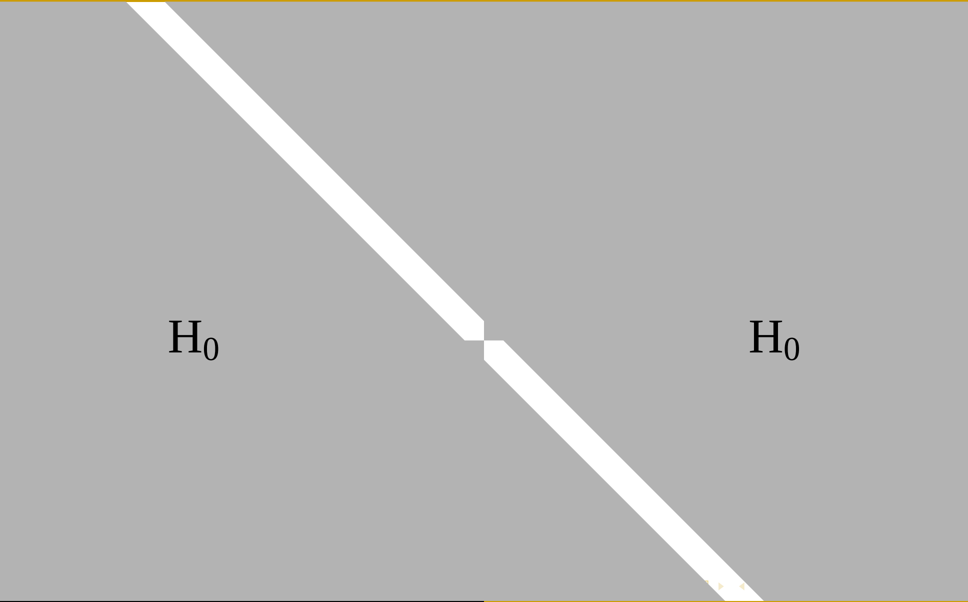
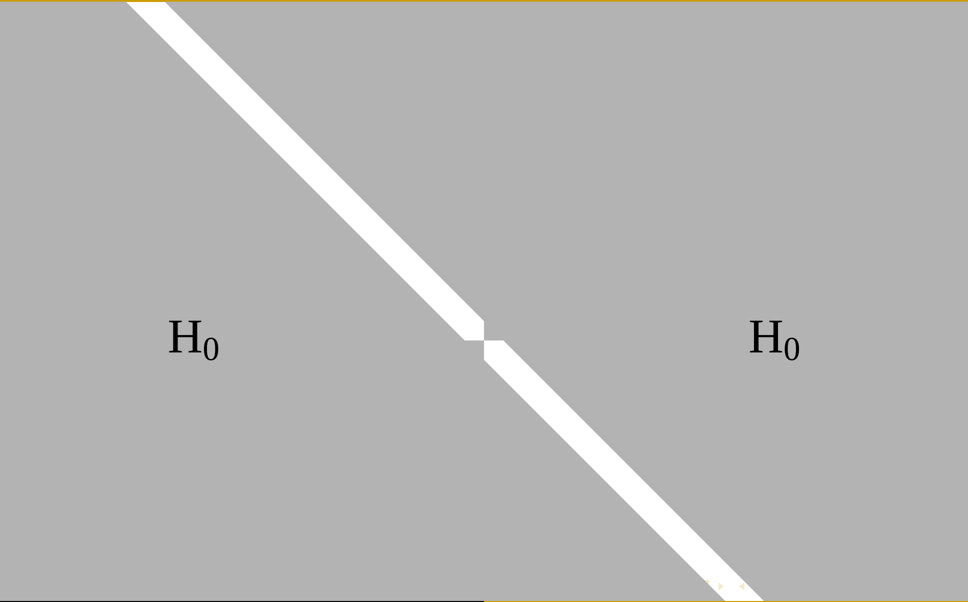
- Further: may depend on distribution, not just shape of Θ_0.
- Examples follow: bivariate normal distribution, unit variances, correlation ρ.

