ANALYTICITY OF LAYER POTENTIALS AND $L^2$ SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR DIVERGENCE FORM ELLIPTIC EQUATIONS WITH COMPLEX $L^\infty$ COEFFICIENTS

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Abstract. We consider divergence form elliptic operators of the form $L = -\text{div}(A(x)\nabla)$, defined in $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}, n \geq 2\}$, where the $L^\infty$ coefficient matrix $A$ is $(n + 1) \times (n + 1)$, uniformly elliptic, complex and $t$-independent. We show that for such operators, boundedness and invertibility of the corresponding layer potential operators on $L^2(\mathbb{R}^n) = L^2(\partial \mathbb{R}^{n+1})$, is stable under complex, $L^1$ perturbations of the coefficient matrix. Using a variant of the $Tb$ Theorem, we also prove that the layer potentials are bounded and invertible on $L^2(\mathbb{R}^n)$ whenever $A(x)$ is real and symmetric (and thus, by our stability result, also when $A$ is complex, $\|A - A^0\|_{L^\infty}$ is small enough and $A^0$ is real, symmetric, $L^\infty$ and elliptic). In particular, we establish solvability of the Dirichlet and Neumann (and Regularity) problems, with $L^2$ (resp. $\dot{L}^2$) data, for small complex perturbations of a real symmetric matrix. Previously, $L^2$ solvability results for complex (or even real but non-symmetric) coefficients were known to hold only for perturbations of constant matrices (and then only for the Dirichlet problem), or in the special case that the coefficients $A_{j,n+1} = 0 = A_{n+1,j}$, $1 \leq j \leq n$, which corresponds to the Kato square root problem.

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1. Introduction, statement of results, history

In this paper, we consider the solvability of boundary value problems for divergence form complex coefficient equations $Lu = 0$, where

$$L = -\text{div} A \nabla \equiv - \sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_i} (A_{i,j} \frac{\partial}{\partial x_j})$$

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is defined in $\mathbb{R}^{n+1} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : n \geq 2\}$ (we use the notational convention that $x_{n+1} = t$), and where $A = A(x)$ is an $(n+1) \times (n+1)$ matrix of complex-valued $L^\infty$ coefficients, defined on $\mathbb{R}^n$ (i.e., independent of the $t$ variable) and satisfying the uniform ellipticity condition

$$\lambda|\xi|^2 \leq \Re e (A(x)\xi, \xi) \equiv \Re e \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_i \xi_j, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda,$$

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^{n+1}$, $x \in \mathbb{R}^n$. The divergence form equation is interpreted in the weak sense, i.e., we say that $Lu = 0$ in a domain $\Omega$ if $u \in W^{1,2}_{\text{loc}}(\Omega)$ and

$$\int \Delta u \cdot \nabla \Psi = 0$$

for all complex valued $\Psi \in C^\infty_0(\Omega)$.

The boundary value problems that we consider are classical. To state them, we first recall the definitions of the non-tangential maximal operators $N_+, \bar{N}_+$. Given $x_0 \in \mathbb{R}^n$, define the cone $\gamma(x_0) = \{(x, t) \in \mathbb{R}^{n+1} : |x_0 - x| < t\}$. Then for $U$ defined in $\mathbb{R}^{n+1}_+$,

$$N_+U(x_0) \equiv \sup_{(x,t) \in \gamma(x_0)} |U(x,t)|, \quad \bar{N}_+U(x_0) \equiv \sup_{(x,t) \in \gamma(x_0)} \left( \int \int_{|t-s| < t/2} |U(y,s)|^2 dy ds \right)^{1/2}.$$

Here, and in the sequel, the symbol $\int$ denotes the mean value, i.e., $\int_E f \equiv |E|^{-1} \int_E f$. We use the notation $u \to f$ n.t. to mean that for a.e. $x \in \mathbb{R}^n$, $\lim_{y \to x}$ in the weak sense, $u(y,t) = f(x)$, where the limit runs over $(y,t) \in \gamma(x)$.

We shall consider the Dirichlet problem

\[ (D2) \]

\[
\begin{cases}
Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ = \{(x, t) \in \mathbb{R}^n \times (0, \infty)\} \\
\lim_{t \to 0} u(\cdot, t) \to f \text{ in } L^2(\mathbb{R}^n) \text{ and n.t.} \\
\sup_{t > 0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty,
\end{cases}
\]

the Neumann problem

\[ (N2) \]

\[
\begin{cases}
Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\
\frac{\partial u}{\partial \nu}(x, 0) \equiv - \sum_{j=1}^{n+1} A_{n+1,j}(x) \frac{\partial u}{\partial x_j}(x, 0) = g(x) \in L^2(\mathbb{R}^n) \\
N_+(\nabla u) \in L^2(\mathbb{R}^n),
\end{cases}
\]

and the Regularity problem

\[ (R2) \]

\[
\begin{cases}
Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\
u(\cdot, t) \to f \in L^2_1(\mathbb{R}^n) \text{ n.t.} \\
\bar{N}_+(\nabla u) \in L^2(\mathbb{R}^n).
\end{cases}
\]

Our solutions will be unique among the class of solutions satisfying the stated $L^2$ bounds (in the case of (N2) and (R2), this uniqueness will hold modulo constants). The homogeneous Sobolev space $L^2_1$ is defined as the completion of $C^\infty_0$ with respect to the semi-norm $\|\nabla F\|_2$. For $n \geq 3$ this space can be identified (modulo constants) with the space $I_1(L^2) \equiv \Delta^{-1/2}(L^2) \subset L^2$, where $2^* = 2n/(n - 2)$; for $n = 2$, the identification with $I_1(L^2)$ is

1. Our uniform $L^2$ estimate for solutions of (D2) can be improved to an $L^2$ bound for $N_+u$, given certain $L^p$ estimates for the layer potentials. The fourth named author and M. Mitrea will present the $L^p$ theory in a forthcoming publication. In the present paper, we shall be content with a weak-$L^2$ bound for $N_+u$.

2. We shall elaborate in section 4 the precise nature by which the co-normal derivative assumes the prescribed data.
still valid, but in that case the fractional integral $I_1 f$ must itself be defined modulo constants for $f \in L^2$, and the space embeds in $\text{BMO}$.

We remark that for the class of operators that we consider, solvability of these boundary value problems in the half-space may readily be generalized to the case of domains given by the region above a Lipschitz graph, and even to the case of star-like Lipschitz domains. We shall return to this point later. We shall also discuss later the significance of our assumption that the coefficients are $t$-independent.

In order to state our main results, we shall need to recall a few definitions and facts. We say that $u$ is locally H"older continuous in a domain $\Omega$ if there is a constant $C$ and an exponent $\alpha > 0$ such that for any ball $B = B(x, R)$, of radius $R$, whose concentric double $2B \equiv B(x, 2R)$ is contained in $\Omega$, we have that

\begin{equation}
|u(Y) - u(Z)| \leq C \left( \frac{|Y - Z|}{R} \right)^\alpha \left( \int_{2B} |u|^2 \right)^{\frac{\alpha}{2}},
\end{equation}

(1.2)

whenever $Y, Z \in B$. Observe that any $u$ satisfying (1.2) also satisfies Moser’s “local boundedness” estimate [M]

\begin{equation}
\sup_{Y \in B} |u(Y)| \leq C \left( \int_{2B} |u|^2 \right)^{\frac{1}{2}}.
\end{equation}

(1.3)

By the classical De Giorgi-Nash Theorem [DeG, N], (1.2) and hence also (1.3) hold, with $C$ and $\alpha$ depending only on dimension and the ellipticity parameters, whenever $u$ is a solution of $Lu = 0$ in $\Omega \subset \mathbb{R}^{n+1}$, if in addition the coefficient matrix $A$ is real (for this result, it need not be $t$-independent). Moreover, it is shown in [A] (see also [AT, HK]), that property (1.2) is stable under complex, $L^\infty$ perturbations.

We now recall the method of layer potentials. For $L$ as above, let $\Gamma, \Gamma^*$ denote the fundamental solutions\(^3\) for $L$ and $L^*$ respectively, in $\mathbb{R}^{n+1}$, so that

\begin{equation}
L_{\gamma,t} \Gamma(x, t, y, s) = \delta_{(t,y)}, \quad L^*_{\gamma,t} \Gamma^*(y, s, x, t) \equiv L^*_{\gamma,s} \Gamma(x, t, y, s) = \delta_{(x,t)},
\end{equation}

(1.4)

where $\delta_X$ denotes the Dirac mass at the point $X$, and $L^*$ is the hermitian adjoint of $L$. By the $t$-independence of our coefficients, we have that

\begin{equation}
\Gamma(x, t, y, s) = \Gamma(x, t - s, y, 0).
\end{equation}

We define the single and double layer potential operators, by

\begin{equation}
S_{\gamma} f(x) \equiv \int_{\mathbb{R}^n} \Gamma(x, t, y, 0) f(y) \, dy, \quad t \in \mathbb{R},
\end{equation}

(1.5)

\begin{equation}
D_{\gamma} f(x) \equiv \int_{\mathbb{R}^n} \partial_{\gamma} \Gamma^*(y, 0, x, t) f(y) \, dy, \quad t \neq 0,
\end{equation}

where $\partial_{\gamma}$ is the adjoint exterior conormal derivative; i.e., if $A^*$ denotes the hermitian adjoint of $A$, then

\begin{equation}
\partial_{\gamma_j} \Gamma^*(y, 0, x, t) = - \sum_{j=1}^{n+1} A^*_{n+1,j}(y) \frac{\partial \Gamma^*}{\partial y_j}(y, 0, x, t) = - \epsilon_{n+1} \cdot A^*(y) \nabla_{\gamma_j} \Gamma^*(y, s, x, t) \big|_{s=0}
\end{equation}

\(^3\)See [HK2] for a construction of the fundamental solution.
have that denote the Hermitian adjoint of an operator (1.10)

| \frac{\partial}{\partial y}(x, 0, y, 0) f(y) dy |

where \( \frac{\partial}{\partial y} \) denotes the exterior conormal derivative in the \((x, t)\) variables. Classically, \( \overline{K} \) is often denoted \( K^* \), but we avoid this notation here as \( \overline{K} \) need not be the adjoint of \( K \) unless \( L \) is self-adjoint. Rather, for us, \( K^*, S^* \) and \( \mathcal{D}' \) will denote the analogues of \( K, S \) and \( \mathcal{D} \) corresponding to \( L^* \) (although sometimes we shall write \( K^* \), etc., when we wish to emphasize the dependence on a particular operator), and we use the notation \( \text{adj}(T) \) to denote the Hermitian adjoint of an operator \( T \) acting in \( \mathbb{R}^n \). With these conventions, we have that \( \overline{K} = \text{adj}(K^*) \), as the reader may verify. We apologize for this departure from tradition, but the context of complex coefficients seems to require it.

For sufficiently smooth coefficients, the following “jump relation” formulae have been established in [MMT]. We refer to Section 4 our discussion of the jump formulae, and the nature of their “non-tangential” realization, in the non-smooth case. We have

\[
\mathcal{D}_{x,t} f \rightarrow \left( \pm \frac{1}{2} I + \overline{K} \right) f
\]

(1.7)

\[
(\nabla S_t) |_{u=\pm x} f \rightarrow \pm \frac{1}{2} f(x) \frac{1}{A_{n+1, n+1}(x)} \nu_{n+1} + T f,
\]

(1.8)

(1.9)

(1.10)

Then, as usual\(^4\), one obtains solvability of \( (D2) \) in the upper (resp. lower) half space by establishing boundedness on \( L^2(\mathbb{R}^n) \) of \( f \rightarrow \mathcal{D}_{x,t} f \), uniformly in \( t \), and invertibility of \( -\frac{1}{2} I + K \) (resp. \( \frac{1}{2} I \)). Similarly, solvability of \( (N2) \) and \( (R2) \) follows from \( L^2 \) boundedness of \( f \rightarrow \mathcal{N}(\nabla S_{x,t} f) \), and (for \( (N2) \)) invertibility of \( L^2 \) of \( \pm \frac{1}{2} I + \overline{K} \), and (for \( (R2) \)) invertibility of the mapping \( S_0 = S_{1, 1} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \). We now set some convenient terminology: we shall say that an operator \( L \) for which all of the above hold has “Bounded and Invertible Layer Potentials”. If in addition we have the square function estimate

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\partial_2^2 S_t f(x)|^2 \frac{dxdt}{|t|^1} \leq C||f||_{L^2}^2,
\]

then we shall say that \( L \) has “Good Layer Potentials”. Finally, we shall refer to the constant in (1.10), together with all of the constants arising in the estimates for the boundedness and invertibility of the layer potentials, collectively as the “Layer Potentials Constants” for \( L \).

In this paper, we prove the following theorems. In the sequel we assume always that our \((n+1) \times (n+1) \) coefficient matrices are \(t\)-independent, complex, and satisfy the ellipticity condition (1.1) and the De Giorgi-Nash-Moser estimates (1.2) and (1.3).

\(^4\) For non-smooth coefficients, some care should be taken to define the “principal value” operators on the boundary - see Section 4.

\(^5\) In the setting of non-smooth coefficients, some rather extensive preliminaries are required in order to apply the layer potential method to obtain solvability; see Section 4.
Theorem 1.11. Suppose that $L_0 = -\div A^0 \nabla$ and $L_1 = -\div A^1 \nabla$ are operators of the type described above, and that solutions $u_0$, $w_0$ of $L_0 u_0 = 0$, $L_0^* w_0 = 0$ satisfy the De Giorgi-Nash-Moser estimates (1.2) and (1.3). Suppose also that $L_0$ and $L_0^*$ have “Good Layer Potentials”. Then $L_1$ and $L_1^*$ have Good Layer Potentials, provided that

\[ \|A^0 - A^1\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon_0, \]

where $\epsilon_0$ is sufficiently small depending only on dimension and on the various constants associated to $L_0$ and $L_0^*$, specifically: the ellipticity parameters, the De Giorgi-Nash-Moser constants (1.2) and (1.3), and the Layer Potential Constants.

We observe that it is not clear whether the property that $L$ has “Good Layer Potentials” is preserved under regularization of the coefficients. For this reason, we shall be forced to prove Theorem 1.11 without recourse to the usual device of making an \textit{a priori} assumption of smooth coefficients. We also note that we shall use the invertibility of the layer potentials associated to $L_0$ and $L_0^*$ even to establish the boundedness of the layer potentials associated to $L_1$ (see Section 7 below).

Theorem 1.12. Suppose that $L = -\div A \nabla$ is an operator of the type defined above, and in addition, suppose that $A$ is real and symmetric. Then $L$ has Good Layer Potentials, and its Layer Potential Constants depend only on dimension and on the ellipticity parameters in (1.1).

We remark that while Theorem 1.12 yields in particular the solvability of (D2), (N2) and (R2) in the case that $A$ is real and symmetric, it is only the fact that this solvability is obtainable via layer potentials that is new here, the solvability of (D2) having been previously obtained by Jerison and Kenig [JK1], and that of (N2) and (R2) by Kenig and Pipher [KP], without the use of layer potentials. The essential missing ingredient had been the boundedness of the layer potentials.

The previous two theorems are our main results. As corollaries, we obtain

Theorem 1.13. Suppose that $L_1 = -\div A^1 \nabla$ is an operator of the type defined above, and that $\|A^1 - A^0\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon_0$, for some real, symmetric, $t$-independent uniformly elliptic matrix $A^0 \in L^\infty(\mathbb{R}^n)$. Then (D2), (N2) and (R2) are all solvable for $L_1$, provided that $\epsilon_0$ is sufficiently small, depending only on dimension and the ellipticity parameters for $A^0$. The solution of (D2) is unique among the class of solutions $u$ for which $\sup_{x \in \mathbb{R}^n} ||u(\cdot, t)||_{L^1(\mathbb{R}^n)} < \infty$, and the solutions of (N2) and (R2) are unique modulo constants among the class of solutions for which $\tilde{N}_u(\nabla u) \in L^2$.

Theorem 1.14. The conclusion of Theorem 1.13 holds also in the case that $\|A^1 - A^0\|_{\infty}$ is sufficiently small, where $A^0$ is now a constant, elliptic complex matrix.

The last theorem follows from Theorem 1.11, and the fact that constant coefficient operators have Good Layer Potentials (see the appendix, Section 10).

We note that by a standard device, Theorems 1.11, 1.12 and 1.13 all extend readily to the case where $\Omega = \{(x, t) : t > F(x)\}$, with $F$ Lipschitz. Indeed, by “pulling back” under the mapping $\rho : \mathbb{R}^{n+1}_+ \rightarrow \Omega$ defined by

\[ \rho(x, t) = (x, F(x) + t), \]

we may reduce to the case of the half-space. The pull-back operators are of the same type, and, in particular, the coefficients remain $t$-independent. Moreover, if the original coefficients are real and symmetric, then so are those of the pull-back operator. In this setting, the parameter $\epsilon_0$ will also depend on $\|\nabla F\|_{\infty}$. In addition, our results may be
further extended to the setting of star-like Lipschitz domains (which would seem to be the most general setting in which the notion of “radial independence” of the coefficients makes sense). The idea is to use a partition of unity argument, as in [MMT], to reduce to the case of a Lipschitz graph. We omit the details.

Let us now briefly review the history of work in this area, which falls broadly into two categories, depending on whether or not the \( t \)-independent coefficient matrix is self-adjoint. We discuss the former category first, and we mention only the case of a single equation, although results for certain constant coefficient self-adjoint systems in a Lipschitz domain are known, see e.g. [K, K2] for further references. (Moreover, the present setting of complex coefficients may be viewed in the context of 2\( \times \)2 systems, and indeed this provides part of our motivation to consider the complex case). For Laplace’s equation in a Lipschitz domain, the solvability of (D2) was obtained by Dahlberg [D], and that of (N2) and (R2) by Jerison and Kenig [JK2]; solvability of the same problems via harmonic layer potentials is due to Verchota [V], using the deep result of Coifman, McIntosh and Meyer [CMcM] concerning the \( L^2 \) boundedness of the Cauchy integral operator on a Lipschitz curve. The results of [V] and [CMcM] are subsumed in our Theorem 1.12 via the pull-back mechanism discussed above. Moreover, as mentioned above, for \( A \) real, symmetric and \( t \)-independent, the solvability of (D2) was obtained in [JK1], and that of (N2) and (R2) in [KP], but those authors did not use layer potentials. The case of real symmetric coefficients with some smoothness has been treated via layer potentials in [MMT].

In the “non self-adjoint” setting, previous results had been obtained in three special cases. First, it was known that (D2) is solvable for small, complex perturbations of constant elliptic matrices. This is due to Fabes, Jerison and Kenig [FJK] via the method of multilinear expansions. To our knowledge, (R2) and (N2) had not been treated in this setting.

Second, one has solvability of (D2), (N2) and (R2) in the special case that the matrix \( A \) is of the “block” form

\[
\begin{pmatrix}
0 & B \\
\vdots & 0 \\
0 & 0 \\
\end{pmatrix}
\]

where \( B = B(x) \) is a \( n \times n \) matrix. In this case, (D2) is an easy consequence of the semigroup theory, while (R2) amounts to solving the Kato square root problem for the \( n \)-dimensional operator

\[
J = -\text{div}_x B(x) \nabla_x,
\]

and (N2) amounts to \( L^2 \) boundedness of the Riesz transforms \( \nabla J^{-\frac{1}{2}} \) (equivalently, to solving the Kato problem for the adjoint operator \( \text{adj}(J) \)). Moreover, the boundedness of the Riesz transform \( \nabla J^{-\frac{1}{2}} \) can also be interpreted as the statement that the single layer potential is bounded from \( L^2 \) into \( L^1_1 \). These results were obtained in [CMcM] (\( n = 1 \)), [HMc] (\( n = 2 \)), [AHLT] (when \( B \) is a perturbation of a real, symmetric matrix), [HLMc] (when the kernel of the heat semi-group \( e^{-tJ} \) has a Gaussian upper bound) and [AHLMcT] in general\(^6\).

\(^6\)We remark that Theorem 1.11 may be combined with these results for block matrices (1.15) to allow perturbations of the block case, but we do not pursue this point here; see, however, [AAH], where this is done without imposing De Giorgi-Nash-Moser bounds, and where also extensions of Theorems 1.13 and 1.14 will be presented, via the development of a functional calculus for certain Dirac type operators.
Third, Kenig, Koch, Pipher and Toro [KKPT] have obtained solvability of (Dp) (the problems (Dp), (Np) and (Rp) are defined analogously to (D2), (N2) and (R2), but with $L^2$ bounds replaced by $L^p$) in the case $n = 1$ (that is, in $\mathbb{R}^2$), for $p$ sufficiently large depending on $L$, in the case that $A(x)$ is real, but non-symmetric. Moreover, they construct a family of examples in $\mathbb{R}^2$ in which solvability of (Dp) may be destroyed for any specified $p$ by taking $A(x)$ to be an appropriate perturbation of the $2 \times 2$ identity matrix. Very recently, in the same setting of real, non-symmetric coefficients in two dimensions (that is, in $\mathbb{R}^2$), Kenig and Rule [KR] have obtained solvability of (Nq) and (Rq), where $q$ is dual to the [KKPT] exponent. Their result uses boundedness, but not invertibility, of the layer potentials.

The main purpose, then, of the present paper is to develop, to the extent possible, an $L^2$ theory of boundary value problems for full coefficient matrices with complex (including also real, not necessarily symmetric) entries. In fact, in the setting of $L^2$ solvability with $t$-independent coefficients, the counter-example of [KKPT] shows that our perturbation results are in the nature of best possible.

A word about $t$-independence is in order. It has been observed by Caffarelli, Fabes and Kenig [CFK] that some regularity in the transverse direction is necessary, in order to deduce solvability of (D2). More precisely, they show that given any function $\omega(\tau)$ with $\int_0^1 (\omega(\tau))^2 d\tau/\tau = +\infty$, there exists a real, symmetric elliptic matrix $A(x, t)$, whose modulus of continuity in the $t$ direction is controlled by $\omega$, but for which the corresponding elliptic-harmonic measure and the Lebesgue measure on the boundary are mutually insignificant. On the other hand, it is shown in [FJK] that (D2) does hold, assuming that the transverse modulus of continuity $\omega(\tau) \equiv \sup_{c \in \mathbb{R}^n, 0 < c < t} |A(x, t) - A(x, 0)|$ satisfies the square Dini condition $\int_0^1 (\omega(\tau))^2 d\tau/\tau < \infty$, provided that $A(x, 0)$ is sufficiently close to a constant matrix $A_{const}$. It seems likely that the methods of the present paper would allow us to obtain a similar result, but with the constant matrix $A_{const}$ replaced by an $L^\infty$ matrix $A^0(x)$ satisfying the hypotheses of Theorem 1.11 (in particular, real, symmetric). However, we have not pursued this variant here, in part because we conjecture that somewhat sharper estimates should be true. To explain this point of view, we recall that a more refined, scale invariant version of the square Dini condition has been introduced by R. Fefferman, Kenig and Pipher [FKP], and Kenig and Pipher [KP, KP2], to prove perturbation results in which one assumes (roughly) that $|A^1(x, t) - A^0(x, t)|^2 \frac{d\omega}{\omega}$ is a Carleson measure (actually, their condition is slightly stronger, but in the same spirit). Note that this condition requires that $A^1 = A^0$ on the boundary. Our work provides a complement to [FKP] and [KP, KP2], in that we allow the coefficients to differ at the boundary. At present, the results of [FKP] and [KP, KP2] apply only to the case of real coefficients. It is an interesting open problem to extend the theorems of [FKP] and [KP, KP2] to the case of complex coefficients, even in the case of small Carleson norm. Given such an extension, along with our results here, one could specialize to the case $A^1(x, t) = A(x, t)$, $A^0(x, t) = A(x, 0)$, with $A(x, 0)$ close enough to a “good” (e.g., real, symmetric) matrix, to obtain a rather complete picture of the situation for $L^2$ solvability.

Let us now set some notation that will be used throughout the paper. We shall use $\operatorname{div}$ and $\nabla$ to denote the full $n + 1$ dimensional divergence and gradient, respectively. At times, we shall need to consider the $n$-dimensional gradient and divergence, acting only in $x$, and these we denote either by $\nabla_\parallel$ and $\operatorname{div}_\parallel$, or by $\nabla_\perp$ and $\operatorname{div}_\perp$; i.e.,

$$
\nabla_\parallel = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right) = \nabla_x
$$
and for $\mathbb{R}^n$-valued $\vec{w}$, $\text{div}_x \vec{w} \equiv \nabla_x \cdot \vec{w}$. Similarly, given an $(n+1) \times (n+1)$ matrix $A$, we shall let $A_{ij}$ denote the $n \times n$ sub-matrix with entries $(A_k)_i,j \equiv A_{i,j}$, $1 \leq i,j \leq n$, and we define the corresponding elliptic operator acting in $\mathbb{R}^n$ by

$$L \equiv \text{div}_x A \nabla_x.$$  

We shall also use the notation

$$D_j \equiv \frac{\partial}{\partial x_j} = \partial_{x_j}, \quad 1 \leq j \leq n + 1$$

bearing in mind that $x_{n+1} = t$. Points in $\mathbb{R}^{n+1}$ may sometimes be denoted by capital letters, e.g. $X = (x,t), Y = (y,s)$. Balls in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ will be denoted respectively by $B(X,r) \equiv \{ Y : |X - Y| < r \}$ and $\Delta_{n}(x) \equiv \{ y : |x - y| < r \}$. We shall often encounter operators whose kernels involve derivatives applied to the second set of variables in the fundamental solution $\Gamma(x,t,y,s)$. We shall indicate this by grouping the operators with appropriate parentheses, thus:

$$(S_j \nabla) f(x) \equiv \int_{\mathbb{R}^n} \nabla_{y_j} \Gamma(x,t,y,s) |_{s=0} f(y) \, dy.$$  

Hence, one then has

$$(S_j \nabla) \cdot \vec{f} = -S_j \left( \text{div}_x \vec{f} \right), \quad (S_j D_{n+1}) = -\partial_t S_j,$$

where in the second identity we have used (1.4).

Given a cube $Q$, we denote the side length of $Q$ by $\ell(Q)$. Furthermore, given a positive number $r$, we let $rQ$ denote the concentric cube with side length $r\ell(Q)$.

We shall use $P_t$ to denote a nice approximate identity, acting on functions defined on $\mathbb{R}^n$; i.e. $P_t f(x) = \phi_t \ast f$, where $\phi_t(x) = \ell^{-n} \phi(x/t)$, $\phi \in C_0^\infty(\{ |x| < 1 \})$, $0 \leq \phi$ and $\int_{\mathbb{R}^n} \phi = 1$.

Following [FJK], we introduce a convenient norm for dealing with square functions (although we warn the reader that our measure differs from that used in [FJK]):

$$\| F \|_n \equiv \left( \int_{\mathbb{R}^n} |F(x,t)|^2 \frac{dx\,dt}{|t|} \right)^{1/2}, \quad \| F \|_{00} \equiv \left( \int_{\mathbb{R}^n} |F(x,t)|^2 \frac{dx\,dt}{|t|} \right)^{1/2}.$$  

For a family of operators $U_t$, we write

$$\| U_t \|_{1,op} \equiv \sup_{\| f \|_{2,op} = 1} \| U_t f \|_1,$$

and similarly for $\| \cdot \|_{1,op}$ and $\| \cdot \|_{00,op}$. Sometimes, we may drop the “+” sign when it is clear that we are working in the upper $1/2$-space. As usual, we allow generic constants $C$ to depend upon dimension and ellipticity, and, in the proof of the perturbation result, upon the constants associated to the “good” operator $L_0$. Specific constants, still depending on the same parameters, will be denoted $C_1, C_2, \text{etc}.$.

The paper is organized as follows. In sections 2 and 3, we prove some useful technical estimates. In section 4 we discuss the boundary behavior and uniqueness of our solutions. The next five sections are the heart of the matter, in which we prove Theorem 1.11 (sections 5, 6 and 7), and Theorem 1.12 (sections 8 and 9). Section 10 is an appendix, in which we briefly discuss the constant coefficient case.

**Acknowledgements.** The fourth named author thanks M. Mitrea for helpful conversations concerning several of the topics treated in this work, including constant coefficient operators, the boundary behavior of layer potentials, and in particular, for suggesting the approach used here in Lemma 4.18 to obtain the analogue of the classical jump relation formulae.
2. SOME CONSEQUENCES OF DE GIORGI-NASH-MOSER BOUNDS

Throughout this section, and throughout the rest of the paper, we suppose always that our differential operators satisfy our “standard assumptions”: that is, divergence form elliptic, with ellipticity parameters $\lambda$ and $\Lambda$, defined in $\mathbb{R}^{n+1}$, $n \geq 2$, with complex coefficients that are bounded, measurable and $t$-independent; moreover, we suppose that solutions of $Lu = 0$ satisfy the De Giorgi-Nash-Moser estimates (1.2) and (1.3). We now prove some technical estimates using rather familiar arguments. In the sequel, $\Gamma$ will denote the fundamental solution of $L$, and we set

$$K_{m,t}(x,y) \equiv (\partial_t)^{m+1} \Gamma(x,t,y,0)$$

**Lemma 2.2.** Suppose that $L$ and $L^*$ satisfy the “standard assumptions” as above. Then there exists a constant $C_1$ depending only on dimension, ellipticity and (1.2) and (1.3), such that for every integer $m \geq -1$, for all $t \in \mathbb{R}$, and $x, y \in \mathbb{R}^n$, we have

$$|K_{m,t}(x,y)| \leq CC_{1}^{m+1} (|t| + |x-y|)^{-m}$$

(2.3)

$$\left|\left(\partial^{\delta} K_{m,t}(\cdot, y)\right)(x) + \left(\partial^{\delta} K_{m,t}(x, \cdot)\right)(y)\right| \leq CC_{1}^{m+1} \frac{|h|^\delta}{(|t| + |x-y|)^{m+\delta}}.$$  

(2.4)

wherever $2|h| \leq |x-y|$ or $|h| < 20|t|$, for some $\alpha > 0$, where $\left(\partial^{\delta} f\right)(x) \equiv f(x+h) - f(x)$.

**Sketch of proof.** The case $m = -1$ of (2.3) follows from its parabolic analogue in [AT], Section 1.4; alternatively, the reader may consult [HK2] for a direct proof in the elliptic case. The case $m = 0$ may be treated by applying (1.3) to the solution $u(x, t) = \partial_t \Gamma(x, t, y, 0)$ in the ball $B((x, t), R/2)$, with $R = \sqrt{|t|^2 + |x-y|^2}$, and then using Caccioppoli’s inequality to reduce to the case $m = -1$. The case $m > 0$ is obtained by iterating the previous argument, and (2.4) follows from (1.2) and (2.3).

We remark that, by taking more care with the Caccioppoli argument, using a ball of appropriately chosen radius $c_m R$ rather than $R/2$, one may obtain the natural growth bound $m! C_{1}^{m}$ in (2.3) and (2.4). We leave the details to the interested reader.

