ROUGH BILINEAR SINGULAR INTEGRALS

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ABSTRACT. We study the rough bilinear singular integral, introduced by Coifman and Meyer [9],

\[ T_\Omega(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(y,z)|^{-2n} \Omega((y,z)/|y,z|) f(x-y) g(x-z) dydz, \]

when \( \Omega \) is a function in \( L^q(S^{2n-1}) \) with vanishing integral and \( 2 \leq q \leq \infty \).

When \( q = \infty \) we obtain boundedness for \( T_\Omega \) from \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) when \( 1 < p_1, p_2 < \infty \) and \( 1/p = 1/p_1 + 1/p_2 \). For \( q = 2 \) we obtain that \( T_\Omega \) is bounded from \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \). For \( q \) between 2 and infinity we obtain the analogous boundedness on a set of indices around the point \((1/2, 1/2, 1)\). To obtain our results we introduce a new bilinear technique based on tensor-type wavelet decompositions.

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1. INTRODUCTION

Singular integral theory was initiated in the seminal work of Calderón and Zygmund [3]. The study of boundedness of rough singular integrals of convolution type has been an active area of research since the middle of

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the twentieth century. Calderón and Zygmund [4] first studied the rough singular integral

$$L_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x - y) \, dy$$

where $\Omega$ is in $L \log L(\mathbb{S}^{n-1})$ with mean value zero and showed that $L_\Omega$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The same conclusion under the less restrictive condition that $\Omega$ lies in $H^1(\mathbb{S}^{n-1})$ was obtained by Coifman and Weiss [10] and Connett [11]. The weak type $(1, 1)$ boundedness of $L_\Omega$ when $n = 2$ was established by Christ and Rubio de Francia [7] and independently by Hofmann [20], both inspired by Christ’s work [6]. Additionally, in unpublished work, Christ and Rubio de Francia extended this result to all dimensions $n \leq 7$. The weak type $(1, 1)$ property of $L_\Omega$ was proved by Seeger [28] in all dimensions and was later extended by Tao [30] to situations in which there is no Fourier transform structure. Several questions remain concerning the endpoint behavior of $L_\Omega$, such as if the condition $\Omega \in L \log L(\mathbb{S}^{n-1})$ can be relaxed to $\Omega \in H^1(\mathbb{S}^{n-1})$, or merely $\Omega \in L^1(\mathbb{S}^{n-1})$ when $\Omega$ is an odd function. On the former there is a partial result of Stefanov [29] but not much is still known about the latter.

The bilinear counterpart of the rough singular integral linear theory is notably more intricate. To fix notation, we fix $1 < q \leq \infty$ and we let $\Omega$ in $L^q(\mathbb{S}^{2n-1})$ with $\int_{\mathbb{S}^{2n-1}} \Omega \, d\sigma = 0$, where $\mathbb{S}^{2n-1}$ is the unit sphere in $\mathbb{R}^{2n}$. Coifman and Meyer [9] introduced the bilinear singular integral operator associated with $\Omega$ by

$$T_\Omega(f, g)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x - y, x - z) f(y) g(z) \, dy \, dz,$$

where $f, g$ are functions in the Schwartz class $S(\mathbb{R}^n)$,

$$K(y, z) = \Omega((y, z)'/|y, z|^{2n}),$$

and $x' = x/|x|$ for $x \in \mathbb{R}^{2n}$. General facts about bilinear operators can be found in [26, Chapter 13], [17, Chapter 7], and [27]. If $\Omega$ possesses some smoothness, i.e. if is a function of bounded variation on the circle, Coifman and Meyer [9, Theorem 1] showed that $T_\Omega$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$. In higher dimensions, it was shown Grafakos and Torres [19], via a bilinear $T1$ condition, that if $\Omega$ a Lipschitz function on $\mathbb{S}^{2n-1}$, then $T_\Omega$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 < p_1, p_2 < \infty$, $1/2 < p < \infty$, and $1/p = 1/p_1 + 1/p_2$. But if $\Omega$ is rough, the situation is significantly more complicated, and the boundedness of $T_\Omega$ remained unresolved until this work, except when in situations when it reduces to the uniform boundedness of bilinear Hilbert transforms. If $\Omega$ is merely integrable function on
$\mathbb{S}^1$, but is odd, the operator $T_\Omega$ is intimately connected with the celebrated (directional) bilinear Hilbert transform

$$\mathcal{H}_{\theta_1,\theta_2}(f_1,f_2)(x) = \int_{-\infty}^{\infty} f_1(x-t\theta_1)f_2(x-t\theta_2) \frac{dt}{t}$$