Thm(iv): example of size depending on ρ



Thm(iv): size is 0% if $\rho = 1$ ($\mathrm{P}(H_0 \mid \mathbf{X}) = 1$, $\forall \mathbf{X}$)



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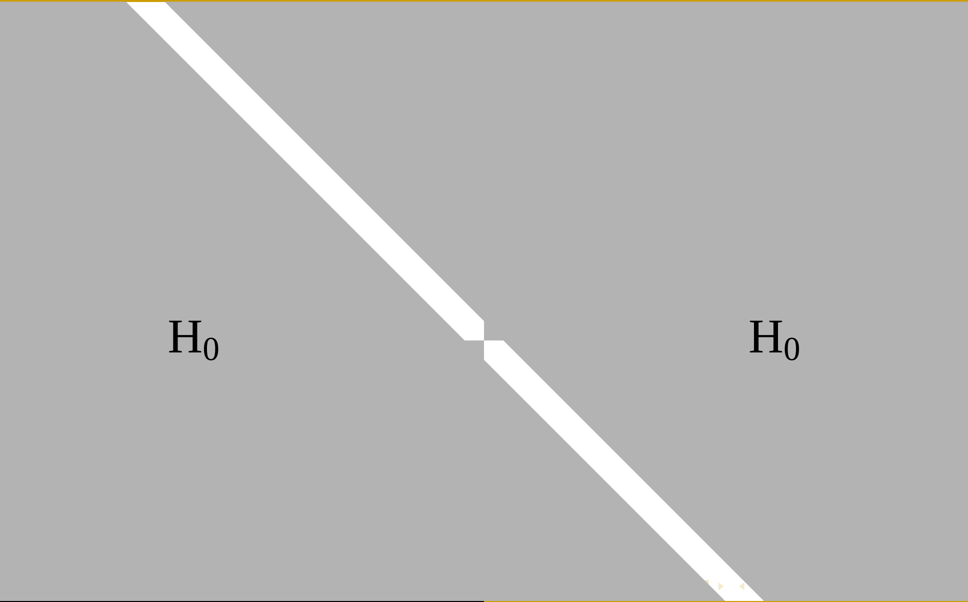
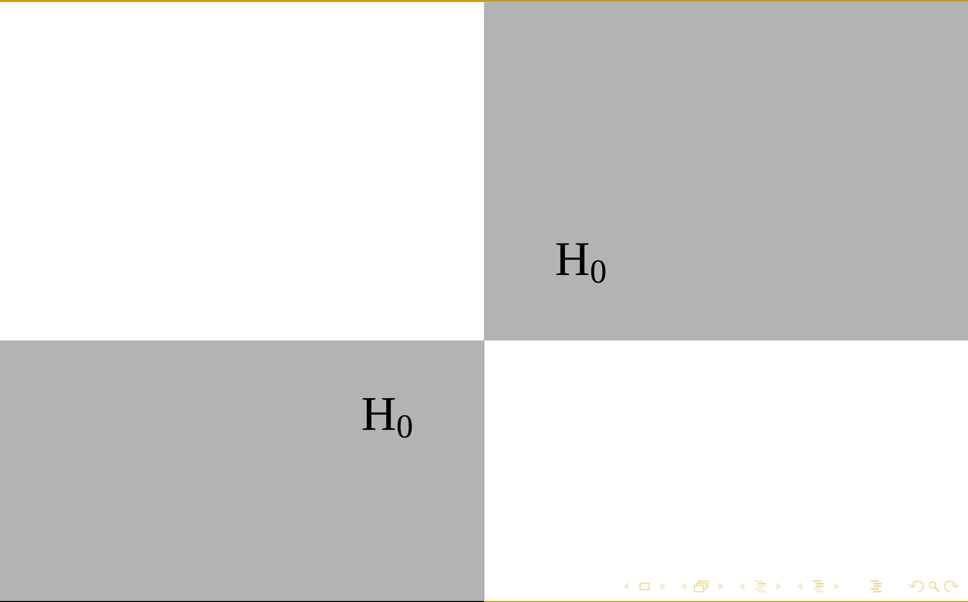


