ANALYTICITY OF LAYER POTENTIALS AND $L^2$ SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR DIVERGENCE FORM ELLIPTIC EQUATIONS WITH COMPLEX $L^\infty$ COEFFICIENTS

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Abstract. We consider divergence form elliptic operators of the form $L = -\text{div} A(x) \nabla$, defined in $\mathbb{R}^{n+1} = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}, \; n \geq 2\}$, where the $L^\infty$ coefficient matrix $A$ is $(n+1) \times (n+1)$, uniformly elliptic, complex and $t$-independent. We show that for such operators, boundedness and invertibility of the corresponding layer potential operators on $L^2(\mathbb{R}^n) = L^2(\partial \mathbb{R}^{n+1})$, is stable under complex, $L^\infty$ perturbations of the coefficient matrix. Using a variant of the Tb Theorem, we also prove that the layer potentials are bounded and invertible on $L^2(\mathbb{R}^n)$ whenever $A(x)$ is real and symmetric (and thus, by our stability result, also when $A$ is complex, $\|A - A^0\|_{L^\infty}$ is small enough and $A^0$ is real, symmetric, $L^\infty$ and elliptic). In particular, we establish solvability of the Dirichlet and Neumann (and Regularity) problems, with $L^2$ (resp. $L^2_1$) data, for small complex perturbations of a real symmetric matrix. Previously, $L^2$ solvability results for complex (or even real but non-symmetric) coefficients were known to hold only for perturbations of constant matrices (and then only for the Dirichlet problem), or in the special case that the coefficients $A_{j,n+1} = 0 = A_{n+1,j}$, $1 \leq j \leq n$, which corresponds to the Kato square root problem.

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1. Introduction, statement of results, history

In this paper, we consider the solvability of boundary value problems for divergence form complex coefficient equations \( Lu = 0 \), where

\[
L = - \text{div} A \nabla := - \sum_{i,j=1}^{n+1} \frac{\partial}{\partial x_i} \left( A_{i,j} \frac{\partial}{\partial x_j} \right)
\]

is defined in \( \mathbb{R}^{n+1} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} \} \), \( n \geq 2 \) (we use the notational convention that \( x_{n+1} = t \)), and where \( A = A(x) \) is an \((n+1) \times (n+1)\) matrix of complex-valued \( L^\infty \) coefficients, defined on \( \mathbb{R}^n \) (i.e., independent of the \( t \) variable) and satisfying the uniform ellipticity condition

\[
(1.1) \quad \lambda |\xi|^2 \leq \Re e \langle A(x)\xi, \xi \rangle := \Re e \sum_{i,j=1}^{n+1} A_{i,j}(x)\xi_i \xi_j, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \Lambda,
\]

for some \( \lambda > 0, \Lambda < \infty \), and for all \( \xi \in \mathbb{C}^{n+1}, x \in \mathbb{R}^n \). The divergence form equation is interpreted in the weak sense, i.e., we say that \( Lu = 0 \) in a domain \( \Omega \) if \( u \in W^{1,2}_{\text{loc}}(\Omega) \) and

\[
\int A\nabla u \cdot \nabla \Psi = 0
\]

for all complex valued \( \Psi \in C_0^\infty(\Omega) \).

The boundary value problems that we consider are classical. To state them, we first recall the definitions of the non-tangential maximal operators \( N_\gamma, \widetilde{N}_\gamma \). Given \( x_0 \in \mathbb{R}^n \), define the cone \( \gamma(x_0) = \{(x, t) \in \mathbb{R}^{n+1} : |x_0 - x| < t \} \). Then for \( U \) defined in \( \mathbb{R}^{n+1} \),

\[
N_\gamma U(x_0) := \sup_{(x,t)\in\gamma(x_0)} |U(x,t)|, \quad \widetilde{N}_\gamma U(x_0) := \sup_{(x,t)\in\gamma(x_0)} \left( \int_0^t \left( \int_{|x-y|<r} |U(y,s)|^2 dy ds \right)^{1/2} \right)^{1/2}.
\]

Here, and in the sequel, the symbol \( \int_0^t \) denotes the mean value, i.e., \( \int_0^t f := |E|^{-1} \int_E f \). We use the notation \( u \to f \) n.t. to mean that for a.e. \( x \in \mathbb{R}^n \), \( \lim_{(x,t)\to(x,0)} u(y,t) = f(x) \), where the limit runs over \( (y,t) \in \gamma(x) \).

We shall consider the Dirichlet problem\(^1\)

(D2) \[ \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ = \{(x, t) \in \mathbb{R}^n \times (0, \infty)\} \\ \lim_{t \to 0} u(., t) = f \text{ in } L^2(\mathbb{R}^n) \text{ and n.t.} \\ \sup_{t > 0} \|u(., t)\|_{L^2(\mathbb{R}^n)} + \left( \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,t)|^2 t dt dx \right)^{1/2} < \infty, \end{cases} \]

the Neumann problem\(^2\)

(N2) \[ \begin{cases} Lu = 0 \text{ in } \mathbb{R}^{n+1}_+ \\ \frac{\partial u}{\partial \nu}(x, 0) := - \sum_{j=1}^{n+1} A_{n+1,j}(x) \frac{\partial u}{\partial x_j}(x, 0) = g(x) \in L^2(\mathbb{R}^n) \\ \widetilde{N}_\gamma(\nabla u) \in L^2(\mathbb{R}^n), \end{cases} \]

\(^1\)Our uniform \( L^2 \) estimate for solutions of (D2) can be improved to an \( L^2 \) bound for \( N_u \), given certain \( L^p \) estimates for the layer potentials. The fourth named author and M. Mitrea will present the \( L^p \) theory in a forthcoming publication. In the present paper, we shall be content with a weak-\( L^2 \) bound for \( N_u \).

\(^2\)We shall elaborate in section 4 the precise nature by which the co-normal derivative assumes the prescribed data.
and the Regularity problem
\[
\begin{align*}
Lu &= 0 \text{ in } \mathbb{R}_+^{n+1} \\
\Gamma(t, \cdot) &\to f \in L^2_0(\mathbb{R}^n) \text{ n.t.} \\
\mathcal{N}(\nabla u) &\in L^2(\mathbb{R}^n).
\end{align*}
\]

Our solutions will be unique among the class of solutions satisfying the stated $L^2$ bounds (in the case of (N2) and (R2), this uniqueness will hold modulo constants). The homogeneous Sobolev space $L^2_0$ is defined as the completion of $C^\infty_0$ with respect to the semi-norm $\|\nabla F\|_2$. For $n \geq 3$, this space can be identified (modulo constants) with the space $I_1(L^2) := \Delta^{-1/2}(L^2) \subset L^2$, where $2^* := 2n/(n-2)$; for $n = 2$, the identification with $I_1(L^2)$ is still valid, but in that case the fractional integral $I_1 f$ must itself be defined modulo constants for $f \in L^2$, and the space embeds in $BMO$.

We remark that for the class of operators that we consider, solvability of these boundary value problems in the half-space may readily be generalized to the case of domains given by the region above a Lipschitz graph, and even to the case of star-like Lipschitz domains. We shall return to this point later. We shall also discuss later the significance of our assumption that the coefficients are $t$-independent.

In order to state our main results, we shall need to recall a few definitions and facts. We say that $u$ is locally Hölder continuous in a domain $\Omega$ if there is a constant $C$ and an exponent $\alpha > 0$ such that for any ball $B = B(x, R)$, of radius $R$, whose concentric double $2B = B(x, 2R)$ is contained in $\Omega$, we have that
\[
|u(Y) - u(Z)| \leq C \left( \frac{|Y - Z|}{R} \right)^\alpha \left( \frac{\int_{2B} |u|^2}{2B} \right)^\frac{\alpha}{2},
\]
whenever $Y, Z \in B$. Observe that any $u$ satisfying (1.2) also satisfies Moser’s “local boundedness” estimate [M]
\[
\sup_{Y \in B} |u(Y)| \leq C \left( \frac{\int_{2B} |u|^2}{2B} \right)^\frac{1}{2}.
\]

By the classical De Giorgi-Nash Theorem [DeG, N], (1.2) and hence also (1.3) hold, with $C$ and $\alpha$ depending only on dimension and the ellipticity parameters, whenever $u$ is a solution of $Lu = 0$ in $\Omega \subseteq \mathbb{R}^{n+1}$, if in addition the coefficient matrix $A$ is real (for this result, it need not be $t$-independent). Moreover, it is shown in [A] (see also [AT, HK]), that property (1.2) is stable under complex, $L^\infty$ perturbations. In an appendix, Section 11 below, we will also observe that the De Giorgi-Nash bound (1.2) holds in the half-space for $t$-independent complex equations $Lu = \nabla \cdot (A(x)\nabla u) = 0$ in dimension $n + 1 = 3$.

We now recall the method of layer potentials. For $L$ as above, let $\Gamma, L^*$ denote the fundamental solutions$^3$ for $L$ and $L^*$ respectively, in $\mathbb{R}^{n+1}$, so that
\[
L_{x,s} \Gamma(x, t, y, s) = \delta(x,y), \quad L_{y,t}^* \Gamma^*(y, s, x, t) := L_{y,t}^* \overline{\Gamma(x, t, y, s)} = \delta(x,t),
\]
where $\delta_X$ denotes the Dirac mass at the point $X$, and $L^*$ is the hermitian adjoint of $L$. By the $t$-independence of our coefficients, we have that
\[
\Gamma(x, t, y, s) = \Gamma(x, t - s, y, 0).
\]

$^3$See [HK2] for a construction of the fundamental solution.
We define the single and double layer potential operators, by

\[ S_t f(x) := \int_{\mathbb{R}^n} \Gamma(x, t, y, 0) f(y) \, dy, \quad t \in \mathbb{R} \]

\[ D_t f(x) := \int_{\mathbb{R}^n} \partial_n \Gamma^\ast(y, 0, x, t) f(y) \, dy, \quad t \neq 0, \]

where \( \partial_n \) is the adjoint exterior conormal derivative; i.e., if \( A^\ast \) denotes the hermitian adjoint of \( A \), then

\[ \partial_n \Gamma^\ast(y, 0, x, t) = -\sum_{j=1}^{g+1} A^\ast_{n+1,j}(y) \frac{\partial \Gamma^\ast}{\partial y_j}(y, 0, x, t) = -\epsilon_{n+1} \cdot A^\ast(y) \nabla_y \Gamma^\ast(y, x, t) \big|_{s=0} \]

(recall that \( y_{n+1} = s \)). We define (loosely\(^4\), for the moment) boundary singular integrals

\[ K f(x) := ”p.v.” \int_{\mathbb{R}^n} \partial_n \Gamma^\ast(y, 0, x, 0) f(y) \, dy \]

\[ \bar{K} f(x) := ”p.v.” \int_{\mathbb{R}^n} \partial \Gamma(x, 0, y, 0) f(y) \, dy \]

where \( \partial \) denotes the exterior conormal derivative in the \( (x, t) \) variables. Classically, \( \bar{K} \) is often denoted \( K^\ast \), but we avoid this notation here as \( \bar{K} \) need not be the adjoint of \( K \) unless \( L \) is self-adjoint. Rather, for us, \( K^\ast, S^\ast \) and \( D^\ast \) will denote the analogues of \( K, S \) and \( D \) corresponding to \( L^\ast \) (although sometimes we shall write \( K^L \), etc., when we wish to emphasize the dependence on a particular operator), and we use the notation \( \text{adj}(T) \) to denote the Hermitian adjoint of an operator \( T \) acting in \( \mathbb{R}^n \). With these conventions, we have that \( \bar{K} = \text{adj}(K^\ast) \), as the reader may verify. We apologize for this departure from tradition, but the context of complex coefficients seems to require it.

For sufficiently smooth coefficients, the following “jump relation” formulae have been established in [MMT]. We defer to Section 4 our discussion of the jump formulae, and the nature of their “non-tangential” realization, in the non-smooth case. We have

\[ D_{ss} f \to \left( \mp \frac{1}{2} I + K \right) f \]

\[ (\nabla S_{f}) \mid_{t=ss} \to \left( \mp \frac{1}{2} \frac{f(x)}{A_{n+1,n+1}(x)} \epsilon_{n+1} + T f, \right. \]

(these convergence statements must be interpreted properly - see Section 4) where

\[ T f(x) \equiv ”p.v.” \int_{\mathbb{R}^n} \nabla \Gamma(x, 0, y, 0) f(y) \, dy. \]

Then, as usual\(^5\), we shall obtain solvability of (D2) in the upper (resp. lower) half space by establishing boundedness on \( L^2(\mathbb{R}^n) \) of \( f \to D_{ss} f \), uniformly in \( t \), and invertibility of \( -\frac{1}{2} I + K \) (resp. \( 1 I + K \)). Similarly, solvability of (N2) and (R2) follows from \( L^2 \) boundedness of \( f \to \bar{N}, (\nabla S_{ss} f) \), and (for (N2)) invertibility on \( L^2 \) of \( \pm \frac{1}{2} I + \bar{K} \), and (for (R2)) invertibility of the mapping \( S_0 = S_{f} \mid_{t=0} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \). We now set some convenient terminology:

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\(^4\)For non-smooth coefficients, some care should be taken to define the “principal value” operators on the boundary - see Section 4.

\(^5\)In the setting of non-smooth coefficients, some rather extensive preliminaries are required in order to apply the layer potential method to obtain solvability; see Section 4.
we shall say that an operator $L$ for which all of the above hold has “Bounded and Invertible Layer Potentials”. If in addition we have the square function estimate

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |\partial^2_t S_t f(x)|^2 \frac{dxdt}{|t|} \leq C\|f\|_2^2,$$

then we shall say that $L$ has “Good Layer Potentials”. Finally, we shall refer to the constant in (1.10), together with all of the constants arising in the estimates for the boundedness and invertibility of the layer potentials, collectively as the “Layer Potentials Constants” for $L$.

It is clear that the “Good Layer Potentials” property implies existence of solutions to (D2), (N2) and (R2), with the desired estimates, with the possible exception of the square function bound in the statement of the problem (D2). However, we shall prove in the sequel that if $L$ and $L^*$ have Good Layer Potentials, then (D2) is solvable with square function bounds; indeed, in that case there exist solutions to (D2) of the form $u(\cdot, t) = D_t f$, and by Corollary 5.20 below, every such solution satisfies

$$\int_{\mathbb{R}^{n+1}} |
abla u(x,t)|^2 tdtdx < \infty.$$

In this paper, we prove the following theorems. In the sequel we assume always that our $(n + 1) \times (n + 1)$ coefficient matrices are $t$-independent, complex, and satisfy the ellipticity condition (1.1) and the De Giorgi-Nash-Moser estimates (1.2) and (1.3).

**Theorem 1.12.** Suppose that $L_0 = -\text{div} A^0 \nabla$ and $L_1 = -\text{div} A^1 \nabla$ are operators of the type described above, and that solutions $u_0$, $w_0$ of $L_0 u_0 = 0$, $L_0^* w_0 = 0$ satisfy the De Giorgi-Nash-Moser estimates (1.2) and (1.3). Suppose also that $L_0$ and $L_0^*$ have “Good Layer Potentials”. Then $L_1$ and $L_1^*$ have Good Layer Potentials, provided that

$$\|A^0 - A^1\|_{L^{n}(\mathbb{R}^n)} \leq \epsilon_0,$$

where $\epsilon_0$ is sufficiently small depending only on dimension and on the various constants associated to $L_0$ and $L_0^*$, specifically: the ellipticity parameters, the De Giorgi-Nash-Moser constants (1.2) and (1.3), and the Layer Potential Constants.

We observe that it is not clear whether the property that $L$ has “Good Layer Potentials” is preserved under regularization of the coefficients. For this reason, we shall be forced to prove Theorem 1.12 without recourse to the usual device of making an a priori assumption of smooth coefficients. We also note that we shall use the invertibility of the layer potentials associated to $L_0$ and $L_0^*$ even to establish the boundedness of the layer potentials associated to $L_1$ (see Section 7 below).

**Theorem 1.13.** Suppose that $L = -\text{div} A \nabla$ is an operator of the type defined above, and in addition, suppose that $A$ is real and symmetric. Then $L$ has Good Layer Potentials, and its Layer Potential Constants depend only on dimension and on the ellipticity parameters in (1.1).

We remark that while Theorem 1.13 yields in particular the solvability of (D2), (N2) and (R2) in the case that $A$ is real and symmetric, it is only the fact that this solvability is obtainable via layer potentials that is new here, the solvability of (D2) having been previously obtained by Jerison and Kenig [JK1], and that of (N2) and (R2) by Kenig and Pipher [KP], without the use of layer potentials. The essential missing ingredient had been the boundedness of the layer potentials.

The previous two theorems are our main results. As corollaries, we obtain
Theorem 1.14. Suppose that $L_1 = - \text{div} A^1 \nabla$ is an operator of the type defined above, and that $\|A^1 - A^0\|_{L^\infty(\mathbb{R}^n)} \leq \epsilon_0$, for some real, symmetric, $t$-independent uniformly elliptic matrix $A^0 \in L^\infty(\mathbb{R}^n)$. Then (D2), (N2) and (R2) are all solvable for $L_1$, provided that $\epsilon_0$ is sufficiently small, depending only on dimension and the ellipticity parameters for $A^0$. The solution of (D2) is unique among the class of solutions $u$ for which $\sup_{t > 0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} < \infty$, and the solutions of (N2) and (R2) are unique modulo constants among the class of solutions for which $\tilde{N}_t(\nabla u) \in L^2$.

Theorem 1.15. The conclusion of Theorem 1.14 holds also in the case that $\|A^1 - A^0\|_\infty$ is sufficiently small, where $A^0$ is now a constant, elliptic complex matrix.

The last theorem follows from Theorem 1.12, and the fact that constant coefficient operators have Good Layer Potentials (see the appendix, Section 10).

We note that by a standard device, Theorems 1.12, 1.13 and 1.14 all extend readily to the case where $\Omega = \{(x, t) : t > F(x)\}$, with $F$ Lipschitz. Indeed, by “pulling back” under the mapping $\rho : \mathbb{R}^{n+1} \rightarrow \Omega$ defined by $\rho(x, t) = (x, F(x) + t)$, we may reduce to the case of the half-space. The pull-back operators are of the same type, and, in particular, the coefficients remain $t$-independent. Moreover, if the original coefficients are real and symmetric, then so are those of the pull-back operator. In this setting, the parameter $\epsilon_0$ will also depend on $\|\nabla F\|_\infty$. In addition, our results may be further extended to the setting of star-like Lipschitz domains (which would seem to be the most general setting in which there is a distinguished “radial direction”). The idea is to use a partition of unity argument, as in [MMT], to reduce to the case of a Lipschitz graph. We omit the details. In this context, see also [AA2].

Let us now briefly review the history of work in this area, which falls broadly into two categories, depending on whether or not the $t$-independent coefficient matrix is self-adjoint. We discuss the former category first, and we mention only the case of a single equation, although results for certain constant coefficient self-adjoint systems in a Lipschitz domain are known, see e.g. [K, K2] for further references. (Moreover, the present setting of complex coefficients may be viewed in the context of $2 \times 2$ systems, and indeed this provides part of our motivation to consider the complex case). For Laplace’s equation in a Lipschitz domain, the solvability of (D2) was obtained by Dahlberg [D], and that of (N2) and (R2) by Jerison and Kenig [JK2]; solvability of the same problems via harmonic layer potentials is due to Verchota [V], using the deep result of Coifman, McIntosh and Meyer [CMcM] concerning the $L^2$ boundedness of the Cauchy integral operator on a Lipschitz curve. The results of [V] and [CMcM] are subsumed in our Theorem 1.13 via the pull-back mechanism discussed above. Moreover, as mentioned above, for $A$ real, symmetric and $t$-independent, the solvability of (D2) was obtained in [JK1], and that of (N2) and (R2) in [KP], but those authors did not use layer potentials. The case of real symmetric coefficients with some smoothness has been treated via layer potentials in [MMT].

In the “non self-adjoint” setting, previous results had been obtained in three special cases. First, it was known that (D2) is solvable for small, complex perturbations of constant elliptic matrices. This is due to Fabes, Jerison and Kenig [FJK] via the method of multilinear expansions. To our knowledge, (R2) and (N2) had not been treated in this setting.
Second, one has solvability of (D2), (N2) and (R2) in the special case that the matrix $A$ is of the "block" form

$$
\begin{bmatrix}
  B & 0 & \cdots & 0 \\
  0 & B & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1
\end{bmatrix}
$$

where $B = B(x)$ is a $n \times n$ matrix. In this case, (D2) is an easy consequence of the semigroup theory, while (R2) amounts to solving the Kato square root problem for the $n$-dimensional operator

$$
J = -\text{div}_x B(x) \nabla_x,
$$

and (N2) amounts to $L^2$ boundedness of the Riesz transforms $\nabla J^{-\frac{1}{2}}$ (equivalently, to solving the Kato problem for the adjoint operator $\text{adj}(J)$). Moreover, the boundedness of the Riesz transform $\nabla J^{-\frac{1}{2}}$ can also be interpreted as the statement that the single layer potential is bounded from $L^2$ into $\dot{L}^2$. These results were obtained in [CMcM] ($n = 1$), [HMc] ($n = 2$), [AHLT] (when $B$ is a perturbation of a real, symmetric matrix), [HLMc] (when the kernel of the heat semi-group $e^{-tJ}$ has a Gaussian upper bound) and [AHLMcT] in general.

We mention also that in the case $n = 1$, where $L^2$ solvability for block matrices appeared in [CMcM] long before the higher dimensional case was treated, there were furthermore $L^p$ and Hardy space solvability results established in [AT2].

Third, Kenig, Koch, Pipher and Toro [KKPT] have obtained solvability of (Dp) (the problems (Dp), (Np) and (Rp) are defined analogously to (D2), (N2) and (R2), but with $L^2$ bounds replaced by $L^p$) in the case $n = 1$ (that is, in $\mathbb{R}^2$), for $p$ sufficiently large depending on $L$, in the case that $A(x)$ is real, but non-symmetric. Moreover, they construct a family of examples in $\mathbb{R}^2$ in which solvability of (Dp) may be destroyed for any specified $p$ by taking $A(x)$ to be an appropriate perturbation of the $2 \times 2$ identity matrix. In the same setting of real, non-symmetric coefficients in two dimensions (that is, in $\mathbb{R}^2$), Kenig and Rule [KR] have obtained solvability of (Nq) and (Rq), where $q$ is dual to the [KKPT] exponent. Their result exploits boundedness of the layer potentials in that setting. Very recently, A. Barton [Bar] has obtained a two-dimensional $L^p$ analogue of our Theorem 1.12, showing that the results of [KKPT] and of [KR] are stable under small complex perturbations.

The main purpose, then, of this paper is to develop, to the extent possible, an $L^2$ theory of boundary value problems for full coefficient matrices with complex (including also real, not necessarily symmetric) entries, and to do so via the method of layer potentials. We remark that, in the setting of $L^2$ solvability with $t$-independent coefficients, the counterexample of [KKPT] shows that our perturbation results are in the nature of best possible. We point out that a parallel program to the present one, but without the De Giorgi-Nash assumption, has been subsequently elaborated in recent work of three of the present authors [AAH], and of two of those authors and A. McIntosh [AAMc], in which $L^2$ solvability results similar to ours (and more generally including the case of $t$-independent systems) are obtained via the construction of a functional calculus for certain Dirac type operators, which is used in lieu of the layer potential method developed here. In dimensions $n + 1 \geq 3$, the layer potential approach remains unique to this paper.

A word about $t$-independence is in order. It has been observed by Caffarelli, Fabes and Kenig [CFK] that some regularity in the transverse direction is necessary, in order to deduce solvability of (D2). More precisely, they show that given any function $\omega(\tau)$ with $\int_0^1 (\omega(\tau))^2 d\tau/\tau = +\infty$, there exists a real, symmetric elliptic matrix $A(x, t)$, whose modulus
of continuity in the $t$ direction is controlled by $\omega$, but for which the corresponding elliptic-harmonic measure and the Lebesgue measure on the boundary are mutually singular. On the other hand, it is shown in [FJK] that (D2) does hold, assuming that the transverse modulus of continuity $\omega(\tau) \equiv \sup_{x \in \mathbb{R}^n, 0 < r < \tau} |A(x, t) - A(x, 0)|$ satisfies the square Dini condition
\[ \int_0^1 (\omega(\tau))^2 d\tau/\tau < \infty, \] provided that $A(x, 0)$ is sufficiently close to a constant matrix $A_{const}$. A more refined, scale invariant version of the square Dini condition has been introduced by Dahlberg [D2], and developed further by R. Fefferman, Kenig and Pipher [FKP], and Kenig and Pipher [KP, KP2], to prove perturbation results in the setting of real coefficients, and in [KP] have been partially extended to the setting of complex coefficients, and more generally, to elliptic systems, by two of the present authors [AA], under the assumption that $A^0 = A^0(x)$ is $t$-independent, that $L_0$ is $L^2$ solvable, and that one has sufficient smallness of the Carleson norm of the measure in (1.17). They then deduce $L^2$ solvability for $L_1$. By the examples of [FKP], such a smallness restriction is necessary to preserve $L^2$ solvability for $L_1$, even for real coefficients. Note that the condition (1.17) requires that $A^1 = A^0$ on the boundary. Our work here (and that of [AAH] and [AAMc]) provides a complement to that of [AA] in that we allow the coefficients to differ at the boundary. Moreover, our results (and again those of [AAH] and [AAMc]) provide a rich class of $t$-independent operators to which the perturbation theory of [AA] may be applied, so that, in combination, these papers establish a rather complete picture of the situation for $L^2$ solvability.

Let us now set some notation that will be used throughout the paper. We shall use div and $\nabla$ to denote the full $n+1$ dimensional divergence and gradient, respectively. At times, we shall need to consider the $n$-dimensional gradient and divergence, acting only in $x$, and these we denote either by $\nabla_x$ and $\text{div}_x$, or by $\nabla$ and $\text{div}$; i.e.
\[ \nabla_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right) = \nabla \]
and for $\mathbb{R}^n$-valued $\vec{w}$, $\text{div}_x \vec{w} \equiv \nabla \cdot \vec{w}$. Similarly, given an $(n+1) \times (n+1)$ matrix $A$, we shall let $A_{ij}$ denote the $n \times n$ sub-matrix with entries $(A_{ij})_{k,l} \equiv A_{k,l}, \quad 1 \leq i, j \leq n$, and we define the corresponding elliptic operator acting in $\mathbb{R}^n$ by
\[ L_{\vec{w}} \equiv -\text{div}_x A_{\vec{w}} \nabla_x. \]
We shall also use the notation
\[ D_j \equiv \frac{\partial}{\partial x_j} = \partial_{x_j}, \quad 1 \leq j \leq n+1 \]
bearing in mind that $x_{n+1} = t$. Points in $\mathbb{R}^{n+1}$ may sometimes be denoted by capital letters, e.g. $X = (x, t)$, $Y = (y, s)$. Balls in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ will be denoted respectively by $B(X, r) \equiv \{ Y : |X - Y| < r \}$ and $\Delta_n(x) \equiv \{ y : |x - y| < r \}$. We shall often encounter operators whose kernels involve derivatives applied to the second set of variables in the fundamental solution $\Gamma(x, t, y, s)$. We shall indicate this by grouping the operators with appropriate parentheses, thus:
\[ (S, \nabla) f(x) \equiv \int_{\mathbb{R}^n} \nabla_{x_n} \Gamma(x, t, y, s) |_{t=0} f(y) \, dy. \]
Hence, one then has
\[(S, \nabla_0) \cdot \tilde{f} = -S_j (\text{div}_j \tilde{f}), \quad (S, D_{n+1}) = -\partial_i S_i,\]
where in the second identity we have used (1.4)

Given a cube $Q$, we denote the side length of $Q$ by $\ell(Q)$. Furthermore, given a positive number $r$, we let $rQ$ denote the concentric cube with side length $r\ell(Q)$.

We shall use $P_t$ to denote a nice approximate identity, acting on functions defined on $\mathbb{R}^n$; i.e. $P_t f(x) = \phi_t + f$, where $\phi_t(x) = t^{-n} \phi(x/t)$, $\phi \in C_0^\infty(\{|x| < 1\})$, $0 \leq \phi$ and $\int_{\mathbb{R}^n} \phi = 1$.

Following [FJK], we introduce a convenient norm for dealing with square functions (although we warn the reader that our measure differs from that used in [FJK]):
\[\|F\|_{L^2} \equiv \left( \int_{\mathbb{R}^{n+1}} |F(x,t)|^2 \frac{dxdt}{|t|} \right)^{1/2}, \quad \|F\|_{L^1} \equiv \left( \int_{\mathbb{R}^{n+1}} |F(x,t)|^2 \frac{dxdt}{|t|} \right)^{1/2}.
\]

For a family of operators $U_t$, we write
\[\|U_t\|_{+,op} \equiv \sup_{|t|_{2,\mathbb{R}^n} = 1} \|U_t f\|_+,
\]
and similarly for $\|\cdot\|_{-,op}$ and $\|\cdot\|_{|\cdot|_{op}}$. Sometimes, we may drop the “+” sign when it is clear that we are working in the upper $\frac{1}{2}$-space. As usual, we allow generic constants $C$ to depend upon dimension and ellipticity, and, in the proof of the perturbation result, upon the constants associated to the “good” operator $L_0$. Specific constants, still depending on the same parameters, will be denoted $C_1$, $C_2$, etc..

