IMPROVED QUANTILE INFERENCE VIA FIXED-SMOOTHING ASYMPTOTICS AND EDGEWORTH EXPANSION: APPENDIX OF PROOFS FOR UNIVARIATE CASE

DAVID M. KAPLAN

Contents

Appendix A. Calculation of type II error rate (Proposition 6)

Let $\gamma \neq 0$ so that the null hypothesis is false, where as before $H_0 : \xi_p = \beta$ with $\xi_p = \beta - \gamma/\sqrt{n}$. Letting $S_0 \equiv 1/f(\xi_p)$,

$$T_{m,n} = \frac{\sqrt{n}(X_{n,r} - \xi_p) - \gamma}{S_{m,n}\sqrt{p(1-p)}} = \frac{\sqrt{n}(X_{n,r} - \xi_p)}{S_{m,n}\sqrt{p(1-p)}} - \frac{\gamma}{\sqrt{p(1-p)}S_0}\left(\frac{S_0}{S_{m,n}} + 1 - 1\right),$$

$$\begin{aligned}
&P(T_{m,n} < z) \\
&= P\left(\frac{\sqrt{n}(X_{n,r} - \xi_p)}{S_{m,n}\sqrt{p(1-p)}} - \frac{\gamma}{\sqrt{p(1-p)}S_0}\left(\frac{S_0}{S_{m,n}} + 1 - 1\right) < z\right) \\
&= P\left(\frac{\sqrt{n}(X_{n,r} - \xi_p)}{S_{m,n}\sqrt{p(1-p)}} - \frac{\gamma}{\sqrt{p(1-p)}S_0}\left(\frac{S_0}{S_{m,n}} - 1\right) < z + C\right) \\
&= P\left(\frac{\sqrt{n}(X_{n,r} - \xi_p) + \gamma(S_{m,n}/S_0 - 1)}{S_{m,n}\sqrt{p(1-p)}} < z + C\right). \qquad (1)
\end{aligned}$$

If the true S_0 were known and used in $T_{m,n}$ instead of its estimator $S_{m,n}$, this would be simply the distribution from Hall and Sheather (1988) with a shift of the critical value by $C \equiv \gamma/[S_0\sqrt{p(1-p)}]$, which is γ normalized by the true (hypothetically known) variance.

Date: July 9, 2014; first version available online January 14, 2011.

But $S_{m,n}$ is random, so the Hall and Sheather (1988) expansion is insufficient and Theorem 2 is needed.

The type II error is the probability of not rejecting when H_0 is false. For a two-sided symmetric test, this is

$$P(|T_{m,n}| < z) = P(T_{m,n} < z) - P(T_{m,n} < -z).$$

Letting the corrected critical value $z_{\alpha,m} = z_{1-\alpha/2} + z^3_{1-\alpha/2}/(4m)$ as in (9), and expanding via (1) and (12), for $C > 0$,

$$\begin{aligned}
P(|T_{m,n}| < z) &= L^+ + H_1^+ - H_2^+ + O(n^{-1/2}) + o(m^{-1} + (m/n)^2) \quad \text{with} \\
L^+ &= \Phi(z_{\alpha,m} + C) - \Phi(-z_{\alpha,m} + C), \\
H_1^+ &= \phi(z_{\alpha,m} + C)\Big[m^{-1}u_{2,\gamma}(z_{\alpha,m} + C) + (m/n)^2 u_{3,\gamma}(z_{\alpha,m} + C)\Big], \\
H_2^+ &= \phi(-z_{\alpha,m} + C)\Big[m^{-1}u_{2,\gamma}(C - z_{\alpha,m}) + (m/n)^2 u_{3,\gamma}(C - z_{\alpha,m})\Big].
\end{aligned}$$

I write $O(n^{-1/2})$ for the $n^{-1/2}$ terms since they do not depend on m and thus do not affect the optimization problem for selecting m. Define L^-, H_1^-, and H_2^- similarly but with $-C < 0$ instead of $C > 0$, and thus $-\gamma = -C\sqrt{p(1-p)}/f(\xi_p)$ instead of γ:

$$\begin{aligned}
L^- &= \Phi(z_{\alpha,m} - C) - \Phi(-z_{\alpha,m} - C), \\
H_1^- &= \phi(z_{\alpha,m} - C)\Big[m^{-1}u_{2,-\gamma}(z_{\alpha,m} - C) + (m/n)^2 u_{3,-\gamma}(z_{\alpha,m} - C)\Big], \\
H_2^- &= \phi(-z_{\alpha,m} - C)\Big[m^{-1}u_{2,-\gamma}(-C - z_{\alpha,m}) + (m/n)^2 u_{3,-\gamma}(-C - z_{\alpha,m})\Big].
\end{aligned}$$

I calculate average power where the alternatives $+C$ and $-C$ each have 0.5 probability. Using $\phi(-x) = \phi(x)$,

$$\begin{aligned}
P(|T_{m,n}| < z_{\alpha,m}) = \frac{1}{2}\Big\{&(L^+ + L^-) + (H_1^+ + H_1^-) - (H_2^+ + H_2^-) \\
&+ O(n^{-1/2}) + o(m^{-1} + (m/n)^2)\Big\}. \qquad (2)
\end{aligned}$$

For the first-order term L^+,

$$\begin{aligned}
&\Phi(z_{\alpha,m}+C) - \Phi(-z_{\alpha,m} + C) \\
&\qquad = \Phi(C + z_{1-\alpha/2} + z^3_{1-\alpha/2}/(4m)) - \Phi(C - z_{1-\alpha/2} - z^3_{1-\alpha/2}/(4m)) \\
&\qquad = 0.5 + m^{-1}\frac{1}{4}z^3_{1-\alpha/2}[\phi(z_{1-\alpha/2} + C) + \phi(C - z_{1-\alpha/2})] \qquad (3)
\end{aligned}$$

since in Proposition 6, C solves $0.5 = \Phi(z_{1-\alpha/2} + C) - \Phi(-z_{1-\alpha/2} + C)$. Since the fixed-$m$ critical value is larger than the standard normal critical value, this term contributes additional type II error in the m^{-1} term. Similarly,

$$
\begin{aligned}
L^- &= \Phi(z_{\alpha,m} - C) - \Phi(-z_{\alpha,m} - C) \\
&= 0.5 + m^{-1}\frac{1}{4}z^3_{1-\alpha/2}[\phi(C - z_{1-\alpha/2}) + \phi(-z_{1-\alpha/2} - C)] = L^+.
\end{aligned}
$$

Within the m^{-1} and $(m/n)^2$ terms, anything $o(1)$ will end up in the $o(m^{-1} + (m/n)^2)$ remainder. Thus, for those terms,

$$
z_{\alpha,m} = z_{1-\alpha/2} + O(m^{-1}), \quad \phi(z_{\alpha,m} + C) = \phi(z_{1-\alpha/2} + C) + O(m^{-1}).
$$

For the m^{-1} terms, since $\phi(x) = \phi(-x)$ and $C \equiv \gamma f(\xi_p)/\sqrt{p(1-p)}$, and letting $d_1 \equiv z_{\alpha,m} + C$, $d_2 \equiv z_{\alpha,m} - C$,

$$
\begin{aligned}
&u_{2,\gamma}(d_1) - u_{2,-\gamma}(-d_1) \\
&\quad = -\frac{1}{4}d_1^3 - \frac{1}{4}C^2 d_1 + \frac{1}{4}2Cd_1^2 - \left[-\frac{1}{4}(-d_1)^3 - \frac{1}{4}(-C)^2(-d_1) + \frac{1}{4}2(-C)(-d_1)^2\right] \\
&\quad = -\frac{1}{2}(d_1^3 + C^2 d_1 - 2Cd_1^2) = -\frac{1}{2}(z_{1-\alpha/2} + C)z^2_{1-\alpha/2} + O(m^{-1}), \\
&u_{2,-\gamma}(d_2) - u_{2,\gamma}(-d_2) = -\frac{1}{2}(d_2^3 + C^2 d_2 + 2Cd_2^2) = -\frac{1}{2}(z_{1-\alpha/2} - C)z^2_{1-\alpha/2} + O(m^{-1}), \\
&\phi(z_{\alpha,m} + C)m^{-1}(u_{2,\gamma}(z_{\alpha,m} + C) - u_{2,-\gamma}(-z_{\alpha,m} - C)) \\
&\quad + \phi(z_{\alpha,m} - C)m^{-1}(u_{2,-\gamma}(z_{\alpha,m} - C) - u_{2,\gamma}(-z_{\alpha,m} + C)) \\
&\quad = -\frac{1}{2}m^{-1}\left[\phi(z_{1-\alpha/2} + C)(z_{1-\alpha/2} + C)z^2_{1-\alpha/2} + \phi(z_{1-\alpha/2} - C)(z_{1-\alpha/2} - C)z^2_{1-\alpha/2}\right] \\
&\qquad + O(m^{-2}).
\end{aligned} \tag{4}
$$

For the $(m/n)^2$ terms, writing $f(\xi_p)$ and its derivatives as f, f', f'', and letting $M \equiv [3(f')^2 - ff'']/(6f^4)$,

$$
\begin{aligned}
&u_{3,\gamma}(z_{\alpha,m} + C) - u_{3,-\gamma}(-z_{\alpha,m} - C) \\
&\quad = M(z_{\alpha,m} + C - C) - M(-z_{\alpha,m} - C - (-C)) = 2Mz_{1-\alpha/2} + O(m^{-1}), \\
&u_{3,-\gamma}(z_{\alpha,m} - C) - u_{3,\gamma}(-z_{\alpha,m} + C) \\
&\quad = M(z_{\alpha,m} - C - (-C)) - M(-z_{\alpha,m} + C - C) = 2Mz_{1-\alpha/2} + O(m^{-1}), \\
&\frac{m^2}{n^2}\big\{\phi(z_{\alpha,m} + C)\left[u_{3,\gamma}(z_{\alpha,m} + C) - u_{3,-\gamma}(-z_{\alpha,m} - C)\right] \\
&\qquad + \phi(z_{\alpha,m} - C)\left[u_{3,-\gamma}(z_{\alpha,m} - C) - u_{3,\gamma}(-z_{\alpha,m} + C)\right]\big\} \\
&\quad = (m/n)^2 2\frac{3(f')^2 - ff''}{6f^4}z_{1-\alpha/2}\left[\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)\right] + O(m/n^2).
\end{aligned} \tag{5}
$$

Combining (3), (4), and (5), (2) becomes

$$
\begin{aligned}
P(|T_{m,n}| < z) = \frac{1}{2}\Big\{ & 2\left(0.5 + m^{-1}(1/4)z_{1-\alpha/2}^3[\phi(C + z_{1-\alpha/2}) + \phi(C - z_{1-\alpha/2})]\right) \\
& - 2(1/4)m^{-1}[\phi(z_{1-\alpha/2} + C)(z_{1-\alpha/2} + C)z_{1-\alpha/2}^2 \\
& \qquad + \phi(z_{1-\alpha/2} - C)(z_{1-\alpha/2} - C)z_{1-\alpha/2}^2] \\
& + 2(m/n)^2 \frac{3(f')^2 - ff''}{6f^4} z_{1-\alpha/2}\left[\phi(z_{1-\alpha/2} + C) + \phi(z_{1-\alpha/2} - C)\right]\Big\} \\
& + O(n^{-1/2}) + o(m^{-1} + (m/n)^2),
\end{aligned}
$$

which simplifies to the forms given in Proposition 6.

Appendix B. Higher-order corrected critical value

Extending the main text's results, noting that the fourth and fifth central moments of χ^2_{4m} are respectively $192m(m+1)$ and $O(m^2)$,

$$
\begin{aligned}
P(T_{m,\infty} < z) &= \Phi(z) - \frac{z^3\phi(z)}{4m} + (1/6)\Phi'''(z)z^3\frac{32m}{64m^3} + (1/24)\Phi''''(z)z^4\frac{E[(\chi^2_{4m} - 4m)^4]}{(4m)^4} + O(m^2/m^5) \\
&= \Phi(z) - \frac{z^3\phi(z)}{4m} + (1/6)\Phi'''(z)z^3\frac{32m}{64m^3} + (1/24)(3z - z^3)\phi(z)z^4(192m^2/256m^4) + O(m^{-3}) \\
&= \Phi(z) - \frac{z^3\phi(z)}{4m} + \frac{\phi(z)(z^5 - z^3 + (9/8)z^5 - (3/8)z^7)}{12m^2} + O(m^{-3}) \\
&= \Phi(z) - \frac{z^3\phi(z)}{4m} + \frac{\phi(z)(17z^5 - 8z^3 - 3z^7)}{96m^2} + O(m^{-3}).
\end{aligned}
$$

Let $z = z_{1-\alpha} + c_1/m + c_2/m^2$. We can verify the result with $c_1 = z_{1-\alpha}^3/4$ and $c_2 = [z_{1-\alpha}^5 + 8z_{1-\alpha}^3]/96$:

$$
\begin{aligned}
& P(T_{m,\infty} < z_{1-\alpha} + c_1/m + c_2/m^2) \\
&= \Phi(z_{1-\alpha} + c_1/m + c_2/m^2) - (4m)^{-1}\phi(z_{1-\alpha} + c_1/m + c_2/m^2)(z_{1-\alpha} + c_1/m + c_2/m^2)^3 \\
&\quad + (1/96)m^{-2}\phi(z_{1-\alpha} + c_1/m + c_2/m^2)[17z_{1-\alpha}^5 - 8z_{1-\alpha}^3 - 3z_{1-\alpha}^7 + O(1/m)] \\
&= \Phi(z_{1-\alpha}) + (c_1/m + c_2/m^2)\phi(z_{1-\alpha}) + (1/2)\phi'(z_{1-\alpha})c_1^2/m^2 \\
&\quad - (4m)^{-1}\left\{[\phi(z_{1-\alpha}) + \phi'(z_{1-\alpha})c_1/m][z_{1-\alpha} + c_1/m]^3\right\} \\
&\quad + (1/96)m^{-2}\phi(z_{1-\alpha})[17z_{1-\alpha}^5 - 8z_{1-\alpha}^3 - 3z_{1-\alpha}^7] + O(m^{-3}) \\
&= 1 - \alpha + m^{-1}c_1\phi(z_{1-\alpha}) + m^{-2}\left[c_2\phi(z_{1-\alpha}) - (1/2)c_1^2 z_{1-\alpha}\phi(z_{1-\alpha})\right] \\
&\quad - (4m)^{-1}\phi(z_{1-\alpha})z_{1-\alpha}^3 - (1/4)m^{-2}\left\{3\phi(z_{1-\alpha})z_{1-\alpha}^2 c_1 - c_1 z_{1-\alpha}^4\phi(z_{1-\alpha})\right\} \\
&\quad + (1/96)m^{-2}\phi(z_{1-\alpha})[17z_{1-\alpha}^5 - 8z_{1-\alpha}^3 - 3z_{1-\alpha}^7] + O(m^{-3}) \\
&= 1 - \alpha + m^{-1}\phi(z_{1-\alpha})\left\{c_1 - (1/4)z_{1-\alpha}^3\right\}
\end{aligned}
$$

$$+ m^{-2}\phi(z_{1-\alpha})\Big\{c_2 - (1/2)c_1^2 z_{1-\alpha} - (3/4)z_{1-\alpha}^2 c_1 + (1/4)c_1 z_{1-\alpha}^4 + (1/96)(17z_{1-\alpha}^5 - 8z_{1-\alpha}^3 - 3z_{1-\alpha}^7)\Big\} + O(m^{-3}).$$

In order to zero the m^{-1} term, we must have $c_1 = (1/4)z_{1-\alpha}^3$. To zero the m^{-2} term,

$$\begin{aligned}
0 &= c_2 - (1/2)z_{1-\alpha}(1/16)z_{1-\alpha}^6 - (3/4)z_{1-\alpha}^2(1/4)z_{1-\alpha}^3 + (1/16)z_{1-\alpha}^7 \\
&\quad + (17/96)z_{1-\alpha}^5 - (8/96)z_{1-\alpha}^3 - (3/96)z_{1-\alpha}^7 \\
&= c_2 - (1/12)z_{1-\alpha}^3 + z_{1-\alpha}^5[(17/96) - (3/16)] - z_{1-\alpha}^7[(1/32) + (1/32) - (1/16)], \\
c_2 &= (1/96)[8z_{1-\alpha}^3 + z_{1-\alpha}^5].
\end{aligned}$$

Appendix C. Accuracy of approximation of fixed-m critical values

I tried three alternative approximations, but the simplest (as used throughout the paper) is quite accurate for all but the very smallest m, as Tables 1, 2, and 3 show. The second approximation alternative adds the $O(m^{-2})$ term to the approximation, so that the error of the approximated critical value is $O(m^{-3})$. The third alternative uses the critical value from the Student's t-distribution with the degrees of freedom chosen to match the variance. The latter two approximations are more accurate, but the difference is small. To compare them, for various m and nominal test size α, I simulated the two-sided rejection probability if a given critical value was used for the statistic $T_{m,\infty}$. To gauge simulation error, I also include in the tables the critical values given in Goh (2004) (who also ran 500,000 simulation replications for each m and α).

Appendix D. Sign of $u_{3,0}(z)$ for common distributions

The method is the same for the example distributions below. I have worked out normal, t-, exponential, χ^2, and Fréchet distributions, all with the same result of being positive. The value of $u_{3,0}$ for the uniform distribution is equal to zero, since its pdf is flat and thus derivatives are all zero.

The pdf and derivatives of the normal distribution are

$$\begin{aligned}
f_n(x) &= \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)} \\
f_n'(x) &= -(x-\mu)/\sigma^2 \cdot f_n(x) \\
f_n''(x) &= -f_n(x)/\sigma^2 - f_n'(x)(x-\mu)/\sigma^2 \\
&= -f_n(x)/\sigma^2 - [-(x-\mu)/\sigma^2 \cdot f_n(x)](x-\mu)/\sigma^2 \\
&= -f_n(x)/\sigma^2 + (x-\mu)^2/\sigma^4 \cdot f_n(x) \\
&= f_n(x)[(x-\mu)^2/\sigma^4 - 1/\sigma^2]
\end{aligned}$$

Table 1. Rejection probabilities (%) for different critical value approximations, for test statistic $T_{m,\infty}$ when $\alpha = 10\%$, based on simulated distribution

	rejection probability for critical value			
m	Goh (2004, simulated)	including m^{-1}	including m^{-2}	t
1	10.00	13.38	10.54	12.33
2	10.12	11.36	10.17	10.44
3	10.08	10.72	10.06	10.12
4	10.15	10.44	10.03	10.03
5	9.93	10.18	9.89	9.87
6	10.03	10.24	10.02	9.99
7	10.04	10.23	10.07	10.02
8	9.99	10.02	9.89	9.86
9	10.06	10.16	10.06	10.02
10	10.00	10.14	10.06	10.03
11	9.97	10.11	10.03	10.00
12	10.02	10.12	10.06	10.02
13	9.92	10.00	9.95	9.92
14	10.06	10.04	9.98	9.95
15	10.07	10.05	10.00	9.97
20	10.04	10.03	10.01	9.99
25	10.08	10.09	10.07	10.06
30	9.99	10.04	10.03	10.01
50	10.10	10.03	10.02	10.01

Then,

$$\begin{aligned}
&3f_n'(x)^2 - f_n(x)f_n''(x) \\
&\quad = 3[-(x-\mu)/\sigma^2 \cdot f_n(x)]^2 - f_n(x) \cdot f_n(x)[(x-\mu)^2/\sigma^4 - 1/\sigma^2] \\
&\quad = 3[f_n(x)]^2(x-\mu)^2/\sigma^4 - [f_n(x)]^2[(x-\mu)^2/\sigma^4 - 1/\sigma^2] \\
&\quad = [f_n(x)]^2\{3(x-\mu)^2/\sigma^4 - (x-\mu)^2/\sigma^4 + 1/\sigma^2\} \\
&\quad = [f_n(x)]^2\{2(x-\mu)^2/\sigma^4 + 1/\sigma^2\} \geq 0,
\end{aligned}$$

since again everything is to an even power and positive. The sign of $3f_n'(\xi_p)^2 - f_n(\xi_p)f_n''(\xi_p)$ is thus positive at any quantile for any parameters μ and σ^2. Thus the sign of $u_{3,0}(z)$ is always the sign of z for the normal distribution at any quantile.

The pdf of the t-distribution is

$$f_t(x) = \frac{\Gamma((v+1)/2)}{\sqrt{v\pi}\,\Gamma(v/2)}\left(1 + x^2/v\right)^{-(v+1)/2}.$$

Writing the derivatives of the pdf in terms of the pdf,

$$f_t'(x) = f_t(x) \cdot \frac{-v-1}{2}\frac{2x}{v}(1 + x^2/v)^{-1}$$

TABLE 2. Rejection probabilities (%) for different critical value approximations, for test statistic $T_{m,\infty}$ when $\alpha = 5\%$, based on simulated distribution

	rejection probability for critical value			
m	Goh (2004, simulated)	including m^{-1}	including m^{-2}	t
1	4.93	8.21	5.84	6.87
2	5.03	6.25	5.20	5.47
3	5.05	5.70	5.10	5.28
4	5.15	5.46	5.06	5.21
5	4.92	5.23	4.96	5.08
6	5.04	5.25	5.04	5.16
7	4.98	5.16	5.00	5.12
8	4.97	5.11	4.99	5.09
9	4.96	5.10	4.99	5.09
10	5.01	5.14	5.06	5.14
11	4.96	5.09	5.01	5.10
12	5.14	5.18	5.12	5.20
13	4.98	5.03	4.98	5.05
14	5.05	5.04	4.99	5.06
15	4.98	5.02	4.98	5.05
20	5.04	5.01	4.99	5.04
25	5.08	5.07	5.06	5.11
30	4.96	5.02	5.01	5.04
50	5.06	5.05	5.05	5.07

$$
\begin{aligned}
f_t''(x) &= \frac{f_t'(x)}{x} + f_t'(x)\cdot\frac{-v-3}{2}\frac{2x}{v}(1+x^2/v)^{-1} \\
&= f_t(x)\cdot\left[\frac{-v-1}{2}\frac{2}{v}(1+x^2/v)^{-1} + \frac{-v-1}{2}\frac{-v-3}{2}(2x/v)^2(1+x^2/v)^{-2}\right] \\
&\equiv f_t(x)\cdot[A].
\end{aligned}
$$

Then,

$$
\begin{aligned}
&3f_t'(x)^2 - f_t(x)f_t''(x) \\
&\quad = 3[f_t(x)]^2\cdot((-v-1)/2)^2(2x/v)^2(1+x^2/v)^{-2} - [f_t(x)]^2\cdot A \\
&\quad = [f_t(x)]^2\cdot\big\{-((-v-1)/2)(2/v)(1+x^2/v)^{-1} \\
&\qquad\quad + ((-v-1)/2)(2x/v)^2(1+x^2/v)^{-2}(3((-v-1)/2)-(-v-3)/2)\big\} \\
&\quad = [f_t(x)]^2\cdot\big\{((v+1)/v)(1+x^2/v)^{-1} \\
&\qquad\quad + 2((v+1)/v^2)x^2(1+x^2/v)^{-2}(3v+3-v-3)/2\big\} \\
&\quad = [f_t(x)]^2\left[((v+1)/v)(1+x^2/v)^{-1} + 2((v+1)/v)x^2(1+x^2/v)^{-2}\right] \\
&\quad = [f_t(x)]^2\cdot((v+1/v)(1+x^2/v)^{-1}[1+2x^2(1+x^2/v)^{-1}] \geq 0,
\end{aligned}
$$

TABLE 3. Rejection probabilities (%) for different critical value approximations, for test statistic $T_{m,\infty}$ when $\alpha = 1\%$

	rejection probability for critical value			
m	Goh (2004, simulated)	including m^{-1}	including m^{-2}	t
1	0.97	3.19	1.83	1.68
2	1.00	1.73	1.21	1.12
3	1.01	1.42	1.11	1.13
4	1.07	1.28	1.09	1.15
5	1.01	1.20	1.05	1.14
6	1.05	1.17	1.05	1.15
7	0.97	1.10	1.01	1.10
8	0.96	1.06	0.99	1.08
9	0.97	1.05	0.99	1.09
10	1.01	1.06	1.01	1.10
11	0.96	1.04	0.99	1.08
12	0.98	1.05	1.01	1.09
13	1.01	1.02	0.99	1.07
14	1.04	1.05	1.03	1.10
15	0.99	1.02	1.00	1.07
20	1.02	1.05	1.03	1.09
25	0.97	0.98	0.97	1.03
30	0.98	1.00	0.99	1.03
50	1.01	1.02	1.01	1.04

since x only appears as x^2. The sign of $3f_t'(\xi_p)^2 - f_t(\xi_p)f_t''(\xi_p)$ is thus positive at any quantile for any degrees of freedom v. Thus the sign of $u_{3,0}(z)$ is always the sign of z for the t-distribution at any quantile.

