Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets

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Abstract

This work deals with various questions concerning Fourier multipliers on L^p, Schur multipliers on the Schatten class S^p as well as their completely bounded versions when L^p and S^p are viewed as operator spaces. We use for this aim subsets of $\mathbb{Z}$ enjoying the non-commutative $\Lambda(p)$-property which is a new analytic property much stronger than the classical $\Lambda(p)$-property. We start by studying the notion of non-commutative $\Lambda(p)$-sets in the general case of an arbitrary discrete group before turning to the group $\mathbb{Z}$.

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Contents

0 Introduction, background and notation

$M(L^p)$ stands for the algebra of Fourier multipliers on the space L^p, and $M_{cb}(L^p)$ for the algebra of Fourier multipliers which are completely bounded on L^p when the latter is endowed with its natural operator space structure. $M(S^p)$ denotes the algebra of Schur multipliers on the Schatten class S^p, and $M_{cb}(S^p)$ the algebra of all Schur multipliers which are completely bounded on S^p when the latter is equipped with its natural operator space structure. Our first motivation was to show that the following contractive inclusion maps

$$M_{cb}(L^q) \subset M_{cb}(L^p), \ M(S^q) \subset M(S^p),$$
$$\Big(M(L^\infty), M(L^2)\Big)_\theta \subset M_{cb}(L^p), \ \Big(M(S^\infty), M(S^2)\Big)_\theta \subset M_{cb}(S^p)$$

where in the three first inclusions p is an even integer and $2 < p < q \leq \infty$ while in the two last ones $0 < \theta < 1$ is arbitrary and $p = \frac{2}{\theta}$ are all strict. The reader should note that the embeddings in which we are interested as well as the isomorphisms we consider in this work are the natural ones meaning those which send a given element simply to itself. For this aim we introduce and study a non-commutative version of the usual $\Lambda(p)$-sets. The idea behind all the proofs is the existence for each even integer $2 < p < \infty$ of a non-commutative $\Lambda(p)$-set which is not a $\Lambda(q)$-set for any $q > p$.

For the rest of this section, we recall all the facts we need along with the notations we use in the sequel.

Section 1 is devoted to the study of the non-commutative $\Lambda(p)$-property in an arbitrary discrete group G. This is a new analytic property more restrictive in general than the classical $\Lambda(p)$-property. We start the section by recalling the definition of $\Lambda(p)$-sets and we point out their relationship with the set $M\big(L^p(\tau_0)\big)$ of all Fourier multipliers on $L^p(\tau_0)$. Here the space $L^p(\tau_0)$ denotes the non-commutative L^p-space associated to the discrete group G equipped with its usual trace τ_0. Then we introduce the non-commutative $\Lambda(p)$-sets. We point out their relationship with the set $M_{cb}\big(L^p(\tau_0)\big)$ of all completely bounded Fourier multipliers on $L^p(\tau_0)$ when the latter is endowed with its natural operator space structure. This justifies the terminology "$\Lambda(p)_{cb}$-sets" we use for "non-commutative $\Lambda(p)$-sets". The links between $\Lambda(p)$-sets and the algebra $M\big(L^p(\tau_0)\big)$ on one hand and between $\Lambda(p)_{cb}$-sets and $M_{cb}\big(L^p(\tau_0)\big)$ on the other are proved by using the non-commutative version of Khintchine inequalities proved in [26] (see also [27]). Then we consider for integers p two combinatorial properties defined on subsets of G namely the $B(p)$-property and the $Z(p)$-property. We show that the $B(p)$-property ensures the $Z(p)$-property and that the $Z(p)$-property ensures the $\Lambda(2p)_{cb}$-property; the latter result is the crucial point of this work.

In Section 2, we consider the $\Lambda(p)_{cb}$-property in the particular case of the group $\mathbb{Z}$. We prove that this property is very different from the usual $\Lambda(p)$-property. More precisely, we prove that there exists a set which is $\Lambda(p)$ for each $2 < p < \infty$ but not $\Lambda(p)_{cb}$ for any $2 < p < \infty$. Then we show that for each even integer $p > 2$ there exists a $\Lambda(p)_{cb}$-set which is not a $\Lambda(q)$-set for any $q > p$; this kind of particular sets will play the key rôle in the proofs of the results announced in the following sections.

In section 3, we focus on Fourier multipliers. We prove that given $2 \leq p < \infty$ an even integer, $M_{cb}(L^p)$ cannot embed continuously into $M(L^q)$ for any $p < q \leq \infty$. Recall that for $p = \frac{2}{\theta}$ and $0 < \theta < 1$, the embedding of $\Big(M(L^\infty), M(L^2)\Big)_\theta$ into $M(L^p)$ is strict (see [45], see also [40]). Then since as we will recall the embedding of $M_{cb}(L^p)$ into $M(L^p)$ is strict for any $2 < p < \infty$, it is natural to wonder whether the embedding of $\Big(M(L^\infty), M(L^2)\Big)_\theta$ into $M_{cb}(L^p)$ is again strict. We prove that this is still the case. More precisely, we show that $M_{cb}(L^p)$ does not embed continuously into the interpolated space $\Big(M(L^\infty), M(L^2)\Big)_\theta$ for any $0 < \theta < 1$.

In Section 4, we introduce and study the so-called $\sigma(p)$-sets and $\sigma(p)_{cb}$-sets. These are subsets of $\mathbb{N} \times \mathbb{N}$ playing for $M(S^p)$ and $M_{cb}(S^p)$ a rôle analogous to the one played by $\Lambda(p)$-sets and $\Lambda(p)_{cb}$-sets for $M(L^p)$ and $M_{cb}(L^p)$ respectively. We will see that from any given $\Lambda(p)_{cb}$-set, we can obtain a $\sigma(p)_{cb}$-set and thus we get for even integers p special $\sigma(p)_{cb}$-sets. Indeed, we prove that for any even integer $p > 2$, there is a $\sigma(p)_{cb}$-set $A \subset \mathbb{N} \times \mathbb{N}$ which is not a $\sigma(q)$-set for any $q > p$.

Section 5 is devoted to Schur multipliers. For each even integer $2 < p < \infty$, we prove the existence of an idempotent Schur multiplier which is completely bounded on S^p but not bounded on S^q for any $p < q \leq \infty$. In fact, our idempotent Schur multiplier is not even bounded on the subspace of S^q formed of all Hankelian operators denoted $\mathfrak{S}^q$ in the sequel. This answers a question raised by J. Erdos. Therefore, the embeddings $M(S^q) \subset M(S^p)$ and $M_{cb}(S^q) \subset M_{cb}(S^p)$ are strict whenever $2 \leq p < q \leq \infty$ and p is an even integer. On the other hand, we show that for each given $2 < p < \infty$, the set $M_{cb}(S^p)$ does not embed continuously into the interpolated space $\Big(M(S^\infty), M(S^2)\Big)_\theta$ for any $0 < \theta < 1$. This answers a question raised by V. Peller. We will take the opportunity in this section to establish links between Fourier and Schur multipliers as follows. Let $M(H^p)$ be the algebra of Fourier multipliers on the Hardy space H^p, $M_{cb}(H^p)$ be the algebra of completely bounded Fourier multipliers on H^p, $M(\mathfrak{S}^p)$ be the algebra of Schur multipliers on $\mathfrak{S}^p$ and $M_{cb}(\mathfrak{S}^p)$ be the algebra of completely bounded Schur multipliers on $\mathfrak{S}^p$. The spaces H^p and $\mathfrak{S}^p$ are viewed as operator subspaces of L^p and S^p respectively. We show that $M(H^p)$ can be injected continuously into $M(\mathfrak{S}^p)$ in the same way $M_{cb}(H^p)$ is injected into $M_{cb}(S^p)$. For this purpose, we are led to characterize the multipliers of $M(\mathfrak{S}^p)$ and $M_{cb}(\mathfrak{S}^p)$ (our characterizations are easy consequences of [29], [30]).

Section 6 is included for the sake of completeness. Using probabilistic ideas, we exhibit a very "large" $Z(2)$-set roughly the "largest" possible one which enjoys some additional properties. On the other hand, we introduce on the subsets of $\mathbb{N} \times \mathbb{N}$ some simple combinatorial properties ensuring the $\sigma(4)_{cb}$-one, which we call property (C) and property (R). Then by using similar probabilistic ideas, we exhibit "large" sets satisfying one of these combinatorial properties.

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We now review the standard notation that we use. Let E and F be two Banach spaces. By $E\otimes F$, we denote the algebraic tensor product of E and F. By $\mathcal{B}(E,F)$ we denote the set of all bounded operators from E to F. $\mathcal{B}(E,E)$ is simply denoted $\mathcal{B}(E)$. B_E stands for the open unit ball of E. id_E denotes the identity map on E. If E_i, F_i are Banach spaces and u_i is in $\mathcal{B}(E_i,F_i)$ for $i=0,1$ then $u_0\otimes u_1$ denotes the operator which carries $x\otimes y$ in $E_0\otimes E_1$ to $u_0(x)\otimes u_1(y)$ in $F_0\otimes F_1$ extended linearly.
A contractive map $u: E\longrightarrow F$ is said to be μ-surjective if the set $u(\mu B_E)$ contains B_F. 1-surjective maps are called metric surjections.
For $1\leq p\leq\infty$, (Ω,ν) a measurable space and E an arbitrary Banach space, we let $L^p(\Omega,d\nu,E)$ be the set of all E-valued functions f on Ω which are Bochner measurable and such that

$$\big\|f\big\|_{L^p(\Omega,d\nu,E)} := \Big(\int_\Omega \big\|f(t)\big\|^p d\nu\Big)^{\frac{1}{p}} < \infty\cdot$$

When Ω is the torus $\mathbb{T}$ and ν is the normalized Lebesgue measure, the space $L^p(\mathbb{T},d\nu,E)$ is simply denoted by $L^p(E)$ and we let $H^p(E)$ be the E-valued Hardy space namely this is the set of all f in $L^p(E)$ such that the Fourier coefficients $\widehat{f}(n)=0$ for all integers $n<0$. $L^p(\mathbb{C})$ and $H^p(\mathbb{C})$ are simply denoted L^p and H^p respectively.
More generally, let M be a von Neumann algebra given with a normal, faithful and semi-finite trace $\tau_{_M}$. For $1\leq p<\infty$, $L^p(\tau_{_M})$ denotes the non-commutative L^p-space associated to M equipped with $\tau_{_M}$. By definition, this is the Banach space obtained from the space of all x in M satisfying $\big\|x\big\|_{L^p(\tau_{_M})} := \tau_{_M}\big((x^*x)^{\frac{p}{2}}\big)^{\frac{1}{p}} < \infty$ after completion with respect to the norm $\big\|\,.\,\big\|_{L^p(\tau_{_M})}$ (*cf.* [16], [28], [38]). By convention, $L^\infty(\tau_{_M})$ denotes M. The non-commutative L^p-space associated to $\mathcal{B}(H)$ where H denotes a separable Hilbert space, equipped with its usual trace is nothing but the p-Schatten class on H. It will be denoted by $S^p(H)$ when $1\leq p<\infty$. $S^\infty(H)$ stands for the set of all compact operators on H. In the particular case $H=\ell_2$ (resp. ℓ_2^n the n-dimensional Hilbert space), the usual trace on $\mathcal{B}(\ell_2)$ (resp. $M_n:=\mathcal{B}(\ell_2^n)$) is denoted tr (resp. $tr_{_n}$) and the space $S^p(H)$ is simply denoted S^p (resp. S_n^p) for each $1\leq p\leq\infty$.
If $\tau_{_M}$ and $\tau_{_N}$ are normal, faithful and semi-finite traces given on the von Neumann algebras M and N respectively, then we let $\tau_{_M}\otimes\tau_{_N}$ denote the trace on the von Neumann algebra generated by $M\otimes N$ defined as follows. $\tau_{_M}\otimes\tau_{_N}(x\otimes y):=\tau_{_M}(x)\tau_{_N}(y),\ \forall x\in M, y\in N$. $\tau_{_M}\otimes\tau_{_N}$ is still normal, faithful and semi-finite thus we can consider unambiguously the space $L^p(\tau_{_M}\otimes\tau_{_N})$.
Given a discrete group G, λ denotes the left regular representation of G into $\mathcal{B}\big(l_2(G)\big)$, $L^p(\tau_0)$ denotes the non-commutative L^p-space associated to the von Neumann algebra generated by $\lambda(G)$ with respect to its usual trace denoted τ_0, and $L^p(\tau)$ denotes the non-commutative L^p-space associated to the von Neumann algebra generated by $\lambda(G)\otimes\mathcal{B}(l_2)$ with respect to the trace $\tau=\tau_0\otimes tr$.
Given M and $\tau_{_M}$ as above, the spaces $L^\infty(\tau_{_M}\otimes tr)$ and $L^1(\tau_{_M}\otimes tr)$ form a compatible couple of complex interpolation for which we have isometrically (*cf.* [20])

$$\forall 1<p<\infty,\quad L^p(\tau_{_M}\otimes tr)=\big(L^\infty(\tau_{_M}\otimes tr),L^1(\tau_{_M}\otimes tr)\big)_{\frac{1}{p}}\cdot$$

This allows us to see the Banach spaces $L^p(\tau_{_M})$ in the recent view point of operator spaces in a natural way (*cf.* [33], [34]).

0.1 Complex interpolation

Let (E_0, E_1) be a pair of compatible Banach spaces *i.e.* E_0 and E_1 are both continuously injected into the same topological space. Let

$$\begin{aligned} \Delta &:= \{z \in \mathbb{C} \mid 0 \leq \mathcal{R}e(z) \leq 1\} \\ \Delta_j &:= \{z \in \mathbb{C} \mid \mathcal{R}e(z) = j\} \end{aligned}$$

for $j = 0, 1$. Then let $\mathcal{G}(E_0, E_1)$ be the set of all functions f of the form $f = \sum_{finite} f_k x_k$ where the x_k's are in $E_0 \cap E_1$, the functions $f_k : \Delta \longrightarrow \mathbb{C}$ are continuous on Δ analytic on its interior and vanishing at infinity. Denote by $\mathcal{F}(E_0, E_1)$ the completion of $\mathcal{G}(E_0, E_1)$ for the norm

$$\|f\| := \max\left\{ \sup_{z \in \Delta_0} \left\| f(z) \right\|_{E_0}, \sup_{z \in \Delta_1} \left\| f(z) \right\|_{E_1} \right\}.$$

For $0 < \theta < 1$, consider the subset $\mathcal{N}_\theta(E_0, E_1)$ of $\mathcal{G}(E_0, E_1)$ of all functions which vanish on θ and let $\mathcal{S}_\theta(E_0, E_1)$ be its closure in $\mathcal{F}(E_0, E_1)$. By definition, the intermediate space E_θ obtained by complex interpolation between E_0 and E_1 corresponding to the value θ is the Banach space $\mathcal{F}(E_0, E_1)/\mathcal{S}_\theta(E_0, E_1)$ equipped with the quotient norm denoted $\|.\|_\theta$. We refer the reader to [39] to make sure that the definition we chose for the complex interpolation coincides with the one given in [2].

Lemma 0.1 *Let (E_0, E_1) and (F_0, F_1) be two compatible couples of interpolation such that $E_0 \cap E_1$ is dense in both E_0 and E_1. Then the space $\Big(\mathcal{B}(E_0, F_0), \mathcal{B}(E_1, F_1)\Big)_\theta$ embeds contractively into $\mathcal{B}(E_\theta, F_\theta)$ for each $0 < \theta < 1$.*

Proof: Let E be the completion of $E_0 \cap E_1$ for the norm $\| x \|_E = \max\Big\{ \| x \|_{E_0}, \| x \|_{E_1} \Big\}$ and F be any Banach space containing continuously F_0 and F_1. The assumption on the pair (E_0, E_1) permits to inject continuously both $\mathcal{B}(E_0, F_0)$ and $\mathcal{B}(E_1, F_1)$ into $\mathcal{B}(E, F)$. Thus they form a compatible couple of interpolation.
By density of $\mathcal{B}(E_0, F_0) \cap \mathcal{B}(E_1, F_1)$ in $\Big(\mathcal{B}(E_0, F_0), \mathcal{B}(E_1, F_1)\Big)_\theta$ and $E_0 \cap E_1$ in E_θ, we are reduced to show that $\left\|Tx\right\|_\theta \leq \left\|T\right\|_\theta \left\|x\right\|_\theta$ for each T in $\mathcal{B}(E_0, F_0) \cap \mathcal{B}(E_1, F_1)$ and each x in $E_0 \cap E_1$. To check this, let φ be in $\mathcal{G}\Big(\mathcal{B}(E_0, F_0), \mathcal{B}(E_1, F_1)\Big)$, f in $\mathcal{G}(E_0, E_1)$ such that $\varphi(\theta) = T$ and $f(\theta) = x$ and consider the function g which takes z in Δ to $\varphi(z)\big(f(z)\big)$ in $F_0 \cap F_1$. Clearly g belongs to $\mathcal{G}(F_0, F_1)$ with $g(\theta) = Tx$. Moreover, its norm satisfies

$$\begin{aligned} \left\|g\right\| &= \max_{j=0,1} \sup_{z \in \Delta_j} \left\{ \left\| \varphi(z)\big(f(z)\big) \right\|_{F_j} \right\} \leq \max_{j=0,1} \sup_{z \in \Delta_j} \left\{ \left\| \varphi(z) \right\|_{\mathcal{B}(E_j, F_j)} \left\| f(z) \right\|_{E_j} \right\} \\ \left\|g\right\| &\leq \left(\max_{j=0,1} \sup_{z \in \Delta_j} \left\| \varphi(z) \right\|_{\mathcal{B}(E_j, F_j)} \right) \left(\max_{j=0,1} \sup_{z \in \Delta_j} \left\| f(z) \right\|_{E_j} \right) = \|\varphi\| \; \|f\| \cdot \end{aligned}$$

This gives the required inequality after taking the infimum over all such φ's and f's. ∎

0.2 Operator spaces

Concretely, by an operator space, we mean a closed subspace of $\mathcal{B}(H)$ for some Hilbert space H. Such an object has natural norms $\|.\|_n$ on $M_n(E)$ the set of $n \times n$ matrices with entries in E. Indeed, $M_n(E)$ can be viewed as a subspace of $\mathcal{B}(\ell_2^n(H))$ via the natural identification between $M_n(\mathcal{B}(H))$ and $\mathcal{B}(\ell_2^n(H))$. This sequence of norms satisfy Ruan's axioms, that is

$$\forall a, b \in M_n, \forall x \in M_n(E) \text{ we have } \big\|a \cdot x \cdot b\big\|_n \leq \big\|a\big\|_{M_n} \big\|x\big\|_n \big\|b\big\|_{M_n}$$
$$\forall x \in M_n(E), \forall y \in M_m(E) \text{ we have } \big\|x \oplus y\big\|_{n+m} = \max\Big\{ \big\|x\big\|_n, \big\|y\big\|_m \Big\}.$$

Here the norm on M_n is the operator norm, the $\oplus$ denotes the direct sum of matrix and the dot denotes the matrix product following the usual rules of calculation.
In the operator settings, a map $u : E \longrightarrow F$ is said to be *c.b.* — short for completely bounded — if the maps

$$\begin{array}{rcl} u^n : M_n(E) & \longrightarrow & M_n(F) \\ (x_{ij})_{i,j} & \longmapsto & \Big(u(x_{ij})\Big)_{i,j} \end{array}$$

are uniformly bounded. We let $\mathcal{CB}(E, F)$ stand for the space of all *c.b.* maps endowed with the norm

$$\big\|u\big\|_{cb} = \sup_n \big\|u^n\big\|.$$

$\mathcal{CB}(E)$ will stand for $\mathcal{CB}(E, E)$. An operator u is said to be a complete contraction (resp. isometry) if each map u^n is contractive (resp. isometric).
Z. Ruan gave an abstract characterization of an operator space as a Banach space given with a sequence of norms on the $M_n(E)$'s which satisfy Ruan's axioms (see [36]). This abstract characterization allows to define for operator spaces the notion of duality, complex interpolation...
The standard dual of an operator space E is the usual Banach space E^* with the norms corresponding to the isometric identifications of $M_n(E^*)$ with $\mathcal{CB}(E, M_n)$ as in [4] and [17].
The complex interpolated space between two operator spaces E_0 and E_1 compatible as Banach spaces is the usual Banach space E_θ with the norms corresponding to the isometric identifications $M_n(E_\theta) := \Big(M_n(E_0), M_n(E_1)\Big)_\theta$ (see [33]).
When E and F are two operator spaces, $\mathcal{CB}(E, F)$ is an operator space for the structure corresponding to the isometric identifications $M_n\Big(\mathcal{CB}(E, F)\Big) := \mathcal{CB}\Big(E, M_n(F)\Big)$.
Note that the min. — short for minimal — tensor product is a very useful tool to describe entirely the operator space structure of an operator space as well as the *c.b.* maps between operator spaces. Let $E \subset \mathcal{B}(H)$ be a concrete operator space. By $S^\infty \otimes_{\min} E$, we mean the completion of $S^\infty \otimes E$ for the norm induced by $\mathcal{B}(\ell_2(H))$. Then the operator space structure of the interpolated space E_θ and the one of the operator space dual E^* are entirely described by the following isometric relations:

$$S^\infty \otimes_{\min} E^* \subset \mathcal{CB}(E, S^\infty) S^\infty \otimes_{\min} E_\theta = \big(S^\infty \otimes_{\min} E_0, S^\infty \otimes_{\min} E_1\big)_\theta.$$

A map $u : E \longrightarrow F$ is *c.b.* if and only if $id_{S^\infty} \otimes u$ extends to a bounded operator from $S^\infty \otimes_{\min} E$ into $S^\infty \otimes_{\min} F$ and we have $\big\|u\big\|_{cb} = \big\|id_{S^\infty} \otimes u : S^\infty \otimes_{\min} E \longrightarrow S^\infty \otimes_{\min} F\big\|$.

Lemma 0.2 *Let (E_0, E_1) and (F_0, F_1) be two compatible couples of interpolation. Assume that $E_0 \cap E_1$ is dense in both E_0 and E_1. Then $\Big(\mathcal{CB}(E_0, F_0), \mathcal{CB}(E_1, F_1)\Big)_\theta$ embeds completely contractively into $\mathcal{CB}(E_\theta, F_\theta)$ for each $0 < \theta < 1$.*

Proof: For arbitrary operator spaces E and F we may view $\mathcal{CB}(E, F)$ as a subspace of $\mathcal{B}(S^\infty \otimes_{\min} E, S^\infty \otimes_{\min} F)$ via the isometric embedding which carries an operator T in $\mathcal{CB}(E, F)$ to the operator $id_{S^\infty} \otimes T$ in $\mathcal{B}(S^\infty \otimes_{\min} E, S^\infty \otimes_{\min} F)$. Now let E_0, E_1, F_0 and F_1 be as above. Lemma 0.1 applied to the new pairs $\big(S^\infty \otimes_{\min} E_0, S^\infty \otimes_{\min} E_1\big)$ and $\big(S^\infty \otimes_{\min} F_0, S^\infty \otimes_{\min} F_1\big)$ implies that for each real number $0 < \theta < 1$, the space $\Big(\mathcal{B}\big(S^\infty \otimes_{\min} E_0, S^\infty \otimes_{\min} F_0\big), \mathcal{B}\big(S^\infty \otimes_{\min} E_1, S^\infty \otimes_{\min} F_1\big)\Big)_\theta$ embeds contractively into the space $\mathcal{B}\big(S^\infty \otimes_{\min} E_\theta, S^\infty \otimes_{\min} F_\theta\big)$. This implies that $\Big(\mathcal{CB}(E_0, F_0), \mathcal{CB}(E_1, F_1)\Big)_\theta$ embeds contractively into $\mathcal{CB}(E_\theta, F_\theta)$. Actually, the embedding is completely contractive. Indeed, this gives for each integer $n \geq 1$

$$M_n\big(\mathcal{CB}(E_0, F_0), \mathcal{CB}(E_1, F_1)\big)_\theta = \Big(M_n\big(\mathcal{CB}(E_0, F_0)\big), M_n\big(\mathcal{CB}(E_1, F_1)\big)\Big)_\theta =$$

$$\Big(\mathcal{CB}\big(E_0, M_n(F_0)\big), \mathcal{CB}\big(E_1, M_n(F_1)\big)\Big)_\theta \subset \mathcal{CB}\Big(E_\theta, \big(M_n(F_0), M_n(F_1)\big)_\theta\Big) = \mathcal{CB}\big(E_\theta, M_n(F_\theta)\big)\cdot$$

Thus the embedding $M_n\big(\mathcal{CB}(E_0, F_0), \mathcal{CB}(E_1, F_1)\big)_\theta \subset M_n\Big(\mathcal{CB}\big(E_\theta, F_\theta\big)\Big)$ is contractive. ■

G. Pisier proved in [34] that in fact the theory of operator spaces can be developped equivalently using other sequences of norms on the $M_n(E)$'s. Indeed, let $1 \leq p \leq \infty$ be a fixed number, let E be an operator space and let E^* be its dual operator space. For an integer $n \geq 1$, we let $S_n^p[E]$ denote the space $M_n(E)$ but equipped with the norm below

$$S_n^p[E] := \Big(S_n^\infty[E], S_n^1[E]\Big)_\theta$$

where $S_n^\infty[E]$ denotes $M_n(E)$ for convenience only, $S_n^1[E]$ is the space $M_n(E)$ viewed as a subspace of $\Big(M_n(E^*)\Big)^*$ and $\theta = \frac{1}{p}$. Note that $S_n^p[E]$ embeds isometrically into $S_{n+1}^p[E]$ thus we set $S^p[E]$ for the completion of $\bigcup_{n \geq 1} S_n^p[E]$.