Thm(iv): example of $H_0\colon \theta_1\theta_2 \geq 0$



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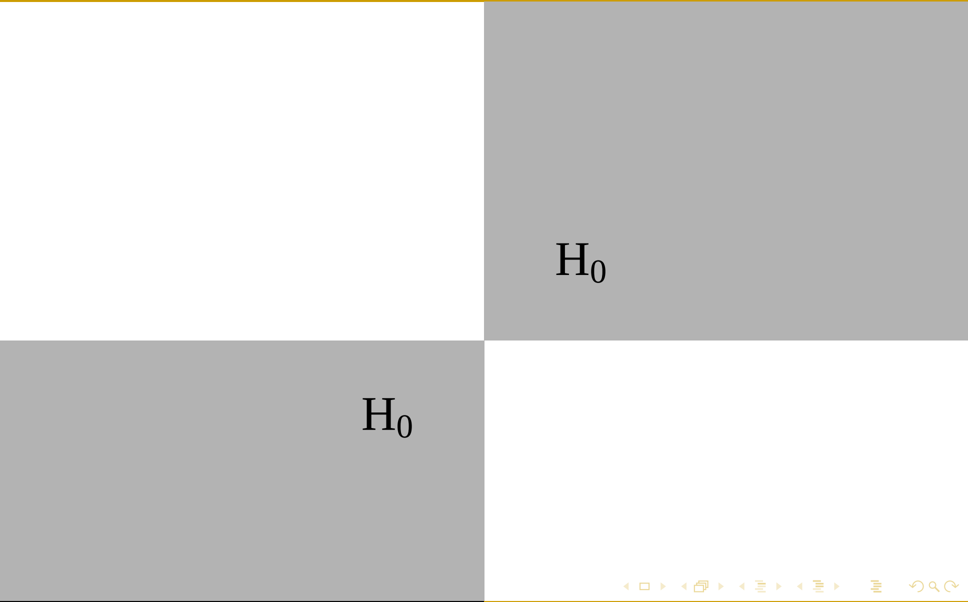
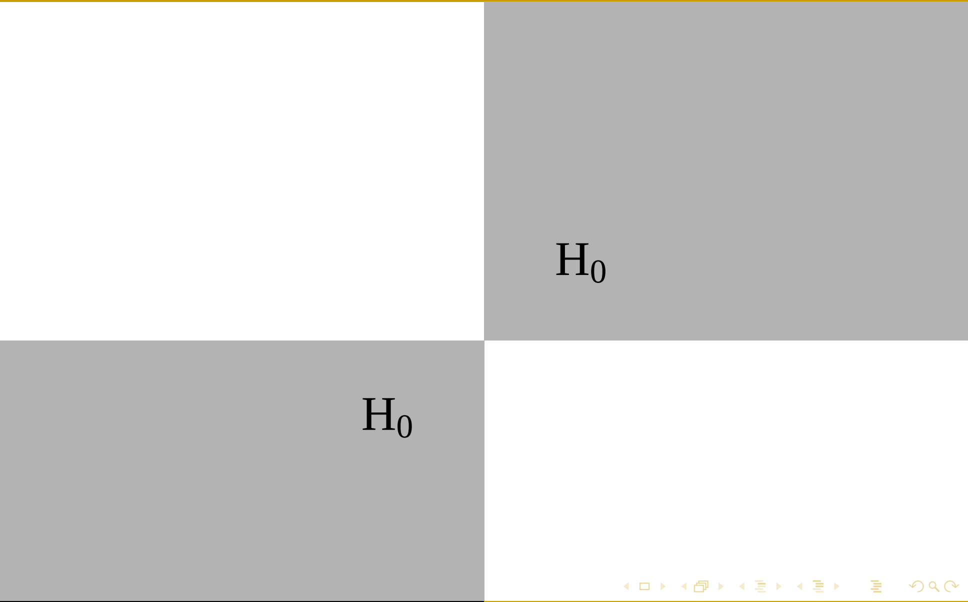


Thm(iv): size is α if $\rho = 1$ (let $\theta_1 \to \infty$, $\theta_2 = 0$)



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- Part (iii) is partly due to small prior $\mathrm{P}(H_0)$.
- Flat prior on $\boldsymbol{\theta}$ means $\mathrm{P}(H_0)$ changes with H_0.
- Would be interesting to examine $\mathrm{P}(H_0) = 1/2$ like J. Berger et al.