The pdf and derivatives of the exponential distribution are

$$\begin{aligned} f_e(x) &= \lambda e^{-\lambda x} \\ f_e'(x) &= -\lambda \cdot f_e(x) \\ f_e''(x) &= \lambda^2 \cdot f_e(x) \end{aligned}$$

Then,

$$\begin{aligned} &3f_e'(x)^2 - f_e(x)f_e''(x) \\ &\quad = 3\lambda^2[f_e(x)]^2 - \lambda^2[f_e(x)]^2 \\ &\quad = 2\lambda^2[f_e(x)]^2 \geq 0, \end{aligned}$$

since again everything is to an even power and positive. The sign of $3f_e'(\xi_p)^2 - f_e(\xi_p)f_e''(\xi_p)$ is thus positive at any quantile for any parameter λ. Thus the sign of $u_{3,0}(z)$ is always the sign of z for the exponential distribution at any quantile.

The pdf and derivatives of the χ^2_k distribution are

$$\begin{aligned}
f_\chi(x) &= \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2} \\
f'_\chi(x) &= f_\chi(x) \cdot [(k/2-1)x^{-1} - (1/2)] \\
f''_\chi(x) &= f'_\chi(x)[(k/2-1)x^{-1} - (1/2)] + f_\chi(x)(1-k/2)x^{-2} \\
&= f_\chi(x)\{(1-k/2)x^{-2} + [(k/2-1)x^{-1} - (1/2)]^2\}
\end{aligned}$$

Then,

$$\begin{aligned}
&3f'_\chi(x)^2 - f_\chi(x)f''_\chi(x) \\
&\quad = 3[f_\chi(x)]^2[(k/2-1)x^{-1} - (1/2)]^2 \\
&\qquad - [f_\chi(x)]^2\{(1-k/2)x^{-2} + [(k/2-1)x^{-1} - (1/2)]^2\} \\
&\quad = [f_\chi(x)]^2\{2[(k/2-1)x^{-1} - (1/2)]^2 - (1-k/2)x^{-2}\} \\
&\quad = [f_\chi(x)]^2\{2[(k/2-1)^2x^{-2} + (1/4) - (k/2-1)x^{-1}] - (1-k/2)x^{-2}\} \\
&\quad = [f_\chi(x)]^2\{2(k/2-1)^2x^{-2} + (1/2) - 2(k/2-1)x^{-1} - (1-k/2)x^{-2}\} \\
&\quad = [f_\chi(x)]^2\{(1/2) + (k-2+1)(k/2-1)x^{-2} - (k-2)x^{-1}\} \\
&\quad = [f_\chi(x)]^2\{(1/2) - (k-2)x^{-1} + (k-1)(k/2-1)x^{-2}\}.
\end{aligned}$$

Ignoring the $f_\chi(x)$, the roots of the last expression are $\{\sqrt{2-k} + k - 2, k - 2 - \sqrt{2-k}\}$. These are imaginary if $k > 2$; if $k = 2$, they are both zero, and if $k = 1$, they are $\{0, -2\}$. Since χ^2 has non-negative support only, this means that there are no quantiles ξ_p that would yield a negative value. So, again, the sign of $u_{3,0}(z)$ is the same as the sign of the critical value z, for any quantile ξ_p of any χ^2_k distribution.

The pdf and derivatives of the Fréchet distribution are

$$\begin{aligned}
f_{Fr}(x) &= \alpha x^{-1-\alpha} e^{-x^{-\alpha}} \\
f'_{Fr}(x) &= (-1-\alpha)x^{-1}x^{-1-\alpha}\alpha e^{-x^{-\alpha}} + \alpha x^{-1-\alpha}\alpha x^{-\alpha-1}e^{-x^{-\alpha}} \\
&= f_{Fr}(x) \cdot [(-1-\alpha)x^{-1} + \alpha x^{-\alpha-1}] \\
f''_{Fr}(x) &= f'_{Fr}(x)[(-1-\alpha)x^{-1} + \alpha x^{-\alpha-1}] + f_{Fr}(x)[(1+\alpha)x^{-2} - \alpha(\alpha+1)x^{-\alpha-2}] \\
&= f_{Fr}(x)[(-1-\alpha)x^{-1} + \alpha x^{-\alpha-1}]^2 + f_{Fr}(x)[(1+\alpha)x^{-2} - \alpha(\alpha+1)x^{-\alpha-2}]
\end{aligned}$$

Then,

$$\begin{aligned}
&3f'_{Fr}(x)^2 - f_{Fr}(x)f''_{Fr}(x) \\
&\quad = 3f'_{Fr}(x)^2 - f'_{Fr}(x)^2 - f_{Fr}(x)^2[(1+\alpha)x^{-2} - \alpha(\alpha+1)x^{-\alpha-2}] \\
&\quad = 2f_{Fr}(x)^2[(-1-\alpha)x^{-1} + \alpha x^{-\alpha-1}]^2 - f_{Fr}(x)^2[(1+\alpha)x^{-2} - \alpha(\alpha+1)x^{-\alpha-2}]
\end{aligned}$$

$$
\begin{aligned}
&= f_{Fr}(x)^2\{2[(-1-\alpha)^2x^{-2}+\alpha^2x^{-2\alpha-2}+2(-1-\alpha)x^{-1}\alpha x^{-\alpha-1}] \\
&\quad -[(1+\alpha)x^{-2}-\alpha(\alpha+1)x^{-\alpha-2}]\} \\
&= f_{Fr}(x)^2\{[2(1+\alpha)^2x^{-2}+2\alpha^2x^{-2\alpha-2}-4(1+\alpha)\alpha x^{-\alpha-2}] \\
&\quad -(1+\alpha)x^{-2}+\alpha(\alpha+1)x^{-\alpha-2}\} \\
&= f_{Fr}(x)^2x^{-2}\{2\alpha^2+4\alpha+2-1-\alpha+x^{-\alpha}[\alpha(\alpha+1)-4(1+\alpha)\alpha]+2\alpha^2x^{-2\alpha}\} \\
&= f_{Fr}(x)^2x^{-2}\{2\alpha^2+3\alpha+1-x^{-\alpha}3\alpha(\alpha+1)+2\alpha^2x^{-2\alpha}\}.
\end{aligned}
$$

This is always positive; there are no values of $\{\alpha, x\}$ where this is zero. So for the Fréchet distribution, too, $u_{3,0}(z)$ has sign equal to the sign of the critical value argument z.

Appendix E. Proof of Edgeworth expansion of SBG test statistic under true local alternative hypothesis

Note that, as in HS88, (a/bc) means $(\frac{a}{bc})$; for instance, $(a_2/2a_1)$ means $(\frac{a_2}{2a_1})$, and $(g''(p)/6g(p))$ means $(\frac{g''(p)}{6g(p)})$. "HS87" refers to Hall and Sheather (1987), the working paper preceding HS88, which still refers to Hall and Sheather (1988). Equation references are to equations labeled within this appendix unless explicitly said to refer to the paper. In general, (intermediate) results are stated before their derivations/calculations, so that the derivation of any given result may be skipped over if desired. The other version of this appendix skips most of the calculations.

This proof of the Edgeworth expansion theorem from the paper uses the same setup and definitions as the main text of this paper. HS88 showed the Edgeworth expansion for the test statistic $T_{m,n}$ when $\gamma = 0$; I want to show the expansion under any γ, which includes the result of HS88 as a special case ($\gamma = 0$).

As in Section 2 of the paper, the null hypothesis is $H_0 : \xi_p = \beta$, and the true $\xi_p = \beta - \gamma/\sqrt{n}$. I will continue from equation (18) in the paper, which showed that

$$
P(T_{m,n} < z) = P\left(\frac{\sqrt{n}(X_{n,r}-\xi_p)+\gamma(S_{m,n}/S_0-1)}{S_{m,n}\sqrt{p(1-p)}} < z + C\right),
$$

where $C \equiv \gamma f(\xi_p)/\sqrt{p(1-p)}$, $S_0 \equiv 1/f(\xi_p)$. I want to derive a higher-order expansion around the (shifted) standard normal distribution.

As in HS88, I must first deal with centering $X_{n,r} = X_{n,\lfloor np\rfloor+1}$ since as n increases, $\lfloor np\rfloor$ increases in discrete steps of one. Since this doesn't depend on γ, the result is the same as HS88, but I show more of the calculations here. To account for these discrete jumps in r, instead of $X_{n,r} - \xi_p$ there will be $X_{n,r} - \eta_p$, where

$$
\eta_p \equiv F^{-1}\left[\exp\left(-\sum_r^n j^{-1}\right)\right]. \tag{6}
$$

As HS88 note,

$$\sum_{r}^{n} j^{-1} = \ln(n/r) - (n-r)(2nr)^{-1} + O(n^{-2})$$
$$= \ln(p^{-1}) - (np)^{-1}[\epsilon_n - 1 + \frac{1}{2}(1-p)] + O(n^{-2}),$$

the first identity following from identity 0.131 of Gradshteyn and Ryzhik (1965).

For the intermediate steps, letting $\epsilon_n \equiv [np] + 1 - np$ (as in HS88),

$$\begin{aligned}\ln(n/r) &= \ln(n/[np]) = \ln(n/(np+\epsilon_n - 1)) = \ln(n) - \ln(np + \epsilon_n - 1)\\ &= \ln(n) - [\ln(np) + (\epsilon_n - 1)\frac{1}{np} + O(n^{-2})]\\ &= \ln(n) - \ln(n) - \ln(p) - (np)^{-1}(\epsilon_n - 1) + O(n^{-2})\\ &= \ln(p^{-1}) - (np)^{-1}(\epsilon_n - 1) + O(n^{-2}), \text{ and}\end{aligned}$$

$$\begin{aligned}(n-r)(2nr)^{-1} &= 1/(2r) - 1/(2n) = \frac{1}{2}(1/(np+\epsilon_n - 1) - p/(pn))\\ &= \frac{1}{2}\left((np)^{-1} - \frac{\epsilon_n - 1}{n}(np)^{-1} + (-(\epsilon_n - 1)/n)^2(np)^{-1} + \ldots - p(np)^{-1}\right)\\ &= \frac{1}{2}\left((np)^{-1} + O(n^{-2}) - p(np)^{-1}\right)\\ &= (np)^{-1}(\frac{1}{2}(1-p)) + O(n^{-2}).\end{aligned}$$

Putting these together gets

$$\sum_{r}^{n} j^{-1} = \ln(p^{-1}) - (np)^{-1}[\epsilon_n - 1 + \frac{1}{2}(1-p)] + O(n^{-2}). \tag{7}$$

HS88 conclude that if f has a bounded derivative in a neighborhood of ξ_p and if $f(\xi_p) > 0$,

$$\eta_p = \xi_p + n^{-1}[\epsilon_n - 1 + \frac{1}{2}(1-p)]/f(\xi_p) + O(n^{-2}), \tag{8}$$

which is derived by plugging into the definition of η_p and taking a first-order Taylor expansion around p, noting that $F^{-1}(pc) = F^{-1}(p) + \frac{1}{f(F^{-1}(p))}p(c-1) + \ldots$ and $F^{-1}(p) \equiv \xi_p$, so

$$\begin{aligned}\eta_p &\equiv F^{-1}\left[\exp\left(-\sum_{r}^{n} j^{-1}\right)\right]\\ &= F^{-1}\left[\exp\left(-\left(\ln(p^{-1}) - (np)^{-1}[\epsilon_n - 1 + \frac{1}{2}(1-p)] + O(n^{-2})\right)\right)\right]\\ &= \xi_p + \frac{1}{f(\xi_p)}\left[\exp(-\ln(p^{-1}))\left(\exp\left\{(np)^{-1}(\epsilon_n - 1 + (1/2)(1-p)) + O(n^{-2})\right\} - 1\right)\right]\end{aligned}$$

$$
\begin{aligned}
&= \xi_p + \frac{1}{f(\xi_p)}\left[p\left(\exp\left\{(np)^{-1}(\epsilon_n - 1 + (1/2)(1-p)) + O(n^{-2})\right\} - 1\right)\right] \\
&= \xi_p + \frac{1}{f(\xi_p)}\left[p\left((np)^{-1}(\epsilon_n - 1 + (1/2)(1-p)) + O(n^{-2})\right)\right], \text{ as } e^x - 1 = x + O(x^2), \\
&= \xi_p + \frac{1}{f(\xi_p)} n^{-1}(\epsilon_n - 1 + (1/2)(1-p)) + O(n^{-2})
\end{aligned}
$$

HS88 continue from (8) and the fact that $S_{m,n} = 1/f(\xi_p) + O_p[m^{-1/2} + (m/n)^2]$ (see Bloch and Gastwirth (1968)), deducing

$$n^{1/2}(X_{n,r} - \xi_p)/\hat{\tau} = n^{1/2}(X_{n,r} - \eta_p)/\hat{\tau} + n^{-1/2}w_n + O_p[n^{-1/2}m^{-1/2} + (m/n)^2 n^{-1/2}],$$

where $w_n \equiv [\epsilon_n - 1 + \frac{1}{2}(1-p)]/[p(1-p)]^{1/2}$, which can be derived by writing

$$
\begin{aligned}
\frac{\sqrt{n}(\eta_p - \xi_p)}{\hat{\tau}} &= \frac{\sqrt{n}(\eta_p - \xi_p)}{\sqrt{p(1-p)}S_{m,n}} \\
&= \frac{\sqrt{n}}{\sqrt{p(1-p)}S_{m,n}}\Big(\frac{1}{n}\frac{\epsilon_n - 1 + (1/2)(1-p)}{f(\xi_p)} \\
&\quad + O(n^{-2})\Big) \text{ (using above)} \\
&= n^{-1/2}\frac{\epsilon_n - 1 + (1/2)(1-p)}{\sqrt{p(1-p)}}\left[1 + O_p\left(m^{-1/2} + (m/n)^2\right)\right] \\
&\quad + O(n^{-3/2}) \text{ (using Bloch and Gastwirth (1968))} \\
&= n^{-1/2}w_n + O_p\left(n^{-1/2}m^{-1/2} + n^{-1/2}(m/n)^2\right) \qquad (9)
\end{aligned}
$$

Then (following HS88) the remainder $n^{-1/2}m^{-1/2} + (m/n)^2 n^{-1/2} = o[m^{-1} + (m/n)^2]$, and so

$$
\begin{aligned}
P[n^{1/2}(X_{n,r} - \xi_p)/\tau \le z] &= P[n^{1/2}(X_{n,r} - \eta_p)/\tau \le z - n^{-1/2}w_n] + O(n^{-3/2}) \\
&= P[n^{1/2}(X_{n,r} - \eta_p)/\tau \le z] - n^{-1/2}w_n\phi(z) + O(n^{-1}),
\end{aligned}
$$

$$
\begin{aligned}
P\left[\frac{n^{1/2}(X_{n,r} - \xi_p) + \gamma(S_{m,n}/S_0 - 1)}{\hat{\tau}} \le z\right] &= P\left[\frac{n^{1/2}(X_{n,r} - \eta_p) + \gamma(S_{m,n}/S_0 - 1)}{\hat{\tau}} \le z\right] \\
&\quad - n^{-1/2}w_n\phi(z) + o[m^{-1} + (m/n)^2].
\end{aligned}
$$

(As HS88 note, this is the so-called 'delta method' for deducing Edgeworth expansions; see Bhattacharya and Ghosh (1978), p. 438. It may be made rigorous by pursuing arguments similar to those given here.) Therefore the theorem will follow if, instead of proving equation

(12) from the paper, I prove that

$$\sup_{-\infty<z<\infty}\left| P\left[\frac{n^{1/2}(X_{n,r}-\eta_p)+\gamma(S_{m,n}/S_0-1)}{\hat{\tau}}\le z\right]\right.$$
$$\left. -\left[\Phi(z)+n^{-1/2}u_{1,\gamma}^{\dagger}(z)\phi(z)+m^{-1}u_{2,\gamma}(z)\phi(z)+(m/n)^2u_{3,\gamma}(z)\phi(z)\right]\right|$$
$$= o[m^{-1}+(m/n)^2], \tag{10}$$

where

$$u_{1,\gamma}^{\dagger}(z)\equiv\frac{1}{6}\left(\frac{p}{1-p}\right)^{1/2}\frac{1+p}{p}(z^2-1)-\frac{\gamma f(\xi_p)}{p}\left(1-\frac{pf'(\xi_p)}{[f(\xi_p)]^2}-\frac{1}{2(1-p)}\right)z$$
$$-\frac{1}{2}\left(\frac{p}{1-p}\right)^{1/2}\left(1+\frac{f'(\xi_p)}{[f(\xi_p)]^2}(1-p)\right)z^2,$$
$$u_{2,\gamma}(z)\equiv\frac{1}{4}\left[\frac{2\gamma f(\xi_p)}{[p(1-p)]^{1/2}}z^2-\frac{\gamma^2[f(\xi_p)]^2}{p(1-p)}z-z^3\right],$$
$$u_{3,\gamma}(z)\equiv\frac{f''(\xi_p)f(\xi_p)+3[f'(\xi_p)]^2}{6[f(\xi_p)]^4}\left(\frac{\gamma f(\xi_p)}{[p(1-p)]^{1/2}}-z\right).$$

Now to prove the above. As defined in HS88, let

$$G\equiv F^{-1}, g\equiv G',\ \text{and}\ H(x)\equiv F^{-1}(e^{-x}), \tag{11}$$

and let $W_1, W_2, \ldots,$ be independent exponential random variables with unit mean. The sequence $\{X_{n,s}, 1\le s\le n\}$ has the same joint distribution as $\{H(\sum_{s\le j\le n} j^{-1}W_j), 1\le s\le n\}$ (David (1981), p. 21), and in a slight abuse of notation as in HS88, write $X_{n,s}=H(\sum_{s\le j\le n} j^{-1}W_j)$. Let $V_j\equiv W_j-1$,

$$\Delta_1\equiv\sum_{r-m}^{r-1}j^{-1}V_j,\quad \Delta_2\equiv\sum_{r}^{r+m-1}j^{-1}V_j,\quad \Delta_3\equiv\sum_{r+m}^{n}j^{-1}V_j,\quad a_k\equiv H^{(k)}\left(\sum_{r}^{n}j^{-1}\right),$$
$$\begin{aligned} Z&\equiv[p(1-p)]^{1/2}[n^{1/2}(X_{n,r}-\eta_p)+\gamma(S_{m,n}/S_0-1)]/\hat{\tau}\\ &=[n^{1/2}(X_{n,r}-\eta_p)+\gamma(S_{m,n}/S_0-1)][(n/2m)(X_{n,r+m}-X_{n,r-m})]^{-1}.\end{aligned} \tag{12}$$

Note that Δ_1 and Δ_2 are $O_p(n^{-1}m^{1/2})$ and Δ_3 is $O_p(n^{-1/2})$. Informally, Δ_1 and Δ_2 are sums of m terms, and asymptotically j is $O(n)$ since the smallest j ever gets is $r-m=np-m$ and $(np-m)/n=p-m/n\to p$. To be $O_p(n^{-1}m^{1/2})$, multiplying by $nm^{-1/2}$ should yield some non-degenerate asymptotic distribution. So, $nm^{-1/2}\Delta_1\approx\sqrt{m}\frac{1}{m}\sum_{np-m}^{np-1}(n/j)V_j\approx\sqrt{m}\overline{V_j}\to_d N(0,\nu^2)$ and indeed the statement for the first two holds. Similarly for Δ_3, note that $\sqrt{n}\Delta_3\approx\sqrt{n}\frac{1}{n}\sum_{np+m}^{n}(n/j)V_j\approx n^{-1/2}(n(1-p))^{1/2}(n(1-p))^{-1/2}\sum_{j=1}^{n(1-p)}V_j\to_d(1-p)^{1/2}N(0,1)=N(0,(1-p))$.

Alternatively, the variances can be calculated. This is already done later in (58) and (57), where it is shown that $E(\Delta_3^2) = O(1/n)$ and $E(\Delta_1^2) = O(m/n^2)$, as with Δ_2. Taking square roots, then,

$$\Delta_3 = O(1/\sqrt{n}) \text{ and} \tag{13}$$
$$\Delta_2 = O(m^{1/2}n^{-1}) \tag{14}$$

(as with Δ_1).

Following the structure of HS88 again, with the above and Taylor expansions, $Z = Y + R$ where

$$Y \equiv -pn^{1/2}[(1+\delta)(\Delta_2 + \Delta_3) + B], \quad \delta \equiv -(m/n)^2 g''(p)[6g(p)]^{-1}, \tag{15}$$

$$\begin{aligned} B \equiv{}& \delta\Psi + (n/m)(b_1\Delta_1 + b_2\Delta_2)\Delta_2 + (n/m)b_3(\Delta_1 + \Delta_2)(\Delta_3 + \Psi) \\ &+ (n/m)^2 b_4(\Delta_1 + \Delta_2)^2(\Delta_3 + \Psi) + b_5\Delta_3(\Delta_3 + 2\Psi), \end{aligned} \tag{16}$$

$b_1 \equiv -p/2, b_2 \equiv -p/2, b_3 \equiv -p/2 - (m/n)(a_2/2a_1), b_4 \equiv (p/2)^2, b_5 \equiv -a_2/2a_1, \Psi \equiv \gamma/(pg(p)\sqrt{n})$, and

$$R = O_p[n^{-1/2}m^{-1/2} + n^{-3/2}m + m^{-3/2} + (m/n)^{2+\epsilon}]. \tag{17}$$

Here, Y is of the same form as HS88, but with Ψ now additionally showing up in the higher-order B terms.

Note the Taylor expansion

$$\begin{aligned} H\left(\sum W_j/j\right) &= H\left(\sum 1/j\right) + \left(\sum W_j/j - \sum 1/j\right) H'\left(\sum 1/j\right) \\ &\quad + (1/2)\left(\sum W_j/j - \sum 1/j\right)^2 H''\left(\sum 1/j\right) + \ldots, \\ H\left(\sum W_j/j\right) - H\left(\sum 1/j\right) &= \left(\sum V_j/j\right) H'\left(\sum 1/j\right) \\ &\quad + (1/2)\left(\sum V_j/j\right)^2 H''\left(\sum 1/j\right) + \ldots, \end{aligned} \tag{18}$$

where all the sums are over the same range.