Proposition 0.3 ([34]) *For all x in $M_n(E)$, we have $\Big\| \, x \, \Big\|_{M_n(E)} = \sup \Big\{ \, \Big\| a.x.b \Big\|_{S_n^p[E]} \Big\}$ where the supremum runs over all a, b in the unit ball of S_n^{2p}. Therefore an operator $u : E \longrightarrow F$ is c.b. if and only if the maps $u^n : S_n^p[E] \longrightarrow S_n^p[F]$ are uniformly bounded in which case we have $\| \, u \, \|_{cb} = \sup_{n \geq 1} \Big\{ \big\| u^n : S_n^p[E] \longrightarrow S_n^p[F] \big\| \Big\}$.*

Now let us go back to the case of non-commutative L^p-spaces. If M is a von Neumann algebra given with a normal, faithful and semi-finite trace τ_M then since $L^\infty(\tau_M)$ is a C^*-algebra, it has a natural operator space structure given by any concrete realization as a C^*-subalgebra of some $\mathcal{B}(H)$. Since $L^1(\tau_M)$ coincides with the predual of $L^\infty(\tau_M)$, it appears also as an operator space in a natural way. Indeed, it is a subspace of the standard dual of $L^\infty(\tau_M)$. Hence the spaces $L^p(\tau_M)$ are also canonically endowed with an operator space structure, the one obtained by complex interpolation in the operator spaces category. Applying Prop. 0.3 we get a nice and a simple characterization of the *c.b.* maps between these spaces since for each integer $n \geq 1$, we have the following natural identifications

$$S_n^{\infty}\big[L^{\infty}(\tau_M)\big] = L^{\infty}(\tau_M \otimes tr_n), \quad S_n^{1}\big[L^{1}(\tau_M)\big] = L^{1}(\tau_M \otimes tr_n)\cdot$$

These imply that we have isometrically

$$S_n^{p}\big[L^{p}(\tau_M)\big] = \Big(S_n^{\infty}\big[L^{\infty}(\tau_M)\big], S_n^{1}\big[L^{1}(\tau_M)\big]\Big)_{\theta} = \big(L^{\infty}(\tau_M \otimes tr_n), L^{1}(\tau_M \otimes tr_n)\big)_{\theta} = L^{p}(\tau_M \otimes tr_n)\cdot$$

Therefore a density argument yields $S^p\big[L^p(\tau_M)\big] = L^p(\tau_M \otimes tr)$ isometrically. Thus Prop. 0.3 implies

Proposition 0.4 *Let $1 \leq p < \infty$, $L^p(\tau_M)$ and $L^p(\tau_N)$ be two non-commutative L^p-spaces and $E \subset L^p(\tau_M), F \subset L^p(\tau_N)$ arbitrary operator subspaces. Then an operator $u : E \longrightarrow F$ is c.b. if and only if the operator $u \otimes id_{S^p} : E \otimes S^p \longrightarrow F \otimes S^p$ which takes $x \otimes y$ to $u(x) \otimes y$ where $x \in E$ and $y \in S^p$, extends to a bounded operator from $\overline{E \otimes S^p}^{L^p(\tau_M \otimes tr)}$ into $\overline{F \otimes S^p}^{L^p(\tau_N \otimes tr)}$. Moreover we have $\big\|u\big\|_{cb} = \Big\|u \otimes id_{S^p} : \overline{E \otimes S^p}^{L^p(\tau_M \otimes tr)} \longrightarrow \overline{F \otimes S^p}^{L^p(\tau_N \otimes tr)}\Big\|$.*

0.3 Non-commutative Khintchine inequalities

Let $\varepsilon_n : \{-1,1\}^{\mathbb{N}} \longrightarrow \{-1,1\}$ be the n-th coordinate projection, ν the uniform probability measure on $\{-1,1\}^{\mathbb{N}}$ and $1 \leq p < \infty$ an arbitrary real number. In the commutative case, the classical Khintchine inequalities say that there exists a constant $k_p > 0$ depending only on p such that for all integers $n \geq 1$ and all scalars $x_1, x_2, ...x_n$ we have

$$\begin{aligned} \Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|_{L^p(\{-1,1\}^{\mathbb{N}},\nu)} &\geq k_p \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}} \text{ when } 1 \leq p \leq 2 \\ \Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|_{L^p(\{-1,1\}^{\mathbb{N}},\nu)} &\leq k_p \left(\sum_{j=1}^{n} |x_j|^2\right)^{\frac{1}{2}} \text{ when } 2 \leq p < \infty \end{aligned} \tag{0.1}$$

See *e.g.* [23] for the proof. Later on these inequalities were generalized to the non-commutative case by F. Lust–Piquard for $1 < p < \infty$ (*cf.* [26]) and by F. Lust–Piquard and G. Pisier for $p = 1$ (*cf.* [27]) as follows. Let M be a von Neumann algebra given with a normal, faithful and semi-finite trace τ_M. For each $1 \leq p < \infty$, there exists a positive constant $K_{L^p(\tau_M)}$ depending only on the pair (M, τ_M) and p such that for all $n \geq 1$ in $\mathbb{N}$ and all $x_1, x_2, ...x_n$ in $L^p(\tau_M)$ we have in the case of $1 \leq p \leq 2$

$$\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|_{L^p(\{-1,1\}^{\mathbb{N}},\nu,L^p(\tau_M))} \geq K_{L^p(\tau_M)} \inf\left\{\Big\|\Big(\sum_{j=1}^{n} y_j y_j^*\Big)^{\frac{1}{2}}\Big\|_{L^p(\tau_M)} + \Big\|\Big(\sum_{j=1}^{n} z_j^* z_j\Big)^{\frac{1}{2}}\Big\|_{L^p(\tau_M)}\right\} \tag{0.2}$$

where the infimum runs over all decompositions of the x_j's in $L^p(\tau_M)$ as $y_j + z_j$, while

$$\Big\|\sum_{j=1}^{n} \varepsilon_j x_j\Big\|_{L^p(\{-1,1\}^{\mathbb{N}},\nu,L^p(\tau_M))} \leq K_{L^p(\tau_M)} \max\left\{\Big\|\Big(\sum_{j=1}^{n} x_j x_j^*\Big)^{\frac{1}{2}}\Big\|_{L^p(\tau_M)}, \Big\|\Big(\sum_{j=1}^{n} x_j^* x_j\Big)^{\frac{1}{2}}\Big\|_{L^p(\tau_M)}\right\} \tag{0.3}$$

in the case of $2 \leq p < \infty$. In the particular case of S^p the constant K_{S^p} will be denoted K_p for simplicity.

0.4 An explicit description of the spaces $S^{p,unc}$ and $S^{p,unc}(S^p)$

An operator in $\mathcal{B}(\ell_2)$ will be frequently identified with its corresponding matrix relatively to the canonical basis of ℓ_2. Let Ω_0 be the set $\{-1,1\}^{\mathbb{N}\times\mathbb{N}}$, ν be the uniform probability measure on Ω_0 and let $\varepsilon_{ij}:\Omega_0\longrightarrow\{-1,1\}$ be the (i,j)-th coordinate projection. For $1\le p\le\infty$, $S^{p,unc}$ denotes the space of all operators $x=\big(x_{ij}\big)_{i,j}$ in S^∞ such that the operators $\big(\varepsilon_{ij}x_{ij}\big)_{i,j}$ belong to S^p for almost all choices of signs $\big(\varepsilon_{ij}\big)_{i,j}$ on $\mathbb{N}\times\mathbb{N}$, equipped with the norm below

$$\big\|x\big\|_{S^{p,unc}}:=\Big\|\big(\varepsilon_{ij}x_{ij}\big)_{i,j}\Big\|_{L^p(\Omega_0,\nu,S^p)}.$$

We mean by $S^p(S^p)$ the set of all matrices $x=\big(x_{ij}\big)_{i,j}$ with entries x_{ij} in S^p and which are — viewed as operators on $\ell_2(\ell_2)$ — in the p-Schatten class on the Hilbert space $\ell_2(\ell_2)$, equipped with the inherited norm (Note that $S^p(S^p)$ is exactly the p-Schatten class on $\ell_2(\ell_2)$ via the identification mentionned above). Then similarly, we let $S^{p,unc}(S^p)$ be the set of all operators $x=\big(x_{ij}\big)_{i,j}$ in $S^\infty(S^\infty)$ with entries in S^∞ such that the operators $\big(\varepsilon_{ij}x_{ij}\big)_{i,j}$ are in $S^p(S^p)$ for almost all choices of signs $\big(\varepsilon_{ij}\big)_{i,j}$ on $\mathbb{N}\times\mathbb{N}$, equipped with the norm below

$$\big\|x\big\|_{S^{p,unc}(S^p)}:=\Big\|\big(\varepsilon_{ij}x_{ij}\big)_{i,j}\Big\|_{L^p\big(\Omega_0,\nu,S^p(S^p)\big)}.$$

The next result essentially goes back to F. Lust–Piquard [26].

Lemma 0.5 *There is an explicit description of the space $S^{p,unc}$ $\big($resp. $S^{p,unc}(S^p)\big)$ for each $1\le p<\infty$ as follows. For all $x=\big(x_{ij}\big)_{i,j}$ in $S^{p,unc}$ $\big($resp. $S^{p,unc}(S^p)\big)$ we have when $2\le p<\infty$*

$$\big\|x\big\|_{S^{p,unc}}\cong\max\left\{\Big(\sum_i\Big(\sum_j|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac1p},\Big(\sum_j\Big(\sum_i|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac1p}\right\}$$

$$\left(\textit{resp. }\big\|x\big\|_{S^{p,unc}(S^p)}\cong\max\left\{\Big(\sum_j\Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{\frac12}\Big\|_{S^p}^p\Big)^{\frac1p},\Big(\sum_i\Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{\frac12}\Big\|_{S^p}^p\Big)^{\frac1p}\right\}\right)$$

while for $1\le p\le 2$

$$\big\|x\big\|_{S^{p,unc}}\cong\inf\left\{\Big(\sum_i\Big(\sum_j|y_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac1p}+\Big(\sum_j\Big(\sum_i|z_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac1p}\right\}$$

$$\left(\textit{resp. }\big\|x\big\|_{S^{p,unc}(S^p)}\cong\inf\left\{\Big(\sum_j\Big\|\Big(\sum_i y_{ij}^*y_{ij}\Big)^{\frac12}\Big\|_{S^p}^p\Big)^{\frac1p}+\Big(\sum_i\Big\|\Big(\sum_j z_{ij}z_{ij}^*\Big)^{\frac12}\Big\|_{S^p}^p\Big)^{\frac1p}\right\}\right)$$

where the infimum runs over all possible decompositions of x as a sum of $y=\big(y_{ij}\big)_{i,j}$ and $z=\big(z_{ij}\big)_{i,j}$ both in S^p $\big($resp. $S^p(S^p)\big)$.

Proof: We prove the lemma for $S^{p,unc}(S^p)$ when $2\le p<\infty$ only, the other cases are quite similar and left to the reader. We start by recalling that for each $2\le p\le\infty$ and each $x=(x_{ij})_{i,j}$ in $S^p(S^p)$ we have

$$\big\|x\big\|_{S^p(S^p)}\ge\max\left\{\Big(\sum_j\Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{\frac12}\Big\|_{S^p}^p\Big)^{\frac1p},\Big(\sum_i\Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{\frac12}\Big\|_{S^p}^p\Big)^{\frac1p}\right\}\qquad(0.4)$$

Indeed, this holds for $p = 2$ and $p = \infty$. Then using complex interpolation, it is also satisfied for all $2 < p < \infty$. Now let $2 \leq p < \infty$ be fixed. When $x = \big(x_{ij}\big)_{i,j}$ belongs to $S^{p,unc}(S^p)$, the $\infty \times \infty$ matrices $\big(\varepsilon_{ij}x_{ij}\big)_{i,j}$ satisfy (0.4) for almost all the choices of signs $\big(\varepsilon_{ij}\big)_{i,j}$ on Ω_0 since the matrices $\big(\varepsilon_{ij}x_{ij}\big)_{i,j}$ belong to $S^p(S^p)$ almost surely. Thus after integrating over all these choices of signs, we get for each $x = \big(x_{ij}\big)_{i,j}$ in $S^{p,unc}(S^p)$

$$\big\|x\big\|_{S^{p,unc}(S^p)} \geq \max\Big\{\Big(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}}, \Big(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}}\Big\}.$$

The converse inequality is obtaind with the help of the inequality (0.3). Indeed, we have

$$\big\|x\big\|_{S^{p,unc}(S^p)} = \Big(\int_{\Omega_0} \Big\| \sum_{(i,j)\in\mathbb{N}\times\mathbb{N}} \varepsilon_{ij}x_{ij}\otimes e_{ij}\Big\|_{S^p(S^p)}^p d\nu\Big)^{\frac{1}{p}}$$

$$\leq K_p \max\Big\{\Big\|\Big(\sum_{i,j}(x_{ij}\otimes e_{ij})^*(x_{ij}\otimes e_{ij})\Big)^{\frac{1}{2}}\Big\|_{S^p(S^p)}, \Big\|\Big(\sum_{i,j}(x_{ij}\otimes e_{ij})(x_{ij}\otimes e_{ij})^*\Big)^{\frac{1}{2}}\Big\|_{S^p(S^p)}\Big\}$$

$$= K_p \max\Big\{\Big\|\Big(\sum_{i,j} x_{ij}^* x_{ij}\otimes e_{jj}\Big)^{\frac{1}{2}}\Big\|_{S^p(S^p)}, \Big\|\Big(\sum_{i,j} x_{ij} x_{ij}^*\otimes e_{ii}\Big)^{\frac{1}{2}}\Big\|_{S^p(S^p)}\Big\}$$

$$= K_p \max\Big\{\Big\|\sum_j \Big(\sum_i x_{ij}^* x_{ij}\Big)^{\frac{1}{2}}\otimes e_{jj}\Big\|_{S^p(S^p)}, \Big\|\sum_i \Big(\sum_j x_{ij} x_{ij}^*\Big)^{\frac{1}{2}}\otimes e_{ii}\Big\|_{S^p(S^p)}\Big\}$$

$$\big\|x\big\|_{S^{p,unc}(S^p)} \leq K_p \max\Big\{\Big(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}}, \Big(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}}\Big\} \quad \blacksquare$$

0.5 Additional non standard notations

For the rest of this paper, we will use frequently the following definitions and notations. A given matrix $x = \big(x_{kl}\big)_{k,l}$ where the entries x_{kl} belong to some fixed set is said to be Hankelian if it satisfies $x_{kl} = x_{k'l'}$ whenever $k + l = k' + l'$.
For $1 \leq p < \infty$, $\mathfrak{S}^p$ $\big($resp. $\mathfrak{S}^p(S^p)\big)$ will stand for the subspace of S^p $\big($resp. $S^p(S^p)\big)$ formed of all Hankelian matrices $x = \big(x_{kl}\big)_{k,l}$ in S^p $\big($resp. $S^p(S^p)\big)$.
For a set Λ, $1\!\!1_{\Lambda}$ stands for its indicator function and $|\Lambda|$ stands for the cardinality of Λ.
If Λ is a subset of a discrete group G and if $\mathcal{F}$ denotes either $L^p(\tau_0)$ or $L^p(\tau)$ for some $1 \leq p \leq \infty$ then we let

$$\mathcal{F}_\Lambda := \Big\{f \in \mathcal{F} \mid \widehat{f}(t) = 0,$$

Recall that for f in $L^p(\tau_0)$ $\big($resp. $L^p(\tau)\big)$ the Fourier coefficient $\widehat{f}(t)$ is defined as follows

$$\widehat{f}(t) = \tau_0\big[\lambda(t^{-1})f\big] \quad \Big(\text{resp. } \widehat{f}(t) = \tau_0 \otimes id_{S^p}\big[(\lambda(t^{-1})\otimes id_{\ell_2})f\big]\Big).$$

We denote simply $\mathcal{F}_G$ by $\mathcal{F}$ and when $G = \mathbb{Z}$ and $\Lambda = \mathbb{N}$, $L^p_{\mathbb{N}}$ will be still denoted by H^p.
Similarly when $\mathcal{F}$ is a class of $\infty \times \infty$ matrices and A is a subset of $\mathbb{N}\times\mathbb{N}$ we let

$$\mathcal{F}_A := \Big\{x = \big(x_{kl}\big)_{k,l} \in \mathcal{F} \mid x_{kl} = 0,$$

$\mathcal{F}_{\mathbb{N}\times\mathbb{N}}$ is denoted simply by $\mathcal{F}$. Moreover when $\mathcal{F}$ is a Banach or an operator space the sets $\mathcal{F}_\Lambda$ and $\mathcal{F}_A$ are automatically viewed as Banach or operator subspaces of $\mathcal{F}$.
When $\Lambda \subset \mathbb{N}$, $\widehat{\Lambda} := \Big\{(k,l) \in \mathbb{N}\times\mathbb{N} \mid k+l \in \Lambda\Big\}$ and any subset of $\mathbb{N}\times\mathbb{N}$ which can be written as $\widehat{\Lambda}$ for some set $\Lambda \subset \mathbb{N}$ is called a Hankelian set. Then given a map $\varphi : \Lambda \longrightarrow \mathbb{C}$ we let

$$\begin{array}{rccl} \widehat{\varphi}: & \widehat{\Lambda} & \longrightarrow & \mathbb{C} \\ & (k,l) & \longmapsto & \varphi(k+l). \end{array}$$

For an analytic function f on $\mathbb{T}$ we let for all z in $\mathbb{T}$ $f_{_{(0)}}(z) := \widehat{f}(0)$ while for all integers $n \geq 1$

$$f_{_{(n)}}(z) := \sum_{k=2^{n-1}}^{2^n-1} \widehat{f}(k) z^k.$$

Similarly given an $\infty\times\infty$ matrix x we let $x_{_{(0)}} := \big(x_{_{00}}\big)$ while for all integers $n \geq 1$ we let

$$x_{_{(n)}} := x 1\!\!1_{\{(k,l) \ \mid \ 2^{n-1} \leq k+l < 2^n\}} \qquad \text{(Schur product)}.$$

0.6 Peller's theorem

The aim of Peller's theorem is to realize $\mathfrak{S}^p$ and more generally $\mathfrak{S}^p(S^p)$ as a space of functions on the torus $\mathbb{T}$ which we will describe here for $1 < p < \infty$ only. Consider the Banach spaces (Besov spaces)

$$\begin{aligned} \mathcal{A}^p &:= \Big\{ f : \mathbb{T} \longrightarrow \mathbb{C} \text{ analytic } \mid \big\|f\big\|_{\mathcal{A}^p} < \infty \Big\} \\ \mathcal{A}^p(S^p) &:= \Big\{ g : \mathbb{T} \longrightarrow S^p \text{ analytic } \mid \big\|g\big\|_{\mathcal{A}^p(S^p)} < \infty \Big\} \end{aligned}$$

where

$$\big\|f\big\|_{\mathcal{A}^p} := \Big(\sum_{n=0}^{\infty} 2^n \Big\|f_{_{(n)}}\Big\|_{L^p}^p\Big)^{\frac{1}{p}} \big\|g\big\|_{\mathcal{A}^p(S^p)} := \Big(\sum_{n=0}^{\infty} 2^n \Big\|g_{_{(n)}}\Big\|_{L^p(S^p)}^p\Big)^{\frac{1}{p}}.$$

Theorem 0.6 *The following maps are well defined, bounded and bijective.*

$$\left[\begin{array}{rcl} \mathcal{A}^p & \longrightarrow & \mathfrak{S}^p \\ f & \longmapsto & \Big(\widehat{f}(k+l)\Big)_{k,l\geq 0} \end{array}\right] \ \textit{and} \ \left[\begin{array}{rcl} \mathcal{A}^p(S^p) & \longrightarrow & \mathfrak{S}^p(S^p) \\ g & \longmapsto & \Big(\widehat{g}(k+l)\Big)_{k,l\geq 0} \end{array}\right].$$

In other words, as Banach spaces, $\mathcal{A}^p$ is isomorphic to $\mathfrak{S}^p$ and $\mathcal{A}^p(S^p)$ is isomorphic to $\mathfrak{S}^p(S^p)$ in a canonical way.

In the case of $\mathfrak{S}^p$ we refer the reader to Section 2 of the paper [29], the norm of $\mathcal{A}^p$ as described above is given explicitly in page 450 while we refer to Section 3 of [30] for the case of $\mathfrak{S}^p(S^p)$, the norm of $\mathcal{A}^p(S^p)$ described above is then implicit. Therefore we have

$$\forall x \in \mathfrak{S}^p, \Big\|x_{_{(n)}}\Big\|_{\mathfrak{S}^p} \cong 2^{\frac{n}{p}} \Big\|\sum_{k=2^{n-1}}^{2^n-1} x_{0k} z^k\Big\|_{L^p} \big\|x\big\|_{\mathfrak{S}^p} \cong \Big(\sum_{n=0}^{\infty} 2^n \Big\|\sum_{k=2^{n-1}}^{2^n-1} x_{0k} z^k\Big\|_{L^p}^p\Big)^{\frac{1}{p}}$$

and similarly for all x in $\mathfrak{S}^p(S^p)$, we have

$$\left\|x_{(n)}\right\|_{\mathfrak{S}^p(S^p)} \cong 2^{\frac{n}{p}}\Big\|\sum_{k=2^{n-1}}^{2^n-1} x_{0k}z^k\Big\|_{L^p(S^p)} \left\|x\right\|_{\mathfrak{S}^p(S^p)} \cong \Big(\sum_{n=0}^{\infty} 2^n\Big\|\sum_{k=2^{n-1}}^{2^n-1} x_{0k}z^k\Big\|^p_{L^p(S^p)}\Big)^{\frac{1}{p}}$$

($x_{0k}z^k := 0$ when $k = \frac{1}{2}$). These descriptions provide $\mathfrak{S}^p$ and $\mathfrak{S}^p(S^p)$ with very useful equivalent norms as follows.

Corollary 0.7 *i) For each fixed $1 < p < \infty$, the following are equivalent norms on $\mathfrak{S}^p$*

$$\left\|x\right\|_{\mathfrak{S}^p} \cong \Big(\sum_{n=0}^{\infty}\left\|x_{(n)}\right\|^p_{\mathfrak{S}^p}\Big)^{\frac{1}{p}}, \ \forall x \in \mathfrak{S}^p.$$

ii) For each fixed $1 < p < \infty$, the following are equivalent norms on the space $\mathfrak{S}^p(S^p)$

$$\left\|x\right\|_{\mathfrak{S}^p(S^p)} \cong \Big(\sum_{n=0}^{\infty}\left\|x_{(n)}\right\|^p_{\mathfrak{S}^p(S^p)}\Big)^{\frac{1}{p}}, \ \forall x \in \mathfrak{S}^p(S^p).$$

0.7 Some suitable operator norms inequalities

Proposition 0.8 *Consider $1 \le q \le \infty$, $\alpha, \beta > 1$ such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, y a positive operator in $S^{q\alpha}$ and $\big(x_n\big)_n$ a finite sequence of operators each in $S^{2q\beta}$, then we have*

$$\Big\|\sum_n x_n^* y x_n\Big\|_{S^q} \le \left\|y\right\|_{S^{q\alpha}} \max\Big\{\Big\|\sum_n x_n^* x_n\Big\|_{S^{q\beta}}, \Big\|\sum_n x_n x_n^*\Big\|_{S^{q\beta}}\Big\}.$$

This proposition goes back to [26] when $\big(x_n\big)_n$ is a family of self-adjoint operators. The general case for which the proof uses basically the three line lemma can be found in [35]. The next corollary follows easily by a reiteration argument.

Corollary 0.9 *Let $1 \le q \le \infty$, $r \ge 1$ and for each $1 \le j \le r$, let I_j be a finite set of indexes, $\alpha_j > 1$ with $\sum_{j=1}^{r}\frac{1}{\alpha_j} = 1$ and $\Big(x^{(j)}_{n_j}\Big)_{n_j \in I_j}$ be a family of operators each in $S^{2q\alpha_j}$. Then we have*

$$\Big\|\sum_{\substack{n_j \in I_j\\ 1\le j\le r}} x^{(r)*}_{n_r}...x^{(2)*}_{n_2}x^{(1)*}_{n_1}x^{(1)}_{n_1}x^{(2)}_{n_2}...x^{(r)}_{n_r}\Big\|_{S^q} \le \prod_{j=1}^{r}\max\Big\{\Big\|\sum_{n_j\in I_j} x^{(j)*}_{n_j}x^{(j)}_{n_j}\Big\|_{S^{q\alpha_j}}, \Big\|\sum_{n_j\in I_j} x^{(j)}_{n_j}x^{(j)^*}_{n_j}\Big\|_{S^{q\alpha_j}}\Big\}.$$

0.8 Fourier multipliers

A scalar valued map φ on $\Lambda \subset G$ is said to be a Fourier multiplier on $L^p_\Lambda(\tau_0)$ if the associated operator

$$\begin{array}{rccc} M_\varphi : & \mathrm{span}\Big\{\lambda(t), t\in\Lambda\Big\} & \longrightarrow & \mathrm{span}\Big\{\lambda(t), t\in\Lambda\Big\} \\ & \lambda(t) & \longmapsto & \varphi(t)\lambda(t) \end{array}$$

extends to a bounded operator on $L^p_\Lambda(\tau_0)$ (M_φ will still denote the extension to $L^p_\Lambda(\tau_0)$) and we let $M(L^p_\Lambda(\tau_0))$ stand for the set of all such maps. Then $M(L^p_\Lambda(\tau_0))$ is a unital Banach algebra for the pointwise product and the following norm

$$\Big\|\varphi\Big\|_{M(L^p_\Lambda(\tau_0))} := \Big\|M_\varphi : L^p_\Lambda(\tau_0) \longrightarrow L^p_\Lambda(\tau_0)\Big\|.$$

Let $M_{cb}(L^p_\Lambda(\tau_0))$ be the subalgebra of all Fourier multipliers φ on $L^p_\Lambda(\tau_0)$ which are *c.b.* *i.e.* the corresponding operators M_φ are *c.b.*, equipped with the norm below

$$\Big\|\varphi\Big\|_{M_{cb}(L^p_\Lambda(\tau_0))} := \Big\|M_\varphi : L^p_\Lambda(\tau_0) \longrightarrow L^p_\Lambda(\tau_0)\Big\|_{cb}.$$

By Prop. 0.4, a multiplier φ belongs to $M_{cb}(L^p_\Lambda(\tau_0))$ if and only if the operator $M_\varphi \otimes id_{S^p}$ is bounded on $L^p_\Lambda(\tau_0) \otimes S^p$ as a subspace of $L^p(\tau)$ with $\Big\|\varphi\Big\|_{M_{cb}(L^p_\Lambda(\tau_0))} = \Big\|M_\varphi \otimes id_{S^p}\Big\|$. By duality, it is very easy to see that for all $1 \le p, q \le \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$M(L^p(\tau_0)) = M(L^q(\tau_0)),\ M_{cb}(L^p(\tau_0)) = M_{cb}(L^q(\tau_0))$$

isometrically. Note that the duality $< f, g >= \tau_0(f\check{g})$ for f in $L^p(\tau_0)$, g in $L^q(\tau_0)$ and where $\check{g} := \sum_{t\in G} \widehat{g}(t^{-1}) \curlywedge (t)$ is the suitable choice to have the previous identifications via the identity map. Therefore we can restrict ourselves to the case where $2 \le p \le \infty$. We see easily that

$$M(L^2(\tau_0)) = M_{cb}(L^2(\tau_0)) \ = \ \ell_\infty(G)$$

isometrically. Since $M_{cb}(L^\infty(\tau_0)) \subset M(L^\infty(\tau_0)) \subset M(L^2(\tau_0)) = M_{cb}(L^2(\tau_0))$ contractively, we get by using the results of complex interpolation

$$M(L^\infty(\tau_0)) \subset M(L^p(\tau_0)) \subset M(L^2(\tau_0))$$
$$M_{cb}(L^\infty(\tau_0)) \subset M_{cb}(L^p(\tau_0)) \subset M(L^2(\tau_0))$$

contractively. By repeating the same argument, we see that for all $2 \le q < p \le \infty$ we have

$$M(L^p(\tau_0)) \subset M(L^q(\tau_0)),\ M_{cb}(L^p(\tau_0)) \subset M_{cb}(L^q(\tau_0))$$

contractively. Thus $\Big(\Big\{M(L^p(\tau_0))\Big\}\Big)_{2\le p\le\infty}$ and $\Big(\Big\{M_{cb}(L^p(\tau_0))\Big\}\Big)_{2\le p\le\infty}$ are two decreasing families of algebras.

Now assume moreover that G is Abelian and equip its dual group $\widehat{G}$ which is compact with its Haar measure. In this case, the von Neumann algebra generated by $\curlywedge(G)$ in $\mathcal{B}\big(l_2(G)\big)$ coincides with $L^\infty(\widehat{G})$, $L^p(\tau_0)$ coincides with $L^p(\widehat{G})$ and $L^p(\tau)$ coincides with $L^p(\widehat{G}, S^p)$. This applies *e.g.* for the group $\mathbb{Z}$ which will be discussed later.

Remark 0.10 It follows from well known results (*cf. e.g.* [7], [8]) that the canonical Hilbert transform defines a *c.b.* multiplier on L^p for $1 < p < \infty$. Therefore the natural projections of L^p onto L^p_Λ which send f to $\sum_{k\in\Lambda} \widehat{f}(k)z^k$ are uniformly completely bounded when Λ runs over all intervals of $\mathbb{Z}$. In other words, the spaces L^p_Λ where $\Lambda \subset \mathbb{Z}$ is an arbitrary interval are uniformly complemented in L^p as operator spaces. According to this, we see that for $1 < p < \infty$ the following inclusion maps

$$\begin{array}{ccc} M_{cb}(L^p_\Lambda) & \hookrightarrow & M_{cb}(L^p) \\ \varphi & \longmapsto & \tilde{\varphi} \end{array}$$

where $\tilde{\varphi}$ is the trivial extension of φ equal to zero outside Λ are uniformly bounded when Λ runs over all intervals of $\mathbb{Z}$.

0.9 Schur multipliers

Let $\{e_{kl}\}_{k,l}$ be the canonical basis of S^p, $1 \leq p \leq \infty$ and A be a subset of $\mathbb{N} \times \mathbb{N}$. A scalar map φ defined on A is said to be a Schur multiplier on S^p_A if the associated operator

$$\begin{array}{rcl} T_\varphi : \operatorname{span}\Big\{e_{kl}, (k,l) \in A\Big\} & \longrightarrow & \operatorname{span}\Big\{e_{kl}, (k,l) \in A\Big\} \\ e_{kl} & \longmapsto & \varphi(k,l)\ e_{kl} \end{array}$$

extends to a bounded operator on S^p_A (T_φ still denotes the extension of φ to S^p_A) and we let $M(S^p_A)$ stand for the set of all Schur multipliers on S^p_A. Then $M(S^p_A)$ is a Banach algebra for the pointwise product and the norm

$$\Big\|\varphi\Big\|_{M(S^p_A)} := \Big\|T_\varphi : S^p_A \longrightarrow S^p_A\Big\|.$$

We will denote by $M_{cb}(S^p_A)$ the algebra of all Schur multipliers φ on S^p_A which are *c.b.*, equipped with the norm

$$\Big\|\varphi\Big\|_{M_{cb}(S^p_A)} = \Big\|T_\varphi : S^p_A \longrightarrow S^p_A\Big\|_{cb}.$$

We will denote by $M^{\mathcal{H}}(S^p_A)$ and $M^{\mathcal{H}}_{cb}(S^p_A)$ the subalgebras of $M(S^p_A)$ and $M_{cb}(S^p_A)$ respectively formed of all Schur multipliers on S^p_A which have a Hankelian form (a multiplier φ is viewed as an $\infty \times \infty$ matrix).

