LATTICE POINTS INSIDE RANDOM ELLIPSOIDS

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Abstract. Let \( N_a(t) = \# \{ t\Omega_a \cap \mathbb{Z}^d \} \), with \( \Omega_a = \left\{ (a_1^{-\frac{1}{2}}x_1, a_2^{-\frac{1}{2}}x_2, \ldots, a_d^{-\frac{1}{2}}x_d) : x \in \Omega \right\} \), where \( \Omega \) is the unit ball. Let \( E_a(t) = N_a(t) - t^d|\Omega_a| \). We give a simple proof of the fact that

\[
\left( \int_{\frac{1}{2}}^{2} \cdots \int_{\frac{1}{2}}^{2} |E_a(t)|^2 da_1 da_2 \cdots da_d \right)^{\frac{1}{2}} \lesssim t^{d-\frac{1}{2}}
\]

in 2 and 3 dimensions.

Introduction

Let

\[
N_a(t) = \# \{ t\Omega_a \cap \mathbb{Z}^d \},
\]

where

\[
\Omega_a = \left\{ (a_1^{-\frac{1}{2}}x_1, a_2^{-\frac{1}{2}}x_2, \ldots, a_d^{-\frac{1}{2}}x_d) : x \in \Omega \right\},
\]

with \( \frac{1}{2} \leq a_j \leq 2 \), where \( \Omega \) is the unit ball.

Let

\[
N_a(t) = t^d|\Omega_a| + E_a(t).
\]

A classical result due to Landau says that

\[
|E_a(t)| \lesssim t^{d-2+\frac{2}{d+1}},
\]

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where here and throughout the paper, \( A \lesssim B \) means that there exists a positive constant \( C \) such that \( A \leq C B \). Similarly, \( A \lesssim_{t} B, \) with a parameter \( t, \) means that given \( \delta > 0 \) there exists \( C_{\delta} > 0 \) such that \( A \leq C_{\delta} t^{2} B. \)

A number of improvements over (0.4) have been obtained over the years in two and three dimensions. The best known result in three dimensions, to the best of our knowledge is

\[
|E_{a}(t)| \lesssim t^{\frac{d}{2}}
\]

proved by Heath-Brown ([H-B97]), improving on an earlier breakthrough due to Vinogradov ([Vinograd63]). It is proved by Szego in [Szego26] that

\[
(0.5) \quad \left| E_{1,1,1}(t) - \frac{4\pi}{3} t^{3} \right| \gtrsim t \log(t).
\]

In two dimensions, the best known result is

\[
|E_{a}(t)| \lesssim t^{\frac{d}{2}}
\]

due to Huxley ([Huxley96]). A classical result due to Hardy says that

\[
|E_{1,1}(t) - \pi t^{2}| \gtrsim t^{\frac{d}{2}} \log^{\frac{1}{2}}(t).
\]

Thus it is reasonable to conjecture that the estimate

\[
(0.7) \quad |E_{a}(t)| \lesssim t^{\frac{d-1}{2}}
\]

holds in \( \mathbb{R}^{2} \) and \( \mathbb{R}^{3}. \)

In higher dimensions, the problem of point-wise estimate of \( E_{a}(t) \) is completely solved. It is a result of Walfisch that if \( d \geq 4, \) then \( |E_{a}(t)| \lesssim t^{d-2}, \) and logarithm may be removed in dimension 5 and greater. It is also known that if the eccentricities \( (a_{1}, \ldots, a_{d}) \) are rational, then this estimate is essentially sharp.

It is not known if there exists a single \( a = (a_{1}, a_{2}, \ldots, a_{d}) \) such that \( |E_{a}(t)| \lesssim t^{\frac{d-1}{2}} \) in any dimension. The question of finding such an \( a \) was posed by Sarnak in a two-dimensional setting a number of years ago. Sarnak’s question would be answered by the following estimate.

**Conjecture 0.2.** Given any \( \delta > 0, \)

\[
(0.8) \quad \sup_{t \geq 1} t^{\frac{d-1}{2} - \delta} |E_{\langle \rangle}(t)| \in L^{p}\left( \left[ \frac{1}{2}, 2 \right] \times \left[ \frac{1}{2}, 2 \right] \times \cdots \times \left[ \frac{1}{2}, 2 \right] \right),
\]

for some \( p \geq 1 \) with a constant depending on \( \delta. \)

In fact, (0.8) would, of course, imply that the estimate \( |E_{a}(t)| \lesssim t^{\frac{d-1}{2}} \) holds for almost every \( a \in \left( \left[ \frac{1}{2}, 2 \right] \times \left[ \frac{1}{2}, 2 \right] \times \cdots \times \left[ \frac{1}{2}, 2 \right] \right). \) We hope to address this issue in a subsequent paper.