The paper is organized as follows. In sections 2 and 3, we prove some useful technical estimates. In section 4 we discuss the boundary behavior and uniqueness of our solutions. The next five sections are the heart of the matter, in which we prove Theorem 1.12 (sections 5, 6 and 7), and Theorem 1.13 (sections 8 and 9). Sections 10 and 11 are appendices, in which we discuss the constant coefficient case, and establish De Giorgi/Nash bounds in 3 dimensions for solutions of $t$-independent equations in the half-space.

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2. Some consequences of De Giorgi-Nash-Moser bounds and $t$-independence

We begin with a simple consequence of $t$-independence of the coefficients, which in particular allows for a “Caccioppoli-type” estimate on horizontal slices.

Let $L := -\nabla \cdot A \nabla$ be a second order divergence form complex elliptic operator with bounded measurable coefficients, defined on $\mathbb{R}^{n+1} := \{(x,t): x \in \mathbb{R}^n, t \in \mathbb{R}\}$.

**Proposition 2.1.** Suppose that the matrix $A$ is $t$-independent, i.e., $A = A(x)$. Then there is a uniform constant $\epsilon > 0$ depending only on $n$ and ellipticity, and for every $p \in [2, 2 + \epsilon)$, a uniform constant $C_p$ such that, for each fixed cube $Q \subset \mathbb{R}^n$, and $t \in \mathbb{R}$, if $Lu = 0$ in the box $I_Q := 4Q \times (t - \ell(Q), t + \ell(Q))$, then we have the following estimates

\[
\left( \frac{1}{|Q|} \int_Q |\nabla u(x,t)|^p dx \right)^{1/p} \leq C_p \left( \frac{1}{|Q|} \int_Q \left| \nabla u(x,t) \right|^2 dx dt \right)^{1/2}, \tag{2.2}
\]

\[
\left( \frac{1}{|Q|} \int_Q |u(x,t)|^p dx \right)^{1/p} \leq C_p \left( \frac{1}{|\ell(Q)|} \int_{\ell(Q)} \left| u(x,t) \right|^2 dx dt \right)^{1/2}, \tag{2.3}
\]
Remark 1: We can replace the box \( I_Q \) by \((1 + \delta)Q \times (t - \delta t(Q), t + \delta t(Q))\), for any fixed \( \delta > 0 \), in which case the constant \( C_p \) and the dimensions of \( Q^* \) and \( Q^{**} \) should be adjusted depending upon \( \delta \).

Remark 2: Clearly, we can also replace the cube \( Q \) by an annular region \( 2Q \setminus Q \), and \( I_Q \) by \((4Q \setminus \frac{1}{2}Q) \times (t - \ell(Q), t + \ell(Q))\), with appropriate adjustments to \( Q^* \) and \( Q^{**} \) and to \( C_p \).

Remark 3: The only features of \( L \) that we use are uniform ellipticity (in the complex sense) and \( t \)-independence of the coefficients. In particular, we do not use De Giorgi-Nash-Moser estimates.

Proof of Proposition 2.1. Fix a cube \( Q \subset \mathbb{R}^n \), and \( t \in \mathbb{R} \). It suffices to prove (2.2), since the latter implies (2.3) by the standard interior Caccioppoli estimate.

We have

\[
\left( \int_Q |\nabla u(x, t)|^p \, dx \right)^{1/p} \leq \left( \int_Q \left| \nabla u(x, t) - \int_t^{t + \ell(Q)/8} \nabla u(x, s) \, ds \right|^p \, dx \right)^{1/p} + \left( \int_Q \int_t^{t + \ell(Q)/8} |\nabla u(x, s)|^p \, ds \, dx \right)^{1/p} =: I + II.
\]

By Hölder’s inequality,

\[
II \leq \left( \int_Q \int_t^{t + \ell(Q)/8} |\nabla u(x, s)|^p \, ds \, dx \right)^{1/p} \leq \left( \frac{1}{|Q^*|} \int_{Q^*} |\nabla u(x, s)|^2 \, dx \, ds \right)^{1/2},
\]

where in the last line we have used the classical result of N. Meyers [Me] concerning the higher integrability of the gradient of solutions of divergence form elliptic equations, whose proof requires only Caccioppoli’s inequality, Sobolev’s inequality, and the self-improvement of reverse Hölder inequalities, and which is valid for \( p \in [2, 2 + \epsilon) \), depending only upon ellipticity and dimension. Also,

\[
I = \left( \int_Q \left( \int_t^{t + \ell(Q)/8} |\nabla \partial_\tau u(x, \tau)| \, d\tau \, ds \right)^p \, dx \right)^{1/p} \leq \ell(Q) \left( \int_Q \int_t^{t + \ell(Q)/8} |\nabla \partial_\tau u(x, \tau)|^p \, d\tau \, dx \right)^{1/p} \leq C \left( \frac{1}{|Q^*|} \int_{Q^*} |\partial_\tau u(x, \tau)|^2 \, dx \, d\tau \right)^{1/2},
\]

where in the last line we have used \( t \)-independence of the coefficients and the estimate of N. Meyers, and then Caccioppoli.

Throughout the remainder of this section, and throughout the rest of the paper, we suppose always that our differential operators satisfy the “standard assumptions”: that is, divergence form elliptic, with ellipticity parameters \( \lambda \) and \( \Lambda \), defined in \( \mathbb{R}^{n+1} \), \( n \geq 2 \), with complex coefficients that are bounded, measurable and \( t \)-independent; moreover, we suppose that solutions of \( Lu = 0 \) satisfy the De Giorgi-Nash-Moser estimates (1.2) and (1.3).
We now prove some technical estimates using rather familiar arguments. In the sequel, $\Gamma$ will denote the fundamental solution of $L$, and we set

$$K_{m,t}(x,y) \equiv (\partial_t)^{-m+1}\Gamma(x,t,y,0)$$

**Lemma 2.5.** Suppose that $L$ and $L^*$ satisfy the “standard assumptions” as above. Then for each integer $m \geq -1$, there exists a constant $C_m$ depending only on $m$, dimension, ellipticity and (1.2) and (1.3), such that for all $t \in \mathbb{R}$, and $x, y \in \mathbb{R}^n$, we have

$$|K_{m,t}(x,y)| \leq C_m(|t| + |x - y|)^{-m-1}$$

whenever $|2h| \leq |x - y|$ or $|h| < 20|t|$, for some $\alpha > 0$, where $(\square^h f)(x) \equiv f(x + h) - f(x)$.

**Sketch of proof.** The case $m = -1$ of (2.6) follows from its parabolic analogue in [AT], Section 1.4; alternatively, the reader may consult [HK2] for a direct proof in the elliptic case. The case $m = 0$ may be treated by applying (1.3) to the solution $u(x,t) = \partial_t \Gamma(x,t,y,0)$ in the ball $B(x,t), R/2$, with $R = \sqrt{|t|^2 + |x - y|^2}$, and then using Caccioppoli’s inequality to reduce to the case $m = -1$. The case $m > 0$ is obtained by iterating the previous argument, and (2.7) follows from (1.2) and (2.6).

We remark that, by taking more care with the Caccioppoli argument, using a ball of appropriately chosen radius $c_mR$ rather than $R/2$, one may obtain the natural growth bound $m!C_m^m$ in (2.6) and (2.7). We leave the details to the interested reader.

**Lemma 2.8.** Suppose that $L, L^*$ satisfy the standard assumptions. Then for each integer $m \geq -1$ and for each $\rho > 1$, there exists constants $C_m, C_{m,\rho}$ (with the same dependence as the constants in the previous lemma), such that for every cube $Q \subseteq \mathbb{R}^n$, for all $x \in \mathbb{Q}$, and for all integers $k \geq 1$, we have

(i) $\int_{\mathbb{R}^n} |2^k \ell(Q)^m (\partial_t)^{m+1} \nabla_x \Gamma(x,t,y,0)|^2 dy \leq C_m (2^k \ell(Q))^{-n-2}, \quad \forall t \in \mathbb{R}$

(ii) $\int_{\mathbb{R}^n} |\ell(Q)^m (\partial_t)^{m+1} \nabla_x \Gamma(x,t,y,0)|^2 dy \leq C_{m,\rho} \ell(Q)^{2m-2} (\frac{\ell(Q)}{\rho})^2, \quad \frac{\ell(Q)}{\rho} < |t| < \rho \ell(Q)$.

**Sketch of Proof.** With $(x, t)$ fixed as in the hypotheses of the lemma, bearing in mind Remarks 1 and 2 following Proposition 2.1, we apply (2.3) with $p = 2$ to the adjoint solution

$u(y,s) := (\partial_t)^{m+1}\Gamma(x,t,y,s),$

and then use (1.4) and (2.6). We leave the routine details to the reader.

As a Corollary of the previous two Lemmata we deduce

**Lemma 2.9.** Suppose that $L, L^*$ satisfy the standard assumptions, and let $f : \mathbb{R}^n \to C^{m+1}$. Then for every cube $Q$ and for all integers $k \geq 1$ and $m \geq -1$, we have

(i) $\|\ell(Q)^m (S,\nabla) \cdot (f|_{2^k\ell(Q)}|_{2^k\ell(Q)}^2 \|_{L^2(Q)} \leq C_m 2^{-m(k+2)} (2^k \ell(Q))^{2m-2} \|f\|_{L^2(2^k\ell(Q))}^2, \quad t \in \mathbb{R}$

(ii) $\|\ell(Q)^m (S,\nabla) \cdot (f|_{2^k\ell(Q)}|_{2^k\ell(Q)}^2 \|_{L^2(Q)} \leq C_{m,\rho} \ell(Q)^{2m-2} (\frac{\ell(Q)}{\rho})^2, \quad \frac{\ell(Q)}{\rho} < |t| < \rho \ell(Q)$.

**Proof.** We consider estimate (i). Let $x \in Q$. Then

$$\|\ell(Q)^m (S,\nabla) \cdot (f|_{2^k\ell(Q)}|_{2^k\ell(Q)}^2 (x) \leq \|\ell(Q)^m \nabla_x \Gamma(x,t,y,s) |_{t=0} \cdot f(y)|dy\|^2 \leq \|\ell(Q)^m \nabla_x \Gamma(x,t,y,s) |_{t=0} \|^2 \|f\|^2_{L^2(2^k\ell(Q))}^2 \leq C_m (2^k \ell(Q))^{-n-2m-2} \|f\|^2_{L^2(2^k\ell(Q))}^2.$$
where in the last step we have used Lemma 2.8(i) (to handle $\nabla \tau$) and (2.6) (to treat $\partial \tau$). We now obtain (i) by integrating over $Q$. The proof of (ii) is similar, and is omitted. □

Lemma 2.10. Suppose that $L, L^*$ satisfy the standard assumptions, and let $f : \mathbb{R}^n \to \mathbb{C}$. Then for every $t \in \mathbb{R}$, and for every integer $m \geq 0$, we have

(i) \[ \| \tau^m \partial_t^{m+1} (S_t \nabla) \cdot f \|_{L^2(\mathbb{R}^n)} \leq C_m \| f \|_{L^2}. \]

(ii) \[ \| \tau^m \partial_t^{m+1} (\nabla S_t) f \|_{L^2(\mathbb{R}^n)} \leq C_m \| f \|_{L^2}. \]

Proof. Fix $t \in \mathbb{R}$ and $m \geq 0$. It is enough to prove (i), since (ii) follows by duality and the fact that $\text{adj} S_t = S_t^*$, where $S_t^*$ is the single layer potential operator associated to $L^*$. We may further suppose that $t \neq 0$, since otherwise the left hand side of the inequality vanishes. Set $\theta_t = \tau^m \partial_t^{m+1} (S_t \nabla)$. We write

\[ \| \theta_t \|_{L^2} = \left( \sum_{Q} \int_Q |\theta_t|^2 \right)^{1/2}, \]

where the sum runs over the dyadic grid of cubes with $|t| = 2^n$. With $Q$ fixed, we decompose $f$ into $f_{12Q}$ plus a sum of dyadic “annular” pieces $(f_{1,2^{n-1}Q}|_{2^{n-1}Q})$. The bound (i) now follows from Lemma 2.9. We omit the details. □

We now prove a “2-sided” version of Lemma 2.10.

Lemma 2.11. Suppose that $L, L^*$ satisfy the standard assumptions, and let $f : \mathbb{R}^n \to \mathbb{C}$. Then for every $t \in \mathbb{R}$, and for every integer $m \geq 0$, we have

(2.12) \[ \| \tau^m (S_t \nabla) \cdot f \|_{L^2} \leq C_m \| f \|_{L^2}. \]

Moreover, if in addition $L$ and $L^*$ have Good (or even merely Bounded) Layer Potentials, then (2.12) remains true in the case $m = -1$.

Proof. By density, it is enough to assume a priori that $f \in C_0^\infty$, and by Lemma 2.10 (ii) and $t$–independence, we may replace $(S_t \nabla) \cdot f$ by $(S_t \nabla) \cdot f^\tau = -S_t \div f^\tau$, where $\bar{f} ^\tau \in C_0^\infty(\mathbb{R}^n, \mathbb{C})$.

Fix $t \in \mathbb{R}$. We may suppose that $t \neq 0$, since otherwise the left hand side of (2.12) vanishes, and we may further suppose that $t > 0$, since the case $t < 0$ may be handled in the same way. We subdivide $\mathbb{R}^n$ into a grid $G(t)$ of cubes of sidelength $\ell(Q) = t$, and write

\[ \| \tau^m (S_t \nabla) \cdot f \|_{L^2}^2 = \sum_{Q \in G(t)} \int_Q \left| \tau^m (S_t \nabla) \cdot f^\tau \right|^2 dx \]

\[ \leq C \sum_{Q \in G(t)} \int_{t/2}^{3t/2} \int_{Q \setminus \partial Q} \left| \tau^m (S_t \nabla) \cdot f^\tau \right|^2 dx d\tau \]

\[ = C \int_{t/2}^{3t/2} \sum_{Q \in G(t)} \int_{Q \setminus \partial Q} \left| \tau^m (S_t \nabla) \cdot f^\tau \right|^2 dx d\tau \]

\[ \leq C \sup_{\tau > 0} \int_{\mathbb{R}^n} \left| \tau^m (S_t \nabla) \cdot f^\tau \right|^2 dx, \]

where in the second line we have applied (2.3) with $p = 2$ to the solution

\[ u(x, t) = -\tau^m S_t \div f^\tau = \tau^m (S_t \nabla) \cdot f^\tau, \]

and in the last line we have used that the cubes $3Q$ have bounded overlaps. The conclusion of the lemma now follows immediately from Lemma 2.10 (i), or from the hypothesis of Bounded Layer Potentials if $m = -1$. □
Lemma 2.13. Suppose that $L, L'$ satisfy the standard assumptions. Fix a cube $Q \subset \mathbb{R}^n$, and suppose that $y, y' \in Q$. For $(x, t) \in \mathbb{R}^{n+1}$, set

$$u(x, t) := \Gamma(x, t, y, 0) - \Gamma(x, t, y', 0).$$

If $\alpha$ is the H"older exponent in (2.7), then for every integer $k \geq 4$, we have

$$\int_{2^k Q} |\nabla u(x, t)|^2 \, dx \leq C 2^{-2k\alpha} \left(2^k(t(Q))^{-n} \right).$$

Sketch of Proof. Bearing in mind Remark 2 (following the statement of Proposition 2.1), we apply (2.3) to the solution $u(x, t)$, and then use (2.7) with $m = -1$. We omit the routine details. \hfill \Box

The next lemma says that

$$L = L_0 - \sum_{j=1}^{n+1} A_{n+1, j} D_{n+1} D_j - \sum_{j=1}^{n} D_j A_{j, n+1} D_{n+1}$$

in an appropriate weak sense on each “horizontal” cross-section.

Lemma 2.15. Let $L$ satisfy the standard assumptions of this paper. Suppose that $Lu = g$ in the strip $a < t < b$, where $g \in C_0^\infty(\mathbb{R}^{n+1})$. Suppose also that $\nabla u, \nabla \bar{\partial} u \in L^2(\mathbb{R}^n)$, uniformly in $t \in (a, b)$, with norms depending continuously on $t \in (a, b)$. Then for every $F \in L^2(\mathbb{R}^n) \cap L^2_2(\mathbb{R}^n)$, and for all $t \in (a, b)$, we have that

$$\int_{\mathbb{R}^n} A_{ij}(x) \nabla_x u(x, t) \nabla_x F(x) \, dx = \sum_{j=1}^{n+1} \int_{\mathbb{R}^n} A_{n+1, j}(x) \partial_x \partial_t u(x, t) F(x) \, dx$$

$$- \sum_{j=1}^{n} \int_{\mathbb{R}^n} A_{j, n+1}(x) \bar{\partial}_x u(x, t) \partial_j F(x) \, dx + \int_{\mathbb{R}^n} g(x, t) F(x) \, dx. \hfill (2.16)$$

Proof. Let $t \in (a, b)$, and let $\eta < \min(t - a, b - t)$. Set $\varphi_\eta(s) = \frac{1}{\eta} \varphi(s/\eta)$, where $\varphi \in C_0^\infty(\frac{1}{2}, \frac{1}{2})$, $0 \leq \varphi$, $\int \varphi = 1$. Define

$$F_{1, \eta}(x, s) \equiv F(x) \varphi_\eta(t - s).$$

Then by the definition of weak solutions, and $t$-independence, we have

$$\int_{\mathbb{R}^n} A_{ij}(x) \nabla_x u(x, s) \nabla_x F_{1, \eta}(x, s) \, dxds = \sum_{j=1}^{n+1} \int_{\mathbb{R}^n} A_{n+1, j}(x) \partial_x \partial_t u(x, s) F_{1, \eta}(x, s) \, dxds$$

$$- \sum_{j=1}^{n} \int_{\mathbb{R}^n} A_{j, n+1}(x) \bar{\partial}_x u(x, s) \partial_j F_{1, \eta}(x, s) \, dxds + \int_{\mathbb{R}^n} g(x, s) F_{1, \eta}(x, s) \, dxds.$$

By our hypotheses, the functions of $t$ defined by the four integrals in (2.16), are all continuous in $(a, b)$. The conclusion of the lemma then follows if we let $\eta \to 0$. \hfill \Box

In the sequel, we shall find it useful to consider approximations of the single layer potential. The bounds in the following lemma will not be used quantitatively, but will serve rather to justify certain formal manipulations. For $\eta > 0$, set

$$S_\eta \equiv \int_{\mathbb{R}} \varphi_\eta(t - s) S_x \, ds,$$

where $\varphi_\eta \equiv \bar{\varphi}_\eta \ast \varphi_\eta$, $\bar{\varphi}_\eta \in C_0^\infty(-\eta/2, \eta/2)$ is non-negative and even, with $\int \bar{\varphi}_\eta = 1$ and $\bar{\varphi}_\eta(t) \equiv \eta^{-1} \varphi(t/\eta)$. \hfill (2.17)
Lemma 2.18. Suppose that $L, L^*$ satisfy the standard assumptions, and let $S_t$ denote the single layer potential operator associated to $L$. Then for each $\eta > 0$ and for every $f \in L^2(\mathbb{R}^n)$ with compact support, we have

(i) $\|\partial_t S^\eta_t f\|_{L^2_{t,\infty}(\mathbb{R}^{n+2})} \leq C_\eta \|f\|_{L^2(\mathbb{R}^n)}$, $0 < \beta < 1$.

(ii) $\|\nabla S^\eta_t f\|_{L^2_{t,\infty}(\mathbb{R}^{n+1})} \leq C_\eta \|f\|_{L^2(\mathbb{R}^n)}$, $0 < \beta < 1$.

(iii) $\|\partial_t^2 S^\eta_t f\| \leq C_\eta \|f\|_{L^2(\mathbb{R}^n)}$, $0 < \beta < 1$.

(iv) $\|\nabla \left( S^\eta_t - S_t \right) f\|_{L^2_{t,\infty}(\mathbb{R}^{n+1})} \leq C_\eta \|f\|_{L^2(\mathbb{R}^n)}$, $\eta < |t|/2$.

(v) $\lim_{\eta \to 0} \int_0^\infty \int_{\mathbb{R}^n} |\nabla \partial_t \left( S^\eta_t - S_t \right) f|^2 \frac{dt}{t} = 0$, $0 < \beta < 1$.

(vi) For each cube $Q \subset \mathbb{R}^n$, $\|\partial_t S^\eta_t f\|_{L^2_t(Q) \to L^2_{t,\infty}(\mathbb{R}^{n+1})} \leq C_{\eta, t}(Q)$.

Proof. (i). We observe that

$$\partial_t S^\eta_t f(x) = \int_{\mathbb{R}^n} k_t(x,y) f(y) dy,$$

where $k_t(x,y) \equiv \partial_t \left( \varphi_\eta * \Gamma(x,\cdot,y,0) \right)(t)$. Thus, by Lemma 2.5

$$|k_t(x,y)| \leq C \min \left\{ |x-y|^{-\beta}, \eta^{-1} |x-y|^{-\beta} \right\} \leq C \eta^{-\beta} |x-y|^{-\beta}, \quad 0 < \beta < 1.$$

Estimate (i) now follows by the fractional integral theorem.

(ii). We first note that

$$S^\eta_t f(x) = \int_{\mathbb{R}^n} C^\eta(x,t,s) f(y)\varphi_\eta(s) du,$$

where $f(y,s) \equiv f(y)\varphi_\eta(s)$. Let $\tilde{g} \in C^\infty_0(\mathbb{R}^n, C^\infty)$, with $\|\tilde{g}\|_2 = 1$, and set $\check{g}_\eta(x,\sigma) \equiv \tilde{g}(x)\varphi_\sigma$. Then

$$|\langle \tilde{g}, \nabla S^\eta_t f \rangle| = \left| \int_{\mathbb{R}^n} \nabla \tilde{g}_\eta(x,\sigma) \nabla L^{-1} f_h(x,\sigma) dx \right| \leq \|\tilde{g}_\eta\|_{L^2(\mathbb{R}^{n+1})} \|\nabla L^{-1} f_h\|_{L^2(\mathbb{R}^{n+1})} \leq C \eta^{-1/2} \|f_h\|_{L^2(\mathbb{R}^{n+1})} \leq C \eta^{-1/2} \|\nabla \varphi_\sigma\|_{L^2(\mathbb{R}^{n+1})} \|f\|_{L^2(\mathbb{R}^{n+1})}.$$

where $2s = (2n + 2)/(\alpha + 3)$, since $L^2(\mathbb{R}^{n+1}) \hookrightarrow L^2_2(\mathbb{R}^{n+1}) \equiv \left( L^2(\mathbb{R}^{n+1}) \right)^*$, and $\nabla L^{-1} \text{div} : L^2(\mathbb{R}^{n+1}) \to L^2_{t,\infty}(\mathbb{R}^{n+1})$. Estimate (ii) now follows.

(iii). We proceed as for estimate (i), and write

$$t \partial^2_t S^\eta_t f(x) = \int_{\mathbb{R}^n} h_t(x,y) f(y) dy,$$

where $h_t(x,y) \equiv \partial^2_t \left( \varphi_\eta * \Gamma(x,\cdot,y,0) \right)(t)$, so that, by Lemma 2.5,

$$|h_t(x,y)| \leq C t \min \left\{ |x-y|^{-\beta}, \eta^{-2} |x-y|^{-\beta} \right\} \leq C t \eta^{-1-\beta} |x-y|^{-\beta}, \quad 0 < \beta < 1.$$

Moreover, if $t > 2\eta$, we have the sharper estimate

$$|h_t(x,y)| \leq C \frac{t}{(t + |x-y|)^{\beta+1}} \leq C \eta^{-\beta} |x-y|^{-\beta}, \quad 0 < \beta < 1.$$

Thus,

$$\|t \partial^2_t S^\eta_t f\| \leq C \left( \int_0^{2\eta} \eta^{-2-2\beta} dt + \int_{2\eta}^{\infty} t^{-1-2\beta} dt \right) \|f\|_{L^2(\mathbb{R}^{n+1})},$$

and (iii) follows.
(iv). We suppose that \( \eta < |t|/2 \). Then
\[
\| \nabla (S_\eta^n - S_\eta) f \|_{L^2(\mathbb{R}^n)} \leq \varphi_\eta \ast \| \nabla (S_\eta - S_\eta) f \|_{L^2(\mathbb{R}^n)}.
\]
But for \( |s - t| < \eta < |t|/2 \), we have by the mean value theorem and Lemma 2.10(ii) that
\[
\| \nabla (S_s - S_t) f \|_{L^2(\mathbb{R}^n)} \leq \frac{\eta}{|t|} \sup_{|r-t|<\eta/2} \| r \nabla \partial_\eta S_r f \|_{L^2(\mathbb{R}^n)} \leq C \frac{\eta}{|t|} \| f \|_2.
\]

(v). We take \( \eta < \varepsilon/2 \), and write
\[
\int_\varepsilon^\infty \int_{\mathbb{R}^n} |r \nabla \partial_\eta (S_\eta^n - S_\eta) f|^2 \frac{dx \, dt}{t} = \int_\varepsilon^\infty \int_{\mathbb{R}^n} \varphi_\eta \ast |r \nabla D_{n+1} (S_\eta - S_\eta) f|^2 \frac{dx \, dt}{t} - \int_\varepsilon^\infty \varphi_\eta \ast \| r \nabla D_{n+1} (S_\eta - S_\eta) f \|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t}.
\]
(2.19)

We claim that the last expression converges to 0, as \( \eta \to 0 \). Indeed, for \( |s - t| < \eta < t/2 \), we have that
\[
\| r \nabla D_{n+1} (S_s - S_t) f \|_{L^2(\mathbb{R}^n)} \leq \eta \sup_{|r-t|<\eta/2} \| r \nabla \partial_\eta S_r f \|_{L^2(\mathbb{R}^n)} \leq C \frac{\eta}{|t|} \| f \|_2
\]
by Lemma 2.10(ii). Thus, for \( \eta < \varepsilon/2 \), (2.19) is bounded by \( C \eta \varepsilon^2 \| f \|_2^2 \), and the claim follows.

(vi). Estimate (vi) follows from (i) and Hölder’s inequality. \( \square \)

3. Some consequences of “off-diagonal” decay estimates

Here, we prove some estimates that hold in general for operators satisfying the conclusions of Lemmas 2.9 and 2.10. For the sake of notational convenience, we observe that part (i) of the former conclusion can be reformulated as

\[
\| \theta_t (f 1_{Q^{n+1}}) \|_{L^2(Q^{n+1})}^2 \leq C_m 2^{-nk} \left( \frac{|t|}{2^k |Q^{n+1}|} \right)^{2m+2} \| f \|_{L^2(Q^{n+1})}^2
\]
where \( \theta_t = \tau^{m+1} \partial_t^{m+1} (S_\eta \nabla) \). We now consider generic operators \( \theta_t \) which satisfy (3.1) for some integer \( m \geq 0 \). The next lemma is essentially due to Fefferman and Stein [FS]. We omit the well known proof.

**Lemma 3.2.** Suppose that \( \{ \theta_t \}_{t \in \mathbb{R}} \) is a family of operators which satisfies (3.1) for some integer \( m \geq 0 \) and in every cube \( Q \), whenever \( |t| \leq C |Q| \). If \( \| \theta_t \|_{op} \leq C \), then
\[
\int_{|t|} |\theta_t b(x)|^2 \frac{dx \, dt}{|t|}
\]
is a Carleson measure for every \( b \in L^\infty \).

**Lemma 3.3.** Suppose that \( \{ \theta_t \}_{t \in \mathbb{R}} \) is a family of operators satisfying (3.1) for some integer \( m \geq 0 \), as well as the bound
\[
\sup_{t \in \mathbb{R}} \| \theta_t f \|_{L^2(\mathbb{R}^n)} \leq C \| f \|_2.
\]
Suppose that \( \{ \Lambda_t \}_{t \in \mathbb{R}} \) is a family of operators satisfying the bounds
\[
\sup_{t \in \mathbb{R}} \| \Lambda_t f \|_2 \leq C \| f \|_2, \quad \sup_{t \in \mathbb{R}} \| \Lambda_t f \|_{L^2(Q)} \leq C \exp \left( \frac{-\text{dist}(E, E')}{|t|} \right) \| f \|_{L^2(E')}
\]
whenever (in the latter estimate) support \( f \subseteq E' \). Then \( \theta_t \Lambda_t \) also satisfies (3.1), whenever \( |t| \leq C |Q| \).
Proof. We may suppose that \( k \geq 4 \), otherwise, subdivide \( Q \) dyadically to reduce to this case. Given \( Q \), set \( \bar{Q} \equiv 2^{k-2}Q \). Then

\[
\theta_i \Lambda_i = \theta_i 1_{\bar{Q}} \Lambda_i + \theta_i 1_{\mathbb{R}^n \setminus \bar{Q}} \Lambda_i.
\]

For the first term, we have the bound

\[
\| \theta_i 1_{\bar{Q}} \Lambda_i \|_{L^2(\bar{Q})} \leq \| \theta_i \|_{L^2} \| 1_{\bar{Q}} \Lambda_i \|_{L^2(\bar{Q})} \leq \| \theta_i \|_{L^2} \exp \left\{ \frac{-2^k \ell(Q)}{C} \right\} \| f \|_{L^2(2^{k+1}Q, 2^{k}Q)}
\]

which in particular yields (3.1) for this term, if \( |t| \leq C \ell(Q) \). Next, we consider the second term in (3.4), which equals

\[
\sum_{j \geq k-2} \theta_i 1_{2^{j+1}Q \setminus 2^{j}Q} \Lambda_i.
\]

The desired bound follows for this term by applying (3.1) for each \( j \) fixed, and summing the resulting geometric series. \( \square \)

**Lemma 3.5.** (i) Suppose that \( \{R_t\}_{t \in \mathbb{R}} \) is a family of operators satisfying (3.1), for some \( m \geq 1 \), and for all \( |t| \leq C \ell(Q) \), and suppose also that \( \sup_t |R_t|_{L^2} \leq C \), and that \( R_1 = 0 \) for all \( t \in \mathbb{R} \) (our hypotheses allow \( R_1 \) to be defined as an element of \( L^2_{\text{loc}} \)). Then for \( h \in L^2_1(\mathbb{R}^n) \),

\[
(3.6) \quad \int_{\mathbb{R}^n} |R_t h|^2 \leq C^2 \int_{\mathbb{R}^n} |

\nabla \cdot h|^2.
\]

(ii) If, in addition, \( \|R_t \div \|_{L^2} \leq C/|t| \), then also

\[
(3.7) \quad \||R_t f|| \leq C||f||_{L^2}.
\]

**Proof.** We suppose that \( t > 0 \), and show that (3.6) implies (3.7). The latter follows from

\[
(3.8) \quad \|R_t (s^2 \Delta e^{r \Delta})\|_{L^2} \leq C \min \left\{ \frac{s}{t}, \frac{t}{s} \right\},
\]

by a standard orthogonality argument. In turn, (3.8) is easy to prove: the case \( t < s \) is just (3.6), and the case \( s < t \) follows by hypothesis from the factorization \( \Delta = \div \nabla \).