When A has a Hankelian form (*i.e.* $A = \widehat{\Lambda}$ for some set $\Lambda \subset \mathbb{N}$), we let $M(\mathfrak{S}^p_A)$ $\big($resp. $M_{cb}(\mathfrak{S}^p_A)\big)$ be the algebra of all scalar maps φ defined on A such that the corresponding operators map $\mathfrak{S}^p_A$ boudedly (resp. completely boundedly) into itself. Note that a multiplier on $\mathfrak{S}^p_A$ has necessarily a Hankelian form.

For an example of *c.b.* Hankelian Schur multipliers on S^p, we can quote the following. Fix z in $\mathbb{T}$ and consider the map $\varphi_z : (k,l) \longmapsto z^{k+l}$. Then the corresponding operator is by definition

$$\begin{array}{rccl} T_{\varphi_z} : & S^p & \longrightarrow & S^p \\ & \big(x_{kl}\big)_{k,l} & \longmapsto & \big(z^{k+l} x_{kl}\big)_{k,l} = D_z x D_z \end{array}$$

where D_z is the unitary operator below

$$D_z = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & z & 0 & \dots \\ 0 & 0 & z^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Clearly T_{φ_z} is an isometry and in fact T_{φ_z} is a complete isometry. Therefore φ_z belongs to $M_{cb}^{\mathcal{H}}(S^p)$ for all $1 \leq p \leq \infty$.
For the study of the spaces $M(S^p)$ and $M_{cb}(S^p)$, we can again reduce to the case where $2 \leq p \leq \infty$ since for $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$M(S^p) = M(S^q),\ M_{cb}(S^p) = M_{cb}(S^q)$$

isometrically. As noticed previously, these identifications can be done via the identity map if we wish by making a suitable choice for the duality between S^p and S^q. Namely, we set

$$\forall x \in S^p, \forall y \in S^q \ \ < x, y >:= tr({}^t xy)\cdot$$

There is a nice description of $M(S^p)$ for and only for $p = 2$ and $p = 1$ or $p = \infty$. Indeed

$$\begin{array}{lclcl} \bullet\ M(S^2) & = & M_{cb}(S^2) & = & \ell_\infty(\mathbb{N} \times \mathbb{N}) \\ \bullet\ M(S^\infty) & = & M_{cb}(S^\infty) & = & \Gamma_2(\ell_1, \ell_\infty) \end{array}$$

isometrically, where $\Gamma_2(\ell_1, \ell_\infty)$ is the space of all operators from ℓ_1 to ℓ_∞ which factor through a Hilbert space, equipped with the usual factorization norm. The case $p = 2$ is trivial while the case $p = \infty$ for which [32] gives the precise statement below and which goes in essence to Grothendieck is not (see [32] for more references).

Theorem 0.11 *For $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$, the following are equivalent.*
i) φ is a Schur multiplier on S^∞ with norm less than 1.
ii) There exist a Hilbert space H, sequences of vectors $(h_n)_n$ and $(k_m)_m$ in the unit ball of H such that $\varphi(n, m) =< h_n, k_m > \forall n, m$ in $\mathbb{N}$.
iii) The operator $u_\varphi : \ell_1 \longrightarrow \ell_\infty$ which takes e_n to $\sum_m \varphi(n, m)e_m$ belongs to $\Gamma_2(\ell_1, \ell_\infty)$ with norm less than 1. Here $\{e_n\}_n$ denotes the canonical basis of ℓ_1.
iv) φ is a c.b. Schur multiplier on S^∞ with c.b. norm less than 1.

It is useful to note that this description provides a class $\mathcal{C}$ of very simple multipliers in the unit ball of $M(S^\infty)$, those of the form $\varphi(n, m) = a_n b_m$ where $a = \big(a_n\big)_n, b = \big(b_m\big)_m$ are in B_{ℓ_∞}. The interest of this class $\mathcal{C}$ comes from the fact that there exists a universal constant $K > 0$ such that $\overline{\mathrm{conv}}(\mathcal{C}) \subset B_{M(S^\infty)} \subset K\overline{\mathrm{conv}}(\mathcal{C})$ where $\overline{\mathrm{conv}}(\mathcal{C})$ is the closure of the convex set generated by $\mathcal{C}$ in the simple convergence topology on $\mathbb{N} \times \mathbb{N}$ (*cf.* [32]).
• For $2 < p < \infty$ we have contractive inclusions $M(S^\infty) \subset M_{cb}(S^p) \subset M(S^p) \subset M(S^2)$. Indeed $M(S^\infty)$ embeds contractively into $M(S^2)$ and we use the complex interpolation. More generally we see that for $2 < p < q < \infty$ we have the following contractive embeddings

$$M(S^\infty) \subset M(S^q) \subset M(S^p) \subset M(S^2)$$
$$M(S^\infty) \subset M_{cb}(S^q) \subset M_{cb}(S^p) \subset M(S^2)\cdot$$

Therefore $\Big(\Big\{M(S^p)\Big\}\Big)_{2\leq p\leq\infty}$ and $\Big(\Big\{M_{cb}(S^p)\Big\}\Big)_{2\leq p\leq\infty}$ are two decreasing families of sets.

1 Non-commutative $\Lambda(p)$-sets in discrete groups

In this section G denotes an arbitrary discrete group with unit e, λ denotes the left regular representation of G into $\mathcal{B}\big(l_2(G)\big)$, $L^p(\tau_0)$ denotes the non-commutative L^p-space associated to the von Neumann algebra generated by $\lambda(G)$ with respect to the usual trace τ_0 and $L^p(\tau)$ denotes the non-commutative L^p-space associated to the von Neumann algebra generated by $\lambda(G) \otimes \mathcal{B}(l_2)$ with respect to the trace $\tau = \tau_0 \otimes tr$ where tr denotes the usual trace on the Schatten class S^p.

Definition 1.1 *Let $2 < p < \infty$ be fixed and let $\Lambda \subset G$ be a given subset. We say that Λ is a $\Lambda(p)$-set if the spaces $L^p_\Lambda(\tau_0)$ and $L^2_\Lambda(\tau_0)$ are isomorphic, equivalently, there exists a constant $\lambda > 0$ such that for all finitely supported families of scalars a_t we have*

$$\Big\| \sum_{t\in\Lambda} a_t \lambda(t) \Big\|_{L^p(\tau_0)} \le \lambda \Big(\sum_{t\in\Lambda} |a_t|^2 \Big)^{\frac{1}{2}}.$$

We let $\lambda_p(\Lambda)$ or sometimes simply λ_p stand for the smallest constant λ for which this happens.

The reader is requested to see Subsection 0.8 for the definition of the algebra of Fourier multipliers $M(L^p(\tau_0))$ as well as its subalgebra $M_{cb}(L^p(\tau_0))$.

Definition 1.2 *A set $\Lambda \subset G$ is said to be an interpolation set for $M\big(L^p(\tau_0)\big)$ for some $1 \le p \le \infty$ if the restriction map below*

$$\begin{array}{rcl} \mathcal{Q} : M(L^p(\tau_0)) & \longrightarrow & \ell_\infty(\Lambda) \\ \varphi & \longmapsto & \big(\varphi(t)\big)_{t\in\Lambda} \end{array}$$

is μ-surjective for some constant μ. We let $\mu_p(\Lambda)$ or simply μ_p be the smallest constant μ for which this happens.

The following result shows that $\Lambda(p)$-sets can be viewed as classes of interpolation sets.

Proposition 1.3 *Let $2 < p < \infty$ and $\Lambda \subset G$ be fixed. The assertions below are equivalent.*
i) Λ is a $\Lambda(p)$-set.
ii) Λ is an interpolation set for $M\big(L^p(\tau_0)\big)$.
Moreover we have $\mu_p(\Lambda) \le \lambda_p(\Lambda) \le K_{L^p(\tau_0)} \mu_p(\Lambda)$ where $K_{L^p(\tau_0)}$ is the constant defined in the inequality (0.3).

Proof: Assume that Λ is a $\Lambda(p)$-set. For $\varepsilon = \big(\varepsilon_t\big)_t$ in $\ell_\infty(\Lambda)$ we let $\tilde{\varepsilon}$ be its trivial extension to $\ell_\infty(G)$ equal to zero outside Λ. Then for any f in $L^p(\tau_0)$

$$\Big\| \sum_{t\in G} \tilde{\varepsilon}_t \widehat{f}(t) \lambda(t) \Big\|_{L^p(\tau_0)} = \Big\| \sum_{t\in\Lambda} \varepsilon_t \widehat{f}(t) \lambda(t) \Big\|_{L^p(\tau_0)} \le \lambda_p \Big(\sum_{t\in\Lambda} |\varepsilon_t \widehat{f}(t)|^2 \Big)^{\frac{1}{2}}$$

$$\le \lambda_p \,\big\| \varepsilon \big\|_{\ell_\infty(\Lambda)} \Big(\sum_{t\in G} |\widehat{f}(t)|^2 \Big)^{\frac{1}{2}} = \lambda_p \big\| \varepsilon \big\|_{\ell_\infty(\Lambda)} \big\| f \big\|_{L^2(\tau_0)} \le \lambda_p \big\| \varepsilon \big\|_{\ell_\infty(\Lambda)} \big\| f \big\|_{L^p(\tau_0)}.$$

Thus $\tilde{\varepsilon}$ is in $M(L^p(\tau_0))$ and it satisfies $\left\| \tilde{\varepsilon} \right\|_{M(L^p(\tau_0))} \leq \lambda_p \left\| \varepsilon \right\|_{\ell_\infty(\Lambda)}$. This means $\mu_p \leq \lambda_p$. Conversely assume that Λ is an interpolation set for $M(L^p(\tau_0))$. Then for any $\delta > 0$, each choice of signs ε on Λ admits a lifting $\tilde{\varepsilon}$ in $M(L^p(\tau_0))$ with $\left\| \tilde{\varepsilon} \right\|_{M(L^p(\tau_0))} \leq \mu_p + \delta$. This implies that for any f in $L^p_\Lambda(\tau_0)$, say with finitely supported Fourier transform $\widehat{f}$, we have

$$\left\| f \right\|_{L^p(\tau_0)} \leq (\mu_p + \delta) \Big\| \sum_{t \in \Lambda} \varepsilon_t \widehat{f}(t) \lambda(t) \Big\|_{L^p(\tau_0)}$$

since $f = M_{\tilde{\varepsilon}}(M_{\tilde{\varepsilon}} f)$. Then we integrate the inequality over all the choices of signs on Λ

$$\begin{aligned} \left\| f \right\|_{L^p(\tau_0)} &\leq (\mu_p + \delta) \Big(\int_{\{-1,1\}^\Lambda} \Big\| \sum_{t \in \Lambda} \varepsilon_t \widehat{f}(t) \lambda(t) \Big\|^p_{L^p(\tau_0)} d\nu(\varepsilon) \Big)^{\frac{1}{p}} \\ &= (\mu_p + \delta) \Big\| \sum_{t \in \Lambda} \varepsilon_t \widehat{f}(t) \lambda(t) \Big\|_{L^p\left(\{-1,1\}^\Lambda, \nu, L^p(\tau_0)\right)}. \end{aligned}$$

Now we apply the non-commutative version of Khintchine inequalities 0.3 and let δ tend to 0 to obtain $\left\| f \right\|_{L^p(\tau_0)} \leq K_{L^p(\tau_0)} \mu_p \left\| f \right\|_{L^2(\tau_0)}$. Hence Λ is a $\Lambda(p)$-set with $\lambda_p \leq K_{L^p(\tau_0)} \mu_p$. ■

Remarks 1.4 (i) Since the embeddings $L^\infty(\tau_0) \subset L^q(\tau_0) \subset L^p(\tau_0) \subset L^2(\tau_0)$ are bounded for all real numbers $2 < p < q < \infty$, we see that the $\Lambda(q)$-property implies the $\Lambda(p)$-property. Thus we have a decreasing family of sets $\Big(\Big\{ \Lambda \subset G \mid \Lambda \text{ is a } \Lambda(p)\text{-set} \Big\} \Big)_{2<p<\infty}$.
(ii) Although no significantly new examples are known, it is useful to consider also the case $1 < p \leq 2$. A set Λ is called a $\Lambda(p)$-set in this case if $L^p_\Lambda(\tau_0)$ and $L^q_\Lambda(\tau_0)$ are equivalent Banach spaces for some and thus any $1 \leq q < p$. Similarly we denote by $\lambda_p(\Lambda)$ the smallest constant $\lambda > 0$ such that for any f in $L^p_\Lambda(\tau_0)$ we have $\left\| f \right\|_{L^p(\tau_0)} \leq \lambda \left\| f \right\|_{L^1(\tau_0)}$. With this terminology it is known using an extrapolation argument that if $q > 2$ and Λ is a $\Lambda(q)$-set then Λ is a $\Lambda(2)$-set. Conversely if Λ is a $\Lambda(2)$-set then Λ is a $\Lambda(q)$-set if and only if its indicator function $1\!\!1_\Lambda$ belongs to $M(L^q(\tau_0))$. Moreover for each set Λ we have

$$\lambda_2(\Lambda) \leq \lambda_q(\Lambda) \leq \lambda_2(\Lambda) \left\| 1\!\!1_\Lambda \right\|_{M(L^q(\tau_0))} \tag{1.5}$$

Now we extend the previous definitions and results to the non-commutative case. Namely we define subsets of G playing for the sets $M_{cb}(L^p(\tau_0))$ a rôle similar to the one played by $\Lambda(p)$-sets for $M(L^p(\tau_0))$.

Definition 1.5 *Let $2 < p < \infty$ be fixed and let $\Lambda \subset G$ be a given set. We say that Λ is a $\Lambda(p)_{cb}$-set if there exists a constant $C > 0$ such that for all finitely supported families of operators x_t in S^p we have*

$$\Big\| \sum_{t \in \Lambda} \lambda(t) \otimes x_t \Big\|_{L^p(\tau)} \leq C \max \Big\{ \Big\| \Big(\sum_{t \in \Lambda} x_t^* x_t \Big)^{\frac{1}{2}} \Big\|_{S^p}, \Big\| \Big(\sum_{t \in \Lambda} x_t x_t^* \Big)^{\frac{1}{2}} \Big\|_{S^p} \Big\}.$$

Then we let $\lambda_p^{cb}(\Lambda)$ stand for the smallest constant C for which the inequality above holds.

Remarks 1.6 (i) Using Jensen's inequality it is very easy to see that when $p \geq 2$, any f in $L^p(\tau)$ satisfies

$$\max\left\{\left\|\left(\sum_{t\in G}\widehat{f}(t)^*\widehat{f}(t)\right)^{\frac{1}{2}}\right\|_{S^p}, \left\|\left(\sum_{t\in G}\widehat{f}(t)\widehat{f}(t)^*\right)^{\frac{1}{2}}\right\|_{S^p}\right\} \leq \left\|f\right\|_{L^p(\tau)} \tag{1.6}$$

Hence the $\Lambda(p)_{cb}$-property means simply that the norms $\left\|\,.\,\right\|_{L^p(\tau)}$ and $|||.|||$ are equivalent on $L^p_\Lambda(\tau)$ where for any f in $L^p(\tau)$

$$|||f||| := \max\left\{\left\|\left(\sum_{t\in G}\widehat{f}(t)^*\widehat{f}(t)\right)^{\frac{1}{2}}\right\|_{S^p}, \left\|\left(\sum_{t\in G}\widehat{f}(t)\widehat{f}(t)^*\right)^{\frac{1}{2}}\right\|_{S^p}\right\}.$$

Therefore whenever Λ is a $\Lambda(p)_{cb}$-set the indicator function $1\!\!1_{\Lambda}$ is in $M_{cb}(L^p(\tau_0))$ *i.e.* the natural projection of $L^p(\tau_0)$ onto $L^p_\Lambda(\tau_0)$ is *c.b.* with *c.b.* norm less or equal to $\lambda_p^{cb}(\Lambda)$.
(ii) Given two $\Lambda(p)_{cb}$-subsets Λ_1 and Λ_2 of G, the set $\Lambda_1\cup\Lambda_2$ has clearly the $\Lambda(p)_{cb}$-property with $\lambda_p^{cb}(\Lambda_1\cup\Lambda_2) \leq \lambda_p^{cb}(\Lambda_1) + \lambda_p^{cb}(\Lambda_2)$.

Definition 1.7 *Given $1 \leq p \leq \infty$, a subset Λ of G is said to be an interpolation set for $M_{cb}(L^p(\tau_0))$ if the restriction map below*

$$\begin{array}{rcl}\mathcal{Q} : M_{cb}(L^p(\tau_0)) & \longrightarrow & \ell_\infty(\Lambda) \\ \varphi & \longmapsto & \Big(\varphi(t)\Big)_{t\in\Lambda}\end{array}$$

is surjective then it is μ-surjective for some constant μ and we let $\mu_p^{cb}(\Lambda)$ or simply μ_p^{cb} be the smallest constant μ for which this happens.

The following result shows that in this more general setting, $\Lambda(p)_{cb}$-sets can be also viewed as classes of interpolation sets.

Proposition 1.8 *Let $2 < p < \infty$ and $\Lambda \subset G$ be fixed. The next assertions are equivalent.*
i) Λ is a $\Lambda(p)_{cb}$-set.
ii) Λ is an interpolation set for $M_{cb}\big(L^p(\tau_0)\big)$.
Moreover we have $\mu_p^{cb}(\Lambda) \leq \lambda_p^{cb}(\Lambda) \leq K_{L^p(\tau)}\mu_p^{cb}(\Lambda)$ where $K_{L^p(\tau)}$ is the constant defined in the inequality (0.3).

Proof: Assume that Λ has the $\Lambda(p)_{cb}$-property. For $\varepsilon = \big(\varepsilon_t\big)_t$ in $\ell_\infty(\Lambda)$ we let $\tilde{\varepsilon}$ be its extension to $\ell_\infty(G)$ trivially by adding zeros. Then for any $f = \sum_{t\in G}\lambda(t)\otimes x_t$ in $L^p(\tau)$, say with finitely many non-zero operators x_t, we have

$$\begin{aligned}\left\|\sum_{t\in\Lambda}\varepsilon_t\lambda(t)\otimes x_t\right\|_{L^p(\tau)} &\leq \lambda_p^{cb}\ \max\left\{\left\|\left(\sum_{t\in\Lambda}|\varepsilon_t|^2x_t^*x_t\right)^{\frac{1}{2}}\right\|_{S^p}, \left\|\left(\sum_{t\in\Lambda}|\varepsilon_t|^2x_tx_t^*\right)^{\frac{1}{2}}\right\|_{S^p}\right\} \\ &\leq \lambda_p^{cb}\ \left\|\varepsilon\right\|_{\ell_\infty(\Lambda)}\max\left\{\left\|\left(\sum_{t\in G}x_t^*x_t\right)^{\frac{1}{2}}\right\|_{S^p}, \left\|\left(\sum_{t\in G}x_tx_t^*\right)^{\frac{1}{2}}\right\|_{S^p}\right\}.\end{aligned}$$

Hence using (1.6) we get

$$\Big\| \sum_{t\in G} \tilde{\varepsilon}_t \lambda(t) \otimes x_t \Big\|_{L^p(\tau)} \leq \lambda_p^{cb} \,\Big\| \varepsilon \Big\|_{\ell_\infty(\Lambda)} \Big\| f \Big\|_{L^p(\tau)}.$$

Thus $\tilde{\varepsilon}$ is in $M_{cb}(L^p(\tau_0))$ with $\Big\| \tilde{\varepsilon} \Big\|_{M_{cb}(L^p(\tau_0))} \leq \lambda_p^{cb} \Big\| \varepsilon \Big\|_{\ell_\infty(\Lambda)}$ which means that $\mu_p^{cb} \leq \lambda_p^{cb}$. Conversely let Λ be an interpolation set for $M_{cb}(L^p(\tau_0))$ and δ be a fixed positive number. Any choice of signs ε on Λ admits a lifting $\tilde{\varepsilon}$ in $M_{cb}(L^p(\tau_0))$ with $\Big\| \tilde{\varepsilon} \Big\|_{M_{cb}(L^p(\tau_0))} \leq \mu_p^{cb} + \delta$. This implies (since $\varepsilon_t^2 = 1$, $\forall t \in \Lambda$) that for any $f = \sum_{t\in\Lambda} \lambda(t) \otimes x_t$ in $L^p_\Lambda(\tau)$ we have

$$\Big\| f \Big\|_{L^p(\tau)} \leq (\mu_p^{cb} + \delta) \Big\| \sum_{t\in\Lambda} \varepsilon_t \lambda(t) \otimes x_t \Big\|_{L^p(\tau)}.$$

After letting δ tend to 0 we see that each f in $L^p_\Lambda(\tau)$ satisfies for each choice of signs ε

$$\Big\| f \Big\|_{L^p(\tau)} \leq \mu_p^{cb} \Big\| \sum_{t\in\Lambda} \varepsilon_t \lambda(t) \otimes x_t \Big\|_{L^p(\tau)}.$$

We integrate the right inside of the inequality above over all the choices of signs on Λ

$$\begin{aligned} \Big\| f \Big\|_{L^p(\tau)} &\leq \mu_p^{cb} \Big(\int_{\{-1,1\}^\Lambda} \Big\| \sum_{t\in\Lambda} \varepsilon_t \lambda(t) \otimes x_t \Big\|^p_{L^p(\tau)} d\nu(\varepsilon) \Big)^{\frac{1}{p}} \\ &= \mu_p^{cb} \Big\| \sum_{t\in\Lambda} \varepsilon_t \lambda(t) \otimes x_t \Big\|_{L^p\left(\{-1,1\}^\Lambda, \nu, L^p(\tau)\right)}. \end{aligned}$$

Now we apply the non-commutative version of Khintchine inequalities 0.3 to obtain

$$\Big\| f \Big\|_{L^p(\tau)} \leq K_{L^p(\tau)} \mu_p^{cb} \max \left\{ \Big\| \Big(\sum_{t\in\Lambda} x_t x_t^* \Big)^{\frac{1}{2}} \Big\|_{S^p}, \Big\| \Big(\sum_{t\in\Lambda} x_t^* x_t \Big)^{\frac{1}{2}} \Big\|_{S^p} \right\}.$$

That is to say Λ is a $\Lambda(p)_{cb}$–set with $\lambda_p^{cb} \leq K_{L^p(\tau)} \mu_p^{cb}$. ■

Remark. Since the embeddings $M_{cb}(L^\infty(\tau_0)) \subset M_{cb}(L^q(\tau_0)) \subset M_{cb}(L^p(\tau_0)) \subset M(L^2(\tau_0))$ where $2 < p < q < \infty$ are bounded, we see that the $\Lambda(q)_{cb}$-property implies the $\Lambda(p)_{cb}$-property. Thus the family of sets $\Big(\Big\{ \Lambda \subset G \mid \Lambda \text{ is a } \Lambda(p)_{cb}\text{-set} \Big\} \Big)_{2<p<\infty}$ is decreasing for each fixed discrete group G. On the other hand, the $\Lambda(p)_{cb}$-property trivially implies the $\Lambda(p)$-property. Moreover we have for any set $\Lambda \subset G$ and any $2 < p < q < \infty$

$$\lambda_p^{cb}(\Lambda) \leq \lambda_q^{cb}(\Lambda),\ \mu_p(\Lambda) \leq \mu_p^{cb}(\Lambda),\ \lambda_p(\Lambda) \leq \lambda_p^{cb}(\Lambda).$$

Comments 1.9 Clearly, we can naturally extend our definitions to the case of $1 < p \leq 2$. We say that a set $\Lambda \subset G$ is a $K(p)_{cb}$-set if there exists a constant $c > 0$ such that for any sequence $(x_t)_{t\in\Lambda}$ of operators in S^p, say a finitely supported sequence, we have

$$c^{-1} \inf \left\{ \Big\| \Big(\sum_{t\in\Lambda} y_t y_t^* \Big)^{\frac{1}{2}} \Big\|_{S^p} + \Big\| \Big(\sum_{t\in\Lambda} z_t^* z_t \Big)^{\frac{1}{2}} \Big\|_{S^p} \right\} \leq \Big\| \sum_{t\in\Lambda} \lambda(t) \otimes x_t \Big\|_{L^p(\tau)}$$

where the infimum runs over all decompositions of the x_t's in S^p as $x_t = y_t + z_t$. We let $K_p^{cb}(\Lambda)$ stand for the smallest constant c for which this holds. Recall that the converse inequality (with constant 1 instead of c) is satisfied by any set Λ and note that the $K(2)_{cb}$-property is trivial.

Let $1 < p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for a given set $\Lambda \subset G$, the following are equivalent.
i) Λ is a $K(p)_{cb}$-set and each *c.b.* multiplier on $L^p_\Lambda(\tau_0)$ extends to a *c.b.* multiplier on $L^p(\tau_0)$.
ii) Λ is a $\Lambda_{cb}(p')$-set.

Indeed, assume *i)*. By Prop. 1.8, we need to prove that Λ is an interpolation set for $M_{cb}(L^{p'}(\tau_0)) = M_{cb}(L^p(\tau_0))$. Equivalently, we need to prove that the choices of signs on Λ extend uniformly completely boundedly to multipliers on $L^p(\tau_0)$. Let $\varepsilon = \big(\varepsilon_t\big)_{t\in\Lambda}$ be an arbitrary choice of signs on Λ. Then, for any finitely supported sequence $\big(x_t\big)_{t\in\Lambda}$ of operators in S^p, we have

$$\Big\|\sum_{t\in\Lambda}\varepsilon_t \lambda(t)\otimes x_t\Big\|_{L^p(\tau)} \le \inf_{\substack{\varepsilon_t x_t = y_t+z_t\\ y_t,z_t\in S^p}} \left\{\Big\|\Big(\sum_{t\in\Lambda} y_t y_t^*\Big)^{\frac12}\Big\|_{S^p} + \Big\|\Big(\sum_{t\in\Lambda} z_t^* z_t\Big)^{\frac12}\Big\|_{S^p}\right\}$$
$$= \inf_{\substack{x_t = y_t+z_t\\ y_t,z_t\in S^p}} \left\{\Big\|\Big(\sum_{t\in\Lambda} y_t y_t^*\Big)^{\frac12}\Big\|_{S^p} + \Big\|\Big(\sum_{t\in\Lambda} z_t^* z_t\Big)^{\frac12}\Big\|_{S^p}\right\}.$$

Thus by our assumption on Λ we get

$$\Big\|\sum_{t\in\Lambda}\varepsilon_t \lambda(t)\otimes x_t\Big\|_{L^p(\tau)} \le K_p^{cb}(\Lambda)\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}.$$

This means that ε defines a *c.b.* multiplier on $L^p_\Lambda(\tau_0)$ with $\big\|\varepsilon\big\|_{M_{cb}(L^p_\Lambda(\tau_0))} \le K_p^{cb}(\Lambda)$. The second assumption on Λ says that the restriction map

$$\begin{array}{ccc} M_{cb}(L^p(\tau_0)) & \longrightarrow & M_{cb}(L^p_\Lambda(\tau_0)) \\ \varphi & \longmapsto & \varphi_{|\Lambda} \end{array}$$

is surjective, hence μ-surjective for some constant $\mu > 0$. Then, for all $\delta > 0$, each choice of signs ε on Λ extends to a *c.b.* multiplier on $L^p(\tau_0)$ with norm less or equal to $\mu K_p^{cb}(\Lambda) + \delta$. Therefore we are done and we have $\mu_{p'}^{cb}(\Lambda) = \mu_p^{cb}(\Lambda) \le \mu K_p^{cb}(\Lambda)$.
Conversely, assume *ii)*. Then by Prop. 1.8, Λ is an interpolation set for $M_{cb}(L^{p'}(\tau_0)) = M_{cb}(L^p(\tau_0))$. A fortiori, each *c.b.* multiplier on $L^p_\Lambda(\tau_0)$ extends to a *c.b.* multiplier on $L^p(\tau_0)$ since every multiplier on $L^p_\Lambda(\tau_0)$ is in particular a bounded sequence on Λ. On the other hand, let $\delta > 0$ be fixed. Then since every choice of signs ε on Λ admits a lifting $\tilde\varepsilon$ with $\big\|\tilde\varepsilon\big\|_{M_{cb}(L^p(\tau_0))} \le \mu_p^{cb}(\Lambda) + \delta$, we get for every $f = \sum_{t\in\Lambda}\lambda(t)\otimes x_t$ in $L^p_\Lambda(\tau)$

$$\Big\|\sum_{t\in\Lambda}\varepsilon_t \lambda(t)\otimes x_t\Big\|_{L^p(\tau)} \le (\mu_p^{cb}(\Lambda)+\delta)\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}.$$

Hence if we let δ tend to zero, integrate over all these choices of signs and apply the non-commutative version of Khinchine's inequalities (0.2), we obtain

$$K_{L^p(\tau)} \inf_{\substack{x_t = y_t+z_t\\ y_t,z_t\in S^p}} \left\{\Big\|\Big(\sum_{t\in\Lambda} y_t y_t^*\Big)^{\frac12}\Big\|_{S^p} + \Big\|\Big(\sum_{t\in\Lambda} z_t^* z_t\Big)^{\frac12}\Big\|_{S^p}\right\} \le \mu_p^{cb}(\Lambda)\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^p(\tau)}$$

where $K_{L^p(\tau)}$ is the constant of (0.2). This means that Λ is a $K(p)_{cb}$-set. Moreover we have $K_p^{cb}(\Lambda) \leq \frac{\mu_p^{cb}(\Lambda)}{K_{L^p(\tau)}} = \frac{\mu_{p'}^{cb}(\Lambda)}{K_{L^p(\tau)}}$. ■

Remark. Let $1 \leq p_1 < p < 2 < p' < \infty$. Then, it is well known that any $\Lambda(p')$-set is a $\Lambda(p_1)$-set. However, we do not know whether a $\Lambda(p')_{cb}$-set must necessarily be a $K(p_1)_{cb}$-set.