Other types of square averages of lattice point discrepancy functions have been studied in the past and in recent years. For example, a classical result due to Kendall says that

\[
(0.9) \quad \int_{T^{2}} \# \{(t\Omega + \tau) \cap \mathbb{Z}^{d} \} - t^{d}|\Omega|^{2} \, d\tau \lesssim t^{\frac{d-1}{2}},
\]
for every convex domain where the boundary has everywhere non-vanishing Gaussian curvature.

This result was recently sharpened up by Magyar and Seeger. They proved that the estimate (0.9) still holds in \( \mathbb{R}^d \) if the exponent 2 is replaced by \( p \leq \frac{2d}{d-1} \).

Another type of average is studied in [ISS02]. The authors prove that

\[
(0.10) \quad \left( \frac{1}{h} \int_{R}^{R+h} \left| \# \{ t \Omega \cap \mathbb{Z}^d \} - t^d \Omega \right|^2 dt \right)^{\frac{1}{2}} \lesssim R^{\alpha_d},
\]

where

\[
(0.11) \quad \alpha_2 = \frac{1}{2}, \text{ with } h \geq \log(R),
\]

and

\[
(0.12) \quad \alpha_d = d - 2, \text{ with } h \approx R,
\]

for \( d \geq 4 \). When \( d = 3 \), \( \alpha_d = 1 \) and an additional factor \( \log(R) \) is present. These results improve results previously obtained by Muller ([Muller97]). See also [Huxley96] and [ISS02] and references contained therein.

Using (0.10), (0.11), and (0.12) and their proofs one can deduce the following result.

**Theorem 0.1.** Let \( E_a(t) \) be as above. Then

\[
(0.13) \quad \int_{\frac{1}{2}}^{2} \int_{\frac{1}{2}}^{2} \cdots \int_{\frac{1}{2}}^{2} |E_a(t)|^2 dt \lesssim R^{\alpha_d},
\]

where \( \alpha_d \) is exactly as above, and the additional \( \log(t) \) factor is still present in three dimensions.

The purpose of this paper is to give a simple and transparent proof of Theorem 0.1 in two and three dimensions. Similar two-dimensional results have recently been obtained by different methods by Toth and Petridis in [TothPetridis02]. We believe that it is likely that our approach will lead to a better estimate in higher dimensions where we conjecture that (0.13) holds with \( \alpha_d = \frac{d-1}{2} \). We hope to address this issue in a subsequent paper.

**SECTION I: PROOF OF THEOREM 0.1 IN \( \mathbb{R}^2 \) AND \( \mathbb{R}^3 \)**

We shall give a proof in three dimensions. We shall then indicate how a two-dimensional proof follows from a simpler version of the same argument.
We start with the following standard reduction. Let ρ₀ ∈ C⁰⁰⁺(1/4, 4) with ρ₀ ≡ 1 on [1, 2], and let ρ be the radial extension of ρ₀ such that ∫ ρ(x)dx = 1.

ρₑ(x) = e⁻³ρ (x/2).

Let

\[ N(t) = \sum_{k \in Z} \chi_{\Omega}(k) \rho(t) = t^3|\Omega_a| + t^3 \sum_{k \neq (0,0,0)} \widehat{\chi_{\Omega}}(tk) \widehat{\rho}(ek) = t^3|\Omega_a| + E_a(t). \]

(1.1)

It is not hard to see that there exists C > 0 such that

\[ N_a(t - C) \leq N_a(t) \leq N_a(t + C). \]

(1.2)

It follows that

\[ \int_{[1/2] \times [1/2] \times [1/2]} |E_a(t)|^2 da \leq \int_{[1/2] \times [1/2] \times [1/2]} |E_a(t)|^2 da + t^4 \epsilon^2. \]

(1.3)

We conclude that it suffices to establish estimates for E_a(t) with ε = t⁻¹.