We now turn to the proof of (3.6). Let \( D(t) \) denote the grid of dyadic cubes with \( \ell(Q) \leq |t| \leq 2\ell(Q) \). For convenience of notation we set \( m_0 h \equiv \int_Q h \). Then

\[
\left( \int_{\mathbb{R}^n} |R_t h|^2 \right)^{\frac{1}{2}} = \left( \sum_{Q \in D(t)} \int_Q |R_t h|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{Q \in D(t)} \int_Q |R_t (h - m_0 h)|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{Q \in D(t)} \int_Q |(h - m_0 h)_{12Q}|^2 \right)^{\frac{1}{2}} + \left( \sum_{Q \in D(t)} \int_Q |(h - m_0 h)_{12Q^c}|^2 \right)^{\frac{1}{2}} \equiv I + II.
\]

Since \( R_t : L^2 \to L^2 \), we have by Poincaré’s inequality that

\[
I \leq C \left( \sum_{Q \in D(t)} \int_{2Q} |h - m_0 h|^2 \right)^{\frac{1}{2}} \leq C |t| \left( \sum_{Q \in D(t)} \int_{2Q} |\nabla h|^2 \right)^{\frac{1}{2}} \leq C |t| \|\nabla h\|_{L^2}.
\]
Moreover, we are given that $R_t$ satisfies (3.1). Thus,

$$H \leq \sum_{k=1}^{\infty} \left( \sum_{Q \in Q(t)} \int_Q |R_t((h - m_2Qh)1_{2^{k+1}Q \cap 2^kQ})|^2 \right)^{1/2} \leq C \sum_{k=1}^{\infty} \left( \sum_{Q \in Q(t)} 2^{-k(n+4)} \int_{2^{k+1}Q} |h - m_2Qh|^2 \right)^{1/2} \leq C \sum_{k=1}^{\infty} \sum_{j=1}^{k} \left( \sum_{Q \in Q(t)} \int_{Q} 2^{-k}2^{-jn} \int_{2^{j+1}Q} |h - m_2Qh|^2 \right)^{1/2},$$

where in the last step we have used that

$$h - m_2Qh = h - m_2Qh + m_2Qh - m_2Qh + \cdots + m_4Qh - m_2Qh.$$ 

By Poincaré’s inequality, since $j \leq k$ we obtain in turn the bound

$$C|t| \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left( \sum_{Q \in Q(t)} 2^{-jn} \int_{2^{j+1}Q} |\nabla x h|^2 \right)^{1/2} \leq C|t| \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left( \sum_{Q \in Q(t)} \int_{Q} 2^{-jn} |\nabla x h|^2 \right)^{1/2} \leq C|t| \sum_{k=1}^{\infty} 2^{-k} \sum_{j=1}^{k} \left( \int_{\mathbb{R}^n} \int_{|x-y| \leq C2^j} |\nabla x h(x)|^2 x \, dy \right)^{1/2} = C|t|\|\nabla x h\|_2.$$

**Lemma 3.9.** Given $\{R_t\}_{t \in \mathbb{R}_+}$, as in part (i) of the previous lemma, we have that

$$\|r^{-1}R_tF\| \leq C\|\nabla_x F\|_{L^2(\mathbb{R}^n)},$$

provided that \(\|r^{-1}R_tF\|_{L^2(\mathbb{R}^n)}\) is a Carleson measure, where $\Phi(x) \equiv x$.

**Proof.** We may assume that $F \in C_0^\infty$, and that $t > 0$. Let $\mathbb{D}_j$ denote the dyadic grid of cubes of side length $2^{-j}$. Then

$$\|r^{-1}R_tF\| = \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{Q}_{\mathbb{D}_j}} \int_Q |r^{-1}R_tF|^2 \, dy \, dt$$

(3.10)

$$= \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{Q}_{\mathbb{D}_j}} \int_{2^jQ} \int_Q |r^{-1}R_tF|^2 \, dy \, dt.$$

We now use an idea taken from [J] and [Ch2, pp. 32-33]. For $(x, t)$ fixed, set

$$G_{x,t}(z) = F(z) - F(x) - (z - x) \cdot P_t(\nabla_x F)(x),$$

where as usual $P_t$ is an approximate identity. Since $R_t1 = 0$, we have, for any fixed $x$,

$$\frac{1}{t}R_tF(y) = \frac{1}{t}R_t(G_{x,t})(y) + \frac{1}{t}R_tF(y) \cdot P_t(\nabla_x F)(x) \equiv I + II.$$ 

The contribution of $II$ to (3.10) is bounded by

$$\sum_{j=-\infty}^{\infty} \sum_{Q \in \mathbb{Q}_{\mathbb{D}_j}} \int_{2^jQ} \int_Q \left|P_t(\nabla_x F)(x)\right|^2 \left|\int_{Q} \frac{1}{t}R_tF(y) \cdot P_t(\nabla_x F)(x)\right|^2 \, dy \, dx \, dt \leq C \int_0^\infty \int_{\mathbb{R}^n} |P_t(\nabla_x F)(x)|^2 \left\{ \int_{B(x,|C|)} \left|\frac{1}{t}R_tF(y)\right|^2 \, dy \right\} \, dx \, dt \leq C\|\nabla_x F\|_{L^2(\mathbb{R}^n)} \|u\|_C.$$
by Carleson’s Lemma, where
\[
\|d\|_{L^2} \equiv \sup_Q \int_0^{\infty} \left\{ \int_Q \left| \frac{1}{t} R_t \Phi(y) \right|^2 \, dy \right\} \, dx \, dt \leq C \sup_Q \int_0^{\infty} \left\{ \int_Q \left| \frac{1}{t} R_t \Phi(y) \right|^2 \, dy \right\} \, dx \, dt \leq C \sup_Q \int_0^{\infty} \left\{ \int_Q \left| \frac{1}{t} R_t \Phi(y) \right|^2 \, dy \right\} \, dx \, dt.
\]

Next we consider the contribution of $I$ to (3.10). For $Q \in \mathbb{D}_-$, and $x \in Q$, we have
\[
I = R_t \left( t^{-1} G_{x,t} 1_{2Q} \right)(y) + \sum_{k=1}^{\infty} R_t \left( t^{-1} G_{x,t} 1_{2^{k+1} Q \setminus 2^k Q} \right)(y) \equiv I_0 + \sum_{k=1}^{\infty} I_k.
\]
Since $R_t : L^2 \to L^2$, we obtain the bound
\[
\|I_0\| \leq C \sum_{j=-\infty}^{\infty} \left\{ \int_Q \int_{2^j Q} \frac{|G_{x,t}(y)|^2}{t^2} \, dy \, dx \, dt \right\} \leq C \int_0^{\infty} \int_{\mathbb{R}^2} (\beta(x,t))^2 \, dx \, dt \leq C \|\nabla F\|_{L^2(\mathbb{R}^2)},
\]
where $(\beta(x,t))^2 = \int_{|y| \in C_t} t^{-2} |G_{x,t}(y)|^2 \, dy$, and where the last step is a well known consequence of Plancherel’s Theorem, see, e.g. [Ch2, pp. 32-33] or [H, pp. 249-250]. Furthermore, since $R_t$ satisfies (3.1) for some $m \geq 1$, whenever $t \approx \ell(Q)$, we have that
\[
C^{-1} \sum_{k=1}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} \left\{ \int_Q \int_{2^j Q} \frac{|G_{x,t}(y)|^2}{t^2} \, dy \, dx \, dt \right\} \right\} \leq C \sum_{k=1}^{\infty} \left\{ \sum_{j=-\infty}^{\infty} \left\{ \int_Q \int_{2^j Q} \frac{|G_{x,t}(y)|^2}{t^2} \, dy \, dx \, dt \right\} \right\} \equiv \sum_{k=1}^{\infty} 2^{-k} \|\beta_k\|,
\]
where, after making the change of variable $t \to t/2^k$, \[
\beta_k(x,t) = \left( \int_{|y| \in C_t} \frac{|F(y) - F(x) - (y-x) \cdot P_{2^k t} (\nabla F)(x)|^2}{t^2} \, dy \right)^{1/2}.
\]
We now claim that $\|\beta_k\| \leq C \sqrt{t} \|\nabla F\|_2$, from which the conclusion of the lemma trivially follows. By Plancherel’s Theorem, the definition of $P_t$ and the change of variable $x - y = h$, we have
\[
\|\beta_k\|^2 = \int_0^{\infty} \int_{|h| < C_t} \left( \int_{|y| < C_t} \frac{|e^{ih \cdot \xi} - 1 - (ih \cdot \xi) \hat{\phi}(2^{-k} t \xi)|^2}{t^2 |\xi|^2} \, d\xi \right) \, dh \, dt,
\]
where $\phi \in C_0^\infty(|x| < 1)$ and $\int_{\mathbb{R}^2} \Phi = 1$. By the change of variable $h \to th$, we have
\[
\|\beta_k\|^2 = \int_0^{\infty} \int_{|h| < C_t} \left( \int_{|y| < C_t} \frac{|e^{ih \cdot \xi} - 1 - (ih \cdot \xi) \hat{\phi}(2^{-k} t \xi)|^2}{t^2 |\xi|^2} \, d\xi \right) \, dh \, dt.
\]
Since $\hat{\phi} \in S$ and $\Phi(0) = 1$, we have that
\[
\frac{|e^{ih \cdot \xi} - 1 - (ih \cdot \xi) \hat{\phi}(2^{-k} t \xi)|}{t |\xi|} \leq C \min \left( \frac{|\xi|}{t}, 1, \frac{2^k}{t |\xi|} \right).
\]
Indeed, if \( |t| \xi | \leq 1 \), then
\[
\left| e^{it \xi} - 1 - (ih \cdot t \xi) \right|_{\xi t} \leq \frac{|e^{it \xi} - 1 - (ih \cdot t \xi)|}{|t| \xi} + \frac{|ih \cdot t \xi (1 - \hat{\phi}(2^{-k}t \xi))|}{|t| \xi}
\]
\[
\leq C(|t| \xi + 2^{-k}|t|) \leq C|t| \xi.
\]
On the other hand, if \( |t| \xi | > 1 \), then
\[
\left| e^{it \xi} - 1 \right|_{\xi t} \leq \frac{2}{|t| \xi},
\]
and
\[
\left| (ih \cdot t \xi) \hat{\phi}(2^{-k}t \xi) \right|_{\xi t} \leq C|\hat{\phi}(2^{-k}t \xi)| \leq \frac{C}{1 + 2^{-k}|t|} \leq C \min\left(1, \frac{2^k}{|t|} \right).
\]
We then obtain the bound \( ||b||^2 \leq Ck ||\nabla f||^2 \) as claimed.

**Lemma 3.11.** Suppose that \( \theta_t \) satisfies (3.1) for some \( m \geq 0 \), whenever \( 0 < t \leq C \), and that \( ||\theta||_{L^2} \leq C \). Let \( b \in L^\infty(\mathbb{R}^n) \), and let \( \mathcal{A}_t \) denote a self-adjoint averaging operator whose kernel \( \varphi_t(x, y) \) satisfies \( |\varphi_t(x, y)| \leq C r^m 1_{|x-y| < Ct}, \varphi_t \geq 0, \int \varphi_t(x, y) dy = 1 \). Then
\[
\sup_{t \in \mathbb{R}_+} ||(\theta_t b, \mathcal{A}_t f)||_2 \leq C ||b||_{\infty} ||f||_2.
\]

**Proof.** Since we do not assume that \( \theta_t \) is bounded on \( L^2 \) uniformly in \( t \), this requires a bit of an argument. Observe that
\[
||\theta_t b, \mathcal{A}_t f||_2 \leq ||f||_2 ||\mathcal{A}_t((\theta_t b)^2 \mathcal{A}_t f)||_2 \leq ||f||^2 ||\mathcal{K}(x, y)||_{L^1(\mathbb{R}^n)},
\]
where \( \mathcal{K}(x, y) \) is the kernel of the self-adjoint operator \( f \rightarrow \mathcal{A}_t((\theta_t b)^2 \mathcal{A}_t f) \), i.e.,
\[
\mathcal{K}(x, y) = \int_{\mathbb{R}^n} \varphi_t(x, z) \varphi_t(b(z)) \varphi_t(z, y) dz.
\]
Consequently,
\[
||\mathcal{K}(x, y)||_{L^1} = \int_{\mathbb{R}^n} \varphi_t(x, z) \varphi_t(b(z)) dz \leq C r^m \int_{|x-y| < Ct} |\theta_t b(z)| dz.
\]
Thus, by (3.1) and the fact that \( \theta_t \) is bounded on \( L^2 \) uniformly in \( t \), we have that
\[
||\mathcal{K}(x, y)||^2_{L^1(\mathbb{R}^n)} \leq C \left\{ \left( \int_{Q(x, 4Ct)} |b|^2 \right)^{1/2} + \sum_{k=2}^\infty 2^{k} \left( \int_{Q(x, 2^kCt \cup Q(x, 2^{k+1}Ct))} |b|^2 \right)^{1/2} \right\} \leq C ||b||_{\infty},
\]
where \( Q(x, R_t) \) is the cube centered at \( x \) with side length \( R_t \). This proves the lemma.

**Lemma 3.12.** Suppose that
\[
\Omega = \int_0^\delta \left( \frac{s}{t} \right)^\delta \mathcal{W}_{1, t} \theta_t ds s,
\]
for some \( \delta > 0 \), where \( \sup_{t} ||\mathcal{W}_{1, t}||_{L^2} \leq C \). Then
\[
||\Omega||_{L^p} \leq C ||\theta||_{L^p}.
\]

**Proof.** This is a standard Schur type argument. Indeed, if \( ||G(x, t)|| \leq 1 \), then
\[
\left| \int_0^\infty \int_{\mathbb{R}^n} G(x, t) \mathcal{W}_{1, t} \theta_t f(x) dx dt \right| \leq \int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{W}_{1, t} \theta_t f(x) dx \right|^2 dt ds \leq C \int_0^\infty \int_{\mathbb{R}^n} |\theta_t f(x)|^2 \int_0^\infty \left( \frac{s}{t} \right)^\delta dt ds dx
\]
\[
\leq C ||\theta||_{L^p} ||f||_{L^p}.
\]
We first observe that

\[ L \text{ is enough to establish the bound } 20 M. \]

\[ \nabla \text{ need only consider } (4.2) \sup_{t>0} \| \nabla u(t) \|_2 \leq C \| \nabla_0 (\nabla u) \|_2. \]

**Proof.** The desired bound for \( \partial_t u \) follows readily from \( t \)-independence and (1.3). Thus, we need only consider \( \nabla u \). Let \( \tilde{\psi} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^m) \), with \( \| \tilde{\psi} \|_2 = 1 \). For \( t_0 > 0 \) fixed, it will then be enough to establish the bound

\[ \left| \int_{\mathbb{R}^n} u(\cdot, t_0) \, \text{div}_x \tilde{\psi} \right| \leq C \| \nabla_0 (\nabla u) \|_2. \]

To this end, we write

\[ \int_{\mathbb{R}^n} u(\cdot, t_0) \, \text{div}_x \tilde{\psi} = \int_{\mathbb{R}^n} \left( u(x, t_0) - \int_{t_0/2}^{0} u(x, t) \, dt \right) \, \text{div}_x \tilde{\psi}(x) \, dx \]

\[ + \int_{\mathbb{R}^n} \int_{t_0/2}^{t_0} u(x, t) \, \text{div}_x \tilde{\psi}(x) \, dt \, dx \equiv I + II. \]

We first observe that

\[ |I| = \left| \int_{\mathbb{R}^n} \int_{t_0/2}^{t_0} \partial_t u(x, s) \, ds \, \text{div}_x \tilde{\psi}(x) \, dx \right| \]

by Cauchy-Schwarz and Fubini’s Theorem. Moreover,

\[ |I| = \left| \int_{\mathbb{R}^n} \int_{t_0/2}^{t_0} \partial_t u(x, s) \, ds \, \text{div}_x \tilde{\psi}(x) \, dx \right| \]

\[ = \int_{t_0/2}^{t_0} \int_{\mathbb{R}^n} \nabla_x \partial_t u(x, s) \, \tilde{\psi}(x) \, dx \, ds \]

\[ \leq C \left( \int_{t_0/2}^{t_0} \int_{\mathbb{R}^n} |\nabla \partial_t u(x, s)|^2 \, dx \, ds \right)^{1/2} \]

\[ \leq C \left( \int_{t_0/2}^{t_0} \int_{\mathbb{R}^n} |\partial_s u(x, s)|^2 \, dx \, ds \right)^{1/2}, \]

where in the last step we have split \( \mathbb{R}^n \) into cubes of side length \( \approx t_0 \) and used Caccioppoli’s inequality. The conclusion of the lemma follows since the bound already holds for \( \partial_t u \). \( \Box \)

We now discuss some trace results. The following lemma is the analogue of Theorem 3.1 of [KP]. We recall that \( u \to f \) n.t. means that \( \lim_{(y, t) \to (x, 0)} u(y, t) = f(x) \), for a.e. \( x \in \mathbb{R}^n \), where the limit runs over \( (y, t) \in \gamma(x) \). As usual, \( P_{\gamma} \) will denote a self-adjoint approximate identity acting in \( \mathbb{R}^n \). We shall denote by \( W^{1,2}_c \) the subspace of compactly supported elements of the usual Sobolev space \( W^{1,2} \).

**Lemma 4.3.** Suppose that \( L, L^* \) satisfy the standard assumptions. If \( Lu = 0 \) in \( \mathbb{R}^{n+1}_+ \) and \( \nabla_0 (\nabla u) \in L^2(\mathbb{R}^n) \), then there exists \( f \in L^2_1(\mathbb{R}^n) \) such that

\[ (i) \ |\nabla f| \leq C \| \nabla_0 (\nabla u) \|_2, \text{ and } u \to f \text{ n.t., with } |u(y, t) - f(x)| \leq C t \| \nabla_0 (\nabla u) \|_2 \text{ whenever } (y, t) \in \gamma(x). \]

\[ (ii) \ \nabla |u(\cdot, t)| \to \nabla f \text{ weakly in } L^2(\mathbb{R}^n) \text{ as } t \to 0. \]

If \( Lu = 0 \) in \( \mathbb{R}^n \times (0, \rho) \), where \( 0 < \rho \leq \infty \), and \( \sup_{0 \leq t \leq \rho} \| \nabla u(\cdot, t) \|_{L^2(\mathbb{R}^n)} < \infty \), then there exists \( g \in L^2(\mathbb{R}^n) \) such that \( g = \partial u / \partial y \) in the variational sense, i.e.,
Thus, we may define a bounded linear functional on $H^1_0$.

(Here, $\tilde{N} = -e_{n+1}$ is the unit outer normal to $\mathbb{R}^{n+1}$).

Of course, the analogous results hold for the lower half space.

Proof. The existence of $f \in L^2_0(\mathbb{R}^n)$ satisfying (i) may be obtained by following *mutatis mutandis* the corresponding argument in [KP] pp. 451-462. We omit the details.

(ii) We first establish convergence in the sense of distributions. Let $\tilde{\psi} \in C^\infty_0(\mathbb{R}^n, \mathbb{C}^m)$. Then by (i),

$$\left| \int_{\mathbb{R}^n} (\nabla u(\cdot, t) - \nabla f) \tilde{\psi} \right| = \left| \int_{\mathbb{R}^n} (u(\cdot, t) - f) \text{div} \tilde{\psi} \right| \leq C t \|\nabla u(\cdot, t)\|_2 \|\text{div} \tilde{\psi}\|_2 \to 0.$$

By the density of $C^\infty_0$ in $L^2$, the weak convergence in $L^2$ then follows readily from (4.2).

(iii) We follow [KP], with some modifications owing to the unboundedness of our domain. We treat only the case $\rho = \infty$, and leave it to the reader to check the details in the case of finite $\rho$. Fix $0 < R < \infty$ and set $B_R = B(0, R) \equiv \{ x \in \mathbb{R}^{n+1} : |x| < R \}$, $B_R^c \equiv B_R \cap \mathbb{R}^{n+1}_-$ and $\Delta_R = B_R \cap \{ t = 0 \}$. Define a linear functional on $W^{1,2}_0(B_R)$ (the closure of $C^\infty_0$ in $W^{1,2}_0(B_R)$) by

$$\Lambda_R(\Psi) \equiv \int_{B_R^c} \nabla u \cdot \nabla \Psi, \quad \Psi \in W^{1,2}_0(B_R).$$

Clearly, $\|\Lambda_R\| \leq CR^{1/2} \sup_{t \neq 0} \|\nabla u(\cdot, t)\|_2$. By trace theory, $tr\left(W^{1,2}_0(B_R)\right) \subset H^{1/2}(\Delta_R)$, defined as the closure in $H^{1/2}(\mathbb{R}^n)$ of $C^\infty_0(\Delta_R)$. Here, $\|f\|_{H^{s}(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} + \|\zeta \cdot \hat{f}\|_{L^2(\mathbb{R}^n)}$, for $0 \leq s \leq 1$. On the other hand, suppose that $\psi \in H^{1/2}_0(\Delta_R)$. We extend $\psi$ to $\psi_{ext} \in W^{1,2}_0(B_R)$ by solving the problems

(D+D-)

$$\begin{cases}
\sum_{i=1}^{n+1} \partial_i^2 \psi^\pm_{ext} = 0 & \text{in } B_R \\
\psi^\pm_{ext}|_{\partial \Delta_R} = \psi, & \psi^+_\text{ext}|_{B^+_{2R}} = 0
\end{cases}$$

We set $\psi_{ext} = \psi^+_\text{ext}|_{B_R^c} + \psi^-_{ext}|_{B_R^c}$, and by standard theory of harmonic functions we have

$$\|\nabla \psi_{ext}\|_{L^2(B_R)} \leq C \|\psi\|_{H^{1/2}(\partial \Delta_R)}. $$

Thus, we may define a bounded linear functional on $H^{1/2}_0(\Delta_R)$ by $\Xi_R(\psi) \equiv \Lambda_R(\psi_{ext})$. Since $\Lambda_R(\Psi) = 0$ whenever $\Psi \in W^{1,2}_0(B_R)$, then $\Xi_R(\psi) \equiv \Lambda_R(\Psi)$ for every extension $\Psi \in W^{1,2}_0(B_R)$ with $tr(\Psi) = \psi$. Thus, there exists a unique $g_R \in H^{-1/2}(\Delta_R)$ with

$$\int_{B_R^c} A\nabla u \cdot \nabla \Psi = \langle g_R, tr(\Psi) \rangle, \quad \forall \Psi \in W^{1,2}_0(B_R).$$

Now suppose that $R_1 < R_2$, and construct $g_{R_k}$ corresponding to $B_k = B(0, R_k), k = 1, 2$. Then, since $W^{1,2}_0(B_1) \subset W^{1,2}_0(B_2)$ (if we extend elements in the former space to be 0 outside of $B_1$), we have that $g_{R_k} = g_{R_k}$ in $H^{-1/2}(\Delta_R)$. Thus, $\langle g_{R_k}, \psi \rangle = \langle g_{R_k}, \psi \rangle$, whenever $\psi \in H^{1/2}_0(\partial \Delta_R)$, and $B_1, B_2$ contain the support of $\psi$. It follows that $g := \lim_{R \to \infty} g_R$ exists in the sense of distributions, and that

(4.4) $$\int_{\mathbb{R}^{n+1}} A\nabla u \cdot \nabla \Psi = \langle g, tr(\Psi) \rangle, \quad \forall \Psi \in W^{1,2}_c(\mathbb{R}^{n+1}).$$

To complete the proof of (iii), it remains only to establish that $g \in L^2$. The bound

$$\|g\|_{L^2} \leq C \sup_{t \neq 0} \|\nabla u(\cdot, t)\|_2$$
will be an immediate consequence of (iv), to which we now turn our attention.

(iv). Again we present only the case \( \rho = \infty \). Since \( \sup_{t>0} ||\nabla u(\cdot, t)||_2 < \infty \), it is enough to verify the weak convergence for test functions in \( C_0^\infty \). Let \( \Psi \in C_0^\infty(\mathbb{R}^{n+1}) \), \( \psi := \Psi|_{t=0} \). By (4.4), it is enough to show that

\[
\int_{\mathbb{R}^n} \check{N} \cdot A\nabla u(\cdot, t) \psi \rightarrow \int_{\mathbb{R}^{n+1}} A\nabla u \cdot \nabla \Psi,
\]
as \( t \rightarrow 0 \). Integrating by parts, we see that for each \( \epsilon > 0 \),

\[
(4.5) \quad \int_{\mathbb{R}^n} \check{N} \cdot P_\epsilon(A\nabla u(\cdot, t)) \psi = \int_{\mathbb{R}^{n+1}} P_\epsilon(A\nabla u(\cdot, t+s)) (x) \cdot \nabla \Psi(x, s) dx ds,
\]
since \( Lu = 0 \) and our coefficients are \( t \)-independent. By dominated convergence, we may pass to the limit as \( \epsilon \rightarrow 0 \) in (4.5) to obtain

\[
(4.6) \quad \int_{\mathbb{R}^n} \check{N} \cdot A\nabla u(\cdot, t) \psi = \int_{\mathbb{R}^{n+1}} A(x) \nabla u(x, t+s) \cdot \nabla \Psi(x, s) dx ds,
\]

It therefore suffices to show that

\[
\int_{\mathbb{R}^{n+1}} A(x)(\nabla u(x, t+s) - \nabla u(x, s)) \cdot \nabla \Psi(x, s) dx ds = O\left( \sqrt{t} \right), \quad t \rightarrow 0.
\]

To this end, let \( R \) denote the radius of a ball centered at the origin which contains the support of \( \Psi \). We split the integral into \( \int_0^R \int_{|x|<R} + \int_R^\infty \int_{|x|<R} \). Since \( \sup_{x \in \mathbb{R}^n} ||\nabla u(\cdot, t)||_2 < \infty \), the first of these contributes at most \( O(t) \), while the second is dominated by

\[
C ||\nabla \Psi||_2 t \left( \int_R^\infty \||\nabla \partial_t u(\cdot, s)||^2_{L^2(\mathbb{R}^n)} ds \right)^{1/2} \leq C_\phi t \left( \int_R^\infty \frac{ds}{s^2} \right)^{1/2} \sup_{t>0} ||\nabla u(\cdot, t)||_2,
\]

where in the last step we have used Cacciopoli’s inequality in Whitney cubes in the \( 1/2 \)-space. The desired conclusion follows.

Next we discuss the boundedness of non- tangential maximal functions of layer potentials. We recall that \( S^\eta_t \) is defined in (2.17), and that \( P_t \) denotes a smooth approximate identity acting in \( \mathbb{R}^n \). In the sequel, given an operator \( T \), we shall use the notation

\[
(4.7) \quad ||T||_{op,Q} := ||T||_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} := \sup_{||f||_2} \frac{||TF||_{L^2(\mathbb{R}^n)}}{||f||_{L^2(\mathbb{R}^n)}},
\]

where the supremum runs over all \( f \) supported in \( Q \) with \( ||f||_2 > 0 \).

**Lemma 4.8.** Let \( L, L^* \) satisfy the standard assumptions. Then for \( 1 < p < \infty \), we have

(i) \( ||N_* (\partial_t S_{\eta f})||_p \leq C_p \left( \sup_{t>0} ||\partial_t S_{\eta f}||_{L^p} + 1 \right) ||f||_p \).

(ii) \( ||\check{N}_* (\nabla S_{\eta f})||_p \leq C_p \left( \sup_{t>0} ||\nabla S_{\eta f}||_p + ||N_* (\partial_t S_{\eta f})||_p \right) \).