In the sequel, we are interested in finding properties simpler and stronger than the $\Lambda(p)_{cb}$-property. This aim is achieved for even integers by introducing two combinatorial properties called the $B(p)$ and the $Z(p)$-properties. For the group $\mathbb{Z}$, the $B(p)$-property was firstly considered by W. Rudin in [37] while the $Z(2)$-property was introduced by A. Zygmund in his work [46] which justifies the name of the latter property. Thus our properties are nothing but an adaptation of Rudin's property to the case of arbitrary discrete groups, and a generalization of Zygmund's property for arbitrary positive integers and arbitrary discrete groups.

Definition 1.10 *Let $p \geq 2$ be an arbitrary integer. A subset $\Lambda \subset G$ has the $B(p)$-property if for all p-tuples $(t_1, t_2, ..., t_p)$ and $(s_1, s_2, ..., s_p)$ in Λ^p, $t_1^{-1}s_1t_2^{-1}s_2...t_p^{-1}s_p = e$ (e is the unit of G) holds if and only if $\{t_1, t_2, ..., t_p\} = \{s_1, s_2, ..., s_p\}$*[1]*.*

Example. When G is the free group, every free subset Λ of G has the $B(p)$-property. Indeed, let $(t_1, t_2, ..., t_p), (s_1, s_2, ..., s_p)$ in Λ^p be such that $t_1^{-1}s_1t_2^{-1}s_2...t_p^{-1}s_p$ is the empty word and assume $\{t_1, t_2, ..., t_p\} \neq \{s_1, s_2, ..., s_p\}$. Denote by i_0 the first index such that $t_{i_0} \neq s_j$ for all $1 \leq j \leq p$ and i_1 the last index for which $t_{i_0} = t_{i_1}$. Then, we would have $t_{i_0}^{-1}s_{i_0}...s_{i_1-1}t_{i_1}^{-1} = s_{i_0-1}^{-1}...t_1s_p^{-1}...t_{i_1+1}s_{i_1}^{-1}$. The reduced word of $t_{i_0}^{-1}s_{i_0}...s_{i_1-1}t_{i_1}^{-1}$ is expressed with letters in Λ and contains necessarily the letter $t_{i_0}^{-1}$ while the reduced word of $s_{i_0-1}^{-1}...t_1s_p^{-1}...t_{i_1+1}s_{i_1}^{-1}$ which is also expressed with letters in Λ does not contain the letter $t_{i_0}^{-1}$. This means that there exists a word which has two different reduced expressions, both with letters belonging to Λ which contradicts the freeness of Λ. We will see later that this property which is adapted to free groups is well adapted to the case of the group $\mathbb{Z}$.

Definition 1.11 *Let $p \geq 2$ be an arbitrary integer. For each $1 \leq i \leq p$, we set $\nu_i = 1$ when i is even and $\nu_i = -1$ otherwise. Then, we say that a set Λ has the $Z(p)$-property if $Z_p(\Lambda) < \infty$, where*

$$Z_p(\Lambda) := \sup_{\gamma \in G} \left| \left\{ (t_1, t_2, ..., t_p) \in \Lambda^p \middle| \forall i \neq j,\ t_i \neq t_j \,\&\, t_1^{\nu_1}t_2^{\nu_2}...t_p^{\nu_p} = \gamma \right\} \right|.$$

Proposition 1.12 *Let $2 \leq p < \infty$ be an arbitrary integer. Then, the $B(p)$-property implies the $Z(p)$-property. Moreover, each $B(p)$-subset Λ of a discrete group G satisfies $Z_p(\Lambda) \leq (\frac{p}{2}!)^2$ if p is even and $Z_p(\Lambda) \leq (\frac{p+1}{2}!)^2$ if p is odd.*

[1]In each set, elements are repeated a number of times equal to their multiplicity in the corresponding sequence.

Proof: Let $(t_1, t_2, ..., t_p), (s_1, s_2, ..., s_p)$ be p-tuples in Λ^p such that $t_1^{\nu_1} t_2^{\nu_2} ... t_p^{\nu_p} = s_1^{\nu_1} s_2^{\nu_2} ... s_p^{\nu_p}$ with $\big(\nu_j\big)_{1 \leq j \leq p}$ as in Definition 1.11 and $t_i \neq t_j$, $s_i \neq s_j$ for all $1 \leq i \neq j \leq p$. Then, we get $t_1^{\nu_1} t_2^{\nu_2} ... t_p^{\nu_p} s_p^{-\nu_p} ... s_2^{-\nu_2} s_1^{-\nu_1} = e$. Since Λ has the $B(p)$-property, we have necessarily[2]

$$\{t_i; 1 \leq i \leq p, i \text{ odd}\} \cup \{s_i; 1 \leq i \leq p, i \text{ even}\} = \{t_i; 1 \leq i \leq p, i \text{ even}\} \cup \{s_i; 1 \leq i \leq p, i \text{ odd}\}\cdot$$

But $t_i \neq t_j$ and $s_i \neq s_j$ for all $1 \leq i \neq j \leq p$, therefore we have

$$\begin{aligned} \{t_i; 1 \leq i \leq p, i \text{ even}\} &= \{s_i; 1 \leq i \leq p, i \text{ even}\} \\ \{t_i; 1 \leq i \leq p, i \text{ odd}\} &= \{s_i; 1 \leq i \leq p, i \text{ odd}\} \end{aligned}$$

and it is easy to deduce from this the announced control of the constant $Z_p(\Lambda)$. ■

Theorem 1.13 *Let $2 \leq p < \infty$ be an integer and let G be a discrete group. Then, every subset Λ of G with the $Z(p)$-property is a $\Lambda(2p)_{cb}$-set. Moreover, there exists a constant C_p depending only on p such that for each set $\Lambda \subset G$, we have $\lambda_{2p}^{cb}(\Lambda) \leq 3 \max\{Z_p(\Lambda)^{\frac{1}{2p}}, C_p\}$.*

The proof of Theorem 1.13 is much easier to follow in the particular case $p = 2$ for which the proof appears in the Appendix (Prop. 6.1) and we urge the reader to look at it first before studying the proof of Theorem 1.13.

For the proof of Theorem 1.13, it will be convenient to set the following definitions and to use the inequality of Prop. 1.14 which was observed by G. Pisier.

Given a partition $\mathbf{P}$ of $\{1, 2, ..., p\}$, we set $k \equiv l$ $(\mathbf{P})$ for each $1 \leq k, l \leq p$ if k and l belong to a same element in $\mathbf{P}$. Now given two partitions $\mathbf{P}_1$ and $\mathbf{P}_2$ of $\{1, 2, ..., p\}$, we set $\mathbf{P}_1 \leq \mathbf{P}_2$ if for each $1 \leq k, l \leq p$, $k \equiv l$ $(\mathbf{P}_1)$ whenever $k \equiv l$ $(\mathbf{P}_2)$ and we set $\mathbf{P}_1 < \mathbf{P}_2$ if $\mathbf{P}_1 \leq \mathbf{P}_2$ and $|\mathbf{P}_1| < |\mathbf{P}_2|$. This provides the set of all the partitions on $\{1, 2, ..., p\}$ with a partial order for which $\mathbf{P}_{\max} = \big\{\{1\}, \{2\}, ..., \{p\}\big\}$ is a (unique) maximal element and $\mathbf{P}_{\min} = \big\{\{1, 2, ..., p\}\big\}$ is a (unique) minimal one. Finally, the partition $\mathbf{P}_\xi$ on $\{1, 2, ..., p\}$ associated to a given p-tuple $\xi = (\xi_1, \xi_2, ... \xi_p)$ in I^p where I is an arbitrary set, is defined as the unique partition such that for all $1 \leq k, l \leq p$, $k \equiv l$ $(\mathbf{P}_\xi)$ if and only if $\xi_k = \xi_l$.

Proposition 1.14 *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and $(\varepsilon_i)_{i \in I}$ a family of independent random variables with $\mathbb{P}\{\varepsilon_i = 1\} = \frac{1}{2} = \mathbb{P}\{\varepsilon_i = -1\}$ for each $i \in I$. Let $p \geq 2$ be an arbitrary integer and let for $1 \leq j \leq p$, E_j be Banach spaces, $f_j : I \longrightarrow E_j$ be finitely supported functions and $\varphi : E_1 \times E_2 \times ... \times E_p \longrightarrow F$ be a p-linear map of norm less or equal to 1, where F is a given Banach space. Fix a partition $\mathbf{P}$ of the set $\{1, 2, ..., p\}$ and set $A_{\mathbf{P}} := \Big\{j \in \{1, 2, ..., p\} \,|\, \{j\} \in \mathbf{P}\Big\}$. Then we have*

$$\Big\| \sum_{\substack{\xi \in I^p, \mathbf{P}_\xi \leq \mathbf{P} \\ \xi = (\xi_1, \xi_2, ..., \xi_p)}} \varphi\big(f_1(\xi_1), f_2(\xi_2), ..., f_p(\xi_p)\big) \Big\|_F \leq \prod_{j \in A_{\mathbf{P}}} \Big\| \sum_{i \in I} f_j(i) \Big\|_{E_j} \prod_{\substack{j \notin A_{\mathbf{P}} \\ 1 \leq j \leq p}} \Big(\int_\Omega \Big\| \sum_{i \in I} \varepsilon_i f_j(i) \Big\|_{E_j}^p d\mathbb{P} \Big)^{\frac{1}{p}}.$$

[2] An element is repeated in a given set a number of times equal to its multiplicity in this set.

Proof of Prop. 1.14: We start by noticing the following. Given a finite set of indexes $\alpha = \{j_1, j_2, ..., j_s\}$ with s elements ($s \geq 2$), we consider $(s-1)$-independent copies of the family $\big(\varepsilon_i\big)_{i\in I}$ on $(\Omega, \Sigma, \mathbb{P})$ assumed large enough, denoted by $\big(Y_{j_1}(\alpha, i)\big)_{i\in I}$, $\big(Y_{j_2}(\alpha, i)\big)_{i\in I}$, ..., $\big(Y_{j_{s-1}}(\alpha, i)\big)_{i\in I}$. Then we set

$$\begin{array}{rcl} Z_{j_1}(\alpha, i) & = & Y_{j_1}(\alpha, i) \\ Z_{j_k}(\alpha, i) & = & Y_{j_{k-1}}(\alpha, i) Y_{j_k}(\alpha, i), \ \forall 2 \leq k \leq s-1 \\ Z_{j_s}(\alpha, i) & = & Y_{j_{s-1}}(\alpha, i)\cdot \end{array}$$

Clearly, each of the families $\big(Z_{j_1}(\alpha, i)\big)_{i\in I}$, $\big(Z_{j_2}(\alpha, i)\big)_{i\in I}$, ... , $\big(Z_{j_s}(\alpha, i)\big)_{i\in I}$ has the same distribution as the family $\big(\varepsilon_i\big)_{i\in I}$. Moreover, using successively the orthonormality of each of the families $\big(Y_{j_k}(\alpha, i)\big)_{i\in I}$, we check easily that for any function $\eta : \alpha \longrightarrow I$, the integral $\displaystyle\int_\Omega \prod_{k=1}^{s} Z_{j_k}\big(\alpha, \eta(j_k)\big) d\mathbb{P}$ is equal to 1 if the function η is constant on α and 0 otherwise.
Now if we are given a partition $\mathbf{P}$ of $\{1, 2, ..., p\}$ say $\mathbf{P} = \{\alpha_1, \alpha_2, ...\alpha_N\}$ then for each set α_k ($1 \leq k \leq N$) with $|\alpha_k| \geq 2$, we can define as above a family $\Big(Z_j(\alpha_k, i)\Big)_{\substack{i\in I\\ j\in\alpha_k}}$. Moreover we can construct these families so that they are mutually independent. A simple verification shows that

$$\sum_{\substack{\xi\in I^p, \mathbf{P}_\xi \leq \mathbf{P}\\ \xi=(\xi_1,\xi_2,...,\xi_p)}} \varphi\big(f_1(\xi_1), f_2(\xi_2), ..., f_p(\xi_p)\big) = \int_\Omega \varphi\big(\phi_1(\omega), \phi_2(\omega), ..., \phi_p(\omega)\big) d\mathbb{P}(\omega)$$

where we have set for each integer $1 \leq j \leq p$

$$\forall \omega \in \Omega, \ \phi_j(\omega) := \begin{cases} \displaystyle\sum_{i\in I} Z_j(\alpha_k, i)(\omega) f_j(i) & \text{if } j \in \alpha_k \text{ with } |\alpha_k| \geq 2 \\ \displaystyle\sum_{i\in I} f_j(i) & \text{if } j \in A_{\mathbf{P}}. \end{cases}$$

Hence by using Hölder's inequality we get

$$\Big\| \sum_{\substack{\xi\in I^p, \mathbf{P}_\xi \leq \mathbf{P}\\ \xi=(\xi_1,\xi_2,...,\xi_p)}} \varphi\big(f_1(\xi_1), f_2(\xi_2), ..., f_p(\xi_p)\big) \Big\|_F \leq \int_\Omega \big\|\phi_1(\omega)\big\|_{E_1} \big\|\phi_2(\omega)\big\|_{E_2} ... \big\|\phi_p(\omega)\big\|_{E_p} d\mathbb{P}(\omega)$$

$$\leq \prod_{j\in A_{\mathbf{P}}} \|\phi_j\|_{E_j} \prod_{\substack{j\notin A_{\mathbf{P}}\\ 1\leq j\leq p}} \Big(\int_\Omega \|\phi_j\|^p_{E_j} d\mathbb{P} \Big)^{\frac{1}{p}} = \prod_{j\in A_{\mathbf{P}}} \Big\| \sum_{i\in I} f_j(i) \Big\|_{E_j} \prod_{\substack{j\notin A_{\mathbf{P}}\\ 1\leq j\leq p}} \Big(\int_\Omega \Big\| \sum_{i\in I} Z_j(\alpha_k, i) f_j(i) \Big\|^p_{E_j} d\mathbb{P} \Big)^{\frac{1}{p}}$$

$$= \prod_{j\in A_{\mathbf{P}}} \Big\| \sum_{i\in I} f_j(i) \Big\|_{E_j} \prod_{\substack{j\notin A_{\mathbf{P}}\\ 1\leq j\leq p}} \Big(\int_\Omega \Big\| \sum_{i\in I} \varepsilon_i f_j(i) \Big\|^p_{E_j} d\mathbb{P} \Big)^{\frac{1}{p}} \qquad \blacksquare$$

Proof of Theorem 1.13: Let $f = \displaystyle\sum_{t\in\Lambda} \lambda(t) \otimes x_t$ with $t \longmapsto x_t$ finitely supported. Then

$$\Big\|f\Big\|^{2p}_{L^{2p}(\tau)} = \tau\big((f^*f)^p\big) = \Big\|f^{\mu_1} f^{\mu_2} ... f^{\mu_p}\Big\|^2_{L^2(\tau)} = \Big\| \sum_{\gamma\in G} \lambda(\gamma) \otimes \sum_{\substack{\xi_1,\xi_2,...,\xi_p\in\Lambda\\ \xi_1^{\nu_1}\xi_2^{\nu_2}...\xi_p^{\nu_p}=\gamma}} x^{\mu_1}_{\xi_1} x^{\mu_2}_{\xi_2} ... x^{\mu_p}_{\xi_p} \Big\|^2_{L^2(\tau)}$$

where, for each $1 \leq k \leq p$, we have set

$$\begin{cases} \mu_k = 1, \quad \nu_k = 1 & \text{if } k \text{ is even} \\ \mu_k = *, \quad \nu_k = -1 & \text{if } k \text{ is odd.} \end{cases}$$

Then we have

$$\Big\|f\Big\|_{L^{2p}(\tau)}^{2p} = \sum_{\gamma\in G}\Big\| \sum_{\substack{\xi_1,\xi_2,\dots,\xi_p\in\Lambda \\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2 = \sum_{\gamma\in G}\Big\| \sum_{\substack{\mathbf{P}\text{ partition} \\ \text{of }\{1,2,\dots,p\}}} \sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi=\mathbf{P} \\ \xi=(\xi_1,\xi_2,\dots,\xi_p) \\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2$$

$$\leq 2\sum_{\gamma\in G}\Big\| \sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi=\mathbf{P}_{\max} \\ \xi=(\xi_1,\xi_2,\dots,\xi_p) \\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2 + C_p \sum_{\substack{\mathbf{P}\text{ partition} \\ \text{of}\{1,2,\dots,p\} \\ \mathbf{P}\neq\mathbf{P}_{\max}}} \Bigg(\sum_{\gamma\in G}\Big\| \sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi=\mathbf{P} \\ \xi=(\xi_1,\xi_2,\dots,\xi_p) \\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2\Bigg)$$

where C_p is a constant depending only on p and more precisely on the number of partitions of the set $\{1, 2, ..., p\}$. Henceforth, all the constants which will appear during the proof and which depend on p only will be denoted by C_p for simplicity. On the other hand, let

$$\mathcal{S} := \max\Big\{\Big\|\Big(\sum_{t\in\Lambda} x_t^* x_t\Big)^{\frac{1}{2}}\Big\|_{S^{2p}}, \Big(\sum_{t\in\Lambda} x_t x_t^*\Big)^{\frac{1}{2}}\Big\|_{S^{2p}}\Big\}$$

and for each partition $\mathbf{P}$, let

$$\mathcal{S}(\mathbf{P}) := \sum_{\gamma\in G}\Big\| \sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi=\mathbf{P} \\ \xi=(\xi_1,\xi_2,\dots,\xi_p) \\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2.$$

With these notations, the inequality above becomes

$$\Big\|f\Big\|_{L^{2p}(\tau)}^{2p} \leq 2\mathcal{S}(\mathbf{P}_{\max}) + C_p \sum_{\substack{\mathbf{P}\text{ partition of} \\ \{1,2,\dots,p\},\mathbf{P}\neq\mathbf{P}_{\max}}} \mathcal{S}(\mathbf{P}) \tag{1.7}$$

Our aim is to prove that $\mathcal{S}(\mathbf{P}_{\max}) \leq Z_p(\Lambda)\mathcal{S}^{2p}$ and that there exists C_p such that for each partition $\mathbf{P} \neq \mathbf{P}_{\max}$, we have

$$\mathcal{S}(\mathbf{P}) \leq C_p \mathcal{S}^2 \Big\|f\Big\|_{L^{2p}(\tau)}^{2p-2} \tag{1.8}$$

Step 1. The assumption Λ has the $Z(p)$-property ensures $\mathcal{S}(\mathbf{P}_{\max}) \leq Z_p(\Lambda)\mathcal{S}^{2p}$. Indeed

$$\mathcal{S}(\mathbf{P}_{\max}) = \sum_{\gamma\in G}\Big\| \sum_{\substack{\xi_1,\xi_2,\dots,\xi_p\in\Lambda \\ \xi_i\neq\xi_j,\forall 1\leq i\neq j\leq p \\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2 \leq Z_p(\Lambda)\sum_{\gamma\in G} \sum_{\substack{\xi_1,\xi_2,\dots,\xi_p\in\Lambda \\ \xi_i\neq\xi_j,\forall 1\leq i\neq j\leq p \\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}} \Big\|x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2$$

$$\leq Z_p(\Lambda) \sum_{\xi_1,\xi_2,\dots,\xi_p\in\Lambda} \Big\|(x_{\xi_p}^{\mu_p})^*\dots(x_{\xi_2}^{\mu_2})^*(x_{\xi_1}^{\mu_1})^* x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^1}$$

$$= Z_p(\Lambda)\Big\| \sum_{\xi_1,\xi_2,\dots,\xi_p\in\Lambda} (x_{\xi_p}^{\mu_p})^*\dots(x_{\xi_2}^{\mu_2})^*(x_{\xi_1}^{\mu_1})^* x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^1}$$

$$\leq Z_p(\Lambda)\prod_{i=1}^{p}\max\Big\{\Big\|\sum_{\xi_i\in\Lambda}x_{\xi_i}^*x_{\xi_i}\Big\|_{S^p},\Big\|\sum_{\xi_i\in\Lambda}x_{\xi_i}x_{\xi_i}^*\Big\|_{S^p}\Big\}$$

where for the last inequality we applied Corollary 0.9. Therefore $\mathcal{S}(\mathbf{P}_{\max})\leq Z_p(\Lambda)\mathcal{S}^{2p}$.

Step 2. Given an integer $1\leq k\leq p-2$, we show that if (1.8) is satisfied for all the partitions $\mathbf{P}$ with $|\mathbf{P}|\leq k$, then it is also satisfied for all the partitions $\mathbf{P}$ with $|\mathbf{P}|\leq k+1$. Indeed, let $\mathbf{P}_0$ be a fixed partition with $|\mathbf{P}_0|=k+1$, then

$$\mathcal{S}(\mathbf{P}_0)=\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}-\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi<\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2$$

$$\mathcal{S}(\mathbf{P}_0)\leq 2\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2+2\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi<\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2$$

$$\leq 2\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2+2C_p\sum_{\substack{\mathbf{P}\text{ partition of}\\ \{1,2,\dots,p\}\\ \mathbf{P}<\mathbf{P}_0}}S(\mathbf{P}).$$

According to the induction hypothesis, each $\mathbf{P}<\mathbf{P}_0$ satisfies (1.8) since its cardinal satisfies $|\mathbf{P}|<|\mathbf{P}_0|=k+1$, hence we are reduced to prove the following inequality

$$\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2\leq C_p\mathcal{S}^2\Big\|f\Big\|_{L^{2p}(\tau)}^{2p-2}.$$

$$\sum_{\gamma\in G}\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big\|_{S^2}^2=\Big\|\sum_{\gamma\in G}\lambda(\gamma)\otimes\Big(\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\dots\xi_p^{\nu_p}=\gamma}}x_{\xi_1}^{\mu_1}x_{\xi_2}^{\mu_2}\dots x_{\xi_p}^{\mu_p}\Big)\Big\|_{L^2(\tau)}^2$$

$$=\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)}}\big(\lambda(\xi_1)\otimes x_{\xi_1}\big)^{\mu_1}\big(\lambda(\xi_2)\otimes x_{\xi_2}\big)^{\mu_2}\dots\big(\lambda(\xi_p)\otimes x_{\xi_p}\big)^{\mu_p}\Big\|_{L^2(\tau)}^2$$

$$=\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)}}f_1(\xi_1)f_2(\xi_2)\dots f_p(\xi_p)\Big\|_{L^2(\tau)}^2$$

where for each $1\leq j\leq p$, f_j is defined on Λ by setting $f_j(t)=(\lambda(t)\otimes x_t)^{\mu_j},\forall t\in\Lambda$. The f_j's belong to $L^{2p}(\tau)$. At this level, we apply Prop. 1.14 to $\{-1,1\}^{\mathbb{N}}$ equipped with the counting probability ν, to the n_t-th coordinate projection on $\{-1,1\}^{\mathbb{N}}$ denoted by ε_t where $\{n_t,t\in\Lambda\}$ is an enumeration of the set Λ, to the functions $f_j:\Lambda\longrightarrow L^{2p}(\tau)$ above and to the p-linear contractive map which is the product from $L^{2p}(\tau)\times L^{2p}(\tau)\dots\times L^{2p}(\tau)$ (p times) into $L^2(\tau)$. Hence, letting $A_{\mathbf{P}_0}:=\Big\{j\in\{1,2,\dots,p\}\,|\,\{j\}\in\mathbf{P}_0\Big\}$, we get

$$\Big\|\sum_{\substack{\xi\in\Lambda^p,\mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\dots,\xi_p)}}f_1(\xi_1)f_2(\xi_2)\dots f_p(\xi_p)\Big\|_{L^2(\tau)}$$

$$\leq \prod_{j\in A_{\mathbf{P}_0}} \Big\| \sum_{t\in\Lambda} f_j(t)\Big\|_{L^{2p}(\tau)} \prod_{\substack{1\leq j\leq p\\ j\notin A_{\mathbf{P}_0}}} \Big(\int_{\{-1,1\}^{\mathbb{N}}} \Big\| \sum_{t\in\Lambda} \varepsilon_t f_j(t)\Big\|^p_{L^{2p}(\tau)} d\nu\Big)^{\frac{1}{p}}$$

$$= \Big\| f\Big\|^{|A_{\mathbf{P}_0}|}_{L^{2p}(\tau)} \Big(\int_{\{-1,1\}^{\mathbb{N}}} \Big\| \sum_{t\in\Lambda} \varepsilon_t \curlywedge(t)\otimes x_t\Big\|^p_{L^{2p}(\tau)} d\nu\Big)^{\frac{p-|A_{\mathbf{P}_0}|}{p}}$$

since for each $1 \leq j \leq p$, we have

$$\sum_{t\in\Lambda} f_j(t) = \Big(\sum_{t\in\Lambda} \curlywedge(t)\otimes x_t\Big)^{\mu_j} = f^{\mu_j}, \quad \sum_{t\in\Lambda} \varepsilon_t f_j(t) = \Big(\sum_{t\in\Lambda} \varepsilon_t \curlywedge(t)\otimes x_t\Big)^{\mu_j}.$$

On the other hand, applying Jensen's inequality followed by the non-commutative version of Khintchine inequalities proved in [26] and [27], we get for each integer $1 \leq j \leq p$

$$\Big(\int_{\{-1,1\}^{\mathbb{N}}} \Big\| \sum_{t\in\Lambda} \varepsilon_t \curlywedge(t)\otimes x_t\Big\|^p_{L^{2p}(\tau)} d\nu\Big)^{\frac{1}{p}} \leq \Big(\int_{\{-1,1\}^{\mathbb{N}}} \Big\| \sum_{t\in\Lambda} \varepsilon_t \curlywedge(t)\otimes x_t\Big\|^{2p}_{L^{2p}(\tau)} d\nu\Big)^{\frac{1}{2p}}$$

$$\leq K_{L^{2p}(\tau)} \max\Big\{ \Big\| \Big(\sum_{t\in\Lambda} (\curlywedge(t)\otimes x_t)^* (\curlywedge(t)\otimes x_t)\Big)^{\frac{1}{2}} \Big\|_{L^{2p}(\tau)}, \Big\| \Big(\sum_{t\in\Lambda} (\curlywedge(t)\otimes x_t)(\curlywedge(t)\otimes x_t)^*\Big)^{\frac{1}{2}} \Big\|_{L^{2p}(\tau)} \Big\}$$

$$= K_{L^{2p}(\tau)} \max\Big\{ \Big\| \Big(e\otimes \sum_{t\in\Lambda} x_t^* x_t\Big)^{\frac{1}{2}} \Big\|_{L^{2p}(\tau)}, \Big\| \Big(e\otimes \sum_{t\in\Lambda} x_t x_t^*\Big)^{\frac{1}{2}} \Big\|_{L^{2p}(\tau)} \Big\}$$

$$= K_{L^{2p}(\tau)} \max\Big\{ \Big\| \Big(\sum_{t\in\Lambda} x_t^* x_t\Big)^{\frac{1}{2}} \Big\|_{S^{2p}}, \Big\| \Big(\sum_{t\in\Lambda} x_t x_t^*\Big)^{\frac{1}{2}} \Big\|_{S^{2p}} \Big\} = K_{L^{2p}(\tau)} \mathcal{S}.$$

This means that

$$\Big\| \sum_{\substack{\xi\in\Lambda^p, \mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\ldots,\xi_p)}} f_1(\xi_1) f_2(\xi_2)\ldots f_p(\xi_p)\Big\|_{L^2(\tau)} \leq (K_{L^{2p}(\tau)}\mathcal{S})^{p-|A_{\mathbf{P}_0}|} \Big\| f\Big\|^{|A_{\mathbf{P}_0}|}_{L^{2p}(\tau)}.$$

Therefore we have

$$\sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p, \mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\ldots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1} x_{\xi_2}^{\mu_2}\ldots x_{\xi_p}^{\mu_p}\Big\|^2_{S^2} \leq (K_{L^{2p}(\tau)}\mathcal{S})^{2(p-|A_{\mathbf{P}_0}|)} \Big\| f\Big\|^{2|A_{\mathbf{P}_0}|}_{L^{2p}(\tau)}.$$

Now since the inequality $\mathcal{S} \leq \Big\| f\Big\|_{L^{2p}(\tau)}$ always holds and since $\mathbf{P}_0 \neq \mathbf{P}_{\max}$ so that $|A_{\mathbf{P}_0}| < p$, we get from the inequality above the following one

$$\sum_{\gamma\in G} \Big\| \sum_{\substack{\xi\in\Lambda^p, \mathbf{P}_\xi\leq\mathbf{P}_0\\ \xi=(\xi_1,\xi_2,\ldots,\xi_p)\\ \xi_1^{\nu_1}\xi_2^{\nu_2}\ldots\xi_p^{\nu_p}=\gamma}} x_{\xi_1}^{\mu_1} x_{\xi_2}^{\mu_2}\ldots x_{\xi_p}^{\mu_p}\Big\|^2_{S^2} \leq K^{2p}_{L^{2p}(\tau)} \mathcal{S}^2 \Big\| f\Big\|^{2p-2}_{L^{2p}(\tau)}.$$

Step 3. The minimal partition satisfies the inequality (1.8). Indeed, Prop. 1.14 applied similarly to $\mathbf{P}_{\min}$ instead of $\mathbf{P}_0$ gives the much more precise inequality $\mathcal{S}(\mathbf{P}_{\min}) \leq K^{2p}_{L^{2p}(\tau)} S^{2p}$.