**Lemma 2.5.** Suppose that $L, L^*$ satisfy the standard assumptions. Then, there exists a constant $C_2$, and for each $\rho > 1$ a constant $C_{\rho}$, such that for every cube $Q \subseteq \mathbb{R}^n$, for all $x \in Q$, and for all integers $k \geq 1$ and $m \geq -1$, we have

(i) \(\int_{Q \cap 2^{k}Q} \left(2^k \ell(Q) \right)^{m+1} \left|\partial_j \nabla \Gamma(x, t, y, 0)\right|^2 dy \leq C C_{\rho}^{m+1} \left(2^k \ell(Q)\right)^{-2}, \quad \forall t \in \mathbb{R}\)

(ii) \(\int_{Q} \left|\ell(Q) \right|^m \left|\partial_j \frac{\ell(Q)}{\rho} \nabla \Gamma(x, t, y, 0)\right|^2 dy \leq C \rho^{m+1} \left(\frac{\ell(Q)}{\rho} \right)^{-2}, \quad \forall \rho > 1\)

**Proof.** We first suppose that $A \in C^{\infty}$; we shall remove this restriction at the end of the proof. Of course, our quantitative bounds will not depend on smoothness. Let us consider estimate (i) first. We shall actually prove that for $C_2$ large enough we have

$$\sum_{m=0}^{\infty} C_{2}^{m} \left\|2^k \ell(Q) \right|^m \left(\partial_j \nabla \Gamma(x, t, \cdot, 0)\right\|^2_{L^{2}((2^{k+1}Q, 2^{k}Q))} \leq C(2^k \ell(Q))^{-2}.$$  

(2.6)

Fix $x \in Q$. Let $\varphi_k \in C^{\infty}_{0}, \varphi_k \equiv 1$ on $2^{k+1}Q \backslash 2^{k}Q$, $\supp \varphi_k \subseteq \frac{3}{2}2^{k+1}Q \setminus \frac{1}{2}2^{k+1}Q$, with $\|\nabla \varphi_k\|_{\infty} \leq C(2^k \ell(Q))^{-1}$.
We observe that
\[ I_m = \int \left[ (\partial_y)^{m+1} \nabla_y \Gamma(x, t, y, 0) \right]^2 \varphi_n^2(y) dy \]

\[ \leq C \Re \int A_1^* \nabla_y (\partial_y)^{m+1} \Gamma(x, t, y, 0) \cdot \nabla_y (\partial_y)^{m+1} \Gamma(x, t, y, 0) \varphi_n^2(y) dy \]

(where \( A_1^* \) is the adjoint of the \( n \times n \) matrix \( A_1 \) defined by \( (A_1)_{ij} = A_{ij} \), \( 1 \leq i, j \leq n \))

\[ = C \Re \int (L_{m+1}^*) (\partial_y)^{m+1} \Gamma(x, t, y, 0) \cdot (\partial_y)^{m+1} \Gamma(x, t, y, 0) \varphi_n^2(y) dy \]

\[ - C \Re \int A_1^* \nabla_y (\partial_y)^{m+1} \Gamma(x, t, y, 0) \cdot \nabla_y \varphi_n^2(y) dy \]

\[ = I_m' + I_m'' \]

where \( L_{m+1}^* \equiv -\text{div}_y A_1^* \nabla_y \). For each integer \( m \geq -1 \), define

\[ a_m = a_m(x) \equiv \| (2^{k} \ell(Q))^m (D_{n+1})^{m+1} \nabla_y \Gamma(x, t, \cdot, 0) \varphi_n \|_2 = (\ell(Q)^m I_m')^{1/2} \]

Since \( \Gamma(x, t, \cdot, 0) \) is a solution of \( L^* \) away from \( x, t \), we have that

\[ (L_{m+1}^*) \Gamma(x, t, y, 0) = \sum_{i=1}^{n} D_i A_{i+n+1}^{*} D_{n+1} + \sum_{j=1}^{n+1} A_{n+1, j}^{*} D_{n+1} \Gamma, \]

where in the second term we have used \( t \)-independence. We designate the respective contribution of these two terms to \( I_m' \) by \( I_{m,1}' \) and \( I_{m,2}' \). Now,

\[ |I_{m,2}'| \leq C \int \left| \nabla_{y,t} (D_{n+1})^{m+1} \Gamma^2 \right| (D_{n+1})^{m+1} \Gamma \varphi_n^2 \]

\[ \leq C \left( \| \nabla_y (D_{n+1})^{m+2} \Gamma \varphi_n \|_2 + \| (D_{n+1})^{m+3} \Gamma \varphi_n \|_2 \right) \| (D_{n+1})^{m+1} \Gamma \varphi_n \|_2 \]

\[ \leq CC_1 \left( 2^{(m+1)} a_{m+1} + C_1^{(m+2)} 2^{(m+1)} \right) \left( 2^{(m+1)} \right)^{\delta^{-1}} \]

(\text{where we have used (2.3)})

\[ \leq CC_1 \left( a_{m+1} 2^{2n-m-2} \right) + C_1^{(m+2)} \left( 2^{(m+1)} \right)^{2n-m-2} \]

\[ \leq C \delta a_{m+1} 2^{(2n-m)} + C_1^{(m+2)} \left( 2^{(m+1)} \right)^{2n-m-2}, \]

where \( \delta > 0 \) is at our disposal. Also, after integrating by parts

\[ I_{m,1}' = -C \Re \sum_{i=1}^{n} \int A_{i+n+1}^* (\partial_y)^{m+2} \Gamma(x, t, y, 0) (\partial_y)^{m+1} D_i \Gamma(x, t, y, 0) \varphi_n^2(y) dy \]

\[ - C \Re \sum_{i=1}^{n} \int A_{i+n+1}^* (\partial_y)^{m+2} \Gamma(x, t, y, 0) (\partial_y)^{m+1} D_i \varphi_n^2(y) dy. \]

By Cauchy’s inequality, (2.3) and the bound for \( \| \nabla \varphi_n \|_\infty \), we obtain

\[ |I_{m,1}'| \leq C I_m + CC_1^{(m+1)} \left( \delta^{-1} + 1 \right) \left( 2^{(k \ell(Q))^{2m-n-2}}. \right. \]

Similarly,

\[ |I_{m,2}| \leq C I_m + CC_1^{(m+1)} \left( \delta^{-1} + 1 \right) \left( 2^{(k \ell(Q))^{2m-n-2}}. \right. \]

Collecting our estimates for \( I_{m,1}', I_{m,2}', \text{ and } I_{m,2}'' \), we obtain for \( \delta \) small enough that

\[ \left( 2^{(k \ell(Q))^{2m}} I_m = a_m^2 \leq C \delta^2 a_{m+1}^2 + CC_1^{(m+2)} \delta^{-1} \left( 2^{(k \ell(Q))^{2m-n-2}}. \right. \]
Thus,
\[
\sum_{m=1}^{\infty} C_2^{-\alpha_3^2} \delta_m^2 \leq \sum_{m=1}^{\infty} C_2^{-\alpha_3^2} C_0^2 \delta_m^2 + \sum_{m=1}^{\infty} C_2^{-\alpha_3^2} C_1^2 (2^{\alpha_3^2} t(Q))^{-\alpha_3^2}.
\]

We now choose \( \delta = \delta_m = \frac{1}{2} C_2^{-2m-1} \), so that the right side of the last inequality equals
\[
\frac{1}{2} \sum_{m=1}^{\infty} C_2^{-(m+1)^2} \delta_m^2 + 2C \sum_{m=1}^{\infty} C_2^{-m^2+2m+1} C_1^2 (2^{\alpha_3^2} t(Q))^{-\alpha_3^2}.
\]

Choosing now \( C_2 = C_1^2 \), we obtain (2.6), under the a priori assumption that
\[
\sum_{m=1}^{\infty} C_2^{-\alpha_3^2} \delta_m^2 < \infty.
\]

The latter holds if \( A(x) \in C^\infty \), for in that case \( (\partial_t)^{m+1} \nabla_y \Gamma(x, t, y) \) satisfies point-wise bounds analogous to (2.3), possibly depending on the regularization of the coefficients. The constants in (2.6) and in the conclusion of Lemma 2.5 are independent of this regularization.

The proof of estimate (ii) is similar, except that we replace the cut-off function \( \varphi_k \) by \( \varphi \in C^\infty_0 (3Q) \), with \( \varphi \equiv 1 \) on \( 2Q \). We omit the details.

To finish the proof of the lemma, it remains to remove the a priori assumption of smoothness of the coefficients. To this end, fix a cube \( Q \), and let \( g \in C^\infty_0 (1Q) \), \( f \in C^\infty_0 (R_0(Q), C^\infty) \), where \( R_0(Q) \equiv 2Q \), and \( R_0(Q) \equiv 2^{k+1}Q \), \( k \in \mathbb{N} \). It is enough to prove the estimate
\[
|\langle g, (D_{n+1})^{m+1} S_t (\nabla \varphi) \rangle| \leq CC^{m/2} (2^k t(Q))^{-\frac{1}{2}-m-1} ||g||_1 ||\varphi||_1,
\]
with \( t > 0 \), and, when \( k = 0 \), \( t \leq t \leq \rho(t(Q)) \), with the constants depending upon \( \rho \) in the latter situation. The case \( t < 0 \) may be handled by an identical argument, which we omit. We define
\[
A_\varepsilon \equiv P_\varepsilon A \equiv A * \varepsilon \varphi_{\varepsilon},
\]
where \( \varphi_{\varepsilon}(x) \equiv \varepsilon^{-n} \varphi(x/\varepsilon) \), and \( \varphi \in C^\infty_0 (||x| < 1) \) is non-negative and even, with \( \int_{R_0} \varphi = 1 \). Then \( A_\varepsilon \to A \) a.e.. Set
\[
L_\varepsilon := - \nabla A_\varepsilon \nabla,
\]
and let \( \Gamma_\varepsilon \) denote the corresponding fundamental solution. We note that
\[
L_\varepsilon^{-1} - L^{-1} = L_\varepsilon^{-1} L L^{-1} - L_\varepsilon^{-1} L_\varepsilon L^{-1} = L_\varepsilon^{-1} \nabla A_\varepsilon \nabla L^{-1} = L_\varepsilon^{-1} \nabla A \nabla L^{-1}.
\]

We choose a non-negative even cut-off function \( \varphi \in C^\infty_0 (01, 1) \), with \( \int_{-1}^1 \varphi = 1 \). Fix \( t > 0 \) (or \( t \in (\rho^{-1}(Q), \rho(Q)) \) if \( k = 0 \)). For \( \delta > 0 \), set \( \varphi_{\delta}(s) \equiv \delta^{-1} \varphi(s/\delta) \), and define
\[
\tilde{f}_{\delta}(y, s) \equiv \tilde{f}(y) \varphi_{\delta}(s) \quad \tilde{g}_{\delta}(x, \tau) \equiv g(x) \varphi_{\delta}(t - \tau).
\]

Now, fix \( \varepsilon > 0 \) and suppose that \( 0 < \delta < t/8 \). Then for \( |t - t| < \delta \), we have
\[
(D_{n+1})^{m+1} L_\varepsilon^{-1} \nabla \tilde{f}_{\delta}(x, \tau) = \int \int (\partial_x)^{m+1} \Gamma_\varepsilon(x, \tau, y, s) \nabla \tilde{f}(y) \varphi_{\delta}(s) d y d s = \int \varphi_{\delta}(s)(D_{n+1})^{m+1} S_{\varepsilon}(x, \tau) d s,
\]
where \( S_{\varepsilon} \) denotes the single layer potential operator associated to \( L_\varepsilon \). Thus,
\[
|\langle g_{\delta}, (D_{n+1})^{m+1} L_\varepsilon^{-1} \nabla \tilde{f}_{\delta} \rangle| \leq \int \int \varphi_{\delta}(\tau) \varphi_{\delta}(s) \langle g, (D_{n+1})^{m+1} S_{\varepsilon}(x, \tau) \nabla \rangle (\tilde{f}) d s d t \leq CC^{m/2} ||g||_1 ||\tilde{f}||_1 (2^k t(Q))^{-\frac{1}{2}-m-1}.
\]
by the a priori bound obtained for smooth coefficients, since $|\tau + s| < 2\delta \leq t/4$ and $|\phi||1 = 1$. Moreover, 
\[
\langle g_{t,\delta}, (D_{n+1})^{m+1}(L_{x}^{-1} - L_{x}^{-1}) \text{div}_{\delta} \mathcal{F}_{0} \rangle = \langle (D_{n+1})^{m+1} g_{t,\delta}, L_{x}^{-1} \text{div}(A_{x} - A)\nabla L_{x}^{-1} \text{div}_{\delta} \mathcal{F}_{0} \rangle
\]
\[
= \langle \nabla (L_{x}^{-1})^{-1} (D_{n+1})^{m+1} g_{t,\delta}, (A_{x} - A)\nabla L_{x}^{-1} \text{div}_{\delta} \mathcal{F}_{0} \rangle,
\]
which converges to 0 as $\varepsilon \to 0$, for each fixed $\delta > 0$, by dominated convergence, since 
\[
\nabla (L_{x}^{-1})^{-1} (D_{n+1})^{m+1} g_{t,\delta}, \nabla L_{x}^{-1} \text{div}_{\delta} \mathcal{F}_{0} \in L^{2}(\mathbb{R}^{n+1}).
\]
(For the first term, the case $m = -1$ uses that $C_{0}^{\infty} \subset L^{2} \hookrightarrow L_{x}^{2,1}$, where $2r = 2(n+1)/(n+3)$ is the lower Sobolev exponent in $n+1 \geq 3$ dimensions.) Thus, 
\[
\langle g_{t,\delta}, (D_{n+1})^{m+1}L_{x}^{-1} \text{div}_{\delta} \mathcal{F}_{0} \rangle \leq CC_{2}^{m+1/2} |g||\mathcal{F}|_{2} (2^{k}t(\Omega))^{-m-1}.
\]
The conclusion of the lemma now follows from the observation that 
\[
\langle g_{t,\delta}, (D_{n+1})^{m+1}S_{t} \text{div}_{\delta} \mathcal{F} \rangle \text{ is continuous (even } C^{\infty}) \text{ in } (0, \infty).
\]

As a Corollary of the previous two Lemmata we deduce

**Lemma 2.7.** Suppose that $L, L^{*}$ satisfy the standard assumptions, and let $f : \mathbb{R}^{n} \to \mathbb{C}^{n+1}$. Then for every cube $Q$ and for all integers $k \geq 1$ and $m \geq -1$, we have

(i) $\|\partial_{t}^{m+1}(S, \nabla) \cdot (f I_{2^{m+1}Q(n+2)Q})\|_{L^{2}(Q)} \leq CC_{2}^{m+1} 2^{-mk}(2^{k}t(\Omega))^{-2m-2}||f||_{2}^{2}$

(ii) $\|\partial_{t}^{m+1}(S, \nabla) \cdot (f I_{2Q})\|_{L^{2}(Q)} \leq CC_{2}^{m+1} t(\Omega)^{-2m-2}||f||_{2}^{2}$, \( \frac{\rho^{m}}{\rho} < |t| < \rho t(\Omega) \).

**Proof.** We consider estimate (i). Let $x \in Q$. Then 
\[
\|\partial_{t}^{m+1}(S, \nabla) \cdot (f I_{2^{m+1}Q(n+2)Q})\|_{L^{2}(Q)}^{2} \leq \left\| \int_{2^{m+1}Q(n+2)Q} \partial_{t}^{m+1} \nabla S_{t}(x, t, y, s) |_{s=0} \cdot f(y)dy \right\|_{2}^{2}
\]
\[
\leq \left\| \int_{2^{m+1}Q(n+2)Q} \nabla S_{t}(x, t, y, s) |_{s=0} \cdot f(y)dy \right\|_{2}^{2} \leq CC_{2}^{m+1} (2^{k}t(\Omega))^{-2m-2} ||f||_{2}^{2}
\]
where in the last step we have used Lemma 2.5(i) and (2.3). The bound (i) now follows from an integration over $Q$. The proof of (ii) is similar, and is omitted.

**Lemma 2.8.** Suppose that $L, L^{*}$ satisfy the standard assumptions, and let $f : \mathbb{R}^{n} \to \mathbb{C}^{n+1}, f : \mathbb{R}^{n} \to \mathbb{C}$. Then for every $t \in \mathbb{R}$, and for every integer $m \geq 0$, we have

(i) $\|\partial_{t}^{m+1}(S, \nabla) \cdot \nabla S_{t} f\|_{L^{2}(\Omega)} \leq CC_{2}^{m+1/2} ||f||_{2}$

(ii) $\|\partial_{t}^{m+1}(S, \nabla) \cdot \nabla S_{t} f\|_{L^{2}(\Omega)} \leq CC_{2}^{m+1/2} ||f||_{2}$.

**Proof.** Fix $t \in \mathbb{R}$ and $m \geq 0$. It is enough to prove (i), since (ii) follows by duality and the fact that $\text{adj} S_{t} = S_{t}^{*}$, where $S_{t}^{*}$ is the single layer potential operator associated to $L^{*}$. We may further suppose that $t \neq 0$, since otherwise the left hand side of the inequality vanishes. Set $\theta_{t} = t^{m+1} \partial_{t}^{m+1}(S, \nabla)$. We write 
\[
||\theta_{t} f||_{2}^{2} = \left( \sum_{Q} \int_{Q} |\theta_{t} f|^{2} \right)^{1/2} = \left( \sum_{Q} \int_{Q} \int_{Q} |\theta_{t} f|^{2} \right)^{1/2},
\]
where the sum runs over the dyadic grid of cubes with $\ell(Q) \approx |t|$. With $Q$ fixed, we decompose $f$ into $f_{12\Omega}$ plus a sum of dyadic “annular” pieces $(f_{22t-1\Omega},2\Omega)$. The bound (1) now follows from Lemma 2.7. We omit the details.

The next lemma says that

$$ L = L_1 - \sum_{j=1}^{n+1} A_{n+1,j} D_{n+1} D_j - \sum_{j=1}^{n} D_j A_{n+1} D_{n+1} $$

in an appropriate weak sense on each “horizontal” cross-section.

**Lemma 2.9.** Let $L$ satisfy the standard assumptions of this paper. Suppose that $Lu = g$ in the strip $a < t < b$, where $g \in C^0_\omega(\mathbb{R}^{n+1})$. Suppose also that $\nabla u, \nabla \partial_t u \in L^2(\mathbb{R}^n)$, uniformly in $t \in (a, b)$, with norms depending continuously on $t \in (a, b)$. Then for every $F \in L^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and for all $t \in (a, b)$, we have that

$$ \int_{\mathbb{R}^n} A_{ij}(x) \nabla x u(x, t) \nabla x F(x) \, dx = \sum_{j=1}^{n+1} \int_{\mathbb{R}^n} A_{n+1,j}(x) \partial_x \partial_t u(x, t) F(x) \, dx 
- \sum_{j=1}^{n} \int_{\mathbb{R}^n} A_{n+1,j}(x) \partial_j u(x, t) \partial_x F(x) \, dx + \int_{\mathbb{R}^n} g(x, t) F(x) \, dx. $$

(2.10)

**Proof.** Let $t \in (a, b)$, and let $\eta = \min(t - a, b - t)$. Set $\varphi_\eta(s) = \eta^{-1} \varphi(s/\eta)$, where $\varphi \in C^0_\omega(\mathbb{R}, \mathbb{R})$, $0 \leq \varphi$, $\int \varphi = 1$. Define

$$ F_{\varphi \eta}(x, s) = F(x) \varphi_\eta(t - s). $$

Then by the definition of weak solutions, and $t$-independence, we have

$$ \iint A_{ij}(x) \nabla x u(x, s) \nabla x F_{\varphi \eta}(x, s) \, dx \, ds = \sum_{j=1}^{n+1} \iint A_{n+1,j}(x) \partial_x \partial_t u(x, s) F_{\varphi \eta}(x, s) \, dx \, ds 
- \sum_{j=1}^{n} \iint A_{n+1,j}(x) \partial_j u(x, s) \partial_x F_{\varphi \eta}(x, s) \, dx \, ds + \iint g(x, s) F_{\varphi \eta}(x, s) \, dx \, ds. $$

By our hypotheses, the functions of $t$ defined by the four integrals in (2.10), are all continuous in $(a, b)$. The conclusion of the lemma then follows if we let $\eta \to 0$. 

We may now prove a “2-sided” version of Lemma 2.8.

**Lemma 2.11.** Suppose that $L, L^*$ satisfy the standard assumptions, and let $f : \mathbb{R}^n \to \mathbb{C}^{n+1}$. Then for every $t \in \mathbb{R}$, and for every integer $m \geq 0$, we have

$$ ||| f |^m \nabla | \partial_t^{m+1} (S_t \nabla) \cdot f ||_2 \leq C_m ||f||_2. $$

**Proof.** Fix $t \in \mathbb{R}$. We may suppose that $t \neq 0$, since otherwise the left hand side vanishes.

By Lemma 2.8 (ii) and $t$-independence, we may replace $(S_t \nabla) \cdot f$ by $(S_t \nabla) \cdot f' = -S_t \, \text{div} f'$, where $f' \in C^0(\mathbb{R}, \mathbb{C}^n)$. Then it follows from Lemma 2.8 (ii) that

$$ \beta_m(t) \equiv ||| f |^m \nabla | \partial_t^{m+1} (S_t \nabla) \cdot f' ||_2 \leq C \, \beta_m ||f'||_2. $$

This last bound will not appear in our final quantitative estimates. Rather, the point is that the left hand side is a priori finite with some (non-optimal) quantitative control.
Thus, taking \(14\) and Lemma \(2.8\) \((i)\), we have that

\[
\beta_m(t) \leq C^{2m+4} Re \left( A_{ij} \nabla_i (\partial_t^{m+1} S_j) \nabla_i f \right) \\
= CRe \sum_{j=1}^{m+1} \langle A_{ij+1,j} \partial_i^{m+2} S_j, \nabla_i f \rangle \\
- CRe \sum_{j=1}^{m} \langle A_{ij+1,i} \partial_i^{m+2} S_j, \nabla_i f \rangle \\
\leq C\delta^{-1} C_2^{-m} \|f\|^2_{L^2} + C\delta C_2^{-m} \|f\|^2_{L^2} + C\delta \beta_{m+1}(t) + C\delta^{-1} C_2^{-m+1} \|f\|^2_{L^2} + C\delta \beta_m(t),
\]

where \(\delta\) is at our disposal. Choosing \(\delta\) small enough, we may hide the last term, so that

\[
\beta_m(t) \leq C\delta^{-1} C_2^{-(m+1)} \|f\|^2_{L^2} + C\delta \beta_{m+1}(t).
\]

Thus, taking \(\delta = \delta_0 C_2^{-2m}\), with \(\delta_0\) small, we have

\[
\sum_{m=0}^{\infty} C_2^{-3m} C_2^{-m+2} \beta_m(t) \leq C \sum_{m=0}^{\infty} C_2^{-3m} \left( \delta_0^{-1} \|f\|^2_{L^2} + C C_2^{-m} \delta_m \beta_{m+1}(t) \right) \\
\leq C \sum_{m=0}^{\infty} \left( \delta_0^{-1} C_2^{-m} \|f\|^2_{L^2} + \delta_0 C_2^{-3m} \beta_{m+1}(t) \right) \\
\leq C \|f\|^2_{L^2} + \frac{1}{2} \sum_{m=1}^{\infty} C_2^{-3m} C_2^{-m+2} \beta_m(t),
\]

by choice of \(\delta_0\) small enough. By (2.12), the series converges, so the last term may be hidden on the left side of the inequality. In particular, we conclude that

\[
\beta_m(t) \leq CC_2^{-(m+1) + 3m} \|f\|^2_{L^2}.
\]

\(\square\)

**Lemma 2.13.** Suppose that \(L, L^*\) satisfy the standard assumptions. Fix a cube \(Q \subset \mathbb{R}^n\), and suppose that \(y, y' \in Q\). For \((x, t) \in \mathbb{R}^{n+1}\), set

\[
u(x, t) \equiv \Gamma(x, t, y, 0) - \Gamma(x, t, y', 0).
\]

If \(\alpha\) is the Hölder exponent in (2.4), then for every integer \(k \geq 4\), we have

\[
(2.14) \quad \int_{Q \cap \mathbb{R}^n} |\nabla \nu(x, t)|^2 dx \leq C2^{-k \alpha} \left( 2^{k \ell(Q)} \right)^{-n}.
\]

**Proof.** By (2.4), it is enough to prove (2.14) with \(\nabla_x\) in place of \(\nabla\). Let \(\varphi_k \in C_0^\infty(3 \cdot 2^k Q \setminus 3 \cdot 2^{k-2} Q)\), with \(\varphi_k \equiv 1\) on \(2^{k+1} Q \setminus 2^k Q\) and \(\|\nabla_x \varphi_k\|_\infty \leq C(2^{k \ell(Q)})^{-1}\). Then the left hand side of (2.14) is bounded by an acceptable term involving a \(t\) derivative, plus

\[
\int |\nabla_x \nu(x, t)|^2 (\varphi_k(x))^2 dx \leq CRe \int A_{ij} \nabla_i u \cdot \nabla_j \varphi_k^2 \\
= CRe \int A_{ij} \nabla_i u \cdot \nabla_j \left( \varphi_k \right) - CRe \int A_{ij} \nabla_i u \cdot \nabla_j \left( \varphi_k^2 \right) \equiv I + II.
\]

By Lemma 2.2, for \(y, y' \in Q\) and \(x \in (2^k Q)^c\), we have

\[
(2.15) \quad |\nu(x, t)| \leq C2^{-k \alpha} \left( 2^{k \ell(Q)} \right)^{1-n}.
\]

and also

\[
(2.16) \quad |\partial_t \nu(x, t)| \leq C2^{-k \alpha} \left( 2^{k \ell(Q)} \right)^{-n}.
\]
Using the first of these, we obtain
\[ |I| \leq C 2^{-ka} \left(2^k \ell(Q)\right)^{1-n} \|\nabla u\|_{\infty} \int |\nabla u|^2 \varphi_k \]
\[ \leq C 2^{-ka} \left(2^k \ell(Q)\right)^{1/2} \left(\int |\nabla u|^2 \varphi_k^2\right)^{1/2} \leq C \varepsilon^{-1} 2^{-2ka} \left(2^k \ell(Q)\right)^{-n} + \varepsilon \int |\nabla u|^2 \varphi_k^2, \]
where \( \varepsilon \) is at our disposal. Moreover, by Lemma 2.9,
\[ I = -C \Re \sum_{j=1}^{m} \left\{ \int A_{i,j+1} \partial u D_j u \varphi_k^2 + \int A_{i,j+1} \partial u D_j \varphi_k^2 \right\} \]
\[ + C \Re \sum_{j=1}^{n+1} \int A_{n+1,j} D_j \varphi_k^2 \equiv I_1 + I_2 + I_3. \]
Now, \( I_1 \) satisfies exactly the same bound as term \( II \), and by essentially the same argument, except that we use (2.16) in place of (2.15). Moreover, using (2.15), (2.16), and the properties of \( \varphi_k \), we see that
\[ |I_2| \leq C 2^{-2ka} \left(2^k \ell(Q)\right)^{-n}. \]
To handle \( I_3 \), we note first that the case \( m = 0 \) of Lemma 2.5(i) (with the roles of \( x \) and \( y \) reversed), applied separately for \( y \) and \( y' \), implies that
\[ \int |\partial u \nabla u(x,t)|^2 \varphi_k^2 dx \leq C \left(2^k \ell(Q)\right)^{-n-2}. \]
Thus, using also (2.15), we have
\[ |I_3| \leq C 2^{-2ka} \left(2^k \ell(Q)\right)^{-n}. \]
Collecting these estimates, choosing \( \varepsilon \) sufficiently small, and hiding the small term on the left hand side of the inequality, we obtain the desired bound.

In the sequel, we shall find it useful to consider approximations of the single layer potential. The bounds in the following lemma will not be used quantitatively, but will serve rather to justify certain formal manipulations. For \( \eta > 0 \), set
\[ (2.17) \quad S_\eta \equiv \int_R \varphi(\eta - s) S_s \, ds, \]
where \( \varphi, \tilde{\varphi}, \varphi_\eta \in C^\infty_0(-\eta/2, \eta/2) \) is non-negative and even, with \( \int \varphi = 1 \) and \( \tilde{\varphi}(t) \equiv \eta^{-1} \varphi(t/\eta) \).

**Lemma 2.18.** Suppose that \( L, L^* \) satisfy the standard assumptions, and let \( S_t \) denote the single layer potential operator associated to \( L \). Then for each \( \eta > 0 \) and for every \( f \in L^2(\mathbb{R}^n) \) with compact support, we have
\[ \begin{align*}
(\text{i}) & \quad \|\partial_t S_\eta f\|_{L^2(\mathbb{R}^n)} \leq C_{\eta,0}\|f\|_{L^{20\eta+20}(\mathbb{R}^n)}, \quad 0 < \beta < 1. \\
(\text{ii}) & \quad \|\nabla S_\eta f\|_{L^2(\mathbb{R}^n)} \leq C_{\eta,0}\|f\|_{L^{20\eta+20}(\mathbb{R}^n)}, \\
(\text{iii}) & \quad \|\partial_t^2 S_\eta f\|_{L^2(\mathbb{R}^n)} \leq C_{\eta,0}\|f\|_{L^{20\eta+20}(\mathbb{R}^n)}, \quad 0 < \beta < 1. \\
(\text{iv}) & \quad \|\nabla \left( S_\eta - S_{\eta/2} \right) f\|_{L^2(\mathbb{R}^n)} \leq C_{\eta,0}\|f\|_{L^2}, \quad \eta < |t|/2. \\
(\text{v}) & \quad \lim_{\eta \to 0} \int_0^\infty \int_{\mathbb{R}^n} |\nabla \partial_t \left( S_\eta - S_{\eta/2} \right) f|^2 \, dt \, dx = 0, \quad 0 < \eta < 1. \\
(\text{vi}) & \quad \text{For each cube } Q \subset \mathbb{R}^n, \quad \|\partial_t S_\eta f\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C_{\eta,\delta} \ell(Q). 
\end{align*} \]
Proof. (i). We observe that

$$\partial_t S^\eta f(x) = \int_{\mathbb{R}^n} k_t(x,y) f(y) dy,$$

where $k_t(x,y) \equiv \partial_t (\varphi_\eta * \Gamma(x,\cdot,y,0))(t)$. Thus, by Lemma 2.2

$$|k_t(x,y)| \leq C \min \{ |x-y|^{-n}, \eta^{-1} |x-y|^{1-n} \} \leq C \eta^{-\beta} |x-y|^{\beta-n}, \quad 0 < \beta < 1.$$  

Estimate (i) now follows by the fractional integral theorem.