(iii) \( ||N_* (P_t(\nabla S_{\eta f}))||_p \leq C \left( \sup_{t>0} ||\nabla S_{\eta f}||_p + ||N_* (\partial_t S_{\eta f})||_p \right) \).

(iv) \( \sup_{t \geq 0} ||N_* (P_t(\partial_t S_{\eta f}))||_{L^2} \leq C \left( \sup_{t \geq 0} ||\partial_t S_{\eta f}||_{L^2} + 1 \right) ||f||_2, \quad \eta > 0, \ supp f \subset Q \).

(v) \( ||N_* (S_{\nabla, \eta} f)||_{L^{2-\infty}} \leq C \left( \sup_{t \geq 0} ||(S_{\nabla} f)||_{L^2} + 1 \right) ||f||_2 \).

(vi) \( ||N_* (\partial_\tau f)||_{L^{2-\infty}} \leq C \left( \sup_{t \geq 0} ||(\partial_\tau f)||_{L^2} + 1 \right) ||f||_2 \).

where \( L^{2-\infty} \) denotes the usual weak-\( L^2 \) space.

**Proof.** By Lemma 2.5, the kernel \( K_\Gamma(x,y) = \partial_t \Gamma(x, t, y, 0) \) is a standard Calderón-Zygmund kernel with bounds independent of \( t \). We may then prove (i) by a familiar argument involving Cotlar’s inequality for maximal singular integrals. We omit the details (but see the proof of (iv) below, which is similar). Estimate (ii) may be obtained by following the
We claim that the following holds for all \( f \) and we define a maximal singular integral

\[
P_t(\nabla_t S_t f)(x) = \nabla_t P_t(S_t f)(x) = t^{-1} \tilde{Q}_t(S_t f)(x)
\]

where we have used that \( t \nabla_t P_t \equiv \tilde{Q}_t \) annihilates constants. But

\[
\tilde{Q}_t \left( t^{-1} \int_0^t \partial_s S_s f ds + S_0 f - \int_{\Delta_0(x_0)} S_0 f \right)(x) \leq CM(N_*(\partial_t S_t f))(x_0).
\]

and, by Poincaré’s inequality,

\[
\left| t^{-1} \tilde{Q}_t \left( S_0 f - \int_{\Delta_0(x_0)} S_0 f \right)(x) \right| \leq CM(\nabla_t S_0 f)(x_0).
\]

(iii). The proof is similar to that of estimate (ii), and we follow [KP]. Fix \( x_0 \in \mathbb{R}^n \), and suppose that \( |x - x_0| < t \). It is enough to replace \( \nabla \) by \( \nabla_t \). We have

\[
P_t(\nabla_t S_t f)(x) = \nabla_t P_t(S_t f)(x) = t^{-1} \tilde{Q}_t(S_t f)(x)
\]

where \( t \nabla_t P_t \equiv \tilde{Q}_t \) annihilates constants. But

\[
\tilde{Q}_t \left( t^{-1} \int_0^t \partial_s S_s f ds + S_0 f - \int_{\Delta_0(x_0)} S_0 f \right)(x) \leq CM(N_*(\partial_t S_t f))(x_0).
\]

and, by Poincaré’s inequality,

\[
\left| t^{-1} \tilde{Q}_t \left( S_0 f - \int_{\Delta_0(x_0)} S_0 f \right)(x) \right| \leq CM(\nabla_t S_0 f)(x_0).
\]

(iv). We suppose that \( \eta << t(Q) \), and that \( Q \) is centered at \( 0 \), as it is only this case that we shall encounter in the sequel. We shall deduce (iv) as a consequence of the following refinement of Cotlar’s inequality for maximal singular integrals. Let \( T \) be a singular integral operator associated to a standard Calderón-Zygmund kernel \( K(x,y) \). As usual, we define truncated singular integrals

\[
T_\varepsilon f(x) = \int_{|x-y|<\varepsilon} K(x,y)f(y)dy,
\]

and we define a maximal singular integral

\[
T^*_\varepsilon f = \sup_{0<\varepsilon<R} |T_\varepsilon f|.
\]

We claim that the following holds for all \( f \) supported in a cube \( Q \):

\[
T^*_\varepsilon(Q)f(x) \leq C (C_K + \|T\|_{op,Q}) Mf(x) + CM(T f)(x),
\]

where \( C_K \) depends on the Calderón-Zygmund kernel conditions. Momentarily taking this claim for granted, we proceed to prove (iv).

Let \( K^\eta_t(x,y) \) denote the kernel of \( \partial_t S^\eta_t \) (see (2.17)), i.e.,

\[
K^\eta_t(x,y) \equiv \partial_t \left( \varphi_{\eta} \ast \Gamma(x, \cdot, y, 0) \right)(t).
\]

Then by Lemma 2.5 we have for all \( t \geq 0 \), uniformly in \( t_0 \geq 0 \),

\[
|K^\eta_{t+t_0}(x,y) - K^\eta_{t_0}(x,y)| \leq C \left( \frac{1_{|x-y|+t_0>40\eta}}{t + |x-y|^{\alpha}} + \frac{1_{|x-y|+t_0>40\eta}}{\eta |x-y|^{1-\alpha}} \right)
\]

(4.11)

\[
|K^\eta_{t+t_0}(x+h,y) - K^\eta_{t_0}(x,y)| \leq C \frac{|h|^2}{(t + |x-y|)^{\alpha+\eta}}, \quad |x-y| + t > 10\eta
\]

where the last bound holds whenever \( |x-y| > 2|h| \) or \( 2t > |h| \). Of course, we also have a similar estimate concerning Hölder continuity in the \( y \) variable. In particular, \( K^\eta_{t+t_0}(x,y) \) is a standard Calderón-Zygmund kernel, uniformly in \( t, t_0 \) and \( \eta \).

We begin by showing that for each fixed \( x_0 \in \mathbb{R}^n \) and \( t_0 \geq 0 \),

\[
N_* \left( P_t \partial_t S^\eta_{t+t_0} f \right)(x_0) \leq \sup_{t \geq 0} |\partial_t S^\eta_{t_0} f(x_0)| + CM(Mf)(x_0).
\]

(4.12)
Thus, taking
\[
|P(\partial_t S^t_{\gamma t} f)(x) - \partial_t S^t_{\gamma t} f(x_0)| \leq C r^n \int_{|u| < 2r} \int_{\mathbb{R}^n} |K^\eta_{\gamma t}(z, y) - K^\eta_{\gamma t}(x_0, y)||f(y)|dydz,
\]
for which, in the case \( t > 10\eta \), we obtain immediately the bound \( CM f(x_0) \) by applying (4.11). In the case \( t \leq 10\eta \), we split the inner integral into
\[
\int_{|y-z| > 10\eta} + \int_{|y-z| \leq 10\eta} \leq CM f(x_0) + C (M f(z) + M f(x_0)),
\]
where we have applied (4.11) to bound the first term, and (4.10) to handle the second. The estimate (4.12) now follows readily.

Next, we observe that for \( f \) supported in a cube \( Q \) centered at 0, with \( \ell(Q) \gg \eta \),
\[
\sup_{r < 0} |\partial_t S^t_{\gamma t} f(x)| \leq \sup_{0 < r < \ell(Q)} |\partial_t S^t_{\gamma t} f(x)| + CM f(x).
\]
Indeed, suppose that \( t \geq \ell(Q) \gg \eta \). Then
\[
|\partial_t S^t_{\gamma t} f(x)| \leq \int |K^\eta_{\gamma t}(x, y)| f(y)|dy \leq CM f(x),
\]
by (4.10), since for \( y \in Q \), we have \(|x - y| \approx |x| \), if \(|x| > C t \), and \(|x - y| < C t \), if \(|x| < C t \).

Combining (4.12) and (4.13), we see that it is enough to treat \( \sup_{0 < r < \ell(Q)} |\partial_t S^t_{\gamma t} f(x)| \). To this end, fix \( x_0 \) and \( t \in (0, \ell(Q)) \), and set \( p \equiv \max(t, 2\eta) \). Then
\[
\partial_t S^t_{\gamma t} f(x_0) = \int_{|y| > 5p} K^\eta_{\gamma t}(x_0, y) - K^\eta_{\gamma t}(x_0, y_0) f(y)dy
\]
\[
+ \int_{|y| \leq 5p} K^\eta_{\gamma t}(x_0, y) f(y)dy - \int_{5p > |x - y| > p} K^\eta_{\gamma t}(x_0, y) f(y)dy
\]
\[
+ \int_{|y| > p} K^\eta_{\gamma t}(x_0, y) f(y)dy = I + II + III + IV.
\]
Then \(|I| + |II| + |III| \leq CM f(x_0) \), by Lemma 2.5 and by (4.10). Also,
\[
|IV| \leq \sup_{0 < r < \ell(Q)} \left| \int_{|y| > x} K^\eta_{\gamma t}(x_0, y) f(y)dy \right|.
\]
Thus, taking \( T \) in (4.9) to be the singular integral operator with kernel \( K^\eta_{\gamma t}(x, y) \), we obtain (iv), modulo the proof of (4.9).

We now turn to the proof of (4.9). The argument is a variant of the standard one. Suppose that \( f \) is supported in a cube \( Q \), and fix \( \varepsilon \in (0, \ell(Q)) \) and \( x_0 \in \mathbb{R}^n \). Set \( \Delta \equiv \Delta_{\varepsilon/2}(x_0), 2\Delta \equiv \Delta_{\varepsilon}(x_0) \). Let \( f_1 \equiv f 1_{2\Delta}, f_2 \equiv f - f_1 \). Then for \( x \in \Delta \), we have
\[
|T_{\gamma t} f(x_0)| = |T_{\gamma t} f_2(x_0)| = |T_{\gamma t} f_2(x_0) - T_{\gamma t} f_2(x) + T_{\gamma t} f(x) - T_{\gamma t} f_1(x)|
\]
\[
\leq C_k M f(x_0) + |T_{\gamma t} f(x)| + |T_{\gamma t} f_1(x)|.
\]
Let \( r \in (0, 1) \), and take an \( L' \) average of this last inequality over \( \Delta \). Note that \( f_1 = 0 \) unless \( 2\Delta \subset 5Q \), since \( \text{diam}(2\Delta) \leq 2\ell(Q) \). We therefore obtain
\[
|T_{\gamma t} f(x_0)| \leq C K M f(x_0) + M(T f)^{1/r}(x_0) + \left( \int_\Delta |T_{\gamma t} f_1|^r \right)^{1/r}
\]
\[
\leq C \left( C_k + \|T\|_{L'(Q) \rightarrow L'(5Q)} M f(x_0) + M(T f)(x_0) \right).
\]
where we have used Kolmogorov’s weak-$L^1$ criterion, and $L^{1,\infty}$ is the usual weak-$L^1$ space. But by a localized version of the Calderón-Zygmund Theorem,

$$
\|T\|_{L^1(Q)\to L^{1,\infty}(S_Q)} \leq C \left( C_K + \|T\|_{L^1(Q)\to L^2(S_Q)} \right) \leq C \left( C_K + \|T\|_{L^1(Q)\to L^2(\mathbb{R}^n)} \right),
$$

and (4.9) follows.

(v). By (i) and $t$-independence, we may replace $\nabla$ by $\nabla_s$. The desired estimate is an immediate consequence of the following pointwise bound. For convenience of notation set $K \equiv \sup_{t>0} \| (S,\nabla) \|_{L\to 2}$. Let $\tilde{f} \in C^\infty_c(\mathbb{R}^n, \mathbb{C}^n)$. We shall prove $^6$

$$
N_s((S,\nabla)\cdot \tilde{f})(x) \leq C \left( M((S)_t \cdot \nabla) \cdot \tilde{f} \right)(x) + (K + 1) (M(\|\tilde{f}\|_1^2))^{1/2}(x)
$$

To this end, we fix $(x_0, t_0) \in \mathbb{R}^{n+1}$ and suppose that $|x_0 - x| < 2t, |t_0 - s| < 2t$ and that $k \geq 4$. We claim that

$$
\int_{2t < |x-s| < 2^{k+1}t} |\nabla_s (\Gamma(x, s, y, 0) - \Gamma(x, t_0, y, 0))|^2 dy \leq C 2^{-ka} (2^k t)^{-n}.
$$

Indeed, the special case $s = t_0$ is essentially a reformulation of Lemma 2.13, but with the roles of $x$ and $y$ reversed. In general, we write

$$
\Gamma(x, s, y, 0) - \Gamma(x, t_0, y, 0) = [\Gamma(x, s, y, 0) - \Gamma(x, 0, s, y, 0)] + [\Gamma(x, 0, s, y, 0) - \Gamma(x, t_0, y, 0)].
$$

The first expression in brackets is the case $s = t_0$, while the horizontal gradient of the second equals

$$
\int_{t_0}^\infty \nabla_s \partial_t \Gamma(x_0, \tau, y, 0) d\tau.
$$

We may handle the contribution of the latter term via Lemma 2.8. This proves the claim.

We set $u(\cdot, t) \equiv (S,\nabla) \cdot \tilde{f}$, and we split $u = u_0 + \sum_{k=4}^\infty u_k \equiv u_0 + \tilde{u}$, where

$$
u_0 \equiv (S,\nabla) \cdot \tilde{f}_0, \quad u_k \equiv (S,\nabla) \cdot \tilde{f}_k, \quad \tilde{u} \equiv \sum_{k=4}^\infty u_k,
$$

and $\tilde{f}_0 \equiv \tilde{f} 1_{[|x_0| - 2^{-1}t < \cdot < 2t]}$, $\tilde{f}_k \equiv \tilde{f} 1_{R_k}$, and $R_k \equiv \{ y : 2^k t \leq |x_0 - y| < 2^{k+1}t \}$. By (4.15), for $s \in [-2t, 2t]$ and $|x_0 - x| < 2t$, we have that

$$
|u_k(x, s) - u_k(x_0, 0)| \leq C 2^{-ka/2} \left( \int_{R_k} |\tilde{f}|^2 \right)^{1/2} \leq C 2^{-ka/2} (M(\|\tilde{f}\|_1^2))^{1/2}(x_0).
$$

Summing in $k$, we obtain

$$
|\tilde{u}(x, s) - \tilde{u}(x_0, 0)| \leq C (M(\|\tilde{f}\|_1^2))^{1/2}(x_0).
$$

Moreover, since $L u_0 = 0$, by (1.3) it follows that

$$
|u_0(x, t)| \leq C \left( \int_{B(x,\tau, t/2)} |u_0|^2 \right)^{1/2} \leq C \tau^{-n/2} \sup_{\tau > 0} \| (S,\nabla) \cdot \tilde{f}_0 \|_2 \leq C K (M(\|\tilde{f}\|_1^2))^{1/2}(x_0).
$$

Taking $s = t$ in (4.16), we therefore need only establish the bound

$$
|\tilde{u}(x_0, 0)| \leq C (K + 1) (M(\|\tilde{f}\|_1^2))^{1/2}(x_0) + C M (u(\cdot, 0)) (x_0)
$$

$^6$The bound for the last term in (4.14) may be improved to $(M(\|\tilde{f}\|_1^2))^{1/4}(x)$, for some $q < 2$ depending on dimension and ellipticity, as the fourth named author will show in a forthcoming paper with M. Mitrea.
The proof of (4.17) is based on that of the well known Cotlar inequality for maximal singular integrals. Set \( \Delta_0 = \{|x - x_0| < t\} \), and let \( x \in \Delta_0 \). We write
\[
\|\tilde{u}(x_0, 0)\| \leq \|\tilde{u}(x, 0) - \tilde{u}(x_0, 0)\| + \|\tilde{u}(x_0, 0)\|
\]
\[
\leq \|\tilde{u}(x, 0) - \tilde{u}(x_0, 0)\| + |u_0(x, 0)| + |u(x, 0)|
\]
\[
\leq C \left( M(\|f\|^2) \right)^{1/2} (x_0) + |u_0(x, 0)| + |u(x, 0)|,
\]
where in the last step we have used (4.16) with \( s = 0 \). Averaging over \( \Delta_0 \), we obtain
\[
|\tilde{u}(x_0, 0)| \leq C \left( M(\|f\|^2) \right)^{1/2} (x_0) + \left( \int_{\Delta_0} |u_0(x, 0)|^2 \, dx \right)^{1/2} + M (u(_, 0)) (x_0).
\]
Since the \( L^2 \) average of \( u_0 \) is bounded by \( CK \left( M(\|f\|^2) \right)^{1/2} (x_0) \), we obtain (4.17). \( \square \)

We are now ready to discuss the jump relations and traces of the layer potentials. We recall that \( S_t^r, D_t^r \) denote the single and double layer potentials associated to \( L_t^r \).

**Lemma 4.18.** Suppose that \( L, L_t^* \) satisfy the standard assumptions, and that the single layer potentials \( S_t^r, S_t^* \) satisfy
\[
(4.19) \quad \sup_{r \in \mathbb{N}} \|\nabla S_t^{r}\|_{L^2 \rightarrow L^2} + \sup_{r \in \mathbb{N}} \|\nabla S_t^{*r}\|_{L^2 \rightarrow L^2} < \infty.
\]
Then there exist \( L^2 \) bounded operators \( K, \tilde{K}, T \) with the following properties: for all \( f \in L^2(\mathbb{R}^n) \), we have
\[
(i) \quad \left( \pm \frac{1}{2} I + \tilde{K} \right) f = \partial_r u^r \quad \text{and} \quad \tilde{N} \cdot A \nabla u^r(\cdot, t) \rightarrow \left( \pm \frac{1}{2} I + \tilde{K} \right) f \quad \text{weakly in} \ L^2.
\]
\[
(ii) \quad D_{x=0} f \rightarrow \left( \frac{1}{2} I + \tilde{K} \right) f \quad \text{weakly in} \ L^2.
\]
\[
(iii) \quad (\nabla S_t^{r})|_{t=\mp 0} f \rightarrow \left( \frac{1}{2} + \tilde{K} \right) f \quad \text{weakly in} \ L^2.
\]

**Proof.** It is enough to prove (i). Indeed, if we define
\[
K := \text{adj} (\tilde{K}^*),
\]
then (ii) follows from (i) and the observation that \( D_t = \text{adj} (\tilde{N} \cdot A^* \nabla S_t^*) \). To obtain (iii), we first use (4.19), Lemma 4.8, Lemma 4.3 and the formula
\[
(4.20) \quad -A_{n+1, r+1} \partial_r S_t = \tilde{N} \cdot A \nabla S_t + \sum_{j=1} A_{n+1, j} D_j S_t,
\]
along with part (i), to deduce that \( \nabla S_t f \) converges weakly in \( L^2 \), as \( t \to 0 \). Then (iii) follows from (4.20) and part (i), for an appropriate choice of \( T \), since \( \nabla S_t f \) does not jump across the boundary.

To prove (i), we apply Lemma 4.3 (iii) in both \( \mathbb{R}^{n+1}_+ \), to obtain \( g^r \in L^2(\mathbb{R}^n) \), with \( g^r = \partial_r u^r \) in the weak sense. We now define \( \tilde{K} \) by
\[
(4.21) \quad \left( \frac{1}{2} I + \tilde{K} \right) f := g^r, \quad \left( -\frac{1}{2} I + \tilde{K} \right) f := g^r.
\]

\( ^7 \)We are indebted to M. Mitrea for suggesting this approach.
and to show that this operator is well defined, we need only verify that $g^+-g^- = f$. It is enough to prove that

$$\int_{\mathbb{R}_+^{n+1}} A\nabla u^+ \cdot \nabla \Psi dx dt + \int_{\mathbb{R}_+^{n+1}} A\nabla u^- \cdot \nabla \Psi dx dt = \int_{\mathbb{R}^n} f \Psi dx,$$  

for all $\Psi \in C_c^\infty(\mathbb{R}^{n+1}_+)$. To this end, set $u_{\eta}^+ \equiv S_{\eta}^f f$, where $S_{\eta}^f$ is defined in (2.17), so that

$$u_{\eta}^+ = \int_{\mathbb{R}^{n+1}_+} \Gamma(x,t,y,s)f_{\eta}(y,s)dy ds, \quad t \in \mathbb{R}_+,$$

where $f_{\eta}(y,s) \equiv f(y)\varphi_{\eta}(s)$ and $\varphi_{\eta}$ is the kernel of a smooth approximate identity acting in 1 dimension. Let $U_{\eta} \equiv u_{\eta}^+ 1_{\mathbb{R}_+^{n+1}} + u_{\eta}^- 1_{\mathbb{R}_+^{n+1}}$. Since $L\Gamma = \delta$, we have that

$$\int_{\mathbb{R}_+^{n+1}} A\nabla u_{\eta}^+ \cdot \nabla \Psi + \int_{\mathbb{R}_+^{n+1}} A\nabla u_{\eta}^- \cdot \nabla \Psi = \int_{\mathbb{R}^n} A\nabla U_{\eta} \cdot \nabla \Psi = \int_{\mathbb{R}_+^{n+1}} f_{\eta} \Psi \rightarrow \int_{\mathbb{R}^n} f \Psi,$$

as $\eta \to 0$. On the other hand, fixing $\epsilon$ momentarily, we have that

$$\int_{\mathbb{R}_+^{n+1}} A\nabla(u_{\eta}^+ - u^+) \cdot \nabla \Psi = \int_{\mathbb{R}_+^{n+1}} A\nabla u_{\eta}^- \cdot \nabla \Psi + \int_{\mathbb{R}_+^{n+1}} A\nabla u_{\eta}^- \cdot \nabla \Psi = \int_{\mathbb{R}_+^{n+1}} f_{\eta} \Psi \rightarrow \int_{\mathbb{R}^n} f \Psi,$$

Fix a number $R$ greater than the diameter of $\text{supp}(\Psi)$. Then

$$|I_{\epsilon}| \leq C_{\Psi} \int_{\mathbb{R}_+^{n+1}} \sup_{\eta < \epsilon} \|\nabla(S_{\eta}^f - S_{\epsilon}^f)\|L^2(\mathbb{R}^n) \to 0,$$

as $\eta \to 0$, by Lemma 2.18. Moreover,

$$\sup_{\eta > 0} |I_{\epsilon}| \leq C_{\Psi} \epsilon \sup_{\eta > 0} \|\nabla S_{\eta}^f\|_2 \leq C_{\Psi} \epsilon \|f\|_2,$$

where we have used that $\sup_{\eta > 0} \|\nabla S_{\eta}^f\|_2 \leq \sup_{\eta > 0} \|\nabla S_{\eta} f\|_2$, by construction of $S_{\eta}^f$ (2.17). The analogous convergence result for the lower half-space concludes the proof of (4.22). Thus, $\tilde{K}$ is well defined by (4.21), and (i) now follows immediately from Lemma 4.3 (iii) and (iv).

We turn now to the issues of non-tangential and strong $L^2$ convergence for $D_t$.

**Lemma 4.23.** Suppose that $L, L^*$ satisfy the standard assumptions, that the single layer potentials $S_t, S_t^*$ satisfy (4.19), and that $S_0^0 \equiv S_0^*: L^2(\mathbb{R}^n) \to \tilde{L}^2_{\Gamma} (\mathbb{R}^n)$ is bijective. Then for every $f \in L^2(\mathbb{R}^n)$, we have the following:

$$D_{\epsilon} f \to \left( \frac{1}{2} I + K \right) f \quad \text{n.t. and in } L^2.$$

We first require a special case of the Gauss-Green formula.

**Lemma 4.24.** Let $L, L^*$ satisfy the standard assumptions, and suppose that $Lu = 0$, $L^*w = 0$ in $\mathbb{R}_+^{n+1}$ with

$$\sup_{\epsilon > 0} \|\nabla u(\cdot, t]\|_2 + \|\nabla w(\cdot, t]\|_2 < \infty,$$  

(4.25)
and $\partial_\nu \overline{w}(\cdot,0), \overline{\partial_\nu w}(\cdot,0) \in L^1(\mathbb{R}^n)^k$. Suppose also that there exist $R_0, \beta > 0$ such that for all $R > R_0$, we have

$$
(4.26) \quad \iint_{\mathbb{R}^{n+1} \cap (B(0,2R) \setminus B(0,R))} |\nabla u| |\nabla w| + |\nabla u| R^{-1} |w| + |\nabla w| R^{-1} |u| = O\left(R^{-\beta}\right).
$$

Then

$$
\int_{\mathbb{R}^n} \partial_\nu u \overline{w} = \int_{\mathbb{R}^n} u \overline{\partial_\nu w}.
$$

Of course, the analogous result holds in $\mathbb{R}^{n+1}$.

**Proof.** By the symmetry of our hypotheses, it is enough to show that

$$
(4.27) \quad \iint_{\mathbb{R}^{n+1}} A \nabla u \cdot \nabla \overline{w} = \int_{\mathbb{R}^n} \partial_\nu u \overline{w}.
$$

To this end, for $R_0 < R < \infty$, let $\Omega_R(X) \equiv \Theta(X/R)$, where $\Theta \in C_0^\infty(B(0,2))$ and $\Theta \equiv 1$ in $B(0,1)$. We set $w_R \equiv w\Theta_R$. Then by Lemma 4.3, we have that

$$
\iint_{\mathbb{R}^{n+1}} A \nabla u \cdot \nabla \overline{w_R} = \int_{\mathbb{R}^n} \partial_\nu u \overline{w_R}.
$$

A simple limiting argument completes the proof. □

**Corollary 4.28.** Let $L, L^*$ satisfy the standard assumptions, and suppose that the respective single layer potentials $S_i, S_i^*$ satisfy (4.19). Further suppose that $u(\cdot, \tau) = S_*^\psi$ in $\mathbb{R}^{n+1}$, where $\psi \in C_0^\infty(\mathbb{R}^n)$. Then setting $u_0 \equiv u(\cdot,0)$, we have

$$
(4.29) \quad \mathcal{D}_i u_0 = S_i(\partial_\nu u).
$$

**Proof.** It is enough to show that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, we have

$$
\int_{\mathbb{R}^n} \mathcal{D}_i u_0 \varphi = \int_{\mathbb{R}^n} S_i(\partial_\nu u) \varphi.
$$

Note that $\text{adj}(\mathcal{D}_i) = \overline{N} \cdot A^*(\nabla S_i^*)_{|\tau=1}$, and that $\text{adj}(S_i) = S_i^* \varphi_i$. Set $u^*(\cdot, \tau) \equiv S_i^* \varphi_i$, so that $L^* u^* = 0$ in $\mathbb{R}^{n+1} \setminus \{\tau = 0\}$. It suffices to verify the hypotheses of Lemma 4.24, in the lower half-space, for $u, w$, with $w(\cdot, s) \equiv u^*(\cdot, s-t)$, $s \leq 0$. Estimate (4.25) is immediate by (4.19). By Lemma 2.5, we have

$$
(4.30) \quad |u(X)| + |w(X)| = O(|X|^{-n+1}) \quad \text{as } |X| \to \infty.
$$

Also, $Lu = 0, L^* w = 0$ in $\mathbb{R}^{n+1} \setminus B(0,R_0)$, if $R_0$ is chosen large enough, since $\varphi, \psi$ have compact support. Thus, by Caccioppoli,

$$
\iint_{\mathbb{R}^{n+1} \cap (B(0,2R) \setminus B(0,R))} |\nabla u|^2 \leq C \iint_{\mathbb{R}^{n+1} \cap (B(0,3R) \setminus B(0,2R))} \left(\frac{|u|^2}{R}\right)^2 = O\left(R^{-n+1}\right),
$$

for $R > 4R_0$, and similarly for $w$. Estimate (4.26) follows. Finally, the boundary integrability of $\partial_\nu u \overline{w}$ and $\overline{\partial_\nu w} u$ follows readily from Cauchy-Schwarz, the fact that $n \geq 2$, and two observations: first, that by Lemma 2.9 and duality, we have

$$
\int_{\lambda_\tau(0), \lambda_{\nu}(0)} |\partial_\nu u|^2 + |\overline{\partial_\nu w}|^2 = O(R^n);
$$

\footnote{Here, $\partial_\nu$ and $\overline{\partial_\nu}$ are the exterior conormal derivatives, corresponding to the matrices $A$ and $A^*$ respectively, which exist in the weak sense of Lemma 4.3.}
Indeed, by hypothesis and duality, $(4.30)$ implies that
\[ \int_{\Delta \tau(0) \Delta \theta(0)} |u|^2 + |w|^2 = O(R^{2-n}). \]

We leave the remaining details to the reader. \(\Box\)

Proof of Lemma 4.23. Since we have already obtained the limits \((\mp \frac{1}{2} I + K)f\) in the weak sense (Lemma 4.18), it is enough here merely to establish existence of n.t. and strong \(L^2\) limits, without concern for their precise values. We give the proof only in the case of the upper half-space, as the proof in the other case is the same.

We begin with the matter of non-tangential convergence. Observe that \(adj(S, \nabla) = (\nabla S^* )|_{r \to -}\), so by (4.19) and Lemma 4.8(vi), it is enough to establish n.t. convergence for \(f\) in a dense class in \(L^2\). We claim now that \(\{S_0 \text{ div}_k \vec{g} : \vec{g} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n)\}\) is dense in \(L^2\).

Indeed, by hypothesis and duality, \(S_0 : \dot{L}^2_{-1} \to L^2\) is bijective. Thus, \(L^2 = \{S_0 \text{ div}_k \vec{g} : \vec{g} \in L^2\}\). The density of \(C_0^\infty\) in \(L^2\) establishes the claim.