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- Flat prior on $\boldsymbol{\theta}$ means $\mathrm{P}(H_0)$ changes with H_0.
- Would be interesting to examine $\mathrm{P}(H_0) = 1/2$ like J. Berger et al.
- But: not the only factor; shape still important.
- Ex: $H_0\colon \theta_1\theta_2 \geq 0$ vs. $H_0\colon \theta_1 \leq 0$.

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- Theorem applies to $\Theta_0 = \{\boldsymbol{\theta} : g(\boldsymbol{\theta}) \leq g_0\}$ as special case, but $g(\cdot)$ not restricted to directional differentiability, etc.

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- Theorem applies to $\Theta_0 = \{\boldsymbol{\theta} : g(\boldsymbol{\theta}) \leq g_0\}$ as special case, but $g(\cdot)$ not restricted to directional differentiability, etc.
- Another special case: if multiple linear inequalities, then size strictly above α. (If single, then size α.)
- It matters greatly whether $H_0\colon \boldsymbol{\theta} \in \Theta_0$ or $H_0\colon \boldsymbol{\theta} \notin \Theta_0$: if part (iii) applies to $H_0\colon \boldsymbol{\theta} \in \Theta_0$, then part (iv) applies to $H_0\colon \boldsymbol{\theta} \notin \Theta_0$.
- Ex: stochastic dominance.

Outline

1. Motivation: health inequality/dispersion in ordinal data
2. Frequentist size of Bayesian inequality tests
   - Setting
   - Theorem
   - Examples
3. Ordinal data again
4. Conclusion

First-order stochastic dominance (SD1)

- One-sample SD1 (see paper for two-sample): $F_X(\cdot) \leq F_0(\cdot) \iff \sqrt{n}(F_X(\cdot) - F_0(\cdot)) \leq 0(\cdot)$.

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- Bernstein–von Mises thm: Lo (1983, 1987) or Castillo and Nickl (2014, Thm. 4).
- Like before, $X_n(\cdot) = \sqrt{n}(\hat{F}_X(\cdot) - F_{0,n}(\cdot))$, $\theta_n(\cdot) = \sqrt{n}(F_X(\cdot) - F_{0,n}(\cdot))$.
- Limit experiment: $H_0\colon \theta(\cdot) \leq 0(\cdot)$, $X(\cdot) - \theta(\cdot) \mid \theta(\cdot) \sim B(F_X(\cdot))$.
- Theorem part (iii) applies here, when H_0 is SD1. (Bayesian test of necessary condition $(\theta(r_1), \theta(r_2)) \leq (0, 0)$ has size above α already.) But, (iv) applies if H_0 is non-SD1.

SD1: analytic results

- Prop. 2: $\mathrm{P}(\mathrm{SD}_1 \mid X(\cdot) = 0(\cdot)) = 0$. (Similar to p-value comparisons in Kline (2011).)
- Prop. 3: Bayesian test's type I error rate is 100% when $\theta(\cdot) = 0(\cdot)$.
- Cor. 4: rejection probability is zero for non-SD1 when $\theta(\cdot) = 0(\cdot)$.

SD1: simulations (fixed dataset)

Posterior probs, $X_i = i/(n+1)$ vs. Unif(0,1) (or vs. $Y_i = i/n$).