(ii). We first note that

$$S^\eta f(x) = \int \int \Gamma(x, t - s - \sigma, y, 0) f(y) dy \varphi_\eta(s) \varphi_\eta(\sigma) d\sigma ds$$

where $f(y,s) \equiv f(y) \varphi_\eta(s)$. Let $\tilde{g} \in C_0^\infty(\mathbb{R}^n, C^\omega)$, with $\|\tilde{g}\|_2 = 1$, and set $\tilde{g}_\eta(x,\sigma) \equiv \tilde{g}(x) \varphi_\eta(\sigma)$. Then

$$|\langle \tilde{g}, \nabla S^\eta f \rangle| = \left| \int \int \text{div} \tilde{g}_\eta(x,\sigma) (L^{-1} f_\eta)(x, t - \sigma) d\sigma dx \right|$$

$$\leq \|\tilde{g}_\eta\|_{L^2(\mathbb{R}^{n+1})} \|\nabla L^{-1} f_\eta\|_{L^2(\mathbb{R}^{n+1})} \leq C \eta^{-1/2} \|f_\eta\|_{L^2(\mathbb{R}^{n+1})} \leq C \eta^{-1/2} \|\varphi_\eta\|_{L^1(\mathbb{R})} \|f\|_{L^1(\mathbb{R})},$$

where $2_+ = (2n + 2)/(n + 3)$, since $L^2(\mathbb{R}^{n+1}) \hookrightarrow L^2_{2_+}(\mathbb{R}^{n+1}) \equiv (L^2(\mathbb{R}^{n+1}))^\prime$, and $\nabla L^{-1} \text{div} : L^2(\mathbb{R}^{n+1}) \to L^1(\mathbb{R}^{n+1})$. Estimate (ii) now follows.

(iii). We proceed as for estimate (i), and write

$$t \partial_t^2 S^\eta f(x) = \int_{\mathbb{R}^n} h_t(x, y) f(y) dy,$$

where $h_t(x,y) \equiv t \partial_t (\varphi_\eta * \Gamma(x,\cdot,y,0))(t)$, so that, by Lemma 2.2,

$$|h_t(x,y)| \leq Ct \min \{ |x-y|^{-n-1}, \eta^{-2} |x-y|^{1-n} \} \leq Ct \eta^{-1-\beta} |x-y|^{\beta-n}, \quad 0 < \beta < 1.$$  

Moreover, if $t > 2\eta$, we have the sharper estimate

$$|h_t(x,y)| \leq C \frac{t}{(t + |x-y|)^{n+1}} \leq Ct^{-\beta} |x-y|^{\beta-n}, \quad 0 < \beta < 1.$$  

Thus,

$$\|t \partial_t^2 S^\eta f\|_2^2 \leq C \left( \int_0^{2\eta} \eta^{-2-2\beta} t^2 dt + \int_{2\eta}^{\infty} t^{-1-2\beta} dt \right) \|f\|_{L^2(\mathbb{R}^{n+1})},$$

and (iii) follows.

(iv). We suppose that $\eta < |t|/2$. Then

$$\|\nabla (S^\eta - S_t) f\|_{L^2(\mathbb{R}^n)} \leq \varphi_\eta * \|\nabla (S_{t-} - S_t) f\|_{L^2(\mathbb{R}^n)},$$

But for $|s - t| < \eta < |t|/2$, we have by the mean value theorem and Lemma 2.8(ii) that

$$\|\nabla (S_s - S_t) f\|_{L^2(\mathbb{R}^n)} \leq \frac{\eta}{|t|} \sup_{|r-s|<|t|/2} \|\nabla \partial_r S_r f\|_{L^2(\mathbb{R}^n)} \leq C \frac{\eta}{|t|} \|f\|_2.$$
(v). We take \( \eta < \varepsilon/2 \), and write

\[
\int_0^\infty \int_{\mathbb{R}^n} |\nabla \partial_1 \left( S_\nu^t - S_\nu^1 \right) f| \frac{dx \, dt}{t} = \int_0^\infty \int_{\mathbb{R}^n} |\varphi_\eta * t\nabla D_{n+1} \left( S_\nu^t - S_\nu^1 \right) f| \frac{dx \, dt}{t} \leq \int_0^\infty \varphi_\eta * |t\nabla D_{n+1} \left( S_\nu^t - S_\nu^1 \right) f| \frac{dt}{t}
\]

(2.19)

We claim that the last expression converges to 0, as \( \eta \to 0 \). Indeed, for \(|s - t| < \eta < \eta/2\), we have that

\[
|t\nabla D_{n+1} \left( S_\nu^t - S_\nu^1 \right) f| \lesssim \eta \sup_{|s - t| < \eta/2} \|r\nabla^2 \partial_2 \mathcal{S}_r f\|_{L^2(\mathbb{R}^n)} \leq C_\eta \eta^{-2} \|f\|_2
\]

by Lemma 2.8(ii). Thus, for \( \eta < \varepsilon/2 \), (2.19) is bounded by \( C\eta^2 \varepsilon^{-2} \|f\|_2^2 \), and the claim follows.

(vi). Estimate (vi) follows from (i) and Hölder’s inequality. \( \square \)

3. Some consequences of “off-diagonal” decay estimates

Here, we prove some estimates that hold in general for operators satisfying the conclusions of Lemmas 2.7 and 2.8. For the sake of notational convenience, we observe that part (i) of the former conclusion can be reformulated as

\[
\|\theta_t(f 1_{2^{k+1}Q} \mathcal{S} f)\|_{L^2(Q)}^2 \lesssim C_m 2^{-nk} \left( \frac{|t|}{2^{k} \mathcal{L}(Q)} \right)^{2m+2} \|f\|_{L^2(2^{k+1}Q)}^2,
\]

where \( \theta_t = e^{it \partial_1} \theta_{2^{-k+1}}(S, \nabla) \). We now consider generic operators \( \theta_t \), which satisfy (3.1) for some integer \( m \geq 0 \). The next lemma is essentially due to Fefferman and Stein [FS]. We omit the well known proof.

**Lemma 3.2.** Suppose that \( \{\theta_t\}_{t \in \mathbb{R}} \) is a family of operators which satisfies (3.1) for some integer \( m \geq 0 \) and in every cube \( Q \), whenever \(|t| \leq C \mathcal{L}(Q)\). If \( \|\theta_t\|_{\text{op}} \leq C \), then

\[
\left| \int \theta_t b(x) \frac{dx \, dt}{|t|} \right| \lesssim C |b| C_m.
\]

**Lemma 3.3.** Suppose that \( \{\theta_t\}_{t \in \mathbb{R}} \) is a family of operators satisfying (3.1) for some integer \( m \geq 0 \), as well as the bound

\[
\sup_{t \in \mathbb{R}} \|\theta_t f\|_{L^2(\mathbb{R}^n)} \lesssim C \|f\|_2.
\]

Suppose that \( \{\Lambda_t\}_{t \in \mathbb{R}} \) is a family of operators satisfying the bounds

\[
\sup_{t \in \mathbb{R}} \|\Lambda_t f\|_2 \leq C \|f\|_2, \quad \|\Lambda_t f\|_{L^2(E')} \leq C \exp \left( \frac{-\text{dist}(E, E')}{C |t|} \right) \|f\|_{L^2(E')}
\]

whenever (in the latter estimate) support \( f \subseteq E' \). Then \( \theta_t \Lambda_t \) also satisfies (3.1), whenever \(|t| \leq C \mathcal{L}(Q)\).

**Proof.** We may suppose that \( k \geq 4 \), otherwise, subdivide \( Q \) dyadically to reduce to this case. Given \( Q \), set \( Q' = 2^{k-2} Q \). Then

\[
\theta_t \Lambda_t = \theta_1 \mathcal{Q} \Lambda_t + \sum_{r \in \mathcal{Q} \setminus \overline{Q}} \Theta r \Lambda_t.
\]

For the first term, we have the bound

\[
\|\theta_t \mathcal{Q} \Lambda_t(f 1_{2^{k+1}Q} \mathcal{S} f)\|_{L^2(Q)} \leq \|\theta_t\|_{L^2} \|\Lambda_t(f 1_{2^{k+1}Q} \mathcal{S} f)\|_{L^2(Q)} \leq \|\theta_t\|_{L^2} \exp \left( \frac{-2^k \mathcal{L}(Q)}{C |t|} \right) \|f\|_{L^2(2^{k+1}Q, 2^k Q)}
\]

and the second term
which in particular yields (3.1) for this term, if $|t| \leq C|Q|$. Next, we consider the second term in (3.4), which equals

$$\sum_{j=2}^{k-2} \theta_j 1_{2^j Q \cap Q} A_j.$$

The desired bound follows for this term by applying (3.1) for each $j$ fixed, and summing the resulting geometric series.

**Lemma 3.5.** (i) Suppose that $\{R_t\}_{t \in \mathbb{R}}$ is a family of operators satisfying (3.1), for some $m \geq 1$, and for all $|t| \leq C|Q|$, and suppose also that $\sup_{R} \|R\|_{L^2 \rightarrow L^2} \leq C$, and that $R_1 \equiv 0$ for all $t \in \mathbb{R}$ (our hypotheses allow $R_1$ to be defined as an element of $L^2_{\text{loc}}$). Then for $h \in L^2_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |R_t h|^2 \leq C t^2 \int_{\mathbb{R}^n} |\nabla x h|^2,$$

(ii) If, in addition, $\|R_t \text{ div } x\|_{L^2 \rightarrow L^2} \leq C/t$, then also

$$\|R_t f\| \leq C \|f\|_2.$$

**Proof.** We suppose that $t > 0$, and show that (3.6) implies (3.7). The latter follows from

$$\|R_t (s^2 \Delta e^{-\lambda t})\|_{L^2 \rightarrow L^2} \leq C \min \left( \frac{t}{s}, \frac{s}{t} \right),$$

by a standard orthogonality argument. In turn, (3.8) is easy to prove: the case $t < s$ is just (3.6), and the case $s < t$ follows by hypothesis from the factorization $\Delta = \text{div } x \nabla x$.

We now turn to the proof of (3.6). Let $D(t)$ denote the grid of dyadic cubes with $|t| \leq 2\ell(Q)$. For convenience of notation we set $m_Q h \equiv \int_Q h$. Then

$$\left( \int_{\mathbb{R}^n} |R_t h|^2 \right)^{\frac{1}{2}} = \left( \sum_{Q \in D(t)} \int_Q |R_t h|^2 \right)^{\frac{1}{2}} = \left( \sum_{Q \in D(t)} \int_Q |R_t(h - m_Q h)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{Q \in D(t)} \int_Q |R_t(h - m_Q h)^1_{2Q}|^2 \right)^{\frac{1}{2}} + \left( \sum_{Q \in D(t)} \int_Q |R_t(h - m_Q h)^1_{2Q}|^2 \right)^{\frac{1}{2}} \equiv I + II.$$

Since $R_t : L^2 \rightarrow L^2$, we have by Poincaré’s inequality that

$$I \leq C \left( \sum_{Q \in D(t)} \int_{2Q} |h - m_Q h|^2 \right)^{\frac{1}{2}} \leq C |t| \left( \sum_{Q \in D(t)} \int_{2Q} |\nabla x h|^2 \right)^{\frac{1}{2}} \leq C |t| \|\nabla x h\|_2.$$

Moreover, we are given that $R_t$ satisfies (3.1). Thus,

$$II \leq \sum_{k=1}^{\infty} \left( \sum_{Q \in D(t)} \int_Q |R_t(h - m_Q h)^1_{2^k Q \cap Q}|^2 \right)^{\frac{1}{2}} \leq C \sum_{k=1}^{\infty} \left( \sum_{Q \in D(t)} 2^{-k(n+1)} \int_{2^k Q} |h - m_{2^k Q} h|^2 \right)^{\frac{1}{2}} \leq C \sum_{k=1}^{\infty} \sum_{j=1}^{k} \left( \sum_{Q \in D(t)} 2^{-4k} 2^{-j} \int_{2^j Q} |h - m_{2^j Q} h|^2 \right)^{\frac{1}{2}},$$

where in the last step we have used that

$$h - m_{2^k Q} h = h - m_{2^k Q} h + m_{2^k Q} h - m_{2^k Q} h + \cdots + m_{4Q} h - m_{2Q} h.$$
By Poincaré’s inequality, since \( j \leq k \) we obtain in turn the bound

\[
C|t|^{2-k} \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left( \sum_{Q \in \mathbb{D}_j(t)} \int_{2^j Q} |\nabla h|^2 \right)^{\frac{1}{2}} \leq C|t|^{2-k} \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left( \int_{2^j Q} \int_{2^{j+1} Q} |\nabla h|^2 \right)^{\frac{1}{2}} \leq C|t|^{2-k} \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left( \int_{\mathbb{R}^2} \int_{1-|y| \leq 2^j R} |\nabla h(x)|^2 \, dx \, dy \right)^{\frac{1}{2}} = C|t|\|\nabla h\|_2.
\]

\[\Box\]

**Lemma 3.9.** Given \( \{R_i\}_{i \in \mathbb{N}_+} \) as in part (i) of the previous lemma, we have that

\[
\|r^{-1} R_i F\| \leq C\|\nabla_i F\|_{L^2(\mathbb{R}^3)},
\]

provided that \( \|r R_i \Phi(x)\|^2 \frac{d\omega}{d\mu} \) is a Carleson measure, where \( \Phi(x) \equiv x \).

**Proof.** We may assume that \( F \in C_0^\infty \), and that \( t > 0 \). Let \( \mathbb{D}_j \) denote the dyadic grid of cubes of side length \( 2^{-j} \). Then

\[
\|r^{-1} R_i F\|^2 = \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{D}_j} \int_Q \left| r^{-1} R_i F(y) \right|^2 \, dy \, \frac{dt}{t} = \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{D}_j} \int_Q \int_{\mathbb{R}^2} \left| r^{-1} R_i F(y) \right|^2 \, dy \, \frac{dt}{t}.
\]

We now use an idea taken from [J] and [Ch2, pp. 32-33]. For \( (x, t) \) fixed, set

\[
G_{x,t}(z) \equiv F(z) - F(x) - (z-x) \cdot P_t(\nabla F)(x),
\]

where as usual \( P_t \) is an approximate identity. Since \( R_i 1 = 0 \), we have, for any fixed \( x \),

\[
\frac{1}{t} R_i F(y) = \frac{1}{t} R_i (G_{x,t})(y) + \frac{1}{t} R_i \Phi(y) \cdot P_t(\nabla F)(x) \equiv I + II.
\]

The contribution of \( II \) to (3.10) is bounded by

\[
\sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{D}_j} \int_{\mathbb{R}^2} \left| P_t(\nabla F)(x) \right|^2 \int_Q \int_{\mathbb{R}^2} \left| \frac{1}{t} R_i \Phi(y) \right|^2 \, dy \, \frac{dt}{t} \leq C \int_{\mathbb{R}^2} \left| P_t(\nabla F)(x) \right|^2 \left( \int_{\mathbb{R}^2} \left| \frac{1}{t} R_i \Phi(y) \right|^2 \, dy \right) \, dx \, \frac{dt}{t} \leq C\|\nabla F\|_{L^2(\mathbb{R}^3)}^2 \|\mu\|_C,
\]

by Carleson’s Lemma, where

\[
\|\mu\|_C \equiv \sup_Q \int_Q \left( \int_{\mathbb{R}^2} \left| \frac{1}{t} R_i \Phi(y) \right|^2 \, dy \right) \, dx \, \frac{dt}{t} \leq C \sup_Q \int_Q \left( \int_{|y| \leq C} \left| \frac{1}{t} R_i \Phi(y) \right|^2 \, dy \right) \, dx \, \frac{dt}{t} \leq C \sup_Q \int_Q \left( \int_{|y| \leq C} \left| \frac{1}{t} R_i \Phi(y) \right|^2 \, dy \right) \, dx \, \frac{dt}{t}.
\]

Next we consider the contribution of \( I \) to (3.10). For \( Q \in \mathbb{D}_j \), and \( x \in Q \), we have

\[
I = R_i \left( r^{-1} G_{x,t} 1_{2Q} \right)(y) + \sum_{k=1}^{\infty} R_i \left( r^{-1} G_{x,t} 1_{2^{k+1} Q \cap 2^k Q} \right)(y) \equiv I_0 + \sum_{k=1}^{\infty} I_k.
\]
Since \( R_t : L^2 \to L^2 \), we obtain the bound

\[
\|I_0\|^2 \leq C \sum_{j=\infty}^{\infty} \sum_{Q \ni 2^j, 2^j} \int_0^{2^j} \int_0^{2^j} \int_0^{2^j} \int_0^{2^j} \frac{|G_{x,t}(y)|^2}{t^2} \, dy \, dx \, dt
\]

\[
\leq C \int_0^{\infty} \int_{\mathbb{R}^2} (\beta(x,t)) \frac{z \, dx \, dt}{t} \leq C \|\nabla_q F\|_{L^2(\mathbb{R}^n)},
\]

where \((\beta(x,t))^2 = \int_{|y| < t} r^{-2} |G_{x,t}(y)|^2 \, dy\), and where the last step is a well known consequence of Plancherel’s Theorem, see, e.g. [Ch2, pp. 32-33] or [H, pp. 249-250]. Furthermore, since \( R_t \) satisfies (3.1) for some \( m \geq 1 \), whenever \( t \approx \ell(Q) \), we have that

\[
C^{-1} \sum_{k=1}^{\infty} ||I_k|| \leq \sum_{k=1}^{\infty} \left( \sum_{j=\infty}^{\infty} \sum_{Q \ni 2^j, 2^j} \int_0^{2^j} \int_0^{2^j} \int_0^{2^j} \int_0^{2^j} \frac{|G_{x,t}(y)|^2}{t^2} \, dy \, dx \, dt \right) \]

\[
= \sum_{k=1}^{\infty} 2^{-k} \left( \int_0^{\infty} \int_{\mathbb{R}^2} \int_{|y| \leq 2^{-k} \ell} \frac{|G_{x,t}(y)|^2}{t^2} \, dy \, dx \, dt \right) \]

where, after making the change of variable \( t \to t/2^k \),

\[
\beta_k(x,t) = \left( \int_{|y| \leq \ell} |F(y) - F(x) - P_{2^{-k}}(\nabla_q F)(x)|^2 \, dy \right)^{1/2}.
\]

We now claim that \(||\beta_k|| \leq C \sqrt{\ell \|\nabla_q F\|_{L^2}}\), from which the conclusion of the lemma trivially follows. By Plancherel’s Theorem, the definition of \( P_k \), and the change of variable \( x - y = h \) we have

\[
||\beta_k||^2 = \int_0^{\infty} \int_{|y| < \ell} \int_{\mathbb{R}^n} \frac{|e^{i \xi \cdot h} - 1 - (ih \cdot \xi) \hat{\phi}(2^{-k} \xi)|^2}{t^2 |\xi|^2} \, d\xi \, dh \, dt.
\]

where \( \phi \in C_0^\infty(|x| < 1) \) and \( \int \phi = 1 \). By the change of variable \( h \to th \), we have

\[
||\beta_k||^2 = \int_0^{\infty} \int_{|y| < \ell} \int_{\mathbb{R}^n} \frac{|e^{i \xi \cdot h} - 1 - (ih \cdot \xi) \hat{\phi}(2^{-k} \xi)|^2}{t^2 |\xi|^2} \, d\xi \, dh \, dt.
\]

Since \( \hat{\phi} \in \mathcal{S} \) and \( \hat{\phi}(0) = 1 \), we have that

\[
|e^{i \xi \cdot h} - 1 - (ih \cdot \xi) \hat{\phi}(2^{-k} \xi)| \leq C \min \left( |\xi|, 1, \frac{2^k}{|\xi|} \right)
\]

Indeed, if \( |\xi| \leq 1 \), then

\[
|e^{i \xi \cdot h} - 1 - (ih \cdot \xi) \hat{\phi}(2^{-k} \xi)| \leq \frac{|e^{i \xi \cdot h} - 1 - ih \cdot \xi|}{|\xi|} + \frac{|ih \cdot \xi (1 - \hat{\phi}(2^{-k} \xi))|}{|\xi|}
\]

\[
\leq C (|\xi| + 2^{-k} |\xi|) \leq C |\xi|.
\]

On the other hand, if \( |\xi| > 1 \), then

\[
|e^{i \xi \cdot h} - 1| \leq \frac{2}{|\xi|},
\]

and

\[
|ih \cdot \xi \hat{\phi}(2^{-k} \xi)| \leq C |\hat{\phi}(2^{-k} \xi)| \leq \frac{C}{1 + 2^{-k} |\xi|} \leq C \min \left( 1, \frac{2^k}{|\xi|} \right).
\]

We then obtain the bound \(||\beta_k||^2 \leq C \sqrt{\ell \|\nabla_q F\|_{L^2}}\) as claimed. \(\square\)
Lemma 3.11. Suppose that $\theta_i$ satisfies (3.1) for some $m \geq 0$, whenever $0 < t \leq C/t(Q)$ and that $||\theta_i||_{2-2} \leq C$. Let $b \in L^\infty(\mathbb{R}^n)$, and let $\mathcal{A}_i$ denote a self-adjoint averaging operator whose kernel $\varphi_i(x, y)$ satisfies $|\varphi_i(x, y)| \leq C r^{-n} 1_{|x-y|<Ct}$, $\varphi_i \geq 0$, $\int \varphi_i(x, y) dy = 1$. Then

$$\sup_r ||(\theta_i b)\mathcal{A}_i f||_2 \leq C ||b||_\infty ||f||_2.$$

Proof. Since we do not assume that $\theta_i : L^\infty \to L^\infty$, this requires a bit of an argument. Observe that

$$||(\theta_i b)\mathcal{A}_i f||_2^2 \leq ||f||_2 ||\mathcal{A}_i (\theta_i b^2)\mathcal{A}_i f||_2 \leq ||f||_2 ||\mathcal{K}_i(x, \cdot)||_{2'}(\mathbb{R}^n),$$

where $\mathcal{K}_i(x, y)$ is the kernel of the self-adjoint operator $f \to \mathcal{A}_i (\theta_i b^2)\mathcal{A}_i f$, i.e.,

$$\mathcal{K}_i(x, y) = \int_{\mathbb{R}^n} \varphi_i(x, z)\theta_i b(z)^2 \varphi_i(z, y) dz.$$ 

Consequently,

$$||\mathcal{K}_i(x, \cdot)||_2 = \int_{\mathbb{R}^n} \varphi_i(x, z)|\theta_i b(z)|^2 dz \leq C r^{-n} \int_{|x-z|<Ct} |\theta_i b(z)|^2 dz.$$

Thus, by (3.1) and the fact that $\theta_i$ is bounded on $L^2$ uniformly in $t$, we have that

$$||\mathcal{K}_i(x, \cdot)||_2^2 \leq C \left\{ \left( \int_{Q(x, 2\varrho \cdot Ct)} |b|^2 \right)^{1/2} + \sum_{k=2}^\infty 2^{-k} \left( \int_{Q(x, 2^{k+1} \varrho \cdot Ct)} |b|^2 \right)^{1/2} \right\} \leq C ||b||_\infty,$$

where $Q(x, Rt)$ is the cube centered at $x$ with side length $Rt$. This proves the lemma. \qed

Lemma 3.12. Suppose that

$$\Omega_\delta = \int_0^\varrho \left( \frac{s}{\varrho} \right)^\delta W_{s, \varrho} \theta_\delta \frac{ds}{s}.$$

for some $\delta > 0$, where $sup_{s} ||W_{s, \varrho}||_{2-2} \leq C$. Then

$$||\Omega_\delta||_{op} \leq C ||\theta_\delta||_{op}.$$

Proof. This is a standard Schur type argument. Indeed, if $||G(x, t)|| \leq 1$, then

$$\left| \int_0^\infty \int_{\mathbb{R}^n} G(x, t) \theta f(x) dx dt \right| = \left| \int_0^\infty \int_0^\infty \int_{\mathbb{R}^n} 1_{|x-y|<Ct} \left( \frac{s}{t} \right)^\delta G(x, t) \theta f(x) dx dt ds \right|$$

$$\leq \left( \int \int_{\mathbb{R}^n} |G(x, t)|^2 \int_0^\varrho \left( \frac{s}{\varrho} \right)^\delta \frac{ds dx dt}{t} \right)^{1/2} \left( C \int_0^\infty \int_{\mathbb{R}^n} |\theta f(x)|^2 \int_0^\infty \left( \frac{s}{t} \right)^\delta \frac{dt dx ds}{t} \right)^{1/2}$$

$$\leq C ||\theta f||_2.$$

\qed

4. Traces, Jump relations, and Uniqueness

We begin by proving a useful technical lemma.

Lemma 4.1. Let $L, L'$ satisfy the standard assumptions. Suppose that $Lu = 0$ and that $\mathcal{N}_\varrho (\nabla u) \in L^2(\mathbb{R}^n)$. Then

$$\sup_{t > 0} ||\nabla u(\cdot, t)||_2 \leq C ||\mathcal{N}_\varrho (\nabla u)||_2.$$
Proof. The desired bound for \( \partial_t u \) follows readily from \( t \)-independence and (1.3). Thus, we need only consider \( \nabla_i u \). Let \( \tilde{\psi} \in C_0^\infty(\mathbb{R}^n) \), with \( \|\tilde{\psi}\|_2 = 1 \). For \( t_0 > 0 \) fixed, it will then be enough to establish the bound
\[
\left| \int_{\mathbb{R}^n} u(\cdot, t_0) \text{div}_x \tilde{\psi} \right| \leq C\|\tilde{N}_x(\nabla u)\|_2.
\]
To this end, we write
\[
\int_{\mathbb{R}^n} u(\cdot, t_0) \text{div}_x \tilde{\psi} = \int_{\mathbb{R}^n} \left( u(x, t_0) - \int_{t_0/2}^{t_0} u(x, t) \, dt \right) \text{div}_x \tilde{\psi}(x) \, dx
\]
\[
+ \int_{\mathbb{R}^n} \int_{t_0/2}^{t_0} u(x, t) \text{div}_x \tilde{\psi}(x) \, dt \, dx = I + II.
\]
We first observe that
\[
|I| = \int_{\mathbb{R}^n} \int_{t_0/2}^{t_0} \left( \int_{|t-j|<\epsilon} \nabla_i u(x, t) \tilde{\psi}(x) \, dx \right) \, dt \, dx \leq C\|\tilde{N}_x(\nabla u)\|_2,
\]
by Cauchy-Schwarz and Fubini’s Theorem. Moreover,
\[
|I| = \int_{\mathbb{R}^n} \int_{t_0/2}^{t_0} \int_{s}^{t_0} \partial_t u(x, s) \, ds \text{div}_x \tilde{\psi}(x) \, dt \, dx
\]
\[
\leq C \int_{t_0/2}^{t_0} \int_{\mathbb{R}^n} \nabla_x \partial_t u(x, s) \tilde{\psi}(x) \, dx \, ds \left( \int_{t_0/2}^{t_0} \left( \nabla \partial_t u(x, s) \right)^2 \, ds \right)^{1/2}
\]
\[
\leq C \int_{t_0/2}^{t_0} \int_{\mathbb{R}^n} \left( \nabla \partial_t u(x, s) \right)^2 \, ds \, dx \left( \int_{t_0/2}^{t_0} \left( \nabla \partial_t u(x, s) \right)^2 \, ds \right)^{1/2},
\]
where in the last step we have split \( \mathbb{R}^n \) into cubes of side length \( \approx t_0 \) and used Caccioppoli’s inequality. The conclusion of the lemma follows since the bound already holds for \( \partial_t u \). □

We now discuss some trace results. The following lemma is the analogue of Theorem 3.1 of [KP]. We recall that \( u \to f \text{ n.t.} \) means that \( \lim_{(y,t)\to(x,0)} u(y,t) = f(x) \), for a.e. \( x \in \mathbb{R}^n \), where the limit runs over \( (y,t) \in \gamma(x) \). As usual, \( P_\epsilon \) will denote a self-adjoint approximate identity acting in \( \mathbb{R}^n \). We shall denote by \( W^{1,2}_\epsilon \) the subspace of compactly supported elements of the usual Sobolev space \( W^{1,2} \).

Lemma 4.3. Suppose that \( L, L^* \) satisfy the standard assumptions. If \( Lu = 0 \in \mathbb{R}^{n+1} \) and \( \tilde{N}_x(\nabla u) \in L^2(\mathbb{R}^n) \), then there exists \( f \in L^2(\mathbb{R}^n) \) such that
\[
(i) \quad \|\nabla f\|_2 \leq C\|\tilde{N}_x(\nabla u)\|_2, \text{ and } u \to f \text{ n.t., with } |u(y,t) - f(x)| \leq C t \tilde{N}_x(\nabla u)(x) \text{ whenever } (y,t) \in \gamma(x).
\]
\[
(ii) \quad \nabla u(t, \cdot) \to \nabla f \text{ weakly in } L^2(\mathbb{R}^n) \text{ as } t \to 0.
\]
If \( Lu = 0 \in \mathbb{R}^n \times (0,\rho) \), where \( 0 < \rho \leq \infty \), and \( \sup_{0 \leq s \leq \rho} \|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)} < \infty \), then there exists \( g \in L^2(\mathbb{R}^n) \) such that \( g = \partial u/\partial v \) in the variational sense, i.e.,
\[
(iii) \quad \int_{\mathbb{R}^{n+1}} A\nabla u \cdot \nabla \Phi \, dx \, dt = \int_{\mathbb{R}^n} g \Phi \, dx, \quad \forall \Phi \in W^{1,2}_\epsilon(\mathbb{R}^{n} \times (-\rho,\rho)).
\]
\[
(iv) \quad \tilde{N}_x \cdot A\nabla u(\cdot,t) \to g \text{ weakly in } L^2(\mathbb{R}^n) \text{ as } t \to 0.
\]
(Here, \( \tilde{N} \equiv -e_n + 1 \) is the unit outer normal to \( \mathbb{R}^{n+1} \)).

Of course, the analogous results hold for the lower half space.

Proof. The existence of \( f \in L^2(\mathbb{R}^n) \) satisfying (i) may be obtained by following mutatis mutandis the corresponding argument in [KP] pp. 461-462.
(ii). We first establish convergence in the sense of distributions. Let \( \tilde{\psi} \in C_0^\infty(\mathbb{R}^n, C^n) \). Then by (i),
\[
\left| \int_{\mathbb{R}^n} (\nabla \tilde{\mu}(\cdot, t) - \nabla \tilde{\mu}) \tilde{\psi} \right| = \left| \int_{\mathbb{R}^n} (\tilde{\mu}(\cdot, t) - \tilde{\mu}) \div \tilde{\psi} \right| \leq C \| \nabla \tilde{\mu} \|_2 \| \div \tilde{\psi} \|_2 \to 0.
\]
By the density of \( C_0^\infty \) in \( L^2 \), the weak convergence in \( L^2 \) then follows readily from (4.2).