Using the standard asymptotic formula for the Fourier transform of the characteristic function of a bounded smooth convex domain where the Gaussian curvature of the boundary is non-vanishing, (see e.g. [Hertz62]), we see that \( \widehat{\chi_{\Omega}}(tk) \) is a sum of two terms of the form

\[ e^{2\pi i t|k|_{a}} t^{-2}|k|_{a}^{-2} + O(t|k|^{-3}), \]

(1.4)

where

\[ |k|_{a} = \sqrt{a_1 k_1^2 + a_2 k_2^2 + a_3 k_3^2}. \]

(1.5)

It follows that

\[ E_a(t) = t \sum_{k \neq (0,0,0)} e^{2\pi i t|k|_{a}} |k|_{a}^{-2} \widehat{\rho}(ek) + t^3 \sum_{k \neq (0,0,0)} O(t|k|^{-3}) \widehat{\rho}(ek) = I + II. \]

(1.6)

Since we can easily handle II point-wise, we turn our attention to I. Squaring, integrating in a, and replacing the limits of integration in a by a smooth cutoff function, we get

\[ t^2 \sum_{k,l \neq (0,0,0)} |k|^{-2} |l|^{-2} \widehat{\rho}(ek) \widehat{\rho}(el) \int e^{2\pi i t(|k|_{a} - |l|_{a})} \psi_{k,l}(a) da \]

(1.7)

= t^2 \sum_{k,l \neq (0,0,0)} |k|^{-2} |l|^{-2} \widehat{\rho}(ek) \widehat{\rho}(el) I_{k,l}(t)

where

\[ \psi_{k,l}(a) = \left( \frac{|k|}{|l|_{a}} \right)^{2} \left( \frac{|l|}{|l|_{a}} \right)^{2} \psi(a), \]

(1.8)

where \( \psi \) is a positive smooth cutoff function, supported in [1/4, 4] and identically equal to 1 on [1/2, 2]. Observe that when k ≠ (0, 0, 0) and l ≠ (0, 0, 0), \( \psi_{k,l} \in C_0^\infty \) with constants uniform in k and l. It suffices to show that (1.7) is bounded above by \( C_4 t^{2+\delta} \) for any \( \delta > 0. \)
SECTION II: PRELIMINARY REDUCTIONS

This section contains some simple observations that we shall make use of in Section III where the main result of the paper is proved.

**Lemma 2.1.** Let \( \delta > 0 \). Let \( N > \frac{1}{\delta} + 1 \). Then

\[ (2.1) \sum_{|k| > \varepsilon^{-1-\delta}} |k|^{-2} |e_k|^{-N} \lesssim 1. \]

**Proof of Lemma 2.1.** We have

\[ \sum_{|k| > \varepsilon^{-1-\delta}} |k|^{-2} |e_k|^{-N} \lesssim \varepsilon^{-N} \int_{|x| > \varepsilon^{-1-\delta}} |x|^{-2-N} \, dx \]

\[ \lesssim \varepsilon^{-N} \varepsilon^{-1-\delta} \varepsilon^\delta = \varepsilon^{-\frac{1}{\delta}} \]

if \( N > \frac{1}{\delta} + 1 \).

Since \( |\hat{\rho}(ek)| \lesssim (1 + |ek|)^{-N} \) for any \( N > 0 \), and \( |I_{k,l}(t)| \lesssim 1 \), Lemma 2.1 shows that in estimating (1.7) we may sum over \( |k|, |l| \lesssim \varepsilon^{-1-\delta} \), \( \delta > 0 \). In particular, this means that we may sum over \( |k_j|, |l_j| \lesssim \varepsilon^{-1-\delta} \).

**Lemma 2.2.** Let \( S, S' \) be subsets of \( \{1, 2, 3\} \) of cardinality at most 2. Then

\[ (2.3) \sum_{1 \leq |k|, |l| \leq \varepsilon^{-1-\delta}; i \in S, j \in S'} |k|^{-2} |l|^{-2} \lesssim t^2. \]

**Proof of Lemma 2.2.** The proof is immediate since we are down to at most 2 variables in \( k \) and \( l \), so the power \( -2 \) suffices, up to logarithms.

