**Lines:**

Describe the line through a given point \( P_0 (x_0, y_0, z_0) \) and parallel to a given vector \( \vec{V} = \langle a, b, c \rangle \).

\[ \vec{V} = \langle a, b, c \rangle \]

A point \( P(x, y, z) \) lies on this line:

\[ \vec{P_0P} \parallel \vec{V} \]

\[ \vec{P_0P} = t \vec{V} \] for some scalar \( t \)

\[ \vec{OP} - \vec{OP_0} = t \vec{V} \] (origin)

\[ \vec{OP} = \vec{OP_0} + t \vec{V} \]

This is the vector equation of the line through \( P_0 \) parallel to \( \vec{V} \).

Now, \( P = (x, y, z) \)

\[ P_0 = (x_0, y_0, z_0) \]

\[ \vec{V} = \langle a, b, c \rangle \]

Thus, the vector equation means:

\[ \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \]

\[ = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle \]

\[ \iff \]

\[ x = x_0 + ta \]

\[ y = y_0 + tb \]

\[ z = z_0 + tc \]

\[ -\infty < t < \infty \]

These are called the parametric equations of the line through point \((x_0, y_0, z_0)\), parallel to \( \langle a, b, c \rangle \).

**RK:** When \( a, b, c \) are all \( \neq 0 \), the parametric equations are equivalent to:

\[ \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \]

These are the symmetric equations of the line through point \((x_0, y_0, z_0)\) parallel to vector \( \langle a, b, c \rangle \).

**Planes:**

Describe the plane through a given point \( P_0 (x_0, y_0, z_0) \) and perpendicular to a given vector \( \vec{N} = \langle a, b, c \rangle \).
Point \( \vec{P} \) lies on the plane

\[ \iff \quad \vec{P}_0 \cdot \vec{n} = 0 \]

[Diagram showing a plane with a point \( \vec{P}_0 \) and a normal vector \( \vec{n} \).]

Thus, the vector equation of the plane containing \( \vec{P}_0 \) is

\[ \vec{r} \cdot \vec{n} = \vec{r}_0 \cdot \vec{n} \]

This is the vector equation of the plane.

Ex. 1. Find the parametric eqs of the line through \( (1,2,-1) \) parallel to \(-2 \vec{i} + 6 \vec{j} + 5 \vec{k}\).

Solution:

\[ \vec{P}_0 = (1,2,-1) \]
\[ \vec{V} = (-2,6,5) \]

Thus, the parametric eq of this line are:

\[
\begin{align*}
  x &= 1 - 2t \\
  y &= 2 + 6t \\
  z &= -1 + 5t \\
\end{align*}
\]

Ex. 2. Parametrize the line segment from \( A (-1,1,7) \) to \( B (2,4,-3) \).

Solution:

First, parametrize \( AB \):

\[ \vec{P}_0 = A = (-1,1,7) \]
\[ \vec{V} = \vec{AB} = (3,3,-10) \]

Thus,

\[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]

This is the scalar equation of the plane. It can be rewritten as:

\[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]
Ex. 4 Find the distance from \((0, -1, 1)\) to the line \(x = 1 + t, \ y = -2t, \ z = 3 + 5t, \ -\infty < t < \infty\).

**Sol. 4**

\(S = (0, -1, 1)\)

\(P_0 = (1, 2, 3)\)

\(\vec{v} = <1, -2, 5>\)

\(\vec{P_0S} = <-1, -3, -2>\)

\[\vec{P_0S} \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ 1 & -3 & -2 \\ 1 & -2 & 5 \end{vmatrix} = 10 \vec{e}_x - 17 \vec{e}_y + 5 \vec{e}_z\]

\[|\vec{P_0S} \times \vec{v}| = \sqrt{10^2 + 17^2 + 5^2} = \sqrt{395}\]

\[|\vec{v}| = \sqrt{1^2 + (-2)^2 + 5^2} = \sqrt{30}\]

Thus: \(d = \frac{\sqrt{395}}{\sqrt{30}}\).

Ex. 5 Find the scalar equation of the plane containing \((0, 0, \frac{1}{2})\) that is normal to \(2\vec{i} + 3\vec{j} - \vec{k}\).

**Sol. 5**

\(P_0 = (0, 0, \frac{1}{2})\)

\(\vec{n} = <2, 3, -1>\)

The scalar eq. of this plane is

\[2(x - 0) + 3(y - 0) + (-1)(z - \frac{1}{2}) = 0\]

\[2x + 3y - z + \frac{1}{2} = 0\]
Ex. 6 Find the equation of the plane containing \((1,2,3), (2,3,5),\) and \((1,0,0).\)

**Sol. 6.**

\[
\vec{AB} \times \vec{AC} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
0 & 2 & 3 \\
1 & 3 & 5 \\
\end{vmatrix}
\]

\[
= \hat{i}(15 - 9) - \hat{j}(0 - 3) + \hat{k}(0 - 2)
\]

\[
= <1, 3, 3> = \vec{n}
\]

Thus, the scalar eq. of the plane is:

\[
1.(x-1) + 3(y-2) - 2(z-3) = 0
\]

\[
x + 3y - 2z - 1 = 0
\]

Ex. 7. Where does the line \(x = 2 - 2t, y = 1 + 3t, z = -1 + t\) \((-\infty < t < \infty)\) intersect the plane \(3x + 2y + z - 10 = 0?\)

**Sol. 7**

\[
3(2-2t) + 2(1+3t) + (-1+t) - 10 = 0
\]

\[
6 - 6t + 2 + 6t - 1 + t - 10 = 0
\]

\[
t = 3
\]

\[
\Rightarrow \quad x = 2 - 6 = -4
\]

\[
y = 1 + 9 = 10
\]

\[
z = -1 + 3 = 2
\]