(iii). We follow [KP], with some modifications owing to the unboundedness of our domain. We treat only the case \( \rho = \infty \), and leave it to the reader to check the details in the case of finite \( \rho \). Fix \( 0 < R < \infty \) and set \( B_R = B(0, R) \equiv \{ X \in \mathbb{R}^{n+1} : |X| < R \} \), \( B^*_R \equiv B_R \cap \mathbb{R}^{n+1} \) and \( \Delta_R = B_R \cap \{ t = 0 \} \). Define a linear functional on \( W_0^{1,2}(B_R) \) (the closure of \( C_0^\infty \) in \( W_1^{1,2}(B_R) \)) by
\[
\Lambda_R(\Psi) = \int B_R \nabla u \cdot \nabla \Psi, \quad \Psi \in W_0^{1,2}(B_R).
\]
Clearly, \( \| \Lambda_R \| \leq CR^{1/2} \sup_{r > 0} \| \nabla u(\cdot, t) \|_2 \). By trace theory, \( \text{tr} \left( W_0^{1,2}(B_R) \right) \subset H^{1/2}(\Delta_R) \), defined as the closure in \( H^{1/2}(\mathbb{R}^{n+1}) \) of \( C_0^\infty(\Delta_R) \). Here, \( \| f \|_{H^{1/2}(\mathbb{R}^{n+1})} \equiv \| f \|_{L^2(\mathbb{R}^{n+1})} + \| | \nabla f | \|_{L^2(\mathbb{R}^{n+1})} \) for \( 0 \leq s \leq 1 \). On the other hand, suppose that \( \psi \in H^{1/2}(\Delta_R) \). We extend \( \psi \) to \( \psi_{ext} \in W_0^{1,2}(B_R) \) by solving the problems
\[
(D+, D-) \quad \begin{cases} \sum_{i=1}^{n+1} \frac{1}{2} \frac{\partial^2}{\partial t^2} \tilde{\psi}^+_ext = 0 \text{ in } B^*_R \\ \psi^+_ext|_{\Delta_R} = \psi, \quad \psi^-ext|_{\partial B^*_R \cap \mathbb{R}^{n+1}} = 0 \end{cases}
\]
We set \( \psi_{ext} \equiv \psi^+_ext 1_{B^*_R} + \psi^-ext 1_{B^*_R} \), and by standard theory of harmonic functions we have
\[
\| \nabla \psi_{ext} \|_{L^2(\Delta_R)} \leq C \| \psi \|_{H^{1/2}(\Delta_R)}.
\]
Thus, we may define a bounded linear functional on \( H^{1/2}(\Delta_R) \) by \( \Xi_R(\psi) \equiv \Lambda_R(\psi_{ext}) \). Since \( \Lambda_R(\tau) = 0 \) whenever \( \tau \in W_0^{1,2}(B^*_R) \), then \( \Xi_R(\psi) = \Lambda_R(\Psi) \) for every extension \( \Psi \in W_0^{1,2}(B_R) \) with \( \text{tr} \Psi = \psi \). Thus, there exists a unique \( g_R \in H^{-1/2}(\Delta_R) \) with
\[
\int_{B^*_R} A \nabla u \cdot \nabla \Psi = \langle g_R, \text{tr} \Psi \rangle, \quad \forall \Psi \in W_0^{1,2}(B_R).
\]
Now suppose that \( R_1 < R_2 \), and construct \( g_{R_k} \) corresponding to \( B_k \equiv B(0, R_k) \), \( k = 1, 2 \). Then, since \( W_0^{1,2}(B_k) \subset W_0^{1,2}(B_2) \), we have that \( g_{R_k} = g_{R_k} \in H^{-1/2}(\Delta_{R_k}) \). Thus, \( \langle g_{R_k}, \psi \rangle = \langle g_{R_k}, \psi \rangle \), whenever \( \psi \in H^{1/2}(\mathbb{R}^{n+1}) \), and \( B_1, B_2 \) contain the support of \( \psi \). It follows that \( g \equiv \lim_{R \to 0} g_R \) exists in the sense of distributions, and that
\[
\text{(4.4)} \quad \int_{\mathbb{R}^{n+1}} A \nabla u \cdot \nabla \Psi = \langle g, \text{tr} \Psi \rangle, \quad \forall \Psi \in W_0^{1,2}(\mathbb{R}^{n+1}).
\]
To complete the proof of (iii), it remains only to establish that \( g \in L^2 \). The bound
\[
\| g \|_2 \leq C \sup_{t > 0} \| \nabla u(\cdot, t) \|_2
\]
will be an immediate consequence of (iv), to which we now turn our attention.

(iv). Again we present only the case \( \rho = \infty \). Since \( \sup_{t > 0} \| \nabla u(\cdot, t) \|_2 < \infty \), it is enough to verify the weak convergence for test functions in \( C_0^\infty \). Let \( \Psi \in C_0^\infty(\mathbb{R}^{n+1}) \), \( \psi \equiv \Psi|_{|x| = 0} \). By (4.4), it is enough to show that
\[
\int_{\mathbb{R}^n} \nabla u(\cdot, t) \psi \to \int_{\mathbb{R}^{n+1}} A \nabla u \cdot \nabla \Psi,
\]
as \( t \to 0 \). Integrating by parts, we see that for each \( \varepsilon > 0 \),

\[
\int_{\mathbb{R}^n} \vec{N} \cdot P_\varepsilon (A \nabla u(\cdot, t)) \psi = \int_{\mathbb{R}^{n+1}} P_\varepsilon (A \nabla u(\cdot, t + s)) (x) \cdot \nabla \Psi(x, s) \; dx \; ds,
\]

since \( Lu = 0 \) and our coefficients are \( t \)-independent. By dominated convergence, we may pass to the limit as \( \varepsilon \to 0 \) in (4.5) to obtain

\[
\int_{\mathbb{R}^n} \vec{N} \cdot A \nabla u(\cdot, t) \psi = \int_{\mathbb{R}^{n+1}} A(x) \nabla u(x, t + s) \cdot \nabla \Psi(x, s) \; dx \; ds,
\]

It therefore suffices to show that

\[
\int_{\mathbb{R}^{n+1}} A(x)(\nabla u(x, t + s) - \nabla u(x, s)) \cdot \nabla \Psi(x, s) \; dx \; ds = O(\sqrt{t}), \quad \text{as} \quad t \to 0.
\]

To this end, let \( R \) denote the radius of a ball centered at the origin which contains the support of \( \Psi \). We split the integral into \( \int_{R^2} \int_{|y| < R} + \int_{2R} \int_{|y| < R} \). Since \( \sup_{t > 0} ||\nabla u(\cdot, t)||_2 < \infty \), the first of these contributes at most \( O(t) \), while the second is dominated by

\[
C ||\nabla \Psi||_2 \left( \int_{R^2} \left( \int_{R} ||\nabla \partial_t u(\cdot, s)||_{L^2(\mathbb{R}^n)}^2 ds \right)^{1/2} \right)^{1/2} \leq C_0 \left( \int_{R} \left( \int_{R} \frac{ds}{s^2} \right)^{1/2} \right) \sup_{t > 0} ||\nabla u(\cdot, t)||_2,
\]

where in the last step we have used Caccioppoli’s inequality in Whitney cubes in the 1/2-space. The desired conclusion follows.

Next we discuss the boundedness of non-tangential maximal functions of layer potentials. We recall that \( S_\gamma^p \) is defined in (2.17), and that \( P_\gamma \) denotes a smooth approximate identity acting in \( \mathbb{R}^n \). In the sequel, given an operator \( T \), we shall use the notation

\[
||T||_{op,Q} \equiv ||T||_{L^2(Q) \to L^2(\mathbb{R}^n)} \equiv \sup \frac{||Tf||_{L^2(\mathbb{R}^n)}}{||f||_{L^2(Q)}},
\]

where the supremum runs over all \( f \) supported in \( Q \) with \( ||f||_2 > 0 \).

**Lemma 4.8.** Let \( L, L^* \) satisfy the standard assumptions. Then for \( 1 < p < \infty \), we have

(i) \( ||N(\partial_t S_\gamma f)||_p \leq C_p (\sup_{t > 0} ||\partial_t S_\gamma f||_{p-p} + 1) ||f||_p \).

(ii) \( ||N(S_\gamma f)||_p \leq C_p (\sup_{t > 0} ||S_\gamma f||_p + ||N(\partial_t S_\gamma f)||_p) \).

(iii) \( ||N(\partial_t S_\gamma f)||_p \leq C_p (\sup_{t > 0} ||\partial_t S_\gamma f||_p + ||N(S_\gamma f)||_p) \).

(iv) \( \sup_{\eta > 0} ||N(\partial_t S_\gamma f)||_{L^2(\mathbb{R}^n)} \leq C(\sup_{\eta > 0} ||\partial_t S_\gamma f||_{op,Q} + 1) ||f||_2 \), \( \eta > 0 \), supp \( f \subset Q \).

(v) \( ||N((S_\gamma f) \cdot \Gamma)||_{L^2(\mathbb{R}^n)} \leq C(\sup_{\eta > 0} ||(S_\gamma f)\partial_t S_\gamma f||_{op,Q} + 1) ||f||_2 \).

(vi) \( ||N(\partial_t f)||_{L^2(\mathbb{R}^n)} \leq C(\sup_{\eta > 0} ||(S_\gamma f)\partial_t S_\gamma f||_{op,Q} + 1) ||f||_2 \).

where \( L^2,\infty \) denotes the usual weak-\( L^2 \) space.

**Proof.** By Lemma 2.2, the kernel \( K_t(x, y) \equiv \partial_t \Gamma(x, t, y, 0) \) is a standard Calderón-Zygmund kernel with bounds independent of \( t \). We may then prove (i) by a familiar argument involving Cotlar’s inequality for maximal singular integrals. We omit the details (but see the proof of (iv) below, which is similar). Estimate (ii) may be obtained by following the argument in [KP], p. 494 (again we omit the details) and (vi) follows from (v). It remains to prove (iii), (iv) and (v)
(iii). The proof is similar to that of estimate (ii), and we follow [KP]. Fix \( x_0 \in \mathbb{R}^n \), and suppose that \( |x - x_0| < t \). It is enough to replace \( \nabla \) by \( \nabla_{t} \). We have

\[
P_t (\nabla fS_0 f)(x) = \nabla_{t} P_t (\nabla fS_0 f)(x) = t^{-1} \tilde{Q}_t S_0 f(x)
\]

where we have used that \( \nabla_{t} P_t \equiv \tilde{Q}_t \) annihilates constants. But

\[
|t^{-1} \tilde{Q}_t \left( \int_0^1 \partial_t S_0 fds + S_0 f - \int_{\Delta_t(x_0)} S_0 f \right)(x)| \leq CM (N_\infty (\partial_t S_0 f))(x_0),
\]

and, by Poincaré’s inequality,

\[
|t^{-1} \tilde{Q}_t \left( S_0 f - \int_{\Delta_t(x_0)} S_0 f \right)(x)| \leq CM(\nabla_{t} S_0 f)(x_0).
\]

(iv). We suppose that \( \eta \ll f(Q) \), and that \( Q \) is centered at 0, as it is only this case that we shall encounter in the sequel. We shall deduce (iv) as a consequence of the following refinement of Cotlar’s inequality for maximal singular integrals. Let \( T \) be a singular integral operator associated to a standard Calderón-Zygmund kernel \( K(x, y) \). As usual, we define truncated singular integrals

\[
T_s f(x) \equiv \int_{|x-y| < s} K(x, y)f(y)dy,
\]

and we define a maximal singular integral

\[
T_s f \equiv \sup_{0 < s < R} |T_s f|.
\]

We claim that the following holds for all \( f \) supported in a cube \( Q \):

\[
(4.9) \quad T_s(Q) f(x) \leq C \left( C_K + ||T||_p, Q \right) M f(x) + CM(T f)(x),
\]

where \( C_K \) depends on the Calderón-Zygmund kernel conditions. Momentarily taking this claim for granted, we proceed to prove (iv).

Let \( K^\eta_s (x, y) \) denote the kernel of \( \partial_t S^\eta f \) (see (2.17)), i.e.,

\[
K^\eta_s (x, y) \equiv \partial_t \left( \varphi_\eta \ast \Gamma(x, y, 0) \right)(t).
\]

Then by Lemma 2.2 we have for all \( t \geq 0 \), uniformly in \( t_0 \geq 0 \),

\[
(4.10) \quad |K^\eta_{t+t_0} (x, y)| \leq C \left( \frac{1}{(t + |x-y|)^n} + \frac{1}{\eta |x-y|^{n-1}} \right),
\]

\[
(4.11) \quad |K^\eta_{t+t_0} (x + h, y) - K^\eta_{t+t_0} (x, y)| \leq C_{\eta} \left( \frac{|h|^a}{(t + |x-y|)^{n+a}}, \frac{1}{\eta |x-y|^{n-1}} \right), \quad |x-y| + t > 10\eta
\]

where the last bound holds whenever \( |x - y| > 2|h| \) or \( 2t > |h| \). Of course, we also have a similar estimate concerning Hölder continuity in the \( y \) variable. In particular, \( K^\eta_{t+t_0} (x, y) \) is a standard Calderón-Zygmund kernel, uniformly in \( t \), \( t_0 \) and \( \eta \).

We begin by showing that for each fixed \( x_0 \in \mathbb{R}^n \) and \( t_0 \geq 0 \),

\[
(4.12) \quad N_\infty (\partial_t S^\eta f)(x_0) \leq \sup_{t > 0} \left| \partial_t S^\eta f(x_0) \right| + CM(M f)(x_0).
\]

To see this, let \( |x - x_0| < t \), and note that

\[
|P_t (\partial_t S^\eta f)(x_0) - \partial_t S^\eta f(x_0)| \leq CT^n \int_{|z| < 2r} \int_{\mathbb{R}^n} |K^\eta_{t+t_0} (z, y) - K^\eta_{t+t_0} (x_0, y)| |f(y)|dydz.
\]
for which, in the case \( t > 10\eta \), we obtain immediately the bound \( CMf(x_0) \) by applying (4.11). In the case \( t \leq 10\eta \), we split the inner integral into
\[
\int_{|x-y|>10\eta} + \int_{|x-y|\leq10\eta} \leq CMf(x_0) + C(Mf(z) + Mf(x_0)),
\]
where we have applied (4.11) to bound the first term, and (4.10) to handle the second. The estimate (4.12) now follows readily.

Next, we observe that for \( f \) supported in a cube \( Q \) centered at 0, with \( \ell(Q) >> \eta \),
\[
(4.13) \quad \sup_{r>0} |\partial_\xi S^0_\theta f(x)| \leq \sup_{0<r<\ell(Q)} |\partial_\xi S^0_\theta f(x)| + CMf(x).
\]
Indeed, suppose that \( t \geq \ell(Q) >> \eta \). Then
\[
|\partial_\xi S^0_\theta f(x)| \leq \int |K^\theta_\xi(x, y) f(y)| dy \leq CMf(x),
\]
by (4.10), since for \( y \in Q \), we have \(|x-y| = |x|\), if \(|x| > Ct\), and \(|x-y| < Ct\), if \(|x| < Ct\).
Combining (4.12) and (4.13), we see that it is enough to treat \( \sup_{0<r<\ell(Q)} |\partial_\xi S^0_\theta f(x)| \). To this end, fix \( x_0 \) and \( t \in (0, \ell(Q)) \), and set \( \rho \equiv \max(t, 2\eta) \). Then
\[
\partial_\xi S^0_\theta f(x_0) = \int_{|x-y|>\rho} \left( K^\theta_\xi(x_0, y) - K^\theta_\xi(x_0, y) \right) f(y) dy
\]
\[
+ \int_{|x-y|\leq\rho} K^\theta_\xi(x_0, y) f(y) dy - \int_{5\rho>|x-y|>\rho} K^\theta_\xi(x_0, y) f(y) dy
\]
\[
+ \int_{|x-y|\leq\rho} K^\theta_\xi(x_0, y) f(y) dy \equiv I + II + III + IV.
\]
Then \(|I| + |II| + |III| \leq CMf(x_0)\), by Lemma 2.2 and by (4.10). Also,
\[
|IV| \leq \sup_{0<r<\ell(Q)} \left| \int_{|x-y|>\rho} K^\theta_\xi(x_0, y) f(y) dy \right|.
\]
Thus, taking \( T \) in (4.9) to be the singular integral operator with kernel \( K^\theta_\xi(x, y) \), we obtain (iv), modulo the proof of (4.9).

We now turn to the proof of (4.9). The argument is a variant of the standard one. Suppose that \( f \) is supported in a cube \( Q \), and fix \( \epsilon \in (0, \ell(Q)) \) and \( x_0 \in \mathbb{R}^n \). Set \( \Delta \equiv \Delta_{\epsilon/2}(x_0) \), \( 2\Delta \equiv \Delta_{\epsilon}(x_0) \). Let \( f_1 \equiv f_{1,2\Delta} \), \( f_2 \equiv f - f_1 \). Then for \( x \in \Delta \), we have
\[
|T\epsilon f(x_0)| = |Tf_1(x_0)| = |Tf_2(x_0) - Tf_2(x) + Tf(x) - Tf_1(x)| \leq CMf(x_0) + |Tf(x)| + |Tf_1(x)|.
\]
Let \( r \in (0, 1) \), and take an \( L^r \) average of this last inequality over \( \Delta \). Note that \( f_1 = 0 \) unless \( 2\Delta \subset 5Q \), since \( \text{diam}(2\Delta) \leq 2\ell(Q) \). We therefore obtain
\[
|T\epsilon f(x_0)| \leq CMf(x_0) + M(|Tf|^r)^{1/r}(x_0) + \left( \int_{\Delta} |Tf_1|^r \right)^{1/r} \leq C(CK + \|T\|_{L^r(\mathbb{R}^n)} Mf(x_0) + M(Tf)(x_0),
\]
where we have used Kolmogorov’s weak-\( L^1 \) criterion, and \( L^{1,\infty} \) is the usual weak-\( L^1 \) space. But by a localized version of the Calderón-Zygmund Theorem,
\[
\|T\|_{L^r(\mathbb{R}^n)} \leq C \left( CK + \|T\|_{L^r(\mathbb{R}^n)} Mf(x_0) \right) \leq C \left( CK + \|T\|_{L^r(\mathbb{R}^n)} Mf(x_0) \right),
\]
and (4.9) follows.
(v) By (i) and \( t \)-independence, we may replace \( \nabla \) by \( \nabla_x \). The desired estimate is an immediate consequence of the following pointwise bound. For convenience of notation set 
\[
 K \equiv \sup_{\gamma > 0} \| (S, \nabla_\gamma) \|_{L^2}.
\]
Let \( f \in C^0_0 (\mathbb{R}^n, \mathbb{C}^n) \). We shall prove \( ^{2} \)
\[
 N_c((S, \nabla_x) \cdot \tilde{f})(x) \leq C \left( M(|(S, \nabla_x) \cdot \tilde{f}|^2) + (K + 1) (M(|(S, \nabla_x) \cdot \tilde{f}|^2))^{1/2} (x) \right) \tag{4.14}
\]
To this end, we fix \( (x_0, t_0) \in \mathbb{R}^{n+1} \) and suppose that \( |x_0 - x| < 2t, |t_0 - t| < 2t \) and that \( k \geq 4 \).
We claim that
\[
(4.15) \quad \int_{|y_0 - y| < 2^{k+1}t} |\nabla_x (\Gamma(x, s, y, 0) - \Gamma(x_0, t_0, y, 0))|^2 \, dy \leq C 2^{-k} (2^k t)^n. \]
Indeed, the special case \( s = t_0 \) is essentially a reformulation of Lemma 2.13, but with the roles of \( x \) and \( y \) reversed. In general, we write
\[
\Gamma(x, s, y, 0) - \Gamma(x_0, s, y, 0) = [\Gamma(x, s, y, 0) - \Gamma(x_0, s, y, 0)] + [\Gamma(x_0, s, y, 0) - \Gamma(x_0, t_0, y, 0)].
\]
The first expression in brackets is the case \( s = t_0 \), while the horizontal gradient of the second equals
\[
\int_{t_0}^x \nabla_y \partial_t \Gamma(x_0, t, y, 0) \, dt. \]
We may handle the contribution of the latter term via Lemma 2.5. This proves the claim.
We set \( u(\cdot, t) = (S, \nabla_0) \cdot \tilde{f} \), and we split \( u = u_0 + \sum_{k=4}^\infty u_k \equiv u_0 + \tilde{u} \),\nwhere \[
u_0 \equiv (S, \nabla_0) \cdot \tilde{f}_0, \quad u_k \equiv (S, \nabla_0) \cdot \tilde{f}_k, \quad \tilde{u} \equiv \sum_{k=4}^\infty u_k,
\]
and \( \tilde{f}_0 \equiv \tilde{f}^1_{[x_0 - t < 1]} \), \( \tilde{f}_k = \tilde{f}^1_R \), and \( R_k \equiv \{ y : 2^k t \leq |x_0 - y| < 2^{k+1} t \} \). By (4.15), for \( s \in [-2t, 2t] \) and \( |x_0 - x| < 2t \), we have that
\[
|u_k(x, s) - u_0(x_0, 0)| \leq C 2^{-k} (2^k t)^{n/2} \left( \int_{R_k} |\tilde{f}_k|^2 \right)^{1/2} \leq C 2^{-k} (2^k t)^{n/2} (M(|\tilde{f}_k|^2))^{1/2} (x_0).
\]
Summing in \( k \), we obtain
\[
|\tilde{u}(x, s) - \tilde{u}(x_0, 0)| \leq C \left( M(|\tilde{f}_k|^2) \right)^{1/2} (x_0). \tag{4.16}
\]
Moreover, since \( LU_0 = 0 \), by (1.3) it follows that
\[
|u_0(x, t)| \leq C \left( \int \int_{B(x_0, t/2)} |u_0|^2 \right)^{1/2} \leq Ct^{-n/2} \sup_{\gamma > 0} \| (S, \nabla_\gamma) \cdot \tilde{f}_0 \|_2 \leq CK \left( M(|\tilde{f}_k|^2) \right)^{1/2} (x_0).
\]
Taking \( s = t \) in (4.16), we therefore need only establish the bound
\[
|\tilde{u}(x_0, 0)| \leq C(K + 1) \left( M(|\tilde{f}_k|^2) \right)^{1/2} (x_0) + CM \left( u(\cdot, 0) \right)(x_0).
\tag{4.17}
\]
The proof of (4.17) is based on that of the well known Cotlar inequality for maximal singular integrals. Set \( \Delta_0 = \{|x - x_0| < t\} \), and let \( x \in \Delta_0 \). We write
\[
|\tilde{u}(x_0, 0)| \leq |\tilde{u}(x_0, 0) - \tilde{u}(x_0, 0)| + |\tilde{u}(x, 0)| \leq |\tilde{u}(x_0, 0) - \tilde{u}(x, 0)| + |u_0(x_0, 0)| + |u(x, 0)| \leq C \left( M(|\tilde{f}_k|^2) \right)^{1/2} (x_0) + |u_0(x_0, 0)| + |u(x, 0)|,
\]
\footnote{The bound for the last term in (4.14) may be improved to \( (M(|\tilde{f}_k|^q))^{1/q} (x) \), for some \( q < 2 \) depending on dimension and ellipticity, as the fourth named author will show in a forthcoming paper with M. Mitrea.}
where in the last step we have used (4.16) with \( s = 0 \). Averaging over \( \Delta_0 \), we obtain
\[
|\hat{u}(x_0,0)| \leq C \left( M(|f|^2) \right)^{1/2}(x_0) + \left( \int_{\Delta_0} |u_0(x,0)|^2 \, dx \right)^{1/2} + M \langle u(-,0) \rangle(x_0).
\]
Since the \( L^2 \) average of \( u_0 \) is bounded by \( CK \left( M(|f|^2) \right)^{1/2}(x_0) \), we obtain (4.17).

We are now ready to discuss the jump relations and traces of the layer potentials. We recall that \( S^*, D^* \) denote the single and double layer potentials associated to \( L^* \).

**Lemma 4.18.** Suppose that \( L, L^* \) satisfy the standard assumptions, and that the single layer potentials \( S^*, S^* \) satisfy
\[
(4.19) \quad \sup_{t \in \mathbb{R}} \|\nabla S_t\|_{L^2} + \sup_{t \in \mathbb{R}} \|\nabla S^*_t\|_{L^2} < \infty.
\]
Then there exist \( L^2 \) bounded operators \( K, \bar{K}, \mathcal{T} \) with the following properties: for all \( f \in L^2(\mathbb{R}^n) \), we have
\[
(i) \quad \left( \frac{1}{2} I + K \right) f = \partial_n u^+\]
where \( u^+ \equiv S_t f, t \in \mathbb{R}^n \), and \( \partial_n \) denotes the conormal derivative \(-e_{n+1} \cdot \nabla \), interpreted in the weak sense of Lemma 4.3 (iii) and (iv).

(ii) \( D_{n+1} f \rightarrow \left( \frac{1}{2} I + K \right) f \) weakly in \( L^2 \)

(iii) \( \langle \nabla S_t \rangle |_{u^+} f \rightarrow \left( \frac{1}{2} I + e_{n+1} + \mathcal{T} \right) f \) weakly in \( L^2 \).

**Proof.** It is enough to prove (i). Indeed, if we define
\[
K \equiv \text{adj} \left( \bar{K} \right),
\]
then (ii) follows from (i) and the observation that \( D_{n+1} = \text{adj} \left( \bar{N} \cdot A \nabla S^* \right) |_{u^+} \). To obtain (iii), we first use (4.19), Lemma 4.8, Lemma 4.3 and the formula
\[
(4.20) \quad -A_{n+1,n+1} \partial_n S_j = N \cdot A \nabla S_j + \sum_{j=1}^n A_{n+1,j} D_j S_j,
\]
to deduce that \( \partial_n S_j f \) converges weakly in \( L^2 \), as \( t \to 0 \). Thus, we may define
\[
\mathcal{T} f \equiv \text{tr} \left( \nabla S_t f \right).
\]
Then (iii) follows from (4.20) since \( \nabla |u^+| f \) does not jump across the boundary.

To prove (i), we apply Lemma 4.3 (iii) in both \( \mathbb{R}^{n+1} \), to obtain \( g^+ \in L^2(\mathbb{R}^n) \), with \( g^+ = \partial_n u^+ \) in the weak sense. We now define \( \bar{K} \) by
\[
(4.21) \quad \left( \frac{1}{2} I + \bar{K} \right) f \equiv g^+ \quad \left( \frac{1}{2} I + \bar{K} \right) f \equiv g^+,
\]
and to show that this operator is well defined, we need only verify that \( g^+ - g^- = f \). It is enough to prove that
\[
(4.22) \quad \int_{\mathbb{R}^{n+1}} A \nabla u^+ \cdot \nabla \psi \, dx dt + \int_{\mathbb{R}^{n+1}} A \nabla u^- \cdot \nabla \psi \, dx dt = \int_{\mathbb{R}^n} f \psi \, dx,
\]
for all \( \psi \in C_0^\infty(\mathbb{R}^{n+1}) \). To this end, set \( u^\psi_n \equiv S_{n+1}^\psi f \), where \( S^\psi \) is defined in (2.17), so that
\[
(4.23) \quad u^\psi_n = \int_{\mathbb{R}^{n+1}} \Gamma(x, t, y, s) f(y, s) \, dy ds, \quad t \in \mathbb{R}^n
\]
\[8\]We are indebted to M. Mitrea for suggesting this approach.
where \( f_0(y, s) \equiv f(y) \phi_0(s) \) and \( \phi_0 \) is the kernel of a smooth approximate identity acting in 1 dimension. Let \( U_\eta \equiv u_\eta^+ 1_{\mathbb{R}^{n+1}_+} + u_\eta^- 1_{\mathbb{R}^{n+1}_-} \). Since \( L^\ast \Gamma = \delta \), we have that

\[
\int_{\mathbb{R}^{n+1}_+} A \nabla u_\eta^+ \cdot \nabla \Psi + \int_{\mathbb{R}^{n+1}_-} A \nabla u_\eta^- \cdot \nabla \Psi = \int_{\mathbb{R}^{n+1}} A \nabla U_\eta \cdot \nabla \Psi
\]

\[
= \int_{\mathbb{R}^{n+1}} f_\eta \Psi \to \int_{\mathbb{R}^n} f \Psi,
\]
as \( \eta \to 0 \). On the other hand, fixing \( \varepsilon \) momentarily, we have that

\[
\int_{\mathbb{R}^{n+1}_+} A \nabla (u_\eta^+ - u^+) \cdot \nabla \Psi = \int_\varepsilon \int_{\mathbb{R}^n} + \int_0^\infty \int_{\mathbb{R}^n} \equiv I_\varepsilon + II_\varepsilon.
\]

Fix a number \( R \) greater than the diameter of \( \text{supp}(\Psi) \). Then

\[
|I_\varepsilon| \leq C_\Psi \int_\varepsilon^R \sup_{t \in [-R]} \|\nabla (S^\eta_f - S_\varepsilon f)\|_{L^2(\mathbb{R}^n)} \to 0
\]
as \( \eta \to 0 \), by Lemma 2.18. Moreover,

\[
\sup_{\eta > 0} |I_\varepsilon| \leq C_\Psi \varepsilon \sup_{t \in [-R]} \|\nabla S_\varepsilon f\|_2 \leq C_\Psi \varepsilon \|f\|_2,
\]

where we have used that \( \sup_{\eta > 0} \|\nabla S_\varepsilon f\|_2 \leq \sup_{t \in [-R]} \|\nabla S_\varepsilon f\|_2 \), by construction of \( S_\varepsilon^\eta \) (2.17). The analogous convergence result for the lower half-space concludes the proof of (i). \( \square \)

We turn now to the issues of non-tangential and strong \( L^2 \) convergence for \( \mathcal{D}_f \).

**Lemma 4.23.** Suppose that \( L, L^\ast \) satisfy the standard assumptions, that the single layer potentials \( S^\eta, S^\ast_\varepsilon \) satisfy (4.19), and that \( S^\varepsilon_{\ast,\delta,0} \equiv S^\ast_{\varepsilon,\delta,0} : L^2(\mathbb{R}^n) \to L^2_\varepsilon(\mathbb{R}^n) \) is bijective. Then for every \( f \in L^2(\mathbb{R}^n) \), we have the following:

\[
\mathcal{D}_f f \to \left( \frac{1}{2} I + K \right) f \quad \text{n.t. and in } L^2.
\]

We first require a special case of the Gauss-Green formula.

**Lemma 4.24.** Let \( L, L^\ast \) satisfy the standard assumptions, and suppose that \( Lu = 0 \), \( L^\ast w = 0 \) in \( \mathbb{R}^{n+1}_+ \) with

\[
\sup_{t \geq 0} (\|\nabla u(\cdot, t)\|_2 + ||\nabla w(\cdot, t)\|_2) < \infty,
\]

and \( \partial_\nu u \mathcal{R} \rightarrow 0 \), \( \partial_\nu \mathcal{R} w \rightarrow 0 \) \in \( L^1(\mathbb{R}^n)^2 \). Suppose also that there exist \( R_0, \beta > 0 \) such that for all \( R > R_0 \), we have

\[
\int_{\mathbb{R}^{n+1} \cap (B(0,2R) \setminus B(0,R))} |\nabla u||\nabla w| + |\nabla u||R^{-1}|w| + |\nabla w||R^{-1}|u| = O \left( R^{-\beta} \right).
\]

Then

\[
\int_{\mathbb{R}^n} \partial_\nu u \mathcal{R} = \int_{\mathbb{R}^n} u \partial_\nu \mathcal{R} w.
\]

Of course, the analogous result holds in \( \mathbb{R}^{n+1}_- \).