We now set \(f = u_0 = S_0(\text{div}_k \vec{g})\), with \(\vec{g} \in C_0^\infty\), and let \(u(\cdot, \tau) = S_\tau(\text{div}_k \vec{g}), \tau < 0\). We may then apply Corollary 4.28 to obtain that \(\mathcal{D}_t f = S_\tau(\partial_\nu u)\). Moreover, (4.19), Lemma 4.8, and Lemma 4.3 imply that \(\partial_\nu u \in L^2\), and hence also that \(S_\tau(\partial_\nu u)\) converges n.t., from which fact the non-tangential part of (ii) now follows.

We turn now to the issue of strong convergence in \(L^2\). By (4.19), we have in particular that \(L^2\) bounds hold, uniformly in \(t > 0\), for \(\mathcal{D}_t\). Thus, it is once again enough to establish convergence in a dense class. To this end, choose \(u_0, u\) as above. It suffices to show that \(\mathcal{D}_t u_0\) is Cauchy convergent in \(L^2\), as \(t \to 0\). Suppose that \(0 < t' < t \to 0\), and observe that, by Corollary 4.28, (4.19) and our previous observation that \(\partial_\nu u \in L^2\),

\[ \|\mathcal{D}_t u_0 - \mathcal{D}_{t'} u_0\|_2 = \| \int_{t'}^t \partial_\nu S_\tau(\partial_\nu u) \, ds \|_2 \leq (t - t')\|\partial_\nu S_\tau(\partial_\nu u)\|_2 \to 0. \]

\(\Box\)

We now turn to the matter of uniqueness.

Lemma 4.31. (Uniqueness). Suppose that \(L, L^*\) satisfy the standard assumptions, and that we have existence of solutions to (D2) and (R2). Then those solutions are unique, in the following sense:

(i) If \(u\) solves (D2), and if \(u(\cdot, t) \to 0\) in \(L^2\), as \(t \to 0\), then \(u \equiv 0\).

(ii) If \(u\) solves (R2), and \(u \to 0\) n.t., then \(u \equiv 0\).\(^9\)

If, in addition, \(L\) and \(L^*\) have “Good Layer Potentials”, then the solution to (N2) is unique, in the sense that:

(iii) If \(u\) solves (N2), with \(\partial u/\partial \nu = 0\) in the sense of Lemma 4.3 (iii) and (iv), then \(u \equiv 0\) (modulo constants).

Proof. Consider first uniqueness in (D2). We begin by constructing Green’s function. By Lemma 2.8 with \(m = -1\), for each fixed \((x, t) \in \mathbb{R}^{n+1}\), we have \(\Gamma(x, t, \cdot, 0) \in L^2_{t^1}\), with

\[ (4.32) \|\nabla ||\Gamma(x, t, \cdot, 0)||_{L^2(\mathbb{R}^n)} \leq C t^{-n/2}. \]

\(^9\)Our data in the problem (R2) belongs to \(L^2_{t^1}\), whose elements are defined modulo constants; thus, uniqueness in this context must be interpreted correspondingly. We assume here that we have chosen a particular realization of the data equal to 0 a.e. on the boundary.
Thus, by (R2), there exists \( w = w_{x,t} \) solving

\[
\begin{aligned}
Lw = 0 \text{ in } \mathbb{R}_t^{n+1} \\
\|N_s(\nabla w)\|_{L^2(\Sigma')} \leq Cr^{-n/2}.
\end{aligned}
\]

(R2)

Set

\[ G(x, t, y, s) = \Gamma(x, t, \cdot, 0) n.t. \quad (\text{4.35}) \]

and note that

\[ G(x, t, y, s) - w_{x,t}(y, s), \]

and that

\[ \sup_{x \in \mathbb{R}_t^{n+1}} \|\nabla G(x, t, \cdot, s)\|_{L^2(\Sigma')} \leq Cr^{-n/2}. \]

Let \( \theta \in C^0_0(\mathbb{R}_t^{n+1}), \) with \( \theta \equiv 1 \) in a neighborhood of \((x, t)\). Then, since \( Lu = 0 \), we have

\[
\begin{aligned}
\phi = \phi & = \int A^\theta \nabla y, G(x, t, t, s) \cdot \nabla (u \theta) dy ds \\
& = \int \nabla G \cdot A \nabla y, u + \int \nabla G \cdot A \nabla \theta u \equiv I + II.
\end{aligned}
\]

We now choose \( \phi \in C^0_0(-2, 2), \) \( \phi \equiv 1 \) in \((-1, 1), \) with \( 0 \leq \phi \leq 1, \) and set \( \theta(y, s) \equiv [1 - \phi(s/\varepsilon)] \phi(s/100R) \phi(|x - y|/R), \) with \( \varepsilon < t/8, R > 8r. \) With this choice of \( \theta, \) the domains of integration in \( I \) and \( II \) are contained in a union \( \Omega_1 \cup \Omega_2 \cup \Omega_3, \) where

1. \( \Omega_1 \subset \Delta_{2R}(x) \times \{ \varepsilon < s < 2\varepsilon \}, \) with \( \|\nabla \theta\|_{L^2(\Omega_1)} \leq C\varepsilon^{-1}. \)
2. \( \Omega_2 \subset \Delta_{2R}(x) \times \{100R < s < 200R\}, \) with \( \|\nabla \theta\|_{L^2(\Omega_2)} \leq CR^{-1}. \)
3. \( \Omega_3 \subset (\Delta_{2R}(x) \setminus \Delta_R(x)) \times \{0 < s < 200R\}, \) with \( \|\nabla \theta\|_{L^2(\Omega_3)} \leq CR^{-1}. \)

We first consider term \( I. \) Having fixed \((x, t), \) we set \( \Omega_i := \mathbb{R}^{n+1}_t \cap \{\delta (\cdot) \geq t/4\}, \) where \( \delta (\cdot) \) denotes the distance to the boundary of the half-space (i.e., the \( s \)-coordinate). We have that

\[
|I| \leq \frac{1}{R} \int_{\Omega_1} |G| \nabla |u| + \frac{1}{R} \int_{\Omega_2 \cup \Omega_3} |G| \nabla |u| =: I_1 + I_2,
\]

and we further split \( I_2 = R^{-1} \int_{\Omega_2 \cup \Omega_3} |G| \nabla |u| =: I'_2, \)

To treat \( I_1, \) we first note that for \( s \leq t/2, \)

\[
|G(x, t, y, s)| \leq C s \left( (|x - y| + t)^{-n} + \tilde{N}_s(\nabla w_{x,t})(y) \right),
\]

by Lemma 2.5, Lemma 4.3 and construction of \( G. \) Consequently, for all \( a \in (0, t/2], \) we have

\[
(\int_{|x|}^\infty |G(x, t, y, s)|^2 dy ds)^{1/2} \leq Ca^{3/2} r^{-n/2}.
\]

Similarly, we note for future reference that

\[
(\int_0^\infty \int_{|x|}^\infty |G(x, t, y, s)|^2 dy ds^{1/2} \leq Ca r^{-n/2}.
\]

Covering \( \Delta_{2R}(x) \) by balls of radius \( \varepsilon \) and using Caccioppoli’s inequality, we have that

\[
\varepsilon^{-1} \left( \int_{\Omega_1} |\nabla u(y, s)|^2 dy ds \right)^{1/2} \leq \varepsilon^{-3/2} \sup_{s \leq 2\varepsilon} |u(\cdot, s)|_{L^2}.
\]

Combining the latter bound with the case \( a = 2\varepsilon \) of (4.36), we deduce that \( I_1 \to 0 \) as \( \varepsilon \to 0, \) since \( u(\cdot, s) \to 0 \) in \( L^2. \)

We now turn to the term \( I_2; \) We recall from [HK2] that

\[
|G(X, Y)| \leq C a (X - Y)^n Y^{-\alpha(1-n)/2} Y^{(1-n)/2},
\]
for all \( X, Y \in \mathbb{R}^{n+1} \), where \( \alpha \) is the De Giorgi/Nash exponent. Therefore,

\[
(4.40) \quad \frac{1}{R} \left\{ \int_{\Omega \cap \Omega_0} \left| \nabla G(y) \right|^2 \frac{dy}{s} \right\}^{1/2} \leq \rho R^{1-\alpha+(1-n)/2} \left( \int_{\Omega_0} \nabla u(x) \cdot \nabla u(y) d\mu(y) \right)^{1/2} \leq \rho R^{1-\alpha+(1-n)/2} R^{-\alpha-1/2}.
\]

Moreover, solvability of (D2) entails that

\[
(4.41) \quad \left( \int_{\mathbb{R}^2} \left| \nabla u(y, s) \right|^2 dy ds \right)^{1/2} < \infty.
\]

Together, (4.40) and (4.41) imply that

\[
I'_R \leq (t/R)^{1/2+\alpha} R^{-n/2} \to 0, \quad \text{as } R \to \infty.
\]

Next, setting \( a = t/4 \) in (4.37), and combining the latter bound with (4.41), we find that

\[
I''_R \leq R^{-1} t^{1-n/2} \to 0, \quad \text{as } R \to \infty.
\]

We now consider term \( II \). By Cauchy-Schwarz and then Caccioppoli’s inequality,

\[
|II| \leq \epsilon^{-1} \int_{\Omega_i} \left| \nabla G \right| |u| + R^{-1} \int_{\Omega_0} \left| \nabla G \right| |u| =: III + II_2
\]

\[
\leq C \epsilon^{-1/2} \left| G(x, t, \cdot, \cdot) \right|_{L^2(\Omega_i)} \sup_{s < \epsilon} \left| u(\cdot, s) \right|_2
\]

\[
+ R^{-3/2} \left| G(x, t, \cdot, \cdot) \right|_{L^2(\Omega_0; \Omega_i)} \sup_{s > 0} \left| u(\cdot, s) \right|_2,
\]

where \( \Omega_i \) is a slightly fattened version of \( \Omega_i \) for each \( i = 1, 2, 3 \), as per the use of Caccioppoli’s inequality. By the case \( a = \epsilon \) of (4.36), term \( III \) \( \to 0 \) since \( u(\cdot, s) \to 0 \) in \( L^2 \). By further splitting the domain of integration in \( ||G||_{L^2(\Omega_i; \Omega_i)} \) according to whether \( s \geq t/4 \) or \( s < t/4 \), and then using either (4.39) or (4.36), we obtain that

\[
(4.43) \quad \left| G(x, t, \cdot, \cdot) \right|_{L^2(\Omega_i; \Omega_i)} \leq \epsilon^2 R^{-\alpha+(1-n)/2} \left( \int_{\Omega_0} \left| \nabla u(x) \right|^2 \right)^{1/2} + \epsilon^{3/2-n/2}
\]

\[
\leq \epsilon^{a+(2-n)/2} R^{-\alpha+1/2} + \epsilon^{3/2-n/2}
\]

(when \( n = 2 \), we multiply by \( \log(R/t) \)). In any case, the factor \( R^{-3/2} \) in (4.42) ensures convergence to 0 as \( R \to \infty \). The proof of uniqueness in (D2) is now complete.

**Uniqueness in (R2).** Suppose now that \( \tilde{N}_i(\nabla u) \in L^2 \); and that \( u \to 0 \) n.t.. Choosing \( \theta \) as above, we split \( u(x, t) = (u\theta)(x, t) \) into the same terms \( I + II \), which we dominate again by \( I_1 + I_2 \) and \( II_1 + II_2 \) as in (4.34) and (4.42), respectively. We now claim that

\[
I_1 + II_1 \leq C \epsilon^{-n/2} \left| \tilde{N}_i(\nabla u) \right|_2 \to 0
\]

as \( \epsilon \to 0 \). For \( I_1 \), this follows from (4.36), with \( a = 2\epsilon \). To handle \( II_1 \), we first note that, by Lemma 4.3(i), \( |u(y, s)| \leq C \epsilon \tilde{N}_i(\nabla u)(y) \) in \( \Omega_i \), since \( u(\cdot, 0) = 0 \) a.e.. The claim then follows from Cauchy-Schwarz and Caccioppoli (applied to \( \nabla G \)), and again (4.36).

Next, using (4.43), we see that (modulo a harmless factor of \( \log(R/t) \) when \( n = 2 \))

\[
I_2 \leq \left( R^{a+(2-n)/2} R^{-\alpha+1/2} + \epsilon^{3/2-n/2} R^{-1} \right) \left( \int_{\Omega_2 \cup \Omega_3} \left| \nabla u \right|^2 \right)^{1/2}
\]

\[
\leq \left( R^{a+(2-n)/2} R^{-\alpha+1/2} + \epsilon^{3/2-n/2} R^{-1} \right) \left| \tilde{N}_i(\nabla u) \right|_{L^2(\mathbb{R})},
\]

which converges to 0 as \( R \to \infty \), since \( \alpha > 0 \).
It remains to treat term $I_2$. We note that $|u(y, s)|/s \leq C\tilde{N}_e(\nabla u)(y)$, by Lemma 4.3(i).

Thus,

$$ I_2 \lesssim R^{-\varepsilon_{\alpha}} \left( \int_{\Omega \setminus \Omega_{2t}} \|\xi_{x, \alpha} G(x, t, y, s)^2 dyds \right)^{1/2} \|\tilde{N}_e(\nabla u)\|_{L^2(\mathbb{R}^n)} $$

$$ \leq \left( t^{n/2} + R^{-\varepsilon_{\alpha}} \right) \left( \|\tilde{N}_e(\nabla u)\|_{L^2(\mathbb{R}^n)} \right), $$

where in the last step we have used Caccioppoli’s inequality in Whitney boxes and (4.43), and where again there is a factor of $\log(R/t)$ when $n = 2$. Since $\varepsilon_{\alpha} > 0$, we obtain convergence to 0.

**Uniqueness in (N2).** Suppose that $\tilde{N}_e(\nabla u) \in L^2$, and that $\partial u/\partial \nu = 0$, where the latter is interpreted in the sense of Lemma 4.3(iii) and (iv). By Lemma 4.3(i), we have that $u \to u_0$ n.t., for some $u_0 \in L^2_1(\mathbb{R}^n)$. By uniqueness in (R2),

$$ u(\cdot, t) = S_t(S_0^{-1}u_0), $$

where $S_0 \equiv S_{|t|=0}$. Thus, by Lemma 4.18,

$$ 0 = \frac{\partial u}{\partial \nu} = \left( \frac{1}{2} I + \tilde{K} \right)(S_0^{-1}u_0). $$

But by hypothesis, $\frac{1}{2} I + \tilde{K} : L^2 \to L^2$ and $S_0 : L^2 \to L^2_1$ are bijective, so that $u_0 = 0$ in the sense of $L^2_1$, i.e., $u_0 \equiv \text{constant a.e.}$. By uniqueness in (R2), $u \equiv \text{constant}$. 

\[ \square \]

As a corollary of uniqueness, we shall obtain the following “Fatou Theorem”.

**Corollary 4.44.** Let $L$, $L'$ satisfy the standard assumptions, and have “Good Layer Potentials”. Suppose also that $Lu = 0$, and that

$$ \sup_{t \to 0} \|u(\cdot, t)\|_2 < \infty. $$

Then $u(\cdot, t)$ converges n.t. and in $L^2$ as $t \to 0$.

**Proof.** By Lemma 4.23, it is enough to show that $u(\cdot, t) = \mathcal{D}_t h$ for some $h \in L^2(\mathbb{R}^n)$. We follow the argument in [St2], pp. 199-200, substituting $\mathcal{D}_t$ for the classical Poisson kernel. For each $\varepsilon > 0$, set $f_\varepsilon \equiv u(\cdot, \varepsilon)$. Let $u_\varepsilon$ be the layer potential solution with data $f_\varepsilon$; i.e.,

$$ u_\varepsilon(x, t) \equiv \mathcal{D}_t \left[ \left( -\frac{1}{2} I + K \right)^{-1} f_\varepsilon \right](x). $$

We claim that $u_\varepsilon(x, t) = u(x, t + \varepsilon)$.

**Proof of Claim.** Set $U_\varepsilon \equiv u(x, t + \varepsilon) - u_\varepsilon(x, t)$. We observe that

1. $LU_\varepsilon = 0$ in $\mathbb{R}^{n+1}_+$ (by $t$-independence of coefficients).
2. (4.45) holds for $U_\varepsilon$, uniformly in $\varepsilon > 0$.
3. $U_\varepsilon(\cdot, 0) = 0$ and $U_\varepsilon(\cdot, t) \to 0$ n.t. and in $L^2$.

(Item (3) relies on interior continuity (1.2) and smoothness in $t$, along with Lemma 4.23). The claim now follows by Lemma 4.31. \[ \square \]
We return now to the proof of the Corollary. By (4.45), $\sup_k \|f_k\|_2 < \infty$. Hence, there exists a subsequence $f_{k_\ell}$ converging in the weak* topology to some $f \in L^2$. For arbitrary $g \in L^2$, set $g_1 = \text{adj} \left( \frac{1}{2} I + K \right)^{-1} \text{adj} (D_t) g$, and observe that

$$
\int_{\mathbb{R}^n} \left[ D_t \left( \frac{1}{2} I + K \right)^{-1} f \right] g = \int_{\mathbb{R}^n} f g_1 = \lim_{k \to \infty} \int_{\mathbb{R}^n} f_{k_\ell} g_1
$$

$$
= \lim_{k \to \infty} \int_{\mathbb{R}^n} \left[ D_t \left( \frac{1}{2} I + K \right)^{-1} f_{k_\ell} \right] g
$$

$$
= \lim_{k \to \infty} \int_{\mathbb{R}^n} \left[ u(\cdot, t + \epsilon_k) \right] g = \int_{\mathbb{R}^n} u(\cdot, t) g.
$$

Since $g$ was arbitrary, the desired conclusion follows. \hfill \Box

We conclude this section with a discussion of $n.t.$ convergence of gradients.

**Lemma 4.46.** Suppose that $L, L^*$ satisfy the standard assumptions, and have “Good Layer Potentials”. Then for all $f \in L^2$, we have

$$P_x ((\nabla S_t)_{|z=0}) f \to \left( \frac{1}{2A_{n+1,n+1}} e_{n+1} + T \right) f \text{ n.t. and in } L^2.$$

**Proof.** We treat only the case of the upper half space, as the proof in the other case is the same. Since the weak limit has already been established (Lemma 4.18) for $\nabla S_t$, it is a routine matter to verify that the strong and $n.t.$ limits for $P_t(\nabla S_t)$ will take the same value, once the existence of those limits has been established. It is to this last point that we therefore turn our attention. By Lemma 4.8 and the dominated convergence theorem, it is enough to establish $n.t.$ convergence.

The non-tangential convergence of $\partial_t S_t$ follows immediately from the “Fatou Theorem” just proved; a simple real variable argument yields the same conclusion for $P_t \partial_t S_t$. We may therefore replace $\nabla$ by $\nabla |_t$. On the other hand, we shall still need to consider the boundary trace of $\partial_t S_t f$, which for the duration of this proof we denote by $V f$. Fix now $x_0 \in \mathbb{R}^n$. For $|x - x_0| < t$, we write

$$P_t (\nabla S_t f)(x) = \nabla_x P_t \left( \int_0^t \partial_t S_t f ds \right)(x) + P_t (\nabla_t S_0 f)(x)$$

$$= \hat{Q}_t \left( \frac{1}{t} \int_0^t \partial_t S_t f ds \right)(x) + P_t (\nabla_t S_0 f)(x) \equiv I + II,$$

where $\hat{Q}_t 1 = 0$. By standard facts for approximate identities, $I \to \nabla |_t S_0 f$ $n.t.$ Also,

$$I = \hat{Q}_t \left( \frac{1}{t} \int_0^t (\partial_t S_t f - V f) ds \right)(x) + \hat{Q}_t (V f - V f(x_0))(x) \equiv I_1 + I_2.$$

It is straightforward to verify that $I_2 \to 0$ as $t \to 0$, if $x_0$ is a Lebesgue point for the $L^2$ function $V f$. The term $I_1$ is more problematic. We first observe that by Lemma 4.3,

$$\left| \hat{Q}_t \left( \frac{1}{t} \int_0^t (S_t f - S_0 f) ds \right)(x) \right| \leq C t \text{M}(\nabla, (\nabla_S f)) (x_0) \to 0$$

for $a.e. \ x_0$. Thus also for $\tilde{f} \in C_0^\infty (\mathbb{R}^n)$, we have

$$\left| \hat{Q}_t \left( \frac{1}{t} \int_0^t (S_t \nabla \cdot \tilde{f} - S_0 \nabla \cdot \tilde{f}) ds \right)(x) \right| \to 0 \text{ n.t.}.$$
By Lemma 4.8(i), the density of \(C^0_0\) in \(L^2\), and the fact that \(\tilde{Q}_t\) is dominated by the Hardy-Littlewood maximal operator which is bounded from \(L^{2,\infty}\) to itself, the latter convergence continues to hold for \(f^t \in L^2\). Moreover, if \(u_0\) belongs to the dense class \(\{S_0 \operatorname{div} \tilde{g} : \tilde{g} \in C^0_0\}\), by Corollary 4.28 and (4.47), we have that

\[
\left| \tilde{Q}_t \left( \frac{1}{t} \int_0^t (D_i u_0 - \text{tr}(D_i u_0)) \, ds \right)(x) \right| \to 0 \text{ n.t.,}
\]

and again this fact remains true for \(u_0\) in \(L^2\), by Lemma 4.8(ii) and our previous observation concerning the action of the maximal operator on weak \(L^2\). Combining (4.48) and (4.49) with the adjoint version of the identity (4.20), we obtain convergence to 0 for the term \(I_1\) since every \(f \in L^2\) can be written in the form \(f = A^*_n \rho_k h, h \in L^2\). \(\square\)

5. Proof of Theorem 1.12: Preliminary Arguments

As noted above, the De Giorgi-Nash estimate (1.2) is stable under \(L^\infty\) perturbation of the coefficients. Thus, for \(\varepsilon_0\) sufficiently small, solutions of \(L_1 u = 0, L_1^* w = 0\) satisfy (1.2) and (1.3). In particular, the results of Section 2 apply to the fundamental solutions and layer potentials \(\Gamma_0, S^0_1, \Gamma_1, S^1_1\) corresponding to \(L_0\) and \(L_1\), respectively.

We claim that the conclusion of Theorem 1.12 will follow, once we have proved

\[
|||\tau \nabla \partial_t S^1_1|||_{op} + \sup_{\tau > 0} |||\nabla S^1_1|||_{L^2} \leq C
\]

(recall that \(\tau \equiv \nabla \cdot D\)). Indeed, by the symmetry of our hypotheses, similar bounds will then hold in the lower half space, and for \(S^L_1\). Moreover, if \(J_f(x, y)\) denotes the kernel of \((S^1_1 \nabla \tilde{g})\), and \(\Gamma_1\) is the fundamental solution for the adjoint operator \(L_1^*\), then the kernel of \(\text{adj}(S^1_1 \nabla \tilde{g})\) is

\[
\overline{J_f}(y, x) = \nabla \cdot \Gamma_1 (0, y, x) = \nabla \cdot \Gamma_1 (x, 0, y, t) = \nabla \cdot \Gamma_1 (x, -t, y, 0).
\]

Consequently, \(\text{adj}(S^1_1 \nabla \tilde{g}) = \nabla \cdot S^L_1\), so that \(L^2\) boundedness of \((S^1_1 \nabla)\) (and hence of \(D^L_1\)) follows from that of \(\nabla S^L_1\). Thus, by Lemma 4.18, we also obtain \(L^2\) bounds for \(K^1, \overline{K}^1\) and \(T^1\). Appropriate non-tangential control follows from Lemma 4.8. Moreover, since we have allowed complex coefficients, analytic perturbation theory implies that

\[
||K^0 - K^1||_{L^2} + ||\overline{K}^0 - \overline{K}^1||_{L^2} + ||T^0 - T^1||_{L^2} \leq C||A^0 - A^1||_\infty.
\]

The method of continuity then yields the invertibility of \(\pm \frac{1}{2} I + K^1 : L^2 \to L^2, \pm \frac{1}{2} I + \overline{K}^1 : L^2 \to L^2\), and \(S^1_1, S^1_1, \rho_0 : L^2 \to L^2\). It therefore suffices to prove (5.1).

**Lemma 5.2.** Suppose that \(L, L^*\) satisfy the standard assumptions. For \(f \in C^\infty_0, \eta > 0, 0, and t_0 \geq 0, we have

\[
\begin{align*}
||\nabla S^k_{\eta/2} f||_{L^2} & \leq C(||\nabla N_{(P, \partial_t S_{t_0 + \eta} f)}||_{L^2} + ||\nabla \partial_t S_{\eta/2} f|| + ||f||_{L^2}) \\
||\nabla S^\eta_{\eta/2} f||_{L^2} & \leq C(||\nabla N_{(P, \partial_t S_{s_0} f)}||_{L^2} + ||\nabla \partial_t S_{\eta/2} f|| + ||f||_{L^2}) \\
||\nabla \partial_t S_{\eta/2} f|| & \leq C(||\partial_t S_{\eta/2} f|| + ||f||_{L^2}) \\
||\nabla \partial_t S_{\eta/2} f|| & \leq C(||\partial_t S_{\eta/2} f|| + ||f||_{L^2})
\end{align*}
\]

The analogous bounds hold also in the lower half-space.
Before proving the lemma, let us use it to reduce the proof of Theorem 1.12 to two main estimates, whose proofs we shall give in the next two sections. We claim that it suffices to prove that for all $f \in C_0^\infty$ and $\eta \in (0, 10^{-10})$, we have

$$
\|\partial_t^2 S_{t}^{1,\eta} f\|_{C^0} \leq C_{f_0} \left( \|\eta \nabla \partial_t S_{t}^{1,\eta} f\|_{C^0} + \|N^\partial_{t} \partial_t S_{t}^{1,\eta} f\|_{L^2} + \sup_{r \in [0,1]} \|\nabla S_{t}^{1,\eta} f\|_{L^2} \right) + C \|f\|_{L^2},
$$

(5.7)

$$
\sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{C^0} \leq C_{f_0} \left( \|\eta \nabla \partial_t S_{t}^{1,\eta} f\|_{C^0} + \|N^\partial_{t} \partial_t S_{t}^{1,\eta} f\|_{L^2} + \sup_{r \in [0,1]} \|\nabla S_{t}^{1,\eta} f\|_{L^2} \right) + C \|f\|_{L^2},
$$

(5.8)

where $N^\partial_{t}$ denotes the non-tangential maximal operator with respect to the double cone $\gamma^+(x) \cup \gamma^-(x) \equiv \{(y, t) \in \mathbb{R}^{n+1} : |x - y| < |t|\}$. Indeed, for $f_0$ sufficiently small, Lemma 2.18 (iii) and (5.6) allow us to hide the small triple bar norm in (5.7), so that

$$
\|\eta \nabla \partial_t S_{t}^{1,\eta} f\|_{C^0} \leq C_{f_0} \left( \|N^\partial_{t} \partial_t S_{t}^{1,\eta} f\|_{L^2} + \sup_{r \in [0,1]} \|\nabla S_{t}^{1,\eta} f\|_{L^2} \right) + C \|f\|_{L^2}.
$$

(5.9)

Using (5.4), (5.9) and hiding the small gradient term via Lemma 2.18 (i, ii), we obtain

$$
\sup_{r \in [0,1]} \|\nabla S_{t}^{1,\eta} f\|_{C^0} \leq C \left( \sup_{t \geq 0} \|N^\partial_{t} \partial_t S_{t}^{1,\eta} f\|_{L^2} + \|\partial_t S_{t}^{1,\eta} f\|_{L^2} \right).
$$

(5.10)

where the notation $N^\partial_{t} \partial_t S_{t}^{1,\eta} f$ is interpreted to mean $t + t_0$ in the upper cone $\gamma^+$, and $t - t_0$ in the lower cone $\gamma^-$. Feeding the latter estimate back into (5.9), we obtain

$$
\|\eta \nabla \partial_t S_{t}^{1,\eta} f\|_{C^0} \leq C_{f_0} \left( \sup_{t \geq 0} \|N^\partial_{t} \partial_t S_{t}^{1,\eta} f\|_{L^2} + \sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{L^2} \right) + C \|f\|_{L^2}.
$$

(5.11)

Combining (5.8), (5.10) and (5.11), we have

$$
\sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{L^2} \leq C \|f\|_{L^2} + C_{f_0} \left( \sup_{t \geq 0} \|N^\partial_{t} \partial_t S_{t}^{1,\eta} f\|_{L^2} + \sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{L^2} \right).
$$

Since $f \in C_0^\infty$, there is a large cube $Q$ centered at 0 containing the support of $f$. By Lemma 4.8 (iv), taking a supremum over all $f \in C_0^\infty(Q)$, with $\|f\|_{L^2(Q)} = 1$, we have

$$
\sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{L^2(Q) \to L^2(\mathbb{R}^n)} \leq C \left( 1 + \epsilon_0 \sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{L^2(Q) \to L^2(\mathbb{R}^n)} \right).
$$

Using Lemma 2.18 (vi), we may hide the small term to obtain

$$
\sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{L^2(Q) \to L^2(\mathbb{R}^n)} \leq C
$$

(5.12)

uniformly in $Q$. Thus, letting $\ell(Q) \to \infty$, and then $\eta \to 0$, we obtain by Lemma 2.18 (iv) that

$$
\sup_{r \in [0,1]} \|\partial_t S_{t}^{1,\eta} f\|_{L^2} \leq C.
$$

(5.13)

In addition, (5.12), Lemma 4.8 (iv) and a limiting argument as $\ell(Q) \to \infty$ imply that

$$
\sup_{t \geq 0} \|N^\partial_{t} \partial_t S_{t}^{1,\eta} f\|_{L^2} \leq C \|f\|_{L^2}, \quad f \in L^2(\mathbb{R}^n).
$$

The latter estimate, (5.11), (5.12) and Lemma 2.18 (v) yield the bound for the first term in (5.1). The bound for the second term in (5.1) follows from (5.3), the bound just established for $\|\nabla S_{t}^{1,\eta} f\|_{C^0}$, the fact that $N_{\gamma}(\partial_t S_{t}^{1,\eta} f) \leq CM (N_{\gamma}(\partial_t S_{t}^{1,\eta} f))$, Lemma 4.8 (i) and (5.13).