Conclusion: By means of Steps 2 and 3, the inequality (1.8) is satisfied for all partitions $\mathbf{P} \neq \mathbf{P}_{\max}$. Therefore, taking into account Step 1, the inequality (1.7) gives

$$\Big\|f\Big\|_{L^{2p}(\tau)}^{2p} \leq 2Z_p(\Lambda)\mathcal{S}^{2p} + C_p\mathcal{S}^2\Big\|f\Big\|_{L^{2p}(\tau)}^{2p-2}.$$

This means, letting $x = \mathcal{S}^{-1}\Big\|f\Big\|_{L^{2p}(\tau)}$ that we have $x^{2p} - C_p x^{2(p-1)} - 2Z_p(\Lambda) \leq 0$ which easily leads to $x \leq 3\max\{Z_p(\Lambda)^{\frac{1}{2p}}, C_p\}$. Hence we are done. ■

As an illustration of Theorem 1.13, we derive the following result already obtained in [19].

Corollary 1.15 *Let $\{g_n; n \in \mathbb{N}\}$ denote an arbitrary free subset of the free group $\mathbb{F}_\infty$. Then for each $2 < p < \infty$, there exists a constant $C_p > 0$ depending only on p such that for each finitely supported sequence $\big(x_n\big)_{n\geq 0}$ of operators each in S^p, we have*

$$\Big\|\sum_{n\in\mathbb{N}} \lambda(g_n)\otimes x_n\Big\|_{L^p(\tau)} \leq C_p \max\left\{\Big\|\Big(\sum_{n\in\mathbb{N}} x_n^* x_n\Big)^{\frac{1}{2}}\Big\|_{S^p}, \Big\|\Big(\sum_{n\in\mathbb{N}} x_n x_n^*\Big)^{\frac{1}{2}}\Big\|_{S^p}\right\} \tag{1.9}$$

Proof: Since $\{g_n; n \in \mathbb{N}\}$ is a free set in $\mathbb{F}_\infty$, it has the $B(p)$-property for all integers $2 \leq p < \infty$. Then, using Theorem 1.13, it has the $\Lambda(2p)_{cb}$-property for all integers $2 \leq p < \infty$. Therefore the inequality (1.9) is satisfied for all real numbers $2 < p < \infty$. ■

Comments 1.16 Taking inverses in the definition of the property $Z(p)$ is compulsory. Indeed, let us say that a subset Λ of G has the property $Z^+(p)$ if the constant $Z_p^+(\Lambda)$ is finite, where

$$Z_p^+(\Lambda) = \sup_{\gamma\in G}\Big|\Big\{(t_1, t_2, ..., t_p) \in \Lambda^p \Big| t_1 t_2 ... t_p = \gamma\Big\}\Big|.$$

When G is Abelian, such a set is certainly a $\Lambda(2p)$-set as shown in [37] but it is not a $\Lambda(2p)_{cb}$-set in general. As an example, take $\Lambda = \{2^i + 2^j; i, j \geq 0\}$ in $G = \mathbb{Z}$. Λ has the $Z^+(p)$-property for all p's but does not have the $\Lambda(p)_{cb}$-property for any $2 < p < \infty$ as we will point out later in Corollary 2.9. However, a $Z^+(p)$-subset of an arbitrary discrete group enjoys a weaker and actually a strictly weaker analytical property namely: for all x_t in S^{2p}, we have

$$\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|_{L^2(\tau)} \leq \sqrt{Z_p^+(\Lambda)}\max\left\{\Big\|\Big(\sum_{t\in\Lambda} x_t x_t^*\Big)^{\frac{p}{2}}\Big\|_{S^2}, \Big(\sum_{t\in\Lambda} x_t^* x_t\Big)^{\frac{p}{2}}\Big\|_{S^2}\right\} \tag{1.10}$$

Indeed,

$$\begin{aligned}\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|_{L^2(\tau)}^2 &= \Big\|\sum_{\gamma\in G}\lambda(\gamma)\otimes\Big(\sum_{\substack{t_1,t_2,...,t_p\in\Lambda\\ t_1t_2...t_p=\gamma}} x_{t_1}x_{t_2}...x_{t_p}\Big)\Big\|_{L^2(\tau)}^2\\
= \sum_{\gamma\in G}\Big\|\sum_{\substack{t_1,t_2,...,t_p\in\Lambda\\ t_1t_2...t_p=\gamma}} x_{t_1}x_{t_2}...x_{t_p}\Big\|_{S^2}^2 &\leq Z_p{}^+(\Lambda)\sum_{\gamma\in G}\sum_{\substack{t_1,t_2,...,t_p\in\Lambda\\ t_1t_2...t_p=\gamma}}\Big\|x_{t_1}x_{t_2}...x_{t_p}\Big\|_{S^2}^2\\
&\leq Z_p{}^+(\Lambda)\Big\|\sum_{t_1,t_2,...,t_p\in\Lambda} x_{t_p}^*...x_{t_2}^*x_{t_1}^*x_{t_1}x_{t_2}...x_{t_p}\Big\|_{S^1}.\end{aligned}$$

Using Corollary 0.9, we get

$$\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|^2_{L^2(\tau)} \leq Z_p^{\,+}(\Lambda)\prod_{j=1}^{p}\max\Big\{\Big\|\sum_{t_j\in\Lambda}x_{t_j}^* x_{t_j}\Big\|_{S^p},\Big\|\sum_{t_j\in\Lambda}x_{t_j}x_{t_j}^*\Big\|_{S^p}\Big\}$$

$$\Big\|\Big(\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big)^p\Big\|^2_{L^2(\tau)} \leq Z_p^{\,+}(\Lambda)\max\Big\{\Big\|\sum_{t\in\Lambda}x_t^* x_t\Big\|^p_{S^p},\Big\|\sum_{t\in\Lambda}x_t x_t^*\Big\|^p_{S^p}\Big\} \qquad \blacksquare$$

In the Abelian case, if a subset Λ has the $Z^+(p)$-property then $L^{2p}_\Lambda(\tau)$ satisfies an inequality analogous to "type 2" *i.e.* for every finitely supported sequence $\big(x_t\big)_{t\in\Lambda}$ with x_t in S^{2p}, we have

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)} \leq \big(Z_p^+(\Lambda)\big)^{\frac{1}{2p}}\Big(\sum_{t\in\Lambda}\Big\|x_t\Big\|^2_{S^{2p}}\Big)^{\frac{1}{2}} \tag{1.11}$$

The proof of the inequality (1.11) we will sketch below is similar to the one given in [43].

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|^{2p}_{L^{2p}(\tau)} = \Big\|\sum_{\gamma\in G}\lambda(\gamma)\otimes\Big(\sum_{\substack{(t_1,t_2,\dots,t_p)\in\Lambda^p\\(\xi_1,\xi_2,\dots,\xi_p)\in\Lambda^p\\t_1\xi_1^{-1}\dots t_p\xi_p^{-1}=\gamma}} x_{t_1}x^*_{\xi_1}x_{t_2}x^*_{\xi_2}\dots x_{t_p}x^*_{\xi_p}\Big)\Big\|_{L^1(\tau)}$$

$$= \Big\|\sum_{\substack{(t_1,t_2,\dots,t_p)\in\Lambda^p\\(\xi_1,\xi_2,\dots,\xi_p)\in\Lambda^p\\t_1\xi_1^{-1}\dots t_p\xi_p^{-1}=e}} x_{t_1}x^*_{\xi_1}x_{t_2}x^*_{\xi_2}\dots x_{t_p}x^*_{\xi_p}\Big\|_{S^1} \leq \sum_{\substack{(t_1,t_2,\dots,t_p)\in\Lambda^p\\(\xi_1,\xi_2,\dots,\xi_p)\in\Lambda^p\\t_1t_2\dots t_p=\xi_1\xi_2\dots\xi_p}}\Big\|x_{t_1}x^*_{\xi_1}x_{t_2}x^*_{\xi_2}\dots x_{t_p}x^*_{\xi_p}\Big\|_{S^1}.$$

For a compact operator y, $\big(s_j(y)\big)_{j\geq 1}$ stands for the decreasing sequence of the eigenvalues[3] of the operator $(y^*y)^{\frac{1}{2}}$. With this notation, we have

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|^{2p}_{L^{2p}(\tau)} \leq \sum_{\substack{(t_1,t_2,\dots,t_p)\in\Lambda^p\\(\xi_1,\xi_2,\dots,\xi_p)\in\Lambda^p\\t_1t_2\dots t_p=\xi_1\xi_2\dots\xi_p}}\sum_{j\geq 1}s_j\big(x_{t_1}x^*_{\xi_1}x_{t_2}x^*_{\xi_2}\dots x_{t_p}x^*_{\xi_p}\big).$$

Then, using the general Horn inequality proved in [43], we obtain the following inequality

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|^{2p}_{L^{2p}(\tau)} \leq \sum_{\substack{(t_1,t_2,\dots,t_p)\in\Lambda^p\\(\xi_1,\xi_2,\dots,\xi_p)\in\Lambda^p\\t_1t_2\dots t_p=\xi_1\xi_2\dots\xi_p}}\sum_{j\geq 1}s_j(x_{t_1})s_j(x^*_{\xi_1})s_j(x_{t_2})s_j(x^*_{\xi_2})\dots s_j(x_{t_p})s_j(x^*_{\xi_p})$$

$$= \sum_{\gamma\in G}\sum_{\substack{(t_1,t_2,\dots,t_p)\in\Lambda^p\\(\xi_1,\xi_2,\dots,\xi_p)\in\Lambda^p\\t_1t_2\dots t_p=\gamma=\xi_1\xi_2\dots\xi_p}}\sum_{j\geq 1}s_j(x_{t_1})s_j(x_{\xi_1})s_j(x_{t_2})s_j(x_{\xi_2})\dots s_j(x_{t_p})s_j(x_{\xi_p})$$

$$= \sum_{j\geq 1}\sum_{\gamma\in G}\Big(\sum_{\substack{t_1,t_2,\dots,t_p\in\Lambda\\t_1t_2\dots t_p=\gamma}} s_j(x_{t_1})s_j(x_{t_2})\dots s_j(x_{t_p})\Big)^2.$$

Using the assumption on Λ, we get

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|^{2p}_{L^{2p}(\tau)} \leq Z_p^+(\Lambda)\sum_{j\geq 1}\sum_{\gamma\in G}\sum_{\substack{t_1,t_2,\dots,t_p\in\Lambda\\t_1t_2\dots t_p=\gamma}} s_j^2(x_{t_1})s_j^2(x_{t_2})\dots s_j^2(x_{t_p})$$

[3]Each eigenvalue is repeated in the sequence a number of times equal to its multiplicity.

$$= Z_p^+(\Lambda)\sum_{j\geq 1}\sum_{t_1,t_2,\dots,t_p\in\Lambda} s_j^2(x_{t_1})s_j^2(x_{t_2})\dots s_j^2(x_{t_p}) = Z_p^+(\Lambda)\sum_{j\geq 1}\Big(\sum_{t\in\Lambda} s_j^2(x_t)\Big)^p.$$

This implies that

$$\Big\|\sum_{t\in\Lambda}\lambda(t)\otimes x_t\Big\|_{L^{2p}(\tau)} \leq \big(Z_p^+(\Lambda)\big)^{\frac{1}{2p}}\bigg(\sum_{j\geq 1}\Big(\sum_{t\in\Lambda} s_j^2(x_t)\Big)^p\bigg)^{\frac{1}{2p}}$$

$$\leq \big(Z_p^+(\Lambda)\big)^{\frac{1}{2p}}\bigg(\sum_{t\in\Lambda}\Big(\sum_{j\geq 1} s_j^{2p}(x_t)\Big)^{\frac{1}{p}}\bigg)^{\frac{1}{2}} \leq \big(Z_p^+(\Lambda)\big)^{\frac{1}{2p}}\bigg(\sum_{t\in\Lambda}\Big\|x_t\Big\|^2_{S^{2p}}\bigg)^{\frac{1}{2}} \qquad \blacksquare$$

2 Non-commutative $\Lambda(p)$-sets in $\mathbb{Z}$

We start by stating the definitions and the results proved in Section 1 for the group $\mathbb{Z}$ in which case $L^p(\tau_0)$ coincides with L^p and $L^p(\tau)$ with $L^p(S^p)$ via the identification for each integer n between the operator $\lambda(n)$ and the function z^n defined on the torus $\mathbb{T}$.

Definition 2.1 *i) Let $2 < p < \infty$. A subset $\Lambda \subset \mathbb{Z}$ is called a $\Lambda(p)$–set $\big($resp. $\Lambda(p)_{cb}$–set$\big)$ if there exists a constant $\lambda > 0$ such that for all f in L^p_Λ $\big($resp. $L^p_\Lambda(S^p)\big)$, say with $\widehat{f}$ finitely supported, we have*

$$\big\|f\big\|_{L^p} \leq \lambda\,\big\|f\big\|_{L^2}$$

$$\Bigg(\textit{resp. } \Big\|f\Big\|_{L^p(S^p)} \leq\ \lambda\ \max\bigg\{\Big\|\Big(\sum_{n\in\Lambda}\widehat{f}(n)^*\widehat{f}(n)\Big)^{\frac{1}{2}}\Big\|_{S^p},\Big\|\Big(\sum_{n\in\Lambda}\widehat{f}(n)\widehat{f}(n)^*\Big)^{\frac{1}{2}}\Big\|_{S^p}\bigg\}\Bigg).$$

We denote by $\lambda_p(\Lambda)$ $\big($resp. $\lambda_p^{cb}(\Lambda)\big)$ or simply λ_p $\big($resp. $\lambda_p^{cb}\big)$ the smallest constant λ for which the inequality above holds.
ii) Let $1 \leq p \leq \infty$. A set $\Lambda \subset \mathbb{Z}$ is said to be an interpolation set for $M(L^p)$ $\big($resp. $M_{cb}(L^p)\big)$ if the restriction map $\mathcal{Q}$ defined on $M(L^p)$ $\big($resp. $M_{cb}(L^p)\big)$ by sending a multiplier φ to the sequence $\big(\varphi(n)\big)_{n\in\Lambda}$ in $\ell_\infty(\Lambda)$, is surjective thus μ–surjective for some constant μ. We let $\mu_p(\Lambda)$ $\big($resp. $\mu_p^{cb}(\Lambda)\big)$ or simply μ_p $\big($resp. $\mu_p^{cb}\big)$ be the smallest constant μ for which this happens.

Proposition 2.2 *Let $2 < p < \infty$. For $\Lambda \subset \mathbb{Z}$, the following properties are equivalent.*
i) Λ is a $\Lambda(p)$–set $\big($resp. $\Lambda(p)_{cb}$–set$\big)$.
ii) Λ is an interpolation set for $M(L^p)$ $\big($resp. $M_{cb}(L^p)\big)$.
Moreover, for each set $\Lambda \subset \mathbb{Z}$, we have

$$\mu_p(\Lambda) \leq \lambda_p(\Lambda) \leq k_p\mu_p(\Lambda)$$
$$\Big(\textit{resp. } \mu_p^{cb}(\Lambda) \leq \lambda_p^{cb}(\Lambda) \leq K_p\mu_p^{cb}(\Lambda)\Big)$$

where k_p (resp. K_p) is the constant defined in the Khintchine inequality (0.1) $\big($*resp.* (0.3)$\big)$.

The following known facts show that there is a restriction on the size of $\Lambda(p)$–sets (thus a fortiori on $\Lambda(p)_{cb}$–sets).

Facts 2.3 *(i)* ([37]) There exists a constant $c_1 > 0$ such that for any $\Lambda(p)$-subset Λ of $\mathbb{Z}$ $(2 < p < \infty)$ and any integers a, b, N with $N \geq 1$, we have

$$\big|\Lambda \cap [a, a+Nb]\big| \leq c_1 \Big(\lambda_p(\Lambda)\Big)^2 N^{\frac{2}{p}}.$$

(ii) ([9], see also [42]) For any fixed $2 < p < \infty$, there exists a constant c_2 such that for any integer $N \geq 1$, there exists $\Lambda_N \subset [0, N]$ satisfying

$$\forall\, N \geq 1, c_2 N^{\frac{2}{p}} \leq \big|\Lambda_N\big| \sup_{N\geq 1} \Big\{\lambda_p(\Lambda_N)\Big\} < \infty.$$

Thus the decreasing family of sets $\Big(\Big\{\Lambda \subset \mathbb{Z} \mid \Lambda \text{ is } \Lambda(p)\Big\}\Big)_{2<p<\infty}$ is in fact strictly decreasing.

In the sequel, we are interested in the size of $\Lambda(p)_{cb}$-sets where p is an even integer. More precisely, our goal is to construct large and actually the largest $\Lambda(p)_{cb}$-sets possible. For this aim, we use the combinatorial properties introduced in Section 1 namely the $B(p)$ and the $Z(p)$-properties which we will recall below.

Definition 2.4 *Let $2 \leq p < \infty$ be an arbitrary integer. We say that a subset Λ of $\mathbb{Z}$ has the $B(p)$-property if for all p-tuples $(n_1, n_2, ..., n_p)$ and $(m_1, m_2, ..., m_p)$ in Λ^p, $\sum_{k=1}^{p} n_k = \sum_{k=1}^{p} m_k$ implies $\{n_k, 1 \leq k \leq p\} = \{m_k, 1 \leq k \leq p\}$*[4]*. We say that Λ has the $Z(p)$-property if $Z_p(\Lambda) < \infty$, where*

$$Z_p(\Lambda) := \sup_{\gamma\in\mathbb{Z}} \Big| \Big\{(n_1, n_2, ..., n_p) \in \Lambda^p \mid \forall 1 \leq i \neq j \leq p, n_i \neq n_j \,\&\, \sum_{k=1}^{p} (-1)^k n_k = \gamma\Big\} \Big| \cdot$$

Theorem 2.5 below was proved previously (*cf.* Section 1) in the more general case of subsets of discrete groups.

Theorem 2.5 *i) If $\Lambda \subset \mathbb{Z}$ has the $B(p)$-property then it has the $Z(p)$-property with $Z_p(\Lambda) \leq (\frac{p}{2}!)^2$ when p is even and $Z_p(\Lambda) \leq (\frac{p+1}{2}!)^2$ when p is odd.*
ii) Every set $\Lambda \subset \mathbb{Z}$ with the $Z(p)$-property has the $\Lambda(2p)_{cb}$-property. Moreover, there exists a constant C_p depending on p only such that $\lambda^{cb}_{2p}(\Lambda) \leq 3 \max\{Z_p(\Lambda)^{\frac{1}{2p}}, C_p\}$.

Corollary 2.6 *For each even integer $p > 2$, there exists a $\Lambda(p)_{cb}$-set which is not a $\Lambda(q)$-set for any $q > p$.*

Proof: Let $p > 2$ be a fixed even integer. By a construction done in [37], there exists a set $\Lambda \subset \mathbb{N}$ which has the $B(\frac{p}{2})$-property and satisfies

$$\overline{\lim_{N\to\infty}} \sup_{a,b\in\mathbb{N}} \frac{\big|\Lambda \cap [a, a+Nb]\big|}{N^{\frac{2}{p}}} > 0\cdot$$

[4]In each set, integers are repeated a number of times equal to their multiplicity in the corresponding sequence.

So there exist sequences of integers $(a_k)_k$, $(b_k)_k$, $(N_k)_k$ with $\lim_{k\to\infty} N_k = \infty$ and a positive constant c such that for each k, we have

$$cN_k^{\frac{2}{p}} \leq \left|\Lambda \cap [a_k, a_k + N_k b_k]\right|\cdot$$

By (i) of Facts 2.3, if Λ has the $\Lambda(q)$-property for some $q > p$ then for each integer k, it satisfies also (c_1 is the constant appearing in this fact)

$$c_1 N_k^{\frac{2}{q}} \geq \left|\Lambda \cap [a_k, a_k + N_k b_k]\right|\cdot$$

Since $q > p$ and N_k can be arbitrary big, we see that this cannot hold. Thus Λ is not a $\Lambda(q)$-set and we are done. ■

We can manage to construct from the Rudin set appearing in the proof above, a sequence of sets as in the following corollary. Corollary 2.7 will be used for the proof of Theorem 4.9 while a reformulation of it will be used to prove Theorem 4.8.

Corollary 2.7 *For each even integer $p > 2$, there exists a sequence of sets $\Lambda_n \subset [2^n, 2^{n+1}[$ such that*

$$\inf_{n\geq 0}\left\{2^{-\frac{2n}{p}}|\Lambda_n|\right\} > 0 \sup_{n\geq 0}\left\{\lambda_p^{cb}(\Lambda_n)\right\} < \infty.$$

The next result shows that the $\Lambda(p)_{cb}$-property is much more restrictive than the usual $\Lambda(p)$-property.

Proposition 2.8 *There exists a numerical constant $\delta > 0$ such that for each $2 < p < \infty$ and each $\Lambda(p)_{cb}$-set Λ, if Λ contains the sum $A + A$ of some arbitrary finite set A then we have $|A| < \delta\big(2\lambda_p^{cb}(\Lambda)\big)^{\frac{8p}{p-2}}$.*

Proof: **Step 1.** If a $\Lambda(p)_{cb}$-set Λ contains the sum $B + B$ of some finite set B satisfying the property $(\star)$ below, then we have necessarily $|B| \leq 2\Big(2\lambda_p^{cb}(\Lambda)\Big)^{\frac{2p}{p-2}}$.

We say that a set of integers B satisfies the property $(\star)$ if given an enumeration of the elements of B say $B = \{b_1, b_2, ...\}$ ($b_k \neq b_l$ whenever $k \neq l$), the following property holds: for each $k \geq 1$, $\beta_k = -1, 0$ or 1 and $\sum_{k\geq 1} |\beta_k| \leq 4$, we have

$$\sum_{k\geq 1} \beta_k b_k = 0 \implies \forall\ k \geq 1,\ \beta_k b_k = 0\cdot$$

Let Λ be a $\Lambda(p)_{cb}$-set and assume that there exists a set $B = \{b_1, b_2, ..., b_n\}$ with cardinality $n \geq 1$ satisfying $(\star)$ and such that Λ contains $B + B = \{b_k + b_l, 1 \leq k, l \leq n\}$. Let $\varepsilon = \big(\varepsilon_{kl}\big)_{1\leq k,l\leq n}$ be such that $\forall 1 \leq k, l \leq n$

$$\begin{aligned}
&\varepsilon_{kl} = \varepsilon_{lk} = \pm 1,\ \forall k \neq l \\
&\varepsilon_{kk} = 1 \text{ if } \ \forall i \neq j, 2b_k \neq b_i + b_j \\
&\varepsilon_{kk} = \varepsilon_{ij} \text{ if } \ \exists i \neq j\ s.\ t.\ 2b_k = b_i + b_j.
\end{aligned}$$

Note that ε is well defined since B satisfies the property $(\star)$. We are interested in controling the norm of the operator T_ε associated to ε defined on S_n^p by sending $x = \big(x_{kl}\big)_{1\leq k,l\leq n}$ to $\big(\varepsilon_{kl}x_{kl}\big)_{1\leq k,l\leq n}$. Let $x = \big(x_{kl}\big)_{1\leq k,l\leq n}$ be a fixed operator in S_n^p and consider the function f_x which takes z in $\mathbb{T}$ to $\Big(z^{b_k+b_l}x_{kl}\Big)_{1\leq k,l\leq n}$ in S_n^p. Such a function f_x is clearly well defined and belongs to $L^p(S_n^p)$. Moreover, we have for all z in $\mathbb{T}$, $f_x(z) = D_z x D_z$ where by definition D_z is the unitary operator below

$$D_z = \begin{pmatrix} z^{b_1} & 0 & 0 & \dots & 0 \\ 0 & z^{b_2} & 0 & \dots & 0 \\ 0 & 0 & z^{b_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & z^{b_n} \end{pmatrix}.$$

Therefore $\big\|f_x\big\|_{L^p(S_n^p)} = \big\|x\big\|_{S_n^p}$. We define the map ν on $\mathbb{Z}$ by setting $\nu(b_k+b_l) := \varepsilon_{kl}$ for all $1 \leq k,l \leq n$ and $\nu(s) := 0$ whenever s belongs to $\mathbb{Z} \setminus B+B$. Note that ν is well defined since B has the property $(\star)$. ν is the trivial extension to $\mathbb{Z}$ of some choice of signs on $B+B$. Hence $\big\|\nu\big\|_{M_{cb}(L^p)} \leq \lambda_p^{cb}(B+B) \leq \lambda_p^{cb}(\Lambda)$. That is to say the operator M_ν associated to the multiplier ν is such that $M_\nu \otimes id_{S^p}$ is bounded on $L^p(S^p)$ with $\Big\|M_\nu \otimes id_{S^p}\Big\| \leq \lambda_p^{cb}(\Lambda)$. We check easily that $M_\nu \otimes id_{S_n^p}(f_x) = f_{T_\varepsilon(x)}$. Thus

$$\Big\|T_\varepsilon(x)\Big\|_{S_n^p} = \Big\|f_{T_\varepsilon(x)}\Big\|_{L^p(S_n^p)} \leq \lambda_p^{cb}(\Lambda)\Big\|f_x\Big\|_{L^p(S_n^p)} = \lambda_p^{cb}(\Lambda)\Big\|x\Big\|_{S_n^p}.$$

Hence the operator T_ε above has norm less or equal to $\lambda_p^{cb}(\Lambda)$. This implies, if we denote by α_m the unconditionality constant of the canonical basis of S_m^p where $m = \frac{n}{2}$ if n is even and $m = \frac{n-1}{2}$ if n is odd, that $\alpha_m \leq \lambda_p^{cb}(\Lambda)$. Indeed, given a choice of signs $\varepsilon = \big(\varepsilon_{kl}\big)_{1\leq k,l\leq m}$ we can consider the map ε' defined on $\{1,2,...n\}\times\{1,2,...n\}$ by setting for each $1 \leq k \leq n$

$$\begin{aligned} &\varepsilon'(k,k) = 1 \text{ if } \forall i \neq j, 2b_k \neq b_i + b_j \\ &\varepsilon'(k,k) = \varepsilon'(i,j) \text{ if } \exists\ 1 \leq i \neq j \leq n\ s.\ t.\ 2b_k = b_i + b_j \end{aligned}$$

and for all $1 \leq l,k \leq m$

$$\begin{aligned} &\varepsilon'(k+m,l) = \varepsilon_{kl},\ \varepsilon'(k,l+m) = \varepsilon_{lk} \\ &\varepsilon'(k,l) = \varepsilon'(k+m,l+m) = 1 \text{ if } k \neq l \end{aligned}$$

when n is even and

$$\begin{aligned} &\varepsilon'(k+m+1,l) = \varepsilon_{kl},\ \varepsilon'(k,l+m+1) = \varepsilon_{lk} \\ &\varepsilon'(k,l) = \varepsilon'(k+m+1,l+m+1) = 1 \text{ if } k \neq l \\ &\varepsilon'(m+1,l) = \varepsilon'(m+1,l+m+1) = 1 \end{aligned}$$

when n is odd. Using what we said before, we see that $\big\|T_{\varepsilon}\big\|_{\mathcal{B}(S_m^p)} \leq \big\|T_{\varepsilon'}\big\|_{\mathcal{B}(S_n^p)} \leq \lambda_p^{cb}(\Lambda)$. By the results of [22] we have $m^{\frac{1}{2}} \leq 2\alpha_m m^{\frac{1}{p}}$. Thus $m^{\frac{1}{2}} \leq 2\lambda_p^{cb}(\Lambda)m^{\frac{1}{p}}$, equivalently $|B| \leq 2\Big(2\lambda_p^{cb}(\Lambda)\Big)^{\frac{2p}{p-2}}$.