For the numerator of Z: since $X_{n,r} = H(\sum_{j\geq r} W_j/j)$ and $\eta_p \equiv H(\sum_r^n 1/j)$, applying the expansion above yields

$$\begin{aligned} X_{n,r} - \eta_p &= \left(\sum_{j=r}^n V_j/j\right) H'\left(\sum_r^n 1/j\right) + \frac{1}{2}\left(\sum_{j=r}^n V_j/j\right)^2 H''\left(\sum_r^n 1/j\right) + O_p(n^{-3/2})a_3 \\ &= (\Delta_2 + \Delta_3)a_1 + \frac{1}{2}(\Delta_2 + \Delta_3)^2 a_2 + O_p(n^{-3/2}), \end{aligned} \tag{19}$$

since $a_3 = O(1)$, shown below following (49).

When $\gamma \neq 0$, the term $\gamma(S_{m,n}/S_0 - 1) \neq 0$, too. The only random element of this expression is $S_{m,n}$, which is the denominator of Z. In (37), the stochastic expansion for $S_{m,n}$ divided by

$g(p)(=S_0$, cf. (44)) is written $1+\nu$ (where ν also contains remainder terms), so the expansion of $S_{m,n}$ is $S_0(1+\nu)$, and consequently $S_{m,n}/S_0-1$ expands to $S_0(1+\nu)/S_0-1=\nu$. Since γ is a constant, this means that the stochastic expansion of the numerator includes the additional term $\gamma\nu$ with ν as defined in (39). Thus,

$$\begin{aligned}&n^{1/2}(X_{n,r}-\eta_p)+\gamma(S_{m,n}/S_0-1)\\ &\qquad=\sqrt{n}\left[(\Delta_2+\Delta_3)a_1+(1/2)(\Delta_2+\Delta_3)^2a_2+O_p(n^{-3/2})+\gamma\nu\right].\end{aligned}\tag{20}$$

For the denominator of Z: the following work, which is no different when $\gamma\neq 0$, rigorously proves the intermediate results in HS87, which are

$$\begin{aligned}&X_{n,r+m}-H\left(\sum_{r+m}^{n}j^{-1}\right)-\left\{X_{n,r-m}-H\left(\sum_{r-m}^{n}j^{-1}\right)\right\}\\ &\quad=-a_1(\Delta_1+\Delta_2)-(m/np)a_2(\Delta_1+\Delta_2+2\Delta_3)+O_p(n^{-3/2}m^{1/2}+n^{-5/2}m^2),\\ &H\left(\sum_{r+m}^{n}j^{-1}\right)-H\left(\sum_{r-m}^{n}j^{-1}\right)\\ &\quad=(2m/n)\left[g(p)+(1/6)(m/n)^2g''(p)+O\{(m/n)^{2+\epsilon}+n^{-1}\}\right].\end{aligned}$$

The denominator of Z in (12) is

$$\begin{aligned}&(n/2m)(X_{n,r+m}-X_{n,r-m})\\ &\qquad=(n/2m)\Bigg[X_{n,r+m}-H\left(\sum_{r+m}^{n}j^{-1}\right)-\left\{X_{n,r-m}-H\left(\sum_{r-m}^{n}j^{-1}\right)\right\}\\ &\qquad\qquad+H\left(\sum_{r+m}^{n}j^{-1}\right)-H\left(\sum_{r-m}^{n}j^{-1}\right)\Bigg].\end{aligned}\tag{21}$$

Define

$$\delta_1\equiv\sum_{j=r-m}^{r-1}j^{-1},\quad \delta_2\equiv\sum_{j=r}^{r+m-1}j^{-1},\quad \delta_3\equiv\sum_{j=r+m}^{n}j^{-1},\tag{22}$$

so that $\Delta_1+\delta_1=\sum_{j=r-m}^{r-1}W_j/j$, etc.

Using Euler-Maclaurin,

$$\begin{aligned}\delta_1&\equiv\sum_{r-m}^{r-1}1/j=\sum_{r-m}^{r}1/j-1/r=\sum_{0}^{m}(r-m+i)^{-1}-1/r\\ &=\int_0^m(r-m+x)^{-1}dx+(1/2)(r^{-1}+(r-m)^{-1})-1/r+O(f^{(1)}(0))\\ &=\ln(r-m+x)|_0^m+(1/2)\frac{r-(r-m)}{r(r-m)}+O(n^{-2})\end{aligned}$$

$$= \ln \frac{r}{r-m} + (1/2)\frac{m}{r^2(1-m/r)} + O(n^{-2})$$
$$= \ln(1 + \frac{m}{r-m}) + O(mn^{-2} + n^{-2})$$
$$= \ln(1) + \frac{1}{1}\frac{m}{r-m} - \frac{1}{2}\frac{1}{1^2}\frac{m^2}{(r-m)^2} + \frac{1}{6}\frac{2}{1^3}\frac{m^3}{(r-m)^3}$$
$$+ O(m^4/n^4) + O(mn^{-2} + n^{-2})$$
$$= \frac{m}{r}\frac{1}{1-m/r} - \frac{1}{2}\frac{m^2}{r^2}\frac{1}{1-(2m/r)+m^2/r^2} + \frac{1}{3}\frac{m^3}{r^3}\frac{1}{1-O(m/n)}$$
$$+ O(m^4/n^4 + mn^{-2} + n^{-2})$$
$$= \frac{m}{r}(1 + (m/r) + (m/r)^2 + O(m^3/n^3)) - \frac{1}{2}\frac{m^2}{r^2}(1 + (2m/r) + O(m^2/n^2))$$
$$+ \frac{1}{3}\frac{m^3}{r^3}(1 + O(m/n)) + O(m^4/n^4 + mn^{-2} + n^{-2})$$
$$= \frac{m}{r} + (m/r)^2 + (m/r)^3 - \frac{1}{2}\frac{m^2}{r^2} - (m/r)^3 + \frac{1}{3}\frac{m^3}{r^3} + O(m^4/n^4 + mn^{-2} + n^{-2})$$
$$= \frac{m}{np} + \frac{1}{2}\frac{m^2}{n^2p^2} + \frac{1}{3}\frac{m^3}{n^3p^3} + O(m^4/n^4 + mn^{-2} + n^{-2}). \tag{23}$$

Again using Euler-Maclaurin,

$$\delta_2 \equiv \sum_{r}^{r+m-1} 1/j = \sum_{r}^{r+m} 1/j - 1/(r+m) = \sum_{0}^{m}(r+i)^{-1} - 1/(r+m)$$
$$= \int_0^m (r+x)^{-1}dx + (1/2)((r+m)^{-1} + (r)^{-1}) - 1/(r+m) + O(f^{(1)}(0))$$
$$= \ln(r+x)|_0^m + (1/2)\frac{(r+m)-r}{r(r+m)} + O(n^{-2})$$
$$= \ln\frac{r+m}{r} + (1/2)\frac{m}{r^2(1+m/r)} + O(n^{-2})$$
$$= \ln(1 + \frac{m}{r}) + O(mn^{-2} + n^{-2})$$
$$= \ln(1) + \frac{1}{1}\frac{m}{r} - \frac{1}{2}\frac{1}{1^2}\frac{m^2}{r^2} + \frac{1}{6}\frac{2}{1^3}\frac{m^3}{r^3} + O(m^4/n^4) + O(mn^{-2} + n^{-2})$$
$$= \frac{m}{np} - \frac{m^2}{2n^2p^2} + \frac{m^3}{3n^3p^3} + O(m^4/n^4 + mn^{-2}), \tag{24}$$

and again using Euler-Maclaurin,

$$\delta_3 \equiv \sum_{r+m}^{n} 1/j = \sum_{0}^{s}(r+m+i)^{-1}$$
$$= \int_0^s (r+m+x)^{-1}dx + (1/2)(n^{-1} + (r+m)^{-1}) + O(f^{(1)}(0))$$

$$
\begin{aligned}
&= \ln(r+m+x)|_0^s + (1/2)\frac{r+m+n}{n(r+m)} + O(n^{-2}) \\
&= \ln\frac{n}{r+m} + (1/2)\frac{n(1+p)}{n^2p(1+(m/np))} + O(mn^{-2}+n^{-2}) \\
&= \ln\frac{n}{r+m} + \frac{1+p}{2np} + O(mn^{-2}) = \ln\frac{n}{r+m} + \ln(1+\frac{1+p}{2np}) + O(n^{-2}) + O(mn^{-2}) \\
&= \ln\left\{\frac{n}{r+m}(1+\frac{1+p}{2np})\right\} + O(mn^{-2}) = \ln\left\{\frac{n}{r+m} + \frac{n(1+p)}{2np(r+m)})\right\} + O(mn^{-2})
\end{aligned}
$$

$$
= \ln\left\{\frac{2n^2p+n(1+p)}{2np(r+m)}\right\} + O(mn^{-2}). \tag{25}
$$

The first needed intermediate result is for $X_{n,r+m}-H(\sum_{r+m}^n j^{-1})-\{X_{n,r-m}-H(\sum_{r-m}^n j^{-1})\}$, which in HS87 is equal to $-a_1(\Delta_1+\Delta_2)-(m/np)a_2(\Delta_1+\Delta_2+2\Delta_3)+O_p(n^{-3/2}m^{1/2}+n^{-5/2}m^2)$. Note: the following assumes that $H'''(\delta_3)$ and $H'''(\delta_1+\delta_2+\delta_3)$ both exist and are $O(1)$. If $m\to\infty, (m/n)\to 0$, then this is true as long as $a_3 = O(1)$, which is shown following (49) to be true assuming $f(\xi_p) > 0$, $f'(\xi_p) < \infty$, and $f''(\xi_p) < \infty$.

In above notation, and using the above expansion,

$$
\begin{aligned}
&X_{n,r+m} - H(\sum_{r+m}^{n} j^{-1}) - \{X_{n,r-m} - H(\sum_{r-m}^{n} j^{-1})\} \\
&\quad = H(\sum_{r+m}^{n} j^{-1}W_j) - H(\sum_{r+m}^{n} j^{-1}) - \{H(\sum_{r-m}^{n} j^{-1}W_j) - H(\sum_{r-m}^{n} j^{-1})\} \\
&\quad = \{(\sum_{r+m}^{n} V_j/j)H'(\sum_{r+m}^{n} 1/j) + (1/2)(\sum_{r+m}^{n} V_j/j)^2 H''(\sum_{r+m}^{n} 1/j) + \ldots\} \\
&\qquad - \{(\sum_{r-m}^{n} V_j/j)H'(\sum_{r-m}^{n} 1/j) + (1/2)(\sum_{r-m}^{n} V_j/j)^2 H''(\sum_{r-m}^{n} 1/j) + \ldots\} \\
&\quad = \{(\Delta_3)[H'(\delta_3)] + (1/2)(\Delta_3)^2[H''(\delta_3)] + O(\Delta_3^3 H'''(\delta_3))\} \\
&\qquad - \{(\Delta_1+\Delta_2+\Delta_3)[H'(\delta_1+\delta_2+\delta_3)] \\
&\qquad\quad + (1/2)(\Delta_1+\Delta_2+\Delta_3)^2[H''(\delta_1+\delta_2+\delta_3)] \\
&\qquad\quad + O(\Delta_3^3 H'''(\delta_1+\delta_2+\delta_3))\} \\
&\quad = \{\Delta_3[H'(\delta_2+\delta_3) + (-\delta_2)H''(\delta_2+\delta_3) + (1/2)\delta_2^2 H'''(\delta_2+\delta_3) + O_p(m^3/n^3)] \\
&\qquad\quad + (1/2)\Delta_3^2[H''(\delta_2+\delta_3) + (-\delta_2)H'''(\delta_2+\delta_3) + O_p(m^2/n^2)] + O_p(n^{-3/2})\} \\
&\qquad - \{(\Delta_1+\Delta_2+\Delta_3) \\
&\qquad\qquad \times [H'(\delta_2+\delta_3) + \delta_1 H''(\delta_2+\delta_3) + (1/2)\delta_1^2 H'''(\delta_2+\delta_3) + O_p(m^3/n^3)] \\
&\qquad\quad + (1/2)(\Delta_1+\Delta_2+\Delta_3)^2 \\
&\qquad\qquad \times [H''(\delta_2+\delta_3) + \delta_1 H'''(\delta_2+\delta_3) + O_p(m^2/n^2)] + O_p(n^{-3/2})\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{\Delta_3 a_1 - \Delta_3\delta_2 a_2 + O_p(n^{-1/2}m^2/n^2 + n^{-1/2}m^3/n^3) \\
&\qquad + (1/2)\Delta_3^2 a_2 - O_p(n^{-1}m/n + m^2/n^3) + O_p(n^{-3/2})\} \\
&\quad - \{(\Delta_1+\Delta_2+\Delta_3)a_1 + (\Delta_1+\Delta_2+\Delta_3)\delta_1 a_2 \\
&\qquad + O_p(n^{-1/2}m^2/n^2 + n^{-1/2}m^3/n^3) \\
&\qquad + (1/2)(\Delta_1+\Delta_2+\Delta_3)^2 a_2 + O_p(n^{-1}m/n + n^{-1}m^2/n^2) + O_p(n^{-3/2})\} \\
&= a_1[\Delta_3 - (\Delta_1+\Delta_2+\Delta_3)] \\
&\quad + a_2[-\Delta_3\delta_2 + (1/2)\Delta_3^2 - (\Delta_1+\Delta_2+\Delta_3)\delta_1 - (1/2)(\Delta_1+\Delta_2+\Delta_3)^2] \\
&\quad + O_p(n^{-3/2}m^{1/2} + n^{-5/2}m^2) \\
&= -a_1(\Delta_1+\Delta_2) \\
&\quad + a_2[-\Delta_3(\delta_2+\delta_1) - (\Delta_1+\Delta_2)\delta_1 + (1/2)\Delta_3^2 - (1/2)\Delta_3^2 + O_p(m^{1/2}n^{-1}n^{-1/2})] \\
&\quad + O_p(n^{-3/2}m^{1/2} + n^{-5/2}m^2) \\
&= -a_1(\Delta_1+\Delta_2) - \delta_1 a_2(\Delta_1+\Delta_2+2\Delta_3) + O_p(n^{-3/2}m^{1/2} + n^{-5/2}m^2), \qquad (26)
\end{aligned}
$$

which verifies the result from HS87 since, as shown in (23), $\delta_1 = (m/np) + O(m^2/n^2)$.

The second needed intermediate result is for $H(\sum_{r+m}^{n} j^{-1}) - H(\sum_{r-m}^{n} j^{-1}) = H(\delta_3) - H(\delta_1+\delta_2+\delta_3)$. From HS87, this should equal $(2m/n)[g(p)+(1/6)(m/n)^2 g''(p)+O\{(m/n)^{2+\epsilon}+n^{-1}\}]$.

Using a Taylor expansion around δ_3 and then directly approximating $H'(\delta_3)$,

$$
\begin{aligned}
H(\delta_1+\delta_2+\delta_3) - H(\delta_3) &= (\delta_1+\delta_2)H'(\delta_3) + (1/2)(\delta_1+\delta_2)^2 H''(\delta_3) \\
&\quad + \frac{1}{6}(\delta_1+\delta_2)^3 H'''(\delta_3) + O(m^4/n^4) \qquad (27)
\end{aligned}
$$

Now to plug in the approximation of δ_3 from (25). Note that for some small ϵ, $e^{-x+\epsilon} = e^{-x} + O(\epsilon(-e^{-x})) = e^{-x} + O(\epsilon)$, and $f(e^{-x}+\epsilon) = f(e^{-x}) + f'(e^{-x})O(\epsilon) = f(e^{-x}) + O(\epsilon)$ if $f'(e^{-x}) = O(1)$. From (25), $\delta_3 = \ln\{\frac{2n^2p+n(1+p)}{2np(r+m)}\} + O(mn^{-2})$.

$$
\begin{aligned}
H'(\delta_3) &= -\frac{\exp(-\delta_3)}{f(F^{-1}(\exp(-\delta_3)))} \\
&= -\frac{\exp(-[\ln\{\frac{2n^2p+n(1+p)}{2np(r+m)}\} + O(mn^{-2})])}{f(F^{-1}(\exp(-[\ln\{\frac{2n^2p+n(1+p)}{2np(r+m)}\} + O(mn^{-2})])))} \\
&= \frac{-\frac{2np(r+m)}{2n^2p+n(1+p)} + O(mn^{-2})}{f(F^{-1}(\frac{2np(r+m)}{2n^2p+n(1+p)} + O(mn^{-2})))} \\
&= \left[-\frac{2np(r+m)}{2n^2p+n(1+p)} + O(mn^{-2})\right]\left[g(\frac{2np(r+m)}{2n^2p+n(1+p)}) + O(mn^{-2})\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \left[-\frac{2pr+2pm}{2np(1+(1+p)/(2np))}\right] g(\cdot) + O(mn^{-2}) \\
&= \left[-\frac{2pnp}{2np}(1-(1+p)/(2np)+O(n^{-2})) - \frac{2pm}{2np}(1-(1+p)/(2np)+O(n^{-2}))\right] \\
&\quad \times g(\cdot) + O(mn^{-2}) \\
&= [-p + (1+p)/(2n) - \frac{m}{n} + O(mn^{-2})]g(\cdot) + O(mn^{-2}) \\
&= [-p - \frac{m}{n} + \frac{1+p}{2n}]g(p + \frac{m}{n} - \frac{1+p}{2n}) + O(mn^{-2}).
\end{aligned}
$$

Continuing,

$$
\begin{aligned}
g(p+\epsilon) &= g(p) + g'(p)\epsilon + (1/2)g''(p)\epsilon^2 + O(\epsilon^3), \\
g(p + \frac{m}{n} - \frac{1+p}{2n}) &= g(p) + g'(p)[\frac{m}{n} - \frac{1+p}{2n}] \\
&\quad + (1/2)g''(p)[\frac{m}{n} - \frac{1+p}{2n}]^2 + O(m^3/n^3), \\
H'(\delta_3) &= [-p - \frac{m}{n} + \frac{1+p}{2n}]g(p + \frac{m}{n} - \frac{1+p}{2n}) \\
&= \left[-p - \frac{m}{n} + \frac{1+p}{2n}\right] \\
&\quad \times \Big[g(p) + g'(p)[\frac{m}{n} - \frac{1+p}{2n}] \\
&\qquad + (1/2)g''(p)[\frac{m^2}{n^2} - O(mn^{-2})] + O(m^3/n^3)\Big] \\
&= -pg(p) - \frac{m}{n}[g(p) + pg'(p)] - \frac{m^2}{n^2}\left[g'(p) + \frac{pg''(p)}{2}\right] + O(n^{-1}) \qquad (28) \\
&\quad + O(mn^{-2} + m^3/n^3).
\end{aligned}
$$

From (47) below, $H''(x)$ can be represented as $g'(e^{-x})e^{-2x} - H'(x)$.

$$
\begin{aligned}
H''(\delta_3) &= g'(e^{-\delta_3})e^{-2\delta_3} - H'(\delta_3) \\
&= g'(e^{-\ln\frac{n}{r+m} + O(n^{-1})})[e^{-\delta_3}]^2 - H'(\delta_3) \\
&= g'(\frac{r+m}{n}(1+O(n^{-1})))[\frac{r+m}{n}(1+O(n^{-1}))]^2 - H'(\delta_3) \\
&= g'(p+m/n)[p+m/n]^2 - H'(\delta_3) \\
&= [g'(p) + (m/n)g''(p) + O(m^2/n^2)][p^2 + 2pm/n + (m/n)^2] - H'(\delta_3) \\
&= g'(p)[p^2 + 2pm/n + (m/n)^2] + (m/n)g''(p)[p^2 + 2pm/n + (m/n)^2] \\
&\quad + O(m^2/n^2) - H'(\delta_3) \\
&= g'(p)p^2 + 2g'(p)pm/n + g'(p)(m/n)^2 + (m/n)g''(p)p^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(m/n)g''(p)pm/n - H'(\delta_3) + O(m^2/n^2) \\
&= (m/n)g''(p)p^2 + g'(p)p^2 + 2g'(p)pm/n - H'(\delta_3) + O(m^2/n^2). \qquad (29)
\end{aligned}
$$

Using results from (48) below,

$$
\begin{aligned}
H'''(\delta_3) &= -g''(\exp(-\delta_3))(\exp(-\delta_3))^3 - 3H''(\delta_3) - 2H'(\delta_3) \\
&= -g''(p)p^3 - 3H''(\delta_3) - 2H'(\delta_3) + O(m/n) \qquad (30)
\end{aligned}
$$

From (23) and (24) above,

$$
\begin{aligned}
\delta_1 + \delta_2 &= \frac{2m}{np} + \frac{m^2}{2n^2p^2} - \frac{m^2}{2n^2p^2} + \frac{2m^3}{3n^3p^3} + O(m^4/n^4 + mn^{-2}) \\
&= \frac{2m}{np} + \frac{2m^3}{3n^3p^3} + O(m^4/n^4 + mn^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
(\delta_1 + \delta_2)H'(\delta_3) &= [\frac{2m}{np} + \frac{2m^3}{3n^3p^3} + O(m^4/n^4 + mn^{-2})] \\
&\quad \times [-pg(p) - \frac{m}{n}[g(p) + pg'(p)] - \frac{m^2}{n^2}[g'(p) + \frac{pg''(p)}{2}] + O(n^{-1} + m^3/n^3)] \\
&= [\frac{2m}{np} + \frac{2m^3}{3n^3p^3} + O(m^4/n^4 + mn^{-2})] \\
&\quad \times [-pg(p) - \frac{m}{n}[g(p) + pg'(p)] - \frac{m^2}{n^2}[g'(p) + \frac{pg''(p)}{2}] + O(n^{-1} + m^3/n^3)] \\
&= \frac{m}{n}[-2g(p)] + \frac{m^2}{n^2}[-\frac{2g(p)}{p} - 2g'(p)] \\
&\quad + \frac{m^3}{n^3}[-\frac{2g(p)}{3p^2} - \frac{2g'(p)}{p} - g''(p)] + O(m^4/n^4 + mn^{-2}). \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}(\delta_1 + \delta_2)^2 H''(\delta_3) &= \frac{1}{2}(\frac{2m}{np} + \frac{2m^3}{3n^3p^3} + O(m^4/n^4 + mn^{-2}))^2 \\
&\quad \times [(m/n)g''(p)p^2 + g'(p)p^2 + 2g'(p)pm/n - H'(\delta_3) + O(m^2/n^2)] \\
&= \frac{1}{2}(\frac{4m^2}{n^2p^2} + O(m^4/n^4)) \\
&\quad \times \{(m/n)g''(p)p^2 + g'(p)p^2 + 2g'(p)pm/n \\
&\qquad - [-pg(p) - \frac{m}{n}[g(p) + pg'(p)] - O(m^2/n^2)] \\
&\qquad + O(n^{-1}) + O(mn^{-2} + m^3/n^3)] + O(m^2/n^2)\} \\
&= (\frac{m^2}{n^2}\frac{2}{p^2})\{[g'(p)p^2 + pg(p)] + \frac{m}{n}[g''(p)p^2 + 2g'(p)p + g(p) + pg'(p)]\} \\
&\quad + O(m^4/n^4 + m^2n^{-3})\}
\end{aligned}
$$

$$= \frac{m^2}{n^2}[2g'(p) + \frac{2g(p)}{p}] + \frac{m^3}{n^3}[2g''(p) + \frac{6g'(p)}{p} + \frac{2g(p)}{p^2}]$$
$$+ O(m^4/n^4 + m^2 n^{-3})$$
$$= \frac{m^2}{n^2}[2g'(p) + \frac{2g(p)}{p}] + \frac{m^3}{n^3}[\frac{2g(p)}{p^2} + \frac{6g'(p)}{p} + 2g''(p)] \tag{32}$$
$$+ O(m^4/n^4 + m^2 n^{-3}).$$

$$\frac{1}{6}(\delta_1+\delta_2)^3 H'''(\delta_3) = \frac{1}{6}\left(\frac{2m}{np} + \frac{2m^3}{3n^3p^3} + O(m^4/n^4 + mn^{-2})\right)^3$$
$$\times [-g''(p)p^3 - 3H''(\delta_3) - 2H'(\delta_3) + O(m/n)]$$
$$= \frac{1}{6}\left(\frac{8m^3}{n^3p^3} + O(m^4/n^4)\right)$$
$$\times [-g''(p)p^3 - 3[g'(p)p^2 - H'(\delta_3) + O(m/n)] - 2H'(\delta_3) + O(m/n)]$$
$$= (\frac{4m^3}{3n^3p^3})[-g''(p)p^3 - 3[g'(p)p^2] + H'(\delta_3)] + O(m^4/n^4)$$
$$= \frac{4m^3}{3n^3p^3}[-g''(p)p^3 - 3g'(p)p^2 - pg(p)] + O(m^4/n^4)$$
$$= -\frac{m^3}{n^3}\left(\frac{4g(p)}{3p^2} + \frac{4g'(p)}{p} + \frac{4g''(p)}{3}\right) + O(m^4/n^4). \tag{33}$$