The estimates (5.7) and (5.8) are the heart of the matter, and will be proved in sections 6 and 7, respectively.
We return now to the proof of the lemma.

**Proof of Lemma 5.2.** We prove (5.5) first. We have that \( \|r \nabla \partial_t f\|^2 = \)

\[
\lim_{\varepsilon \to 0} \|r \nabla \partial_t S, f\|^2(\varepsilon) \equiv \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_0^{1/\varepsilon} \nabla \partial_t S, f \cdot \nabla \partial_t S, f \, dt \]

\[
= -\frac{1}{2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_0^{1/\varepsilon} \partial_t(\nabla \partial_t S, f \cdot \nabla \partial_t S, f) \, t^2 \, dt + \text{"OK"},
\]

where we may use Lemma 2.10(ii) to dominate the "OK" boundary terms by \( C\|f\|^2 \). By Cauchy's inequality, we then obtain that

\[
\|r \nabla \partial_t S, f\|^2(\varepsilon) \leq \delta \|r \nabla \partial_t S, f\|^2(\varepsilon) + \frac{C}{\delta} \|r^2 \nabla \partial_t S, f\|^2(\varepsilon) + C\|f\|^2.
\]

where \( \delta \) is at our disposal. For \( \delta \) small, we can hide the first term. The second term is bounded by \( \|r \nabla \partial_t S, f\| \), as may be seen by splitting \( \mathbb{R}^{n+1} \) into Whitney boxes, and applying Caccioppoli’s inequality. The bound (5.5) now follows.

The proof of (5.6) is similar. We write

\[
\|r \nabla \partial_t S, f\|^2 = \int_0^{2\eta} + \int_{2\eta}^{\infty} \int_{\mathbb{R}^d} \equiv I + II.
\]

Term II may be handled just like (5.5), since by definition (2.17),

\[
|r \nabla \partial_t S, f| \leq C \left( \varphi_{\eta} + \left( 1_{\eta \neq 0} |s \nabla \partial_t S, f| \right) \right) (t), \quad t > 2\eta,
\]

and \( u(x, t) \equiv \partial_t S, f(x) \) solves \( Lu = 0 \) in the half space \( \{ t > \eta \} \). We omit the details. To bound term I, we note that by definition (2.17), \( \partial_t S, f(x) = L^{-1}(D_{n+1} f)(x, t) \), where \( f_{\eta}(y, s) \equiv f(y) \varphi_{\eta}(s) \), so that

\[
|I| \leq C \eta \int \int_{\mathbb{R}^{n+1}} |\nabla L^{-1}(D_{n+1} f)|^2 \, dx \, dt \leq C \eta \left( \int |\varphi_{\eta}(t)|^2 \, dt \right) \|f\|^2 = C\|f\|^2.
\]

where we have used that \( \nabla L^{-1} : L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1}) \).

Next, we prove (5.3). By the ellipticity of the sub-matrix \( A_{\eta} \), we have that

\[
\|\nabla \partial S, f\|_2 \leq C\|A_{\eta} \nabla \partial S, f\|_2.
\]

Now let \( \bar{g} \in C^{\infty}(\mathbb{R}^n, C^n) \), with \( \|\bar{g}\|_2 = 1 \). By the Hodge decomposition [AT, p. 116], we have that \( \bar{g} = \nabla F + \bar{h} \), where \( F \in L^2(\mathbb{R}^n), \|\nabla F\|_2 \leq C\|\bar{g}\|_2 \) (depending only on ellipticity), \( h \in L^2(\mathbb{R}^n) \) and \( \text{div}_0(A_{\eta} \bar{h} + \bar{h}) = 0 \) in the sense that \( \int A_{\eta} \nabla \xi \cdot \bar{h} = 0 \) for all \( \xi \in L^2(\mathbb{R}^n) \). Lemma 2.9, with \( m = -1 \), ensures that \( S_{n+1} f \in L^2(\mathbb{R}^n) \), (albeit without quantitative bounds). Thus, for \( f \in C^{\infty}(\mathbb{R}^n) \), we have

\[
\langle A_{\eta} \nabla \partial S, \bar{g} \rangle = \langle A_{\eta} \nabla \partial S, f, \nabla F \rangle,
\]

and it suffices to bound the latter expression with \( F \in C^{\infty}_0 \). Now,

\[
\langle A_{\eta} \nabla \partial S, f, \nabla F \rangle = - \int_0^\infty \partial_t \langle A_{\eta} \nabla \partial e^{-t L_2} S_{n+1} f, \nabla e^{-t (L_2)^*} F \rangle \, dt
\]

\[
= 2 \int_0^\infty \left\{ \langle A_{\eta} \nabla \partial e^{-t L_2} S_{n+1} f, \nabla e^{-t (L_2)^*} F \rangle + \langle A_{\eta} \nabla \partial e^{-t L_2} S_{n+1} f, \nabla e^{-t (L_2)^*} F \rangle \right\} \, dt
\]

\[
- \int_0^\infty \langle A_{\eta} \nabla e^{-t L_2} \partial S_{n+1} f, \nabla e^{-t (L_2)^*} F \rangle \, dt = I + II - III.
\]
Integrating by parts, we see that

\[(5.14) \quad |I + II| = 4 \int_0^\infty \left| \sum_{i=1}^n \left( L_\eta e^{-\tau L_i} S_{t+t_0} f(x) \right) \left( (L_\eta)^+ e^{-\tau (L_i)^+} F(x) \right) td\eta \right| \]

\[\leq 4 ||te^{-\tau L_i} L_\eta S_{t+t_0} f|| ||(L_\eta)^+ e^{-\tau (L_i)^+} F|| \leq C ||te^{-\tau L_i} L_\eta S_{t+t_0} f|| ||\nabla F||_2,\]

since, by [AHLMcT], applied to \((L_\eta)^+\), we have that \(||(L_\eta)^+ e^{-\tau (L_i)^+} F|| \leq C ||\nabla F||_2\). We consider now the first factor on the right side of (5.14). Since \(u(x, t) \equiv S_{t+t_0} f(x)\) solves \(Lu = 0\), we have

\[L_\eta S_{t+t_0} f = \sum_{i=1}^n D_i A_{i,n+1} D_{n+1} S_{t+t_0} f + \sum_{j=1}^{n+1} A_{n+1,j} D_j D_{n+1} S_{t+t_0} f \equiv \Sigma_1 + \Sigma_2,\]

in the weak sense of Lemma 2.15. Since \(e^{-\tau L_i} : L^2 \to L^2\) uniformly in \(t\), we obtain

\[||te^{-\tau L_i} \Sigma_2|| \leq C ||\nabla \partial_i S_{t+t_0} f|| \leq C ||\nabla \partial_i S_{t} f||\]

which is one of the allowable terms in the bound that we seek. Also,

\[(5.15) \quad te^{-\tau L_i} \Sigma_1 = R_i \partial_i S_{t+t_0} f + \sum_{j=1}^{n} (te^{-\tau L_i} D_j A_{i,n+1}) P_j \partial_i S_{t+t_0} f,\]

where, by the familiar “Gaffney estimate” (e.g., [AHLMcT], pp. 636-637), the operator

\[R_i = \sum_{j=1}^{n} \left( te^{-\tau L_i} D_j A_{i,n+1} - (te^{-\tau L_i} D_j A_{i,n+1}) P_j \right)\]

satisfies the bound (3.1) for every \(m \geq 1\) (indeed, it satisfies a stronger exponential decay estimate). Moreover, \(R_1 = 0\), and \(R_i : L^2 \to L^2\); Thus, by Lemma 3.5 we have

\[||R_i \partial_i S_{t+t_0} f|| \leq C ||\nabla \partial_i S_{t} f||\]

as desired. In addition, by [AHLMcT], we have that \(|te^{-\tau L_i} \text{ div}_f \tilde{\eta}^2 \frac{d\tilde{\eta}}{\tilde{\eta}}|\) is a Carleson measure for all \(f \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)\). Therefore, by Carleson’s Lemma, the triple bar norm of the last term in (5.15) is dominated by \(||N(P, \partial_i S_{t+t_0} f)||_2\).

It remains to handle the term III. Integrating by parts in \(t\), we obtain

\[(5.16) \quad -III = \int_0^\infty \left\langle A_i \nabla \tilde{\eta} e^{-\tau L_i} \partial_i^2 S_{t+t_0} f, \nabla \tilde{\eta} e^{-\tau (L_i)^+} F \right\rangle dt + \text{“easy”},\]

where the two “easy terms” arise when \(\partial_i\) hits either \(e^{-\tau L_i}\) or \(e^{-\tau (L_i)^+}\). These two easy terms may be handled by an argument similar to, but simpler than the one used to treat (5.14) above. The main term in (5.16) is dominated by

\[||te^{-\tau L_i} \partial_i^2 S_{t+t_0} f|| ||(L_\eta)^+ e^{-\tau (L_i)^+} F|| \leq C ||\partial_i^2 S_{t} f|| ||\nabla F||_2,\]

where we have used the \(L^2\) boundedness of \(e^{-\tau L_i}\) to estimate the first factor, and [AHLMcT] to handle the second.

Finally, (5.4) may be proved in the same way as (5.3) with one minor modification. Since \(L^\eta_{t} f(x) = f_\eta(x, t) \equiv f(x) \varphi_\eta(t)\), the application of Lemma 2.15 produces, in addition to the analogues of \(\Sigma_1\) and \(\Sigma_2\), an error term \(f_\eta(\cdot, t + t_0)\). But

\[||te^{-\tau L_i} f_\eta(\cdot, t + t_0)|| \leq C \left( \eta \int |\varphi_\eta(t + t_0)|^2 dt \right)^{1/2} ||f||_{L^2(\mathbb{R}^3)} \leq C ||f||_2,\]

and (5.4) follows.

We finish this section with a variant of the square function estimates.
**Lemma 5.17.** Suppose that $L, L^*$ satisfy the standard assumptions, and have “Good Layer Potentials”. Then for $m \geq 0$, we have the square function bound
\[
\|\partial_t^{m+1} \partial_s^{n+1} (S_t \nabla) \cdot f\| \leq C_m \|f\|_2,
\]
where $f \in L^2(\mathbb{R}^n, C^{m+1})$.

**Proof.** By $t$-independence and Caccioppoli’s inequality in Whitney boxes, we may reduce to the case $m = 0$. By $t$-independence and (1.10), we may replace $\nabla$ by $\nabla_i$. By ellipticity of the $n \times n$ sub-matrix $A_{ij}$, and the Hodge decomposition of [AT, p. 116], as in the proof of Lemma 5.2, it suffices to show that
\[
(\nabla_i)_{ij} \Gamma(x, t, y, s) = \sum_{j=1}^n \frac{\partial}{\partial y_j} \left( A_{ij, n+1}^*(y) \partial_t \Gamma(x, t, y, s) \right) + \sum_{j=1}^n A_{n+1,ij}^*(y) \frac{\partial}{\partial y_j} \Gamma(x, t, y, s).
\]

By $t$-independence, we therefore have that
\[
\partial_t (S_{ij} \nabla) \cdot A_{ij} \nabla F = \sum_{i=1}^n \partial_t^3 S_{ij} A_{n+1,i} D_i F + \partial_t^3 (S_{ij} \overline{\partial} e_i) F,
\]
where $\overline{\partial} e_i = - \sum_{j=1}^n A_{ij, n+1}^* D_j$. We set $u(\cdot, \tau) = S_t \psi$, $\tau < 0$, so that $u(\cdot, 0) \equiv F$. Using “Good Layer Potentials”, we obtain in particular that
\[
||\nabla u(\cdot, 0)||_2 \leq C ||\nabla F||_2.
\]

Since $(S_t, \overline{\partial} e_i) = D_i$, Corollary 4.28 implies that
\[
\partial_t^3 (S_{ij} \overline{\partial} e_i) F = \partial_t^3 S_{ij} (\partial_t u(\cdot, 0)).
\]

Consequently, the left hand side of (5.18) is dominated by
\[
\sum_{i=1}^n ||\partial_t^3 S_{ij} A_{n+1,i} D_i F|| + ||\partial_t^3 S_{ij} (\partial_t u(\cdot, 0))|| \leq C ||\nabla F||_2,
\]
where in the last step we have used (1.10) and (5.19).

**Corollary 5.20.** Suppose that $L, L^*$ satisfy the standard assumptions, and have “Good Layer Potentials”. Then
\[
||\nabla (S_t \nabla) \cdot f|| \leq C \|\| f \|\|_2,
\]
where $f \in L^2(\mathbb{R}^n, C^{m+1})$.

**Sketch of proof.** One may follow almost verbatim the proof of (5.5) in Lemma 5.2, first integrating by parts in $t$ and using the case $m = -1$ of Lemma 2.11 to handle the boundary terms, and then hiding a small term and using Caccioppoli’s inequality in Whitney boxes to bound the left hand side of (5.21) by $C(||F||_2 + ||\partial_t (S_t \nabla) \cdot f||)$. We omit the details. The conclusion of the Corollary now follows immediately from Lemma 5.17.
6. Proof of Theorem 1.12: the square function estimate (5.7)

In this section we prove estimate (5.7). To be precise, suppose that \( \varphi_\delta = \delta^{-1} \varphi(\cdot/\delta) \) is the kernel of a nice approximate identity in 1 dimension, as in the definition of \( S^\delta_1 \) (2.17). We shall prove that, for all \( f \in C^0_0(\mathbb{R}^n) \), for all \( \Psi \in C^0_0(\mathbb{R}^{n+1}) \), with \( ||\Psi|| \leq 1 \), and for all \( \delta > 0 \) sufficiently small, if \( \Psi_\delta(x,t) \equiv \varphi_\delta \ast \Psi(x, \cdot)(t) \), then

\[
\int_{\mathbb{R}^{n+1}} t_0^2 S^\delta_{1,t} f(x) \frac{dx\,dt}{t} \leq C e_0 (M^+ + M^-) + C ||f||_2,
\]

where

\[
M^+ \equiv \left( ||\nabla \partial_\delta S^\delta_{1,t} f||_+ + ||N_z (P_t \partial_\delta S^\delta_{1,t} f) ||_2 + \sup_{t \geq 0} ||\nabla S^\delta_{1,t} f||_2 + ||f||_2 \right),
\]

and \( M^- \) is the corresponding quantity for the lower half-space. The proof of the analogous estimate in \( \mathbb{R}^{n+1} \) is identical, and we omit it. By Lemma 2.18 (iii), we may take first the limit as \( \delta \to 0 \), and then the supremum over all such \( \Psi \) to obtain (5.7).

The proof is by perturbation. Setting \( \epsilon(z) \equiv A^1(z) - A^0(z) \), we have

\[
L_0^1 - L_t = L_0^1 L_{-1} - L_0^1 L_{-1} = - L_0^1 \nabla L_{-1}.
\]

Since \( ||t \partial_{2}^2 S^\delta f|| \leq C ||f||_2 \), we have also that \( \sup_{\tau \geq 0} ||t \partial_{2}^2 S^\delta f|| \leq C ||f||_2 \), as may be seen by arguing as in the proof of (5.6). Thus, it is enough to consider the difference \( t \partial_{2}^2 (S^\delta_{1,t} - S^\delta_{0,t}) \). By definition (2.17),

\[
\partial_\delta S^\delta_{1,t} f(x) = \left( (D_{n+1} \varphi_\delta) \ast S^\delta_{1,t} f(x) \right) (t) = L_t^{-1} (D_{n+1} \varphi_\delta)(x, t), \quad i = 0, 1,
\]

where \( f_{\varphi}(y, s) \equiv f(y) \varphi_\delta(s) \), and \( \varphi_\delta = \eta^{-1} \varphi(\cdot/\eta) \) is as above. We then have

\[
\partial_{2}^2 S^\delta_{1,t} f(x) - \partial_{2}^2 S^\delta_{0,t} f(x) = \partial_\delta \left( L_0^1 \nabla L_{-1} (D_{n+1} \varphi_\delta) \right)(x, t)
\]

so that

\[
\int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \epsilon(y) \nabla \partial_\delta S^\delta_{1,t} f(y) \cdot \nabla \nabla (L_0^1)^{-1} (D_{n+1} \varphi_\delta)(y, s) dy ds.
\]

Essentially following [FJK], and using (6.3), we decompose

\[
\nabla (L_0^1)^{-1} (D_{n+1} \varphi_\delta)(y, s) = \int \nabla \nabla \partial_\delta S^\delta_{1,t} \left( \Psi(\cdot, t) \right)(y) dt
\]

\[
= \int_{t>0} \left\{ \nabla \nabla \partial_\delta S^\delta_{1,t} \left( \Psi(\cdot, t) \right)(y) - \nabla \nabla \partial_\delta S^\delta_{1,t} \bigg|_{t>0} \left( \Psi(\cdot, t) \right)(y) \right\} dt
\]

\[
+ \int_{t>0} \left( \nabla \nabla \partial_\delta S^\delta_{1,t} \left( \Psi(\cdot, t) \right)(y) \right) dt
\]

\[
+ \int_{t>0} \left( \frac{|s|}{t} \right)^{1/2} \nabla \nabla \partial_\delta S^\delta_{1,t} \left( \Psi(\cdot, t) \right)(y) dt
\]

\[
- \int_{t>0} \left( \frac{|s|}{t} \right)^{1/2} \nabla \nabla \partial_\delta S^\delta_{1,t} \left( \Psi(\cdot, t) \right)(y) dt = i + ii + iii + iv - v.
\]
In turn, this induces a corresponding decomposition in (6.4):

\[
I + II + III + IV - V \equiv \int_{\mathbb{R}^{n+1}} \epsilon(y)\nabla \partial_t S^{1,\eta}_t f(y) \cdot \left( I + II + III + IV - V \right) dy ds.
\]

All but term II will be easy to handle, and we shall deal with these easy terms as in [FJK]. The main term here (and in [FJK]) is II, but in our situation, matters are much more delicate, since for us \( A^0 \) is not constant. The approach of [FJK] depends critically on the fact that solutions of constant coefficient equations are, in particular, twice differentiable, a fact which fails utterly in the present setting (unless at least one of the derivatives falls on the \( t \)-variable). We shall require new methods, which exploit the technology of the solution of the Kato problem, to deal with term II.

We dispose of the easy terms in short order. To begin,

\[
IV = \int_{\mathbb{R}^{n+1}} |s|^{1/2} \epsilon(y)\nabla \partial_t S^{1,\eta}_s f(y) \cdot \left( D_{n+1} \left( \frac{\partial_t \Psi}{\sqrt{t}} \right) \right) (y, s) dy ds.
\]

Since \( \nabla L_0^{-1} \div : L^2(\mathbb{R}^{n+1}) \rightarrow L^2(\mathbb{R}^{n+1}) \), we have that \( |IV| \leq C_0 ||\nabla \partial_t S^{1,\eta}_t f||_{\text{alt}} \). Given the following lemma, I, II, and V may be handled by Hardy’s inequality, yielding also the bound \( |I| + |III| + |V| \leq C_0 ||\nabla \partial_t S^{1,\eta}_t f||_{\text{alt}} \). We omit the details.

**Lemma 6.5.** We have

\[
\begin{align*}
(6.6) & \quad \|\nabla D_{n+1} S^{L_0^\delta}_{t-t} - \nabla D_{n+1} S^{L_0^\delta}_{t-t} \|_{L^2} \leq C \frac{|s|}{T}, \quad |s| < t/2, \quad \delta < 1000^{-1} t \\
(6.7) & \quad \|\nabla \partial_t S^{L_0^\delta}_{t-t} \|_{L^2} \leq C \frac{1}{|t|}, \quad \tau \neq 0
\end{align*}
\]

**Proof of the Lemma.** If \( |t| > 100\delta \), estimate (6.7) is essentially just the case \( m = 0 \) of Lemma 2.10. Otherwise, we obtain the better bound \( C\delta^{-1} \), using definition (2.17) and the hypothesis that \( L_0, L_0^* \) have bounded layer potentials. Estimate (6.6) is obtained from the case \( m = 1 \) of Lemma 2.10, and the identity

\[
\nabla D_{n+1} S^{L_0^\delta}_{t-t} - \nabla D_{n+1} S^{L_0^\delta}_{t-t} = \int_0^\infty \nabla \partial^2_t S^{L_0^\delta}_{t-t} d\tau.
\]

It remains to handle II, which equals

\[
\begin{align*}
& \int_{\mathbb{R}^{n+1}} \left\{ \int_{t-t/2}^{t/2} \epsilon(y)\nabla \partial_t S^{1,\eta}_s f(y) ds \right\} \cdot \left( \nabla D_{n+1} S^{L_0^\delta}_{t-t} (\Psi(\cdot, t)) (y) \right) dy dt \\
& \quad = - \int_{\mathbb{R}^{n+1}} \left( \nabla \partial_t S^{1,\eta}_s \Psi \right) \cdot \epsilon \left( S^{1,\eta}_{t/2} f - S^{1,\eta}_{t/2} f \right) (x) \frac{\partial_t \Psi(x, t)}{\partial t} dx dt,
\end{align*}
\]

where we have used that for \( \eta > 0 \), \( \nabla S^{\eta}_t \) does not jump across the boundary. Since \( \Psi \) is compactly supported in \( R^+_t \), for \( \delta \) sufficiently small,

\[
t^{-1/2} |\partial_t \Psi(x, t)| \leq C \int \varphi_s (t-s) |\Psi(x, s)| s^{-1/2} ds.
\]

Thus, it is enough to bound \( ||\nabla \left( \partial_t S^{0}_t \nabla \cdot \epsilon \nabla S^{1,\eta}_{t/2} f \right)||, \) plus a similar term with \( -t/2 \) in place of \( t/2 \), which may be handled in the same way. The desired bound then follows immediately from the change of variable \( t \rightarrow 2t \) and (6.10) below.
Lemma 6.9. Suppose that \( a \in \mathbb{R} \setminus \{0\} \), and define \( M^* \) as in (6.2). Then

\[
\| t \left( \partial_t S_{at}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f \| \leq C(a)_0 M^*
\]
(6.10)

\[
\| t^2 \left( \partial_t^2 S_{at}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f \| \leq C(a)_0 M^*.
\]
(6.11)

Moreover, the analogous bound holds in the lower half space.

Proof of Lemma 6.9. This lemma is the deep fact underlying estimate (5.7), and the proof is rather delicate. For the sake of notational simplicity, we treat only the case \( a = 1 \), as the general case is handled by an almost identical argument. We begin by showing that (6.11) implies (6.10). Set

\[
J(\sigma) \equiv \int_{\sigma}^{1} \int_{\mathbb{R}^n} \left| \partial_t \left( S_{t}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f \right|^2 \, dx \, dt.
\]

After integrating by parts in \( t \), we obtain that

\[
J(\sigma) = -\Re e \int_{\sigma}^{1} \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \left( \left( \partial_t S_{t}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f \right) \left( \left( \partial_t S_{t}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f \right) \, dx \, dt + \text{"OK"},
\]

where by Lemma 2.10 (i), the “OK” boundary terms are dominated by \( C_0^2 \sup_{t>0} \| \nabla S_{t}^{1,\eta} f \|^2 \). By Cauchy’s inequality, modulo the “OK” terms,

\[
J(\sigma) \leq \frac{1}{2} J(\sigma) + \| t \left( \partial_t S_{t}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f \|^2 + \| t^2 \left( \partial_t^2 S_{t}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f \|^2
\]

\[
\equiv \frac{1}{2} J(\sigma) + I + II.
\]

The term \( \frac{1}{4} J(\sigma) \) may be hidden on the left hand side. By Lemma 2.10 (i) with \( m = 0 \), term \( I \) is no larger than \( C_0^2 \| \nabla \partial_t S_{t}^{1,\eta} f \|^2 \). The square root of the main term, \( II \), is estimated in (6.11). Taking the latter for granted momentarily, we obtain (6.10) by letting \( \sigma \to 0 \).

We now turn to the proof of (6.11), again with \( a = 1 \). We make the splitting:

\[
\partial_t^2 \left( S_{t}^0 \nabla \right) \cdot \epsilon \nabla S_{t}^{1,\eta} f = \sum_{j=1}^{n+1} \sum_{p=1}^{n} \partial_t^2 \left( S_{t}^0 D_{p} \right) e_{i} D_{j} S_{t}^{1,\eta} f
\]

\[
+ \sum_{j=1}^{n+1} \partial_t^2 \left( S_{t}^0 D_{n+1} \right) e_{i,n+1} D_{j} S_{t}^{1,\eta} f = V_i f + \check{V}_i f.
\]

We treat \( \check{V}_i \) first. For \( f : \mathbb{R}^n \to C^{a+1} \), set

\[
\theta_{\epsilon} f \equiv \partial_t^2 \left( S_{t}^0 \nabla \right) \cdot \epsilon f,
\]

and let \( \epsilon \equiv (\epsilon_{1,n+1}, \epsilon_{2,n+1}, \ldots, \epsilon_{n+1,n+1}) \). Then, using a well known trick of [CM], we write

\[
\check{V}_i f = \left( \theta_{\epsilon} \check{V}_i \right) \left( \theta_{\epsilon} P_{i} \right) \partial_t S_{t}^{1,\eta} f + \left( \theta_{\epsilon} \check{V}_i \right) \left( \theta_{\epsilon} P_{i} \right) \partial_t S_{t}^{1,\eta} f \equiv R_{\epsilon}^i \partial_t S_{t}^{1,\eta} f + \left( \theta_{\epsilon} \check{V}_i \right) \left( \theta_{\epsilon} P_{i} \right) \partial_t S_{t}^{1,\eta} f,
\]

where as usual \( P_{i} \) is a nice approximate identity. By Lemmas 5.17, 2.9, 3.2 and Carleson’s Lemma, the triple bar norm of the second summand is no larger than \( C_0 \| N_{\epsilon} \left( P_{i} \partial_t S_{t}^{1,\eta} f \right) \|_2 \). In addition, by Lemma 3.5, we have that

\[
\| R_{\epsilon}^i \partial_t S_{t}^{1,\eta} f \| \leq C_0 \| \nabla \partial_t S_{t}^{1,\eta} f \| \leq C_0 \| \nabla \partial_t S_{t}^{1,\eta} f \|.
\]

It remains to control \( \| V_i f \| \), which is the primary difficulty. By definition,

\[
V_i = \partial_t \check{V}_i S_{t}^{1,\eta} \equiv \partial_t^2 \left( S_{t}^0 \nabla \right) \cdot \check{V}_i S_{t}^{1,\eta},
\]
where \( \tilde{\epsilon} \) is the \((n+1) \times n\) matrix \((\epsilon_{ij})_{1 \leq i \leq n+1, 1 \leq j \leq n}\). Recall that \( A^1_{ij} \) is the \(n \times n\) sub-matrix of \( A^1 \) with \((A^1_{ij})_{ij} = A^1_{ij}, 1 \leq i, j \leq n\), and that \((L_1)_{ij} \equiv -\text{div} A^1_{ij} \nabla \). Then

\[
V_t = \theta_t \tilde{\epsilon} \nabla \left( I - (1 + \tilde{t}^2 (L_1)_h)^{-1} \right) S_1^{1,\eta} + \theta_t \tilde{\epsilon} \nabla \left( I + \tilde{t} (L_1)_h \right)^{-1} S_0^{1,\eta} \equiv Y_t + Z_t.
\]

We first consider \( Y_t \). Note that \( \left( I - (1 + \tilde{t}^2 (L_1)_h)^{-1} \right) = \tilde{t} (L_1)_h (I + \tilde{t}^2 (L_1)_h)^{-1} \), so

\[
Y_t = \theta_t \tilde{t} \tilde{\epsilon} \nabla \left( I + \tilde{t}^2 (L_1)_h \right)^{-1} (L_1)_h S_1^{1,\eta}.
\]

As above, set \( f_\eta(x, t) \equiv f(x, \varphi_\eta(t)) \). In the weak sense of Lemma 2.15, we then have

\[
(L_1)_h S_1^{1,\eta} f = \sum_{i=1}^n D_i A^1_{i,n+1} \partial_i S_1^{1,\eta} f + \sum_{i=1}^{n+1} A^1_{i,n+1,j} \partial_j S_1^{1,\eta} f + f_\eta,
\]

and we denote by \( Y_t^{(1)} + Y_t^{(2)} + Y_t^{(3)} \) the corresponding splitting of \( Y_t \). Now, by Lemma 2.10, \( \theta_t : L^2 \rightarrow L^2 \), and it is well known that \( \nabla (I + \tilde{t}^2 (L_1)_h)^{-1} : L^2 \rightarrow L^2 \). Thus

\[
\|Y_t^{(2)} f\| \leq C_0 \| \tilde{t} \nabla S_1^{1,\eta} f \|,
\]

and also, as in the proof of (5.6),

\[
\| Y_t^{(3)} \| \leq C_0 \| t f_\eta \| \leq C_0 \| f \|_{L^2(E)}.
\]