		Comparison distribution	
H_0	n	$\text{Unif}(0,1)$	Y
SD1	10	0.103	0.097
SD1	40	0.028	0.025
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non-SD1	100	0.991	0.990
non-SD1	∞	1.000	1.000

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Curvature: background

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- O'Donnell and Coelli (2005): Bayesian approach appealing for testing curvature due to relative simplicity.
- Here: test concavity of cost function wrt input prices.
- Translog functional form (Christensen, Jorgenson, and Lau, 1973): parametric, but flexible enough to allow violations.

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- Output y; input prices $\mathbf{w} = (w_1, w_2, w_3)$; total cost $C(y, \mathbf{w})$.

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- $C(y, \mathbf{w})$ concave in $\mathbf{w}$ (e.g., Kreps, 1990, §7.3).
- $\implies$ Hessian matrix (of C wrt $\mathbf{w}$) negative semidefinite (NSD).
- Here: test "local" NSD at $(1, 1, 1, 1)$; necessary (not sufficient) for global NSD. (Check signs of principal minors. . . .)

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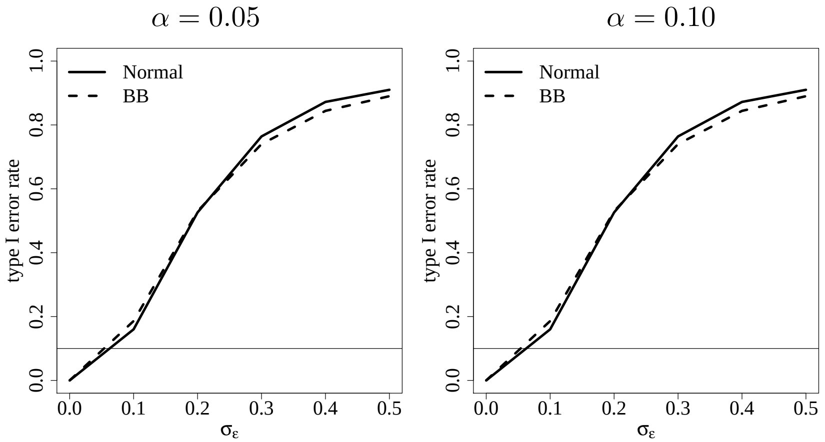
- Posterior: Bayesian bootstrap or OLS-based.
- Other: $n = 100$ observations, 500 simulation replications, 200 posterior draws.

Curvature: simulated type I error rates



Outline

SRHS inequality: SD1

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- Latent first-order stochastic dominance (SD1) would show one distribution is unambiguously better.
- But does ordinal SD1 imply latent SD1? Or: is ordinal SD1 at least a testable implication of latent SD1?

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- (2.iii,iv) To have ordinal SD1 $\Longrightarrow$ latent SD1, need very strong assumption, like pure location shift.

SRHS inequality: SD1

- What if latent location–scale model? Unknown base distribution $F^*(\cdot)$, assume $F^*_X(r) = F^*((r-\mu_X)/\sigma_X)$, $F^*_Y(r) = F^*((r-\mu_Y)/\sigma_Y)$.
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- Latent "restricted SD1" on interval $[a, b]$: Condition I of Atkinson (1987, p. 751), $F^*_X(r) \le F^*_Y(r)$ for all $r \in [a, b]$.
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- See also Davidson and Duclos (2000, 2013).
- (2.v) If same γ_j and location–scale model, then ordinal SD1 $\implies$ latent restricted SD1 on interval $[\gamma_1, \gamma_4]$. (Reason: location–scale model implies at most one crossing of latent CDFs. So, may cross in tails, but can't secretly cross and then cross back within any $[\gamma_{j-1}, \gamma_j]$.)

SRHS inequality: motivation for dispersion

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- "The breakthrough in analyzing inequality with [SRHS] data came from Allison and Foster (2004)."

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- Pure location shift has zero effect on "dispersion." Similarly, γ_j can all shift by some constant Δ_γ without affecting results, unlike for SD1.
- But still can't have each γ_j shift idiosyncratically.
- Lindeboom and van Doorslaer (2004) call these "index shift" and "cut-point shift" (respectively). They and Hernández-Quevedo, Jones, and Rice (2005) find mixed evidence of no shift, index shift, and cut-point shift among different subpopulations in Canadian and British data (respectively).