Adding (31), (32), and (33), the expansion of the second part of the denominator in (27) becomes

$$H(\delta_1+\delta_2+\delta_3) - H(\delta_3)$$
$$= (\delta_1+\delta_2)H'(\delta_3) + (1/2)(\delta_1+\delta_2)^2 H''(\delta_3)$$
$$+ \frac{1}{6}(\delta_1+\delta_2)^3 H'''(\delta_3) + O(m^4/n^4)$$
$$= \frac{m}{n}[-2g(p)] + \frac{m^2}{n^2}[-\frac{2g(p)}{p} - 2g'(p)] + \frac{m^3}{n^3}[-\frac{2g(p)}{3p^2} - \frac{2g'(p)}{p} - g''(p)]$$
$$+ \frac{m^2}{n^2}[2g'(p) + \frac{2g(p)}{p}] + \frac{m^3}{n^3}[\frac{2g(p)}{p^2} + \frac{6g'(p)}{p} + 2g''(p)]$$
$$- \frac{m^3}{n^3}[\frac{4g(p)}{3p^2} + \frac{4g'(p)}{p} + \frac{4g''(p)}{3}] + O(m^4/n^4 + mn^{-2})$$
$$= \frac{m}{n}[-2g(p)] + \frac{m^2}{n^2}[-\frac{2g(p)}{p} - 2g'(p) + 2g'(p) + \frac{2g(p)}{p}]$$
$$+ \frac{m^3}{n^3}[-\frac{2g(p)}{3p^2} - \frac{2g'(p)}{p} - g''(p) + \frac{2g(p)}{p^2} + \frac{6g'(p)}{p}$$

$$+ 2g''(p) - \frac{4g(p)}{3p^2} - \frac{4g'(p)}{p} - \frac{4g''(p)}{3}]$$

$$+ O(m^4/n^4 + mn^{-2})$$

$$= -\frac{2m}{n}g(p) + \frac{m^3}{n^3}[-\frac{g''(p)}{3}] + O(m^4/n^4 + mn^{-2})$$

$$= -(2m/n)[g(p) + (1/6)(m/n)^2 g''(p)] + O(m^4/n^4 + mn^{-2}). \tag{34}$$

For the sole purpose of comparing this work to HS88, note that the full expansion given for Y as defined in (15) is

$$Y \equiv -pn^{1/2}[(1+\delta)(\Delta_2 + \Delta_3) + B]$$

$$= -pn^{1/2}\Big[(1 - (m/n)^2 g''(p)[6g(p)]^{-1})(\Delta_2 + \Delta_3) + (n/m)(b_1\Delta_1 + b_2\Delta_2)\Delta_2$$

$$+ (n/m)b_3(\Delta_1 + \Delta_2)\Delta_3 + (n/m)^2 b_4(\Delta_1 + \Delta_2)^2\Delta_3 + b_5\Delta_3^2\Big]$$

$$= -pn^{1/2}$$

$$\times \Big[(1 - (m/n)^2 g''(p)[6g(p)]^{-1})(\Delta_2 + \Delta_3)$$

$$+ (n/m)(-p/2\Delta_1 + (-p/2 + (m/n)(a_2/2a_1))\Delta_2)\Delta_2$$

$$+ (n/m)(-p/2 - (m/n)(a_2/2a_1))(\Delta_1 + \Delta_2)\Delta_3$$

$$+ (n/m)^2(p/2)^2(\Delta_1 + \Delta_2)^2\Delta_3 - a_2/2a_1\Delta_3^2\Big]. \tag{35}$$

From (12) before, $Z \equiv [p(1-p)]^{1/2}[n^{1/2}(X_{n,r} - \eta_p) + \gamma(S_{m,n}/S_0 - 1)][(n/2m)(X_{n,r+m} - X_{n,r-m})]^{-1}$. I will first expand the Z when $\gamma = 0$, written Z_0, as considered in HS88. Using the expansion that for small ϵ, $(1+\epsilon)^{-1} = 1^{-1} + (-1)1^{-2}\epsilon + 1/2(-2)(-1)1^{-3}\epsilon^2 + 1/6(-3)(-2)(-1)1^{-4}\epsilon^3 + \ldots = 1 - \epsilon + \epsilon^2 + O(\epsilon^3)$, plugging in (19), (26), and (34) yields

$$Z_0 = n^{1/2}(X_{n,r} - \eta_p)[(n/2m)(X_{n,r+m} - X_{n,r-m})]^{-1}$$

$$= n^{1/2}((\Delta_2 + \Delta_3)a_1 + \frac{1}{2}(\Delta_2 + \Delta_3)^2 a_2 + O_p(n^{-3/2}))$$

$$\times \Big[\frac{n}{2m}\big(-a_1(\Delta_1 + \Delta_2) - (m/np)a_2(\Delta_1 + \Delta_2 + 2\Delta_3) + O_p(n^{-3/2}m^{1/2} + n^{-5/2}m^2)\big)$$

$$+ (n/2m)(2m/n)[g(p) + (1/6)(m/n)^2 g''(p) + O\{(m/n)^{2+\epsilon} + n^{-1}\}]\Big]^{-1}$$

$$= -pn^{1/2}((\Delta_2 + \Delta_3)(-a_1/p) - \frac{1}{2p}(\Delta_2 + \Delta_3)^2 a_2 + O_p(n^{-3/2}))$$

$$\times \Big[\frac{n}{2m}\big(-a_1(\Delta_1 + \Delta_2) - (m/n)(a_2/p)(\Delta_1 + \Delta_2 + 2\Delta_3) + O_p(n^{-3/2}m^{1/2} + n^{-5/2}m^2)\big)$$

$$+ g(p) + (m/n)^2\frac{g''(p)}{6} + O\{(m/n)^{2+\epsilon} + n^{-1}\}\Big]^{-1}$$

$$= -pn^{1/2}((\Delta_2 + \Delta_3)(-a_1/pg(p)) - \frac{1}{2pg(p)}(\Delta_2 + \Delta_3)^2 a_2 + O_p(n^{-3/2}))$$

$$
\begin{aligned}
&\times \Big[\frac{n}{2m}\big(- (a_1/g(p))(\Delta_1 + \Delta_2) - (m/n)(a_2/pg(p))(\Delta_1 + \Delta_2 + 2\Delta_3) \\
&\qquad\qquad + O_p(n^{-3/2}m^{1/2} + n^{-5/2}m^2)\big) \\
&\qquad + 1 + (m/n)^2\frac{g''(p)}{6g(p)} + O\{(m/n)^{2+\epsilon} + n^{-1}\}\Big]^{-1} \\
&\equiv -pn^{1/2}\Theta(1+\nu)^{-1} = -pn^{1/2}\Theta(1 - \nu + \nu^2 + O(\nu^3)), \qquad (36)
\end{aligned}
$$

defining Θ and ν for ease of notation.

If (20) is used instead of (19), including the new γ term, and noting that above I had divided through by $g(p)$ in addition to factoring $-pn^{1/2}$ out of the numerator,

$$
\begin{aligned}
Z &= [-pn^{1/2}\Theta + \gamma\nu][1+\nu]^{-1} = -pn^{1/2}(\Theta - \Psi\nu)(1+\nu)^{-1} \\
&= -pn^{1/2}[\Theta - (\Theta + \Psi)\nu + (\Theta+\Psi)\nu^2 + O(n^{-1/2}\nu^3)], \qquad (37)
\end{aligned}
$$

with $\Psi \equiv \gamma/(pg(p)\sqrt{n})$ as in (16). Θ and Ψ are both $O_p(n^{-1/2})$, and $\nu = O_p(m^{-3/2})$, which is in R, so the remainder works out.

In the following, I again first do the simpler case when $\gamma = 0$, and then add in Ψ alongside Θ in the ν and ν^2 terms. Keep in mind that from (17) the final remainder R from $Z = Y + R$ is $R = O_p[n^{-1/2}m^{-1/2} + n^{-3/2}m + m^{-3/2} + (m/n)^{2+\epsilon}]$. I will keep the $n^{1/2}$ out front, so that means that anything in $\Theta - \Theta\nu + \Theta\nu^2$ that is $O_p[n^{-1}m^{-1/2} + n^{-2}m + n^{-1/2}m^{-3/2} + n^{-1/2}(m/n)^{2+\epsilon}]$ will end up in the remainder term, and thus can be suppressed; but anything bigger must be kept. It turns out that every term in $\Theta\nu^3$ (and consequently any higher order terms in the expansion) is small enough to go in the remainder. Also note that the biggest term in Θ is $\Delta_3 = O_p(n^{-1/2})$, shown in (13), so any terms in ν and ν^2 that are $O_p[n^{-1/2}m^{-1/2} + n^{-3/2}m + m^{-3/2} + (m/n)^{2+\epsilon}]$ can be put into the remainder. Also note that $(a_1/(pg(p)) = -1 + O(n^{-1})$ as shown in (45). So some algebra is needed. Starting with the implicit definitions of Θ and ν from (36), the individual terms in Z can be simplified and calculated.

$$
\begin{aligned}
\Theta &\equiv (\Delta_2 + \Delta_3)(-a_1/pg(p)) - \frac{1}{2pg(p)}(\Delta_2+\Delta_3)^2 a_2 + O_p(n^{-3/2}) \\
&= (\Delta_2 + \Delta_3) + \frac{a_2}{2a_1}(\Delta_2+\Delta_3)^2 + O_p(n^{-3/2}), \qquad (38) \\
\nu &\equiv \frac{n}{2m}\big(- (a_1/g(p))(\Delta_1 + \Delta_2) - (m/n)(a_2/pg(p))(\Delta_1 + \Delta_2 + 2\Delta_3) \\
&\qquad\qquad + O_p(n^{-3/2}m^{1/2} + n^{-5/2}m^2)\big) \\
&\quad + (m/n)^2\frac{g''(p)}{6g(p)} + O\{(m/n)^{2+\epsilon} + n^{-1}\}
\end{aligned}
$$

$$
\begin{aligned}
(39) \qquad &= \frac{n}{2m}p(\Delta_1+\Delta_2) + \frac{a_2}{2a_1}(\Delta_1+\Delta_2+2\Delta_3) + (m/n)^2\frac{g''(p)}{6g(p)} \\
&\quad + O_p((m/n)^{2+\epsilon} + n^{-1/2}m^{-1/2} + mn^{-3/2}), \\
\nu^2 &= O((m/n)^4) + (n/m)^2\Big(\frac{a_1}{pg(p)}\frac{p}{2}\Big)^2(\Delta_1+\Delta_2)^2 + \Big(\frac{a_1}{pg(p)}\frac{a_2}{2a_1}\Big)^2 O_p(n^{-1}) + O_p(\cdot)^2 \\
&\quad + O_p((m/n)^2m^{-1/2} + (m/n)^2n^{-1/2} + m^{-1/2}n^{-1/2}) + O_p(\cdot)\times[\text{first 3 terms}\to\text{zero}] \\
&= (n/m)^2\Big(\frac{a_1}{pg(p)}\Big)^2(p/2)^2(\Delta_1+\Delta_2)^2 + R \\
(40) \qquad &= (n/m)^2(p/2)^2(\Delta_1+\Delta_2)^2 + R.
\end{aligned}
$$

Note that crossing out, as in $\cancel{\Delta_2}$, means that all terms it would be a part of will end up in the remainder. Sometimes these appear in HS88, though I believe they should all be in the remainder.

$$
\begin{aligned}
\Theta\nu &= (\Delta_2+\Delta_3)\Big(-\frac{a_1}{pg(p)}\Big) \\
&\quad \times \Big[(m/n)^2\frac{g''(p)}{6g(p)} - (n/m)\frac{a_1}{2g(p)}(\Delta_1+\Delta_2) - (a_2/2pg(p))(\Delta_1+\Delta_2+2\Delta_3) \\
&\qquad + O_p((nm)^{-1/2} + mn^{-3/2} + (m/n)^{2+\epsilon})\Big] \\
&\quad - (\Delta_2+\Delta_3)^2(a_2/2pg(p)) \\
&\qquad \times \Big[(m/n)^2\frac{g''(p)}{6g(p)} - (n/m)\frac{a_1}{2g(p)}(\Delta_1+\Delta_2) - (a_2/2pg(p))(\Delta_1+\Delta_2+2\Delta_3) \\
&\qquad\quad + O_p((nm)^{-1/2} + mn^{-3/2} + (m/n)^{2+\epsilon})\Big] + O_p(n^{-1})\times[\cdot] \\
&= (\Delta_2+\Delta_3)\big[-(m/n)^2(g''(p)/6g(p))(a_1/pg(p))\big] \\
&\quad + (\Delta_2+\Delta_3)(a_1/pg(p))^2p(n/m)(1/2)(\Delta_1+\Delta_2) \\
&\quad + (\Delta_2+\Delta_3)(a_1/pg(p))^2(a_2/2a_1)(\Delta_1+\Delta_2+2\Delta_3) \\
&\quad + O_p((nm)^{-1/2} + mn^{-3/2} + (m/n)^{2+\epsilon}) - [O_p(mn^{-3/2}(m/n)) = O_p(mn^{-3/2})] \\
&\quad + (\cancel{\Delta_2}+\Delta_3)^2(n/m)(a_1a_2/4pg(p)^2)(\Delta_1+\Delta_2) - n^{-1/2}R \\
\Theta\nu^2 &= -(n/m)^2(a_1/pg(p))^3(p/2)^2(\Delta_1+\Delta_2)^2(\Delta_3 + O_p(n^{-1}m^{1/2})) + n^{-1/2}R
\end{aligned}
$$

$$
\begin{aligned}
\Theta - \Theta\nu + \Theta\nu^2 &= -(a_1/pg(p))(\Delta_2+\Delta_3) - (a_1/pg(p))(a_2/2a_1)(\cancel{\Delta_2^2} + 2\Delta_2\Delta_3 + \Delta_3^2) \\
&\quad + (m/n)^2(a_1/pg(p))(g''(p)/6g(p))(\Delta_2+\Delta_3) \\
&\quad - (n/m)(p/2)(a_1/pg(p))^2(\Delta_2+\Delta_3)(\Delta_1+\Delta_2) \\
&\quad - (a_1/pg(p))^2(a_2/2a_1)(\cancel{\Delta_2\Delta_1 + \Delta_2^2} + 3\Delta_2\Delta_3 + \Delta_3\Delta_1 + 2\Delta_3^2) \\
&\quad - (n/m)(a_2/2a_1)(p/2)(a_1/pg(p))^2(\Delta_1+\Delta_2)\Delta_3^2
\end{aligned}
$$

$$
\begin{aligned}
&- (n/m)^2(a_1/pg(p))^3(p/2)^2(\Delta_1+\Delta_2)^2\Delta_3 + n^{-1/2}R \\
&= (\Delta_2+\Delta_3) + (a_2/2a_1)(2\Delta_2\Delta_3+\Delta_3^2) \\
&\quad - (m/n)^2(g''(p)/6g(p))(\Delta_2+\Delta_3) - (n/m)(p/2)(\Delta_2+\Delta_3)(\Delta_1+\Delta_2) \\
&\quad - (a_2/2a_1)(3\Delta_2\Delta_3+\Delta_3\Delta_1+2\Delta_3^2) \\
&\quad - O_p(n^{-1}m^{-1/2}) + (n/m)^2(p/2)^2(\Delta_1+\Delta_2)^2\Delta_3 + n^{-1/2}R \\
&= (\Delta_2+\Delta_3) - (m/n)^2(g''(p)/6g(p))(\Delta_2+\Delta_3) \\
&\quad - (n/m)(p/2)(\Delta_1\Delta_2+\Delta_2^2) - (n/m)(p/2)\Delta_3(\Delta_1+\Delta_2) \\
&\quad + (a_2/2a_1)(2\Delta_2\Delta_3+\Delta_3^2) - (a_2/2a_1)(3\Delta_2\Delta_3+\Delta_3\Delta_1+2\Delta_3^2) \\
&\quad + (n/m)^2(p/2)^2(\Delta_1+\Delta_2)^2\Delta_3 + n^{-1/2}R \\
&= (\Delta_2+\Delta_3) - (m/n)^2(g''(p)/6g(p))(\Delta_2+\Delta_3) \\
&\quad - (n/m)(p/2)(\Delta_1\Delta_2+\Delta_2^2) - (n/m)(p/2)\Delta_3(\Delta_1+\Delta_2) \\
&\quad - (a_2/2a_1)(\Delta_2\Delta_3+\Delta_3^2+\Delta_3\Delta_1) + (n/m)^2(p/2)^2(\Delta_1+\Delta_2)^2\Delta_3 + n^{-1/2}R
\end{aligned} \tag{41}
$$

Note: I think the second b_2 term of $(a_2/2a_1)\Delta_2^2$ is incorrectly included in Hall and Sheather (1988). The term instead should be in R since $n^{1/2}\Delta_2^2 = n^{1/2}O_p(n^{-2}m) = O_p(mn^{-3/2})$, which is in R. Even if the term were included, there is another identical but negative term that would be included, so they would cancel anyway. (The positive term is from Θ, the negative one from $\Theta\nu$.) Of course, my claim is that this is a higher-order term anyway, so the effect of including it should be negligible.

And now with Ψ added,

$$
\begin{aligned}
(\Theta+\Psi)\nu &= \left((\Delta_2+\Delta_3) + \frac{a_2}{2a_1}(\Delta_2+\Delta_3)^2 + O_p(n^{-3/2}) + \Psi\right) \\
&\quad \times \Big[\frac{n}{2m}p(\Delta_1+\Delta_2) + (a_2/2a_1)(\Delta_1+\Delta_2+2\Delta_3) \\
&\qquad + (m/n)^2\frac{g''(p)}{6g(p)} + O_p(n^{-1/2}m^{-1/2}+mn^{-3/2})\Big] \\
&= \frac{n}{m}(p/2)(\Delta_2+\Delta_3+\Psi)(\Delta_1+\Delta_2) + (a_2/2a_1)(\Delta_2+\Delta_3+\Psi)(\Delta_1+\Delta_2+2\Delta_3) \\
&\quad + (\Delta_2+\Delta_3+\Psi)(m/n)^2\frac{g''(p)}{6g(p)} + (n/m)(p/2)(a_2/2a_1)(\Delta_2+\Delta_3)^2(\Delta_1+\Delta_2) \\
&\quad + (a_2/2a_1)^2(\Delta_2+\Delta_3)^2(\Delta_1+\Delta_2+2\Delta_3) + (m/n)^2(a_2/2a_1)\frac{g''(p)}{6g(p)}(\Delta_2+\Delta_3)^2 \\
&\quad + n^{-1/2}O_p(n^{-1/2}m^{-1/2}+mn^{-3/2}) \\
&= \frac{n}{m}(p/2)(\Delta_2+\Delta_3+\Psi)(\Delta_1+\Delta_2) \\
&\quad + (a_2/2a_1)(2\Delta_3^2+3\Delta_3\Delta_2+\Delta_3\Delta_1+\Psi(\Delta_1+\Delta_2+2\Delta_3)+\Delta_2^2+\Delta_1\Delta_2)
\end{aligned}
$$

$$+ (\Delta_2 + \Delta_3 + \Psi)(m/n)^2 \frac{g''(p)}{6g(p)} + n^{-1/2} O_p(n^{-1/2} m^{-1/2} + mn^{-3/2}).$$

$$\begin{aligned}
(\Theta + \Psi)\nu^2 &= \left((\Delta_2 + \Delta_3) + \frac{a_2}{2a_1}(\Delta_2 + \Delta_3)^2 + O_p(n^{-3/2}) + \Psi \right) \\
&\quad \times \Big[\frac{n}{2m} p(\Delta_1 + \Delta_2) + (a_2/2a_1)(\Delta_1 + \Delta_2 + 2\Delta_3) \\
&\qquad + (m/n)^2 \frac{g''(p)}{6g(p)} + O_p(n^{-1/2} m^{-1/2} + mn^{-3/2}) \Big]^2 \\
&= \big(\Delta_3 + \Psi + O_p(m^{1/2} n^{-1})\big) \\
&\quad \times \big[(n/m)^2 (p/2)^2 (\Delta_1 + \Delta_2)^2 + O_p(n^{-1} + (m/n)^4 + m^{-1/2} n^{-1/2} + m^{-1/2}(m/n)^2) \big] \\
&= (\Delta_3 + \Psi)(n/m)^2 (p/2)^2 (\Delta_1 + \Delta_2)^2 + n^{-1/2} R.
\end{aligned}$$

$$\begin{aligned}
&\Theta - (\Theta + \Psi)\nu + (\Theta + \Psi)\nu^2 \\
&= (\Delta_2 + \Delta_3) + \frac{a_2}{2a_1}(\Delta_2 + \Delta_3)^2 + O_p(n^{-3/2}) \\
&\quad - \Big[\frac{n}{m}(p/2)(\Delta_2 + \Delta_3 + \Psi)(\Delta_1 + \Delta_2) \\
&\qquad + (a_2/2a_1)(2\Delta_3^2 + 3\Delta_3\Delta_2 + \Delta_3\Delta_1 + \Psi(\Delta_1 + \Delta_2 + 2\Delta_3) + \Delta_2^2 + \Delta_1\Delta_2) \\
&\qquad + (\Delta_2 + \Delta_3 + \Psi)(m/n)^2 \frac{g''(p)}{6g(p)} + n^{-1/2} O_p(n^{-1/2} m^{-1/2} + mn^{-3/2}) \Big] \\
&\quad + (\Delta_3 + \Psi)(n/m)^2 (p/2)^2 (\Delta_1 + \Delta_2)^2 + n^{-1/2} R \\
&= (\Delta_2 + \Delta_3) \left(1 - (m/n)^2 \frac{g''(p)}{6g(p)} \right) - \Psi (m/n)^2 \frac{g''(p)}{6g(p)} - \frac{n}{m}(p/2)(\Delta_2)(\Delta_1 + \Delta_2) \\
&\quad - \frac{n}{m}(p/2)(\Delta_3 + \Psi)(\Delta_1 + \Delta_2) \\
&\quad + \frac{a_2}{2a_1}(\Delta_2^2 + 2\Delta_2\Delta_3 + \Delta_3^2 - 2\Delta_3^2 - 3\Delta_3\Delta_2 - \Delta_3\Delta_1 \\
&\qquad\quad - \Psi(\Delta_1 + \Delta_2 + 2\Delta_3) - \Delta_2^2 - \Delta_1\Delta_2) \\
&\quad + (n/m)^2 (p/2)^2 (\Delta_1 + \Delta_2)^2 (\Delta_3 + \Psi) + n^{-1/2} R \\
&= (\Delta_2 + \Delta_3) \left(1 - (m/n)^2 \frac{g''(p)}{6g(p)} \right) \\
&\quad - \Psi (m/n)^2 \frac{g''(p)}{6g(p)} - \frac{n}{m}(p/2)(\Delta_1 + \Delta_2)\Delta_2 - \frac{n}{m}(p/2)(\Delta_3 + \Psi)(\Delta_1 + \Delta_2) \\
&\quad + \frac{a_2}{2a_1}(-\Delta_2\Delta_3 - \Delta_3\Delta_1 - \Psi(\Delta_1 + \Delta_2) - \Delta_3^2 - 2\Delta_3\Psi) - n^{-1/2} O_p(mn^{-3/2}) \\
&\quad + (n/m)^2 (p/2)^2 (\Delta_1 + \Delta_2)^2 (\Delta_3 + \Psi) + n^{-1/2} R \\
&= (\Delta_2 + \Delta_3)(1 + \delta) + \Psi\delta + \frac{n}{m}((-p/2)\Delta_1 + (-p/2)\Delta_2)\Delta_2
\end{aligned}$$

$$+\frac{n}{m}(-p/2)(\Delta_1+\Delta_2)(\Delta_3+\Psi)-\frac{a_2}{2a_1}(\Delta_1+\Delta_2)(\Delta_3+\Psi)$$
$$+(n/m)^2(p/2)^2(\Delta_1+\Delta_2)^2(\Delta_3+\Psi)-(a_2/2a_1)(\Delta_3^2+2\Psi\Delta_3)+n^{-1/2}R, \tag{42}$$

where $\delta \equiv -(m/n)^2\frac{g''(p)}{6g(p)}$ as in HS88. This matches (15) and (16).