---

9Here, \( \partial_\nu \) and \( \partial_\nu \) are the exterior conormal derivatives, corresponding to the matrices \( A \) and \( A^\ast \) respectively, which exist in the weak sense of Lemma 4.3.
Proof. By the symmetry of our hypotheses, it is enough to show that
\begin{equation}
\int_{\mathbb{R}^{n+1}} A \nabla u \cdot \nabla w = \int_{\mathbb{R}^n} \partial_r u \overline{w}.
\end{equation}
To this end, for $R_0 < R < \infty$, let $\Theta_0(X) \equiv \Theta(X/R)$, where $\Theta \in C_0^\infty(B(0,2))$ and $\Theta \equiv 1$ in $B(0,1)$. We set $w_R \equiv w\Theta_R$. Then by Lemma 4.3, we have that
\begin{equation}
\int_{\mathbb{R}^{n+1}} A \nabla u \cdot \nabla w_R = \int_{\mathbb{R}^n} \partial_r u \overline{w_R}.
\end{equation}
A simple limiting argument completes the proof. \end{proof}

**Corollary 4.28.** Let $L, L^*$ satisfy the standard assumptions, and suppose that the respective single layer potentials $S_\ast, S_\ast^*$ satisfy (4.19). Further suppose that $u(\cdot, \tau) = S_\ast \psi$ in $\mathbb{R}^{n+1}$, where $\psi \in C_0^\infty(\mathbb{R}^n)$. Then setting $u_0 \equiv u(\cdot, 0)$, we have
\begin{equation}
\mathcal{D}_t u_0 = S_\ast(\partial_r u).
\end{equation}
**Proof.** It is enough to show that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have
\begin{equation}
\int_{\mathbb{R}^n} \mathcal{D}_t u_0 \varphi = \int_{\mathbb{R}^n} S_\ast(\partial_r u) \overline{\varphi}.
\end{equation}
Note that $\text{adj}(\mathcal{D}_t) = \overline{N} \cdot A^* (\nabla S_\ast^*) |_{\tau = 0}$, and that $\text{adj}(S_\ast) = S_\ast^* \varphi$, so that $L^* u^* = 0$ in $\mathbb{R}^{n+1} \setminus \{ \tau = 0 \}$. It suffices to verify the hypotheses of Lemma 4.24, in the lower half-space, for $u, w$, with $w(\cdot, s) \equiv u^*(\cdot, s-\tau)$, $s \leq 0$. Estimate (4.25) is immediate by (4.19). By Lemma 2.2, we have
\begin{equation}
|u(X)| + |w(X)| = O(|X|^{-n+1}) \quad \text{as} \quad |X| \to \infty.
\end{equation}
Also, $Lu = 0, L^* w = 0$ in $\mathbb{R}^{n+1} \setminus B(0,R_0)$, if $R_0$ is chosen large enough, since $\varphi, \psi$ have compact support. Thus, by Caccioppoli,
\begin{equation}
\int_{\mathbb{R}^{n+1} \setminus B(0,2R) \cup B(0,R)} |\nabla u|^2 \leq C \int_{\mathbb{R}^{n+1} \setminus B(0,3R) \cup B(0,R/2)} \frac{|u|^2}{R} = O(R^{-n+1}),
\end{equation}
for $R > 4R_0$, and similarly for $w$. Estimate (4.26) follows. Finally, the boundary integrabil-
ity of $\partial_r u \overline{w}$ and $\overline{w} \partial_r w$ follows readily from Cauchy-Schwarz, the fact that $n \geq 2$, and two observations: first, that by Lemma 2.7 and duality, we have
\begin{equation}
\int_{\Lambda_{2\tau}(0) \setminus \Lambda_\tau(0)} |\partial_r u|^2 + |\partial_r w|^2 = O(R^{-n});
\end{equation}
second, that (4.30) implies that
\begin{equation}
\int_{\Lambda_{2\tau}(0) \setminus \Lambda_\tau(0)} |u|^2 + |w|^2 = O(R^{-2n}).
\end{equation}
We leave the remaining details to the reader. \end{proof}

**Proof of Lemma 4.23.** Since we have already obtained the limits $(\mp \frac{1}{2} I + K)f$ in the weak
sense (Lemma 4.18), it is enough here merely to establish existence of n.t. and strong $L^2$
limits, without concern for their precise values. We give the proof only in the case of the
upper half-space, as the proof in the other case is the same.

We begin with the matter of non-tangential convergence. Observe that $\text{adj}(S_\ast \nabla) = (\nabla S_\ast^*) |_{\tau = -1}$, so by (4.19) and Lemma 4.8(v), it is enough to establish n.t. convergence for $f$ in a dense class in $L^2$. We claim now that $\{ S_\ast \text{div}_\ast \tilde{g} : \tilde{g} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^m) \}$ is dense in $L^2$. 

Indeed, by hypothesis and duality, $S_0 : L^2_{-a} \to L^2$ is bijective. Thus, $L^2 = \{ S_0 \text{ div} \vec{g} : \vec{g} \in L^2 \}$. The density of $C_0^\infty$ in $L^2$ establishes the claim.

We now set $f = u_0 = S_0(\text{div} \vec{g})$, with $\vec{g} \in C_0^\infty$, and let $u(\cdot, \tau) = S_\tau(\text{div} \vec{g})$, $\tau < 0$. We may then apply Corollary 4.28 to obtain that $\mathcal{D}_f = S_\tau(\partial_t u)$. Moreover, (4.19), Lemma 4.8, and Lemma 4.3 imply that $\partial_t u \in L^2$ and hence also that $S_\tau(\partial_t u)$ converges n.t., from which fact the non-tangential part of (ii) now follows.

We turn now to the issue of strong convergence in $L^2$. By (4.19), we have in particular that $L^2$ bounds hold, uniformly in $t > 0$, for $\mathcal{D}_t$. Thus, it is once again enough to establish convergence in a dense class. To this end, choose $u_0, u$ as above. It suffices to show that $\mathcal{D}_t u_0$ is Cauchy convergent in $L^2$, as $t \to 0$. Suppose that $0 < t' < t \to 0$, and observe that, by Corollary 4.28, (4.19) and our previous observation that $\partial_t u \in L^2$,

$$\| \mathcal{D}_t u_0 - \mathcal{D}_t u_0 \|_2 = \| \int_t^{t'} \partial_t S_\tau(\partial_t u) \, ds \|_2 \leq (t-t') \| \partial_t S_\tau(\partial_t u) \|_2 \to 0.$$ 

\[\Box\]

**Lemma 4.31. (Uniqueness).** Suppose that $L, L^*$ satisfy the standard assumptions, and that we have existence of solutions to (D2) and (R2). Then those solutions are unique, in the following sense:

(i) If $u$ solves (D2), with $u(\cdot, t) \to 0$ in $L^2$, as $t \to 0$, then $u \equiv 0$.

(ii) If $u$ solves (R2), and $u \to 0$ n.t., then $u \equiv 0$.\(^{10}\)

If, in addition, $L$ and $L^*$ have “Good Layer Potentials”, then the solution to (N2) is unique, in the sense that:

(iii) If $u$ solves (N2), with $\partial u / \partial n = 0$ in the sense of Lemma 4.3 (iii) and (iv), then $u \equiv 0$ (modulo constants).

**Proof.** Consider first uniqueness in (D2). We begin by constructing Green’s function. By Lemma 2.5 with $m = -1$, for each fixed $(x, t) \in \mathbb{R}^{n+1}$, we have $\Gamma(x, t, \cdot, 0) \in L^2_{\tilde{m}}$, with

$$\| \tilde{\nabla} \|_\Gamma(x, t, \cdot, 0) \|_{L^2(\mathbb{R}^n)} \leq C r^{-\tilde{m}/2}.$$  

Thus, by (R2), there exists $w = w_{x,t}$ solving

$$
\begin{cases}
Lw = 0 \text{ in } \mathbb{R}^{n+1} \\
w(\cdot, s) \to \Gamma(x, t, \cdot, 0) \text{ n.t.} \\
\| \tilde{\nabla} w \|_{L^2(\mathbb{R}^n)} \leq C r^{-\tilde{m}/2}.
\end{cases}
$$

Set

$$G(x, t, y, s) \equiv \Gamma(x, t, y, s) - w_{x,t}(y, s),$$

and note that

$$\sup_{|x-x| < C r / 2} \| \nabla G(x, t, \cdot, s) \|_{L^2(\mathbb{R}^n)} \leq C r^{-\tilde{m}/2}.$$  

Let $\theta \in C_0^\infty(\mathbb{R}^{n+1})$, with $\theta \equiv 1$ in a neighborhood of $(x, t)$. Then, since $Lu = 0$, we have

$$u(x, t) = (u\theta)(x, t) = \int \int \nabla \nabla \theta \cdot \nabla (u \theta) \, dy \, ds$$

$$= - \int \int \nabla \nabla \theta \cdot A \nabla u + \int \int \nabla G \cdot A \nabla \theta u \equiv I + II.$$  

\(^{10}\)Our data in the problem (R2) belongs to $L^2_{\tilde{m}}$, whose elements are defined modulo constants; thus, uniqueness in this context must be interpreted correspondingly. We assume here that we have chosen a particular realization of the data equal to 0 a.e. on the boundary.
We now choose \( \phi \in C_0^\infty(-2, 2) \), \( \phi \equiv 1 \) in \((-1, 1)\), with \( 0 \leq \phi \leq 1 \), and set \( \theta(y, s) \equiv [1 - \phi(s/\epsilon)] \phi(s/(100R)) \phi(|x - y|/R) \), with \( \epsilon < t/8, R > 8t \). With this choice of \( \theta \), the domains of integration in \( I \) and \( II \) are contained in a union \( \Omega_1 \cup \Omega_2 \cup \Omega_3 \), where

1. \( \Omega_1 \subseteq \Delta_{2R}(x) \times \{ |y| < 2\epsilon \} \), with \( ||\nabla \theta||_{L^2(\Omega_1)} \leq C\epsilon^{-1} \).
2. \( \Omega_2 \subseteq \Delta_{2R}(x) \times \{ 100R < |y| < 200R \} \), with \( ||\nabla \theta||_{L^2(\Omega_2)} \leq CR^{-1} \).
3. \( \Omega_3 \subseteq (\Delta_{2R}(x) \setminus \Delta_R(x)) \times \{ 0 < |y| < 200R \} \), with \( ||\nabla \theta||_{L^2(\Omega_3)} \leq CR^{-1} \).

We treat term \( I \) first. We recall from [HK2] that

\[
\|\nabla (X, \cdot)\|_{L^2(\mathbb{R}^{n+1})} \leq Cr^{1-n/2}, \quad \forall r > 0, \quad X \in \mathbb{R}^{n+1}
\]

and that

\[
|G(X, Y)| \leq C|X - Y|^{1-n},
\]

whenever \( |X - Y| \leq \min(\delta(X), \delta(Y)) \), where \( \delta(X) \) denotes the distance to the boundary of the half-space (i.e., the \( t \)-coordinate). Thus, in particular we obtain that

\[
R^{-1}|G(x, t, \cdot)|_{L^2(\Omega_2, \Omega_3)} \leq CR^{1-n/2},
\]

where in proving the bound on \( \Omega_2 \) we have used that \( G \) vanishes on the boundary, to reduce matters to (4.34). We then have that

\[
\frac{1}{C} |I| \leq \epsilon^{-1} \int_{\Omega_2} |\nabla u| + R^{1-n/2} \left( \int_{\Omega_2} |\nabla u|^2 \right)^{1/2} \equiv I_1 + I_2.
\]

Since \( u \) vanishes on \( \{ t = 0 \} \), we may apply Caccioppoli’s inequality in \( \Omega_2 \cup \Omega_3 \) to obtain that \( I_2 \leq CR^{-n/2} \sup_{s \in \partial \Omega} \|u(\cdot, s)\|_2 \rightarrow 0 \) as \( R \rightarrow \infty \).

To treat \( I_1 \), we first note that for \( (y, s) \in \Omega_1 \),

\[
|G(x, t, y, s)| \leq Ce \left( (|x - y| + t)^{-n} + \tilde{N}_e(\nabla w_x)(y) \right),
\]

by Lemma 2.2, Lemma 4.3 and construction of \( G \). Consequently,

\[
\left( \epsilon^{-1} \int_{\mathbb{R}^n} |G(x, t, y, s)|^2 dy ds \right)^{1/2} \leq Ce^{-n/2}.
\]

Thus, using Caccioppoli to estimate the \( L^2 \) norm of \( \nabla u \) in \( \Omega_1 \), we obtain that

\[
I_1 \leq CR^{-n/2} \sup_{s \in \partial \Omega} \|u(\cdot, s)\|_2 \rightarrow 0
\]
as \( \epsilon \rightarrow 0 \), since \( u(\cdot, s) \rightarrow 0 \) in \( L^2 \).

We now consider term \( II \). By Cauchy-Schwarz and then Caccioppoli’s inequality,

\[
|II| \leq \epsilon^{-1} \int_{\Omega_2} |\nabla G| |u| + R^{-1} \int_{\Omega_2 \cap \Omega_3} |\nabla G| |u| \equiv H_1 + H_2
\]

\[
\leq C\epsilon^{-3/2} |G(x, t, \cdot, \cdot)|_{L^2(\Omega_2)} \sup_{s \in \partial \Omega} \|u(\cdot, s)\|_2
\]

\[
+ R^{-3/2} |G(x, t, \cdot, \cdot)|_{L^2(\Omega_2 \cap \Omega_3)} \sup_{s \in \partial \Omega} \|u(\cdot, s)\|_2.
\]

By (4.39), the term \( H_1 \) may be handled exactly like \( I_1 \), and by (4.36), \( H_2 \) yields the same bound as \( I_2 \). The proof of uniqueness in (D2) is now complete.

**Uniqueness in (R2).** Suppose now that \( \tilde{N}_e(\nabla u) \in L^2 \), and that \( u \rightarrow 0 \) n.t.. Choosing \( \theta \) as above, we split \( u(x, t) = (u(\theta))(x, t) \) into the same terms \( I + II \), which we dominate again by \( I_1 + I_2 \) and \( II_1 + II_2 \) as in (4.37) and (4.40), respectively. We now claim that

\[
I_1 + II_1 \leq C\epsilon^{-n/2} \|\tilde{N}_e(\nabla u)\|_2 \rightarrow 0
\]
as $\varepsilon \to 0$. For $I_1$, this follows from Cauchy-Schwarz and (4.39). To handle $H_1$, we first note that, by Lemma 4.3(i), $|u(y, s)| \leq C\tilde{N}_c(\nabla u)(y)$ in $\Omega_1$, since $u(\cdot, 0) = 0$ a.e.. The claim then follows from Cauchy-Schwarz and Caccioppoli (applied to $\nabla u$).

Rewriting the last expression in (4.37), we see that

$$I_2 = R^{(2-n)/2} \left( R^{-1} \int_{\Omega_2 \cup \Omega_3} |\nabla u|^2 \right)^{1/2} \leq CR^{(2-n)/2} ||\tilde{N}_c(\nabla u)||_{L^2(\Delta_{2\varepsilon}(x), \Delta_{2\varepsilon}(y))},$$

by construction of $\Omega_2 \cup \Omega_3$. Moreover, Lemma 4.3(iii) implies that $|u|/R \leq C\tilde{N}_c(\nabla u)$ in $\Omega_3$ and $|u|/R \leq C \inf_{\Delta_{2\varepsilon}(x), \Delta_{2\varepsilon}(y)} \tilde{N}_c(\nabla u)$ in $\Omega_2$. Thus, by (4.34),

$$H_2 \leq CR^{1/2} \left( \int_{\Omega_2 \cup \Omega_3} |\nabla u|^2 \right)^{1/2} \left( R^{-1} \int_{\Omega_2 \cup \Omega_3} |u|^{1/2} \right) \leq CR^{2-n/2} ||\tilde{N}_c(\nabla u)||_{L^2(\Delta_{2\varepsilon}(x), \Delta_{2\varepsilon}(y))}.$$

Since $n \geq 2$, we obtain dominated convergence to 0.

**Uniqueness in (N2).** Suppose that $\tilde{N}_c(\nabla u) \in L^2$, and that $\partial u/\partial y = 0$, where the latter is interpreted in the sense of Lemma 4.3(iii) and (iv). By Lemma 4.3(i), we have that $u \to u_0$ n.t., for some $u_0 \in L^2_1(\mathbb{R}^n)$. By uniqueness in (R2),

$$u(\cdot, t) = S_t(S_0^{-1}u_0),$$

where $S_0 \equiv S_{\mid t=0}$. Thus, by Lemma 4.18,

$$0 = \frac{\partial u}{\partial y} = \left( \frac{1}{2} I + \tilde{K} \right) (S_0^{-1}u_0).$$

But by hypothesis, $\frac{1}{2} I + \tilde{K} : L^2 \to L^2$ and $S_0 : L^2 \to L^2_1$ are bijective, so that $u_0 = 0$ in the sense of $L^2_1$, i.e., $u_0 \equiv \text{constant a.e.}$. By uniqueness in (R2), $u \equiv \text{constant}$.

As a corollary of uniqueness, we shall obtain the following “Fatou Theorem”.

**Corollary 4.41.** Let $L, L^*$ satisfy the standard assumptions, and have “Good Layer Potentials”. Suppose also that $Lu = 0$, and that

$$\sup_{t>0} \|u(\cdot, t)\|_2 < \infty.$$ \hfill (4.42)

Then $u(\cdot, t)$ converges n.t. and in $L^2$ as $t \to 0$.

**Proof.** By Lemma 4.23, it is enough to show that $u(\cdot, t) = D_\varepsilon h$ for some $h \in L^2(\mathbb{R}^n)$. We follow the argument in [Si2], pp. 199-200, substituting $D_\varepsilon$ for the classical Poisson kernel. For each $\varepsilon > 0$, set $f_\varepsilon \equiv u(\cdot, \varepsilon)$. Let $u_\varepsilon$ be the layer potential solution with data $f_\varepsilon$; i.e.,

$$u_\varepsilon(x, t) \equiv D_\varepsilon \left[ \left( \frac{1}{2} I + \tilde{K} \right)^{-1} f_\varepsilon \right] (x).$$

We claim that $u_\varepsilon(x, t) = u(x, t + \varepsilon)$.

**Proof of Claim.** Set $U_\varepsilon \equiv u(x, t + \varepsilon) - u_\varepsilon(x, t)$. We observe that

1. $LU_\varepsilon = 0$ in $\mathbb{R}^{n+1}$ (by $t$-independence of coefficients).
2. (4.42) holds for $U_\varepsilon$, uniformly in $\varepsilon > 0$
3. $U_\varepsilon(\cdot, 0) = 0$ and $U_\varepsilon(\cdot, t) \to 0$ n.t. and in $L^2$.

(Item (3) relies on interior continuity (1.2) and smoothness in $t$, along with Lemma 4.23). The claim now follows by Lemma 4.31. \hfill $\Box$
We return now to the proof of the Corollary. By (4.42), \( \sup_2 \| f_k \|_2 < \infty \). Hence, there exists a subsequence \( f_{k_j} \) converging in the weak* topology to some \( f \in L^2 \). For arbitrary \( g \in L^2 \), set \( g_1 \equiv \text{adj}(\frac{1}{2}I + K)^{-1} \text{adj}(D_i)g \), and observe that

\[
\int_{\mathbb{R}^n} D_i \left( \frac{1}{2}I + K \right)^{-1} f g = \int_{\mathbb{R}^n} f g_1 = \lim_{k \to \infty} \int_{\mathbb{R}^n} f_{k_j} g_1 = \lim_{k \to \infty} \int_{\mathbb{R}^n} D_i \left( \frac{1}{2}I + K \right)^{-1} f_{k_j} g = \int_{\mathbb{R}^n} u(\cdot, t + \epsilon_k) g = \int_{\mathbb{R}^n} u(\cdot, t) g.
\]

Since \( g \) was arbitrary, the desired conclusion follows. \( \square \)

We conclude this section with a discussion of n.t. convergence of gradients.

**Lemma 4.43.** Suppose that \( L, L^* \) satisfy the standard assumptions, and have “Good Layer Potentials”. Then for all \( f \in L^2 \), we have

\[
P_i((\nabla S_i)_{|x=\pm}) f \to \left( \frac{1}{2\Lambda_{a+1,n+1}} e_{n+1} + \mathcal{T} \right) f \quad \text{n.t. and in } L^2.
\]

**Proof.** We treat only the case of the upper half space, as the proof in the other case is the same. Since the weak limit has already been established (Lemma 4.18) for \( \nabla S_i \), it is a routine matter to verify that the strong and n.t. limits for \( P_i(\nabla S_i) \) will take the same value, once the existence of those limits has been established. It is to this last point that we therefore turn our attention. By Lemma 4.8 and the dominated convergence theorem, it is enough to establish n.t. convergence.

The non-tangential convergence of \( \partial S_i \), follows immediately from the “Fatou Theorem” just proved; a simple real variable argument yields the same conclusion for \( P_i \partial S_i \). We may therefore replace \( \nabla \) by \( \nabla \hat{S} \). On the other hand, we shall still need to consider the boundary trace of \( \partial S_i f \), which for the duration of this proof we denote by \( V f \). Fix now \( x_0 \in \mathbb{R}^n \). For \( |x - x_0| < \delta \), we write

\[
P_i(\nabla \hat{S}_{i,f})(x) = \nabla_{\hat{S}} P_i \left( \int_0^\delta \partial S_i f ds \right)(x) + P_i(\nabla \hat{S}_{i,f})(x) = \hat{Q}_i \left( \int_0^\delta \partial S_i f ds \right)(x) + P_i(\nabla \hat{S}_{i,f})(x) \equiv I + II,
\]

where \( \hat{Q}_i 1 = 0 \). By standard facts for approximate identities, \( II \to \nabla \hat{S}_{i,f} \) n.t.. Also,

\[
I = \hat{Q}_i \left( \frac{1}{\delta} \int_0^\delta (\partial S_i f - V f) ds \right)(x) + \hat{Q}_i (V f - V f(x_0)) \equiv I_1 + I_2.
\]

It is straightforward to verify that \( I_2 \to 0 \) as \( t \to 0 \), if \( x_0 \) is a Lebesgue point for the \( L^2 \) function \( V f \). The term \( I_1 \) is more problematic. We first observe that by Lemma 4.3,

\[
\text{lim sup}_{t \to 0} \left| \hat{Q}_i \left( \frac{1}{\delta} \int_0^\delta (S_i f - S_{0,f}) ds \right)(x) \right| \leq C t M(\tilde{N}_i(\nabla S_i f))(x_0) \to 0 \quad \text{for a.e. } x_0.
\]

Thus also for \( \tilde{f} \in C_0^\infty(\mathbb{R}^n) \), we have

\[
\text{lim sup}_{t \to 0} \left| \hat{Q}_i \left( \frac{1}{\delta} \int_0^\delta (S_i \nabla \hat{S}) \cdot \tilde{f} - (S_{0,\nabla \hat{S}}) \cdot \tilde{f}) ds \right)(x) \right| \to 0 \text{ n.t.}
\]

\[
\text{(4.44)} \quad \left| \hat{Q}_i \left( \frac{1}{\delta} \int_0^\delta (S_i \nabla \hat{S}) \cdot \tilde{f} - (S_{0,\nabla \hat{S}}) \cdot \tilde{f}) ds \right)(x) \right| \to 0 \text{ n.t.}
\]
By Lemma 4.8(v), the density of $C_{0}^{\infty}$ in $L^{2}$, and the fact that $\tilde{Q}$ is dominated by the Hardy-Littlewood maximal operator which is bounded from $L^{2,\infty}$ to itself, the latter convergence continues to hold for $\tilde{f} \in L^{2}$. Moreover, if $u_{0}$ belongs to the dense class $\{\mathcal{S}_{0} \, \text{div} \, \tilde{g} : \tilde{g} \in C_{0}^{\infty}\}$, by Corollary 4.28 and (4.44), we have that

\[ (4.46) \quad \left| \tilde{Q} \left( \frac{1}{t} \int_{0}^{t} \mathcal{D}_{\gamma} u_{0} - t \mathcal{D}(\mathcal{D}_{\gamma} u_{0}) \right) \, ds \right| (x) \rightarrow 0 \text{ n.t.,} \]

and again this fact remains true for $u_{0}$ in $L^{2}$, by Lemma 4.8(vi) and our previous observation concerning the action of the maximal operator on weak $L^{2}$. Combining (4.45) and (4.46) with the adjoint version of the identity (4.20), we obtain convergence to 0 for the term $I_{1}$ since every $f \in L^{2}$ can be written in the form $f = A_{n=1,n+1}^* h$, $h \in L^{2}$.

5. PROOF OF THEOREM 1.11: PRELIMINARY ARGUMENTS

As noted above, the De Giorgi-Nash estimate (1.2) is stable under $L^{\infty}$ perturbation of the coefficients. Thus, for $\varepsilon_{0}$ sufficiently small, solutions of $L_{1} u = 0$, $L_{1}^{*} w = 0$ satisfy (1.2) and (1.3). In particular, the results of Section 2 apply to the fundamental solutions and layer potentials $\mathcal{F}_{0}, S_{0}^{1}$ and $\mathcal{F}, S_{1}^{1}$, corresponding to $L_{0}$ and $L_{1}$, respectively.

We claim that the conclusion of Theorem 1.11 will follow, once we have proved

\[ (5.1) \quad \| \| \nabla \partial_{\nu} S_{i}^{1} \|_{op} + \sup_{r>0} \| \nabla S_{1}^{1} \|_{L^{2} \rightarrow 2} \leq C \]

(recall that $\nabla \equiv \nabla_{x,y}$). Indeed, by the symmetry of our hypotheses, similar bounds will then hold in the lower half space, and for $S_{1}^{L_{1}}$. Now, by $\nu$-independence, $-(S_{1}^{1} D_{n+1}) = D_{n+1} S_{1}^{1}$. Moreover, if $\mathcal{F}_{i}(x,y)$ denotes the kernel of $(S_{1}^{1} \nabla_{y})$, and $\Gamma_{i}$ is the fundamental solution for the adjoint operator $L_{1}^{*}$, then the kernel of $adj(S_{1}^{1} \nabla_{y})$ is

\[ \mathcal{F}_{i}(y,x) = \nabla_{y} \Gamma_{i}(y,t,x,0) = \nabla_{y} \Gamma_{i}(x,0,y,t) = \nabla_{y} \Gamma_{i}(x,-t,y,0). \]

Consequently, $adj(S_{1}^{1} \nabla_{y}) = \nabla_{y} S_{1}^{L_{1}}$, so that $L_{1}^{2}$ boundedness of $(S_{1}^{1} \nabla)$ (and hence of $\mathcal{D}_{1}^{1}$) follows from that of $\nabla S_{1}^{L_{1}}$. Thus, by Lemma 4.18, we also obtain $L^{2}$ bounds for $K^{1}, \tilde{K}^{1}$ and $\mathcal{T}^{1}$. Appropriate non-tangential control follows from Lemma 4.8. Moreover, since we have allowed complex coefficients, analytic perturbation theory implies that

\[ \| K^{0} - K^{1} \|_{L^{2} \rightarrow 2} + \| \tilde{K}^{0} - \tilde{K}^{1} \|_{L^{2} \rightarrow 2} + \| \mathcal{T}^{0} - \mathcal{T}^{1} \|_{L^{2} \rightarrow 2} \leq C \| \mathcal{A}^{0} - \mathcal{A}^{1} \|_{L^{\infty}}. \]

The method of continuity then yields the invertibility of $\pm \frac{1}{2} I + K^{1} : L^{2} \rightarrow L^{2}, \pm \frac{1}{2} I + \tilde{K}^{1} : L^{2} \rightarrow L^{2}$ and $S_{0}^{L_{1}} \equiv S_{1}^{L_{1}}|_{t=0} : L^{2} \rightarrow L^{2}_{\mathcal{J}}$. It therefore suffices to prove (5.1).