(in the direction $(\theta_1,\theta_2)$), via the relationship

$$T_\Omega(f_1,f_2)(x) = \frac{1}{2} \int_{\mathbb{S}^{2n-1}} \Omega(\theta_1,\theta_2) \mathcal{H}_{\theta_1,\theta_2}(f_1,f_2)(x) d(\theta_1,\theta_2).$$

The boundedness of $\mathcal{H}_{\theta_1,\theta_2}$ was proved by Lacey and Thiele [22], [23] while the more relevant, for this problem, uniform in $\theta_1,\theta_2$ boundedness of $\mathcal{H}_{\theta_1,\theta_2}$ was addressed by Thiele [31], Grafakos and Li [18], and Li [24]. Exploiting the uniform boundedness of $\mathcal{H}_{\theta_1,\theta_2}$, Diestel, Grafakos, Honzík, Si, and Terwilleger [13] showed that if $n=2$ and the even part of $\Omega$ lies in $H^1(\mathbb{S}^1)$, then $T_\Omega$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1 < p_1, p_2, p < \infty$, $1/p = 1/p_1 + 1/p_2$, and the triple $(1/p_1, 1/p_2, 1/p)$ lies in the open hexagon described by the conditions:

$$\left| \frac{1}{p_1} - \frac{1}{p_2} \right| < \frac{1}{2}, \quad \left| \frac{1}{p_1} - \frac{1}{p} \right| < \frac{1}{2}, \quad \left| \frac{1}{p_2} - \frac{1}{p} \right| < \frac{1}{2}.$$  

This is exactly the region in which the uniform boundedness of the bilinear Hilbert transforms is currently known. It is noteworthy to point out that $T_\Omega$ itself reduces to a bilinear Hilbert transform $\mathcal{H}_{\theta_1,\theta_2}$, if $\Omega$ is the sum of the pointmasses $\delta_{(\theta_1,\theta_2)} + \delta_{-(\theta_1,\theta_2)}$ on $\mathbb{S}^1$.

In this work we provide a proof of the boundedness of $T_\Omega$ on $L^p$ for all $p > 1/2$ in all dimensions. This breakthrough is a consequence of a new technique in this context. We build on the work of Duoandikoetxea and Rubio de Francia [15], but our key idea is to decompose the multiplier in terms of a tensor-type compactly-supported wavelet decomposition and to use combinatorial arguments to group the different pieces together, exploiting orthogonality. One may speculate that other decompositions, such as certain types of frames or the $\phi$-transform of Frazier and Jawerth [16], which has good almost-orthogonality estimates, could be used to attack this problem. But as the combinatorial aspect of our approach heavily depends on the fact that our basis is of tensor type, similar properties of other decompositions will be necessary if they are to be adapted to the techniques of this article.

The main result of this paper is the following theorem.

**Theorem 1.** For all $n \geq 1$, if $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, then for $T_\Omega$ defined in (1), we have

$$\|T_\Omega\|_{L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty$$

whenever $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. 

In the remaining sections we focus on the proof of this result while in the last section we focus on extensions to the case where \( \Omega \) lies in \( L^q(S^{2n-1}) \) for \( q < \infty \).