SRHS inequality: dispersion

- Most fundamentally: we can learn about certain latent interquantile range (IQR) differences (in some cases).
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- Then: stronger assumptions allow extrapolation from these IQR differences to other/all IQR differences.
- If even stronger assumptions, including no γ_j shift: then can even infer latent SD2.

SRHS inequality: dispersion, CDF crossing

- (Prop 3.i) If first ordinal CDF crosses the second from below, then know at least some interquanile ranges are larger in second latent distribution.

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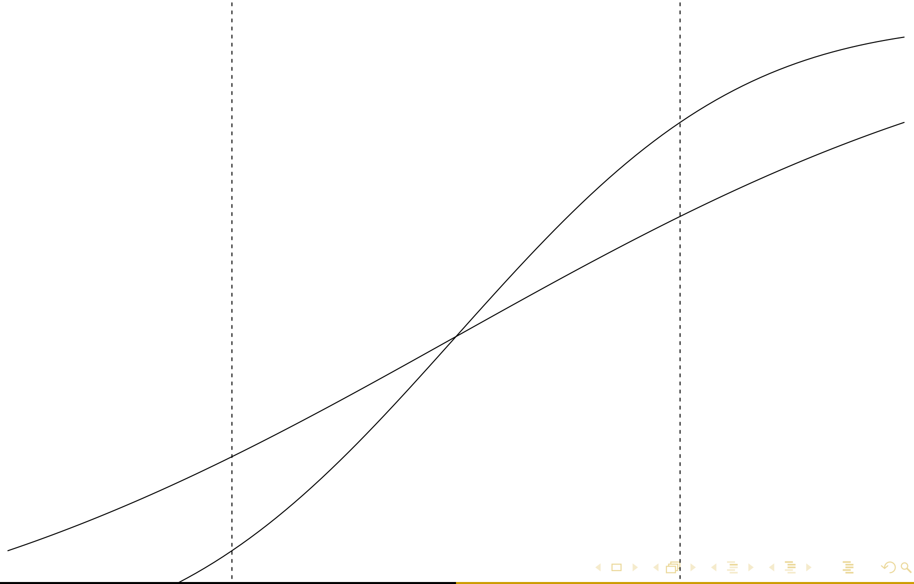
- (Prop 3.i) If first ordinal CDF crosses the second from below, then know at least some interquanile ranges are larger in second latent distribution.
- "Median-preserving spread" of Allison and Foster (2004) is special case of single CDF crossing.
- Ordinal CDF crossing implies no ordinal SD1. But, if γ_j shift, then possibly latent SD1.

SRHS inequality: dispersion, CDF crossing



SRHS inequality: dispersion, CDF crossing

- (3.ii) Assuming latent location–scale model lets you infer the scale parameter is larger if even one IQR is larger, which then implies all IQRs are larger.

SRHS inequality: dispersion, CDF crossing

- (3.ii) Assuming latent location–scale model lets you infer the scale parameter is larger if even one IQR is larger, which then implies all IQRs are larger.
- (3.iii) If also $F^*(\cdot)$ symmetric about zero, and no γ_j shift, and second ordinal distribution's median is strictly lower: then latent SD2. (Reason: strong enough assumptions to characterize SD2 by only μ and σ.)
- We call this a "median-decreasing spread." Even with all these assumptions, if the medians are identical (as in median-preserving spread), then SD2 is ambiguous.