Writing $(a_1/pg(p)) = -1+R_2$, I now show that $R_2 = O(1/n)$, as is used to simplify (38) and (39). Recall from (11) that $g \equiv G'$, where $G \equiv F^{-1}$, where $F(\cdot)$ is the population cdf. Thus, in terms of F and f,

$$g(x) = \frac{d}{dx}F^{-1}(x) = \frac{1}{F'(F^{-1}(x))} = \frac{1}{f(F^{-1}(x))}, \tag{43}$$

and thus

$$g(p) = 1/f(\xi_p). \tag{44}$$

Also recall from (11) that $H(x) \equiv F^{-1}(e^{-x})$, and

$$a_1 = H'(\sum_{j\geq r} j^{-1}) = -\exp(-\sum_{j\geq r} j^{-1})/f(\eta_p)$$

by the chain rule and definition (6) of $\eta_p \equiv F^{-1}(\exp(-\sum_{j\geq r} j^{-1}))$. Last, recall from (7) that $\sum_{j\geq r} j^{-1} = \ln(p^{-1}) - (np)^{-1}[\epsilon_n - 1 + \frac{1}{2}(1-p)] + O(n^{-2})$. Thus,

$$\begin{aligned}
-\exp(-\sum_{j\geq r} j^{-1}) &= -\exp(\ln p + (np)^{-1}[\epsilon_n - 1 + (1/2)(1-p)] - O(n^{-2})) \\
&= -\exp(\ln p)\exp((np)^{-1}[\epsilon_n - 1 + (1/2)(1-p)] - O(n^{-2})) \\
&= -p\left[1 + (np)^{-1}[\epsilon_n - 1 + (1/2)(1-p)] + O(n^{-2})\right], \\
a_1 &= -pg(p)(f(\xi_p)/f(\eta_p))[1+O(n^{-1})],
\end{aligned}$$

and recall from (8) that $\eta_p = \xi_p + n^{-1}[\epsilon_n - 1 + \frac{1}{2}(1-p)]/f(\xi_p) + O(n^{-2})$. Since $f(\eta_p) = f(\xi_p) + f'(\xi_p)[n^{-1}[\epsilon_n - 1 + \frac{1}{2}(1-p)]/f(\xi_p) + O(n^{-2})]$, then if f' is bounded, $f(\eta_p)/f(\xi_p) = 1 + O(n^{-1})$, and consequently (plugging into above) $a_1 = -pg(p)(1+O(n^{-1}))$ and

$$a_1/pg(p) = -1 + O(n^{-1}), \tag{45}$$
$$(a_1/pg(p))^2 = 1 + O(n^{-1}). \tag{46}$$

Next are proofs that $a_2 = O(1)$ and $a_3 = O(1)$.

From (11), $H(x) \equiv F^{-1}(e^{-x}) = G(e^{-x})$, and the first derivative is $H'(x) = g(e^{-x})(-e^{-x})$ using the chain rule and since $g \equiv G'$. Thus,

$$\begin{aligned}
H''(x) &= (-e^{-x}g'(e^{-x}))(-e^{-x}) + g(e^{-x})e^{-x} = g'(e^{-x})e^{-2x} + g(e^{-x})e^{-x} \\
&= g'(e^{-x})e^{-2x} - H'(x), \\
H'''(x) &= g''(e^{-x})(-e^{-x})e^{-2x} + g'(e^{-x})(-2e^{-2x}) - H''(x)
\end{aligned} \tag{47}$$

$$
\begin{aligned}
&= -g''(e^{-x})e^{-3x} - 2g'(e^{-x})e^{-2x} - g'(e^{-x})e^{-2x} + H'(x) \\
&= -g''(e^{-x})e^{-3x} - 3g'(e^{-x})e^{-2x} + H'(x) \\
&= -g''(e^{-x})e^{-3x} - 3H''(x) - 3H'(x) + H'(x) \\
&= -g''(e^{-x})e^{-3x} - 3H''(x) - 2H'(x), \qquad (48)
\end{aligned}
$$

$$a_3 = -g''(e^{-\sum_{j\geq r} j^{-1}})e^{-3\sum_{j\geq r} j^{-1}} - 3a_2 - 2a_1, \qquad (49)$$

$$a_2 = g'(e^{-\sum_{j\geq r} j^{-1}})e^{-2\sum_{j\geq r} j^{-1}} - a_1. \qquad (50)$$

The task is now to show that the g' and g'' terms above are $O(1)$, since $a_1 = O(1)$ per (45) above. Note first that $e^{-x} \in (0,1)$ for $x > 0$, so those objects are both $O(1)$, and thus g' and g'' are the critical terms.

$$
\begin{aligned}
g(x) &= \frac{1}{f(F^{-1}(x))} \qquad \text{from (43)}, \\
g'(x) &= -\frac{1}{[f(F^{-1}(x))]^2} f'(F^{-1}(x))g(x) = -\frac{f'(F^{-1}(x))}{[f(F^{-1}(x))]^3} \\
&= -f'(F^{-1}(x))[f(F^{-1}(x))]^{-3}, \text{ and} \qquad (51) \\
g'(e^{-\sum_{j\geq r} j^{-1}}) &= -\frac{f'(\eta_p)}{[f(F^{-1}(\eta_p))]^3},
\end{aligned}
$$

which is $O(1)$ as long as $f'(\eta_p)/f(\eta_p) = O(1)$, which is true if $f'(\xi_p) < \infty$ and $f(\xi_p) > 0$, as assumed.

$$
\begin{aligned}
g''(x) &= -f''(F^{-1}(x))g(x)[f(F^{-1}(x))]^{-3} \\
&\quad + (-f'(F^{-1}(x)))(-3)[f(F^{-1}(x))]^{-4} f'(F^{-1}(x))g(x) \\
&= [f(F^{-1}(x))]^{-4}(-f''(F^{-1}(x))) + 3[g'(x)]^2 f(F^{-1}(x)), \text{ and} \qquad (52) \\
g''(e^{-\sum_{j\geq r} j^{-1}}) &= -f''(\eta_p)/[f(\eta_p)]^4 + 3[g'(e^{-\sum_{j\geq r} j^{-1}})]^2 f(\eta_p) = -f''(\eta_p)O(1) + O(1)O(1),
\end{aligned}
$$

which is $O(1)$ as long as (additionally) $f''(\xi_p) < \infty$, as assumed.

As stated in HS88 (and with this ϵ different than the one use above with Θ), under the restrictions $n^\eta \leq m \leq n^{1-\eta}$ and $0 < \epsilon \leq 1/6$, (17) "entails

$$R = O_p\{[m^{-1} + (m/n)^2]n^{-\epsilon\eta}\}," \qquad (53)$$

which "may be strengthened to

$$P\{|R| > [m^{-1} + (m/n)^2]n^{-\zeta}\} = O(n^{-\lambda}), \text{ all } 0 < \zeta < \epsilon\eta \text{ and all } \lambda > 0,$$

on noting that by Markov's and Rosenthal's inequalities (Burkholder (1973), p. 40),

$$\sum_{i=1}^{2} P(|nm^{-1/2}\Delta_i| > n^{\epsilon}) + P(|n^{1/2}\Delta_3| > n^{\epsilon}) = O(n^{-\lambda}),\ \text{all } \epsilon > 0 \text{ and all } \lambda > 0.\text{''}$$

They continue, "Therefore the theorem will follow by proving that result for Y rather than for Z. Since Δ_1, Δ_2 and Δ_3 are independent and have smooth distributions, it is elementary (but tedious) to estimate the difference between $\exp(-t^2/2)$ and the characteristic function of Y, and then derive an Edgeworth expansion by following classical arguments, as in Petrov (1975) or Bhattacharya and Ghosh (1978). To simply identify terms in the expansion here, only moments of Y are needed," the result of which is, as stated in (3.2) of HS87,

$$E[(-p^{-1}Y)^{\ell}] = z_1(\ell) + z_2(\ell) + z_3(\ell) + O\{m^{-3/2} + m^{-1/2}(m/n)^2\}, \tag{54}$$
$$z_1 \equiv n^{\ell/2}E[\{(1+\delta)(\Delta_2+\Delta_3)\}^{\ell}], z_2 \equiv \ell n^{\ell/2}E\{(\Delta_2+\Delta_3)^{\ell-1}B\},$$
$$z_3 \equiv \frac{1}{2}\ell(\ell-1)n^{\ell/2}E\{(\Delta_2+\Delta_3)^{\ell-2}B^2\}.$$

To show the above expansion of moments of Y, first recall from (15) that

$$\begin{aligned} Y &\equiv -pn^{1/2}[(1+\delta)(\Delta_2+\Delta_3)+B] \quad \text{and} \\ \delta &\equiv -(m/n)^2 g''(p)[6g(p)]^{-1},\ \text{so} \\ (-p^{-1}Y)^{\ell} &= (n^{1/2}[(1+\delta)(\Delta_2+\Delta_3)+B])^{\ell} \\ &= n^{\ell/2}[(1+\delta)(\Delta_2+\Delta_3)+B]^{\ell}. \end{aligned} \tag{55}$$

Letting $A \equiv (1+\delta)(\Delta_2+\Delta_3)$, and Taylor expanding around A, also noting that $(1+\delta)^{\ell} = 1 + \ell\delta + (1/2)\ell(\ell-1)\delta^2 + \ldots = 1 + O(\delta)$, the expectation of (55) is

$$\begin{aligned} E\left(n^{\ell/2}[A+B]^{\ell}\right) &= E\left(n^{\ell/2}\left[A^{\ell} + \ell A^{\ell-1}B + (1/2)\ell(\ell-1)A^{\ell-2}B^2 + R_2\right]\right) \\ &= E\left(n^{\ell/2}[(1+\delta)(\Delta_2+\Delta_3)]^{\ell}\right) \\ &\quad + E\left(n^{\ell/2}\ell[(\Delta_2+\Delta_3)^{\ell-1}B + O(\delta B(\Delta_2+\Delta_3)^{\ell-1})]\right) \\ &\quad + E\left((1/2)\ell(\ell-1)n^{\ell/2}[(\Delta_2+\Delta_3)^{\ell-2}B^2 + O(\delta B^2(\Delta_2+\Delta_3)^{\ell-2})]\right) + R_2 \\ &= z_1(\ell) + z_2(\ell) + z_3(\ell) + O\{m^{-3/2} + m^{-1/2}(m/n)^2\} \end{aligned}$$

if the remainder terms from the second-to-last equality fit in the remainder from (54). Note from (13), (14), the definition of Ψ, and (16) that Δ_1 and Δ_2 are $O_p(n^{-1}m^{1/2})$, $\Delta_3 = O_p(n^{-1/2})$, $\Psi = O(n^{-1/2})$, all b_i coefficients are $O(1)$, and consequently, starting from its definition in (16), the order of B is

$$\begin{aligned} B &\equiv \delta\Psi + (n/m)(b_1\Delta_1 + b_2\Delta_2)\Delta_2 + (n/m)b_3(\Delta_1+\Delta_2)(\Delta_3+\Psi) \\ &\quad + (n/m)^2 b_4(\Delta_1+\Delta_2)^2(\Delta_3+\Psi) + b_5\Delta_3(\Delta_3+2\Psi) \end{aligned}$$

$$
\begin{aligned}
&= O_p(m^2n^{-5/2} + n^{-1} + m^{-1/2}n^{-1/2} + m^{-1}n^{-1/2} + n^{-1}) \\
&= O_p(m^2n^{-5/2} + m^{-1/2}n^{-1/2}), \qquad (56)
\end{aligned}
$$

since also Δ_3 is of larger order than Δ_2 and thus $(\Delta_2 + \Delta_3) = O_p(n^{-1/2})$. From its definition in (15), clearly $\delta = O((m/n)^2)$. Thus,

$$
\begin{aligned}
\delta B &= O((m/n)^2)O_p(m^2n^{-5/2} + m^{-1/2}n^{-1/2}) \\
&= O_p((m^2n^{-5/2} + n^{-1/2}m^{-1/2})(m/n)^2) \\
n^{\ell/2}\delta B(\Delta_2 + \Delta_3)^{\ell-1} &= n^{\ell/2}O_p((m^2n^{-5/2} + n^{-1/2}m^{-1/2})(m/n)^2)O_p(n^{-(\ell-1)/2}) \\
&= O_p((m/n)^4 + m^{-1/2}(m/n)^2) \\
n^{\ell/2}\delta B^2(\Delta_2 + \Delta_3)^{\ell-2} &= n^{\ell/2}\delta B(\Delta_2 + \Delta_3)^{\ell-1} \times (B/(\Delta_2 + \Delta_3)) \\
&= O_p((m/n)^4 + m^{-1/2}(m/n)^2) \times O_p((m^2n^{-5/2} + m^{-1/2}n^{-1/2})/n^{-1/2}) \\
&= O_p((m^4n^{-4} + m^{3/2}n^{-2})(m^{-1/2} + m^2n^{-2})) \\
&= O_p(m^{3/2}n^{-4} + mn^{-2} + m^6n^{-6} + m^{7/2}n^{-4}) \\
R_2 &= O_p(n^{\ell/2}(\Delta_2 + \Delta_3)^{\ell-3}B^3) \\
&= O_p(n^{3/2}(m^2n^{-5/2} + m^{-1/2}n^{-1/2})^3) \\
&= O_p(n^{3/2}(m^{-3/2}n^{-3/2} + mn^{-7/2} + m^{7/2}n^{-11/2} + m^6n^{-15/2})) \\
&= O_p(m^{-3/2} + mn^{-2} + m^{7/2}n^{-4} + m^6n^{-6}).
\end{aligned}
$$

From above (from HS88), it was shown that $O(mn^{-3/2}) = o(m^{-1} + (m/n)^2)$, and thus $O(\sqrt{n}/m) = o(1)$ and $O(m^2n^{-3/2}) = o(1)$. Thus, $O(m^4n^{-4} = O(m^{3/2}n^{-2}m^{5/2}n^{-2}) = O(m^{3/2}n^{-2}m^2n^{-3/2}(m/n)^{1/2}) = O(m^{3/2}n^{-2})o(1)o(1)$ as desired. Also not immediately apparent is that $O(m^6n^{-6}) = O((m/n)^2(m/n)^4) = o(1) \times O(m^{3/2}n^{-2})o(1)o(1)$ using the previous result.

Now I verify the HS88 statements that Δ_1, Δ_2 and Δ_3 are independent with zero means, and also that $E(\Delta_1^2), E(\Delta_2^2)$ both equal $m(np)^{-2}(1 + O(m/n))$.

Clearly the Δ_i are mean zero since they are sums of mean zero random variables (the V_j); they are independent since they include mutually exclusive sets of the V_j and the V_j are defined to be independent.

For the second moment of Δ_1 (and Δ_2 will approximate similarly),

$$
\begin{aligned}
E(\Delta_1^2) &= E\left[\left(\sum_{r-m}^{r-1} j^{-1}V_j\right)^2\right] \\
&= E\left[\sum_{r-m}^{r-1} \left(j^{-1}V_j\right)^2\right], \text{ since } E(V_iV_j) = E(V_i)E(V_j) = 0, i \neq j
\end{aligned}
$$

$$= \sum_{r-m}^{r-1} j^{-2} \text{ since } E(V_j^2) = \mathrm{Var}(W_j) = 1^2 = 1.$$

Using Euler-Maclaurin,

$$\sum_{r-m}^{r-1} 1/j^2 = \sum_{r-m}^{r} 1/j^2 - 1/r^2 = \sum_{0}^{m} (r-m+i)^{-2} - 1/r^2$$
$$= \int_0^m (r-m+x)^{-2}dx + (1/2)(r^{-2} + (r-m)^{-2}) - 1/r^2 + O(f^{(1)}(0))$$
$$= -(r-m+x)^{-1}|_0^m + (1/2)[\frac{1}{r^2 - 2mr + m^2} - r^{-2}] + O(n^{-3})$$
$$= 1/(r-m) - 1/r + (1/2)[(1/r^2)(1 - 1 + O(m/n))] + O(n^{-3})$$
$$= \frac{r-(r-m)}{r^2(1-m/r)} + O(m/n^3) = \frac{m}{(np)^2}(1 + O(m/n)). \tag{57}$$

The second moment of Δ_3 may be calculated using the same method:

$$E(\Delta_3^2) = \sum_{r+m}^{n} 1/j^2 = \sum_{0}^{s} (r+m+i)^{-2}$$
$$= \int_0^s (r+m+x)^{-2}dx + (1/2)(n^{-2} + (r+m)^{-2}) + O(f^{(1)}(0))$$
$$= -(r+m+x)^{-1}|_0^s$$
$$+ (1/2)[n^{-2}p^{-2}(1 - 2m/(np) + O(m^2/n^2))] + O(n^{-3})$$
$$= 1/(r+m) - 1/n + (1/2)(np)^{-2} + O(mn^{-3} + n^{-3})$$
$$= (n-(r+m))/(n(r+m)) + (1/2)(np)^{-2} + O(mn^{-3})$$
$$= n(1-p)/(n^2p + nm) - O(m/n^2) + O(n^{-2})$$
$$= \frac{(1-p)n}{n^2p(1+m(np)^{-1})} + O(m/n^2)$$
$$= (1-p)/(np) \times (1 + O(m/n)) + O(m/n^2)$$
$$= (1-p)/(np) + O(m/n^2) \to (1-p)/(np). \tag{58}$$

Back to the proof outline in HS88, but with the new Ψ terms, with $D_i \equiv n^{1/2}\Delta_i$,

$$z_2 = \ell\Big\{n^{-1/2}\big[p^{-2}b_2E(D_3^{\ell-1}) + 2b_5E(D_3^{\ell})\Psi\sqrt{n}\big]$$
$$+ 2m^{-1}p^{-2}b_4\left[E(D_3^{\ell}) + E(D_3^{\ell-1})\Psi\sqrt{n}\right] + n^{-1/2}b_5E(D_3^{\ell+1}) + \delta E(D_3^{\ell-1})\Psi\sqrt{n}\Big\}$$
$$+ \ell(\ell-1)n^{-1/2}p^{-2}b_3\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right) + O(mn^{-3/2} + m^{-1/2}n^{-1/2}), \tag{59}$$

$$z_3 = \ell(\ell-1)m^{-1}b_3^2p^{-2}\left[E(D_3^{\ell}) + 2E(D_3^{\ell-1})\Psi\sqrt{n} + E(D_3^{\ell-2})\Psi^2n\right]$$
$$+ O(m^{-1/2}n^{-1/2} + m^{-3/2} + (m/n)^4). \tag{60}$$

To show (59), recall from (54) that $z_2 \equiv \ell n^{\ell/2} E\{(\Delta_2+\Delta_3)^{\ell-1} B\}$, where from (16), $B \equiv \delta\Psi + (n/m)(b_1\Delta_1 + b_2\Delta_2)\Delta_2 + (n/m)b_3(\Delta_1+\Delta_2)(\Delta_3+\Psi) + (n/m)^2 b_4(\Delta_1+\Delta_2)^2(\Delta_3+\Psi) + b_5\Delta_3(\Delta_3+2\Psi)$, with $b_1 \equiv -p/2$, $b_2 \equiv -p/2$, $b_3 \equiv -p/2 - (m/n)(a_2/2a_1)$, $b_4 \equiv (p/2)^2$, $b_5 \equiv -a_2/2a_1$, and $\Psi \equiv \gamma/(pg(p)\sqrt{n})$.

To start, pretend that only the Δ_3 from $(\Delta_2+\Delta_3)^{\ell-1}$ isn't pushed into the remainder (not actually true). Then, remembering that the Δ_i have mean zero and are independent,

$$\begin{aligned}
\ell n^{\ell/2} E\{\Delta_3^{\ell-1} B\} &= \ell n^{\ell/2} E\Big\{\Delta_3^{\ell-1}[\delta\Psi + (n/m)(b_1\Delta_1 + b_2\Delta_2)\Delta_2 \\
&\qquad + (n/m)b_3(\Delta_1+\Delta_2)(\Delta_3+\Psi) \\
&\qquad + (n/m)^2 b_4(\Delta_1+\Delta_2)^2(\Delta_3+\Psi) + b_5\Delta_3(\Delta_3+2\Psi)]\Big\} \\
&= \ell n^{\ell/2} E\Big\{\Delta_3^{\ell-1}\big[\delta\Psi + (n/m)(b_2\Delta_2^2) + (n/m)^2 b_4(\Delta_1^2+\Delta_2^2)(\Delta_3+\Psi) \\
&\qquad + b_5\Delta_3^2 + 2b_5\Delta_3\Psi\big]\Big\} \\
&= \ell\Big\{n^{1/2}E(D_3^{\ell-1})(n/m)b_2E(\Delta_2^2) + (n/m)^2 b_4 E(D_3^{\ell})E(\Delta_1^2+\Delta_2^2) \\
&\qquad + b_5 n^{-1/2}E(D_3^{\ell+1}) + n^{1/2}E(D_3^{\ell-1})\delta\Psi \\
&\qquad + (n/m)^2 b_4 E(D_3^{\ell-1})E(\Delta_1^2+\Delta_2^2)n^{1/2}\Psi + 2b_5E(D_3^{\ell})\Psi\Big\} \\
&= \ell\Big\{n^{3/2}m^{-1}b_2E(D_3^{\ell-1})[m(np)^{-2}(1+O(m/n))] \\
&\qquad + (n/m)^2 b_4 E(D_3^{\ell})2[m(np)^{-2}(1+O(m/n))] + n^{-1/2}b_5E(D_3^{\ell+1}) \\
&\qquad + n^{1/2}E(D_3^{\ell-1})\delta\Psi + 2b_5E(D_3^{\ell})\Psi \\
&\qquad + (n/m)^2 b_4 E(D_3^{\ell-1})2[m(np)^{-2}(1+O(m/n))]n^{1/2}\Psi\Big\} \\
&= \ell\Big\{n^{-1/2}\left[p^{-2}b_2E(D_3^{\ell-1}) + 2b_5E(D_3^{\ell})\Psi\sqrt{n}\right] + O(mn^{-3/2}) \\
&\qquad + 2m^{-1}p^{-2}b_4\left[E(D_3^{\ell}) + E(D_3^{\ell-1})\Psi\sqrt{n}\right] + O(1/n) \\
&\qquad + n^{-1/2}b_5E(D_3^{\ell+1}) + \delta E(D_3^{\ell-1})\Psi\sqrt{n}\Big\}.
\end{aligned} \tag{61}$$

Still to be shown are 1) the order of the third moment of Δ_2, 2) another term in z_2 from taking the terms in $(\Delta_2+\Delta_3)^{\ell-1}$ with one Δ_2, and 3) that everything else is in the remainder.