Step 2. There exists a numerical constant $\delta > 0$ such that each finite subset A of $\mathbb{Z}$ contains a subset B with cardinality $|B| > \frac{1}{\delta}|A|^{\frac{1}{4}}$ and satisfying the property $(\star)$.

We start by picking up an arbitrary b_1 in the set A and assume that for some integer $k \geq 1$, we chose k elements $b_1, b_2, ..., b_k$ in A such that $\{b_1, b_2, ..., b_k\}$ enjoys the property $(\star)$. Then consider the set

$$[b_1, b_2, ..., b_k] := \Big\{ \sum_{l=1}^{k} \beta_l b_l \,\big|\, \forall 1 \leq l \leq k, \beta_l = 0, 1, -1 \,\&\, \sum_{l=1}^{k} |\beta_l| \leq 4 \Big\}.$$

$$\big|[b_1, b_2, ..., b_k]\big| \leq 3^4 \frac{k!}{4!(k-4)!} < \frac{27}{8} k^4.$$

Assume that $|A| > \frac{27}{8}k^4$. Then there exists at least one element b_{k+1} in $A \setminus [b_1, b_2, ..., b_k]$. We check easily that the set $\{b_1, b_2, ..., b_k, b_{k+1}\}$ still has the property $(\star)$. Indeed, suppose that $\sum_{l=1}^{4} \beta_l b_{i_l} = 0$ for some $\beta_l = -1, 0$ or -1 and $1 \leq i_l \leq k+1$. Since $\{b_1, b_2, ..., b_k\}$ has the property $(\star)$, the only case to check is the case $\sum_{l=1}^{3} \beta_l b_{i_l} - b_{k+1} = 0$. But the latter does not hold since it would contradict the assumption b_{k+1} does not belong to $[b_1, b_2, ..., b_k]$. This means that we can find by induction a set $B = \{b_1, b_2, ..., b_n, b_{n+1}\}$ having the property $(\star)$ where n is such that $\frac{27}{8}n^4 < |A| < \frac{27}{8}(n+1)^4$ thus $|B| > \Big(\frac{8}{27}|A|\Big)^{\frac{1}{4}}$.

Conclusion: If Λ contains the sum $A + A$ of some finite set A, then by Step 2 it contains a sum $B + B$ of some set B which has the property $(\star)$ and cardinality $|B| > \Big(\frac{8}{27}|A|\Big)^{\frac{1}{4}}$. Applying Step 1 we get $|B| \leq 2\Big(2\lambda_p^{cb}(\Lambda)\Big)^{\frac{2p}{p-2}}$. Therefore the cardinality of A satisfies

$$|A| < 54\Big(2\lambda_p^{cb}(\Lambda)\Big)^{\frac{8p}{p-2}} := \delta\Big(2\lambda_p^{cb}(\Lambda)\Big)^{\frac{8p}{p-2}}$$

■

Corollary 2.9 *There exists a set which is $\Lambda(p)$ for each $2 < p < \infty$ but not $\Lambda(p)_{cb}$ for any $2 < p < \infty$.*

Proof: Consider the set $\Lambda = \{2^k + 2^l; k, l \geq 0\}$. It is well known that the set Λ is a $\Lambda(p)$-set for all $2 < p < \infty$ (*cf.* [24], see also Comments 1.16). But by Prop. 2.8, Λ cannot be a $\Lambda(p)_{cb}$-set for any $2 < p < \infty$. ■

3 Applications to Fourier multipliers

Proposition 3.1 *For each $2 < p < \infty$, the inclusion map $M_{cb}(L^p) \subset M(L^p)$ is strict.*

Proof: Let $2 < p < \infty$. By Corollary 2.9, there exists a set Λ which is $\Lambda(p)$ but not $\Lambda(p)_{cb}$. Hence, by Prop. 2.2, it is an interpolation set for $M(L^p)$ and is not an interpolation set for $M_{cb}(L^p)$. Thus a fortiori, the embedding of $M_{cb}(L^p)$ into $M(L^p)$ is strict. ■

Comment. It was shown in [21] that a Banach space X is a subspace of a quotient of L^p if and only if there exists a constant c such that, for any bounded operator T on L^p, the operator $T \otimes id_X$ extends to a bounded operator on $L^p(X)$ with $\|T \otimes id_X\| \leq c\, \|T\|$. Prop. 3.1 implies that there exists an operator T on L^p such that the map $T \otimes id_{S^p}$ is not bounded which means that S^p is not a subspace of a quotient of L^p (*cf.* [31]).

Proposition 3.2 *For each $2 < p < q \leq \infty$ where p is an even integer, the inclusion map $M_{cb}(L^q) \subset M_{cb}(L^p)$ is strict. Moreover, $M_{cb}(L^p)$ does not embed continuously into $M(L^q)$.*

Proof: Let $2 < p < \infty$ be an even integer. By Corollary 2.6, there exists a $\Lambda(p)_{cb}$-set Λ which is not a $\Lambda(q)$-set for any $q > p$. Hence, by Prop. 2.2, it is an interpolation set for $M_{cb}(L^p)$ and is not an interpolation set for $M(L^q)$. Thus, $M_{cb}(L^p)$ does not embed continuously into $M(L^q)$. ∎

As a direct application of Lemma 0.2, the interpolated space $\Big(M(L^\infty), M(L^2)\Big)_{\frac{2}{p}}$ embeds contractively into $M_{cb}(L^p)$ for each $2 < p < \infty$. Thus, it is natural to wonder whether we do have equality. The following Lemmas will be used for the study of this converse as well as for the proof of Theorem 5.2.

Lemma 3.3 *Let X, Y be two Banach spaces and $u : X \longrightarrow Y$ be a bounded operator. Assume there exist $c > 0$ and $0 < r < 1$ such that for all y in B_Y , there exists x in cB_X with $\|ux - y\| < r$. Then, the operator u is surjective. More precisely, u is μ-surjective for some constant $\mu \leq \frac{c}{1-r}$.*

Proof: The proof is elementary. Indeed, for y in B_Y, we can produce a sequence of vectors y_k in B_Y with $y_1 = y$, a sequence of vectors x_k in X such that $\|x_k\| < c$, $y_{k+1} = \frac{1}{r}(y_k - ux_k)$ and $\|ux_k - y_k\| < r$. Then let $x = \sum_{k\geq 0} r^k x_{k+1}$. We clearly have $y = ux$ and $\|x\| \leq \frac{c}{1-r}$. This means that u is μ-surjective for some $\mu \leq \frac{c}{1-r}$. ∎

Lemma 3.4 *Let $0 < \theta < 1$ and x in $(X_0, X_1)_\theta$. Then for all $s, \delta > 0$ there exist x_0 in X_0, x_1 in X_1 such that $x = x_0 + x_1$ and $\|x_0\|_{X_0} + s\|x_1\|_{X_1} \leq s^\theta \|x\|_\theta + \delta$.*

Proof: For the proof, the reader is referred to page 103 of [2]. ∎

Lemma 3.5 *Let (X_0, X_1) be a compatible couple of Banach spaces and Y be an arbitrary Banach space. Consider two operators $u_0 : X_0 \longrightarrow Y$, $u_1 : X_1 \longrightarrow Y$ which agree on the intersection $X_0 \cap X_1$. Assume that the operator obtained by complex interpolation $u_\theta : X_\theta \longrightarrow Y$ is μ-surjective. Then u_0 (resp. u_1) is necessarily surjective. Moreover, it is $\alpha_0(\theta,\mu)$-surjective (resp. $\alpha_1(\theta,\mu)$-surjective) for some constant satisfying*

$$\alpha_0(\theta,\mu) \leq \alpha(\theta)\mu^{\frac{1}{1-\theta}} \|u_1\|^{\frac{\theta}{1-\theta}}$$

$$\left(\text{resp. } \alpha_1(\theta,\mu) \leq \alpha(1-\theta)\mu^{\frac{1}{\theta}} \|u_0\|^{\frac{1-\theta}{\theta}}\right)$$

where $\alpha(\theta) = \frac{1}{1-\theta}\theta^{\frac{\theta}{\theta-1}}$.

Proof: It suffices to prove the part of the lemma concerning u_0 since for each $0 < \theta < 1$, $(X_0, X_1)_\theta = (X_1, X_0)_{1-\theta}$. Then by replacing u_0, u_1 and μ by $u_0\|u_1\|^{-1}$, $u_1\|u_1\|^{-1}$ and $\mu\|u_1\|$ respectively, we can assume that u_1 has norm one for the proof ($u_1 \neq 0$). Let y be in B_Y. Since u_θ is μ-surjective, there exists x in X_θ with $\|x\|_\theta < \mu$ and $u_\theta(x) = y$. For a fixed $s > 0$, there exists by Lemma 3.4, a decomposition of x as a sum $x_0 + x_1$ satisfying $\|x_0\|_{X_0} < s^\theta \mu$, $\|x_1\|_{X_1} < s^{\theta-1}\mu$. Let us start from s such that $s^{\theta-1}\mu < 1$ equivalently $s > \mu^{\frac{1}{1-\theta}}$. Then we see that $u_0 : X_0 \longrightarrow Y$ satisfies the conditions of Lemma 3.3 with $r = r(s) = s^{\theta-1}\mu$ and $c = c(s) = s^\theta \mu$. Hence, Lemma 3.3 implies that u_0 is $\mu(s)$-surjective for some constant $\mu(s) \leq \frac{c(s)}{1-r(s)}$. The infimum of $s \longmapsto \frac{c(s)}{1-r(s)}$ when s runs in $]\mu^{\frac{1}{1-\theta}}, \infty[$ is attained in $s_{\min} = \left(\frac{\theta}{\mu}\right)^{\frac{1}{\theta-1}}$ and its value is $\alpha(\theta)\mu^{\frac{1}{1-\theta}}$ with $\alpha(\theta)$ as in the statement of Lemma 3.5. ■

Proposition 3.6 *For each $2 < p < \infty$, the interpolated space $\Big(M(L^\infty), M(L^2)\Big)_{\frac{2}{p}}$ embeds strictly into $M_{cb}(L^p)$. Moreover, the space $M_{cb}(L^p)$ does not embed into any interpolated space $\Big(M(L^\infty), M(L^2)\Big)_\theta$ for any $0 < \theta < 1$.*

Proof: Since $M_{cb}(L^p)$ embeds into $M_{cb}(L^s)$ for each $2 < s < p < \infty$, we can restrict ourselves to even integers p. Then, let $\Lambda \subset \mathbb{Z}$ be any interpolation set for $M_{cb}(L^p)$ which is not an interpolation set for any $M(L^q)$ when $q > p$. Corollary 2.6 implies that such a set Λ exists. This means that the restriction map $\mathcal{Q}$ which carries a multiplier φ in $M_{cb}(L^p)$ to the sequence $\big(\varphi(n)\big)_{n\in\Lambda}$ in $\ell_\infty(\Lambda)$ is surjective and a fortiori if we suppose that $M_{cb}(L^p)$ embeds into $\Big(M(L^\infty), M(L^2)\Big)_\theta$ for some real number $0 < \theta < 1$, the map $\mathcal{Q} : \Big(M(L^\infty), M(L^2)\Big)_\theta \longrightarrow \ell_\infty(\Lambda)$ will be surjective too. Lemma 3.5 implies then that $\mathcal{Q} : M(L^\infty) \longrightarrow \ell_\infty(\Lambda)$ will be surjective. Hence, Λ will be an interpolation set for $M(L^\infty)$ and thus for all $M(L^q)$ which is false. This contradiction completes the proof. ■

4 $\sigma(p)$-sets and $\sigma(p)_{cb}$-sets

Definition 4.1 *Let $2 < p < \infty$. A subset A of $\mathbb{N} \times \mathbb{N}$ is called a $\sigma(p)$-set if there exists a constant $C > 0$ such that for all $x = \big(x_{ij}\big)_{i,j}$ in S^p_A, we have*

$$\|x\|_{S^p} \leq C \max\Big\{\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}, \Big(\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty}|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\}.$$

We denote by $\sigma_p(A)$ the smallest constant $C > 0$ for which the previous inequality holds.

Recall that when $2 \leq p \leq \infty$ the inequality below is satisfied for each $x = (x_{ij})_{i,j}$ in S^p

$$\|x\|_{S^p} \geq \max\Big\{\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}, \Big(\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty}|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\}$$

so that A is a $\sigma(p)$-set if and only if the following are equivalent norms on the space S^p_A

$$\big\|x\big\|_{S^p} \cong \max\Big\{\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}},\Big(\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty}|x_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\},\ \forall x=\big(x_{ij}\big)_{i,j}\in S^p_A.$$

In other words, A is a $\sigma(p)$-set if and only if the spaces S^p_A and $S^{p,unc}_A$ are isomorphic.

Remarks 4.2 As first properties of $\sigma(p)$-sets, we mention the following.
(i) Every subset A_1 of a $\sigma(p)$-set A_2 is a $\sigma(p)$-set with $\sigma_p(A_1)\le\sigma_p(A_2)$.
(ii) If A_1 and A_2 are $\sigma(p)$-sets, then $A_1\cup A_2$ is also a $\sigma(p)$-set. In this case, we have $\sigma_p(A_1\cup A_2)\le\sigma_p(A_1)+\sigma_p(A_2)$.
(iii) A set A is a $\sigma(p)$-set if and only if ${}^tA:=\{(i,j)\in\mathbb{N}\times\mathbb{N}\,|\,(j,i)\in A\}$ is a $\sigma(p)$-set. Moreover, we have $\sigma_p({}^tA)=\sigma_p(A)$.
$(4i)$ Let $\big(n_k\big)_{k\ge1}$ be a strictly increasing sequence of integers and let A_k be a subset of $[n_{k-1},n_k[\times[n_{k-1},n_k[$ for each $k\ge1$. Then $A=\bigcup_{k\ge1}A_k$ is a $\sigma(p)$-set if and only if $\sup_{k\ge1}\sigma_p(A_k)$ is finite. Moreover, we have $\sup_{k\ge1}\sigma_p(A_k)\le\sigma_p(A)\le 2\sup_{k\ge1}\sigma_p(A_k)$.

The reader is requested to consult Subsection 0.9 for the definition of the set of Schur multipliers on S^p.

Proposition 4.3 *Let $2<p<\infty$ and A be a subset of $\mathbb{N}\times\mathbb{N}$. Then, the following assertions are equivalent.*
i) A is a $\sigma(p)$-set.
ii) The canonical basis $\big\{e_{ij}\,|\,(i,j)\in A\big\}$ is an unconditional basis of S^p_A.
iii) The restriction map below is surjective

$$\begin{array}{rcl}\mathcal{Q}:M(S^p) & \longrightarrow & \ell_\infty(A)\\ \varphi & \longmapsto & \varphi_{|_A}.\end{array}$$

Letting $\alpha_p(A)$ denote the unconditionality constant of the canonical basis of S^p_A and $\mu_p(A)$ denote the smallest constant μ for which the above $\mathcal{Q}$ is μ-surjective, we see that for each set $A\subset\mathbb{N}\times\mathbb{N}$, we have $\big(K_p=K_{S^p}$ is the constant defined in the inequality $(0.3)\big)$

$$\begin{array}{c}\alpha_p(A)\le\sigma_p(A)\le K_p\alpha_p(A)\\ \mu_p(A)\le\sigma_p(A)\le K_p\mu_p(A)\cdot\end{array}$$

Proof: The proof which is analogous to the proof of Prop. 1.8 is left to the reader. ■

Definition 4.4 *Let $2<p<\infty$. A subset A of $\mathbb{N}\times\mathbb{N}$ is called a $\sigma(p)_{cb}$-set if there exists a constant $C>0$ such that for all $x=\big(x_{ij}\big)_{i,j}$ in $S^p_A(S^p)$, we have*

$$\big\|x\big\|_{S^p(S^p)}\le C\max\Big\{\Big(\sum_i\Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{\frac12}\Big\|^p_{S^p}\Big)^{\frac1p},\Big(\sum_j\Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{\frac12}\Big\|^p_{S^p}\Big)^{\frac1p}\Big\}.$$

We denote by $\sigma^{cb}_p(A)$ the infimum of the constants C for which the inequality above holds.

Since (0.4) always holds, a set A is a $\sigma(p)_{cb}$-set if and only if for all $x = \big(x_{ij}\big)_{i,j}$ in $S^p_A(S^p)$

$$\Big\|x\Big\|_{S^p(S^p)} \cong \max\Big\{\Big(\sum_i \Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{\frac{1}{2}}\Big\|^p_{S^p}\Big)^{\frac{1}{p}}, \Big(\sum_j \Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{\frac{1}{2}}\Big\|^p_{S^p}\Big)^{\frac{1}{p}}\Big\}.$$

Equivalently, A is a $\sigma(p)_{cb}$-set if and only if the spaces $S^p_A(S^p)$ and $S^{p,unc}_A(S^p)$ are isomorphic. All the properties quoted in Remarks 4.2 have an analogous version for $\sigma(p)_{cb}$-sets. Prop. 4.3 has also a *c.b.* version as follows. We will skip all the proofs.

Proposition 4.5 *Let $2 < p < \infty$ and A be a subset of $\mathbb{N}\times\mathbb{N}$. Then, the following are equivalent.*
i) A is a $\sigma(p)_{cb}$-set.
ii) The restriction map $\mathcal{Q}$ which takes $\varphi \in M_{cb}(S^p)$ to $\varphi_{|_A} \in \ell_\infty(A)$, is surjective.
iii) The operators T_ε where $\varepsilon = \big(\varepsilon_{ij}\big)_{i,j}$ with $\varepsilon_{ij} = 1$ or -1 if $(i,j) \in A$ and $\varepsilon_{ij} = 0$ if not, defined on S^p by sending an operator $x = \big(x_{ij}\big)_{i,j}$ to $T_\varepsilon(x) = \big(\varepsilon_{ij}x_{ij}\big)_{i,j}$ are uniformly c.b..
Letting $\alpha^{cb}_p(A)$ denote the unconditionality constant of the canonical basis of S^p_A viewed as an operator space and $\mu^{cb}_p(A)$ denote the smallest constant μ for which $\mathcal{Q}$ is μ-surjective, we see that for each set $A \subset \mathbb{N}\times\mathbb{N}$, we have $\big(K_p = K_{S^p}$ is the constant defined in (0.3)$\big)$

$$\alpha^{cb}_p(A) \le \sigma^{cb}_p(A) \le K_p\alpha^{cb}_p(A)$$
$$\mu^{cb}_p(A) \le \sigma^{cb}_p(A) \le K_p\mu^{cb}_p(A).$$

Remarks 4.6 *(i)* Since $M(S^q) \subset M(S^p)$, $M_{cb}(S^q) \subset M_{cb}(S^p)$ for all $2 < p < q < \infty$ and the embeddings are both contractive, the $\sigma(q)$-property implies the $\sigma(p)$-property, the $\sigma(q)_{cb}$-property implies the $\sigma(p)_{cb}$-property and we have $\sigma_p(A) \le \sigma_q(A)$, $\sigma^{cb}_p(A) \le \sigma^{cb}_q(A)$ for each set A. On the other hand, the $\sigma(p)_{cb}$-property implies trivially the $\sigma(p)$-property and we have $\sigma_p(A) \le \sigma^{cb}_p(A)$ for each set A.
(ii) Each bounded map $\varphi = \big(\varphi_{ij}\big)_{i,j}$ supported by A defines a Schur multiplier on S^p $\big($resp. a *c.b.* Schur multiplier on $S^p\big)$ whenever A is a $\sigma(p)$-set $\big($resp. $\sigma(p)_{cb}$-set$\big)$ and we have

$$\Big\|\varphi\Big\|_{M(S^p)} \le \sigma_p(A)\Big\|\varphi\Big\|_{\ell_\infty(\mathbb{N}\times\mathbb{N})}\Big(\text{resp. } \Big\|\varphi\Big\|_{M_{cb}(S^p)} \le \sigma^{cb}_p(A)\Big\|\varphi\Big\|_{\ell_\infty(\mathbb{N}\times\mathbb{N})}\Big).$$

This applies in particular for the indicator function $1\!\!1_A$ of a $\sigma(p)$-set $\big($resp. a $\sigma(p)_{cb}$-set$\big)$.
(iii) The preceding results can be extended to the case $p = \infty$, but then the resulting notion is entirely elucidated by the work of N. Varopoulos (*cf.* [44]) who characterized the sets $A \subset \mathbb{N}\times\mathbb{N}$ for which the restriction map

$$\begin{array}{rcl}\mathcal{Q} : M(S^\infty) & \longrightarrow & \ell_\infty(A)\\ \varphi & \longmapsto & \varphi_{|A}\end{array}$$

is surjective. His work shows that this holds if and only if A can be written as a finite union of 1-sections and 2-sections in the following sense. We say that a subset $A \subset \mathbb{N}\times\mathbb{N}$ is a 1-section (resp. 2-section) if the first (resp. second) coordinate projection is injective when restricted to A.
(iv) The definition of $\sigma(p)$-sets and $\sigma(p)_{cb}$-sets can be extended to the case where $1 \le p \le 2$ as follows. Roughly speaking, a subset A of $\mathbb{N}\times\mathbb{N}$ is called a $\sigma(p)$-set if S^p_A is isomorphic

to $S_A^{p,unc}$ and it is called a $\sigma(p)_{cb}$-set if $S_A^p(S^p)$ is isomorphic to $S_A^{p,unc}(S^p)$. Moreover, we let $\sigma_p(A)$ be the smallest constant $C>0$ such that for all $x=\big(x_{ij}\big)_{i,j}$ in S_A^p, we have

$$\Big\|x\Big\|_{S^p}\geq C^{-1}\inf\Big\{\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}|y_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}+\Big(\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty}|z_{ij}|^2\Big)^{\frac{p}{2}}\Big)^{\frac{1}{p}}\Big\}$$

where the infimum runs over all the decompositions of x as $x=y+z$ with $y=(y_{ij})_{i,j}$ and $z=(z_{ij})_{i,j}$ both in S^p. We let $\sigma_p^{cb}(A)$ be the smallest constant $C>0$ such that for all $x=\big(x_{ij}\big)_{i,j}$ in $S_A^p(S^p)$, we have

$$\Big\|x\Big\|_{S^p(S^p)}\geq C^{-1}\inf\Big\{\Big(\sum_{i=1}^{\infty}\Big\|\Big(\sum_{j=1}^{\infty}y_{ij}y_{ij}^*\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}}+\Big(\sum_{j=1}^{\infty}\Big\|\Big(\sum_{i=1}^{\infty}z_{ij}^*z_{ij}\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}}\Big\}$$

where the infimum runs over all the decompositions of x as $x=y+z$ with $y=(y_{ij})_{i,j}$ and $z=(z_{ij})_{i,j}$ both in $S^p(S^p)$. Then, in analogy with Comments 1.9, it is easy to check that if $1\leq p<2$, a subset $A\subset\mathbb{N}\times\mathbb{N}$ is a $\sigma(p')$-set (resp. $\sigma(p')_{cb}$-set) where $\frac{1}{p}+\frac{1}{p'}=1$, if and only if it is a $\sigma(p)$-set (resp. $\sigma(p)_{cb}$-set) in the above sense and its indicator function $\mathbb{1}_A$ defines a bounded (resp. *c.b.*) Schur multiplier on S^p.

Proposition 4.7 *Let Λ be a a subset of $\mathbb{N}$ and let $\widehat{\Lambda}:=\big\{(i,j)\in\mathbb{N}\times\mathbb{N} i+j\in\Lambda\big\}$.*
i) For $1\leq p\leq 2$, $\widehat{\Lambda}$ is a $\sigma(p)_{cb}$-set whenever Λ is a $K(p)_{cb}$-set with $\sigma_p^{cb}\big(\widehat{\Lambda}\big)\leq K_p^{cb}(\Lambda)$.
ii) For $2<p<\infty$, $\widehat{\Lambda}$ is a $\sigma(p)_{cb}$-set whenever Λ is a $\Lambda(p)_{cb}$-set with $\sigma_p^{cb}\big(\widehat{\Lambda}\big)\leq\lambda_p^{cb}(\Lambda)$.