**Lemma 5.2.** Suppose that $L, L^{*}$ satisfy the standard assumptions. For $f \in C_{0}^{\infty}, \eta > 0$, and $t_{0} \geq 0$, we have

\[ (5.3) \quad \| \nabla S_{i,j} \|_{L^{2}} \leq C \left( \| N_{i} (P \partial_{t} S_{i,j},f) \|_{L^{2}} + \| \nabla \partial_{t} S_{i,j} \| + \| f \|_{L^{2}} \right) \]

\[ (5.4) \quad \| \nabla S_{i,j}^{0} \|_{L^{2}} \leq C \left( \| N_{i} (P \partial_{t} S_{i,j}^{0},f) \|_{L^{2}} + \| \nabla \partial_{t} S_{i,j}^{0} \| + \| f \|_{L^{2}} \right) \]

\[ (5.5) \quad \| \nabla \partial_{t} S_{i,j} \| \leq C \| \partial_{x}^{2} S_{i,j} \| + C \| f \|_{L^{2}} \]

\[ (5.6) \quad \| \nabla \partial_{t} S_{i,j}^{0} \| \leq C \| \partial_{x}^{2} S_{i,j}^{0} \| + C \| f \|_{L^{2}}. \]

The analogous bounds hold also in the lower half-space.
Before proving the lemma, let us use it to reduce the proof of Theorem 1.11 to two main estimates, whose proofs we shall give in the next two sections. We claim that it suffices to prove that for all \( f \in C_0^\infty \), and \( \eta \in (0, 10^{-10}) \), we have
\[
(5.7) \quad \left\| \nabla \partial_t S_t^{1,\eta} f \right\|_{\text{all}} \leq C \varepsilon_0 \left( \left\| \nabla \partial_t S_t^{1,\eta} f \right\|_2 + \sup_{t \neq 0} \left\| \nabla S_t^{1,\eta} f \right\|_2 + C \|f\|_2
\]
\[
(5.8) \quad \sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_2 \leq C \varepsilon_0 \left( \left\| \nabla \partial_t S_t^{1,\eta} f \right\|_2 + \sup_{t \neq 0} \left\| \nabla S_t^{1,\eta} f \right\|_2 + C \|f\|_2,
\]
where \( N_s^{db} \) denotes the non-tangential maximal operator with respect to the double cone \( \gamma^{db}(x) \equiv \gamma^*(x) \cup \gamma^-(x) \equiv \{ (y, t) \in \mathbb{R}^{n+1} : |x - y| < |t| \}. \) Indeed, for \( \varepsilon_0 \) sufficiently small, Lemma 2.18 (iii) and (5.6) allow us to hide the small triple bar norm in (5.7), so that
\[
(5.9) \quad \left\| \nabla \partial_t S_t^{1,\eta} f \right\|_{\text{all}} \leq C \varepsilon_0 \left( \left\| \nabla S_t^{1,\eta} f \right\|_2 + \sup_{t \neq 0} \left\| \nabla S_t^{1,\eta} f \right\|_2 + C \|f\|_2.
\]
Using (5.4), (5.9) and hiding the small gradient term via Lemma 2.18 (i, ii), we obtain
\[
(5.10) \quad \sup_{t \neq 0} \left\| \nabla S_t^{1,\eta} f \right\|_2 \leq C \varepsilon_0 \left( \sup_{t \neq 0} \left\| N_s^{db} \left( P_t \partial_t S_t^{1,\eta} f \right) \right\|_2 + \sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_2 + \|f\|_2
\]
where the notation \( N_s^{db} \left( P_t \partial_t S_t^{1,\eta} f \right) \) is interpreted to mean \( t + t_0 \) in the upper cone \( \gamma^* \), and \( t - t_0 \) in the lower cone \( \gamma^- \). Feeding the latter estimate back into (5.9), we obtain
\[
(5.11) \quad \left\| \nabla \partial_t S_t^{1,\eta} f \right\|_{\text{all}} \leq C \varepsilon_0 \left( \sup_{t \neq 0} \left\| N_s^{db} \left( P_t \partial_t S_t^{1,\eta} f \right) \right\|_2 + \sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_2 + C \|f\|_2.
\]
Combining (5.8), (5.10) and (5.11), we have
\[
\sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_2 \leq C \|f\|_2 + C \varepsilon_0 \left( \sup_{t \neq 0} \left\| N_s^{db} \left( P_t \partial_t S_t^{1,\eta} f \right) \right\|_2 + \sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_2
\]
Since \( f \in C_0^\infty \), there is a large cube \( Q \) centered at 0 containing the support of \( f \). By Lemma 4.8 (iv), taking a supremum over all \( f \in C_0^\infty(Q) \), with \( \|f\|_{L^2(Q)} = 1 \), we have
\[
\sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_{L^2(Q) - L^2(\mathbb{R}^n)} \leq C \left( 1 + \varepsilon_0 \sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_{L^2(Q) - L^2(\mathbb{R}^n)} \right).
\]
Using Lemma 2.18 (vii), we may hide the small term to obtain
\[
\sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_{L^2(Q) - L^2(\mathbb{R}^n)} \leq C \varepsilon_0 \sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_{L^2(Q) - L^2(\mathbb{R}^n)} \leq C
\]
uniformly in \( Q \). Thus, letting \( \ell(Q) \to \infty \), and then \( \eta \to 0 \), we obtain by Lemma 2.18 (iv) that
\[
\sup_{t \neq 0} \left\| \partial_t S_t^{1,\eta} f \right\|_{L^2(\mathbb{R}^n)} \leq C.
\]
In addition, (5.12) and Lemma 4.8 (iv) and a limiting argument as \( \ell(Q) \to \infty \) imply that
\[
\sup_{t \neq 0} \left\| N_s^{db} \left( P_t \partial_t S_t^{1,\eta} f \right) \right\|_2 \leq C \|f\|_2, \quad f \in L^2(\mathbb{R}^n).
\]
The latter estimate, (5.11), (5.12) and Lemma 2.18 (v) yield the bound for the first term in (5.1). The bound for the second term in (5.1) follows from (5.3), the bound just established for \( \|\partial_t S_t^{1,\eta} f\|_{L^p} \), the fact that \( N_s \left( P_t \partial_t S_t^{1,\eta} f \right) \leq C M \left( N_s \left( \partial_t S_t^{1,\eta} f \right) \right) \), Lemma 4.8 (v) and (5.13). The estimates (5.7) and (5.8) are the heart of the matter, and will be proved in sections 6 and 7, respectively.
We return now to the proof of the lemma.

Proof of Lemma 5.2. We prove (5.5) first. We have that \( \|\nabla \partial_t S_{t;f}\|^2 \) =

\[
\lim_{\varepsilon \to 0} \left\| \nabla \partial_t S_{t;f} \right\|^2(e) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_0^{1/\varepsilon} \nabla \partial_t S_{t;f} \cdot \nabla \partial_t S_{t;f} dt
\]

\[
= -\frac{1}{2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \int_0^{1/\varepsilon} \partial_t (\nabla \partial_t S_{t;f} \cdot \overline{\nabla \partial_t S_{t;f}}) dt + \text{"OK"},
\]

where we may use Lemma 2.8(ii) to dominate the "OK" boundary terms by \( C\|f\|^2 \). By Cauchy’s inequality, we then obtain that

\[
\|\nabla \partial_t S_{t;f}\|^2(e) \leq \frac{\delta}{2} \|\nabla \partial_t S_{t;f}\|^2(e) + \frac{C}{\delta} \|\nabla \partial_t S_{t;f}\|^2(e) + C\|f\|^2,
\]

where \( \delta \) is at our disposal. For \( \delta \) small, we can hide the first term. The second term is bounded by \( \|\nabla \partial_t S_{t;f}\| \), as may be seen by splitting \( \mathbb{R}^{n+1}_+ \) into Whitney boxes, and applying Caccioppoli’s inequality. The bound (5.5) now follows.

The proof of (5.6) is similar. We write

\[
\|\nabla \partial_t S_{t;f}\|^2 = \int_0^{2\eta} \int_{\mathbb{R}^n} + \int_0^\infty \int_{\mathbb{R}^n} = I + II.
\]

Term II may be handled just like (5.5), since by definition (2.17),

\[
\|\nabla \partial_t S_{t;f}\| \leq C \left( \hat{\varphi}_\eta \ast (1_{\eta < |\nabla \partial_t S_{t;f}|}) \right)(t), \quad t > 2\eta,
\]

and \( u(x, t) \equiv \partial_t S_{t;f} \) solves \( Lu = 0 \) in the half space \( \{ t > \eta \} \). We omit the details. To bound term I, we note that by definition (2.17), \( \partial_t S_{t;f} \) solves \( L^{-1}(D_{n+1} \eta_1)(x, t) \), where \( \eta_1(y, s) \equiv f(y) \hat{\varphi}_\eta(s) \), so that

\[
I \leq C \eta \int_0^{2\eta} \int_{\mathbb{R}^n} |\nabla \partial_t S_{t;f}|^2 dx dt \leq C\eta \int_0^{2\eta} \int_{\mathbb{R}^n} |\nabla \partial_t S_{t;f}|^2 dx dt = C\|f\|_2^2.
\]

where we have used that \( \nabla \partial_t \) \( \text{div} : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1}) \).

Next, we prove (5.3). By the ellipticity of the sub-matrix \( A_{2i} \), we have that

\[
\|\nabla S_{t;f}\|_2 \leq C\|A_t \nabla S_{t;f}\|_2.
\]

Now let \( \tilde{g} \in C_0^{\infty}(\mathbb{R}^n, C^0) \), with \( \|g\|_2 = 1 \). By the Hodge decomposition [AT, p. 116], we have that \( g = \nabla F + \tilde{h} \), where \( F \in L^2_1(\mathbb{R}^n), \|\nabla F\|_2 \leq C\|g\|_2 \) (C depending only on ellipticity), \( h \in L^2_1(\mathbb{R}^n) \) and \( \text{div}(A_1) \tilde{h} = 0 \) in the sense that \( \int A_1 \nabla \zeta \cdot \tilde{h} = 0 \) for all \( \zeta \in L^2_1 \).

Lema 2.9, with \( m = -1 \), ensures that \( S_{t;f} \in L^2_1 \), (albeit without quantitative bounds). Thus, for \( f \in C_0^{\infty}(\mathbb{R}^n) \), we have

\[
\langle A_1 \nabla \|S_{t;f}\|_2, g \rangle = \langle A_1 \nabla \|S_{t;f}\|_2, \nabla F \rangle,
\]

and it suffices to bound the latter expression with \( F \in C_0^{\infty} \). Now,

\[
\langle A_1 \nabla \|S_{t;f}\|_2, \nabla F \rangle = -\int_0^\infty \partial_t \langle A_1 \nabla \|g\| g e^{-\tilde{L}_k t} S_{t+\tilde{t}_k} f, \nabla g e^{-\tilde{L}_k t} F \rangle dt
\]

\[
= 2 \int_0^\infty \left\{ \langle A_1 \nabla t L_k g e^{-\tilde{L}_k t} S_{t+\tilde{t}_k} f, \nabla g e^{-\tilde{L}_k t} F \rangle + \langle A_1 \nabla \|g\| g e^{-\tilde{L}_k t} S_{t+\tilde{t}_k} f, \nabla \|g\| g e^{-\tilde{L}_k t} F \rangle \right\} dt
\]

\[
- \int_0^\infty \langle A_1 \nabla g e^{-\tilde{L}_k t} \partial_t S_{t+\tilde{t}_k} f, \nabla g e^{-\tilde{L}_k t} F \rangle dt = I + II - III.
\]
Integrating by parts, we see that

\[(5.14) \quad |L + IF| = 4 \left| \int_0^\infty \int_{\mathbb{R}^n} \left( L_0 e^{-tL_0} S_{t+\theta} f(x) \right) \left( L_0 f(x) e^{-tL_0} F(x) \right) \, dt \, dx \right| \leq 4 \left| \left| e^{-tL_0} L_0 S_{t+\theta} f \right| \right| \left| \left| (L_0 f(x) e^{-tL_0} F(x)) \right| \right| \leq C \left| \left| e^{-tL_0} L_0 S_{t+\theta} f \left\| \nabla F \right\|_2 \right| \right|,
\]

since, by [AHLMcT], applied to \((L_0 f(x) e^{-tL_0} F(x))\), we have that \(\left| \left| (L_0 f(x) e^{-tL_0} F(x)) \right| \right| \leq C \left\| \nabla F \right\|_2\). We can consider the first factor on the right side of (5.14). Since \(u(x, t) \equiv S_{t+\theta} f(x)\) solves \(Lu = 0\), we have

\[L_0 S_{t+\theta} f = \sum_{i=1}^n D_i A_{i,n+1} D_{n+1} S_{t+\theta} f + \sum_{j=1}^{n+1} A_{n+1,j} D_j D_{n+1} S_{t+\theta} f \equiv \Sigma_1 + \Sigma_2,
\]
in the weak sense of Lemma 2.9. Since \(e^{-tL_0} : L^2 \rightarrow L^2\) uniformly in \(t\), we obtain

\[\left| \left| e^{-tL_0} \Sigma_2 \right| \right| \leq C \left| \left| \nabla \partial_t S_{t+\theta} f \right| \right| \leq C \left| \left| \nabla \partial_t S f \right| \right|
\]
which is one of the allowable terms in the bound that we seek. Also,

\[(5.15) \quad te^{-tL_0} \Sigma_1 = R_0 \partial_t S_{t+\theta} f + \sum_{i=1}^n (te^{-tL_0} D_i A_{i,n+1}) \partial_l D_i S_{t+\theta} f,
\]

where, by the familiar “Gaffney estimate” (e.g., [AHLMcT], pp. 636-637), the operator

\[R_0 \equiv \sum_{i=1}^n (te^{-tL_0} D_i A_{i,n+1} - (te^{-tL_0} D_i A_{i,n+1} P_i))
\]
satisfies the bound (3.1) for every \(m \geq 1\) (indeed, it satisfies a stronger exponential decay estimate). Moreover, \(R_0 1 = 0\), and \(R_0 : L^2 \rightarrow L^2\). Thus, by Lemma 3.5 we have

\[\left| \left| R_0 \partial_t S_{t+\theta} f \right| \right| \leq C \left| \left| \nabla \partial_t S_{t+\theta} f \right| \right| \leq C \left| \left| \nabla \partial_t S f \right| \right|
\]
as desired. In addition, by [AHLMcT], we have that \(\left| te^{-tL_0} \partial_j \partial_i \partial_{\alpha} \right| \) is a Carleson measure for all \(\alpha \in \mathbb{Z}^n\). Therefore, by Carleson’s Lemma, the triple bar norm of the last term in (5.15) is dominated by \(\left| \left| N_j(P_i \partial_l S_{t+\theta} f) \right| \right|_{L^2}\).

It remains to handle the term III. Integrating by parts in \(t\), we obtain

\[(5.16) \quad -III = \int_0^\infty \langle A_{ij} \nabla \partial_j \partial_i e^{-tL_0} S_{t+\theta} f, \nabla \partial_l e^{-tL_0} F(x) \rangle dt + \text{ "easy"},
\]

where the two “easy terms” arise when \(\partial_j\) hits either \(e^{-tL_0}\) or \(e^{-tL_0} F\). These two easy terms may be handled by an argument similar to, but simpler than the one used to treat (5.14) above. The main term in (5.16) is dominated by

\[\left| \left| te^{-tL_0} \partial_j \partial_i e^{-tL_0} S_{t+\theta} f \right| \right| \left| \left| (L_0 f(x) e^{-tL_0} F(x)) \right| \right| \leq C \left| \left| \partial_j \partial_i S f \right| \right| \left| \nabla F \right|_2\]

where we have used the \(L^2\) boundedness of \(e^{-tL_0}\) to estimate the first factor, and [AHLMcT] to handle the second.

Finally, (5.4) may be proved in the same way as (5.3) with one minor modification. Since \(LS^0 f(x) = f_0(x, t) \equiv f(x) \phi_0(t)\), the application of Lemma 2.9 produces, in addition to the analogues of \(\Sigma_1\) and \(\Sigma_2\), an error term \(f_0(t+\theta, t+\theta)\). But

\[\left| \left| te^{-tL_0} f_0(t+\theta, t+\theta) \right| \right| \leq C \left( \eta \int |\phi_0(t+\theta)|^2 dt \right)^{1/2} \left| \left| f \right| \right|_{L^2(\mathbb{R}^n)} = C \left| \left| f \right| \right|_2,
\]
and (5.4) follows. \(\Box\)

We finish this section with a variant of the square function estimates.
Lemma 5.17. Suppose that $L, L^*$ satisfy the standard assumptions, and have “Good Layer Potentials”. Then for $m \geq 0$, we have the square function bound

$$\|\partial_{\xi}^{m+1}(S, \nabla) \cdot f\|_2 \leq C\|f\|_2,$$

where $f \in L^2(\mathbb{R}^n, C^{m+1})$.

**Proof.** By $t$-independence and Caccioppoli’s inequality in Whitney boxes, we may reduce to the case $m = 0$. By $t$-independence and (1.10), we may replace $\nabla$ by $\nabla_1$. By ellipticity of the $n \times n$ sub-matrix $A_k$, and the Hodge decomposition of [AT, p. 116], as in the proof of Lemma 5.2, it suffices to show that

$$\|\partial_{\xi}(S, \nabla) \cdot A_k \|_2 \leq C\|\nabla F\|_2,$$

with $F \in \mathcal{S}_0^1(\mathbb{R}^n)$ (which is dense in $L^2$, by the bijectivity of the mapping $S_0^1 : L^2 \to L^2(n)$). In the weak sense of Lemma 2.9, we have

$$\langle L_0, S(x, t, y, s) \rangle = \sum_{i=1}^{n} \frac{\partial}{\partial y_i} (A_{i, 2}^{\alpha_1}(y) \partial_{\xi}^\alpha \Gamma(x, t, y, s)) + \sum_{j=1}^{n+1} A_{n+1, j}^{\alpha_1}(y) \frac{\partial}{\partial y_j} \partial_{\xi}^\alpha \Gamma(x, t, y, s).$$

By $t$-independence, we therefore have that

$$\partial_{\xi}(S, \nabla) \cdot A_k F = \sum_{i=1}^{n} \partial_{\xi}^\alpha S_i A_{n+1, i} D_i F + \partial_{\xi}^\alpha (S_{\xi} \partial_{\xi} F),$$

where $\partial_{\xi}^\alpha = - \sum_{j=1}^{n+1} A_{n+1, j}^{\alpha_1} D_j$. We set $u(\cdot, \tau) = S \psi, \tau < 0$, so that $u(\cdot, 0) \equiv F$. Using “Good Layer Potentials”, we obtain in particular that

$$\|\nabla u(\cdot, 0)\|_2 \leq C\|\nabla F\|_2.$$ 

Since $(S, \partial_{\xi} \cdot) = D_t$, Corollary 4.28 implies that

$$\partial_{\xi}^\alpha (S_{\xi} \partial_{\xi} F) = \partial_{\xi}^\alpha S_i (\partial_{\xi} u(\cdot, 0)).$$

Consequently, the left hand side of (5.18) is dominated by

$$\sum_{i=1}^{n} \||\partial_{\xi}^\alpha S_i A_{n+1, i} D_i F\|_2 + \||\partial_{\xi}^\alpha S_i (\partial_{\xi} u(\cdot, 0))\|_2 \leq C\|\nabla F\|_2,$$

where in the last step we have used (1.10) and (5.19). \qed

6. **Proof of Theorem 1.11: the square function estimate (5.7)**

In this section we prove estimate (5.7). To be precise, suppose that $\varphi_{\delta} = \varphi_{\delta}^\alpha(\cdot/\delta)$ is the kernel of a nice approximate identity in 1 dimension, as in the definition of $S_{\delta}^\alpha (\cdot)$. We shall prove that, for all $f \in C_0^\infty(\mathbb{R}^n)$, for all $\Psi \in C_0^\infty(\mathbb{R}^{n+1})$, with $||\Psi||_1 \leq 1$, and for all $\delta > 0$ sufficiently small, if $\Psi_{\delta}(x, t) \equiv \varphi_{\delta} * \Psi(x, \cdot)(t)$, then

$$\int_{\mathbb{R}^{n+1}} \partial_t^\delta S_{\delta}^{1, \alpha} f(x) \nabla \Psi_{\delta}(x, t) \frac{dx dt}{t} \leq C\varepsilon_0 (M^+ + M^-) + C\|f\|_2,$$

where

$$M^+ \equiv \left(\||\partial_t \partial_{\xi}^{1, \alpha} f\|_2 + ||N_+ (P_t \partial_t S_{\delta}^{1, \alpha} f)\|_2 + \sup_{t \geq 0} ||\nabla S_{\delta}^{1, \alpha} f\|_2 + ||f\|_2\right),$$

and $M^-$ is the corresponding quantity for the lower half-space. The proof of the analogous estimate in $\mathbb{R}^{n+1}$ is identical, and we omit it. By Lemma 2.18 (iii), we may take first the limit as $\delta \to 0$, and then the supremum over all such $\Psi$ to obtain (5.7).
The proof is by perturbation. Setting \( \epsilon(\zeta) \equiv A^1(\zeta) - A^0(\zeta) \), we have
\[
L_0^{-1} - L_1^{-1} = L_0^{-1} L_1 - L_0^{-1} L_1^{-1} = -L_0^{-1} \mathrm{div} \nabla L_1^{-1}.
\]
Since \( \|\nabla^2 S_{ij}^0 f\| \leq C\|f\|_2 \), we have also that \( \sup_{\partial \Omega} \|\nabla^2 S_{ij}^{0,j} f\| \leq C\|f\|_2 \), as may be seen by arguing as in the proof of (5.6). Thus, it is enough to consider the difference \( \nabla^2 \left( S_{ij}^1 - S_{ij}^0 \right) \). By definition (2.17),
\[
(6.3) \quad \partial_i S_j^{0,j} f(x) = \left( (D_{n+1} \varphi_{\eta}) * S_j^1 f(x) \right)(t) = L_1^{-1} (D_{n+1} f_{\eta})(x,t), \quad i, j = 1, 2,
\]
where \( f_{\eta}(y, s) \equiv f(y)(\varphi_{\eta}(s), \varphi_{\eta} = \eta^{-1} \varphi(-\eta) \) is as above. We then have
\[
\partial_i^2 S_j^{0,j} f(x) - \partial_i^2 S_j^{1,j} f(x) = \partial_i \left( L_0^{-1} \mathrm{div} \nabla L_1^{-1} (D_{n+1} f_{\eta}) \right)(x,t) = \partial_i \left( L_0^{-1} \mathrm{div} \nabla D_{n+1} S_j^{1,j} f \right)(x,t),
\]
so that
\[
\int_{\mathbb{R}^{n+1}} \left( \partial_i^2 S_j^{1,j} f(x) - \partial_i^2 S_j^{0,j} f(x) \right) \Psi_{\delta}(x,t) \frac{dx dt}{t} = \int_{\mathbb{R}^{n+1}} \epsilon(y) \partial_i S_j^{0,j} f(y) \cdot \nabla (L_0^{-1})(D_{n+1} \Psi_{\delta})(y,s) dy ds.
\]
Essentially following [FJK], and using (6.3), we decompose
\[
\nabla (L_0^{-1})(D_{n+1} \Psi_{\delta})(y,s) = \int \nabla_{y,s} S_{ij}^{\eta,\delta} \left( \Psi(\cdot, t) \right)(y) \, dt
\]
\[
= \int_{|s| \leq 2|t|} \left( \nabla_{y,s} S_{ij}^{\eta,\delta} \right) \left. \left( \Psi(\cdot, t) \right)(y) \right|_{s=0} \, dt
\]
\[
+ \int_{|s| > 2|t|} \left( \nabla_{y,s} S_{ij}^{\eta,\delta} \right) \left. \left( \Psi(\cdot, t) \right)(y) \right|_{s=0} \, dt
\]
\[
= \int_{|s| < 2|t|} \frac{\sqrt{t} - \sqrt{|s|}}{\sqrt{t}} \nabla_{y,s} S_{ij}^{\eta,\delta} \left( \Psi(\cdot, t) \right)(y) \, dt
\]
\[
+ \int \left( \frac{|s|}{t} \right)^{1/2} \nabla_{y,s} S_{ij}^{\eta,\delta} \left( \Psi(\cdot, t) \right)(y) \, dt
\]
\[
- \int_{|s| < 2|t|} \left( \frac{|s|}{t} \right)^{1/2} \nabla_{y,s} S_{ij}^{\eta,\delta} \left( \Psi(\cdot, t) \right)(y) \, dt \equiv I + II + III + IV - V.
\]
In turn, this induces a corresponding decomposition in (6.4):
\[
I + II + III + IV - V \equiv \int_{\mathbb{R}^{n+1}} \left( \epsilon(y) \right) \nabla_{y,s} S_{ij}^{0,j} f(y) \cdot \left( \nabla (L_0^{-1})(D_{n+1} \varphi_{\eta} \left( \frac{\Psi}{\sqrt{t}} \right) \right) (y, s) dy ds.
\]

All but term II will be easy to handle, and we shall deal with these easy terms as in [FJK]. The main term here (and in [FJK]) is II, but in our situation, matters are much more delicate, since for us \( A^0 \) is not constant. The approach of [FJK] depends critically on the fact that solutions of constant coefficient equations are, in particular, twice differentiable, a fact which fails utterly in the present setting (unless at least one of the derivatives falls on the \( t \)-variable). We shall require new methods, which exploit the technology of the solution of the Kato problem, to deal with term II.

We dispose of the easy terms in short order. To begin,
\[
IV = \int_{\mathbb{R}^{n+1}} \left( |s|^{1/2} \epsilon(y) \nabla_{y,s} S_{ij}^{0,j} f(y) \cdot \nabla (L_0^{-1})(D_{n+1} \left( \varphi_{\eta} \left( \frac{\Psi}{\sqrt{t}} \right) \right) (y, s) dy ds.
\]
Proof of Lemma 6.9. The general case is handled by an almost identical argument. We begin by showing that (6.11) is rather delicate. For the sake of notational simplicity, we treat only the case $m = 1$ of Lemma 2.8. Otherwise, we obtain the better bound $C\delta^{-1}$, using definition (2.17) and the hypothesis that $L_0, L_0^*$ have bounded layer potentials. Estimate (6.6) is obtained from the following lemma, I, III and V may be handled by Hardy’s inequality, yielding also the bound $|I| + |III| + |V| \leq C\|\nabla \partial_1 S_{1,0}^f\|_a$. We omit the details.

**Lemma 6.5.** We have

\begin{align}
\|\nabla D_{n+1} S_{\tau,\delta}^{\tau,\eta} - \nabla D_{n+1} S_{\tau,\delta}^{\tau,\eta}\|_{L^2_{\tau-2}} &\leq C\frac{\delta}{|\tau|}, \quad |\tau| < t/2, \quad \delta < 1000^{-1}t \\
\|\nabla \partial_1 S_{\tau,\delta}^{\tau,\eta}\|_{L^2_{\tau-2}} &\leq C \tau \neq 0
\end{align}

*Proof of the Lemma.* If $|\tau| > 100\delta$, estimate (6.7) is essentially just the case $m = 0$ of Lemma 2.8. Otherwise, we obtain the better bound $C\delta^{-1}$, using definition (2.17) and the hypothesis that $L_0, L_0^*$ have bounded layer potentials. Estimate (6.6) is obtained from the case $m = 1$ of Lemma 2.8, and the identity

\[
\nabla D_{n+1} S_{\tau,\delta}^{\tau,\eta} - \nabla D_{n+1} S_{\tau,\delta}^{\tau,\eta} = \int_0^t \nabla \partial_1 S_{\tau,\delta}^{\tau,\eta} dt.
\]

It remains to handle II, which equals

\[
\int_{\mathbb{R}^n} \left\{ \int_{t/2}^{t/2} (\partial S_{\tau,\delta}^{\tau,\eta}) \cdot (\nabla \nabla \partial_1 S_{\tau,\delta}^{\tau,\eta}) \right\} \cdot \nabla f(y) \, ds \right\} \cdot (\nabla D_{n+1} S_{\tau,\delta}^{\tau,\eta})(\Psi(\tau, t)) (y) dy dt
\]

\[
= - \int_{\mathbb{R}^n} (\partial S_{\tau,\delta}^{\tau,\eta}) \cdot \nabla f(y) \nabla S_{\tau,\delta}^{\tau,\eta}(\tau, t) \, ds.
\]

where we have used that for $\eta > 0$, $\nabla S_{\tau,\delta}^{\tau,\eta}$ does not jump across the boundary. Since $\Psi$ is compactly supported in $R^*_f$, for $\delta$ sufficiently small,

\[
t^{-1/2} |\Psi_\delta(x, t)| \leq C \int |\partial S_{\tau,\delta}^{\tau,\eta}(x, s)| ds.
\]

Thus, it is enough to bound $\|\partial S_{\tau,\delta}^{\tau,\eta} \cdot \nabla S_{\tau,\delta}^{\tau,\eta}\|_t$, plus a similar term with $-t/2$ in place of $t/2$, which may be handled in the same way. The desired bound then follows immediately from the change of variable $t \rightarrow 2t$ and (6.10) below.

**Lemma 6.9.** Suppose that $a \in \mathbb{R} \setminus \{0\}$, and define $M^+$ as in (6.2). Then

\begin{align}
\| \nabla (\partial S_{\tau,\delta}^{\tau,\eta}) \cdot \nabla S_{\tau,\delta}^{\tau,\eta}\|_t &\leq C(a)_{0} M^+ \\
\|\partial S_{\tau,\delta}^{\tau,\eta} \cdot \nabla S_{\tau,\delta}^{\tau,\eta}\|_t &\leq C(a)_{0} M^+.
\end{align}

Moreover, the analogous bound holds in the lower half space.

*Proof of Lemma 6.9.* This lemma is the deep fact underlying estimate (5.7), and the proof is rather delicate. For the sake of notational simplicity, we treat only the case $a = 1$, as the general case is handled by an almost identical argument. We begin by showing that (6.11) implies (6.10). Set

\[
J(a) = \int_0^{1/\sigma} \int_{\mathbb{R}^n} \left\| \partial (S_{\tau,\delta}^{\tau,\eta}) \cdot \nabla S_{\tau,\delta}^{\tau,\eta}\right\|^2 dx dt.
\]

After integrating by parts in $t$, we obtain that

\[
J(a) = - \Re \int_0^{1/\sigma} \int_{\mathbb{R}^n} \partial_t (S_{\tau,\delta}^{\tau,\eta}) \cdot \nabla S_{\tau,\delta}^{\tau,\eta} f dx dt + "OK"
\]
where by Lemma 2.8 (i), the “OK” boundary terms are dominated by $C_2 \sup_{\gamma > 0} \|\nabla S_1^{1/2} f\|_2^2$. 
By Cauchy’s inequality, modulo the “OK” terms,

$$J'(\sigma) \leq \frac{1}{2} J'(\sigma) + \|\varepsilon \nabla \partial_1 S_1^{1/2} f\|_2^2 + \|\nabla (\partial_1^2 S_1^{1/2} f)\|_2^2 \leq \frac{1}{2} J'(\sigma) + I + II.$$ 

The term $\frac{1}{2} J'(\sigma)$ may be hidden on the left hand side. By Lemma 2.8 (i) with $m = 0$, term $I$ is no larger than $C_2 \|\partial_1 \nabla S_1^{1/2} f\|_2^2$. The square root of the main term, $II$, is estimated in (6.11). Taking the latter for granted momentarily, we obtain (6.10) by letting $\sigma \to 0$.