Some remarks about our notation in this paper: For \( 1 < q < \infty \) we set \( q' = q/(q-1) \) and for \( q = \infty \) we set \( \infty' = 1 \). We denote the the norm of a bounded bilinear operator \( T \) from \( X \times Y \) to \( Z \) by
\[
\|T\|_{X \times Y \to Z} = \sup_{\|f\|_X \leq 1} \sup_{\|g\|_Y \leq 1} \|T(f,g)\|_Z.
\]
This notation was already used in (2). If \( x_1,x_2 \) are in \( \mathbb{R}^n \), then we denote the point \((x_1,x_2)\) in \( \mathbb{R}^{2n} \) by \( \bar{x} \). We denote the set of positive integers by \( \mathbb{N} \) and we set \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). In the sequel, multiindices in \( \mathbb{Z}^{2n} \) are elements of \( \mathbb{N}_0^{2n} \). Finally, we adhere to the standard convention to denote by \( C \) a constant that depends only on inessential parameters of the problem.

2. Estimates of Fourier transforms of the kernels

Let us fix a \( q \) satisfying \( 1 < q \leq \infty \) and a function \( \Omega \in L^q(S^{n-1}) \) with mean value zero. We fix a smooth function \( \alpha \) in \( \mathbb{R}^+ \) such that \( \alpha(t) = 1 \) for \( t \in (0,1], \alpha(t) < 1 \) for \( t \in (1,2) \) and \( \alpha(t) = 0 \) for \( t \geq 2 \). For \( (y,z) \in \mathbb{R}^{2n} \) and \( j \in \mathbb{Z} \) we introduce the function
\[
\beta_j(y,z) = \alpha(2^{-j}(|y|+|z|)) - \alpha(2^{-j+1}(|y|+|z|)).
\]
We write \( \beta = \beta_0 \) and we note that this is a function supported in \([1/2,2]\).

We denote \( \Delta_j \) the Littlewood-Paley operator \( \Delta_j f = \mathcal{F}^{-1}(\beta_j \hat{f}) \). Here and throughout this paper \( \mathcal{F}^{-1} \) denotes the inverse Fourier transform, which is defined via \( \mathcal{F}^{-1}(g)(x) = \int_{\mathbb{R}^n} g(\xi) e^{2\pi i \xi \cdot x} d\xi = \hat{g}(-x) \), where \( \hat{g} \) is the Fourier transform of \( g \). We decompose the kernel \( K \) as follows: we denote \( K^i = \beta_i K \) and we set \( K_j^i = \Delta_{j-i} K^i \) for \( i, j \in \mathbb{Z} \). Then we write
\[
K = \sum_{j=-\infty}^{\infty} K_j,
\]
where
\[
K_j = \sum_{i=-\infty}^{\infty} K_j^i.
\]
We also denote \( m_j = \hat{K}_j \).

Then the operator can be written as
\[
T_\Omega(f,g)(x) = \sum_j \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_j(x-y,x-z)f(y)g(z)dydz =: \sum_j T_j(f,g)(x).
\]
We have the following lemma whose proof is known (see for instance [14]) and is omitted.
Lemma 2. Given $\Omega \in L^q(\mathbb{S}^{2n-1})$, $0 < \delta < 1/q'$ and $\vec{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^{2n}$ we have

$$|\hat{K}^0(\vec{\xi})| \leq C\|\Omega\|_{L^q} \min(|\vec{\xi}|, |\vec{\xi}|^{-\delta})$$

and for all multiindices $\alpha$ in $\mathbb{Z}^{2n}$ with $\alpha \neq 0$ we have

$$|\partial^\alpha \hat{K}^0(\vec{\xi})| \leq C\|\Omega\|_{L^q} \min(1, |\vec{\xi}|^{-\delta}).$$

The following proposition is a consequence of the preceding lemma.

Proposition 3. Let $1 \leq p_1, p_2 < \infty$ and define $p$ via $1/p = 1/p_1 + 1/p_2$. Let $\Omega \in L^q(\mathbb{S}^{2n-1})$, $1 < q \leq \infty$, $0 < \delta < 1/q'$, and for $j \in \mathbb{Z}$ consider the bilinear operator

$$T_j(f,g)(x) = \int_{\mathbb{R}^{2n}} K_j(x-y,y-z)f(y)g(z)dydz.$$ 

If both $p_1, p_2 > 1$, then $T_j$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with norm at most $C\|\Omega\|_{L^q} 2^{(2n-\delta)j}$ if $j \geq 0$ and at most $C\|\Omega\|_{L^q} 2^{-|j|(1-\delta)}$ if $j < 0$. If at least one of $p_1$ and $p_2$ is equal to 1, then $T_j$ maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ with a similar norm.