SRHS inequality: dispersion, fanning out

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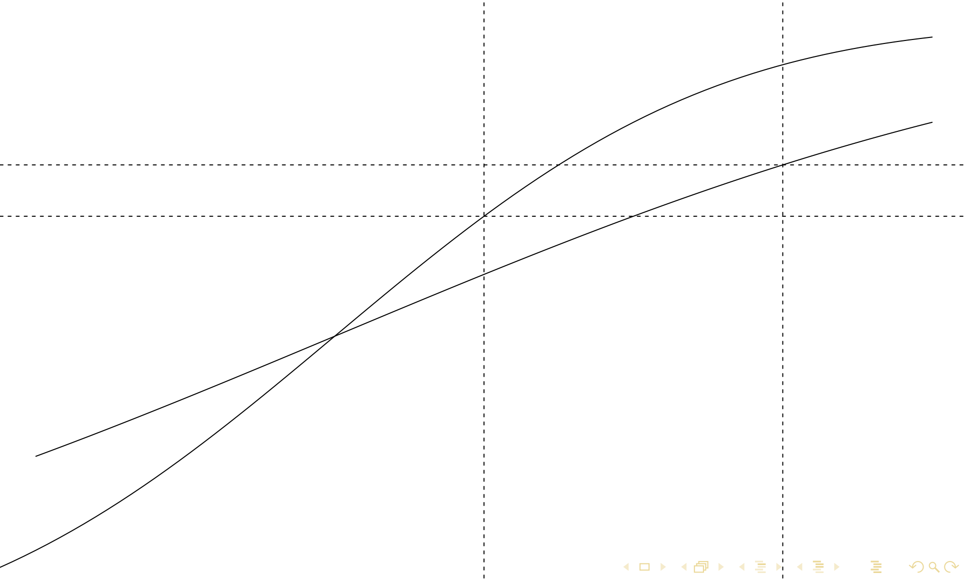
- Yes, with stronger assumptions: symmetric, unimodal latent distributions.
- (Prop 4.i) Since unimodal symmetric implies latent CDFs are concave after their medians, then "fanning out" of ordinal CDFs after median implies a certain IQR is larger in the lower distribution.

SRHS inequality: dispersion, fanning out



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- (4.ii) Similar (mirror image) result for below median.

SRHS inequality: dispersion, fanning out

- (4.ii) Similar (mirror image) result for below median.
- (4.iii) Adding location–scale assumption again allows extrapolation from certain IQRs to all IQRs (via σ).

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- Leads to adjusting posterior by constant and renormalizing; e.g., see Goutis, Casella, and Wells (1996), eqn (9).
- Weird: use different prior to examine ordinal SD1 vs. single-crossing (vs. ...).
- Also weird to have $\mathrm{P}(H_0) = 1/2$ for SD1 if you're just looking at all pairs of US states or something, like Allison and Foster (2004); can't have $1/2$ probability on both $X \ \mathrm{SD}_1 \ Y$ and $Y \ \mathrm{SD}_1 \ X$.
- But, worth trying, simulating (haven't done yet).

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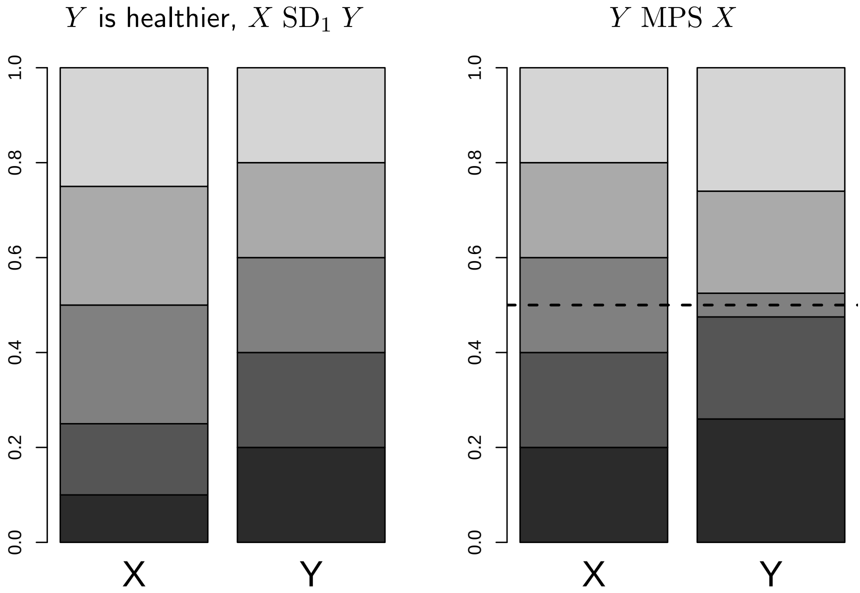
- If not: union of sets of inequalities (union over possible m.)
- m: shared median; i.e., $F_1(m-1) < 1/2 \leq F_1(m)$, same for F_2.
- For $j < m$, $F_1(j) \leq F_2(j)$; for $j \geq m$, $F_1(j) \geq F_2(j)$.
- Let $\theta_j \equiv F_1(j) - F_2(j)$ again.
- MPS is $\theta_j \leq 0$ for $j < m$ and $\theta_j \geq 0$ for $j \geq m$.