We now turn to the proof of (6.11), again with $a = 1$. We make the splitting:

$$\hat{\theta}^2 \partial_1^2 (S_1^{1/2} f) \cdot \varepsilon \nabla S_1^{1/2} f = \sum_{i=1}^{n+1} \sum_{j=1}^n \hat{\theta}^2 \partial_1^2 (S_1^{1/2} f) \epsilon_{ij} D_i S_1^{1/2} f$$

$$+ \sum_{i=1}^{n+1} \hat{\theta}^2 \partial_1^2 (S_1^{1/2} f) \epsilon_{i,n+1} D_{n+1} S_1^{1/2} f \equiv \hat{V}_i f + \hat{V}_t f.$$

We treat $\hat{V}_t$ first. For $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^{n+1}$, set

$$\theta \mathbf{f} \equiv \hat{\theta}^2 \partial_1^2 (S_1^{1/2} f) \cdot \mathbf{f},$$

and let $\mathbf{e} \equiv (\epsilon_{1,n+1}, \epsilon_{2,n+1}, \ldots, \epsilon_{n+1,n+1})$. Then, using a well known trick of [CM], we write

$$\hat{V}_t f = (\hat{\theta}^2 - \hat{\theta} \mathbf{e} P_j \mathbf{e}) \partial_1 S_1^{1/2} f + (\hat{\theta}^2 P_j \mathbf{e}) \partial_1 S_1^{1/2} f \equiv \partial_1^2 \partial_1 S_1^{1/2} f + (\hat{\theta}^2 P_j \mathbf{e}) \partial_1 S_1^{1/2} f,$$

where as usual $P_j$ is a nice approximate identity. By Lemmas 5.17, 2.7, 3.2 and Carleson’s Lemma, the triple bar norm of the second summand is no larger than $C_{20} \|\nabla (P_j \partial_1 S_1^{1/2} f)\|_2^2$. In addition, by Lemma 3.5, we have that

$$\|\partial_1^2 \partial_1 S_1^{1/2} f\| \leq C_{20} \|\nabla \partial_1 S_1^{1/2} f\| \leq C_{20} \|\nabla \partial_1 S_1^{1/2} f\|.$$ 

It remains to control $\|\hat{V}_i f\|$, which is the primary difficulty. By definition,

$$\hat{V}_i = \hat{\theta} \mathbf{e} \nabla S_1^{1/2} \equiv \hat{\theta} \mathbf{e} \nabla S_1^{1/2},$$

where $\mathbf{e}$ is the $(n+1) \times n$ matrix $(\epsilon_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$. Recall that $A_1 i$ is the $n \times n$ sub-matrix of $A_1$ with $(A_1 I)_{ij} = A_1 i j, 1 \leq i, j \leq n$, and that $(L_1) i = - \nabla \cdot A_1 i \nabla$. Then

$$\hat{V}_i = \hat{\theta} \mathbf{e} \nabla \left( I - \left( I + \hat{\theta} \mathbf{e} \nabla \right)^{-1} \right) S_1^{1/2} = \hat{\theta} \mathbf{e} \nabla (I + \hat{\theta} \mathbf{e} \nabla)^{-1} S_1^{1/2} \equiv \hat{Y}_i + \mathbf{Z}_i.$$ 

We first consider $\hat{Y}_i$. Note that $(I - \left( I + \hat{\theta} \mathbf{e} \nabla \right)^{-1}) = \hat{\theta} \mathbf{e} \nabla (I + \hat{\theta} \mathbf{e} \nabla)^{-1} S_1^{1/2}$, so

$$\hat{Y}_i = \hat{\theta} \mathbf{e} \nabla \left( I + \hat{\theta} \mathbf{e} \nabla \right)^{-1} (L_1) i S_1^{1/2}.$$ 

As above, set $f_2(x,t) = f(x \mathbf{e}_y)$. In the weak sense of Lemma 2.9, we then have

$$(L_1) i S_1^{1/2} f = \sum_{i=1}^{n+1} D_1 A_{1,n+1} \partial_1 S_1^{1/2} f + \sum_{j=1}^{n+1} A_{1,j} D_1 \partial_1 S_1^{1/2} f + f_y,$$

and we denote by $Y_1^{(1)} + Y_1^{(2)} + Y_1^{(3)}$ the corresponding splitting of $\hat{Y}_i$. Now, by Lemma 2.8, $\hat{\theta} : L^2 \to L^2$, and it is well known that $\nabla (I + \hat{\theta} \mathbf{e} \nabla)^{-1} : L^2 \to L^2$. Thus

$$\|\hat{Y}_i^{(2)} f\| \leq C_{20} \|\nabla \partial_1 S_1^{1/2} f\|.$$
and also, as in the proof of (5.6),
\[ ||Y_t^{(3)}|| \leq C_0 ||t_f|| \leq C_0 ||f||_{L^2(Z)}, \]

We make a further decomposition of \( Y_t^{(1)} \) as follows:
\[ Y_t^{(1)} = (U_t \tilde{d} - (U_t \tilde{d})P_t \partial \tilde{S}_t^{1,\eta} + (U_t \tilde{d})P_t \partial \tilde{S}_t^{1,\eta} = \tilde{R} \partial \tilde{S}_t^{1,\eta} + (U_t \tilde{d})P_t \partial \tilde{S}_t^{1,\eta}, \]
where
\[ (6.12) \quad U_t \tilde{g} = \theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \nabla_{\|} \tilde{g}, \]
and \( \tilde{d} \equiv (A_{1,n+1}^1, A_{2,n+1}^1, \ldots, A_{n,n+1}^1) \). We now claim that
\[ (6.13) \quad ||U_t||_{op} \leq C_0 \]
Let us momentarily defer the proof of this claim. It is a standard fact that for two sets \( E \) and \( E' \subseteq \mathbb{R}^n \), with \( \tilde{g} \) supported in \( E' \), we have
\[ (6.14) \quad \left\| \tilde{r}^2 \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \nabla_{\|} \tilde{g} \right\|_{L^2(E')} \leq C \exp \left( \frac{-\text{dist}(E, E')}{Ct} \right) \| \tilde{g} \|_{L^2(E)} \]
(the corresponding fact for the operator \( \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \) is proved in [AHL,McT] for example, and (6.14) may be readily deduced from this fact plus the same argument). Thus, by Lemma 3.3, the operator \( U_t \) satisfies (3.1), with a bound on the order of \( C_0 \), whenever \( t \leq c l(Q) \). Therefore, by Lemma 3.2 and Carleson’s Lemma, we have that
\[ \|(U_t \tilde{d})P_t \partial \tilde{S}_t^{1,\eta} f\| \leq C_0 ||N_t(P_t \partial \tilde{S}_t^{1,\eta} f)||_2. \]
Moreover, by Lemmas 3.5 and 3.11, we have that
\[ ||\tilde{R} \partial \tilde{S}_t^{1,\eta} f|| \leq C_0 ||\nabla_{\|} \partial \tilde{S}_t^{1,\eta} f|| \leq C_0 ||\nabla \partial \tilde{S}_t^{1,\eta} f||. \]
To finish our treatment of \( Y_t \), it remains to prove (6.13). We continue to defer the proof of this estimate for the moment, and proceed to discuss the term \( Z_t \). We write
\[
Z_t = \theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} (S_t^{1,\eta} - S_0^{1,\eta})
+ \theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} - I)S_0^{1,\eta} + \theta_t \tilde{e} \nabla_{\|} S_0^{1,\eta} \equiv Z_t^{(1)} + Z_t^{(2)} + Z_t^{(3)}.
\]
By Lemma 5.17 with \( m = 1 \), we have that
\[ ||Z_t^{(3)} f|| \leq C_0 \sup_{r>0} ||\nabla S_0^{1,\eta} f||_2. \]
Also,
\[ Z_t^{(2)} = \theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \tilde{r}^2 \nabla (\tilde{r}_1 A_{1}^1 \nabla) S_0^{1,\eta} \equiv U_t A_{1}^1 \nabla S_0^{1,\eta} \]
(see (6.12)), so by the deferred estimate (6.13) we have that
\[ ||Z_t^{(2)} f|| \leq C_0 \sup_{r>0} ||\nabla S_0^{1,\eta} f||_2. \]
Integrating by parts, we obtain
\[ Z_t^{(1)} = \theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \int_0^1 \partial_s S_0^{1,\eta} ds = -\theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \int_0^1 s \tilde{r}_2^2 S_0^{1,\eta} ds
+ \theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \partial \tilde{S}_t^{1,\eta} \equiv \Omega_t^{(1)} + \Omega_t^{(2)}.
\]
By Lemma 3.3, and the fact that \( \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} = 0 \), we have that the operator
\[ R_t = \theta_t \tilde{e} \nabla_{\|} (I + \tilde{r}^2(L_1))^{-1} \]
satisfies the hypothesis of Lemma 3.5, with a bound on the order of $C_{e_0}$, so that

$$||\Omega_2^{(2)} f|| \leq C_{e_0} ||r\nabla_\cdot \tilde{S}_i^{1,0} f||.$$

Furthermore,

$$\Omega_2^{(1)} = - \int_0^s \frac{s}{I} \partial_x F \left( I + r^2(L_1) \right)^{-1} s \partial_x^{2} \tilde{S}_i^{1,0} \frac{ds}{s},$$

so by Lemma 3.12, we have

$$||\Omega_2^{(1)} f|| \leq C_{e_0} ||s\partial_x^{2} \tilde{S}_i^{1,0} f||.$$

Modulo (6.13), this concludes the proof of Lemma 6.9, and hence also that of (5.7).

We conclude the present section by proving (6.13). The proof will depend on some technology from the proof of the Kato square root conjecture. By ellipticity, it is enough to show that

$$||U_f A_i^1 \tilde{g}|| \leq C_{e_0} ||\tilde{g}||_2$$

for $\tilde{g} \in L^2(\mathbb{R}^n, \mathbb{C}^n)$. But

$$U_f A_i^1 = \theta \tilde{\epsilon} r^2 \nabla_\cdot \left( I + r^2(L_1) \right)^{-1} \mathrm{div}_\cdot A_i^1,$$

so by the Hodge decomposition [AT, p. 116], we may replace $\tilde{g}$ by $\nabla_\cdot F$, where $||\nabla_\cdot F||_2 \leq C||\tilde{g}||_2$. As usual, by density we may suppose that $F \in C_c^\infty$. Now

$$U_f A_i^1 \nabla_\cdot F = - \theta \tilde{\epsilon} \nabla_\cdot \left( I + r^2(L_1) \right)^{-1} \left( r^2(L_1) \right) F = \theta \tilde{\epsilon} \nabla_\cdot \left( I + r^2(L_1) \right)^{-1} F.$$

We recall that $\theta_i = r^2 \partial_i^2 (S_i^{1,0} \nabla_\cdot)$, so by Lemma 5.17 with $m = 1$,

$$||\theta_i \tilde{\epsilon} \nabla_\cdot F|| \leq C_{e_0} ||\nabla_\cdot F||_2.$$

The main term is

$$\theta \tilde{\epsilon} \nabla_\cdot \left( I + r^2(L_1) \right)^{-1} F \equiv \frac{1}{t} R_x F,$$

where by Lemmas 2.7, 2.8, and 3.3, and the fact that $\nabla_\cdot (I + r^2L_1)^{-1} = 0$, we have that $R_x$ satisfies the hypotheses of Lemma 3.9, with a bound on the order of $C_{e_0}$. Therefore, it suffices to prove the Carleson measure estimate

$$\int_0^{\gamma(Q)} \int_Q \frac{1}{t} \left| R_x \Phi(x) \right|^2 \frac{dxdt}{t} \leq C_{e_0} |Q|,$$

where $\Phi(x) \equiv x$. To this end, we write

$$\frac{1}{t} R_x \Phi = \theta \tilde{\epsilon} \nabla_\cdot \left( I + r^2(L_1) \right)^{-1} \Phi + \theta \tilde{\epsilon} \nabla_\cdot \Phi.$$

But $\nabla_\cdot \Phi = I$, the $n \times n$ identity matrix. Thus, Lemmas 5.17, 2.7 and 3.2 yield the bound

$$\int_0^{\gamma(Q)} \int_Q \left| \theta \tilde{\epsilon} \nabla_\cdot \Phi \right|^2 \frac{dxdt}{t} \leq C_{e_0} |Q|.$$

The remaining term in (6.15) equals

$$\theta \tilde{\epsilon} r^2 \nabla_\cdot \left( I + r^2(L_1) \right)^{-1} \mathrm{div}_\cdot A_i \nabla_\cdot \Phi = \theta \tilde{\epsilon} r^2 \nabla_\cdot \left( I + r^2(L_1) \right)^{-1} \mathrm{div}_\cdot A_i \equiv T_i A_i |\cdot|
We now invoke a key fact in the proof of the Kato conjecture. By [AHLMcT], there exists, for each $Q$, a mapping $F_Q = \mathbb{R}^n \to \mathbb{C}^n$ such that

\begin{align}
(i) \quad & \int_{\mathbb{R}^n} |\nabla_i F_Q|^2 \leq C|Q| \\
(ii) \quad & \int_{\mathbb{R}^n} |(L_1)_i F_Q|^2 \leq C\frac{|Q|}{t(Q)^2} \\
(iii) \quad & \sup_Q \int_0^t \int_Q |\tilde{\phi}(x,t)\zeta| \frac{dxdt}{t} \\
& \quad \leq C \sup_Q \int_0^t \int_Q |\tilde{\phi}(x,t)E_i \nabla \|F_Q(x)\|^2 dxdt,
\end{align}

for every function $\tilde{\phi} : \mathbb{R}^{n+1} \to \mathbb{C}^n$, where $E_i$ denotes the dyadic averaging operator, i.e. if $Q(x,t)$ is the minimal dyadic cube (with respect to the grid induced by $Q$) containing $x$, with side length at least $t$, then

$$E_i g(x) \equiv \int_{Q(x,t)} g.$$ 

Here $\nabla_i F_Q$ is the Jacobian matrix $(D_i(F_Q))_{j,i,j=1}^n$, and the product

$$\tilde{\phi} E_i \nabla ||F_Q||$$

is a vector. Given the existence of a family of mappings $F_Q$ with these properties, as in [AT, Chapter 3], we see by (iii), applied with $\tilde{\phi}(x,t) = T_r A_i$, that it is enough to show that

$$\int_0^t \int_Q \left| T_r A_i(x) E_i \nabla ||F_Q(x)||^2 \right| \frac{dxdt}{t} \leq C \varepsilon_0 |Q|.$$ 

But as in [AT], we may exploit the idea of [CM] to write

$$(T_r A_i)E_i \nabla ||F_Q|| = [(T_r A_i) E_i - T_r A_i] \nabla_i F_Q + T_r A_i \nabla_i F_Q$$

$$= (T_r A_i) E_i - (T_r A_i) P_i \nabla_i F_Q + [(T_r A_i) E_i - T_r A_i] \nabla_i F_Q + T_r A_i \nabla_i F_Q$$

$$= R_i^{(1)} \nabla_i F_Q + R_i^{(2)} \nabla_i F_Q + T_r A_i \nabla_i F_Q,$$

where $P_i$ is a nice approximate identity. The last term is easy to handle. We have that

$$T_r A_i \nabla_i F_Q \equiv \theta_i \tilde{\phi} \nabla i \left( I + \tilde{\phi}^2 (L_i)^2 \right)^{-1} (L_i) \nabla i F_Q.$$ 

Therefore, since $\theta_i$ and $\nabla i \left( I + \tilde{\phi}^2 (L_i)^2 \right)^{-1}$ are uniformly bounded on $L^2$, we obtain that

$$\int_0^t \int_Q \left| T_r A_i \nabla i F_Q \right| \frac{dxdt}{t} \leq C \varepsilon_0 \int_0^t \left| (L_i) \nabla F_Q \right|^2 \int_0^t \left| t \right| dtdx \leq C \varepsilon_0 |Q|,$$

where in the last step we have used (6.16)(ii).

It is also easy to handle $R_i^{(1)} \nabla_i F_Q$. Indeed $E_i = E_i^2$, so that

$$R_i^{(1)} = (T_r A_i) E_i (E_i - P_i).$$

By the definition of $T_i$, Lemma 3.3 and Lemma 3.11, we have that

(6.17) \quad \| (T_r A_i) E_i \|_{L^1} \leq C \varepsilon_0.$$

Thus,

$$\int_0^t \int_Q \left| R_i^{(1)} \nabla_i F_Q \right| \frac{dxdt}{t} \leq C \varepsilon_0 \int_0^t \left| (E_i - P_i) \nabla \|F_Q\|^2 \right| \frac{dxdt}{t} \leq C \varepsilon_0 |Q|,$$
where in the last step we have used (6.16)(i), as well as the boundedness on $L^2$ of
\[ g \to \left( \int_0^\infty \| (E_t - P_t) g \|_2^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \]

It remains to treat the contribution of the term $R^{(2)}_I \nabla_t F_Q$. By (6.16)(i), it will be enough to establish the square function bound
\[ ||R^{(2)}_I \nabla_t F_Q|| \leq C_{\varepsilon_0} ||\nabla_t F_Q||_2. \]
To this end, we write
\[ R^{(2)}_I \nabla_t F_Q = R^{(2)}_I (I - P_t) \nabla_t F_Q + R^{(2)}_I P_t \nabla_t F_Q, \]
where $I$ denotes the identity operator. The last term is easy to handle. We note that $R^{(2)}_I 1 = 0$, and therefore by Lemmas 2.7, 2.8, 3.3 and 3.11, the operator $R^{(2)}_I$ satisfies the hypotheses of Lemma 3.5 with bound on the order of $C_{\varepsilon_0}$. Thus,
\[ ||R^{(2)}_I P_t \nabla_t F_Q|| \leq C_{\varepsilon_0} ||\nabla_t P_t \nabla_t F_Q|| \leq C_{\varepsilon_0} ||\nabla_t F_Q||_2, \]
where the last inequality is standard Littlewood-Paley theory.

By the definition of $R^{(2)}_I$, we may further decompose the first summand on the right side of (6.18) as
\[ (T_s A_s) E_i Q_i \nabla_t F_Q - T_s A_s \nabla_t (I - P_t) F_Q \equiv I - \II, \]
where $Q_i = P_t (I - P_t)$ satisfies $||Q_i||_{op} \leq C$. Then by (6.17), we have
\[ ||\II|| \leq C_{\varepsilon_0} ||\nabla_t F_Q||_2. \]

Next, by definition of $T_s$, we see that
\[ \II = \theta_t \hat{\varepsilon} \nabla_t \left( (I + \hat{\Delta} (L_1))^{-1} - I \right) (I - P_t) F_Q = - \theta_t \hat{\varepsilon} \nabla_t F_Q + \theta_t \hat{\varepsilon} \nabla_t \left( (I + \hat{\Delta} (L_1))^{-1} - I \right) P_t F_Q \equiv \II_1 + \II_2 + \II_3. \]
By Lemma 5.17,
\[ ||\II|| \leq C_{\varepsilon_0} ||\nabla_t F_Q||. \]
Moreover, by Lemma 2.8 and the fact that $||\nabla_t (1 + \hat{\Delta} (L_1))^{-1}||_{2 \to 2} \leq C$, we obtain that
\[ ||\II_1|| \leq C_{\varepsilon_0} ||\hat{\tau}^{-1} L_1 (I - P_t) \nabla_t F_Q|| \leq C_{\varepsilon_0} ||\nabla_t F_Q||_2, \]
where $L_1 = (-\Delta)^{-1/2}$ is the fractional integral operator of order one on $\mathbb{R}^n$, and where we have used the Littlewood-Paley inequality
\[ ||\hat{\tau}^{-1} L_1 (I - P_t)||_{op} \leq C. \]
The latter estimate holds by Plancherel’s Theorem, since
\[ (6.19) \quad \left| \frac{1}{|t|} (1 - \hat{\phi}(t \xi)) \right| \leq C \min \left( \frac{1}{|t|}, \frac{1}{|\xi|} \right), \]
if $\phi(x) = \hat{\phi}(x/t)$, the convolution kernel of $P_t$, is chosen so that $\int_{\mathbb{R}^n} x \phi(x) dx = 0$.

Finally, it remains only to consider the term $\II_2$. Now
\[ \II_2 = \theta_t \hat{\varepsilon} P_t \nabla_t F_Q, \]
so we need that $||\theta_t \hat{\varepsilon} P_t||_{op} \leq C_{\varepsilon_0}$. By Lemmas 5.17, 3.2 and 2.7, $||\theta_t \hat{\varepsilon} P_t||_{op} \leq C_{\varepsilon_0}$.

We may choose $P_t$ to be of the form $P_t = \hat{P}_t^2$, where $\hat{P}_t$ is of the same type. Set
\[ R_t = \theta_t \hat{\varepsilon} P_t - (\theta_t \hat{\varepsilon}) \hat{P}_t, \]
where in the last step we have used (6.16)(i), as well as the boundedness on $L^2$ of
\[ g \to \left( \int_0^\infty \| (E_t - P_t) g \|_2^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \]
which satisfies the hypothesis of Lemma 3.5 with bound $C\epsilon_0$. Thus,
\[ ||\partial_\nu P_t - (\partial_\nu \bar{e}) P_t||_{L^\infty} \leq ||R_t P_t||_{L^\infty} \leq C\epsilon_0||\nabla P_t||_{L^2} \leq C\epsilon_0.\]

This concludes the proof of Lemma 6.9, and hence that of the square function bound (5.7).

7. Proof of Theorem 1.11: the singular integral estimate (5.8)

We shall consider separately the cases $t > 0$ and $t < 0$, and since the proof is the same in each case we treat only the former. More precisely, we shall prove
\[ \sup_{0 < q < 10^{-10}} \sup_{r > 0} ||\partial_\nu S_{1,q}^t f||_2 \leq C||f||_2 + C\epsilon_0 (M^+ + M^-), \]
where $M^\pm$ are defined in (6.2). We begin by reducing matters to the case $t \geq 4\eta$. Suppose that $0 \leq t < 4\eta$. We claim that
\[ ||\partial_\nu S_{1,q}^t f(x) - D_{n+1} S_{1,q}^t f(x)|| \leq CMf(x). \]

Indeed, let $K^\eta_t(x,y)$ denote the kernel of $\partial_\nu S_{1,q}^t$, i.e.,
\[ K^\eta_t(x,y) \equiv \partial_\nu \left( \varphi_\eta * \Gamma_t(x - \cdot, y, 0) \right)(t). \]

To prove the claim, it is enough to establish the following estimate:
\[ ||K^\eta_t(x,y) - K_{4\eta}^\eta_t(x,y)|| \leq C \left( \frac{1}{|x-y|^{n+1}} + \frac{\eta}{|x-y|^{n+1} + 1|y|^{10\eta}} \right). \]

In turn, the case $|x-y| \leq 10\eta$ of the latter bound follows directly from (4.10). On the other hand, if $|x-y| > 10\eta$, we have by Lemma 2.2 that
\[ ||K^\eta_t(x,y) - K_{4\eta}^\eta_t(x,y)|| = \left| \int \varphi_\eta(s) (D_{n+1} \Gamma_t(x,t-s,y,0) - D_{n+1} \Gamma_t(x,4\eta - s,y,0)) ds \right| \leq C \int \varphi_\eta(s) \frac{|4\eta-s|}{|x-y|^{n+1}} ds \leq C \frac{\eta}{|x-y|^{n+1}}. \]

Having proved the claim, we fix $t_0 \geq 4\eta$, and use (1.3) to obtain, for each $y \in \mathbb{R}^n$, 
\[ ||D_{n+1} S_{1,q}^{t_0} f(y)|| \leq C \left( \iint_{B(t,y_0,\eta/2)} |\partial_\nu S_{1,q}^{t_0} f(x)|^2 dx \right)^{1/2} \]
\[ \leq C \left( \iint_{B(t,y_0,\eta/2)} |\partial_\nu S_{1,q}^{t_0} f - \partial_\nu S_{0,q}^{t_0} f|^2 dx \right)^{1/2} + \text{“OK”}, \]

where \text{“OK”} \leq ||f||_{L^2} uniformly in $t_0$, by our hypotheses regarding $L_0$, and where we have used that $u_\eta(x,t) \equiv S_{1,q}^{t_0} f(x)$ solves $L_1 u_\eta = 0$ in $[t, \eta]$. Consequently, 
\[ ||D_{n+1} S_{1,q}^{t_0} f||_2^2 \leq C||f||_2 + C \frac{1}{t_0} \int_{\eta/2}^{3\eta/2} \int_{\mathbb{R}^n} |\partial_\nu S_{1,q}^{t_0} f - \partial_\nu S_{0,q}^{t_0} f|^2 dx dt. \]

As in the section 6, 
\[ \partial_\nu S_{1,q}^{t_0} f(x) - \partial_\nu S_{0,q}^{t_0} f(x) = \partial_\nu (L_0^{-1} \text{div} \bar{e} \nabla S_{1,q}^{t_0} \bar{e})(x, \tau). \]
Thus, it is enough to prove that for every δ > 0 sufficiently small, we have

\[
\frac{1}{t_0} \int_{\mathbb{R}^n} \left( \int_{t_0/4}^{4t_0} \frac{1}{t_0} \left( \int_{\mathbb{R}^n} \epsilon(y) \nabla S_{t_0}^{1/2} \cdot \nabla \Psi(y) \cdot \nabla (L_{t_0}^{1/2}(D_{t_0+1} \Psi_\delta))(y, s) dy ds \right) \right) \leq C_{\delta_0}(M^* + M^*),
\]

where again Ψ_δ ∼ Ψ. We may then obtain (7.1) by taking first a limit as δ → 0, and then a supremum over all Ψ.

To prove (7.2), we begin by splitting the integral on the left hand side into

\[
\frac{1}{t_0} \left\{ \int_{t_0/4}^{4t_0} \int_{t_0/4}^{4t_0} \int_{\mathbb{R}^n} |\nabla \Psi(y)| |f(y)| dy ds \right\}^{1/2} \leq C_{\delta_0} \sup_{t > 0} ||\nabla \Psi(y)||_2.
\]

Next we consider terms III and IV. These may be handled in the same way, so we treat only III explicitly. We use (6.3) to write

\[
\nabla (L_{t_0}^{1/2}(D_{t_0+1} \Psi_\delta))(y, s) = \int \nabla_{y_\delta} S_{t_0, r}^{1/2} (\Psi(\cdot, \tau))(y) d\tau,
\]

so that

\[
III = t_0^{-1} \int_{2R} \int_{2R} \left( \nabla_{y_\delta} S_{t_0, r}^{1/2} \right) \cdot \epsilon \nabla \Psi_\delta(x) dx d\tau
\]

In the error term, s = t ∼ s = t ∼ t_0, if δ is sufficiently small, given the support constraints on Ψ. Thus by Cauchy-Schwarz and Lemma 2.8 (i), the absolute value of the error term is bounded by C_{\delta_0} sup_{t > 0} ||\nabla \Psi_\delta||_2. The remaining term is

\[
III = t_0^{-1} \int_{2R} \int_{2R} \left( \nabla_{y_\delta} S_{t_0, r}^{1/2} \right) \cdot \epsilon \nabla \Psi_\delta(x) dx d\tau
\]

where the expression in curly brackets equals

\[
H_R(x, \tau) = -\int_{\tau}^{R} \partial_1 \left( \int_{-\frac{R}{2}}^{2R} \left( \partial_1 S_{t_0, r}^{1/2} \nabla \right) \cdot \epsilon \nabla \Psi_\delta(x) dx \right) dt
\]

\[
= \int_{\tau}^{R} \left( D_{t_0+1} S_{t_0, r}^{1/2} \nabla \right) \cdot \epsilon \nabla \Psi_\delta(x) dx dt - \int_{\tau}^{R} \left( D_{t_0+1} S_{t_0, r}^{1/2} \nabla \right) \cdot \epsilon \nabla \Psi_\delta(x) dx dt
\]

\[
= H_R^{\text{error}}(x, \tau) - H_R^{\text{error}}(x, \tau) - H_R^{\text{error}}(x, \tau).
\]
Lemma 7.5. Let \(a, b\) denote non-zero real constants. We then have that

\[
\sup_{0 \leq \tau, t < \infty} \left\| \int_0^\tau \left( D_{n+1} S_{\tau, \tau}^0 v \right) \cdot \epsilon \nabla S_{\tau, \tau}^{1, \eta} f \, dt \right\|_2 \leq C(a, b) \epsilon \|M^+ + M^-\|.
\]

We defer for the moment the proof of this Lemma, and consider now

\[
H''_R(x, \tau) = \int_\tau^R \int_{2\eta}^{2R} \left( \partial_t S_{\tau, \tau}^0 v \right) \cdot \epsilon \partial_t \nabla S_{\tau, \tau}^{1, \eta} f(x) \, ds dt.
\]

Then for \(h \in L^2(\mathbb{R}^n)\), with \(\|h\|_2 = 1\), we have

\[
|h, H''_R(\cdot, \tau)| = \left| \int_\tau^R \int_{2\eta}^{2R} \left( \nabla D_{n+1} S_{\tau, \tau}^0 h, \epsilon \partial_t \nabla S_{\tau, \tau}^{1, \eta} f \right) \, ds dt \right|.
\]

where we have used that \(adj(S_{\tau, \tau}^0) = S_{\tau, \tau}^{L_0}\). (recall that \(adj\) indicates that we have taken the adjoint in the \(x, y\) variables only, whereas \(S_{\tau, \tau}^{L_0}\) is the single layer potential operator associated to \(L_0\)). Thus, (7.6) is dominated by

\[
C \epsilon_0 \left( \int_0^\infty \int_0^\infty \left\| \nabla \partial_t S_{\tau, \tau}^{L_0} h \right\|^2 \, ds dt \right)^{1/2} \left( \int_0^\infty \left\| \partial_t \nabla S_{\tau, \tau}^{1, \eta} f \right\|^2 \, dt \right)^{1/2} \| \epsilon \| \| M^+ + M^- \| \equiv C \epsilon_0 B_1 \cdot B_2.
\]

Note that \(B_2 = C \|\nabla \partial_t S_{\tau, \tau}^{1, \eta} f\|\). Similarly, the change of variable \(s \mapsto s + t\) yields that \(B_1 = \|s \partial_s S_{\tau, \tau}^{L_0} h\| \leq C \|h\|_2 = C\). A suitable bound follows for the contribution of \(H''_R\).