**Lemma 2.3.** Let \( U = \{k, l \in \mathbb{Z}^3 \times \mathbb{Z}^3 : |k_j|, |l_j| \leq \varepsilon^{-1-\delta}; k_1 = 0, l_1 \neq 0\} \). Then

\[ (2.4) t^2 \sum_U |k|^{-2} |l|^{-2} I_{k,l}(t) \lesssim t^2. \]

**Proof of Lemma 2.3.** Let \( \Phi_{k,l}(a) = |k| - |l| \). We have

\[ (2.5) \nabla \Phi_{k,l}(a) = \frac{1}{2} \left( \frac{k_1^2}{|k|} - \frac{l_1^2}{|l|}, \frac{k_2^2}{|k|} - \frac{l_2^2}{|l|}, \frac{k_3^2}{|k|} - \frac{l_3^2}{|l|} \right). \]
Since $k_1 = 0$, $|\nabla \Phi_{k,l}(a)| \gtrsim \frac{l^2}{|l|}$. Integrating by parts once (see the appendix) shows that

$$|I_{k,l}(t)| \lesssim t^{-1} \frac{|l|}{l_1^2}.$$  \hspace{1cm} (2.6)

We get

$$t^2 t^{-1} \sum_{1 \leq |k_j|, |l_j| \lesssim \varepsilon^{-1-\delta}, k_1 = 0} |k|^{-2} |l|^{-2} |l|^{-2}$$

$$\lesssim t \sum_{1 \leq |l_j| \lesssim \varepsilon^{-1-\delta}} (|l_2| + |l_3|)^{-1} l_1^{-2} \lesssim t \varepsilon^{-1} \lesssim t^2.$$  \hspace{1cm} (2.7)

The same argument works if $k_2 = 0$ and $l_2 \neq 0$, or if $k_3 = 0$ and $l_3 \neq 0$.

The basic idea of these reductions is that we only need to sum up to $|k|, |l| \lesssim \varepsilon^{-1-\delta}$, and that it suffices to consider the case where $k_j, l_j \neq 0$, $j = 1, 2, 3$.

**Section 3:**

The determinant of the Hessian matrix of $\Phi_{k,l}$ with respect to $(a_1, a_2)$ equals

$$-\frac{1}{16} \frac{(k_1^2 t_2^2 - k_2^2 t_1^2)^2}{|k|^3 |l|^3},$$

and its absolute value is bounded from below by a constant multiple of

$$\frac{(k_1^2 t_2^2 - k_2^2 t_1^2)^2}{|k|^3 |l|^3}.$$  \hspace{1cm} (3.1)

It follows that

$$t^2 \sum_{1 \leq |k_j|, |l_j| \lesssim \varepsilon^{-1-\delta}, \left| \frac{k_1}{k_2} \right| - \left| \frac{l_1}{l_2} \right| \neq 0} |k|^{-2} |l|^{-2} I_{k,l}(t)$$

$$\lesssim t \sum_{1 \leq |k_j|, |l_j| \lesssim \varepsilon^{-1-\delta}, \left| \frac{k_1}{k_2} \right| - \left| \frac{l_1}{l_2} \right| \neq 0} |k|^{-\frac{1}{2}} |l|^{-\frac{1}{2}} |k_1^2 t_2^2 - k_2^2 t_1^2|^{-1}$$

$$\lesssim t \sum_{1 \leq |k_j|, |l_j| \lesssim \varepsilon^{-1-\delta}, \left| \frac{k_3}{k_2} \right| - \left| \frac{l_3}{l_2} \right| \neq 0} |k_3|^{-\frac{1}{2}} |l_3|^{-\frac{1}{2}} |k_1^2 t_2^2 - k_2^2 t_1^2|^{-1}$$

$$\lesssim t \sum_{1 \leq |k_j|, |l_j| \lesssim \varepsilon^{-1-\delta}, \left| \frac{k_3}{k_2} \right| - \left| \frac{l_3}{l_2} \right| \neq 0} |k_1^2 t_2^2 - k_2^2 t_1^2|^{-1}.$$
Either $\text{sgn}(k_1l_2) = \text{sgn}(l_1k_2)$ or $\text{sgn}(k_1l_2) = -\text{sgn}(l_1k_2)$. Without loss of generality suppose that $k_j, l_j > 0$. It follows that (3.3) is bounded by the expression of the form

$$t \approx t \sum_{m=0}^{\log(\epsilon^{-2})} 2^{-m} \left| \sum_{1 \leq |k_j|, |l_j| \leq \epsilon^{-1}, j=1,2 \atop 2^m \leq |k_1l_2 - k_2l_1| \leq 2^{m+1}} k_1^{-1}l_2^{-1} \right|.$$  