Proof: We sketch only *ii*). Let $x=\big(x_{ij}\big)_{i,j}$ be in $S_{\widehat{\Lambda}}^p(S^p)$. Using the assumption on Λ, we get

$$\Big\|\big(x_{ij}\big)_{i,j}\Big\|_{S^p(S^p)}=\Big\|\big(z^{i+j}x_{ij}\big)_{i,j}\Big\|_{L^p\big(S^p(S^p)\big)}=\Bigg\|\sum_{n\in\Lambda}z^n\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda},\\ i+j=n}}x_{ij}\otimes e_{ij}\Big)\Bigg\|_{L^p\big(S^p(S^p)\big)}$$
$$\leq\lambda_p^{cb}(\Lambda)\max\Big\{\Big\|\sum_j\Big(\sum_i x_{ij}^*x_{ij}\Big)^{\frac{1}{2}}\otimes e_{jj}\Big\|_{S^p(S^p)},\Big\|\sum_i\Big(\sum_j x_{ij}x_{ij}^*\Big)^{\frac{1}{2}}\otimes e_{ii}\Big\|_{S^p(S^p)}\Big\}$$

since we have

$$\Bigg(\sum_{n\in\Lambda}\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda},\\ i+j=n}}x_{ij}\otimes e_{ij}\Big)^*\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda},\\ i+j=n}}x_{ij}\otimes e_{ij}\Big)\Bigg)^{\frac{1}{2}}=\sum_j\Big(\sum_i x_{ij}^*x_{ij}\Big)^{\frac{1}{2}}\otimes e_{jj}$$
$$\Bigg(\sum_{n\in\Lambda}\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda},\\ i+j=n}}x_{ij}\otimes e_{ij}\Big)\Big(\sum_{\substack{(i,j)\in\widehat{\Lambda},\\ i+j=n}}x_{ij}\otimes e_{ij}\Big)^*\Bigg)^{\frac{1}{2}}=\sum_i\Big(\sum_j x_{ij}x_{ij}^*\Big)^{\frac{1}{2}}\otimes e_{ii}.$$

Hence we get

$$\Big\|\big(x_{ij}\big)_{i,j}\Big\|_{S^p(S^p)}\leq\lambda_p^{cb}(\Lambda)\max\Big\{\Big(\sum_j\Big\|\Big(\sum_i x_{ij}^*x_{ij}\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}},\Big(\sum_i\Big\|\Big(\sum_j x_{ij}x_{ij}^*\Big)^{\frac{1}{2}}\Big\|_{S^p}^p\Big)^{\frac{1}{p}}\Big\}.$$

This implies that $\widehat{\Lambda}$ is a $\sigma(p)_{cb}$-set with $\sigma_p^{cb}(\widehat{\Lambda})\leq\lambda_p^{cb}(\Lambda)$. ■

Theorem 4.8 *Let $2 < p < \infty$ be a fixed even integer. Then, for each $n \geq 1$, we may find a Hankelian subset A_n of $[1, n] \times [1, n]$ satisfying the next conditions*

$$\sup_{n\geq 1} \left\{ \sigma_p^{cb}(A_n) \right\} < \infty \inf_{n\geq 1} \left\{ n^{-(1+\frac{2}{p})} |A_n| \right\} > 0.$$

Proof: A reformulation of Corollary 2.7 implies that there exists for each integer $n \geq 1$, a set $\Lambda_n \subset [0, n]$ such that $\sup_{n\geq 1} \left\{ \lambda_p^{cb}(\Lambda_n) \right\} < \infty$ and $\inf_{n\geq 1} \left\{ n^{-\frac{2}{p}} |\Lambda_n| \right\} > 0$. Then we let for each n, $A_n := \widehat{\Lambda_n}$ thus $A_n \subset [0, n] \times [0, n]$. In order to ensure $|A_n| \geq n|\Lambda_n|$, we can clearly assume that $\Lambda_n \subset [\frac{n}{2}, n]$. Therefore, using Prop. 4.7, we get a sequence of sets $\left(A_n\right)_{n\geq 1}$ satisfying $\sup_{n\geq 1} \left\{ \sigma_p^{cb}(A_n) \right\} < \infty$ along with $\inf_{n\geq 1} \left\{ n^{-(1+\frac{2}{p})} |A_n| \right\} > 0$. ■

Theorem 4.9 *For any even integer $p > 2$, there is a $\sigma(p)_{cb}$-set $A \subset \mathbb{N} \times \mathbb{N}$ which is not a $\sigma(q)$-set for any $q > p$. More precisely, the indicator function of A is not in $M(S^q)$ for any $q > p$.*

Proof: Let $2 < p < \infty$ be a fixed even integer. Then, according to Corollary 2.7, there exist a constant $c > 0$ and sets $\Lambda_n \subset [2^{n-1}, 2^n[$ for each integer $n \geq 1$ satisfying $c\, 2^{\frac{2(n-1)}{p}} \leq \left|\Lambda_n\right|$ and $\sup_{n\geq 1}\left\{ \lambda_p^{cb}(\Lambda_n) \right\} < \infty$. Such sets satisfy necessarily $\sup_{n\geq 1}\left\{ \lambda_q(\Lambda_n) \right\} = \infty$ for all $p < q < \infty$. Indeed, taking into account (i) of Facts 2.3, we see that there exists a constant $c_1 > 0$ such that we have

$$c\, 2^{\frac{2(n-1)}{p}} \leq \left|\Lambda_n\right| \leq c_1\, 2^{\frac{2(n-1)}{q}} \lambda_q^2(\Lambda_n)$$

for all integers $n \geq 1$, that is to say

$$c\, 2^{2(n-1)(\frac{1}{p}-\frac{1}{q})} \leq c_1\, \lambda_q^2(\Lambda_n) \leq c_1 \sup_{k\geq 1} \lambda_q^2(\Lambda_k)\cdot$$

Since $\frac{1}{p} - \frac{1}{q} > 0$ and n can be arbitrary big, $\sup_{k\geq 1} \lambda_q(\Lambda_k) = \infty$. On the other hand, we let

$$A_n := (2^n, 2^n) + \widehat{\Lambda_n} = \left\{ (2^n + i, 2^n + j); i + j \in \Lambda_n \right\},\ \forall n \geq 1\cdot$$

Note that for all $n \geq 1$, we have

$$A_n \subset [2^n, 2^{n+1}[\times[2^n, 2^{n+1}[$$

Then, we consider the set $A := \bigcup_{n\geq 1} A_n$. We apply successively the *c.b.* version of $(4i)$ in Remarks 4.2, Prop. 4.7 and the fact that the $\Lambda(p)_{cb}$-property is stable under translations, to see that A is a $\sigma(p)_{cb}$-set as follows

$$\sigma_p^{cb}(A) \leq 2 \sup_{n\geq 1} \left\{ \sigma_p^{cb}(A_n) \right\} \leq 2 \sup_{n\geq 1} \left\{ \lambda_p^{cb}(2^{n+1} + \Lambda_n) \right\} = 2 \sup_{n\geq 1} \left\{ \lambda_p^{cb}(\Lambda_n) \right\} < \infty\cdot$$

Now we check that the indicator function $1\!\!1_A$ is not in $M(S^q)$ for all $q > p$. Indeed, taking the supremum after applying the inequality (1.5) to each set Λ_n, we get

$$\sup_{n\geq 1} \lambda_q(\Lambda_n) \leq \sup_{n\geq 1} \lambda_2(\Lambda_n) \sup_{n\geq 1} \left\| 1\!\!1_{\Lambda_n} \right\|_{M(L^q)} \leq \sup_{n\geq 1} \lambda_p(\Lambda_n) \sup_{n\geq 1} \left\| 1\!\!1_{\Lambda_n} \right\|_{M(L^q)}$$

$$\sup_{n\geq 1} \lambda_q(\Lambda_n) \leq \sup_{n\geq 1} \lambda_p^{cb}(\Lambda_n) \sup_{n\geq 1} \left\| 1\!\!1_{\Lambda_n} \right\|_{M(L^q)} \cdot$$

Since by our choice of the sequence $\{\Lambda_n\}_{n\geq 1}$, $\sup\limits_{n\geq 1}\lambda_p^{cb}(\Lambda_n) < \infty$ and $\sup\limits_{n\geq 1}\lambda_q(\Lambda_n) = \infty$, we have necessarily $\sup\limits_{n\geq 1}\left\| 1\!\!1_{\Lambda_n} \right\|_{M(L^q)} = \infty$. We can see easily that

$$\left\| 1\!\!1_{A_n} \right\|_{M(S^q)} = \left\| 1\!\!1_{\widehat{\Lambda_n}} \right\|_{M(S^q)} \geq \left\| 1\!\!1_{\widehat{\Lambda_n}} \right\|_{M(\mathfrak{S}^q)}, \ \forall n \geq 1 \cdot$$

Then, using Peller's results (see Subsection 0.6), we get

$$\sup_{n\geq 1} \left\| 1\!\!1_{A_n} \right\|_{M(S^q)} \geq \sup_{n\geq 1} \left\| 1\!\!1_{\widehat{\Lambda_n}} \right\|_{M(\mathfrak{S}^q)} \cong \sup_{n\geq 1} \left\| 1\!\!1_{\Lambda_n} \right\|_{M(L^q)} = \infty \cdot$$

Thus $1\!\!1_A$ does not belong to $M(S^q)$ and we conclude that A is not a $\sigma(q)$-set by using (ii) of Remarks 4.6. ■

5 Applications to Schur multipliers

The last assertion of the following proposition answers a question raised by J. Erdos as Remark 2 in [18] (I am very grateful to Professor E. Katsoulis for transmitting this reference, see [1] for related work).

Theorem 5.1 *For all $2 < p < q \leq \infty$ where p is an even integer, the inclusion maps*

$$\begin{array}{rcl} M_{cb}(S^q) & \subset & M_{cb}(S^p) \\ M(S^q) & \subset & M(S^p) \end{array}$$

are strict. Moreover, there is an idempotent Schur multiplier which is c.b. on S^p but not bounded on S^q for any $q > p$.

Proof: Let $p > 2$ be an even integer then by Theorem 4.9, there exists a $\sigma(p)_{cb}$-set $A \subset \mathbb{N} \times \mathbb{N}$ which is not a $\sigma(q)$-set for any $q > p$. Moreover, the indicator function $1\!\!1_A$ is not in $M(S^q)$. Using (ii) of Remarks 4.6, we see that $1\!\!1_A$ is in $M_{cb}(S^p)$ and we are done. ■

Theorem 5.2 *For $2 < p < \infty$, the following canonical inclusion map is contractive*

$$\Big(M(S^\infty), M(S^2)\Big)_{\frac{2}{p}} \subset M_{cb}(S^p) \cdot$$

Moreover, $M_{cb}(S^p)$ does not embed into any interpolated space $\Big(M(S^\infty), M(S^2)\Big)_\theta$ when θ runs in $]0, 1[$. Therefore, the inclusion above is strict.

Proof: The first part of the statement above follows immediately from Lemma 0.2. For the last part, we can clearly restrict ourselves to even integers p. Thus, fix an even integer $2 < p < \infty$ and let $A \subset \mathbb{N} \times \mathbb{N}$ be a $\sigma(p)_{cb}$-set which is not $\sigma(q)$ for any $q > p$. Such a set exists by using Theorem 4.9. Thus, A is an interpolation set for $M_{cb}(S^p)$ but not for $M(S^q)$ according to Prop. 4.5 *i.e.* the restriction map $\mathcal{Q} : M_{cb}(S^p) \longrightarrow \ell_\infty(A)$ is surjective but $\mathcal{Q} : M(S^q) \longrightarrow \ell_\infty(A)$ is not. If we assume that $M_{cb}(S^p)$ embeds into $\big(M(S^\infty), M(S^2)\big)_\theta$ for some $0 < \theta < 1$, we see that $\mathcal{Q} : \big(M(S^\infty), M(S^2)\big)_\theta \longrightarrow \ell_\infty(A)$ is again surjective. Lemma 3.5 implies then that $\mathcal{Q} : M(S^\infty) \longrightarrow \ell_\infty(A)$ is also surjective. This contradicts the assumption that Λ is not an interpolation set for $M(S^q)$ for any $q > p$. ■

Now we will be interested in $M(\mathfrak{S}^p)$, $M_{cb}(\mathfrak{S}^p)$ and in establishing some links between Fourier and Schur multipliers. We start by recalling few notations. For a set $\Lambda \subset \mathbb{N}$, we define $\widehat{\Lambda} := \{(k,l) \in \mathbb{N} \times \mathbb{N} \mid k + l \in \Lambda\}$ and we say that a map $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$ is Hankelian if it satisfies $\varphi(k,l) = \varphi(k',l')$ whenever $k+l = k'+l'$ for all pairs $(k,l),(k',l')$ in $\mathbb{N}\times\mathbb{N}$. Recall also that for all integers $n \geq 1$ and all $\varphi : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{C}$, we let $\varphi_{(n)} := 1\!\!1_{\widehat{I_n}} \varphi$ (Schur product) where $I_0 := \{0\}$ and $I_n := \{k \in \mathbb{N} \mid 2^{n-1} \leq k < 2^n\}$ for each integer $n \geq 1$.

Proposition 5.3 *For a Hankelian map $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$, the following are equivalent.*
i) φ belongs to $M(\mathfrak{S}^p)$.
ii) The multipliers $\varphi_{(n)}$ are uniformly bounded in $M\big(\mathfrak{S}^p_{\widehat{I_n}}\big)$.
iii) The multipliers $\varphi_{(n)}$ are uniformly bounded in $M(\mathfrak{S}^p)$.
Moreover, the two norms defined below are equivalent norms on $M(\mathfrak{S}^p)$.

$$\Big\|\varphi\Big\|_{M(\mathfrak{S}^p)} \cong \sup_{n\geq 0} \Big\|\varphi_{(n)}\Big\|_{M\left(\mathfrak{S}^p_{\widehat{I_n}}\right)} \cong \sup_{n\geq 0} \Big\|\varphi_{(n)}\Big\|_{M(\mathfrak{S}^p)}.$$

Proof: The equivalence between *ii*) and *iii*) is easy since the spaces $\mathfrak{S}^p_{\widehat{I_n}}$ are uniformly complemented in $\mathfrak{S}^p$. To prove the equivalence between *i*) and *iii*), consider x in $\mathfrak{S}^p$. We have $T_\varphi(x) = \sum_{n\geq 0} T_{\varphi_{(n)}}\big(x_{(n)}\big)$. Thus we get by using *i*) of Corollary 0.7

$$\Big\|T_\varphi(x)\Big\|_{\mathfrak{S}^p} \cong \bigg(\sum_{n\geq 0} \Big\|T_{\varphi_{(n)}}\big(x_{(n)}\big)\Big\|^p_{\mathfrak{S}^p}\bigg)^{\frac{1}{p}} \leq \bigg(\sum_{n\geq 0} \Big\|T_{\varphi_{(n)}}\Big\|^p_{\mathcal{B}(\mathfrak{S}^p)} \Big\|x_{(n)}\Big\|^p_{\mathfrak{S}^p}\bigg)^{\frac{1}{p}}$$

$$\leq \sup_{n\geq 0} \Big\|T_{\varphi_{(n)}}\Big\|_{\mathcal{B}(\mathfrak{S}^p)} \bigg(\sum_{n\geq 0} \Big\|x_{(n)}\Big\|^p_{\mathfrak{S}^p}\bigg)^{\frac{1}{p}} \cong \sup_{n\geq 0} \Big\|\varphi_{(n)}\Big\|_{M(\mathfrak{S}^p)} \Big\|x\Big\|_{\mathfrak{S}^p} \leq \Big\|\varphi\Big\|_{M(\mathfrak{S}^p)} \Big\|x\Big\|_{\mathfrak{S}^p}$$ ■

Proposition 5.4 *For a Hankelian map $\varphi : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}$, the following are equivalent.*
i) φ belongs to $M_{cb}(\mathfrak{S}^p)$.
ii) The multipliers $\varphi_{(n)}$ are uniformly bounded in $M_{cb}\big(\mathfrak{S}^p_{\widehat{I_n}}\big)$.
iii) The multipliers $\varphi_{(n)}$ are uniformly bounded in $M_{cb}(\mathfrak{S}^p)$.
Moreover, the two norms defined below are equivalent norms on $M_{cb}(\mathfrak{S}^p)$.

$$\Big\|\varphi\Big\|_{M_{cb}(\mathfrak{S}^p)} \cong \sup_{n\geq 0} \Big\|\varphi_{(n)}\Big\|_{M_{cb}\left(\mathfrak{S}^p_{\widehat{I_n}}\right)} \cong \sup_{n\geq 0} \Big\|\varphi_{(n)}\Big\|_{M_{cb}(\mathfrak{S}^p)}.$$

Proof: By the characterization of *c.b.* maps given in Prop. 0.4, we can prove Prop. 5.4 exactly in the same way as we did for Prop. 5.3 by applying *ii*) of Corollary 0.7 instead of *i*). ■

Proposition 5.5 *For all $1 < p < \infty$, $M(H^p)$ can be injected continuously into $M(\mathfrak{S}^p)$ via the map which takes $\varphi \in M(H^p)$ to $\widehat{\varphi} \in M(\mathfrak{S}^p)$, where $\widehat{\varphi}$ denotes the map which sends (k,l) in $\mathbb{N} \times \mathbb{N}$ to $\varphi(k+l)$.*

Proof: We assume first that φ has support in I_n. Then by *i*) of Corollary 0.7 we have

$$\begin{aligned} \left\|\widehat{\varphi}\right\|_{M(\mathfrak{S}^p)} \cong \left\|\widehat{\varphi}\right\|_{M\left(\mathfrak{S}^p_{\widehat{I_n}}\right)} = \sup\left\{\left\|T_{\widehat{\varphi}}(x)\right\|_{\mathfrak{S}^p_{\widehat{I_n}}} \left\|x\right\|_{\mathfrak{S}^p_{\widehat{I_n}}} \leq 1\right\} \\ \cong \sup\left\{\left\|M_\varphi(f)\right\|_{\mathcal{A}^p_{I_n}} \mid = \sup\left\{\left\|M_\varphi(f)\right\|_{H^p_{I_n}} \mid \left\|f\right\|_{H^p_{I_n}} \leq 1\right\}\right.\end{aligned}$$

(see Subsection 0.6 for the definition of the space $\mathcal{A}^p$). Applying Remark 0.10, we get

$$\left\|\widehat{\varphi}\right\|_{M(\mathfrak{S}^p)} \cong \left\|\varphi\right\|_{M\left(H^p_{I_n}\right)} \cong \left\|\varphi\right\|_{M(H^p)}.$$

Using Prop. 5.3, we see that for arbitrary φ in $M(H^p)$

$$\left\|\widehat{\varphi}\right\|_{M(\mathfrak{S}^p)} \cong \sup_{n\geq 0}\left\|\widehat{\varphi}_{(n)}\right\|_{M(\mathfrak{S}^p)} = \sup_{n\geq 0}\left\|\widehat{\varphi_{(n)}}\right\|_{M(\mathfrak{S}^p)} \cong \sup_{n\geq 0}\left\|\varphi_{(n)}\right\|_{M(H^p)} \leq \left\|\varphi\right\|_{M(H^p)}$$ ■

Proposition 5.6 *$M_{cb}(H^p)$ can be injected contractively into $M^{\mathcal{H}}_{cb}(S^p)$ via the map which takes $\varphi \in M_{cb}(H^p)$ to $\widehat{\varphi} \in M^{\mathcal{H}}_{cb}(S^p)$.*

Proof: Let φ be fixed in $M_{cb}(H^p)$ and $M_\varphi : H^p \longrightarrow H^p$ be its associated operator. The multiplier φ is *c.b.* hence the operator $M_\varphi \otimes id_{S^p}$ extends to a bounded operator on $H^p(S^p)$. On the other hand, for an operator x in S^p, we consider one more time the function $f_x(z) = D_z x D_z$ defined on $\mathbb{T}$ where D_z denotes the unitary $\infty \times \infty$ matrix

$$D_z = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & z & 0 & \dots \\ 0 & 0 & z^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

f_x lies in $H^p(S^p)$ and satisfies $\left\|f_x\right\|_{H^p(S^p)} = \left\|x\right\|_{S^p}$. Now set $T_{\widehat{\varphi}}(x)$ for the $\infty \times \infty$ matrix $\Big(\varphi(k+l)x_{kl}\Big)_{k,l}$. We see that $M_\varphi \otimes id_{S^p}(f_x)(z) = D_z T_{\widehat{\varphi}}(x) D_z$. Hence, the operator

$$T_{\widehat{\varphi}} : \quad S^p \longrightarrow S^p$$

is well defined, has a Hankelian form and satisfies

$$\left\|T_{\widehat{\varphi}}x\right\|_{S^p} = \left\|M_\varphi \otimes id_{S^p}(f_x)\right\|_{L^p(S^p)} \leq \left\|M_\varphi\right\|_{\mathcal{CB}(H^p)}\left\|f_x\right\|_{H^p(S^p)} = \left\|\varphi\right\|_{M_{cb}(H^p)}\left\|x\right\|_{S^p}.$$

This means that $\widehat{\varphi}$ is in $M^{\mathcal{H}}(S^p)$ with $\left\|\widehat{\varphi}\right\|_{M^{\mathcal{H}}(S^p)} \leq \left\|\varphi\right\|_{M_{cb}(H^p)}$. In a similar way, using Prop. 0.4, we prove that in fact $\widehat{\varphi}$ belongs to $M^{\mathcal{H}}_{cb}(S^p)$ with $\left\|\widehat{\varphi}\right\|_{M^{\mathcal{H}}_{cb}(S^p)} \leq \left\|\varphi\right\|_{M_{cb}(H^p)}$. ■

Remark. The case $p = 1$ is quite interesting since $M_{cb}(H^1)$ and $M^{\mathcal{H}}_{cb}(S^1)$ coincide isometrically. See [32] for the proof. The question seems to be open for the other non trivial values of p.

6 Appendix

For the sake of completeness, we include a different way to show the existence of "large" $\Lambda(4)_{cb}$-sets by using probabilistic ideas to exhibit "large" sets having the combinatorial property $Z(2)$ and satisfying moreover some additional assumptions. We check first that the $Z(2)$-property implies the $\Lambda(4)_{cb}$-property directly (without using Theorem 1.13). Recall that a set $\Lambda \subset \mathbb{Z}$ is said to have the $Z(2)$-property whenever

$$Z_2(\Lambda) := \sup_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \Big| \Big\{(n_1, n_2) \in \Lambda \times \Lambda \mid n_1 - n_2 = k\Big\} \Big| < \infty.$$

Proposition 6.1 *If a set $\Lambda \subset \mathbb{Z}$ has the $Z(2)$-property then it has the $\Lambda(4)_{cb}$-property and we have $\lambda^{cb}_4(\Lambda) \leq \Big(1 + Z_2(\Lambda)\Big)^{\frac{1}{4}}$.*

Proof: Let $f = \sum_{n\in\Lambda} x_n e^{int}$ be in $L^4(S^4)$ say with finitely many $x_n \neq 0$. We have

$$\left\|f\right\|^4_{L^4(S^4)} = \left\|f^* f\right\|^2_{L^2(S^2)} = \sum_{k\in\mathbb{Z}} \left\|\widehat{f^* f}(k)\right\|^2_{S^2}.$$

For all integers k, we have $\widehat{f^* f}(k) = \sum_{\substack{n_1,n_2\in\Lambda,\\ -n_1+n_2=k}} x^*_{n_1} x_{n_2}$. Then, for each $k \neq 0$

$$\left\|\widehat{f^* f}(k)\right\|^2_{S^2} \leq \Bigg(\sum_{\substack{n_1,n_2\in\Lambda,\\ -n_1+n_2=k}} \left\|x^*_{n_1} x_{n_2}\right\|_{S^2}\Bigg)^2 \leq Z_2(\Lambda) \sum_{\substack{n_1,n_2\in\Lambda,\\ -n_1+n_2=k}} \left\|x^*_{n_1} x_{n_2}\right\|^2_{S^2}.$$

Hence, using the trace property, we get

$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left\|\widehat{f^* f}(k)\right\|^2_{S^2} \leq Z_2(\Lambda) \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \sum_{\substack{n_1,n_2\in\Lambda,\\ -n_1+n_2=k}} tr(x^*_{n_2} x_{n_1} x^*_{n_1} x_{n_2})$$
$$= Z_2(\Lambda) \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \sum_{\substack{n_1,n_2\in\Lambda,\\ -n_1+n_2=k}} tr(x_{n_1} x^*_{n_1} x_{n_2} x^*_{n_2}) \leq Z_2(\Lambda) \sum_{-n_1,n_2\in\Lambda} tr(x_{n_1} x^*_{n_1} x_{n_2} x^*_{n_2}).$$
$$\sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \left\|\widehat{f^* f}(k)\right\|^2_{S^2} \leq Z_2(\Lambda) x_{n_1} x^*_{n_1} \sum_{n_2\in\Lambda} x_{n_2} x^*_{n_2}\Bigg) = Z_2(\Lambda) \Big\| \sum_{n\in\Lambda} x_n x^*_n \Big\|^2_{S^2}.$$

Thus, we have

$$\left\|f\right\|^4_{L^4(S^4)} \leq \Big\| \sum_{n\in\Lambda} x^*_n x_n \Big\|^2_{S^2} + Z_2(\Lambda) \Big\| \sum_{n\in\Lambda} x_n x^*_n \Big\|^2_{S^2}$$

$$\leq \Big(1+Z_2(\Lambda)\Big)\max\Big\{\Big\|\sum_{n\in\Lambda}x_n^*x_n\Big\|_{S^2}^2, \Big\|\sum_{n\in\Lambda}x_nx_n^*\Big\|_{S^2}^2\Big\}\cdot$$

Finally, we get

$$\Big\|f\Big\|_{L^4(S^4)} \leq \Big(1+Z_2(\Lambda)\Big)^{\frac{1}{4}}\max\Big\{\Big\|\Big(\sum_{n\in\Lambda}x_n^*x_n\Big)^{\frac{1}{2}}\Big\|_{S^4}, \Big\|\Big(\sum_{n\in\Lambda}x_nx_n^*\Big)^{\frac{1}{2}}\Big\|_{S^4}\Big\} \qquad \blacksquare$$

The following is essentially a known result in the folklore of harmonic analysis but it does not seem to appear in print anywhere.

Proposition 6.2 *For all small $\delta > 0$ and all $C > \frac{2}{\delta}$, there exist constants C_1, C_2 depending only on δ such that for all sequences $(u_n)_{n\geq 1}$ of non-negative integers with $\sum_{n\geq 1} u_n^{-\frac{1}{2}+\delta} < \infty$, there exists a $Z(2)$-set Λ satisfying the following.*

- $Z_2(\Lambda) \leq C$.