We make a further decomposition of \( Y_t^{(1)} \) as follows:

\[
Y_t^{(1)} = (U_{\tilde{\epsilon}} \tilde{\epsilon} - (U_{\tilde{\epsilon}} \tilde{\epsilon} P_\eta) \partial_\eta S_1^{1,\eta} + (U_{\tilde{\epsilon}} \tilde{\epsilon} P_\eta) \partial_\eta S_1^{1,\eta} \equiv \tilde{\rho} \partial_\eta S_1^{1,\eta} + (U_{\tilde{\epsilon}} \tilde{\epsilon} P_\eta) \partial_\eta S_1^{1,\eta},
\]

where

\[
(6.12) \quad U_{\tilde{\epsilon}} \tilde{\epsilon} \equiv \theta_t \tilde{\epsilon} \nabla \left( I + \tilde{t}^2 (L_1)_h \right)^{-1} \text{div} \tilde{g}.
\]

and \( \tilde{\epsilon} \equiv (A^1_{1,n+1}, A^1_{2,n+1}, \ldots, A^1_{n,n+1}) \). We now claim that

\[
\| U_{\tilde{\epsilon}} \|_{L^p(E)} \leq C_0
\]

(6.13)

Let us momentarily defer the proof of this claim. It is a standard fact that for two sets \( E \) and \( E' \subseteq \mathbb{R}^n \), with \( \tilde{g} \) supported in \( E' \), we have

\[
\| \tilde{t} \nabla \left( I + \tilde{t}^2 (L_1)_h \right)^{-1} \text{div} \tilde{g} \|_{L^2(E)} \leq C \exp \left( \frac{-\text{dist}(E, E')}{Ct} \right) \| \tilde{g} \|_{L^2(E')}
\]

(6.14)

(the corresponding fact for the operator \( \tilde{t} \nabla (I + \tilde{t}^2 (L_1)_h)^{-1} \) is proved in [AHLMcT] for example, and (6.14) may be readily deduced from this fact plus the same argument). Thus, by Lemma 3.3, the operator \( U_{\tilde{\epsilon}} \) satisfies (3.1), with a bound on the order of \( C_0 \), whenever \( t \leq c t(Q) \). Therefore, by Lemma 3.2 and Carleson’s Lemma, we have that

\[
\| (U_{\tilde{\epsilon}} \tilde{\epsilon} P_\eta) \partial_\eta S_1^{1,\eta} f \| \leq C_0 \| N_\eta(P_\eta \partial_\eta S_1^{1,\eta} f) \|_{L^2}
\]

Moreover, by Lemmas 3.5 and 3.11, we have that

\[
\| \nabla \partial_\eta S_1^{1,\eta} f \| \leq C_0 \| \nabla \partial_\eta S_1^{1,\eta} f \| \leq C_0 \| \nabla \partial_\eta S_1^{1,\eta} f \|.
\]

To finish our treatment of \( Y_t \), it remains to prove (6.13). We continue to defer the proof of this estimate for the moment, and proceed to discuss the term \( Z_t \). We write

\[
Z_t = \theta_t \tilde{\epsilon} \nabla \left( I + \tilde{t}^2 (L_1)_h \right)^{-1} (S_1^{1,\eta} - S_0^{1,\eta})
\]

\[
+ \theta_t \tilde{\epsilon} \nabla \left( I + \tilde{t}^2 (L_1)_h \right)^{-1} - I \right) S_1^{1,\eta} + \theta_t \tilde{\epsilon} \nabla S_0^{1,\eta} \equiv Z_t^{(1)} + Z_t^{(2)} + Z_t^{(3)}.
\]
By Lemma 5.17 with \( m = 1 \), we have that
\[
\|\|Z_t^{(2)} f\|\| \leq C_{\epsilon_0} \sup_{r > 0} \|\nabla S_{t, \epsilon} f\|_2.
\]
Also,
\[
Z_t^{(2)} = \partial_\epsilon \nabla \| (I + \hat{r}^2(L_1))^{-1} \hat{r}^2 \text{div}_g A^1_\| \nabla g S_{t, \epsilon}^{1, \eta} = U_\epsilon A^1_\| \nabla g S_{t, \epsilon}^{1, \eta}
\]
(see (6.12)), so by the deferred estimate (6.13) we have that
\[
\|\|Z_t^{(2)} f\|\| \leq C_{\epsilon_0} \sup_{r > 0} \|\nabla S_{t, \epsilon} f\|_2.
\]
Integrating by parts, we obtain
\[
Z_t^{(1)} = \partial_\epsilon \nabla \| (I + \hat{r}^2(L_1))^{-1} \int_0^t \partial_\epsilon S_{s, \epsilon}^{1, \eta} ds = -\partial_\epsilon \nabla \| (I + \hat{r}^2(L_1))^{-1} \int_0^t s \partial_\epsilon^2 S_{s, \epsilon}^{1, \eta} ds + \partial_\epsilon \nabla \| (I + \hat{r}^2(L_1))^{-1} t \partial_\epsilon S_{t, \epsilon}^{1, \eta} \equiv \Omega_t^{(1)} + \Omega_t^{(2)}.
\]
By Lemma 3.3, and the fact that \( \nabla g(I + \hat{r}^2(L_1))^{-1} 1 = 0 \), we have that the operator
\[
R_t \equiv \partial_\epsilon t \nabla \| (I + \hat{r}^2(L_1))^{-1} t \partial_\epsilon S_{t, \epsilon}^{1, \eta}
\]
satisfies the hypothesis of Lemma 3.5, with a bound on the order of \( C_{\epsilon_0} \), so that
\[
\|\|\Omega_t^{(2)} f\|\| \leq C_{\epsilon_0} \|\|t \nabla \partial_\epsilon S_{t, \epsilon}^{1, \eta} f\|\|.
\]
Furthermore,
\[
\Omega_t^{(1)} = -\int_0^t \frac{s}{t} \partial_\epsilon t \nabla \| (I + \hat{r}^2(L_1))^{-1} s \partial_\epsilon^2 S_{s, \epsilon}^{1, \eta} ds,
\]
so by Lemma 3.12, we have
\[
\|\|\Omega_t^{(1)} f\|\| \leq C_{\epsilon_0} \|\|t \partial_\epsilon^2 S_{t, \epsilon}^{1, \eta} f\|\|.
\]
Modulo (6.13), this concludes the proof of Lemma 6.9, and hence also that of (5.7).

We conclude the present section by proving (6.13). The proof will depend on some technology from the proof of the Kato square root conjecture. By ellipticity, it is enough to show that
\[
\|\|U_\epsilon A^1_\| g\| \leq C_{\epsilon_0} \|\|g\|\|_2
\]
for \( g \in L^2(\mathbb{R}^n, C^\alpha) \). But
\[
U_\epsilon A^1_\| = \partial_\epsilon \hat{r} \nabla \| (I + \hat{r}^2(L_1))^{-1} \text{div}_g A^1_\|,
\]
so by the Hodge decomposition [AT, p. 116], we may replace \( g \) by \( \nabla g \| F \), where \( \|\nabla g\|_2 \leq C\|g\|_2 \). As usual, by density we may suppose that \( F \in C^\infty_0 \). Now
\[
U_\epsilon A^1_\| \nabla \| F = -\partial_\epsilon \nabla \| (I + \hat{r}^2(L_1))^{-1} (\hat{r}^2(L_1)) F = \partial_\epsilon \nabla \| (I + \hat{r}^2(L_1))^{-1} - I \| F.
\]
We recall that \( \partial_\epsilon = \hat{r}^2 \partial_\epsilon^2(S_{t, \epsilon}^1 \nabla) \cdot \), so by Lemma 5.17 with \( m = 1 \),
\[
\|\|\partial_\epsilon \nabla \| F\|\| \leq C_{\epsilon_0} \|\|\nabla \| F\|\|_2.
\]
The main term is
\[
\partial_\epsilon \nabla \| (I + \hat{r}^2(L_1))^{-1} F \equiv \frac{1}{t} R_t F,
\]
where by Lemmas 2.9, 2.10, and 3.3, and the fact that $\nabla (I + t^2 L_1^{-1}) = 0$, we have that $R_t$ satisfies the hypotheses of Lemma 3.9, with a bound on the order of $C_{\epsilon_0}$. Therefore, it suffices to prove the Carleson measure estimate

$$\int_0^\infty \int_Q |\frac{1}{t} R_t \Phi(x)|^2 \frac{dxdt}{t} \leq C_{\epsilon_0}|Q|,$$

where $\Phi(x) \equiv x$. To this end, we write

$$\begin{align*}
(6.15) \quad & \frac{1}{t} R_t \Phi = \theta_t \tilde{\epsilon} \nabla_{\|} \left( (I + t^2 (L_1))^{-1} - I \right) \Phi + \theta_t \tilde{\epsilon} \nabla_{\|} \Phi.
\end{align*}$$

But $\nabla_{\|} \Phi = I$, the $n \times n$ identity matrix. Thus, Lemmas 5.17, 2.9 and 3.2 yield the bound

$$\int_0^\infty \int_Q |\theta_t \tilde{\epsilon} \nabla_{\|} \Phi|^2 \frac{dxdt}{t} \leq C_{\epsilon_0}|Q|.$$ 

The remaining term in (6.15) equals

$$\theta_t \tilde{\epsilon} t^2 \nabla_{\|} \left( (I + t^2 (L_1))^{-1} \right) \text{div}_{\|} A_{\parallel} \nabla_{\|} \Phi = \theta_t \tilde{\epsilon} t^2 \nabla_{\|} \left( (I + t^2 (L_1))^{-1} \right) \text{div}_{\|} A_{\parallel} \equiv T_t A_{\parallel}$$

We now invoke a key fact in the proof of the Kato conjecture. By [AHLMcT], there exists, for each $Q$, a mapping $F_Q : \mathbb{R}^n \to \mathbb{C}^n$ such that

\begin{align*}
(6.16) & \quad (i) \quad \int_{\mathbb{R}^n} |\nabla_{\parallel} F_Q|^2 \leq C |Q| \\
 & \quad (ii) \quad \int_{\mathbb{R}^n} |(L_1)_{\parallel} F_Q|^2 \leq C \frac{|Q|}{t^2(Q)^2} \\
 & \quad (iii) \quad \sup_Q \int_0^\infty \int_Q |\tilde{\zeta}(x,t)|^2 \frac{dxdt}{t} \\
 & \quad \quad \quad \leq C \sup_Q \int_0^\infty \int_Q |\tilde{\zeta}(x,t) E_t \nabla_{\|} F_Q(x)|^2 \frac{dxdt}{t},
\end{align*}

for every function $\tilde{\zeta} : \mathbb{R}^{n+1} \to \mathbb{C}^n$, where $E_t$ denotes the dyadic averaging operator, i.e. if $Q(x,t)$ is the minimal dyadic cube (with respect to the grid induced by $Q$) containing $x$, with side length at least $t$, then

$$E_t g(x) \equiv \int_{Q(x,t)} g.$$

Here $\nabla_{\|} F_Q$ is the Jacobian matrix $(D_i(F_Q))_{i \leq j \leq n}$, and the product

$$\tilde{\zeta} E_t \nabla_{\|} F_Q = \sum_{i=1}^n \zeta_i E_t D_i F_Q$$

is a vector. Given the existence of a family of mappings $F_Q$ with these properties, as in [AT, Chapter 3], we see by (iii), applied with $\tilde{\zeta}(x,t) = T_t A_{\parallel}$, that it is enough to show that

$$\int_0^\infty \int_Q |T_t A_{\parallel}(x) E_t \nabla_{\|} F_Q(x)|^2 \frac{dxdt}{t} \leq C_{\epsilon_0}|Q|.$$ 

But as in [AT], we may exploit the idea of [CM] to write

$$T_t A_{\parallel} E_t \nabla_{\|} F_Q = [(T_t A_{\parallel}) E_t - T_t A_{\parallel}] \nabla_{\|} F_Q + T_t A_{\parallel} \nabla_{\|} F_Q$$

$$= (T_t A_{\parallel}) (E_t - P_t) \nabla_{\|} F_Q + [(T_t A_{\parallel}) E_t P_t - T_t A_{\parallel}] \nabla_{\|} F_Q + T_t A_{\parallel} \nabla_{\|} F_Q$$

$$\equiv R_t^{(1)} \nabla_{\|} F_Q + R_t^{(2)} \nabla_{\|} F_Q + T_t A_{\parallel} \nabla_{\|} F_Q,$$
where $P_t$ is a nice approximate identity. The last term is easy to handle. We have that
\[
T_t A \nabla F = -\theta_t \bar{e} \nabla \left( (I + \bar{t}^2 (L_1)_{\theta})^{-1} \right) I(L_1)_{\theta} F_Q.
\]
Therefore, since $\theta_t$ and $\bar{t} \nabla \left( (I + \bar{t}^2 (L_1)_{\theta})^{-1} \right)$ are uniformly bounded on $L^2$, we obtain that
\[
\int_0^{\infty} \int_Q |T_t A \nabla F|^2 \frac{dx dt}{t} \leq C \epsilon_0 \int_{\mathbb{R}^n} |(L_1)_{\theta} F_Q|^2 \int_0^{\infty} t dt dx \leq C \epsilon_0 |Q|,
\]
where in the last step we have used (6.16)(ii).

It is also easy to handle $R_t^{(1)} \nabla F$. Indeed $E_t = E_t^{(2)}$, so that
\[
R_t^{(1)} = (T_t A) E_t (E_t - P_t).
\]
By the definition of $T_t$, Lemma 3.3 and Lemma 3.11, we have that
\[
\|(T_t A) E_t\|_{L^2} \leq C \epsilon_0.
\]
Thus,
\[
\int_0^{\infty} \int_Q |R_t^{(1)} \nabla F|^2 \frac{dx dt}{t} \leq C \epsilon_0 \int_{\mathbb{R}^n} |(E_t - P_t) \nabla F_Q|^2 \frac{dx dt}{t} \leq C \epsilon_0 |Q|,
\]
where in the last step we have used (6.16)(ii), as well as the boundedness on $L^2$ of
\[
g \rightarrow \left( \int_0^{\infty} |(E_t - P_t) g|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.
\]
It remains to treat the contribution of the term $R_t^{(2)} \nabla F$. By (6.16)(i), it will be enough to establish the square function bound
\[
\||R_t^{(2)} \nabla F_Q|| \leq C \epsilon_0 \|\nabla F_Q\|_{L^2}.
\]
To this end, we write
\[
R_t^{(2)} \nabla F = R_t^{(2)} (I - P_t) \nabla F + R_t^{(2)} P_t \nabla F,
\]
where $I$ denotes the identity operator. The last term is easy to handle. We note that $R_t^{(2)} 1 = 0$, and therefore by Lemmas 2.9, 2.10, 3.3 and 3.11, the operator $R_t^{(2)}$ satisfies the hypotheses of Lemma 3.5 with bound on the order of $C \epsilon_0$. Thus,
\[
\||R_t^{(2)} P_t \nabla F_Q|| \leq C \epsilon_0 \||P_t \nabla F_Q|| \leq C \epsilon_0 \||\nabla F||_{L^2},
\]
where the last inequality is standard Littlewood-Paley theory.

By the definition of $R_t^{(2)}$, we may further decompose the first summand on the right side of (6.18) as
\[
(T_t A) E_t Q, \nabla F - T_t A \nabla (I - P_t) F_Q \equiv \mathbf{I} - \mathbf{II},
\]
where $Q_t \equiv P_t (I - P_t)$ satisfies $\|Q_t\|_{op} \leq C$. Then by (6.17), we have
\[
\||\mathbf{I}|| \leq C \epsilon_0 \|\nabla F_Q\|_{L^2}.
\]
Next, by definition of $T_t$, we see that
\[
\mathbf{II} = \theta_t \bar{e} \nabla \left( (I + \bar{t}^2 (L_1)_{\theta})^{-1} \right) (I - P_t) F_Q = -\theta_t \bar{e} \nabla F_Q + \theta_t \bar{e} \nabla P_t F_Q + \theta_t \bar{e} \nabla \left( (I + \bar{t}^2 (L_1)_{\theta})^{-1} (I - P_t) F_Q \equiv \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3.
\]
By Lemma 5.17,
\[
\||\mathbf{I}_1|| \leq C \epsilon_0 \|\nabla F_Q\|.
\]
Moreover, by Lemma 2.10 and the fact that \[ \|r\nabla(1 + r^2(L_{1})^{-1})\|_{2\to 2} \leq C, \]
we obtain that
\[ \|\Pi_{1}\| \leq C \|r^{-1} I_{1}(I - P_{1})\sqrt{-\Delta F_{Q}}\| \leq C \|\nabla F_{Q}\|_{2}, \]
where \( I_{1} = (-\Delta)^{-1/2} \) is the fractional integral operator of order one on \( \mathbb{R}^{n} \), and where we have used the Littlewood-Paley inequality
\[ \|r^{-1} I_{1}(I - P_{1})\|_{op} \leq C. \]
The latter estimate holds by Plancherel’s Theorem, since
\[ \left| \frac{1}{|\xi|} (1 - \phi(t\xi)) \right| \leq C \min \left( \frac{1}{|\xi|}, \frac{1}{t |\xi|} \right), \]
if \( \phi_{t}(x) = r^{n} \phi_{t}(x/t) \), the convolution kernel of \( P_{t} \), is chosen so that \( \int_{\mathbb{R}^{n}} x \phi_{t}(x) \, dx = 0. \)
Finally, it remains only to consider the term \( \Pi_{2} \). Now
\[ \Pi_{2} = \theta \tilde{\omega} P_{t} \nabla F_{Q}, \]
so we need that \( \|\theta \tilde{\omega} P_{t}\|_{op} \leq C \). By Lemmas 5.17, 3.2 and 2.9. \( |\theta \tilde{\omega} r^{-1} dx dt| \) is a Carleson measure with norm at most \( C \), so it is enough to bound \( \|\theta \tilde{\omega} P_{t} - (\theta \tilde{\omega}) P_{t}\|_{op} \). We may choose \( P_{t} \) to be of the form \( P_{t} = \tilde{P}_{t} \), where \( \tilde{P}_{t} \) is of the same type. Set
\[ R_{t} = \theta \tilde{\omega} P_{t} - (\theta \tilde{\omega}) P_{t}, \]
which satisfies the hypothesis of Lemma 3.5 with bound \( C \). Thus,
\[ \|\theta \tilde{\omega} P_{t} - (\theta \tilde{\omega}) P_{t}\|_{op} \equiv \|R_{t} \tilde{P}_{t}\|_{op} \leq C \|\nabla \tilde{P}_{t}\|_{op} \leq C. \]
This concludes the proof of Lemma 6.9, and hence that of the square function bound (5.7). □

7. Proof of Theorem 1.12: the singular integral estimate (5.8)
We shall consider separately the cases \( t > 0 \) and \( t < 0 \), and since the proof is the same in each case we treat only the former. More precisely, we shall prove
\[ \sup_{0 < |t| < 10^{-n}} \sup_{\partial \Omega} \|\partial_{t} S^{1, \eta}_{t} f\|_{2} \leq C \|f\|_{2} + C \left( M^{+} + M^{-} \right), \]
where \( M^{\pm} \) are defined in (6.2). We begin by reducing matters to the case \( t \geq 4 \eta \). Suppose that \( 0 \leq t < 4 \eta \). We claim that
\[ |\partial_{t} S^{1, \eta}_{t} f(x) - D_{n+1} S^{1, \eta}_{4 \eta} f(x)| \leq CM f(x). \]
Indeed, let \( K^{\eta}_{t}(x, y) \) denote the kernel of \( \partial_{t} S^{1, \eta}_{t} \), i.e.,
\[ K^{\eta}_{t}(x, y) = \partial_{t} \left( \bar{\phi}_{\eta} + \Gamma_{1}(x, \cdot, y, 0) \right)(t). \]
To prove the claim, it is enough to establish the following estimate:
\[ |K^{\eta}_{t}(x, y) - K^{\eta}_{4 \eta}(x, y)| \leq C \left( \frac{|t|}{|x - y|^{n+1}} + \theta \right) \left( \frac{\eta}{|x - y|^{n+1}} + \frac{1}{|x - y|^{n+1}} \right). \]
In turn, the case \( |x - y| \leq 10 \eta \) of the latter bound follows directly from (4.10). On the other hand, if \( |x - y| > 10 \eta \), we have by Lemma 2.5 that
\[ |K^{\eta}_{t}(x, y) - K^{\eta}_{4 \eta}(x, y)| = \left| \int \phi_{\eta}(s) \left( D_{n+1} \Gamma_{1}(x, t - s, y, 0) - D_{n+1} \Gamma_{1}(x, 4 \eta - s, y, 0) \right) ds \right| \leq C \int |\phi_{\eta}(s)| \left( \frac{4 \eta - \eta}{|x - y|^{n+1}} ds \right) \leq C \frac{\eta}{|x - y|^{n+1}}. \]
Having proved the claim, we fix \( t_0 \geq 4\eta \), and use (1.3) to obtain, for each \( y \in \mathbb{R}^n \),

\[
|D_n^{+1}S_{\eta,t}^{1,0} f(y)| \leq C \left( \iint_{B(y,t_0),|x|/2} |\partial_r S_{\tau}^{1,0} f(x)|^2 dx d\tau \right)^{1/2}
\]

\[
\leq C \left( \iint_{B(y,t_0),|x|/2} |\partial_r S_{\tau}^{1,0} f - \partial_r S_{\tau}^{0,0} f|^2 dx d\tau \right)^{1/2} + \text{“OK”}
\]

where \( \|\text{“OK”}\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_2 \) uniformly in \( t_0 \), by our hypotheses regarding \( L_0 \), and where we have used that \( u_\eta(x,t) \equiv S_{\tau}^{1,0} f(x) \) solves \( L_1 u_\eta = 0 \) in \( |t| > \eta \). Consequently,

\[
\|D_n^{+1}S_{\eta,t}^{1,0} f\|_2^2 \leq C\|f\|_2 + C \frac{1}{t_0} \int_{|x|/2}^{4t_0/2} \int_{\mathbb{R}^n} |\partial_r S_{\tau}^{1,0} f - \partial_r S_{\tau}^{0,0} f|^2 dx d\tau.
\]

As in the section 6,

\[
\partial_r S_{\tau}^{1,0} f(x) - \partial_r S_{\tau}^{0,0} f(x) = \partial_r \left( L_0^{-1} \operatorname{div} e\nabla S_{\tau}^{1,0} f \right)(x, \tau).
\]

Thus, it is enough to prove that for every \( \Psi \in C^0_0 \left( \mathbb{R}^n \times \left( \frac{3\eta}{2}, \frac{3t}{2} \right) \right) \), with \( t_0^{-1/2}\|\Psi\|_2 = 1 \), and for each \( \eta > 0 \) and \( \delta > 0 \) sufficiently small, we have

\[
(7.2) \quad \left| \frac{1}{t_0} \int_{\mathbb{R}^n} e(y) \nabla S_{\tau}^{1,0} f(y) \cdot \nabla \left( L_0^{-1} D_n^{+1} \Psi \right)(y, s) dy ds \right| \leq C e_0 (M^* + M^+),
\]

where again \( \Psi = \varphi \ast \Psi \). We may then obtain (7.1) by taking first a limit as \( \delta \to 0 \), and then a supremum over all such \( \Psi \).

To prove (7.2), we begin by splitting the integral on the left hand side into

\[
(7.3) \quad \frac{1}{t_0} \left\{ \int_{t_0/4}^{t_0/4} \int_{\mathbb{R}^n} + \int_{t_0/4}^{t_0/4} \int_{\mathbb{R}^n} + \int_{t_0/4}^{t_0/4} \int_{\mathbb{R}^n} + \int_{-t_0/4}^{-t_0/4} \int_{\mathbb{R}^n} \right\} \equiv I + II + III + IV.
\]

Since \( \nabla \left( L_0^{-1} \operatorname{div} \right) \) is bounded on \( L^2(\mathbb{R}^{n+1}) \), by Cauchy-Schwarz and our assumptions on \( \Psi \), we have that

\[
|III| \leq C e_0 \left( \int_{t_0/4}^{t_0/4} \int_{\mathbb{R}^n} |\nabla S_{\tau}^{1,0} f(y)|^2 dy ds \right)^{1/2} \leq C e_0 \sup_{|s| \leq r_0} \|\nabla S_{\tau}^{1,0} f\|_2.
\]

Next we consider terms \( III \) and \( IV \). These may be handled in the same way, so we treat only \( III \) explicitly. We use (6.3) to write

\[
(7.4) \quad \nabla \left( L_0^{-1} D_n^{+1} \Psi \right)(y, s) = \int \nabla_{y,s} \partial_r S_{\partial} \left( (\Psi(\cdot, \tau))(y) \right) d\tau,
\]

so that

\[
III = \int_{t_0/4}^{t_0/4} \int_{\mathbb{R}^n} \left( \partial_r S_{\partial} \nabla \right) : e\nabla S_{\tau}^{1,0} f(x) \Psi(y, x, \tau) dx dy d\tau
\]

\[
= \frac{1}{t_0} \int_{2\tau}^{2t} \int_{\mathbb{R}^n} - \frac{1}{t_0} \int_{2\tau}^{2t} \int_{\mathbb{R}^n} \equiv III - \text{error}.
\]

In the error term, \( s - \tau \approx s \approx \tau \approx t_0 \), if \( \delta \) is sufficiently small, given the support constraints on \( \Psi \). Thus by Cauchy-Schwarz and Lemma 2.10 (i), the absolute value of the error term
is bounded by $C \epsilon_0 \sup_{\tau \neq 0} ||\nabla S_{t, \tau}^{1, \eta} f||_2$. The remaining term is
\[
\tilde{II} = t_0^{\frac{1}{2}} \int \int_{\mathbb{R}^2} \left( \partial_t S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) \Psi_{\delta}(x, \tau) dxd\tau
\]
\[
= t_0^{\frac{1}{2}} \lim_{R \to \infty} \int \int_{\mathbb{R}^2} \left( \partial_t S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) dxd\tau
\]
\[
\equiv t_0^{\frac{1}{2}} \lim_{R \to \infty} \int \int_{\mathbb{R}^2} H_R(x, \tau) \Psi_{\delta}(x, \tau) dxd\tau,
\]
where the expression in curly brackets equals
\[
H_R(x, \tau) = -\int_0^\infty \partial_t \left( \int_0^R \left( D_{t+1} S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) d\tau \right) dt
\]
\[
= \int_0^\infty \left( D_{t+1} S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) dt - \int_0^\infty \left( D_{t+1} S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) dt
\]
\[
- \int_0^\infty \left( \int_0^0 \left( D_{t+1} S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) d\tau \right) dt
\]
\[
\equiv H_R''(x, \tau) = H_R''(x, \tau) - H_R''(x, \tau).
\]
Since $|t - 2R| \approx R$, we have that by Lemma 2.10 (i),
\[
\sup_{\tau \neq 0} ||H_R''(x, \tau)||_2 \leq C \epsilon_0 \sup_{\tau \neq 0} ||\nabla S_{t, \tau}^{1, \eta} f||_2,
\]
from which the desired bound for the corresponding part of $\tilde{II}$ follows readily. Similarly, we may treat the contribution of $H''_R(x, \tau)$ by a direct application of the following Lemma, which is really the deep result in this section.