First,

$$\begin{aligned}
E(\Delta_2^3) &= E\left[\left(\sum_{r}^{r+m-1} j^{-1}V_j\right)^3\right] \\
&= E\left[\sum_{r}^{r+m-1}\left(j^{-1}V_j\right)^3\right], \text{ since } E(V_iV_j) = E(V_i)E(V_j) = 0, i \neq j
\end{aligned}$$

$$= \sum_{r}^{r+m-1} j^{-3} E(V_j^3) = 2 \sum_{r}^{r+m-1} j^{-3} \text{ since } E(V_j^3) = 2.$$

Using Euler-Maclaurin,

$$\begin{aligned}
\sum_{r}^{r+m-1} 1/j^3 &= \sum_{r}^{r+m} 1/j^3 - 1/(r+m)^3 = \sum_{0}^{m} (r+i)^{-3} - 1/(r+m)^3 \\
&= \int_0^m (r+x)^{-3} dx + (1/2)(r^{-3} + (r+m)^{-3}) \\
&\quad - 1/(r+m)^3 + O(f^{(1)}(0)) \\
&= -(1/2)(r+x)^{-2}|_0^m + (1/2)[r^{-3} - (r+m)^{-3}] + O(n^{-4}) \\
&= (1/2)r^{-2} - (1/2)(r+m)^{-2} \\
&\quad + (1/2)[(1/r^3) - 1/(r^3(1 + 3m/r + 3m^2/r^2 + m^3/r^3))] + O(n^{-4}) \\
&= (1/2)[r^{-2} - 1/(r^2(1 + 2m/r + m^2/r^2))] \\
&\quad + (1/2)(1/r^3)(1 - 1 + O(m/n)) + O(n^{-4}) \\
&= (1/2)[r^{-2}(1 - 1 + 2m/r + m^2/r^2)] + O(m/n^4 + n^{-4}) \\
&= r^{-2}(m/r) + O(m^2/n^4 + m/n^4 + n^{-4}) \\
&= m/(np)^3 + O(m^2/n^4 + m/n^4 + n^{-4}),
\end{aligned}$$

$$\begin{aligned}
E(\Delta_2^3) &= 2m/(np)^3 + O(m^2/n^4) = O(mn^{-3}), \qquad (62)\\
E(D_2^3) &= E((n^{1/2}\Delta_2)^3) = n^{3/2} E(\Delta_2^3) \\
&= n^{3/2}[m/(np)^3 + O(m^2/n^4 + m/n^4 + n^{-4})] \\
&= mn^{-3/2}p^{-3} + O(m^2/n^{5/2} + m/n^{5/2} + n^{-5/2}) \\
&= O(mn^{-3/2}).
\end{aligned}$$

Note that this is smaller than the result from earlier that $\Delta_2 = O(m^{1/2}n^{-1})$, which would yield $D_2^3 = O(m^{3/2}n^{-3/2})$, an additional $\sqrt{m}$.

Second, to show the extra term in z_2, from taking the terms in $(\Delta_2 + \Delta_3)^{\ell-1}$ with one Δ_2,

$$\begin{aligned}
(\Delta_2 + \Delta_3)^{\ell-1} &= \Delta_3^{\ell-1} + (\ell-1)\Delta_2\Delta_3^{\ell-2} + O(\Delta_2^2\Delta_3^{\ell-3}), \\
&\ell n^{\ell/2} E\{(\ell-1)\Delta_2\Delta_3^{\ell-2} B\} \\
&= \ell n^{\ell/2} E\Big\{(\ell-1)\Delta_2\Delta_3^{\ell-2}\big[\delta\Psi + (n/m)(b_1\Delta_1 + b_2\Delta_2)\Delta_2 \\
&\qquad + (n/m)b_3(\Delta_1 + \Delta_2)(\Delta_3 + \Psi)
\end{aligned}$$

$$
\begin{aligned}
&\qquad + (n/m)^2 b_4 (\Delta_1+\Delta_2)^2(\Delta_3+\Psi) + b_5(\Delta_3^2 + 2\Delta_3\Psi)\Big]\Big\} \\
&= \ell n^{\ell/2} E\Big\{(\ell-1)\Delta_2\Delta_3^{\ell-2}\big[(n/m)b_2\Delta_2^2 + (n/m)b_3\Delta_2(\Delta_3+\Psi) \\
&\quad + (n/m)^2 b_4(\Delta_1^2+\Delta_2^2)(\Delta_3+\Psi) + b_5(\Delta_3^2+2\Delta_3\Psi)\big]\Big\} \\
&= \ell(\ell-1)E\Big\{(n/m)b_2 n^{\ell/2}\Delta_2\Delta_3^{\ell-2}\Delta_2^2 + (n/m)b_3 n^{\ell/2}\Delta_2\Delta_3^{\ell-2}\Delta_2(\Delta_3+\Psi) \\
&\qquad + (n/m)^2 b_4 n^{\ell/2}\Delta_2\Delta_3^{\ell-2}(\Delta_1^2+\Delta_2^2)(\Delta_3+\Psi) \\
&\qquad + b_5 n^{\ell/2}\Delta_2\Delta_3^{\ell-2}(\Delta_3^2+2\Delta_3\Psi)\Big\} \\
&= \ell(\ell-1)E\Big\{m^{-1}b_2 n^{1+\ell/2}\Delta_2^3\Delta_3^{\ell-2} + m^{-1}b_3 n^{1+\ell/2}\Delta_2^2\Delta_3^{\ell-1} \\
&\qquad + m^{-1}b_3 n^{1+\ell/2}\Delta_2^2\Delta_3^{\ell-2}\Psi \\
&\qquad + m^{-2}b_4 n^{2+\ell/2}\Delta_2^3\Delta_3^{\ell-1} + m^{-2}b_4 n^{2+\ell/2}\Delta_2^3\Delta_3^{\ell-2}\Psi\Big\} \\
&= \ell(\ell-1)E\Big\{m^{-1}b_2 n^{1+\ell/2-(\ell-2)/2}\Delta_2^3 D_3^{\ell-2} \\
&\qquad + m^{-1}b_3 n^{1+\ell/2-(\ell-1)/2}\Delta_2^2(D_3^{\ell-1} + D_3^{\ell-2}\Psi\sqrt{n}) \\
&\qquad + m^{-2}b_4 n^{2+\ell/2-(\ell-1)/2}\Delta_2^3(D_3^{\ell-1} + D_3^{\ell-2}\Psi\sqrt{n})\Big\} \\
&= \ell(\ell-1)\Big\{m^{-1}b_2 n^2 E(\Delta_2^3)E(D_3^{\ell-2}) \\
&\qquad + m^{-1}b_3 n^{3/2}E(\Delta_2^2)\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right) \\
&\qquad + m^{-2}b_4 n^{5/2}E(\Delta_2^3)\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right)\Big\},
\end{aligned}
$$

and using (62),

$$
\begin{aligned}
&= \ell(\ell-1)\Big\{m^{-1}b_2 n^2 O(mn^{-3})E(O_p(1)) \\
&\qquad + m^{-1}b_3 n^{3/2} m(np)^{-2}(1+O(m/n))\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right) \\
&\qquad + m^{-2}b_4 n^{5/2}O(mn^{-3}E(O_p(1))\Big\} \\
&= \ell(\ell-1)\Big\{O(n^{-1})O(1) + b_3 n^{-1/2}p^{-2}\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right) \\
&\qquad + O(mn^{-3/2}) + O(m^{-1}n^{-1/2})O(1)\Big\} \\
&= \ell(\ell-1)n^{-1/2}p^{-2}b_3\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right) \\
&\quad + O(n^{-1} + mn^{-3/2} + m^{-1}n^{-1/2}) \\
&= \ell(\ell-1)n^{-1/2}p^{-2}b_3\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right) \\
&\quad + O(mn^{-3/2} + m^{-1/2}n^{-1/2}), \qquad (63)
\end{aligned}
$$

as in (59).

Third, everything else should be in the remainder. The biggest term in B will be the b_3 term. Then,

$$\begin{aligned}
&\ell n^{\ell/2}E\left\{\left(\binom{\ell-1}{2}\Delta_2^2\Delta_3^{\ell-3}\right)(n/m)b_3(\Delta_1+\Delta_2)(\Delta_3+\Psi)\right\}\\
&\quad\sim n^{\ell/2}E(\Delta_2^3)E(\Delta_3^{\ell-2})(n/m)\\
&\quad= (n/m)n^{\ell/2}O(mn^{-3})O(n^{-(\ell-2)/2})\\
&\quad= O(n^{-1}),
\end{aligned}$$

and thus all smaller terms will also be in the remainder.

For z_3, recall from (54) that $z_3 \equiv \frac{1}{2}\ell(\ell-1)n^{\ell/2}E\{(\Delta_2+\Delta_3)^{\ell-2}B^2\}$. Ignoring remainders for now, note that the largest term in B^2 is the square of the b_3 term, which is $(n/m)^2 b_3^2(\Delta_1+\Delta_2)^2\Delta_3^2$. Since $\Delta_1 \perp\!\!\!\perp \Delta_2$ and $E(\Delta_1^2) = E(\Delta_2^2) = m(np)^{-2}(1+O(m/n))$ as shown in (57), then $E(\Delta_1+\Delta_2)^2 = E(\Delta_1^2+\Delta_2^2) = 2m(np)^{-2}(1+O(m/n))$. Since $\Delta_1, \Delta_2 \perp\!\!\!\perp \Delta_3$, this term can be pulled outside the expectation. If all that remains of $(\Delta_2+\Delta_3)^{\ell-2}$ is the $\Delta_3^{\ell-2}$ term, then

$$\begin{aligned}
z_3 &\equiv \frac{1}{2}\ell(\ell-1)n^{\ell/2}E\{(\Delta_2+\Delta_3)^{\ell-2}B^2\}\\
&= \frac{1}{2}\ell(\ell-1)n^{\ell/2}E\{(\Delta_3)^{\ell-2}[(n/m)^2b_3^2(\Delta_1+\Delta_2)^2(\Delta_3+\Psi)^2]\}\\
&= \frac{1}{2}\ell(\ell-1)n^{\ell/2}(n/m)^2b_3^2E\{(\Delta_1+\Delta_2)^2\}E(\Delta_3^\ell+2\Psi\Delta_3^{\ell-1}+\Psi^2\Delta_3^{\ell-2})\\
&= \frac{1}{2}\ell(\ell-1)n^{\ell/2}(n/m)^2b_3^2 2m(np)^{-2}(1+O(m/n))E(\Delta_3^\ell+2\Psi\Delta_3^{\ell-1}+\Psi^2\Delta_3^{\ell-2})\\
&= \ell(\ell-1)m^{-1}b_3^2p^{-2}\left[E(D_3^\ell)+2E(D_3^{\ell-1})\Psi\sqrt{n}+E(D_3^{\ell-2})\Psi^2 n\right]+O(n^{-1}),
\end{aligned}$$

matching (60).

Now it must be checked that all remainders are indeed of the proper order. First, look at the next-biggest terms in B^2. The product of b_3 and b_5 terms is expectation zero due to the $(\Delta_1+\Delta_2)$ in the b_3 term. The b_2 and b_5 terms are of the same order, so just consider b_5. The relative order of magnitude of b_4 and b_5 terms depends on the relation of m and n, so both the b_3b_4 product term and the b_5^2 term should be checked to be rigorous.

For the b_5^2,

$$n^{\ell/2}E(\Delta_3^{\ell-2}b_5^2\Delta_3^4) = E(n^{\ell/2}O_p(n^{-(\ell+2)/2})) = O(n^{-1}) = O(m^{-1/2}n^{-1/2}).$$

For b_3b_4,

$$\begin{aligned}
n^{\ell/2}E(\Delta_3^{\ell-2}b_3(n/m)\Delta_2\Delta_3b_4(n/m)^2\Delta_2^2\Delta_3) &= E(n^{\ell/2}n^3m^{-3}\Delta_2^3\Delta_3^\ell)\\
&= n^3m^{-3}E(\Delta_2^3)E(D_3^\ell)\\
&= O(m^{-2})O(1) = O(m^{-2}) = O(m^{-3/2}).
\end{aligned}$$

Second, the additional terms that come in when $\gamma \neq 0$ must be checked. It is sufficient to check the square of the $\delta\Psi$ term along with its product term with b_3; if these are in the remainder, all other product terms will also be in the remainder since all square terms aside from b_3^2 are in the remainder. For the square term, $(\delta\Psi)^2$ is of the order $O(m^4n^{-5})$, so $n^{\ell/2}E(\Delta_3^{\ell-2})(\delta\Psi)^2 = O(nm^4n^{-5}) = O((m/n)^4) = o(m^2/n^2)$, the last expression being the remainder in (65). For the product term with b_3, from the b_3 term there will be lone $\Delta_1+\Delta_2$, whose expectation is zero, so the term will evaluate to zero.

Third, note that $(\Delta_2+\Delta_3)^{\ell-2} = \Delta_3^{\ell-2} + O(\Delta_2\Delta_3^{\ell-3})$. Checking against the biggest (b_3^2) term in B^2 (thus other terms will only be even smaller),

$$\begin{aligned}
n^{\ell/2}E\{\Delta_2\Delta_3^{\ell-3}(n/m)^2\Delta_2^2\Delta_3^2\} &= m^{-2}n^{2+\ell/2}E(\Delta_2^3)E(\Delta_3^{\ell-1})\\
&= O(m^{-2}n^{2+\ell/2}mn^{-3}n^{-(\ell-1)/2})\\
&= O(m^{-1}n^{2-3+1/2})\\
&= O(m^{-1}n^{-1/2})\\
&= O(m^{-1/2}n^{-1/2}).
\end{aligned}$$

Using results going back to (54) , and noting that $b_3 = b_2 + O(m/n) = -(p/2) + O(m/n)$ and $b_3^2 = b_4 + O(m/n) = (p/2)^2 + O(m/n)$,

$$\begin{aligned}
z_2 + z_3 = \ell\Big\{&n^{-1/2}\left[p^{-2}b_2E(D_3^{\ell-1}) + 2b_5E(D_3^{\ell})\Psi\sqrt{n}\right] + 2m^{-1}p^{-2}b_4\left[E(D_3^{\ell}) + E(D_3^{\ell-1})\Psi\sqrt{n}\right]\\
&+ n^{-1/2}b_5E(D_3^{\ell+1}) + \delta E(D_3^{\ell-1})\Psi\sqrt{n}\Big\}\\
&+ \ell(\ell-1)n^{-1/2}p^{-2}b_3\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right)\\
&+ \ell(\ell-1)m^{-1}b_3^2p^{-2}\left[E(D_3^{\ell}) + 2E(D_3^{\ell-1})\Psi\sqrt{n} + E(D_3^{\ell-2})\Psi^2n\right]\\
&+ O(mn^{-3/2} + m^{-1/2}n^{-1/2} + m^{-3/2} + (m/n)^4)\\
= n^{-1/2}\ell\Big\{&p^{-2}b_2E(D_3^{\ell-1}) + 2b_5E(D_3^{\ell})\Psi\sqrt{n} + b_5E(D_3^{\ell+1})\\
&+ (\ell-1)p^{-2}b_3\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right)\Big\}\\
+ m^{-1}\ell\Big\{&2p^{-2}b_4\left[E(D_3^{\ell}) + E(D_3^{\ell-1})\Psi\sqrt{n}\right]\\
&+ (\ell-1)b_3^2p^{-2}\left[E(D_3^{\ell}) + 2E(D_3^{\ell-1})\Psi\sqrt{n} + E(D_3^{\ell-2})\Psi^2n\right]\Big\}\\
+ \delta\ell E(D_3^{\ell-1})&\Psi\sqrt{n} + O(mn^{-3/2} + m^{-1/2}n^{-1/2} + m^{-3/2} + (m/n)^4)\\
= n^{-1/2}\ell\Big\{&p^{-2}(-p/2)E(D_3^{\ell-1}) + 2b_5E(D_3^{\ell})\Psi\sqrt{n} + b_5E(D_3^{\ell+1})\\
&+ (\ell-1)p^{-2}(-p/2 + (m/n)b_5)\left(E(D_3^{\ell-1}) + E(D_3^{\ell-2})\Psi\sqrt{n}\right)\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+ m^{-1}\ell\Big\{2p^{-2}(p^2/4)\left[E(D_3^{\ell}) + E(D_3^{\ell-1})\Psi\sqrt{n}\right] \\
&\qquad + (\ell-1)(p^2/4 + O(m/n))p^{-2}\left[E(D_3^{\ell}) + 2E(D_3^{\ell-1})\Psi\sqrt{n} + E(D_3^{\ell-2})\Psi^2 n\right]\Big\} \\
&+ \delta\ell E(D_3^{\ell-1})\Psi\sqrt{n} + O(mn^{-3/2} + m^{-1/2}n^{-1/2} + m^{-3/2} + (m/n)^4) \\
= {}& n^{-1/2}\ell\Big\{(\ell-1)(-2p)^{-1}E(D_3^{\ell-2})\Psi\sqrt{n} + [(\ell-1)(-2p)^{-1} + (-2p)^{-1}]E(D_3^{\ell-1}) \\
&\qquad + 2b_5E(D_3^{\ell})\Psi\sqrt{n} + b_5E(D_3^{\ell+1})\Big\} \\
&+ m^{-1}\ell\Big\{(\ell-1)(1/4)E(D_3^{\ell-2})\Psi^2 n + E(D_3^{\ell-1})[(\ell-1)(1/2)\Psi\sqrt{n} + (1/2)\Psi\sqrt{n}] \\
&\qquad + E(D_3^{\ell})[(1/2) + (\ell-1)(1/4)]\Big\} \\
&+ \delta\ell E(D_3^{\ell-1})\Psi\sqrt{n} + O(mn^{-3/2} + m^{-1/2}n^{-1/2} + m^{-3/2} + (m/n)^4) \\
= {}& n^{-1/2}\ell\Big\{-(\ell-1)(2p)^{-1}\Psi\sqrt{n}E(D_3^{\ell-2}) - \ell(2p)^{-1}E(D_3^{\ell-1}) \\
&\qquad - (a_2/a_1)E(D_3^{\ell})\Psi\sqrt{n} - (a_2/2a_1)E(D_3^{\ell+1})\Big\} \\
&+ m^{-1}\ell\Big\{\frac{\ell-1}{4}E(D_3^{\ell-2})\Psi^2 n + \frac{\ell+1}{4}\Psi\sqrt{n}E(D_3^{\ell-1}) + \frac{\ell}{2}E(D_3^{\ell})\Big\} \\
&+ \delta\ell E(D_3^{\ell-1})\Psi\sqrt{n} + O\left(mn^{-3/2} + m^{-1/2}n^{-1/2} + m^{-3/2} + (m/n)^4\right). \qquad (64)
\end{aligned}
$$

Clearly, $O(n^{-1/2}m^{-1/2})$ and $O(m^{-3/2})$ are $o(m^{-1})$, and $O((m/n)^4)$ is $o((m/n)^2)$. From the restrictions given immediately before (53), $O(mn^{-3/2}) = o(m^{-1}+(m/n)^2)$, noting that there was a $O(mn^{-3/2})$ in the original remainder R right above that. This is consistent with the final result from HS88 at least, since the rate of m is faster than $n^{1/2}$, so $mn^{-1/2}$ is bigger than $O(1)$, so $O((m/n)^2) = O(mn^{-3/2}mn^{-1/2}) > O(mn^{-3/2})$.

$$
\begin{aligned}
E[(-p^{-1}Y)^{\ell}] = {}& E[(1+\delta)(D_2+D_3)]^{\ell} \\
&+ n^{-1/2}\ell\Big\{-(\ell-1)(2p)^{-1}\Psi\sqrt{n}E(D_3^{\ell-2}) - \ell(2p)^{-1}E(D_3^{\ell-1}) \\
&\qquad - (a_2/a_1)E(D_3^{\ell})\Psi\sqrt{n} - (a_2/2a_1)E(D_3^{\ell+1})\Big\} \\
&+ m^{-1}\ell\Big\{\frac{\ell-1}{4}E(D_3^{\ell-2})\Psi^2 n + \frac{\ell}{2}\Psi\sqrt{n}E(D_3^{\ell-1}) + \frac{\ell+1}{4}E(D_3^{\ell})\Big\} \\
&+ \delta\ell E(D_3^{\ell-1})\Psi\sqrt{n} + o\left(m^{-1} + (m/n)^2\right). \qquad (65)
\end{aligned}
$$

As in HS88, let

$$K \equiv [p(1-p)]^{-1/2}Y, \qquad L \equiv -[p/(1-p)]^{1/2}(1+\delta)(D_2+D_3), \qquad (66)$$

and as they state,

$$E(D_3^{2k}) = [(1-p)/p]^k(2k)!(k!2^k)^{-1} + O(n^{-1}), \qquad E(D_3^{2k+1}) = O(n^{-1/2}), \qquad (67)$$

and D_3 is asymptotically $N(0,(1-p)/p)$. Multiplying equation (65) by $\{-[p/(1-p)]^{1/2}\}^\ell (it)^\ell/\ell!$ and adding over ℓ, the formal expansion is

$$E(e^{itK}) = E(e^{itL}) + n^{-1/2}\alpha_1(t) + m^{-1}\alpha_2(t) + \delta\alpha_3(t) + o[m^{-1} + (m/n)^2], \tag{68}$$

where

$$\begin{aligned}
\alpha_1(t) &\equiv e^{-t^2/2}\Bigg[\left(\frac{1}{2} - b_5(1-p)\right)(p(1-p))^{-1/2}\left((it)^3 + (it)\right) \\
&\qquad + (it)^2\Psi\sqrt{n}\left(2b_5 - \frac{1}{2(1-p)}\right)\Bigg], \\
\alpha_2(t) &\equiv e^{-t^2/2}\frac{1}{4}\left[(it)^4 + (it)^2\left(3 + \Psi^2 n\frac{p}{1-p}\right) - 2((it)^3 + (it))\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}\right], \\
\alpha_3(t) &\equiv e^{-t^2/2}(it)\left(-\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}\right).
\end{aligned}$$

To see this, the LHS of (65) becomes

$$E[(-p^{-1}Y)^\ell[-\{p/(1-p)\}^{1/2}]^\ell(it)^\ell/\ell! = E[(\{p(1-p)\}^{-1/2}Y)^\ell](it)^\ell/\ell! = E(K^\ell)(it)^\ell/\ell!,$$

so that

$$\begin{aligned}
E(K^\ell)(it)^\ell/\ell! &= [-\{p/(1-p)\}^{1/2}]^\ell(it)^\ell/\ell! \\
&\quad \times \Big\{E[(1+\delta)(D_2+D_3)]^\ell \\
&\quad + n^{-1/2}\ell\Big[-(\ell-1)(2p)^{-1}\Psi\sqrt{n}E(D_3^{\ell-2}) - \ell(2p)^{-1}E(D_3^{\ell-1}) \\
&\qquad - (a_2/a_1)E(D_3^\ell)\Psi\sqrt{n} - (a_2/2a_1)E(D_3^{\ell+1})\Big] \\
&\quad + m^{-1}\ell\Big[\frac{\ell-1}{4}E(D_3^{\ell-2})\Psi^2 n + \frac{\ell}{2}\Psi\sqrt{n}E(D_3^{\ell-1}) + \frac{\ell+1}{4}E(D_3^\ell)\Big] \\
&\quad + \delta\ell E(D_3^{\ell-1})\Psi\sqrt{n} + o[m^{-1} + (m/n)^2]\Big\}, \\
E(L^\ell)(it)^\ell/\ell! &= [-\{p/(1-p)\}^{1/2}]^\ell(it)^\ell/\ell! \times E[(1+\delta)(D_2+D_3)]^\ell, \\
E(K^\ell)(it)^\ell/\ell! &= E(L^\ell)(it)^\ell/\ell! \\
&\quad + \left(-\{p/(1-p)\}^{1/2}\right)^\ell(it)^\ell/\ell! \\
&\quad \times \Big\{n^{-1/2}\ell\Big[-(\ell-1)(2p)^{-1}\Psi\sqrt{n}E(D_3^{\ell-2}) - \ell(2p)^{-1}E(D_3^{\ell-1}) \\
&\qquad - (a_2/a_1)E(D_3^\ell)\Psi\sqrt{n} - (a_2/2a_1)E(D_3^{\ell+1})\Big] \\
&\quad + m^{-1}\ell\Big[\frac{\ell-1}{4}E(D_3^{\ell-2})\Psi^2 n + \frac{\ell}{2}\Psi\sqrt{n}E(D_3^{\ell-1}) + \frac{\ell+1}{4}E(D_3^\ell)\Big]
\end{aligned}$$

$$+\delta\ell E(D_3^{\ell-1})\Psi\sqrt{n}+o[m^{-1}+(m/n)^2]\Big\}. \tag{69}$$

The characteristic function of K is, equivalently, $E(e^{itK})$ or the sum of the LHS immediately above from $\ell=1$ to ∞. On the RHS, the sum of the L terms will be the characteristic function of L, which can also be written $E(e^{itL})$.