It remains to consider the term \(I\) in (7.3), which we shall also treat via Lemma 7.5. Again using (7.4), and that for small \(\delta, \Psi_\delta\) is supported in \([t_0/2 < \tau < 3t_0/2]\), we write

\[
I = t_0^1 \int_{-t_0/2}^{t_0/4} \int_{-t_0/4}^{t_0/4} \int_{\mathbb{R}^n} \left( \partial_t S_{\tau, \tau}^0 \nabla \right) \cdot \epsilon \nabla S_{\tau, \tau}^{1, \eta} f(x) \Psi_\delta(x, \tau) \, dx \, d\tau = \frac{1}{t_0} \int_{-t_0/2}^{t_0/2} \int_{-t_0/2}^{t_0/2} \int_{\mathbb{R}^n} \Psi_\delta(x, \tau) \, dx \, d\tau = I - \text{error}.
\]

By Cauchy-Schwarz and Lemma 2.8 (i), the absolute value of the error term is bounded by \(C \epsilon_0 \sup_{t_0} \|\nabla S_{\tau, \tau}^{1, \eta} f\|_2\), since \(\tau - s \approx \tau \approx t_0\). The remaining term splits into

\[
I = t_0^1 \int_{\mathbb{R}^n} \left\{ \int_0^{t_0/2} \left( \partial_t S_{\tau, \tau}^0 \nabla \right) \cdot \epsilon \nabla S_{\tau, \tau}^{1, \eta} f(x) \, dx \right\} \Psi_\delta(x, \tau) \, dx \, d\tau
\]

\[
\equiv t_0^1 \int_{\mathbb{R}^n} F(x, \tau) \Psi_\delta(x, \tau) \, dx \, d\tau.
\]
plus a similar term $\tilde{f}$, which may be treated by the same arguments, in which the expression in curly brackets has domain of integration $(-\tau/2,0)$. Now,

$$F(\cdot, \tau) = \int_0^\tau \partial_t \left( \int_0^{\tau/2} \left( \partial_t S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f ds \right) dt \right)$$

$$= \int_0^\tau \partial_t \left( \int_0^{\tau/2} \left( D_{n+1} S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f ds \right) dt \right)$$

$$= \int_0^\tau \left( \partial_t S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f dt - \int_0^{\tau/2} \left( D_{n+1} S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f dt \right) \right.$$  

$$+ \int_0^\tau \int_0^{\tau/2} \left( \partial_t S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f ds dt \right) \equiv F' - F'' + F''' .$$

We may estimate the contribution of $F''$ directly via Lemma 7.5. Also,

$$F'(\cdot, \tau) = \left( S_{t}^0 \nabla \cdot e \nabla S_{t}^{1,\eta} f - S_{0}^0 \nabla \cdot e \nabla S_{0}^{1,\eta} f \right),$$

so by our hypotheses concerning $L_0$, we have

$$\sup_\tau \| F'(\cdot, \tau) \|_{L^2(\mathbb{R}^n)} \leq C_0 \sup_{r \geq 0} \| \nabla S_{r}^{1,\eta} f \|_2 .$$

We therefore obtain a permissible bound for the contribution of $F'$. We also have that

$$(7.7) \quad F'''(\cdot, \tau) = \int_0^\tau \int_0^{\tau/2} \partial_t \left( \int_0^{\tau/2} \left( S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f ds \right) dt \right)$$

$$= \int_0^\tau \left( \partial_t S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f dt - \int_0^{\tau/2} \left( \partial_t S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f dt \right) \right.$$  

In turn, the last term equals $-F'$, and the middle summand may be handled via Lemma 7.5. The first summand on the right hand side of (7.7) equals

$$- \int_0^\tau \int_0^{\tau/2} \partial_t^2 (S_{t-s}^0 \nabla \cdot e \nabla S_{t-s}^{1,\eta} f) ds dt.$$

Dualizing against $h \in L^2(\mathbb{R}^n)$, with $\|h\|_2 = 1$, we see it is enough to consider

$$\left| \int_0^\tau \int_0^\infty \int_0^\infty 1_{|r-s| < \tau/2} (\nabla D_{n+1} S_{t-s}^{1,\eta} f, h) ds dr ds dt \right|$$

$$\leq C_0 \left( \int_0^\infty \int_0^\infty \int_0^\infty 1_{|r-s| < \tau/2} (r^{-\frac{1}{2}} t^{-\frac{1}{2}} \|\partial_t S_{t-s}^{1,\eta} f\|_2^2) ds dr ds dt \right)^\frac{1}{2}$$

$$\times \left( \int_0^\infty \int_0^\infty \int_0^\infty 1_{|r-s| < \tau/2} (r^{-\frac{1}{2}} t^{-\frac{1}{2}} \|\partial_t S_{t-s}^{1,\eta} f\|_2^2) ds dr ds dt \right)^\frac{1}{2}$$

$$\equiv C_0 B_4 \cdot B_5 .$$

Now,

$$B_4 = \left( \int_0^\infty \|\partial_t S_{t-s}^{1,\eta} f\|_2^2 \left( \int_0^\tau \int_0^{\tau/2} (r^{-\frac{1}{2}} t^{-\frac{1}{2}}) ds dr \right) ds \right)^\frac{1}{2} = C \|\partial_t S_{t-s}^{1,\eta} f\|.$$
Similarly, the change of variable $t \to t + \sigma$ yields the bound
\[
B_3 = \left( \int_0^\infty \int_0^\infty \int_0^\infty 1_{|x - x(t + \sigma)| < 2}\tilde{S}^{\frac{1}{2}}(t + \sigma)^{\frac{1}{2}} \left\| \nabla D_{h_{\tau+},h_{\tau+}} \tilde{S}_{\tau}^h \right\|^2 \, d\sigma \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_0^\infty t^{\frac{1}{2}} \left\| \nabla \partial_\tau^2 \tilde{S}_{\tau}^h \right\|^2 \int_0^t s^{-\frac{1}{2}} \int_0^s d\sigma ds \, dt \right)^{\frac{1}{2}} = C \left\| t^2 \nabla \partial_\tau^2 \tilde{S}_{\tau}^h \right\| \leq C \|h\|_2 = C,
\]
and the desired estimate for the contribution of $F'''$ now follows.

To complete the proof of estimate (5.8), it therefore remains to prove Lemma 7.5.

**Proof of Lemma 7.5.** For the sake of simplicity of notation, we shall treat the case $a = 2$, $b = 1$, as the general case follows via the same argument.

As above we dualize against $h \in L^2(\mathbb{R}^n)$, so that it is enough to consider
\[
(7.8) \quad \int_{\tau_1}^{\tau_2} \langle \nabla \partial_\tau \tilde{S}_{\tau}^h, e \nabla \tilde{S}_{\tau}^h \rangle \, dt = - \int_{\tau_1}^{\tau_2} \langle \nabla \partial_\tau^2 \tilde{S}_{\tau}^h, e \nabla \tilde{S}_{\tau}^h \rangle \, dt \quad + \quad \text{boundary term}
\]
where we have integrated by parts in $t$, and where the boundary term is dominated by
\[
C_0 \left( \sup_{r > 0} \left\| r \nabla \partial_\tau \tilde{S}_{\tau}^h \right\| \right) \left( \sup_{r > 0} \left\| \nabla \tilde{S}_{\tau}^h \right\| \right) \leq C_0 \sup_{r > 0} \left\| \nabla \tilde{S}_{\tau}^h \right\|_2,
\]
as desired. Here, the last inequality follows from Lemma 2.8 (ii). Moreover, by Cauchy-Schwarz, the middle term on the right hand side of (7.8) is no larger than
\[
C_0 \left\| \nabla \partial_\tau \tilde{S}_{\tau}^h \right\| \cdot \left\| \nabla \partial_\tau \tilde{S}_{\tau}^h \right\| \leq C_0 \left\| \nabla \partial_\tau \tilde{S}_{\tau}^h \right\|.
\]
In the first term on the right hand side of (7.8), we integrate by parts again in $t$, to obtain
\[
(7.9) \quad \frac{1}{2} \int_{\tau_1}^{\tau_2} \langle \nabla \partial_\tau^2 \tilde{S}_{\tau}^h, e \nabla \tilde{S}_{\tau}^h \rangle \, dt \quad + \quad \text{Errors},
\]
where the error terms correspond to the last two terms in (7.8) and are handled in a similar fashion. Turning to the main term in (7.9), we note that
\[
\frac{1}{2} \partial_\tau^2 \tilde{S}_{\tau}^h = \partial_\tau \partial_\tau \tilde{S}_{\tau}^h \big|_{\text{est}}.
\]
Now set $g = \partial_\tau^2 \tilde{S}_{\tau}^h$. Let $u$ solve
\[
\begin{cases}
L_0^h u = 0 & \text{in } \mathbb{R}^{n+1} \\
u(t, 0) = g
\end{cases}
\]
By invertibility of the layer potentials for $L_0^h$, and by uniqueness, we have that
\[
u(t, -s) = \mathcal{D}_{\tau}^h \left( \frac{1}{2} I + K_{\tau}^h \right)^{-1} g.
\]
On the other hand, we also have that $u(t, -s) = \partial_\tau^2 \tilde{S}_{\tau}^h$. Consequently,
\[
\partial_\tau \nabla u(t, -s) = \partial_\tau \nabla \mathcal{D}_{\tau}^h \left( \frac{1}{2} I + K_{\tau}^h \right)^{-1} g = \partial_\tau \nabla \partial_\tau \tilde{S}_{\tau}^h.
\]
Setting $s = t$, we have that
\[
\frac{1}{2} \nabla \partial_i^s L_{-2} \cdot h = - D_{n+1} \nabla D_{\zeta_1} \left( \frac{1}{2} I + K^L \right)^{-1} g = - D_{n+1} \nabla D_{\zeta_1} \left( \frac{1}{2} I + K^L \right)^{-1} \partial_i^s L_{-2} \cdot h.
\]
But, $D_{\zeta_1} = (S_{-2}^L, \overline{\partial}_s)$, where $\overline{\partial}_s$ denotes conjugate exterior co-normal differentiation for $L_0$. Thus,
\[
adj \left( \nabla D_{n+1} D_{\zeta_1} \right) = (\partial_s \partial_{s}^0 S_{0}^0 \nabla).
\]
Therefore, the main term in (7.9) equals in absolute value
\[
2.8 \quad (7.10) \text{by Lemma 2.8,} \quad u
\]
\[\text{Applying Lemma 2.8 (Caccioppoli’s inequality in Whitney boxes to reduce matters to considering 2.8,)} \quad \text{To conclude the proof of Lemma 7.5, it then suffices to prove that}
\]
\[
(7.10) \quad \|r^2 \left( \nabla \partial_s S_{0}^0 \nabla \right) \cdot e \nabla S_{1,0}^{1,0} f \| \le C \omega M^*.
\]
To this end, we first prove a lemma that will allow us to reduce matters to (6.10).

**Lemma 7.11.** For $k \in \mathbb{Z}$, set $t_k \equiv 2^{k-1}$. Then
\[
\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+1}} \int_{\mathbb{R}^n} |\nabla S_{1,0}^{1,0} f(x) - \nabla S_{h_k}^{1,0} f(x)|^2 \frac{dx dt}{t} \le C \|\nabla \partial_s S_{1,0}^{1,0} f \|_2.
\]
Let us momentarily take the lemma for granted, and deduce (7.10). Combining Lemma 2.8 (i), Lemma 2.11 and Lemma 7.11, we may replace the square of the left hand side of (7.10) by
\[
\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+1}} \int_{\mathbb{R}^n} |r^2 \left( \nabla \partial_s S_{0}^0 \nabla \right) \cdot e \nabla S_{h_k}^{1,0} f(x)|^2 \frac{dx dt}{t}.
\]
Since $u_t(\cdot, t) \equiv (\partial_s S_{0}^0 \nabla) \cdot e \nabla S_{h_k}^{1,0} f$ solves $L_0 u_t = 0$ in the upper half space, we may use Caccioppoli’s inequality in Whitney boxes to reduce matters to considering
\[
\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+1}} \int_{\mathbb{R}^n} |r^2 \left( \nabla \partial_s S_{0}^0 \nabla \right) \cdot e \nabla S_{h_k}^{1,0} f(x)|^2 \frac{dx dt}{t}.
\]
Applying Lemma 2.8 (i) and Lemma 7.11 again, along with (6.10), we obtain (7.10).

**Proof of Lemma 7.11.** The left hand side of (7.12) equals
\[
\sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+1}} \int_{\mathbb{R}^n} \left| \frac{1}{\sqrt{t}} \int_{\mathbb{R}^n} \nabla \partial_s S_{1,0}^{1,0} f(x) ds \right|^2 \frac{dx dt}{t}
\]
\[\le C \sum_{k=-\infty}^{\infty} \int_{2^{k-1}}^{2^{k+1}} \left| \int_{\mathbb{R}^n} 1_{\{2^{k-1} \le x < 2^{k-1}\}} \sqrt{t} \nabla \partial_s S_{1,0}^{1,0} f(x) ds \right|^2 \frac{dx dt}{t}.
\]
The desired bound now follows from the Hardy-Littlewood maximal theorem. \qed

This concludes the proof Lemma 7.5, and thus also that of Theorem 1.11.  \qed
8. Proof of Theorem 1.12: boundedness

Let $L \equiv -\text{div} A \nabla$, where $A$ is real, symmetric, $L^\omega$, $t$-independent and uniformly elliptic. In this section, we show that the layer potentials associated to $L$ are bounded; we defer the proof of invertibility to the next section. By the classical de Giorgi-Nash Theorem, estimates (1.2) and (1.3) hold for solutions of $Lu = 0$. By Lemma 5.2 and Lemma 4.8, in order to establish boundedness of the layer potentials, it suffices to prove

$$(8.1) \quad \sup_{x \in \Omega} ||\partial_1 S \psi||_2 \leq C ||f||_2$$

and

$$(8.2) \quad \int_{-\infty}^\infty \int_{\mathbb{R}^n} |\partial_2^2 S \psi|^2 \, dt \, dx \leq C ||f||_2.$$ 

By Lemma 2.2, the kernel $K_r(x, y) \equiv \partial_1 \Gamma(x, t, y, 0)$ satisfies the standard Calderón-Zygmund estimates

$$(8.3a) \quad |K_r(x, y)| \leq \frac{c}{|x-y|^n}$$

$$(8.3b) \quad |K_r(x + h, y) - K_r(x, y)| + |K_r(x + h, y) - K_r(x, y)| \leq C \frac{|h|^\alpha}{|x-y|^{n+\alpha}},$$

uniformly in $t$, where the later inequality holds for some $\alpha > 0$ whenever $|x-y| > 2|h|$. In addition, the kernel

$$\psi_r(x, y) \equiv t \partial_1^2 \Gamma(x, t, y, 0)$$

satisfies the standard Littlewood-Paley kernel conditions

$$(8.4) \quad |\psi_r(x, y)| \leq \frac{|t|}{|t| + |x-y|^{n+1}} C|t| \frac{|h|^\alpha}{|t| + |x-y|^{n+1+\alpha}} \leq C \frac{|h|^\alpha}{|t| + |x-y|^{n+\alpha}}$$

for some $\alpha > 0$, whenever $|h| \leq \frac{1}{4}|x-y|$ or $|h| \leq |t|/2$.

The bound (8.2) will be deduced from the following “local” $Tb$ Theorem for square functions

**Theorem 8.5.** Let $\psi f(x) \equiv \int_{\mathbb{R}^n} \psi(x, y) f(y) \, dy$, where $\psi(x, y)$ satisfies (8.4). Suppose also that there exists a system $\{b_Q\}$ of functions indexed by cubes $Q \subseteq \mathbb{R}^n$ such that for each cube $Q$

1. $\int_{b_Q} |b_Q|^2 \leq C |Q|$
2. $\int_Q \int_{b_Q} |\partial_1 b_Q(x) |^2 \, dx \, dt \leq C |Q|$
3. $\frac{1}{|Q|} \int_Q \text{Re} \int_{b_Q} \, dt$

Then we have the square function bound

$$||\partial_1 f||_2 \leq C ||f||_2.$$ 

We omit the proof here. A direct proof of the present formulation of Theorem 8.5 may be found in [A2] or [H2], although we note that the theorem and its proof were already implicit in the proof of the Kato square root conjecture [HMc], [HLMc] and [AHLMcT]; see also the works [Ch], [S] and [AT] for some important antecedents.

We shall deduce estimate (8.1) as a consequence of the following extension of a local $Tb$ Theorem for singular integrals introduced by M. Christ [Ch] in connection with the theory of analytic capacity. A 1-dimensional version of the present result, valid for "perfect
dyadic” Calderón-Zygmund kernels, appears in [AHMTT]. A self-contained proof of the more general formulation below may be found in [H3]. Alternatively, the result of [AY] may be combined with that of [AHMTT] to deduce the general case (in the slightly sharper form $q = 2$). In the sequel, we let $T^\ast$ denote the transpose of the operator $T$.

**Theorem 8.6.** Let $T$ be a singular integral operator associated to a kernel $K$ satisfying (8.3), and suppose that $K$ satisfies the generalized truncation condition $K(x, y) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose also that there exist pseudo-accretive systems $\{b_Q^1\}, \{b_Q^2\}$ such that $b_Q^1$ and $b_Q^2$ are supported in $Q$, and

(i) $\int_Q (|b_Q^1|^2 + |b_Q^2|^2) \leq C|Q|$, for some $q > 2$

(ii) $\int_Q (|Tb_Q^1|^2 + |T^a b_Q^2|^2) \leq C|Q|$

(iii) $\frac{1}{|Q|} \leq \min (\Re \int_Q b_Q^1, \Re \int_Q b_Q^2)$.

Then $T : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, with bound independent of $\|K\|_{\infty}$.

Let us first show that Theorem 8.5 implies (8.2). As usual, we may restrict our attention to the case $t > 0$. As above let $\psi_t(x, y) \equiv t^d \Gamma(x, t, y, 0)$, so that

$$\partial_t^2 S f(x) \equiv \partial_t f(x) = \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy.$$ 

By Theorem 8.5, it suffices to construct a system $\{b_Q\}$ satisfying the hypotheses (i), (ii) and (iii) of the Theorem.

Our functions $b_Q$ will be normalized Poisson kernels. Given a cube $Q \subset \mathbb{R}^n$, let $x_Q$ denote its center, and let $\ell(Q)$ denote its side length. We define

$$A_Q^+ \equiv (x_Q, \ell(Q)) \in \mathbb{R}^{n+1}_-, \quad A_Q^- \equiv (x_Q, -\ell(Q)) \in \mathbb{R}^{n+1}_+.$$ 

Given $X^+ \in \mathbb{R}^{n+1}_-$, $X^- \in \mathbb{R}^{n+1}_+$, let $k_{X^+}^+(y)$, $k_{X^-}^-(y)$ denote, respectively, the Poisson kernels for $L$ in the upper and lower half spaces, and let $G^+(X, Y)$, $G^-(X, Y)$ denote the corresponding Green functions, so that

$$k_{X^+}^+(y) \equiv \frac{\partial G^+}{\partial \nu^+}(X^+, y, 0), \quad k_{X^-}^-(y) \equiv \frac{\partial G^-}{\partial \nu^-}(X^-, y, 0),$$

where $\frac{\partial}{\partial \nu^+}$, $\frac{\partial}{\partial \nu^-}$ denote the co-normal derivatives at the point $y \in \partial \mathbb{R}^{n+1}_+$, $\partial \mathbb{R}^{n+1}_-$ respectively.

We now set

(8.7) \quad $b_Q \equiv |Q| k_{A_Q^+}^+.$

We recall the following fundamental result of Jerison and Kenig [JK1] (see also [K, pp 63-64]), which amounts to the solvability of (D2) in the lower half-space:

**Theorem 8.8.** [JK1] Suppose that $L = -\div A \nabla$, where $A$ is real, symmetric, $(n+1) \times (n+1)$, $t$-independent, $L^\infty$ and uniformly elliptic. Then there exists $\varepsilon_1 \equiv \varepsilon_1(n, \lambda, A)$ such that for all $0 \leq \varepsilon < \varepsilon_1$ and for every cube $Q$,

(8.9) \quad $$\int_{\mathbb{R}^n} (k_{A_Q^+}^+(y))^2 d y \leq C \varepsilon |Q|^{1+\varepsilon}.$$ 

We remark that (8.9) is usually stated in terms of an integral over $Q$, but in fact the global bound follows from the local one and duality, since by [JK1], [K] the local version of (8.9) and the $L^p$ version of (1.3) yield the estimate

$$|u(A_Q^+)| \leq C \sup_{t > 0} |u(\cdot, t)|_{L^p(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)}$$

where $u(x, t) = \int_{\mathbb{R}^n} k_{X^-}^-(y) g(y) dy$, and $p$ is the dual exponent to $2 + \varepsilon$. 

We now note that hypothesis (i) of Theorem 8.5 follows immediately from (8.7) and (8.9). Moreover, (iii) follows immediately from (8.7) and the following well known estimate of Caffarelli, Fabes, Mortola and Salsa [CFMS] (also [K, Lemma 1.3.2, p. 9]):

\[(8.10) \quad \omega^X_\partial(Q) \geq \frac{1}{C},\]

where \(\omega^X_\partial\) denotes harmonic measure for \(L\) at \(X^- \in \mathbb{R}^{n+1}\).

It remains to verify that \(b_\partial\) as defined in (8.7) satisfies hypothesis (ii) of Theorem 8.5. To this end, let \((x, t) \in Q_t^{\infty} \equiv Q \times (0, t(0))\). Then, since for fixed \((x, t) \in \mathbb{R}^{n+1}\), we have that \(\partial_\alpha \Gamma(x, t, \cdot, \cdot)\) is a solution of \(Lu = 0\) in \(\mathbb{R}^{n+1}\),

\[(8.11) \quad \theta_i b_\partial(x) = |Q| t \int \partial_\alpha \Gamma(x, t, y, 0) k^X_\partial(y) dy = |Q| t \partial_\alpha \Gamma(x, t, A^\partial_\partial),\]

by Theorem 8.8 (i.e., [JK1]) and uniqueness in (D2) (e.g., Lemma 4.31 (i), although of course, uniqueness in the present setting of real symmetric coefficients appears already in [JK1], [K]). Therefore, by (2.3) and translation invariance in \(t\), we have that

\[|\theta_i b_\partial(x)| \leq C \frac{t}{t(0)},\]

from which hypothesis (ii) follows readily. Thus, given Theorem 8.5, we conclude that

\[\int_0^\infty \int_{\mathbb{R}^n} |\partial_\alpha \mathcal{S}_f(x)|^2 \frac{dx dt}{t} \leq C ||f||_{L^2}.\]

The corresponding square function estimate in the lower half-space follows by the same argument, if we replace \(k^X_\partial\) by \(k^{X}_\partial\) in the definition of \(b_\partial\). We then obtain (8.2) as desired.

Next, we show that Theorem 8.6 implies (8.1). We consider only the case \(t > 0\), the other case being handled by a similar argument. Again, it suffices to construct systems \(\{b_t^1\}, \{b_t^2\}\), now with \(b_t^1\) and \(b_t^2\) supported in \(Q\), satisfying hypotheses (i), (ii) and (iii) of Theorem 8.6.

In fact, we shall use the same construction as before, except that we truncate the function outside of \(Q\), i.e. we set

\[(8.12) \quad b_t^1 \equiv |Q| k^{X^1}_\partial 1_Q = b_\partial 1_Q, \quad b_t^2 \equiv |Q| k^{X^2}_\partial 1_Q\]

As before, (iii) and (i) follow immediately from [CFMS], and (8.9), respectively.

It remains to establish (ii). We observe first that, as in (8.11),

[\(\partial_\alpha \mathcal{S}_f b_\partial(x)\) = \(|Q| \int_{\mathbb{R}^n} \partial_\alpha \Gamma(x, t, y, 0) k^X_\partial(y) dy\) = \(|Q| \partial_\alpha \Gamma(x, t, A^\partial_\partial)| \leq C,\]

uniformly in \((x, t) \in \mathbb{R}^{n+1}\), where \(b_\partial\) is defined as in (8.7), and we have used (2.3) and the fact that \(t > 0\). We now claim that, for \(x \in Q\) and \(t > 0\), the same \(L^\infty\) bound holds for \(\partial_\alpha \mathcal{S}_f(\varphi \xi_\partial)(x)\), where \(\varphi \in C_0^\infty, \varphi \equiv 1\) on \(5Q\), \(\sup |\varphi| \leq 6Q\), with \(||\varphi \xi_\partial||_{L^\infty} \leq C/t(0)\). Indeed, fixing \((x, t) \in Q \times (0, \infty),\) and setting \(u = \partial_\alpha \Gamma(x, t, \cdot, \cdot)\), we have that

[\(\partial_\alpha \mathcal{S}_f(\varphi \xi_\partial)(x) = |Q| \int_{\mathbb{R}^n} u(y, 0) \varphi(y) \frac{\partial \varphi}{\partial y^n}(A^\partial_\partial, y, 0) dy\).]

We now extend \(\varphi \xi_\partial\) smoothly into the lower half-space so that \(\varphi \xi_\partial(y, s) \equiv 1\) on \(5Q \times (0, -t(0)/4),\) \(\varphi \xi_\partial(y, s)\) vanishes in \(\mathbb{R}^{n+1}\) \(\setminus (6Q \times (0, -t(0)/2)),\) and

[\(||\varphi \xi_\partial||_{L^\infty(\mathbb{R}^{n+1})} \leq C t(0)^{-1} \).]
Since $G^-(A^{-}_{Q}, \cdot, \cdot)$ and $u$ are both solutions of $Lu = 0$ in $\text{supp} \varphi_{Q}$, we obtain from Green’s formula (whose use may be justified in the sense of Lemma 4.3 (iii)) that

$$\partial_j S_j(\varphi \circ b)(x) = \left| Q \right| \int_{\mathbb{R}^n \times \mathbb{R}^n} A_{x,y} \nabla G^-(A^{-}_{Q}, y, s) \nabla \varphi(y, s) u(y, s) dy ds$$

$$- \left| Q \right| \int_{\mathbb{R}^n \times \mathbb{R}^n} G^{-} \nabla \varphi \cdot A \nabla u dy ds \equiv I + II.$$

We first consider term II. Let $D_Q = \text{supp} \nabla \varphi$. By the definition of $\varphi(y, s)$, a standard estimate for $G^{-}$, and Cauchy-Schwarz, we have that

$$|II| \leq C \ell(Q)^{(a+1)/2} \left( \int_{D_Q} |\nabla \partial_j \Gamma(x, t, y, s)|^2 dy ds \right)^{1/2} \leq C \ell(Q)^{(a-1)/2} \left( \int_{D_Q} |\partial_j \Gamma(x, t, y, s)|^2 dy ds \right)^{1/2},$$

where the last inequality follows by Caccioppoli’s inequality, and where $\tilde{D}_Q$ is a fattened version of $D_Q$. But for $x \in Q$, $t > 0$, and $(y, s) \in \tilde{D}_Q$, we have by (2.3) that

$$|\partial_j \Gamma(x, t, y, s)| \leq C |Q|^{-1},$$

hence $|II| \leq C$. Similarly,

$$|I| \leq C \ell(Q)^{(a-1/2)} \left( \int_{D_Q} |\nabla G^{-}(A^{-}_{Q}, y, s)|^2 dy ds \right)^{1/2} \leq C,$$

again by Caccioppoli. Altogether then, $\sup_{r > 0} \|\partial_j S_j(\varphi \circ b)(Q)\|_{L^2(Q)} \leq C$, and therefore

$$(8.13) \quad \sup_{r > 0} \int_Q |\partial_j S_j(\varphi \circ b)|^2 \leq C |Q|.$$  

To prove (ii), it will be enough to observe that, for any kernel $K(x, y)$ satisfying (8.3)(a), we have

$$(8.14) \quad \int_Q \left| \int_Q K(x, y) 1_{6Q'}(y) f(y) dy \right|^2 dx \leq C \int_{6Q'} |f|^2.$$  

Indeed, given (8.14), we may replace $\varphi \circ b$ by $1_{Q}$ in (8.13) (with controlled error), and (ii) follows. The proof of (8.14) is omitted. Since $\Gamma(x, t, y, 0) = \Gamma(y, -t, x, 0)$, a similar argument yields the corresponding bound for $(\partial_j S_j)^{(b)^{-1}_{Q}}$, and (8.1) now follows.

9. Proof of Theorem 1.12: Invertibility

We now consider invertibility of the layer potentials in the case of real symmetric coefficients. The proof will follow the strategy of Verchota [V], using the well known “Rellich identities” combined with the method of continuity. In our case, the continuity argument will exploit Theorem 1.11.

Proof of Invertibility. From self-adjointness and integration by parts, we obtain the equivalence

$$(9.1) \quad \|\partial_j u\|_{L^2(\mathbb{R}^n)} \approx \|\nabla_j u\|_{L^2(\mathbb{R}^n)}.$$


for solutions of $Lu = 0$ in $\mathbb{R}^{n+1}_+$ for which $\mathcal{N}_e(\nabla u) \in L^2$, where the implicit constants depend only upon ellipticity (see, e.g., [K] for details). In particular, (9.1) holds for $u(\cdot, t) \equiv S_tf$, with $f \in L^2$. By the jump relation formulae Lemma 4.18, (9.1) becomes

$$\|f\|_2 \leq C\|\left(\frac{1}{2}I + \tilde{K}\right)f\|_2$$

Thus, by the triangle inequality and (9.2) we have

$$\|f\|_2 \leq C\|\nabla_x S_0f\|_2$$

and also

$$\|f\|_2 \leq C\|\nabla_x S_0f\|_2,$$

where the constants in (9.3) and (9.4) depend only on ellipticity. Moreover, if we set

$$L_\sigma \equiv -\text{div} A_\sigma \nabla, \quad 0 \leq \sigma \leq 1,$$

where

$$A_\sigma \equiv (1 - \sigma)\mathbb{I} + \sigma A,$$

and $I$ denotes the $(n + 1) \times (n + 1)$ identity matrix, then (9.3) and (9.4) hold, uniformly in $\sigma$, for the layer potentials associated to $L_\sigma$. Indeed, we have uniform control of the ellipticity constants for $A_\sigma$. By the result of Section 8, we have boundedness of the layer potentials associated to $L_\sigma$, again with uniform constants depending only upon ellipticity and dimension. Thus, once we have established invertibility of the layer potentials associated to $L_\sigma$, for a given $\sigma$, the corresponding Layer Potential Constants will depend only upon ellipticity and dimension, since, in particular, the quantitative bounds for the inverses are precisely the constants in (9.3) and (9.4). We may therefore establish invertibility of $\frac{1}{2}I + \tilde{K} : L^2 \rightarrow L^2$ and $S_0 : L^2 \rightarrow L^2_+$ as follows. Since $L_0$ clearly has Good Layer Potentials, we may invoke Theorem 1.11 to deduce that $L_\sigma$ has Good Layer Potentials, for $0 \leq \sigma < \epsilon_0$, for some $\epsilon_0$ depending only upon ellipticity and dimension. By our previous observation concerning the uniform control of the layer potential constants, we may then iterate this procedure, advancing each time by the same distance $\epsilon_0$, so that we reach $A = A_1$ in finitely many steps.

10. Appendix: Constant coefficients

Suppose that $L = -\text{div} a \nabla$, where $a$ is a constant complex elliptic matrix. Following [FJK], we observe that $L$ has Fourier symbol

$$q(\xi, it) = \sum_{j=1}^{n+1} a_{j, j}(\xi) \xi_j \xi_k \tau_\alpha(\tau - \tau_\alpha(\xi)),\quad -1 \leq \alpha \leq 1,$$

where $\xi_{n+1} \equiv \tau$, and $\tau_\alpha : \mathbb{R}^n \rightarrow \mathbb{C}$ are each homogeneous of degree 1, $C^\infty(S^{n-1})$, with

$$\Im \tau_\alpha(\xi) \geq \mu, \quad \Im \tau_{-\alpha}(\xi) \leq -\mu,$$

for some $\mu > 0$. In particular,

$$|\tau_\alpha(\xi) - \tau_{-\alpha}(\xi)| \equiv |\xi|, \quad \xi \in \mathbb{R}^n.$$

The fundamental solution $\Gamma(x, t)$ is a convolution kernel with Fourier symbol $q(it\xi, t^{-1})$. Inverting the Fourier symbol in $t$ only, and then using the method of residues, we obtain

$$\Gamma(x, t)(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\tau \cdot \xi}}{q(it\xi, t^{-1})} d\tau = \frac{e^{i\tau_\alpha(\xi) \tau_{-\alpha}(\xi)}}{itd_{n+1, r+1} \tau_\alpha(\tau_\alpha(\xi) - \tau_{-\alpha}(\xi))}.$$
and (10.2), the same holds for the denominator, and the invertibility follows. Of course, a corresponding Fourier symbol is readily verifies via Plancherel’s Theorem that

$$\sup_{t \in \mathbb{R}} \| \nabla S_t \|_{L^1} \leq C, \quad \| \partial_t^2 S_t \|_{L^1} \leq C.$$ 

Finally, we note that

$$f \to \left( \frac{4}{d} I + \tilde{K} \right) f = \partial_t S_t f |_{t=0}$$

is invertible on $L^2$. Indeed, the corresponding Fourier symbol is

$$- \lim_{t \to 0^+} e_n t \cdot a \mathcal{F} \left( \cdot, t \right)(\xi) = \frac{a_{n+1, n+1} \tau_s(\xi) + \sum_{j=1}^{n} a_{n+1, j} \tilde{\eta}_j}{a_{n+1, n+1} (\tau_s(\xi) - \tau_0(\xi))},$$

and by [AQ], Lemma 4, the modulus of the numerator $\approx |\xi|$. By the accretivity of $a_{n+1, n+1}$ and (10.2), the same holds for the denominator, and the invertibility follows. Of course, a similar observation holds for $-\frac{4}{d} I + \tilde{K}$.

**References**