Y is healthier, X SD_1 Y
Y MPS X
0.0
0.2
0.4
0.6
0.8
1.0
X
Y
X
Y

Empirical: data and methods

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- "Y MPS X" means Y is a MPS of X.
- Frequentist: Andrews and Barwick (2012) refined moment selection (RMS).
- Bayesian: Dirichlet–multinomial model, uninformative prior.

Empirical: SD1 and MPS over age

X: 20–24 years old; Y: age range in table.

p-values and posterior probabilities; **bold** if below 5%.

	$H_0: Y \text{ SD}_1 X$		$H_0: X \text{ SD}_1 Y$	
Y	RMS	Bayes	RMS	Bayes
$[25, 29]$	8.0%	**0.8%**	63.6%	13.1%
$[30, 34]$	100%	42.3%	**3.7%**	**0.1%**
$[35, 39]$	100%	74.7%	**0.1%**	**0.0%**
$[40, 44]$	100%	86.0%	**0.0%**	**0.0%**

Empirical: SD1 and MPS over age

X: 20–24 years old; Y: 25–29 years old.

H_0 in table header. (MPS: same median, "very good.")

	$Y \text{ SD}_1 X$	$X \text{ SD}_1 Y$	$Y \text{ MPS } X$	$X \text{ MPS } Y$
RMS	8.0%	63.6%	37.2%	9.3%
Bayes	**0.8%**	13.1%	7.7%	**0.0%**

Empirical: SD1 and MPS over generation (at same age)

Ages 65–69. X born 1937–41, Y born 1932–36.

B= "black"; W= "white"; M= "male"; F= "female"

	$H_0: Y$ MPS X		$H_0: Y\ \mathrm{SD}_1\ X$		$H_0: X\ \mathrm{SD}_1\ Y$	
Sample	RMS	Bayes	RMS	Bayes	RMS	Bayes
B	100%	25.8%	39.7%	**3.2%**	100%	17.9%
BM	76.4%	11.7%	12.4%	**0.1%**	100%	39.4%
BF	100%	10.3%	100%	25.4%	61.2%	**4.3%**

Empirical: SD1 and MPS over generation (at same age)

Ages 65–69. X born 1947–51, Y born 1942–46.

B= "black"; W= "white"; M= "male"; F= "female"

	$H_0: Y$ MPS X		$H_0: Y$ SD_1 X		$H_0: X$ SD_1 Y	
Sample	RMS	Bayes	RMS	Bayes	RMS	Bayes
M	100%	26.9%	51.4%	9.3%	58.9%	6.8%
BM	100%	38.3%	**4.6%**	**1.2%**	16.4%	**2.0%**
WM	100%	12.4%	100%	15.8%	65.3%	10.3%

Outline

Conclusion

- Summary (Bayes/freq): if null hypothesis is smaller than half-space, then Bayesian test has (asymptotic) size above α; if not, then can depend on sampling distribution, too.
- Summary (ordinal): given continuous latent distributions but no parametric model, certain latent distribution relationships imply certain ordinal relationships, under certain assumptions. Ordinal relationships are combinations of moment inequalities.

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- Thank you!
- (And further questions or comments are welcome)

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