Recall that for a standard normal random variable, the odd moments are zero, and the even moments are the double-factorials $1, 3\cdot 1, 5\cdot 3\cdot 1, \ldots, (\ell-1)!!, \ldots$ for even moment ℓ, noting the equivalent formula $(2k-1)!! = (2k)!/(k!2^k)$.

Next, I look at each additive term separately, first the terms without Ψ, and second the new terms that only appear when $\gamma \neq 0$.

Start with part of the m^{-1} term: $m^{-1}(1/4)((it)^\ell/\ell!)[-\{p/(1-p)\}^{1/2}]^\ell \ell(\ell+1)E(D_3^\ell)$. Using the moments of D_3 from (67), the odd moments are all $O(n^{-1/2})$, so when multiplied by the leading m^{-1}, they will end up in the $o(m^{-1})$ remainder. The $O(n^{-1})$ error from the even moments also ends up in the $o(m^{-1})$ remainder when multiplied by the leading m^{-1}, so I can focus on just the even moments and rest assured that the remainder works out.

Plugging in from (67) and summing over ℓ,

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} m^{-1}\frac{1}{4}\frac{(it)^\ell}{\ell!}\left[-\{p/(1-p)\}^{1/2}\right]^\ell \ell(\ell+1)E(D_3^\ell)\\
&\qquad = m^{-1}\frac{1}{4}\sum_{k=1}^{\infty}\frac{(it)^{2k}}{(2k)!}[p/(1-p)]^k(2k)(2k+1)[(1-p)/p]^k\frac{(2k)!}{k!2^k}\\
&\qquad = m^{-1}\frac{1}{4}\sum_{k=1}^{\infty}(it)^{2k}(2k+1)\frac{2k}{k!2^k} = m^{-1}\frac{1}{4}\sum_{k=1}^{\infty}\frac{(it)^{2k}(2k+1)}{(k-1)!2^{k-1}}.
\end{aligned}$$

$$\begin{aligned}
e^{-t/2} &= E(e^{itN(0,1)}) = 1+\frac{(it)^2}{2!}+\frac{(it)^4}{4!}3+\frac{(it)^6}{6!}15+\ldots\\
&= \sum_{k=0}^{\infty}\frac{(it)^{2k}}{(2k)!}(2k-1)!! = \sum_{k=0}^{\infty}\frac{(it)^{2k}}{(2k)!}\frac{(2k)!}{(k)!2^k}\\
&= \sum_{k=1}^{\infty}\frac{(it)^{2(k-1)}}{(k-1)!2^{k-1}} = \sum_{k=1}^{\infty}\frac{(it)^{2k-2}}{(k-1)!2^{k-1}}, \qquad (70)
\end{aligned}$$

$$(it)^2e^{-t^2/2} = \sum_{k=1}^{\infty}\frac{(it)^{2k}}{(k-1)!2^{k-1}}, \tag{71}$$

$$\sum_{k=1}^{\infty}\frac{(it)^{2k}2(k-1)}{(k-1)!2^{k-1}} = \sum_{k=2}^{\infty}\frac{(it)^{2k}2(k-1)}{(k-1)!2^{k-1}} = \sum_{k=2}^{\infty}\frac{(it)^{2k}}{(k-2)!2^{k-2}} = (it)^4\sum_{k=2}^{\infty}\frac{(it)^{2k-4}}{(k-2)!2^{k-2}}$$

$$= (it)^4 \sum_{k=2}^{\infty} \frac{(it)^{2(k-2)}}{(k-2)!2^{k-2}} = (it)^4 \sum_{k=1}^{\infty} \frac{(it)^{2(k-1)}}{(k-1)!2^{k-1}} = (it)^4 e^{-t^2/2}, \tag{72}$$

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(it)^{2k}(2k+1)}{(k-1)!2^{k-1}} &= \sum_{k=1}^{\infty} \frac{(it)^{2k}2(k-1)}{(k-1)!2^{k-1}} + 3\sum_{k=1}^{\infty} \frac{(it)^{2k}}{(k-1)!2^{k-1}} \\
&= (it)^4 e^{-t^2/2} + 3(it)^2 e^{-t^2/2},
\end{aligned}$$

and thus combining these,

$$\begin{aligned}
\sum_{\ell=1}^{\infty} & m^{-1}\frac{1}{4}\frac{(it)^\ell}{\ell!}[-\{p/(1-p)\}^{1/2}]^\ell \ell(\ell+1)E(D_3^\ell) \\
&= m^{-1}\frac{1}{4}\sum_{k=1}^{\infty} \frac{(it)^{2k}(2k+1)}{(k-1)!2^{k-1}} \\
&= m^{-1}\frac{1}{4}[(it)^4 e^{-t^2/2} + 3(it)^2 e^{-t^2/2}]. \qquad (73)
\end{aligned}$$

The other terms that are the same when $\gamma = 0$ are the $n^{-1/2}$ terms in (69) without Ψ.

$$\begin{aligned}
\sum_{\ell=1}^{\infty} & [-\{p/(1-p)\}^{1/2}]^\ell (it)^\ell/\ell![b_5 \ell E(D_3^{\ell+1}) - (2p)^{-1}\ell^2 E(D_3^{\ell-1})] \\
&= \sum_{k=0}^{\infty} \Big\{ [-\{p/(1-p)\}^{1/2}]^{(2k+1)} (it)^{(2k+1)}/(2k+1)! \\
&\qquad \times [b_5(2k+1)E(D_3^{(2k+1)+1}) - (2p)^{-1}(2k+1)^2 E(D_3^{(2k+1)-1})] \Big\} + O(n^{-1/2}) \\
&= \sum_{k=0}^{\infty} \Big\{ -\{p/(1-p)\}^{(2k+1)/2} (it)^{(2k+1)}/(2k+1)! \\
&\qquad \times \Big[b_5(2k+1)[(1-p)/p]^{k+1}\frac{(2(k+1))!}{(k+1)!2^{k+1}} - (2p)^{-1}(2k+1)^2[(1-p)/p]^k \frac{(2k)!}{k!2^k} \Big] \Big\} \\
&= \sum_{k=0}^{\infty} (it)^{(2k+1)} \Big[(2p)^{-1}(2k+1)^2[(1-p)/p]^{-1/2}\frac{(2k)!}{(2k+1)!k!2^k} \\
&\qquad - b_5(2k+1)[(1-p)/p]^{1/2}\frac{(2k+2)!}{(2k+1)!(k+1)!2^{k+1}} \Big]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} (it)^{(2k+1)} \Big[(2p)^{-1}(2k+1)^2[(1-p)/p]^{-1/2}\frac{1}{(2k+1)k!2^k} \\
&\qquad - b_5(2k+1)[(1-p)/p]^{1/2}\frac{2k+2}{(k+1)!2^{k+1}} \Big]
\end{aligned}$$

$$
\begin{aligned}
&= (it)\sum_{k=0}^{\infty}(it)^{2k}\Bigg[(1/2)(2k+1)[(1-p)p]^{-1/2}\frac{1}{k!2^k} \\
&\qquad\qquad - b_5(2k+1)[(1-p)/p]^{1/2}\frac{2k+2}{(k+1)!2^{k+1}}\Bigg] \\
&= (it)\sum_{k=0}^{\infty}(it)^{2k}[(1-p)p]^{-1/2} \\
&\qquad \times\left[(1/2)(2k+1)\frac{1}{k!2^k} - b_5(1-p)(2k+1)(2k+2)\frac{1}{(k+1)!2^{k+1}}\right] \\
&= [(1-p)p]^{-1/2}(it) \\
&\quad \times\left\{\sum_{k=0}^{\infty}(it)^{2k}(1/2)(2k+1)\frac{1}{k!2^k} - \sum_{k=0}^{\infty}(it)^{2k}b_5(1-p)(2k+1)(2k+2)\frac{1}{(k+1)!2^{k+1}}\right\} \\
&= [(1-p)p]^{-1/2}(it) \\
&\quad \times\Bigg\{\frac{1}{2}\sum_{k=1}^{\infty}(it)^{2(k-1)}\frac{(2(k-1)+1)}{(k-1)!2^{(k-1)}} \\
&\qquad - \sum_{k=1}^{\infty}(it)^{2(k-1)}b_5(1-p)\frac{(2(k-1)+1)(2(k-1)+2)}{((k-1)+1)!2^{(k-1)+1}}\Bigg\} \\
&= [(1-p)p]^{-1/2}(it) \\
&\quad \times\left\{\frac{1}{2}\frac{1}{(it)^2}\sum_{k=1}^{\infty}(it)^{2k}\frac{(2k-1)}{(k-1)!2^{k-1}} - b_5(1-p)\frac{1}{(it)^2}\sum_{k=1}^{\infty}(it)^{2k}\frac{(2k-1)(2k)}{k!2^k}\right\}
\end{aligned}
$$

Recall from (72), (70), and (71) that

$$
\sum_{k=1}^{\infty}\frac{(it)^{2k}2(k-1)}{(k-1)!2^{k-1}} = (it)^4 e^{-t^2/2}, \quad e^{-t^2/2} = \sum_{k=0}^{\infty}\frac{(it)^{2k}}{k!2^k}, \quad (it)^2 e^{-t^2/2} = \sum_{k=1}^{\infty}\frac{(it)^{2k}}{(k-1)!2^{k-1}},
$$

so that

$$
\begin{aligned}
\sum_{k=1}^{\infty}(it)^{2k}\frac{2k-1}{(k-1)!2^{k-1}} &= \sum_{k=1}^{\infty}\frac{(it)^{2k}2(k-1)}{(k-1)!2^{k-1}} + \sum_{k=1}^{\infty}\frac{(it)^{2k}}{(k-1)!2^{k-1}} \\
&= (it)^4 e^{-t^2/2} + (it)^2 e^{-t^2/2} \\
&= (it)^2 e^{-t^2/2}[(it)^2+1],
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{\infty}\frac{(it)^{2k}[2k][2(k-1)]}{k!2^k} &= \sum_{k=2}^{\infty}\frac{(it)^{2k}[2k][2(k-1)]}{k!2^k} = \sum_{k=2}^{\infty}\frac{(it)^{2k}[k][(k-1)]}{k!2^{k-2}} \\
&= \sum_{k=2}^{\infty}\frac{(it)^{2k}}{(k-2)!2^{k-2}} = (it)^4\sum_{k=2}^{\infty}\frac{(it)^{2k-4}}{(k-2)!2^{k-2}}
\end{aligned}
$$

$$
\begin{aligned}
&= (it)^4 \sum_{k=2}^{\infty} \frac{(it)^{2(k-2)}}{(k-2)!2^{k-2}} = (it)^4 \sum_{k=1}^{\infty} \frac{(it)^{2(k-1)}}{(k-1)!2^{k-1}} \\
&= (it)^4 e^{-t^2/2}, \\
\frac{1}{(it)^2} \sum_{k=1}^{\infty} (it)^{2k} \frac{2k-1}{(k-1)!2^{k-1}} &= \sum_{k=0}^{\infty} \frac{(it)^{2k}(2k+1)}{k!2^k} = \frac{1}{(it)^2} (it)^2 e^{-t^2/2}((it)^2+1) \\
&= e^{-t^2/2}((it)^2+1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{k=1}^{\infty} (it)^{2k} \frac{(2k-1)(2k)}{k!2^k} = \sum_{k=0}^{\infty} (it)^{2k} \frac{4k^2-2k}{k!2^k} &= \sum_{k=0}^{\infty} (it)^{2k} \frac{(4k^2-4k)+(2k+1)-(1)}{k!2^k} \\
&= (it)^4 e^{-t^2/2} + e^{-t^2/2}((it)^2+1) - e^{-t^2/2},
\end{aligned}
$$

and additionally,

$$
\begin{aligned}
&[(1-p)p]^{-1/2}(it) \left\{ \frac{1}{2} \frac{1}{(it)^2} \sum_{k=1}^{\infty} (it)^{2k} \frac{(2k-1)}{(k-1)!2^{k-1}} - b_5(1-p) \frac{1}{(it)^2} \sum_{k=1}^{\infty} (it)^{2k} \frac{(2k-1)(2k)}{k!2^k} \right\} \\
&\quad = [(1-p)p]^{-1/2}(it) \\
&\qquad \times \left\{ \frac{1}{2} \frac{1}{(it)^2} \left[(it)^2 e^{-t^2/2}((it)^2+1) \right] \right. \\
&\qquad\qquad \left. - b_5(1-p) \frac{1}{(it)^2} \left[(it)^4 e^{-t^2/2} + e^{-t^2/2}((it)^2+1) - e^{-t^2/2} \right] \right\} \\
&\quad = [(1-p)p]^{-1/2} e^{-t^2/2} \\
&\qquad \times \left\{ \frac{1}{2}(it)((it)^2+1) - b_5(1-p) \left[(it)^3 + (it)^{-1}((it)^2+1) - (it)^{-1} \right] \right\} \\
&\quad = [(1-p)p]^{-1/2} e^{-t^2/2} \\
&\qquad \times \left\{ \frac{1}{2} \left[(it)^3 + (it) \right] - b_5(1-p) \left[(it)^3 + ((it)+(it)^{-1}) - (it)^{-1} \right] \right\} \\
&\quad = [(1-p)p]^{-1/2} e^{-t^2/2} \left\{ \frac{1}{2} \left[(it)^3 + (it) \right] - b_5(1-p) \left[(it)^3 + (it) \right] \right\} \\
&\quad = [(1-p)p]^{-1/2}[(1/2) - b_5(1-p)](it) \left[(it)^2 + 1 \right] e^{-t^2/2}. \qquad (74)
\end{aligned}
$$

Now for the terms involving Ψ,

$$
\begin{aligned}
&\sum_{\ell=1}^{\infty} \frac{(it)^\ell}{\ell!} (-1)^\ell \left(\frac{p}{1-p} \right)^{\ell/2} n^{-1/2} \ell \left[-(\ell-1)(2p)^{-1} \Psi \sqrt{n} E(D_3^{\ell-2}) \right] \\
&\qquad = [-n^{-1/2} \Psi \sqrt{n} (2p)^{-1}] \sum_{\ell=1}^{\infty} \frac{(it)^\ell}{\ell!} (-1)^\ell \left(\frac{p}{1-p} \right)^{\ell/2} \ell(\ell-1) E(D_3^{\ell-2})
\end{aligned}
$$

$$= [A]\sum_{k=1}^{\infty}\frac{(it)^{2k}}{(2k)!}(-1)^{2k}\left(\frac{p}{1-p}\right)^{k}(2k)(2k-1)\cdot\left(\frac{1-p}{p}\right)^{k-1}\frac{(2k-2)!}{(k-1)!2^{k-1}}$$

$$= [A]\frac{p}{1-p}\sum_{k=1}^{\infty}(it)^{2k}\frac{(2k-2)!(2k)(2k-1)}{(2k)!}\frac{1}{(k-1)!2^{k-1}}$$

$$= [A]\frac{p}{1-p}\sum_{k=1}^{\infty}\frac{(it)^{2k}}{(k-1)!2^{k-1}}$$

$$= -n^{-1/2}\Psi\sqrt{n}(2p)^{-1}\frac{p}{1-p}(it)^2e^{-t^2/2}$$

$$= -n^{-1/2}\Psi\sqrt{n}\frac{1}{2(1-p)}(it)^2e^{-t^2/2}, \tag{75}$$

using (70) for the second-to-last line.

Similarly to the derivation of (75),

$$\sum_{\ell=1}^{\infty}\frac{(it)^{\ell}}{\ell!}(-1)^{\ell}\left(\frac{p}{1-p}\right)^{\ell/2}m^{-1}\frac{1}{4}\ell(\ell-1)\Psi^2nE(D_3^{\ell-2})$$

$$= m^{-1}\frac{\Psi^2n}{4}\sum_{\ell=1}^{\infty}\frac{(it)^{\ell}}{\ell!}(-1)^{\ell}\left(\frac{p}{1-p}\right)^{\ell/2}\ell(\ell-1)E(D_3^{\ell-2})$$

$$= m^{-1}\frac{1}{4}\Psi^2n\frac{p}{1-p}(it)^2e^{-t^2/2}. \tag{76}$$

For the final $n^{-1/2}$ term,

$$\sum_{\ell=1}^{\infty}\frac{(it)^{\ell}}{\ell!}(-1)^{\ell}\left(\frac{p}{1-p}\right)^{\ell/2}n^{-1/2}\ell 2b_5\Psi\sqrt{n}E(D_3^{\ell})$$

$$= [n^{-1/2}2b_5\Psi\sqrt{n}]\sum_{\ell=1}^{\infty}\frac{(it)^{\ell}}{\ell!}(-1)^{\ell}\left(\frac{p}{1-p}\right)^{\ell/2}\ell E(D_3^{\ell})$$

$$= [A]\sum_{k=1}^{\infty}\frac{(it)^{2k}}{(2k)!}\left(\frac{p}{1-p}\right)^{k}(2k)\left(\frac{1-p}{p}\right)^{k}\frac{(2k)!}{k!2^k}$$

$$= [A]\sum_{k=1}^{\infty}\frac{(it)^{2k}}{(k-1)!2^{k-1}}$$

$$= n^{-1/2}2b_5\Psi\sqrt{n}(it)^2e^{-t^2/2}, \tag{77}$$

again using (70) for the final line.

For the final m^{-1} term, similar to (74),

$$\sum_{\ell=1}^{\infty}\frac{(it)^{\ell}}{\ell!}(-1)^{\ell}\left(\frac{p}{1-p}\right)^{\ell/2}m^{-1}\frac{1}{2}\ell^2\Psi\sqrt{n}E(D_3^{\ell-1})$$

$$= -(1/2)m^{-1}\Psi\sqrt{n}\sqrt{p/(1-p)}[(it)^3+(it)]e^{-t^2/2}.$$

For the δ term, i.e. the $(m/n)^2$ term,

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \frac{(it)^\ell}{\ell!}(-1)^\ell \left(\frac{p}{1-p}\right)^{\ell/2} \delta\ell\Psi\sqrt{n}E(D_3^{\ell-1}) \\
&\qquad = \delta\Psi\sqrt{n}\sum_{k=0}^{\infty} \frac{(it)^{2k+1}}{(2k+1)!}(-1)\left(\frac{p}{1-p}\right)^{k+\frac{1}{2}}(2k+1)\left(\frac{1-p}{p}\right)^k \frac{(2k)!}{k!2^k} \\
&\qquad = -\delta\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}\sum_{k=0}^{\infty}\frac{(it)^{2k+1}}{k!2^k} \\
&\qquad = -\delta\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}(it)e^{-t^2/2}, \qquad (78)
\end{aligned}$$

using (70) for the last line.

Summing the $n^{-1/2}$ terms yields $\alpha_1(t)$ as in (68), and similarly for the m^{-1} terms to get $\alpha_2(t)$ and δ term to get $\alpha_3(t)$.

The three functions $\alpha_1(t)$, $\alpha_2(t)$, and $\alpha_3(t)$ in (68) are Fourier–Stieltjes transforms of

$$a_1(z) \equiv -\left[\Psi\sqrt{n}\left(2b_5 - \frac{1}{2(p-1)}\right)z + [(1/2) - b_5(1-p)][p(1-p)]^{-1/2}z^2\right]\phi(z), \qquad (79)$$

$$a_2(z) \equiv \frac{1}{4}\left[-\Psi^2 n\frac{p}{1-p}z + 2\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}z^2 - z^3\right]\phi(z), \text{ and} \qquad (80)$$

$$a_3(z) \equiv \left[\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}\right]\phi(z), \qquad (81)$$

respectively. Note that when $\gamma = 0$ and thus $\Psi = 0$, these reduce to the functions found in HS88.

Define $e^{-t^2/2}(= \int_{-\infty}^{\infty}\phi(x)e^{itx}dx)$ as the Fourier–Stieltjes transform of the distribution $\Phi(z)$. Note that the derivatives of $\phi(z)$ are:

$$\begin{aligned}
\phi(z) &= (2\pi)^{-1/2}e^{-z^2/2} \\
\phi'(z) &= -z(2\pi)^{-1/2}e^{-z^2/2} = -z\phi(z) \\
\phi''(z) &= -\phi(z) - z\phi'(z) = -\phi(z) + z^2\phi(z) = \phi(z)(-1+z^2) \\
\phi'''(z) &= -\phi'(z) + 2z\phi(z) + z^2\phi'(z) = z\phi(z) + 2z\phi(z) - z^3\phi(z) = \phi(z)(3z - z^3) \\
\phi''''(z) &= \phi'(z)(3z-z^3) + \phi(z)(3-3z^2) = \phi(z)(3 - 3z^2 - z(3z - z^3)) \\
&= \phi(z)(3 - 6z^2 + z^4)
\end{aligned}$$

Also,

$$\int_{-\infty}^{\infty}\phi'(x)e^{itx}dx = \int_{-\infty}^{\infty}e^{itx}\cdot\phi'(x)dx$$

$$= \int_{-\infty}^{\infty} u \cdot dv = uv|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \cdot du$$
$$= e^{itx}\phi(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi(x)(it)e^{itx}dx = 0 - (it)e^{-t^2/2},$$

and it can be shown similarly that $\int_{-\infty}^{\infty} \phi^{(k)}(x)e^{itx}dx = (-1)^k(it)^k e^{-t^2/2}$.