•• *For all $n \geq n_0$, where n_0 is an integer depending only on the convergence speed of $\sum_{n\geq 1} u_n^{-\frac{1}{2}+\delta}$ $\Big($say e.g. $\sum_{n\geq n_0} u_n^{-\frac{1}{2}+\delta} \leq \frac{C_1}{4}\Big)$, we have $\quad C_1 u_n^{\frac{1}{2}-\delta} \leq \big|\,\Lambda\cap[u_n, 2u_n[\,\big| \leq C_2 u_n^{\frac{1}{2}-\delta}$.*

Proof: Let $\{\xi_k\}_{k\geq 1}$ be a sequence of independent random variables on a standard probability space for example the torus $\mathbb{T}$ equipped with the normalized Lebesgue measure $d\mathbb{P} = \frac{dt}{2\pi}$ such that for each k, ξ_k takes its values in $\{0,1\}$ and has expectation $\mathbb{E}\xi_k = \frac{\beta}{k^\alpha}$ where $\alpha = \frac{1}{2}+\delta$ for simplicity, with $\alpha < 1$ and β is a constant depending only on δ — to be fixed later. Thus $\mathbb{P}(\xi_k = 1) = \frac{\beta}{k^\alpha}$ and $\mathbb{P}(\xi_k = 0) = 1 - \frac{\beta}{k^\alpha}\cdot$

To each ω in $\mathbb{T}$, we can associate the subset $\Lambda_\omega := \Big\{k \in \mathbb{N}^* \mid \xi_k(\omega) = 1\Big\}$. We will show that by a convenient choice of β, most of these random sets Λ_ω satisfy the required properties. Indeed, for $\gamma \in \mathbb{Z}^*$ we set

$$Z_2(\gamma, \Lambda_\omega) := \Big|\Big\{(k,l)\in\Lambda_\omega\times\Lambda_\omega \mid k-l=\gamma\Big\}\Big| = \sum_{\substack{k,l\geq 1\\ k-l=\gamma}}\xi_k(\omega)\xi_l(\omega)$$

$$Z_2(\gamma, \Lambda_\omega) = \sum_{k\geq 1}\xi_k(\omega)\xi_{k+|\gamma|}(\omega) = Z_2(|\gamma|, \Lambda_\omega) \tag{6.12}$$

We split $\mathbb{N}^*$ into J_1 and J_2 where

$$J_1 = \bigcup_{s\geq 0}\Big[1+2s|\gamma|, (1+2s)|\gamma|\Big],\quad J_2 = \bigcup_{s\geq 0}\Big](1+2s)|\gamma|, (2+2s)|\gamma|\Big]$$

in order to have the variables $\Big\{\xi_k\xi_{k+|\gamma|}\Big\}_{k\in J_j}$ independent for each value of j. Then, we let

$$Z_{2,j}(\gamma, \Lambda_\omega) = \sum_{k\in J_j}\xi_k(\omega)\xi_{k+|\gamma|}(\omega)\cdot$$

Step 1. We start by selecting among these sets Λ_ω those satisfying the $Z(2)$-property. Using (6.12), we see that for an integer m

$$\mathbb{P}\Big\{\sup_{\gamma\neq 0} Z_2(\gamma,\Lambda_\omega) \geq 2m\Big\} = \mathbb{P}\Big\{\sup_{\gamma\geq 1} Z_2(\gamma,\Lambda_\omega)\geq 2m\Big\}\cdot$$

Hence, we get

$$\mathbb{P}\Big\{\sup_{\gamma\neq 0} Z_2(\gamma,\Lambda_\omega)\geq 2m\Big\}\leq \mathbb{P}\Big\{\sup_{\gamma\geq 1} Z_{2,1}(\gamma,\Lambda_\omega)\geq m\Big\}+\mathbb{P}\Big\{\sup_{\gamma\geq 1} Z_{2,2}(\gamma,\Lambda_\omega)\geq m\Big\}\cdot$$

For each $\gamma\geq 1$, we have

$$\mathbb{P}\Big\{Z_{2,1}(\gamma,\Lambda_\omega)\geq m\Big\}\leq \sum_{\substack{k_1,..,k_m\in J_1\\ k_i\neq k_j,\forall i\neq j}}\mathbb{P}\Big\{\xi_{k_j}(\omega)\xi_{k_j+\gamma}(\omega)=1;$$

$$=\sum_{\substack{k_1,..,k_m\in J_1\\ k_i\neq k_j,\forall i\neq j}}\prod_{j=1}^m\mathbb{P}\Big\{\xi_{k_j}(\omega)\xi_{k_j+\gamma}(\omega)=1\Big\}\leq \sum_{k_1,..,k_m\in J_1}\prod_{j=1}^m\mathbb{P}\Big\{\xi_{k_j}(\omega)\xi_{k_j+\gamma}(\omega)=1\Big\}\cdot$$

Thus, we have

$$\mathbb{P}\Big\{Z_{2,1}(\gamma,\Lambda_\omega)\geq m\Big\}\leq \sum_{k_1,..,k_m\in J_1}\prod_{j=1}^m\frac{\beta}{k_j^\alpha}\frac{\beta}{(k_j+\gamma)^\alpha}=\beta^{2m}\Big(\sum_{k\in J_1}\frac{1}{k^\alpha(k+\gamma)^\alpha}\Big)^m\cdot$$

Now we use the fact that $\forall K\geq 1$, $\sum\limits_{k=1}^K\frac{1}{k^\alpha}\leq 1+\int_1^K\frac{1}{t^\alpha}dt=1+\frac{1}{1-\alpha}(K^{1-\alpha}-1)$. This gives us (recall that $2\alpha>1$)

$$\begin{aligned}\sum_{k\in J_1}\frac{1}{k^\alpha(k+\gamma)^\alpha}&=\sum_{k=1}^{\gamma}\frac{1}{k^\alpha(k+\gamma)^\alpha}+\sum_{s=1}^{\infty}\Big(\sum_{k=1+2s\gamma}^{(1+2s)\gamma}\frac{1}{k^\alpha(k+\gamma)^\alpha}\Big)\\ &\leq\frac{1}{\gamma^\alpha}\sum_{k=1}^{\gamma}\frac{1}{k^\alpha}+\sum_{s=1}^{\infty}\Big(\sum_{k=1+2s\gamma}^{(1+2s)\gamma}\frac{1}{k^{2\alpha}}\Big)\leq\frac{1}{(1-\alpha)\gamma^{2\alpha-1}}+\sum_{s=1}^{\infty}\frac{\gamma}{(2s\gamma)^{2\alpha}}\\ &\leq\frac{1}{(1-\alpha)\gamma^{2\alpha-1}}+\frac{1}{4^\alpha\ \gamma^{2\alpha-1}}\sum_{s=1}^{\infty}\frac{1}{s^{2\alpha}}\leq\frac{1}{(1-\alpha)\gamma^{2\alpha-1}}+\frac{2\alpha}{4^\alpha\ \gamma^{2\alpha-1}(2\alpha-1)}\\ &\leq\Big(\frac{1}{1-\alpha}+\frac{2\alpha}{2\alpha-1}\Big)\frac{1}{\gamma^{2\alpha-1}}\cdot\end{aligned}$$

Thus, we obtain

$$\sum_{k\in J_1}\frac{1}{k^\alpha(k+\gamma)^\alpha}\leq\frac{4\alpha}{(2\alpha-1)\gamma^{2\alpha-1}}$$

by assuming for simplicity δ small enough, say $\delta\leq\frac{1}{\sqrt{2}}-\frac{1}{2}$ so that $\frac{1}{1-\alpha}\leq\frac{2\alpha}{2\alpha-1}\cdot$ By the same calculation, we get

$$\sum_{k\in J_2}\frac{1}{k^\alpha(k+\gamma)^\alpha}\ \leq\ \frac{2\alpha}{(2\alpha-1)\gamma^{2\alpha-1}}\cdot$$

Hence

$$\mathbb{P}\Big\{Z_{2,1}(\gamma,\Lambda_\omega)\geq m\Big\}\leq\frac{c^m}{\gamma^{(2\alpha-1)m}}\ \text{ and }\ \mathbb{P}\Big\{Z_{2,2}(\gamma,\Lambda_\omega)\geq m\Big\}\leq\frac{c^m}{2^m\gamma^{(2\alpha-1)m}}$$

where $c:=\frac{4\alpha}{2\alpha-1}\beta^2$. This implies that $\mathbb{P}\Big\{Z_2(\gamma,\Lambda_\omega)\geq 2m\Big\}\leq\frac{2c^m}{\gamma^{(2\alpha-1)m}}$ and therefore

$$\mathbb{P}\Big\{\sup_{\gamma\neq 0}Z_2(\gamma,\Lambda_\omega)\geq 2m\Big\}\leq 2c^m\sum_{\gamma\geq 1}\frac{1}{\gamma^{2\delta m}}\cdot$$

By taking $m > \frac{1}{2\delta}$ and β such that $c = \frac{1+2\delta}{\delta}\beta^2 < 1$ say $\beta = \left(\frac{\delta}{8(1+2\delta)}\right)^{\frac{1}{2}}$, we obtain

$$\mathbb{P}\Big\{\sup_{\gamma\neq 0} Z_2(\gamma,\Lambda_\omega) \geq 2m\Big\} \leq 2c^m\left(1 - \frac{1}{1-2\delta m}\right) = 8^{-m}\frac{4\delta m}{2\delta m - 1} := P_1(m)\cdot$$

Therefore, we conclude

$$\mathbb{P}\Big\{Z_2(\Lambda_\omega) < 2m\Big\} \geq 1 - P_1(m) > 0\cdot$$

Note that if we take $C = 2m > \frac{2}{\delta}$ and δ small enough (say $\delta < \delta_0$ for some $\delta_0 > 0$) then we can ensure that $P_1(m) < \frac{1}{2}$.

Step 2. Among these random sets Λ_ω satisfying $Z_2(\Lambda_\omega) < 2m$, we will select the "largest" one in the sense of our condition ••
For each integer $n \geq 1$, I_n sets for the interval $[u_n, 2u_n[$. We have

$$\mathbb{E}|\Lambda_\omega \cap I_n| = \mathbb{E}\left(\sum_{k=u_n}^{2u_n-1} \xi_k(\omega)\right) = \sum_{k=u_n}^{2u_n-1} \frac{\beta}{k^\alpha}\cdot$$

Since

$$C_1(u_n) := \beta\int_{u_n}^{2u_n} \frac{1}{t^\alpha}dt \leq \beta\sum_{u_n}^{2u_n-1}\frac{1}{k^\alpha} \leq C_2(u_n) := \beta\int_{u_n-1}^{2u_n-1}\frac{1}{t^\alpha}dt$$

we get

$$C_1(u_n) \leq \mathbb{E}|\Lambda_\omega \cap I_n| \leq C_2(u_n)\cdot$$

We define the constants C_1 and C_2 depending on δ only as follows.

$$C_1(u_n) = \beta\,\frac{1}{1-\alpha}\,\left((2u_n)^{1-\alpha} - u_n^{1-\alpha}\right) = \beta\,\frac{2^{1-\alpha}-1}{1-\alpha}\,u_n^{1-\alpha} := 2C_1 u_n^{\frac{1}{2}-\delta}\cdot$$
$$C_2(u_n) = \frac{\beta}{1-\alpha}\,\left((2u_n-1)^{1-\alpha} - (u_n-1)^{1-\alpha}\right) \leq \frac{\beta}{1-\alpha}(2u_n)^{1-\alpha}$$
$$C_2(u_n) = \beta\,\frac{2^{1-\alpha}}{1-\alpha}\,u_n^{1-\alpha} := \frac{2}{3}C_2 u_n^{\frac{1}{2}-\delta}\cdot$$

Since $\left(\xi_k - \mathbb{E}\xi_k\right)_k$ is a sequence of centered and independent random variables, the variance of $|\Lambda_\omega \cap I_n|$ is

$$\Big\||\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\Big\|_{L^2}^2 = \Big\|\sum_{k=u_n}^{2u_n-1}(\xi_k - \mathbb{E}\xi_k)\Big\|_{L^2}^2 = \sum_{k=u_n}^{2u_n-1}\Big\|\xi_k - \mathbb{E}\xi_k\Big\|_{L^2}^2\cdot$$

Hence, we get

$$\Big\||\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\Big\|_{L^2}^2 \leq \sum_{k=u_n}^{2u_n-1}\Big\|\xi_k\Big\|_{L^2}^2 = \sum_{k=u_n}^{2u_n-1}\mathbb{E}(\xi_k^2) = \sum_{k=u_n}^{2u_n-1}\frac{\beta}{k^\alpha} = \mathbb{E}|\Lambda_\omega \cap I_n|\cdot$$

Using Tchebytchev's inequality, we obtain

$$\mathbb{P}\Big\{\Big||\Lambda_\omega \cap I_n| - \mathbb{E}|\Lambda_\omega \cap I_n|\Big| \geq \frac{1}{2}\mathbb{E}|\Lambda_\omega \cap I_n|\Big\} \leq \frac{4}{\mathbb{E}|\Lambda_\omega \cap I_n|} \leq \frac{4}{C_1(u_n)} = \frac{2}{C_1}u_n^{-\frac{1}{2}+\delta}.$$

For each fixed integer N, we have

$$\Big\{\exists n\geq N, \Big||\Lambda_\omega\cap I_n|-\mathbb{E}|\Lambda_\omega\cap I_n|\Big|\geq\frac{1}{2}\mathbb{E}|\Lambda_\omega\cap I_n|\Big\}=\bigcup_{n=N}^{\infty}\Big\{\Big||\Lambda_\omega\cap I_n|-\mathbb{E}|\Lambda_\omega\cap I_n|\Big|\geq\frac{1}{2}\mathbb{E}|\Lambda_\omega\cap I_n|\Big\}\cdot$$

Therefore

$$\mathbb{P}\Big\{\exists n\geq N, \Big||\Lambda_\omega\cap I_n|-\mathbb{E}|\Lambda_\omega\cap I_n|\Big|\geq\frac{1}{2}\mathbb{E}|\Lambda_\omega\cap I_n|\Big\}\leq\frac{2}{C_1}\sum_{n=N}^{\infty}u_n^{-\frac{1}{2}+\delta}:=P_2(N)\cdot$$

Then since

$$\Big\{\Big||\Lambda_\omega\cap I_n|-\mathbb{E}\ |\Lambda_\omega\cap I_n|\Big|<\frac{1}{2}\mathbb{E}|\Lambda_\omega\cap I_n|, \forall n\geq N\Big\}\subset\Big\{C_1u_n^{\frac{1}{2}-\delta}\leq|\Lambda_\omega\cap I_n|\leq C_2u_n^{\frac{1}{2}-\delta}, \forall n\geq N\Big\}$$

we get clearly for each fixed integer N

$$\mathbb{P}\Big\{C_1u_n^{\frac{1}{2}-\delta}\leq|\Lambda_\omega\cap I_n|\leq C_2u_n^{\frac{1}{2}-\delta}, \forall n\geq N\Big\}\geq 1-P_2(N)\cdot$$

Thus we finally obtain

$$\mathbb{P}\Big\{Z_2(\Lambda_\omega)<m\ \&\ C_1u_n^{\frac{1}{2}-\delta}\leq|\Lambda_\omega\cap I_n|\leq C_2u_n^{\frac{1}{2}-\delta}, \forall n\geq N\Big\}\geq 1-P_1(m)-P_2(N)\cdot$$

Since by our assumption on $(u_n)_{n\geq1}$, $P_2(N)$ tends to zero at infinity, there exists an integer n_0 such that $1-P_1(m)-P_2(n_0)>0$. Note that $\lim\limits_{m,N\to\infty}\big(1-P_1(m)-P_2(N)\big)=1$.

Conclusion: There exists at least one set complying with the properties required in Prop. 6.2. ■

Consequence. For each $4<p\leq\infty$, there exists an idempotent Hankelian Schur multiplier which is *c.b.* on S^4 but not bounded on $\mathfrak{S}^p$.
Indeed, let $p>4$ and choose $0<\delta<\frac{p-4}{2p}$ small enough. We consider the sequence $\big(u_n\big)_n$ defined by $u_n=2^{n-1}$. According to Prop. 6.1 and Prop. 6.2, there exists a set $\Lambda\subset\mathbb{N}$ which has the $\Lambda(4)_{cb}$-property and satisfies $\forall n\geq n_0$

$$C_1 2^{(n-1)(\frac{1}{2}-\delta)}\leq|\ \Lambda_n\ |\leq C_2 2^{(n-1)(\frac{1}{2}-\delta)}$$

where $\Lambda_n:=\Lambda\cap I_n$ with $I_n:=[2^{n-1},2^n[$, $I_0:=\{0\}$ and where the constants C_1,C_2 and the integer n_0 are defined as in Prop. 6.2. (i) of Facts 2.3 implies that there exists a constant $c_1>0$ such that for all integers $n\geq n_0$ we have

$$C_1 2^{(n-1)(\frac{1}{2}-\delta)}\leq|\Lambda_n|\leq c_1\ 2^{\frac{2(n-1)}{p}}\lambda_p^2(\Lambda_n)\leq c_1\ 2^{\frac{2(n-1)}{p}}\sup_{k\geq n_0}\lambda_p^2(\Lambda_k)$$

that is to say

$$C_1 2^{(n-1)(\frac{1}{2}-\frac{2}{p}-\delta)}\leq c_1\sup_{k\geq0}\lambda_p^2(\Lambda_k)\cdot$$

Since $\frac{1}{2}-\frac{2}{p}-\delta>0$ and n can be arbitrary big, $\sup\limits_{k\geq0}\lambda_p(\Lambda_k)=\infty$. Using Prop. 2.2, we see that $\sup\limits_{k\geq0}\mu_p(\Lambda_k)=\infty$. On each interval Λ_n, we may find a choice of signs ε_n such that its extension to $\mathbb{Z}$ by adding 0's on $\mathbb{Z} \setminus I_n$ and 1's on $I_n \setminus \Lambda_n$ denoted by ξ_n satisfies $\Big\|\xi_n\Big\|_{M(L^p)} \geq \frac{1}{3}\mu_p(\Lambda_n)$. This is clearly possible by using the definition of the constant $\mu_p(\Lambda_n)$ and an extreme points argument. Then we consider $\varepsilon := \sum_{n\geq 0} \xi_n\big|_{\mathbb{N}}$. Note that $\varepsilon(k) = \pm 1$ for each integer $k \geq 0$. Using Prop. 5.3, Peller's results (see Subsection 0.6) together with Remark 0.10, we get

$$\Big\|\,\widehat{\varepsilon}\,\Big\|_{M(\mathfrak{S}^p)} \cong \sup_{n\geq 0}\Big\|\widehat{\varepsilon}_{(n)}\Big\|_{M(\mathfrak{S}^p_{\widehat{I_n}})} \cong \sup_{n\geq 0}\Big\|\xi_n\big|_{I_n}\Big\|_{M(L^p_{I_n})} \cong \sup_{n\geq 0}\Big\|\xi_n\Big\|_{M(L^p)} \geq \frac{1}{3}\sup_{n\geq 0}\mu_p(\Lambda_n) = \infty.$$

Whence $\widehat{\varepsilon}$ does not belong to $M(\mathfrak{S}^p)$. Now we consider $\eta := \frac{1}{2}\big(\varepsilon + 1\!\!1_{\mathbb{N}}\big)$. Recall that $1\!\!1_{\mathbb{N}\setminus\Lambda}$ is a *c.b.* multiplier on H^4 since Λ is a $\Lambda(4)_{cb}$-set and that the constant function $1\!\!1_{\mathbb{N}}$ is trivially a *c.b.* multiplier on H^r for all r. Then, $\eta \in M_{cb}(H^4)$. The idempotent Hankelian multiplier $\widehat{\eta}$ is in $M_{cb}(S^4)$ by using Prop. 5.6 but is not in $M(\mathfrak{S}^p)$ by the above and the fact that the constant function 1 is trivially a *c.b.* multiplier on S^r for all r, so we are done.

Now we show the existence of "large" $\sigma(4)_{cb}$-sets by using probabilistic ideas to exhibit "large" sets having the combinatorial properties (C) or (R) defined below after checking of course that the (R) and (C)-properties imply the $\sigma(4)_{cb}$-property.

Definition 6.3 *We say that a subset A of $\mathbb{N}\times\mathbb{N}$ has the property (C) if $C(A) < \infty$ and that it has the property (R) if $R(A) < \infty$ where*

$$C(A) := \sup_{i\neq i'}\Big|\Big\{j \in \mathbb{N}\big|(i,j) \in A \ \&\ (i',j) \in A\Big\}\Big|$$
$$R(A) := \sup_{j\neq j'}\Big|\Big\{i \in \mathbb{N}\big|(i,j) \in A \ \&\ (i,j') \in A\Big\}\Big|.$$

Remarks 6.4 (i) If $\Lambda \subset \mathbb{N}$ has the $Z(2)$-property then the Hankelian set $\widehat{\Lambda}$ associated to Λ has (C) and (R) with both $C(\widehat{\Lambda})$ and $R(\widehat{\Lambda})$ less than $Z_2(\Lambda)$.
(ii) (C) and (R) are two different combinatorial properties. As an example, the set $A := \mathbb{N}\times\{1\} \cup \mathbb{N}\times\{2\}$ has property (C) but not property (R) (Note that neither (C) nor (R) is stable under finite unions). However, A has property (C) if and only if the set ${}^tA := \big\{(i,j)\ \big|\ (j,i) \in A\big\}$ has property (R) and we have $C(A) = R({}^tA)$.
(iii) 1-sections (resp. 2-sections) are not 2-sections (resp. 1-sections) and are not a finite union of 2-sections (resp. 1-sections) in general but they have both (C) and (R). Assume that $A_1, A_2, ..., A_n$ are 1-sections (resp. 2-sections). Then $A := \bigcup_{i=1}^{n} A_i$ has necessarily the property (C) $\big($resp. $(R)\big)$ with $C(A) \leq n$ $\big($resp. $R(A) \leq n\ \big)$ but has not the property (R) $\big($resp. $(C)\big)$ in general. However, a set with (C) $\big($resp. $(R)\big)$ is not a finite union of 1-sections $\big($resp. 2-sections$\big)$ in general. As an example, consider an increasing sequence of integers $(k_i)_i$ tending to infinity with $k_{i+1} \geq k_i^2$ for each i and let

$$A := \bigcup_{i=1}^{\infty}\Big\{(k_i,l),(k,k_i),(k,k_i^2)\,\big|\,k_i \leq l \leq k_i^2, k_i < k < k_{i+1}\Big\}.$$

A satisfies $C(A) = 2$ but cannot be written as a finite union of 1-sections with moreover $R(A) = \infty$.

Proposition 6.5 *Let $A \subset \mathbb{N} \times \mathbb{N}$. Then, A is a $\sigma(4)_{cb}$-set whenever A has property (C) $\big($resp. property $(R)\big)$. Moreover, we have*

$$\sigma_4^{cb}(A) \leq \big(1 + C(A)\big)^{\frac{1}{4}} \sigma_4^{cb}(A) \leq \big(1 + R(A)\big)^{\frac{1}{4}}\Big).$$

Thus, a finite union of sets having properties either (C) or (R) is necessarily a $\sigma(4)_{cb}$-set.

Proof: Since a set A has property (C) if and only if ${}^tA$ has property (R) with moreover $\sigma_4^{cb}({}^tA) = \sigma_4^{cb}(A)$, we can restrict ourselves to the case where A has property (R). Let $x = \big(x_{ij}\big)_{i,j}$ be in $S^4_A(S^4)$, say with only finitely many non zero entries x_{ij}. We have

$$\Big\|x\Big\|^4_{S^4(S^4)} = tr\bigg(\Big(\sum_{i,j} x_{ij} \otimes e_{ij}\Big)^* \Big(\sum_{i,j} x_{ij} \otimes e_{ij}\Big)\bigg)^2 = tr\bigg(\sum_{i,j,k} x_{ij}^* x_{ik} \otimes e_{jk}\bigg)^2$$
$$= \sum_{i,j,k,r} tr\Big(x_{ij}^* x_{ik} x_{rk}^* x_{rj}\Big) = \sum_{j,k} tr\Big(\sum_i x_{ij}^* x_{ik}\Big)\Big(\sum_r x_{rk}^* x_{rj}\Big) = \sum_{j,k} \Big\|\sum_i x_{ij}^* x_{ik}\Big\|^2_{S^2}$$
$$= \sum_j \Big\|\sum_i x_{ij}^* x_{ij}\Big\|^2_{S^2} + \sum_{j \neq k} \Big\|\sum_i x_{ij}^* x_{ik}\Big\|^2_{S^2}.$$

Using the assumption on A as well as the trace property, we get the following inequalities

$$\Big\|x\Big\|^4_{S^4(S^4)} \leq \sum_j \Big\|\sum_i x_{ij}^* x_{ij}\Big\|^2_{S^2} + R(A) \sum_{j \neq k} \sum_i \Big\|x_{ij}^* x_{ik}\Big\|^2_{S^2}$$
$$\leq \sum_j \Big\|\sum_i x_{ij}^* x_{ij}\Big\|^2_{S^2} + R(A) \sum_i \sum_{j,k} tr\Big(x_{ik}^* x_{ij} x_{ij}^* x_{ik}\Big)$$
$$\Big\|x\Big\|^4_{S^4(S^4)} \leq \sum_j \Big\|\sum_i x_{ij}^* x_{ij}\Big\|^2_{S^2} + R(A) \sum_i \mathrm{tr}\bigg(\Big(\sum_j x_{ij} x_{ij}^*\Big)^* \Big(\sum_j x_{ij} x_{ij}^*\Big)\bigg)$$
$$\leq \sum_j \Big\|\sum_i x_{ij}^* x_{ij}\Big\|^2_{S^2} + R(A) \sum_i \Big\|\sum_j x_{ij} x_{ij}^*\Big\|^2_{S^2}.$$

This implies that for each $x = \big(x_{ij}\big)_{i,j}$ in $S^4_A(S^4)$, we have

$$\Big\|x\Big\|_{S^4(S^4)} \leq \big(1+R(A)\big)^{\frac{1}{4}} \max\bigg\{\bigg(\sum_j \Big\|\Big(\sum_i x_{ij}^* x_{ij}\Big)^{\frac{1}{2}}\Big\|^4_{S^4}\bigg)^{\frac{1}{4}}, \bigg(\sum_i \Big\|\Big(\sum_j x_{ij} x_{ij}^*\Big)^{\frac{1}{2}}\Big\|^4_{S^4}\bigg)^{\frac{1}{4}}\bigg\}.$$

■

Remark. Incidentally, the converse of Prop. 6.5 might be true: perhaps every $\sigma(4)_{cb}$-set is a finite union of sets satisfying either (C) or (R).

Proposition 6.6 *For all small $\delta > 0$ and all $c \geq \frac{1}{\delta}$, there exist constants $n_0, C_1, C_2 > 0$ depending on δ and c only such that for each integer $n \geq n_0$, there exists a subset A_n of $[1, n] \times [1, n]$ satisfying $C(A_n) \leq c$ $\big($resp. $R(A_n) \leq c\big)$ and $C_1 n^{\frac{3}{2}-\delta} \leq \big|A_n\big| \leq C_2 n^{\frac{3}{2}-\delta}$.*

Proof: Let $\{\xi_{ij}\}_{1 \leq i,j \leq n}$ be a sequence of independent random variables on say the torus $\mathbb{T}$ equipped with the normalized Lebesgue measure $d\mathbb{P} = \frac{dt}{2\pi}$ such that for each $1 \leq i, j \leq n$, ξ_{ij} takes its values in $\{0, 1\}$ and has expectation $\mathbb{E}\xi_{ij} = \frac{\beta}{n^{\frac{1}{2}+\delta}}$ where β is a non-negative constant which will be fixed later. For each ω in $\mathbb{T}$, we let

$$A_\omega = \Big\{(i, j) \in [1, n] \times [1, n] \mid \xi_{ij}(\omega) = 1\Big\}.$$

Clearly we have $\left|A_\omega\right| = \sum\limits_{1\le i,j\le n} \xi_{ij}(\omega)$ thus $\mathbb{E}\left|A_\omega\right| = \beta n^{\frac{3}{2}-\delta}$. For each $1 \le i \ne i' \le n$, we let

$$C(i,i',A_\omega) := \left|\left\{ j \in [1,n] \;\middle|\; (i,j),(i',j) \in A_\omega \right\}\right| = \sum_{1\le j\le n} \xi_{ij}(\omega)\xi_{i'j}(\omega)\cdot$$

Then, given an integer $m \ge 1$, we get by using the independence of the ξ_{ij}'s

$$\mathbb{P}\Big\{\omega \mid C(i,i',A_\omega) \ge m\Big\} = \mathbb{P}\Big\{\omega \mid \sum_{1\le j\le n} \xi_{ij}(\omega)\xi_{i'j}(\omega) \ge m\Big\}$$

$$= \mathbb{P}\Big\{\omega \;\Big|\; \exists 1 \le j_1 \ne j_2 \ne ... \ne j_m \le n \text{ such that } \xi_{ij}(\omega)\xi_{i'j}(\omega) = 1, \forall j = j_1, j_2, ..., j_m\Big\}$$

$$= \sum_{1\le j_1\ne ...\ne j_m\le n} \left(\prod_{k=1}^{m} \mathbb{P}\Big\{\omega \mid \xi_{ij_k}(\omega)\xi_{i'j_k}(\omega) = 1\Big\}\right) = \sum_{1\le j_1\ne ...\ne j_m\le n} \left(\prod_{k=1}^{m} \frac{\beta^2}{n^{1+2\delta}}\right) \le \frac{\beta^{2m}}{n^{2m\delta}}\cdot$$

This implies that we have

$$\mathbb{P}\Big\{\omega \mid C(A_\omega) \ge m\Big\} = \mathbb{P}\Big\{\omega \mid \sup_{1\le i\ne i'\le n} C(i,i',A_\omega) \ge m\Big\} \le \sum_{1\le i\ne i'\le n} \frac{\beta^{2m}}{n^{2m\delta}}$$
$$\mathbb{P}\Big\{\omega \mid C(A_\omega) \ge m\Big\} \le n^{2(1-m\delta)}\beta^{2m}\cdot$$

By choosing β such that $\beta^{2c} < \frac{1}{2}$, we get $\mathbb{P}\Big\{\omega \mid C(A_\omega) \ge c\Big\} < \frac{1}{2}$ since $1 - c\delta \le 0$. On the other hand, the sequence $\Big(\xi_{ij} - \mathbb{E}\xi_{ij}\Big)_{i,j}$ is a sequence of centered and independent random variables hence the variance of $\left|A_\omega\right| - \mathbb{E}\left|A_\omega\right|$ is

$$\Big\|\left|A_\omega\right| - \mathbb{E}\left|A_\omega\right|\Big\|_{L^2}^2 \le \beta n^{\frac{3}{2}-\delta} = \mathbb{E}\left|A_\omega\right|\cdot$$

Thus, using Tchebytchev's inequality, we get

$$\mathbb{P}\Big\{\omega \;\Big|\; \big||A_\omega| - \mathbb{E}|A_\omega|\big| \ge \frac{1}{2}\mathbb{E}|A_\omega|\Big\} \le \frac{4}{\mathbb{E}|A_\omega|} = \frac{4}{\beta} n^{-\frac{3}{2}+\delta} := P(n)$$
$$\mathbb{P}\Big\{\omega \mid \frac{\beta}{2} n^{\frac{3}{2}-\delta} \le |A_\omega| \le \frac{3\beta}{2} n^{\frac{3}{2}-\delta}\Big\} \ge 1 - P(n)\cdot$$

This completes the proof since $P(n)$ tends to zero at infinity. ■

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