(3.4)

$$t \approx t \sum_{m=0}^{\log(\epsilon^{-2})} 2^{-m} \left| \int_{1 \leq x, y \leq \epsilon^{-2}, 2^m \leq |x_2 - y_2| \leq 2^{m+1}} x_1^{-1}x_2^{-1}dx dy \right|.$$  

Let

$$u_1 = x_1x_2, \ u_2 = x_2, \ v_1 = y_1y_2, \ \text{and} \ v_2 = y_2.$$  

(3.5)

It follows that

$$du_1 = x_2dx_1 + x_1dx_2, \ du_2 = dx_2, \ dv_1 = y_2dy_1 + y_1dy_2, \ \text{and} \ dv_2 = dy_2.$$  

(3.6)

Also, $x_1 = \frac{u_1}{u_2}$, so $x_1x_2 = u_1$. Combining this with (3.5) and (3.6), we see that (3.4) is bounded by

$$t \approx t \sum_{m=0}^{\log(\epsilon^{-2})} 2^{-m} \left| \int_{1 \leq u_1, v_1 \leq \epsilon^{-2}, 1 \leq u_2, v_2 \leq \epsilon^{-1}, 2^m \leq |u_1 - v_1| \leq 2^{m+1}} u_1^{-1}u_2^{-1}v_1^{-1}v_2^{-1}dudv \right|.$$  

(3.7)

$$t \approx t \sum_{m=0}^{\log(\epsilon^{-2})} 2^{-m} \left| \int_{1 \leq u_1, v_1 \leq \epsilon^{-2}, 2^m \leq |u_1 - v_1| \leq 2^{m+1}} u_1^{-1}dudv \right| \approx t \leq t^2.$$  

Clearly, the same argument works if $\left| \frac{k_1}{k_2} - \frac{l_1}{l_2} \right| \neq 0$ or if $\left| \frac{k_2}{k_3} - \frac{l_2}{l_3} \right| \neq 0$.

SECTION 4: $\left| \frac{k_1}{k_2} - \frac{l_1}{l_2} \right| + \left| \frac{k_1}{k_3} - \frac{l_1}{l_3} \right| + \left| \frac{k_2}{k_3} - \frac{l_2}{l_3} \right| = 0$

In this case

$$\left| \frac{k_1}{l_1} \right| = \left| \frac{k_2}{l_2} \right| = \left| \frac{k_3}{l_3} \right|.$$  

(4.1)
It follows that \( k = \alpha l \). Dominating \(|I_{k,l}(t)|\) by 1, we have

\[
(4.2) \quad t^2 \sum_{1 \leq |k_j|, |l_j| \leq \varepsilon^{-1-\delta}} |k|^{-2} |l|^{-2} I_{k,l}(t).
\]

We are summing over the set where \( l = \alpha k \). Observe that \( \alpha \) must be of the form \( \frac{m}{\gcd(k_1, k_2, k_3)} \). It follows that the expression in (4.2) is bounded by a constant multiple of

\[
\lesssim t^2 \sum_{1 \leq |k| \leq \varepsilon^{-1-\delta}} \sum_{\gcd(k_1, k_2, k_3) = \frac{m}{j}} \alpha^{-2} |k|^{-4}
\]

\[
= t^2 \sum_{1 \leq |k| \leq \varepsilon^{-1-\delta}} \sum_{m=1}^{\varepsilon^{-1-\delta}} \left( \frac{\gcd(k_1, k_2, k_3)}{m^2} \right)^2 |k|^{-4}
\]

\[
\lesssim t^2 \sum_{1 \leq |k| \leq \varepsilon^{-1-\delta}} (\gcd(k_1, k_2, k_3))^2 |k|^{-4}
\]

\[
= t^2 \sum_{n=1}^{\varepsilon^{-1-\delta}} 2^{-4n} \sum_{|k| \approx 2^n} \sum_{j=1}^{\gcd(k_1, k_2, k_3) = 1} j^2
\]

\[
\approx t^2 \sum_{n=1}^{\varepsilon^{-1-\delta}} 2^{-4n} \sum_{|k| \approx \frac{m}{j}} \sum_{j=1}^{\gcd(k_1, k_2, k_3) = 1} j^2
\]

\[
\lesssim t^2 \sum_{n=1}^{\varepsilon^{-1-\delta}} \sum_{j=1}^{\gcd(k_1, k_2, k_3) = 1} 2^{-4n} \frac{\frac{3n}{j^3} j^2}{j^3}
\]

\[
(4.3) \quad = t^2 \sum_{n=1}^{\varepsilon^{-1-\delta}} \sum_{j=1}^{\gcd(k_1, k_2, k_3) = 1} 2^{-n} \frac{1}{j^4} \lesssim t^2.
\]

This completes the three dimensional proof. We now outline the two dimensional argument. The determinant of the Hessian matrix of \( \Phi_{k,l} \) in two dimensions is given by (3.1).