**Lemma 7.5.** Let $a, b$ denote non-zero real constants. We then have that
\[
\sup_{\tau \neq 0} \left\| \int_{\tau}^{\tau + \epsilon} \left( D_{n+1} S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) d\tau \right\|_2 \leq C(a, b) \epsilon_0 \left( M^+ + M^- \right).
\]

We defer for the moment the proof of this Lemma, and consider now
\[
H''''_R(x, \tau) = \int_0^R \int_0^R \partial_t \left( \partial_t S_{t, \tau}^0 \cdot \nabla \right) \cdot \epsilon \nabla S_{t, \tau}^{1, \eta} f(x) dxd\tau.
\]
Then for $h \in L^2(\mathbb{R}^n)$, with $||h||_2 = 1$, we have
\[
(7.6) \quad \left| \langle h, H''''_R(\cdot, \tau) \rangle \right| = \left| \int_0^R \int_0^R \left( \nabla D_{n+1} S_{t, \tau}^{L_0} h \right) \cdot \epsilon \partial_t \nabla S_{t, \tau}^{1, \eta} f(x) dxd\tau \right|,
\]
where we have used that $\text{adj}(S_{t, \tau}^0) = S_{t, \tau}^{L_0}$ (recall that adj indicates that we have taken the adjoint in the $x, y$ variables only, whereas $S_{t, \tau}^{L_0}$ is the single layer potential operator associated to $L_0$). Thus, (7.6) is dominated by
\[
C \epsilon_0 \left( \int_0^\infty \left\| \nabla \partial_t S_{t, \tau}^{L_0} h \right\|_2^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^\infty \left\| \partial_t \nabla S_{t, \tau}^{1, \eta} f \right\|_2^2 \int_0^\infty \int_0^\infty dt d\tau \right)^{\frac{1}{2}} \equiv C \epsilon_0 B_1 \cdot B_2.
\]
Note that $B_2 = C ||s \partial_t \nabla S_{t, \tau}^{L_0} f||$. Similarly, the change of variable $s \rightarrow s + t$ yields that $B_1 = ||s \partial_t \nabla S_{t, \tau}^{L_0} h|| \leq C ||h||_2 = C$. A suitable bound follows for the contribution of $H''''_R$. \]
It remains to consider the term $I$ in (7.3), which we shall also treat via Lemma 7.5. Again using (7.4), and that for small $\delta$, $\Psi_\delta$ is supported in $\{t_0/2 < \tau < 3t_0/2\}$, we write

$$I = \int_0^{t_0} \int_{t_0/4}^{3t_0/4} \int_{\mathbb{R}^n} \left( \partial_t S_{t-\tau}^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f(x) \Psi_\delta(x, \tau) dx \, ds \, d\tau$$

$$= \frac{1}{t_0} \int_{-\tau/2}^{\tau/2} \int_{\mathbb{R}^n} - \frac{1}{t_0} \int_{t_0/4 < |\sigma| < 3t_0/2} \int_{\mathbb{R}^n} \equiv \overline{I} - \text{error}.$$

By Cauchy-Schwarz and Lemma 2.10 (i), the absolute value of the error term is bounded by $C_0 \sup_{\tau > 0} \|\nabla S_s^{1,\eta} f\|_2$, since $\tau - s < \tau < t_0$. The remaining term splits into

$$\overline{I} = \int_0^{t_0} \int_{\mathbb{R}^n} \left\{ \int_0^{\tau/2} \left( \partial_t S_{t-\tau}^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f(x) ds \right\} \Psi_\delta(x, \tau) dx \, d\tau$$

$$\equiv \int_0^{t_0} \int_{\mathbb{R}^n} F(x, \tau) \Psi_\delta(x, \tau) dx \, d\tau,$$

plus a similar term $\overline{I}$, which may be treated by the same arguments, in which the expression in curly brackets has domain of integration $(-\tau/2, 0)$. Now,

$$F(\cdot, \tau) = \int_0^\tau \partial_t \left( \int_0^{\tau/2} \left( \partial_t S_{t-\tau}^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f ds \right) dt$$

$$= \int_0^\tau \partial_t \left( \int_0^{\tau/2} \left( D_{n+1} S_s^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f ds \right) dt$$

$$= \int_0^\tau \left( \partial_t S_s^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f dt - \int_0^\tau \left( D_{n+1} S_s^{0,\tau/2} \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f dt$$

$$+ \int_0^\tau \int_0^{\tau/2} \left( \partial_t S_{t-\tau}^0 \nabla \right) \cdot \epsilon \nabla \partial_s S_s^{1,\eta} f ds \, dt = F' - F'' + F'''.$$

We may estimate the contribution of $F'''$ directly via Lemma 7.5. Also,

$$F'(\cdot, \tau) = \left( S_s^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f - \left( S_s^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f,$$

so by our hypotheses concerning $L_0$,

$$\sup_{\tau} \|F'(\cdot, \tau)\|_{L^2(\mathbb{R}^n)} \leq C_0 \sup_{\tau > 0} \|\nabla S_s^{1,\eta} f\|_2.$$

We therefore obtain a permissible bound for the contribution of $F'$. We also have that

$$F'''(\cdot, \tau) = \int_0^\tau \int_0^{\tau/2} \partial_t \left( \left( S_s^0 \nabla - S_s^{0,\tau/2} \nabla \right) \cdot \epsilon \nabla \partial_s S_s^{1,\eta} f ds \right) dt$$

$$+ \int_0^\tau \left( \partial_t S_s^{0,\tau/2} \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f dt - \int_0^\tau \left( \partial_t S_s^0 \nabla \right) \cdot \epsilon \nabla S_s^{1,\eta} f dt.$$

In turn, the last term equals $-F'$, and the middle summand may be handled via Lemma 7.5. The first summand on the right hand side of (7.7) equals

$$-\int_0^\tau \int_0^{\tau/2} \partial_t^2 \left( S_s^{0,\tau/2} \nabla \right) \cdot \epsilon \nabla \partial_s S_s^{1,\eta} f(x) dx \, d\sigma \, ds \, dt.$$
Dualizing against $h \in L^2(\mathbb{R}^n)$, with $|h|_2 = 1$, we see that it is enough to consider
\[
\left| \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty 1_{\{r < s < t/2\}}(\nabla D_\alpha^2 S_{\sigma,\eta}^f h, e D_\alpha^2 S_{\sigma,\eta}^f h) d\sigma d\tau dt ds \right|
\leq \frac{C_0}{r^\alpha} \left( \int_0^\infty \int_0^\infty 1_{\{r < s < t/2\}}(\nabla D_\alpha^2 S_{\sigma,\eta}^f h)_2^2 d\sigma d\tau dt ds \right)^{\frac{1}{2}}
\times \left( \int_0^\infty \int_0^\infty 1_{\{r < s < t/2\}} s^{-\frac{1}{2}} \|\nabla S_{\alpha,\eta}^f\|_2^2 d\sigma d\tau dt ds \right)^{\frac{1}{2}}
\equiv C_0 B_4 \cdot B_3.
\]
Now,
\[
B_4 = \left( \int_0^\infty \|\partial_s \nabla S_{\alpha,\eta}^f\|_2^2 \left( \int_0^\infty d\sigma \int_{2\tau}^\infty s^\frac{1}{2} \frac{d\tau}{2} dt \right) ds \right)^{\frac{1}{2}} = C\|s \nabla \partial_s S_{\alpha,\eta}^f\|.
\]
Similarly, the change of variable $t \to t + \sigma$ yields the bound
\[
B_3 = \left( \int_0^\infty \int_0^\infty 1_{\{r < s < t/2\}} s^{-\frac{1}{2}} (t + \sigma)^{\frac{1}{2}} \|\nabla D_\alpha^2 S_{\sigma,\eta}^f h\|_2^2 d\sigma d\tau dt ds \right)^{\frac{1}{2}}
\leq C \left( \int_0^\infty t^\frac{1}{2} \|\partial_s^2 \partial_s S_{\sigma,\eta}^f h\|_2^2 \int_0^\infty s^{-\frac{1}{2}} \int_0^\infty d\sigma d\tau dt ds \right)^{\frac{1}{2}} = C\|t^2 \nabla \partial_s^2 S_{\sigma,\eta}^f h\| \leq C\|h\|_2 = C,
\]
and the desired estimate for the contribution of $F'''$ now follows.

To complete the proof of estimate (5.8), it therefore remains to prove Lemma 7.5.

**Proof of Lemma 7.5.** For the sake of simplicity of notation, we shall treat the case $a = 2$, $b = 1$, as the general case follows via the same argument.

As above we dualize against $h \in L^2(\mathbb{R}^n)$, so that it is enough to consider
\[
\int_0^{t_2} \langle \nabla \partial_t S_{\alpha,\eta}^f h, e \nabla S_{\alpha,\eta}^f h \rangle dt = - \int_0^{t_2} \langle \nabla \partial_t S_{\alpha,\eta}^f h, e \nabla S_{\alpha,\eta}^f h \rangle dt - \int_0^{t_2} \langle \nabla \partial_t S_{\alpha,\eta}^f h, e \nabla S_{\alpha,\eta}^f h \rangle dt + \text{boundary},
\]
where we have integrated by parts in $t$, and where the boundary term is dominated by
\[
\frac{C_0}{r^\alpha} \left( \sup_{\tau > \tau_0} \|\nabla \partial_t S_{\alpha,\eta}^f h\|_2 \right) \left( \sup_{\tau > \tau_0} \|\nabla S_{\alpha,\eta}^f h\|_2 \right) \leq C_0 \sup_{\tau > \tau_0} \|\nabla S_{\alpha,\eta}^f h\|_2,
\]
as desired. Here, the last inequality follows from Lemma 2.10 (ii). Moreover, by Cauchy-Schwarz, the middle term on the right hand side of (7.8) is no larger than
\[
C_0 \|\nabla \partial_t S_{\alpha,\eta}^f h\| \cdot \|e \nabla S_{\alpha,\eta}^f h\| \leq C_0 \|\nabla \partial_t S_{\alpha,\eta}^f h\|.
\]
In the first term on the right hand side of (7.8), we integrate by parts again in $t$, to obtain
\[
\frac{1}{2} \int_0^{t_2} \langle \nabla \partial_t^2 S_{\alpha,\eta}^f h, e \nabla S_{\alpha,\eta}^f h \rangle dt + \text{Errors},
\]
where the error terms correspond to the last two terms in (7.8) and are handled in a similar fashion. Turning to the main term in (7.9), we note that
\[
\frac{1}{2} \partial_t^2 S_{\alpha,\eta}^f h = \partial_t \partial_t S_{\alpha,\eta}^f h|_{x = t}.
\]
Now set $g \equiv \partial_t^2 S_{-2}^{L_0^*} h$. Let $u$ solve
\[
\begin{cases}
L_0^* u = 0 & \text{in } \mathbb{R}_{n+1}^n \\
u(\cdot, 0) = g
\end{cases}
\]
By invertibility of the layer potentials for $L_0^*$, and by uniqueness, we have that
\[u(\cdot, -s) = D_2^L \left( \frac{1}{2} I + K_t^L \right)^{-1} g.
\]
On the other hand, we also have that $u(\cdot, -s) = \partial_t^2 S_{-s}^{L_0^*} h$. Consequently,
\[
\partial_s \nabla u(\cdot, -s) = \partial_s \nabla D_2^L \left( \frac{1}{2} I + K_t^L \right)^{-1} g = \partial_s \partial_t^2 S_{-s}^{L_0^*} h.
\]
Setting $s = t$, we have that
\[
\frac{1}{2} \nabla \partial_t^2 S_{-2}^{L_0^*} h = -D_{n+1} \nabla D_2^L \left( \frac{1}{2} I + K_t^L \right)^{-1} g = -D_{n+1} \nabla D_2^L \left( \frac{1}{2} I + K_t^L \right)^{-1} \partial_t^2 S_{-2}^{L_0^*} h.
\]
But, $D_2^L = (S_{-2}^{L_0^*} \overline{\delta_{y_0}})$, where $\overline{\delta_{y_0}}$ denotes conjugate exterior co-normal differentiation for $L_0$. Thus,
\[
\text{adj} \left( \nabla D_{n+1} D_2^L \right) = \left( \partial_n \partial_x S_0^I \nabla \right).
\]
Therefore, the main term in $(7.9)$ equals in absolute value
\[
\left| \int_{\tau_1}^{T_2} \left( \left( \frac{1}{2} I + K_t^L \right)^{-1} \partial_t^2 S_{-2}^{L_0^*} h, \left( \partial_n \partial_x D_{n+1} S_0^I \nabla \right) \cdot \epsilon \nabla S_t^{1, \eta} f \right)^2 dt \right| 
\leq C \| \nabla \partial_t^2 S_{-2}^{L_0^*} h \| \cdot \| \epsilon \nabla D_{n+1} S_0^I \nabla \| \cdot \| \epsilon \nabla S_t^{1, \eta} f \| \| \epsilon \nabla S_t^{1, \eta} f \|.
\]
To conclude the proof of Lemma 7.5, it then suffices to prove that
\[
\| \epsilon \nabla S_t^{1, \eta} f \| \leq C \epsilon \theta \mathbf{M}^r.
\]
To this end, we first prove a lemma that will allow us to reduce matters to $(6.10)$. 

**Lemma 7.11.** For $k \in \mathbb{Z}$, set $t_k \equiv 2^{k-1}$. Then
\[
(7.12) \sum_{k=-\infty}^{\infty} \int_{2^{-1}}^{2^{k+2}} \int_{\mathbb{R}^n} |\nabla S_{t_k}^{1, \eta} f(x) - \nabla S_{t_k}^{1, \eta} f(x)|^2 dxdt \leq C \| \epsilon \nabla \partial_t S_{t_k}^{1, \eta} f \|_{L^2}.
\]
Let us momentarily take the lemma for granted, and deduce $(7.10)$. Combining Lemma 2.10 (i), Lemma 2.11 and Lemma 7.11, we may replace the square of the left hand side of $(7.10)$ by
\[
\sum_{k=-\infty}^{\infty} \int_{2^{-1}}^{2^{k+1}} \int_{\mathbb{R}^n} |\epsilon^2 \left( \nabla_0 \partial_\eta S_0^I \nabla \right) \cdot \epsilon \nabla S_{t_k}^{1, \eta} f(x)|^2 dxdt \leq C \| \epsilon \nabla \partial_t S_{t_k}^{1, \eta} f \|_{L^2}.
\]
Since $u_k(\cdot, t) \equiv \left( \partial_t S_{t}^{0} \nabla \right) \cdot \epsilon \nabla S_{t_k}^{1, \eta} f$ solves $L_0 u_k = 0$ in the upper half space, we may use Caccioppoli’s inequality in Whitney boxes to reduce matters to considering
\[
\sum_{k=-\infty}^{\infty} \int_{2^{-1}}^{2^{k+1}} \int_{\mathbb{R}^n} \left| \epsilon \left( \partial_t S_{t_k}^{0} \nabla \right) \cdot \epsilon \nabla S_{t_k}^{1, \eta} f(x) \right|^2 dxdt.
\]
Applying Lemma 2.10 (i) and Lemma 7.11 again, along with (6.10), we obtain (7.10).
Proof of Lemma 7.11. The left hand side of (7.12) equals
\[
\sum_{k=0}^{\infty} \int_{2^{k-1}}^{2^{k+2}} \int_{\mathbb{R}^n} \left| \frac{1}{\sqrt{t}} \int_{t}^{t'} \nabla \partial_s S_{+}^{\eta} f(x) ds \right|^2 dx dt
\]
\[
\leq C \sum_{k=0}^{\infty} \int_{\mathbb{R}^n} \left| \frac{1}{\sqrt{t}} \int_{t}^{t'} 1_{[2^{k-1} \leq s < 2^{k+2}]} \sqrt{s} \nabla \partial_s S_{+}^{\eta} f(x) ds \right|^2 dt dx.
\]
The desired bound now follows from the Hardy-Littlewood maximal theorem. \(\square\)

This concludes the proof Lemma 7.5, and thus also that of Theorem 1.12. \(\square\)

8. Proof of Theorem 1.13: boundedness

Let \(L \equiv -\text{div } A \nabla\), where \(A\) is real, symmetric, \(L^\infty\), \(t\)-independent and uniformly elliptic. In this section, we show that the layer potentials associated to \(L\) are bounded; we defer the proof of invertibility to the next section. By the classical de Giorgi-Nash Theorem, estimates (1.2) and (1.3) hold for solutions of \(Lu = 0\). By Lemma 5.2 and Lemma 4.8, in order to establish boundedness of the layer potentials, it suffices to prove

\[(8.1) \sup_{t>0} \|\partial_t S_t f\|_2 \leq C \|f\|_2\]

and

\[(8.2) \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |\partial_t^2 S_t f|^2 dx dt \leq C \|f\|_2^2.\]

Let us observe further that (8.2) implies (8.1). Indeed, the \(L^2\) solvability results of [JK1] (cf. Theorem 8.7 below) combined with the square function estimates of [DJK], imply in particular that

\[(8.3) \sup_{t>0} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^{n+1}_t} |\nabla u(x, t)|^2 dx dt,\]

for \(u\) solving \(Lu = 0\) in \(\mathbb{R}^{n+1}_n\). Of course, there is an analogous result for the lower half-space. Applying (8.3) (and its analogue for \(t < 0\)) to the solution \(u = \partial_t S_t f\), and again invoking Lemma 5.2 (specifically (5.5)), we see that (8.2) implies (8.1).

Thus, it is enough to prove (8.2). By Lemma 2.5, the kernel
\[
\psi_t(x, y) \equiv t^2 \partial_t^2 \Gamma(x, t, y, 0)
\]
satisfies the standard Littlewood-Paley kernel conditions

\[(8.4) |\psi_t(x, y)| \leq \frac{|t|}{(|t| + |x-y|)^{n+1}},\]

\[
|\psi_t(x, y + h) - \psi_t(x, y)| \leq \frac{C|h|^\alpha}{(|t| + |x-y|)^{n+1+\alpha}} \leq \frac{C|h|^\alpha}{(|t| + |x-y|)^{n+\alpha}}
\]

for some \(\alpha > 0\), whenever \(|h| \leq \frac{1}{2}|x-y|\) or \(|h| \leq |t|/2\). The bound (8.2) will be deduced from the following “local” \(T_b\) Theorem for square functions.

Theorem 8.5. Let \(b_Q f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y)f(y)dy\), where \(\psi_t(x, y)\) satisfies (8.4). Suppose also that there exists a system \(\{b_Q\}\) of functions indexed by cubes \(Q \subseteq \mathbb{R}^n\) such that for each cube \(Q\)

\[(i) \int_Q |b_Q|^2 \leq C |Q|,\]

\[(ii) \int_0^{t_Q} \int_Q |\partial_t b_Q(x)|^2 dx dt \leq C |Q|\]
Then we have the square function bound

$$\|\theta_t f\| \leq C\|f\|_2.$$  

We omit the proof here. A direct proof of the present formulation of Theorem 8.5 may be found in [A2] or [H2], although we note that the theorem and its proof were already implicit in the proof of the Kato square root conjecture [HMc], [HLMc] and [AHLMcT]; a similar theorem for singular integrals appeared previously in [Ch]. See also [S] and [AT] for some important antecedents.

For cube $Q \subset \mathbb{R}^n$, let $x_Q$ denote its center, and let $\ell(Q)$ denote its side length. We define

$$A^+_Q \equiv (x_Q, \ell(Q)) \in \mathbb{R}^{n+1}_+, \quad A^-_Q \equiv (x_Q, -\ell(Q)) \in \mathbb{R}^{n+1}.$$  

Given $X^+ \in \mathbb{R}^{n+1}_+$, $X^- \in \mathbb{R}^{n+1}$, let $k^+_X(y)$, $k^-_X(y)$ denote, respectively, the Poisson kernels for $L$ in the upper and lower half spaces, and let $G^+(X,Y)$, $G^-(X,Y)$ denote the corresponding Green functions, so that

$$k^+_X(y) \equiv \frac{\partial G^+}{\partial y^+}(X^+, y, 0), \quad k^-_X(y) \equiv \frac{\partial G^-}{\partial y^-}(X^-, y, 0),$$

where $\frac{\partial}{\partial y^+}$, $\frac{\partial}{\partial y^-}$ denote the co-normal derivatives at the point $y \in \partial \mathbb{R}^{n+1}_+$, $\partial \mathbb{R}^{n+1}_-$ respectively.

We now set

$$b_Q \equiv |Q| k^+_Q.$$  

We recall the aforementioned result of Jerison and Kenig [JK1] (see also [K, pp 63-64]), which amounts to the solvability of (D2) in the half-spaces $\mathbb{R}^{n+1}_+$:

**Theorem 8.7.** [JK1] Suppose that $L = -\text{div } A \nabla$, where $A$ is real, symmetric, $(n+1) \times (n+1)$, $t$-independent, $L^{\infty}$ and uniformly elliptic. Then there exists $\varepsilon_1 \equiv \varepsilon_1(n, A, \Lambda)$ such that for all $0 \leq \varepsilon < \varepsilon_1$ and for every cube $Q$,

$$\int_{\mathbb{R}^n} (k^+_Q(y))^2 \, dx \leq C\varepsilon |Q|^{1-\varepsilon}.$$  

We remark that (8.8) is usually stated in terms of an integral over $Q$, but in fact the global bound follows from the local one and duality, since by [JK1], [K] the local version of (1.3) yield the estimate

$$|u(A^+_Q)| \leq C \sup_{s,t>0} |u(s,t)|_{L^p(\mathbb{R}^n)} \leq C\|g\|_{L^p(\mathbb{R}^n)},$$

where $u(x,t) = \int_{\mathbb{R}^n} k^{+t}_x(y)g(y) \, dy$, and $p$ is the dual exponent to $2 + \varepsilon$.

We now note that hypothesis (i) of Theorem 8.5 follows immediately from (8.6) and (8.8). Moreover, (iii) follows immediately from (8.6) and the following well known estimate of Caffarelli, Fabes, Mortola and Salsa [CFMS] (also [K, Lemma 1.3.2, p. 9]):

$$\omega^A_\varepsilon(Q) \geq \frac{1}{C}.$$
where $\omega^X$ denotes harmonic measure for $L$ at $X^+ \in \mathbb{R}^{n+1}$.

It remains to verify that $b_Q$ as defined in (8.6) satisfies hypothesis (ii) of Theorem 8.5. To this end, let $(x, t) \in R^+_{x} \equiv Q \times (0, t(Q))$. Then, since for fixed $(x, t) \in \mathbb{R}^{n+1}$, we have that $\partial_t^2 \Gamma(x, t, \cdot, \cdot)$ is a solution of $Lu = 0$ in $\mathbb{R}^{n+1}$, \( \theta_t b_Q(x) = |Q| \int_{\partial_t^2 \Gamma(x, t, y, 0)} k^{A_Q}(y) dy = |Q| t \partial_t^2 \Gamma(x, t, A_Q) \), by Theorem 8.7 (i.e., [JK1]) and uniqueness in (D2) (e.g., Lemma 4.31 (i), although of course, uniqueness in the present setting of real symmetric coefficients appears already in [JK1], [K]). Therefore, by (2.6) and translation invariance in $t$, we have that
\[
|\theta_t b_Q(x)| \leq C t \ell(Q),
\]
from which hypothesis (ii) follows readily. Thus, given Theorem 8.5, we conclude that \[
\int_0^\infty \int_{\mathbb{R}^n} |\partial_t^2 S_t f(x)|^2 dx dt \leq C \|f\|_2^2.
\]

The corresponding square function estimate in the lower half-space follows by the same argument, if we replace $k^{A_Q}$ by $k^{A_{-Q}}$ in the definition of $b_Q$. We then obtain (8.2) as desired.

9. Proof of Theorem 1.13: Invertibility

We now consider invertibility of the layer potentials in the case of real symmetric coefficients. The proof will follow the strategy of Verchota [V], using the well known “Rellich identities” combined with the method of continuity. In our case, the continuity argument will exploit Theorem 1.12.

Proof of Invertibility. From self-adjointness and integration by parts, we obtain the equivalence
\[
\|\partial_t u\|_{L^2(\mathbb{R}^n)} \approx \|\nabla_x u\|_{L^2(\mathbb{R}^n)},
\]
for solutions of $Lu = 0$ in $\mathbb{R}^{n+1}$ for which $\nabla_x u \in L^2$, where the implicit constants depend only upon ellipticity (see, e.g., [K] for details). In particular, (9.1) holds for $u(\cdot, t) \equiv S_t f$, with $f \in L^2$. By the jump relation formulae Lemma 4.18, (9.1) becomes
\[
\left\| \left( \pm \frac{1}{2} I + \bar{K} \right) f \right\|_2 \approx \|\nabla_x S_0 f\|_2.
\]
Thus, by the triangle inequality and (9.2) we have
\[
\|f\|_2 \leq C \left\| \left( \pm \frac{1}{2} I + \bar{K} \right) f \right\|_2
\]
and also
\[
\|f\|_2 \leq C \|\nabla_x S_0 f\|_2,
\]
where the constants in (9.3) and (9.4) depend only on ellipticity. Moreover, if we set \( L_\sigma \equiv -\text{div} \, A_\sigma \nabla, \quad 0 \leq \sigma \leq 1, \)
where
\[
A_\sigma \equiv (1 - \sigma) I + \sigma A,
\]
and $I$ denotes the $(n + 1) \times (n + 1)$ identity matrix, then (9.3) and (9.4) hold, uniformly in $\sigma$, for the layer potentials associated to $L_\sigma$; indeed, we have uniform control of the ellipticity constants for $A_\sigma$. By the result of Section 8, we of course have boundedness of the layer...
Inverting the Fourier symbol in \( \frac{1}{2} I + \tilde{K} \) are precisely the constants in (9.3) and (9.4). We may therefore establish invertibility of \( L \) upon ellipticity and dimension, since, in particular, the quantitative bounds for the inverses associated to \( L \) and dimension. Thus, once we have established invertibility of the layer potentials associated to \( \Gamma \), we observe that \( S_{\sigma} \) has Fourier symbol \( \langle F, J \rangle \), we establish invertibility of the layer potentials associated to \( L \). Consequently, \( S_{\sigma} \) is bounded and invertible, by Plancherel’s Theorem. One also readily verifies via Plancherel’s Theorem that

\[
\sup_{t \neq 0} \| \nabla S_{\sigma} \|_{\infty} \leq C, \quad \| \tilde{\partial}_t S_{\sigma} \|_{\infty} \leq C.
\]

Finally, we note that \( f \to \frac{1}{2} I + \tilde{K} \) is invertible on \( L^2 \). Indeed, the corresponding Fourier symbol is

\[
\lim_{t \to 0^+} e_{n+1} \cdot a \tilde{\nabla} \Gamma(\cdot, t)(\tilde{\xi}) = \frac{a_{n+1, n+1} \tau_+ (\tilde{\xi}) + \sum_{j=1}^{n} a_{n+1, j} \tilde{\xi}_j}{a_{n+1, n+1} (\tau_+ (\tilde{\xi}) - \tau_- (\tilde{\xi}))},
\]

and by [AQ], Lemma 4, the modulus of the numerator \( = |\tilde{\xi}| \). By the accretivity of \( a_{n+1, n+1} \) and (10.2), the same holds for the denominator, and the invertibility follows. Of course, a similar observation holds for \( -\frac{1}{2} I + \tilde{K} \).

### 10. Appendix 1: constant coefficients

Suppose that \( L = -\text{div} a \nabla \), where \( a \) is a constant complex elliptic matrix. Following [FKJ], we observe that \( L \) has Fourier symbol

\[
q(i\xi, it) = \sum_{j=1}^{n+1} a_{j,k} \xi_j \xi_k = a_{n+1, n+1} (\tau_+ (\xi) - \tau_- (\xi)),
\]

where \( \xi_{n+1} \equiv \tau \), and \( \tau_\pm : \mathbb{R}^n \to \mathbb{C} \) are each homogeneous of degree 1, \( C^\infty (S^{n-1}) \), with

\[
\Im \tau_+ (\xi) \geq \mu, \quad \Im \tau_- (\xi) \leq -\mu,
\]

for some \( \mu > 0 \). In particular,

\[
|\tau_+ (\xi) - \tau_- (\xi)| \approx |\xi|, \quad \xi \in \mathbb{R}^n.
\]

The fundamental solution \( \Gamma(x, \tau) \) is a convolution kernel with Fourier symbol \( q(i\xi, it)^{-1} \). Inverting the Fourier symbol in \( t \) only, and then using the method of residues, we obtain

\[
\tilde{\Gamma}(\cdot, t)(\tilde{\xi}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\tau} \frac{q(i\xi, it)}{|q(i\xi, it)|} d\tau = -\frac{e^{it\tau_+ (\xi)} 1_{(t > 0)} + e^{it\tau_- (\xi)} 1_{(t < 0)}}{ia_{n+1, n+1} (\tau_+ (\xi) - \tau_- (\xi))},
\]

so by (10.2) and the accretivity of \( a_{n+1, n+1} \), we have in particular that

\[
|\tilde{\Gamma}(\cdot, 0)(\tilde{\xi})| \approx |\xi|^{-1}.
\]

Consequently, \( S_0 : L^2 \to L^2 \) is bounded and invertible, by Plancherel’s Theorem. One also readily verifies via Plancherel’s Theorem that

\[
\sup_{t \neq 0} \| \nabla S_{\sigma} \|_{\infty} \leq C, \quad \| \tilde{\partial}_t S_{\sigma} \|_{\infty} \leq C.
\]
11. Appendix 2: $t$-independence implies De Giorgi/Nash in 3 dimensions

In this section we show that for $t$-independent coefficients, the De Giorgi/Nash bound (1.2) holds in ambient dimension $n + 1 = 3$. It is enough to establish Hölder continuity in the horizontal (i.e. $x$) variable. To this end, we first recall the Morrey/M. Weiss inequality:

\[
|f(x) - f(y)| \leq C_p |x - y|^{1-n/p} \left( \int_{|z - y| < |x - y|} |\nabla f(z)|^p \right)^{1/p},
\]

which is valid in $\mathbb{R}^n$ for every $p > n$ (see, e.g., [C], Lemma 1.4). We now apply this inequality with $n = 2$, $p \in (2, 2 + \epsilon)$, and $f(x) := u(x, t)$ for fixed $t > 8|x - y|$, where $u$ is a solution of $Lu := \nabla \cdot (A(x)\nabla u) = 0$ in $\mathbb{R}^{n+1}$, and where $\epsilon > 0$ is chosen so that Proposition 2.1 holds for $p$ in the stated range. Under this scenario, (11.1) yields

\[
|u(x, t) - u(y, t)| \leq C |x - y|^{1-2/p} \left( \int_{|z - y| < |x - y|/4} |\nabla u(z, t)|^p \right)^{1/p}.
\]

Combining the latter inequality with (2.3), we obtain

\[
|u(x, t) - u(y, t)| \leq C \left( \frac{|x - y|}{t} \right) \left( \int_{|z - y| < |x - y|/2} |u(z, t)|^p \right)^{1/2},
\]

with $\alpha := 1 - 2/p > 0$.

References


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