The line \(l\) and the plane intersect at point \((-4, 10, 2).\)

Ex. 8. Determine the distance from \((1,1,1)\) to the plane \(3x + 2y + z - 10 = 0.\)

**Sol. 8**

The distance formula for a point \(P_0\) to a plane \(\vec{n} \cdot \vec{r} = D\) is:

\[
d = \frac{|\vec{n} \cdot \vec{r}_0 - D|}{|\vec{n}|}
\]

Given \(A = (1,1,1),\) \(\vec{n} = <3,3,1>,\) and the plane \(3x + 2y + z - 10 = 0,\)

\[
d = \frac{|<3,3,1> \cdot <1,1,1> - 10|}{\sqrt{3^2 + 3^2 + 1^2}}
\]

\[
d = \frac{|3 + 3 + 1 - 10|}{\sqrt{25}}
\]

\[
d = \frac{1}{5}
\]

\[
\boxed{\text{Distance} = \frac{1}{5}}
\]
Ex. 9: Determine whether the lines $L_1$ and $L_2$ are parallel.

The distance $d$ between two skew lines $L_1$ and $L_2$ is given by:

$$d = \frac{1}{\left| \mathbf{n} \cdot \mathbf{s} \right|},$$

where $\mathbf{n}$ is the direction vector of the lines and $\mathbf{s}$ is a vector between the lines.

Given:

- $L_1: x = 1 + t, y = 3 + 2t, z = -1 + 3t$
- $L_2: x = 2 - s, y = 1 - s, z = 3 + 2s$

Find the vector $\mathbf{s}$ between the lines:

$$\mathbf{s} = \mathbf{r}_2 - \mathbf{r}_1 = (2 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (3 + 1 - 3)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 0\mathbf{k}.$$

The direction vector $\mathbf{n}$ is the cross product of the direction vectors of $L_1$ and $L_2$:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2,$$

where $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

Calculating $\mathbf{n}$:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 2 & -1 & 0 \end{vmatrix} = \begin{pmatrix} 6 \\ 6 \\ -1 \end{pmatrix}.$$

Now, calculate $d$:

$$d = \frac{1}{\left| \begin{pmatrix} 6 \\ 6 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \right|} = \frac{1}{\sqrt{6^2 + 6^2 + (-1)^2}} = \frac{1}{\sqrt{36 + 36 + 1}} = \frac{1}{\sqrt{73}}.$$
\[10x + 2y - 2z = 5; \quad \vec{n}_1 = \langle 10, 2, -2 \rangle \]
\[5x + y - 2 = 1; \quad \vec{n}_2 = \langle 5, 1, -1 \rangle \]
\[\vec{n}_1 \parallel \vec{n}_2 \quad (\vec{n}_1 = 2 \vec{n}_2) \]

- Pick a point \( P_1 \) on plane.
  For example:
  \[ P_1 = (\frac{1}{2}, 10, 0) \]
- Consider line \( L \) through \( P_1 \), and perpendicular to plane.
  (\( \vec{n}_1 \) is then parallel to \( L \).) Its parametric eqs. are:
  \[
  \begin{align*}
  x &= \frac{1}{2} + 10t \\
  y &= 2t \\
  z &= -2t \\
  -\infty < t < \infty
  \end{align*}
  \]
- \( P_2 \) is the point where line \( L \) intersects plane \( \beta \). Its coordinates satisfy:
  \[5 \left( \frac{1}{2} + 10t \right) + 2t - (-2t) = 1\]

\[\frac{5}{2} + 50t + 2t + 2t = 1\]
\[54t = 1 - \frac{5}{2} = -\frac{3}{2}\]
\[t = -\frac{3}{2}, \quad \frac{1}{54} = \frac{1}{36} \]

\[x_B = \frac{1}{2} + 10 \cdot \frac{1}{36} = \frac{2}{9}\]
\[y_B = 2 \cdot \frac{1}{36} = \frac{1}{18}\]
\[z_B = -2 \cdot \frac{1}{36} = \frac{1}{18}\]
\[B = \left( \frac{2}{9}, \frac{1}{18}, \frac{1}{18} \right) \]

**Conclusion:**
Distance \( AB \)
\[= \sqrt{\left( \frac{2}{9} - \frac{1}{2} \right)^2 + \left( \frac{1}{18} \right)^2 + \left( \frac{1}{18} \right)^2} \]
\[= \sqrt{\left( -\frac{5}{18} \right)^2 + \left( \frac{1}{18} \right)^2 + \left( \frac{1}{18} \right)^2} \]
\[= \frac{\sqrt{3}}{6} \]

**Ex. 11**
Find the angle between the planes
\[x - 2y + 2 = 7\]
\[x + y + 2 = 8\]

**Sol. 11**

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11

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12
\[ x - 2y + z = 7 : \mathbf{n}_1 = \langle 1, -2, 1 \rangle \]
\[ x + y + z = 8 : \mathbf{n}_2 = \langle 1, 1, 1 \rangle \]

Note that \( \theta \) is exactly the angle between \( \mathbf{n}_1 \) & \( \mathbf{n}_2 \).

\[ \mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1| \, |\mathbf{n}_2| \, \cos \theta \]

\[ \Rightarrow \quad \cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} \]

\[ \mathbf{n}_1 \cdot \mathbf{n}_2 = 1 \cdot 1 + (-2) \cdot 1 + 1 \cdot 1 = 0 \]

\[ \Rightarrow \quad \cos \theta = 0 \]

\[ \theta = \cos^{-1} (0) = \frac{\pi}{2} \]