When \( \frac{k_1}{k_2} \neq \pm \frac{l_1}{l_2} \), the calculation identical to the one contained in (3.3)-(3.7) does the job. If \( \frac{k_1}{k_2} = \pm \frac{l_1}{l_2} \), we repeat the argument in (4.2), (4.3) as follows

\[
t \sum_{1 \leq |k_j|, |l_j| \leq \varepsilon^{-1-\delta}; \frac{k_1}{k_2} = \pm \frac{l_1}{l_2}} |k|^{-\frac{2}{8}} |l|^{-\frac{2}{8}} I_{k,l}(t)
\]
\[ \lesssim t \sum_{1 \leq |k_1|, |l_1| \leq \epsilon^{-1-\delta}, \frac{k_1}{l_1} = \frac{k_2}{l_2}} |k|^{-1}|l|^{-1} I_{k,l}(t). \]

\[ \lesssim t \sum_{1 \leq |k| \leq \epsilon^{-1-\delta}} \frac{\sum_{|y|=|k_1|=|k_2| \leq \epsilon^{-1-\delta}} \alpha^{-1}|k|^{-2} \gcd(k_1,k_2)}{m} \]

\[ = t \sum_{1 \leq |k| \leq \epsilon^{-1-\delta}} \frac{\gcd(k_1,k_2)}{m} |k|^{-2} \]

\[ \lesssim t \sum_{1 \leq |k| \leq \epsilon^{-1-\delta}} \frac{\gcd(k_1,k_2)}{m} |k|^{-2} \]

\[ = t \sum_{1 \leq |k| \leq \epsilon^{-1-\delta}} \frac{\gcd(k_1,k_2)}{m} |k|^{-2} \]

\[ \approx \frac{\log(\epsilon^{-1-\delta})}{m} \sum_{n=1}^{\approx \epsilon^{-1-\delta}} 2^{-2n} \sum_{|k| \leq 2^n} \sum_{j=1}^{\gcd(k_1,k_2)=j} \]

\[ \approx \frac{\log(\epsilon^{-1-\delta})}{m} \sum_{n=1}^{\approx \epsilon^{-1-\delta}} 2^{-2n} \sum_{|k| \leq 2^n} \sum_{j=1}^{\gcd(k_1,k_2)=1} \]

\[ \lesssim t \sum_{n=1}^{\approx \epsilon^{-1-\delta}} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-2n} \frac{2^n}{j^2} \]

\[ \approx \frac{\log(\epsilon^{-1-\delta})}{m} \sum_{n=1}^{\approx \epsilon^{-1-\delta}} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-2n} \frac{2^n}{j^2} \]

\[ = t \sum_{n=1}^{\approx \epsilon^{-1-\delta}} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-n} j^{-1} \lesssim t. \]

\[ \approx \frac{\log(\epsilon^{-1-\delta})}{m} \sum_{n=1}^{\approx \epsilon^{-1-\delta}} \sum_{j=1}^{\approx \epsilon^{-1-\delta}} 2^{-n} j^{-1} \lesssim t. \]

**Appendix: Oscillatory Integrals of the First Kind**

In this paper we made use of the following basic facts about the oscillatory integrals of the form

\[ I(t) = \int_{\mathbb{R}^d} e^{i t f(x)} \psi(x) dx, \]

where \( \psi \) is a smooth cutoff function and \( f \) is smooth. See, for example [Stein93], [BNW88] for related information.

**Theorem 5.1.** Suppose that \( f \) is convex and finite type, and the hessian matrix of \( f \) contains an \( M \) by \( M \) sub-matrix of determinant \( \geq c_0 \). Then

\[ |I(t)| \lesssim t^{-\frac{M}{2}} c_0^{-\frac{1}{2}}. \]
Theorem 5.2. Suppose that $|\nabla f(a)| \gtrsim c_0$. Then

\[(5.3) \quad |I(t)| \lesssim t^{-1} c_0^{-1}.\]

We note that in both theorems the constants may depend on the upper bounds of derivatives of $f$ and $\psi$. 


REFERENCES