For simplicity of notation, write

$$\alpha_1(t) = e^{-t^2/2}\left[\left((it)^3 + (it)\right)C_1 + (it)^2 C_2\right]$$
$$\alpha_2(t) = \frac{1}{4}e^{-t^2/2}\left[(it)^4 + (it)^2 C_3 - \left((it)^3 + (it)\right)C_4\right]$$
$$\alpha_3(t) = e^{-t^2/2}(it)(-C_5).$$

Since $\phi(z) \to -(it)e^{-t^2/2}$, $C_5\phi(z) \to \alpha_3(t)$, and thus

$$a_3(z) = C_5\phi(z) = \left[\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}\right]\phi(z).$$

From HS88, the inverse Fourier–Stieltjes transform of the terms in $\alpha_1(t)$ without Ψ is

$$-[(1/2) - b_5(1-p)][p(1-p)]^{-1/2}z^2\phi(z).$$

Since the inverse Fourier–Stieltjes transform is linear, the new terms can just be added. Since $\phi'(z) \to (it)^2 e^{-t^2/2}$, then $C_2\phi'(z) = -C_2 z\phi(z) \to e^{-t^2/2}(it)^2 C_2$. Adding together,

$$a_1(z) = -C_1 z^2\phi(z) - C_2 z\phi(z)$$
$$= -\left[\Psi\sqrt{n}\left(2b_5 - \frac{1}{2(p-1)}\right)z + [(1/2) - b_5(1-p)][p(1-p)]^{-1/2}z^2\right]\phi(z).$$

Regarding $a_2(z)$, note that

$$\phi'''(z) \to (it)^4 e^{-t^2/2},$$
$$C_3\phi'(z) \to C_3(it)^2 e^{-t^2/2}, \text{ and}$$
$$C_4[\phi''(z) + \phi(z)] \to -C_4 e^{-t^2/2}\left((it)^3 + (it)\right).$$

Thus,

$$a_2(z) = \frac{1}{4}[\phi'''(z) + C_3\phi'(z) + C_4[\phi''(z) + \phi(z)]]$$
$$= \frac{1}{4}\left[\phi(z)(3z - z^3) - C_3 z\phi(z) + C_4[\phi(z)(-1 + z^2) + \phi(z)]\right]$$
$$= \frac{1}{4}\phi(z)\left[3z - z^3 - C_3 z + C_4 z^2\right]$$
$$= \frac{1}{4}\phi(z)\left[(3 - C_3)z + C_4 z^2 - z^3\right]$$

$$= \frac{1}{4}\phi(z)\left[-\Psi^2 n\frac{p}{1-p}z + 2\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2} z^2 - z^3\right].$$

The characteristic function of L is

$$E(e^{itL}) = \{1 + \delta(it)^2 - n^{-1/2}\frac{1}{6}[p/(1-p)]^{1/2}[(1+p)/p](it)^3\}e^{-t^2/2} + O(\delta^2 + n^{-1}), \tag{82}$$

where the right-hand side is the Fourier–Stieltjes transform of

$$\Phi(z) - \delta z\phi(z) + n^{-1/2}\frac{1}{6}[p/(1-p)]^{1/2}[(1+p)/p](z^2-1)\phi(z) + O(\delta^2 + n^{-1}). \tag{83}$$

To explicitly show the characteristic function of L, I take an expansion of the cumulant generating function, and use the known mean, variance, and third moment to plug in, calculating the third cumulant from the third moment. The fourth cumulant is also needed to show the bound on the continuation of the series expansion. (Higher cumulants are not needed since the mean value theorem can be used to express the rest of the expansion only in terms of the fourth cumulant, as with $f(x+a) = f(x) + af'(x^*)$ for some $x^* \in [x, x+a]$.) Note the recursive formula and that the nth (non-central) moment μ'_n is an nth-degree polynomial in the first n cumulants, $\{\kappa_i\}_1^n$:

$$\begin{aligned}
\kappa_n &= \mu'_n - \sum_{m=1}^{n-1}\binom{n-1}{m-1}\kappa_m\mu'_{n-m}, \text{ so} \\
\kappa_1 &= \mu'_1 \\
\kappa_2 &= \mu'_2 - \binom{1}{0}\kappa_1\mu'_1 = \mu'_2 - (\mu'_1)^2 \\
\kappa_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 \\
\kappa_4 &= \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6(\mu'_1)^4
\end{aligned} \tag{84}$$

$$\begin{aligned}
\mu'_1 &= \kappa_1 \\
\mu'_2 &= \kappa_2 + \kappa_1^2 \\
\mu'_3 &= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3 \\
\mu'_4 &= \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4 \\
\mu'_5 &= \kappa_5 + 5\kappa_4\kappa_1 + 10\kappa_3\kappa_2 + 10\kappa_3\kappa_1^2 + 15\kappa_2^2\kappa_1 + 10\kappa_2\kappa_1^3 + \kappa_1^5.
\end{aligned}$$

$$\begin{aligned}
L &\equiv -[p/(1-p)]^{1/2}(1+\delta)(D_2 + D_3) \quad \text{from (66)}, \\
E(L) &= \mu'_1 = 0 \text{ since } E(D_2) = E(D_3) = E(V_j) = 0, \\
\delta &\equiv -(m/n)^2 g''(p)/(6g(p)) = O((m/n)^2) \quad \text{from (15)}.
\end{aligned} \tag{85}$$

$$E[(\Delta_2+\Delta_3)^2] = E[(\sum_{j=r}^{n} V_j/j)^2] = E[V_j^2]E[\sum_{j=r}^{n} j^{-2}] = \sum_{j=r}^{n} j^{-2},$$

$$\sum_{r}^{n} 1/j^2 = \sum_{0}^{n-r}(r+i)^{-2} = \int_0^{n-r}(r+x)^{-2}dx + (1/2)(n^{-2}+r^{-2}) + O(f^{(1)}(0))$$

$$= -(r+x)^{-1}|_0^{n-r} + O(n^{-2}) + O(n^{-3}) = r^{-1} - n^{-1} + O(n^{-2})$$

$$= 1/n((1/p)-1) + O(n^{-2}) = (1-p)/(np) + O(n^{-2}), \tag{86}$$

and so the second moment is

$$E(L^2) = \mu_2' = (p/(1-p))(1+\delta)^2 E[(D_2+D_3)^2]$$

$$= (p/(1-p))(1+2\delta+\delta^2)nE[(\Delta_2+\Delta_3)^2]$$

$$= (p/(1-p))(1+2\delta+\delta^2)n[(1-p)/(np) + O(n^{-2})]$$

$$= (p/(1-p))(1+2\delta+\delta^2)[(1-p)/p + O(n^{-1})]$$

$$= (p/(1-p))(1+2\delta)[(1-p)/p] + O(n^{-1}) + O(\delta^2)$$

$$= (1+2\delta) + O(n^{-1}+\delta^2). \tag{87}$$

For the third moment,

$$E[(\Delta_2+\Delta_3)^3] = E[(\sum_{j=r}^{n} V_j/j)^3] = E[V_j^3]E[\sum_{j=r}^{n} j^{-3}] = 2\sum_{j=r}^{n} j^{-3},$$

$$\sum_{r}^{n} 1/j^3 = \sum_{0}^{n-r}(r+i)^{-3} = \int_0^{n-r}(r+x)^{-3}dx + (1/2)(n^{-3}+r^{-3}) + O(f^{(1)}(0))$$

$$= -(1/2)(r+x)^{-2}|_0^{n-r} + O(n^{-3}) + O(n^{-4}) = (1/2)r^{-2} - 2n^{-2} + O(n^{-3})$$

$$= (1/2)\frac{1}{n^2}((1/p^2)-1) + O(n^{-3}) = (1/2)\frac{1}{n^2}(1-p^2)/p^2 + O(n^{-3})$$

$$= (1/2)\frac{1}{n^2}(1-p)(1+p)/p^2 + O(n^{-3}),$$

$$E[(D_2+D_3)^3] = 2n^{3/2}[(1/2)\frac{1}{n^2}(1-p)(1+p)/p^2 + O(n^{-3})]$$

$$= n^{-1/2}(1-p)(1+p)/p^2 + O(n^{-3/2}), \tag{88}$$

and so

$$E(L^3) = \mu_3' = -(p/(1-p))^{3/2}(1+\delta)^3 E[(D_2+D_3)^3]$$

$$= -(p/(1-p))^{3/2}(1+O(\delta))[n^{-1/2}(1-p)(1+p)/p^2 + O(n^{-3/2})]$$

$$= -p^{3/2}(1-p)^{-3/2}n^{-1/2}(1-p)(1+p)/p^2 + O(\delta n^{-1/2} + n^{-3/2})$$

$$= -(1+p)p^{3/2}/p^2(1-p)(1-p)^{-3/2}n^{-1/2} + O(\delta^2 + n^{-1} + n^{-3/2})$$

$$= -n^{-1/2}(1+p)p^{-1/2}(1-p)^{-1/2} + O(\delta^2 + n^{-1}). \tag{89}$$

For the fourth moment,

$$E[(\Delta_2+\Delta_3)^4] = E\left[\left(\sum_{j=r}^{n} V_j/j\right)^4\right] = E\left[\sum_{j=r}^{n} V_j^4/j^4 + B\right],$$

$$B \equiv \sum_{i=r}^{n}\sum_{j=i+1}^{n}\binom{4}{2}\frac{V_i^2}{i^2}\frac{V_j^2}{j^2} = 6\sum_{i=r}^{n}\sum_{j=i+1}^{n}\frac{V_i^2V_j^2}{i^2j^2},$$

$$\begin{aligned}
E(B) &= 6\sum_{i=r}^{n}\sum_{j=i+1}^{n}\frac{E(V_i^2)E(V_j^2)}{i^2j^2} = 6\sum_{i=r}^{n}\sum_{j=i+1}^{n}\frac{1}{i^2j^2}\\
&= 6\int_r^n\int_x^n \frac{1}{y^2x^2}dydx + O(n^{-5})\\
&= 6\int_r^n\left(-\frac{1}{x^2y}|_x^n\right)dx = 6\int_r^n(\frac{1}{x^3}-\frac{1}{x^2n})dx\\
&= 6\{-(1/2)x^{-2}|_r^n - (1/n)(-1/x)|_r^n\}\\
&= 6[(1/2)r^{-2} - (1/2)n^{-2} - (1/n)(1/r) + (1/n)(1/n)]\\
&= 6\left[(1/n)(1/n^2)\left(\frac{1}{p^2}-1\right) - (1/n^2)\left(\frac{1}{p}-1\right)\right]\\
&= 6\left[(1/n^2)\left(\frac{1-p^2}{2p^2}-\frac{1-p}{p}\right)\right]\\
&= \frac{6}{n^2}\frac{1-p^2-2p(1-p)}{2p^2} = \frac{6}{n^2}\frac{1-2p+p^2}{2p^2} = 3n^{-2}\left(\frac{1-p}{p}\right)^2,
\end{aligned}$$

$$\begin{aligned}
\sum_r^n 1/j^4 &= \sum_0^{n-r}(r+i)^{-4} = \int_0^{n-r}(r+x)^{-4}dx + (1/2)(n^{-4}+r^{-4}) + O(f^{(1)}(0))\\
&= -(1/3)(r+x)^{-3}|_0^{n-r} + O(n^{-4}) + O(n^{-5})\\
&= (1/3)r^{-3} - 3n^{-3} + O(n^{-4})\\
&= (1/3)\frac{1}{n^3}((1/p^3)-1) + O(n^{-4})\\
&= (1/3)\frac{1}{n^3}(1-p^3)/p^3 + O(n^{-4}) = O(n^{-3}),
\end{aligned}$$

$$E[(D_2+D_3)^4] = n^2[O(n^{-3}) + \left[3n^{-2}\left(\frac{1-p}{p}\right)^2 + O(n^{-5})\right]$$

$$= 3\left(\frac{1-p}{p}\right)^2 + O(n^{-1}), \tag{90}$$

and so

$$\begin{aligned}
E(L^4) &= \mu_4' = (p/(1-p))^2(1+\delta)^4 E[(D_2+D_3)^4] \\
&= (p/(1-p))^2(1+4\delta+O(\delta^2))\left[3\left(\frac{1-p}{p}\right)^2 + O(n^{-1})\right] \\
&= (1+4\delta)(3) + O(\delta^2+n^{-1}) \qquad (91)
\end{aligned}$$

From above in (85), (87), and (89) were the results $E(L)=0, E(L^2)=1+2\delta+O(\delta^2+n^{-1}), E(L^3)=-n^{-1/2}(1+p)p^{-1/2}(1-p)^{-1/2}+O(\delta^2+n^{-1})$. Note, then, that $O((\mu_2'-1)^2)=O(\delta^2)$. Expanding the log of the characteristic function of L, which is the cumulant generating function,

$$\begin{aligned}
\ln E(e^{itL}) &= \sum_{j=1}^{\infty}\frac{(it)^j}{j!}\kappa_j, \quad \kappa_j \equiv \text{cumulants} \\
&= (it)\kappa_1 - (t^2/2)\kappa_2 + \frac{(it)^3}{6}\kappa_3 + \frac{(it)^4}{4!}\kappa_4 + \dots \\
&= (it)\mu_1' - (t^2/2)(\mu_2'-(\mu_1')^2) + \frac{(it)^3}{6}(\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3) \\
&\quad + \frac{(it)^4}{4!}(\mu_4' - 4\mu_3'\mu_1' - 3(\mu_2')^2 + 12\mu_2'(\mu_1')^2 - 6(\mu_1')^4) + \dots \\
&= 0 - (t^2/2)\mu_2' + \frac{(it)^3}{6}(\mu_3') + \frac{(it)^4}{4!}(\mu_4' - 3(\mu_2')^2) + \dots, \text{ so} \\
E(e^{itL}) &= \exp\{-(t^2/2)(1+(\mu_2'-1)) + \frac{(it)^3}{6}(\mu_3') + \frac{(it)^4}{4!}(\mu_4'-3(\mu_2')^2) + \dots\} \\
&= \exp\{-(t^2/2)\} \\
&\quad \times \exp\{((it)^2/2)(\mu_2'-1) + \frac{(it)^3}{6}(\mu_3') + \frac{(it)^4}{4!}(\mu_4'-3(\mu_2')^2) + \dots\} \\
&= e^{-t^2/2}\left[1 + ((it)^2/2)(\mu_2'-1) + \frac{(it)^3}{6}(\mu_3') + \frac{(it)^4}{4!}(\mu_4'-3(\mu_2')^2) + \dots\right] \\
&\quad + O((\mu_2'-1)^2) \quad (\text{since } e^x = e^0 + xe^0 + O(x^2)) \\
&= e^{-t^2/2}\Big[1 + ((it)^2/2)(2\delta + O(\delta^2+n^{-1})) \\
&\qquad + \frac{(it)^3}{6}(n^{-1/2}(1+p)p^{-1/2}(1-p)^{-1/2} + O(\delta^2+n^{-1})) \\
&\qquad + O\Big\{\frac{(it)^4}{4!}([(1+4\delta)(3)+O(\delta^2+n^{-1})] \\
&\qquad\quad - 3(1+2\delta+O(\delta^2+n^{-1}))^2)\Big\}\Big] + O(\delta^2) \\
&= e^{-t^2/2}\Big[1 + ((it)^2/2)(2\delta) + \frac{(it)^3}{6}(-n^{-1/2}(1+p)p^{-1/2}(1-p)^{-1/2})
\end{aligned}$$

$$
\begin{aligned}
&\qquad + O\Big\{3(1+4\delta) - 3(1+4\delta+O(\delta^2+n^{-1})) + \delta^2 + n^{-1}\Big\}\Big] \\
&\quad + O(n^{-1}+\delta^2) \\
&= e^{-t^2/2}\Big[1 + \delta(it)^2 - n^{-1/2}\frac{1}{6}(p^{1/2}(1-p)^{-1/2}p^{-1}(1+p)(it)^3) + O\{\delta^2+n^{-1}\}\Big] \\
&\quad + O(n^{-1}+\delta^2), \qquad (92)
\end{aligned}
$$

verifying (82).

When simplifying terms, it can easily be verified that the Fourier–Stieltjes transform of the expression in (83),

$$
\begin{aligned}
\Phi(z) - \delta z\phi(z) + n^{-1/2}(1/6)(p/(1-p))^{1/2}((1+p)/p)(z^2-1)\phi(z) \\
= \Phi(z) - \delta z\phi(z) + C(z^2-1)\phi(z) \\
= \Phi(z) + \delta\phi'(z) + C\phi''(z),
\end{aligned}
$$

is $e^{-t^2/2} + \delta(-1)^2(it)^2e^{-t^2/2} + C(-1)^3(it)^3e^{-t^2/2} = e^{-t^2/2}(1+\delta(it)^2 - C(it)^3)$, matching (82). I trust from the proof in HS88 that the conditions are such that the remainder will be the same in both equations.

Combining results going back to (68) and noting that $b_5 = -(a_2/2a_1) = \frac{1}{2} + pg'(p)[2g(p)]^{-1} + O(n^{-1})$, the distribution function of K admits the Edgeworth expansion on the RHS of (10),

(93)
$$
P(K \le z) = \Phi(z) + \{n^{-1/2}u^{\dagger}_{1,\gamma}(z) + m^{-1}u_{2,\gamma}(z) + (m/n)^2u_{3,\gamma}(z)\}\phi(z) + o\{m^{-1} + (m/n)^2\},
$$

where

$$
\begin{aligned}
u^{\dagger}_{1,\gamma}(z) &\equiv \frac{1}{6}\left(\frac{p}{1-p}\right)^{1/2}\frac{1+p}{p}(z^2-1) - \frac{\gamma f(\xi_p)}{p}\left(1 - \frac{pf'(\xi_p)}{[f(\xi_p)]^2} - \frac{1}{2(1-p)}\right)z \\
&\quad - \frac{1}{2}\left(\frac{p}{1-p}\right)^{1/2}\left(1 + \frac{f'(\xi_p)}{[f(\xi_p)]^2}(1-p)\right)z^2, \\
u_{2,\gamma}(z) &\equiv \frac{1}{4}\left[\frac{2\gamma f(\xi_p)}{[p(1-p)]^{1/2}}z^2 - \frac{\gamma^2[f(\xi_p)]^2}{p(1-p)}z - z^3\right], \text{ and} \\
u_{3,\gamma}(z) &\equiv \frac{3[f'(\xi_p)]^2 - f(\xi_p)f''(\xi_p)}{6[f(\xi_p)]^4}\left(z - \frac{\gamma f(\xi_p)}{[p(1-p)]^{1/2}}\right),
\end{aligned}
$$

where the RHS of (93) is identical to the expansion indicated by (10).

Note that this is for η_p, not ξ_p (hence $u^{\dagger}_{1,\gamma}$ instead of $u_{1,\gamma}$), but when combined with equation (9), implies the final result in Theorem 2 in the paper.

To show the final result, the inverse Fourier–Stieltjes transformed functions can be added to get the final answers since it is a linear transform (and consequently linear inverse).

Starting at (68) , adding the inverse Fourier–Stieltjes transformed functions of the RHS leads to the distribution (cdf) of K:

$$\begin{aligned}
P(K \le z) &= \Phi(z) - \delta z\phi(z) + n^{-1/2}\frac{1}{6}\{p/(1-p)\}^{1/2}\{(1+p)/p\}(z^2-1)\phi(z) \\
&\quad + O(\delta^2 + n^{-1}) + n^{-1/2}a_1(z) + m^{-1}a_2(z) + \delta a_3(z) + o\{m^{-1} + (m/n)^2\} \\
&= \Phi(z) + \delta\phi(z)\left(\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2} - z\right) \\
&\quad + n^{-1/2}\phi(z)\Bigg[\frac{1}{6}\left(\frac{p}{1-p}\right)^{1/2}\frac{1+p}{p}(z^2-1) - \Psi\sqrt{n}\left(2b_5 - \frac{1}{2(1-p)}\right)z \\
&\qquad\qquad - ((1/2) - b_5(1-p))(p(1-p))^{-1/2}z^2\Bigg] \\
&\quad + m^{-1}\phi(z)\frac{1}{4}\left[-\Psi^2 n\frac{p}{1-p}z + 2\Psi\sqrt{n}\left(\frac{p}{1-p}\right)^{1/2}z^2 - z^3\right],
\end{aligned}$$

and using $\delta \equiv -(m/n)^2\frac{g''(p)}{6g(p)}$ and $\Psi = \gamma/(pg(p)\sqrt{n})$,

$$\begin{aligned}
&\quad + o\{m^{-1} + (m/n)^2\} \\
&= \Phi(z) + (m/n)^2\frac{g''(p)}{6g(p)}\phi(z)\left(z - \frac{\gamma}{g(p)[p(1-p)]^{1/2}}\right) \\
&\quad + n^{-1/2}\phi(z)\Bigg[\frac{1}{6}\left(\frac{p}{1-p}\right)^{1/2}\frac{1+p}{p}(z^2-1) \\
&\qquad\qquad - \frac{\gamma}{pg(p)}\left(1 + \frac{pg'(p)}{g(p)} - \frac{1}{2(1-p)}\right)z \\
&\qquad\qquad - \frac{1}{2}\left(\frac{p}{1-p}\right)^{1/2}\left(1 - \frac{g'(p)}{g(p)}(1-p)\right)z^2\Bigg] \\
&\quad + m^{-1}\phi(z)\frac{1}{4}\left[-\frac{\gamma^2}{[g(p)]^2p(1-p)}z + \frac{2\gamma}{g(p)[p(1-p)]^{1/2}}z^2 - z^3\right] \\
&\quad + o\{m^{-1} + (m/n)^2\},
\end{aligned}$$

and using $g(p) = 1/f(F^{-1}(p))$, $g'(p) = -f'(F^{-1}(p))/[f(F^{-1}(p))]^3$, and $g''(p) = [-f''(\xi_p)f(\xi_p) + 3[f'(\xi_p)]^2]/[f(\xi_p)]^5$ from (43), (51), and (52), and thus $g'(p)/g(p) = -f'(\xi_p)/[f(\xi_p)]^2$,

$$= \Phi(z) + (m/n)^2\phi(z)\frac{3[f'(\xi_p)]^2 - f(\xi_p)f''(\xi_p)}{6[f(\xi_p)]^4}\left(z - \frac{\gamma f(\xi_p)}{[p(1-p)]^{1/2}}\right)$$

$$
\begin{aligned}
&+ n^{-1/2}\phi(z)\Bigg[\frac{1}{6}\left(\frac{p}{1-p}\right)^{1/2}\frac{1+p}{p}(z^2-1) \\
&\qquad\qquad - \frac{\gamma f(\xi_p)}{p}\left(1 - \frac{pf'(\xi_p)}{[f(\xi_p)]^2} - \frac{1}{2(1-p)}\right) z \\
&\qquad\qquad - \frac{1}{2}\left(\frac{p}{1-p}\right)^{1/2}\left(1 + \frac{f'(\xi_p)}{[f(\xi_p)]^2}(1-p)\right) z^2\Bigg] \\
&+ m^{-1}\phi(z)\frac{1}{4}\left[\frac{2\gamma f(\xi_p)}{[p(1-p)]^{1/2}}z^2 - \frac{\gamma^2[f(\xi_p)]^2}{p(1-p)}z - z^3\right] \\
&+ o\{m^{-1} + (m/n)^2\} \\
= {}& \Phi(z) + n^{-1/2}u^{\dagger}_{1,\gamma}(z)\phi(z) + (m/n)^2 u_{3,\gamma}(z)\phi(z) + m^{-1}u_{2,\gamma}(z)\phi(z) \\
&+ o\{m^{-1} + (m/n)^2\}
\end{aligned}
$$

as in (93).

References

Bloch, D. A. and Gastwirth, J. L. (1968). On a simple estimate of the reciprocal of the density function. *The Annals of Mathematical Statistics*, 39(3):1083–1085.

Goh, S. C. (2004). *Smoothing choices and distributional approximations for econometric inference*. PhD thesis, UC Berkeley.

Hall, P. and Sheather, S. J. (1987). On the distribution of a studentized quantile. Technical Report 71, Department of Statistics, Pennsylvania State University.

Hall, P. and Sheather, S. J. (1988). On the distribution of a Studentized quantile. *Journal of the Royal Statistical Society. Series B (Methodological)*, 50(3):381–391.

Department of Economics, University of Missouri–Columbia