Proof. We prove the assertion by showing that the multiplier $m_j = \hat{K}_j$ associated with $T_j$ satisfies the conditions of the Coifman-Meyer multiplier theorem [8], which was extended to the case $p < 1$ by Kenig and Stein [21] and by Grafakos and Torres [19]. To be able to use this theorem, we need to show that $m_j$ is a $C^\infty$ function on $\mathbb{R}^{2n} \setminus \{0\}$ that satisfies

$$|\partial^\alpha m_j(\vec{\xi})| \leq C\|\Omega\|_{L^q} |\vec{\xi}|^{-|\alpha|}$$

for all multiindices $\alpha$ in $\mathbb{Z}^{2n}$ with $|\alpha| \leq 2n$ and all $\vec{\xi} \in \mathbb{R}^{2n} \setminus \{0\}$, where $Q(j) = 2^{(2n-\delta)j}$ if $j \geq 0$ and $Q(j) = 2^{-|j|(1-\delta)}$ if $j < 0$. Then we may use Theorem 7.5.3 in [17] to deduce the claimed boundedness. It is not hard to verify that

$$m_j(\vec{\xi}) = \sum_{i=-\infty}^{\infty} \beta(2^{i-j}|\vec{\xi}|)\hat{K}^0(2^i\vec{\xi})$$

If $|\vec{\xi}| \approx 2^l$, then since $\beta$ is supported in $[1/2, 2]$, $2^l$ must be comparable to $2^{j-l}$ in (3). Using Lemma 2 we have the estimate

$$|m_j(\vec{\xi})| \leq \sum_{i \in F} |\hat{K}^0(2^i\vec{\xi})| \leq C\|\Omega\|_{L^q} \sum_{i \in F} \min \{2^{|i-j|\delta}, (2^{|i-j|}|\vec{\xi}|)^{-\delta}\} \leq C\|\Omega\|_{L^q} I(j),$$

where $F$ is a finite set of $i$'s near $j - l$ and $I(j) = 2^{-|j|\delta}$ if $j \geq 0$ whereas $I(j) = 2^{-|j|}$ if $j < 0$. For an $\alpha$th derivative of $m_j$ with $1 \leq |\alpha| \leq 2n$, using
that $|\partial^\alpha \hat{K}^0(\vec{\xi})| \leq C_\alpha \|\Omega\|_{L^q} |\vec{\xi}|^{-\delta}$, we obtain

$$\sum_{i \in F} |\partial^\alpha (\hat{\hat{K}}^0(2^i \vec{\xi}) \beta_j(2^i \vec{\xi}))| \leq \|\Omega\|_{L^q} \sum_{i \in F} C_\alpha 2^i |\alpha| (2^i |\vec{\xi}|)^{-\delta}$$

and this is at most $C \|\Omega\|_{L^q} 2^{(2n-\delta)j}$ if $j \geq 0$.

If $j < 0$, we have

$$\sum_{i \in F} |\partial^\alpha (\hat{\hat{K}}^0(2^i \vec{\xi}) \beta_j(2^i \vec{\xi}))| \leq \|\Omega\|_{L^q} \sum_{i \in F} C_\alpha 2^i |\alpha| 2^j (1-|\alpha|)$$

$$\leq C \|\Omega\|_{L^q} 2^j |\vec{\xi}|^{-|\alpha|}.$$  

□

The operators $T_j$ associated with the multipliers $\hat{K}_j$ are bounded with bounds that grow in $j$ since the smoothness of the symbol is getting worse with $j$. We certainly have that $\|\hat{K}_j\|_{L^\infty} \leq C 2^{-|j|\delta}$, but there is no good estimate available for the derivatives of $\hat{K}_j$, and moreover, a good $L^\infty$ estimate for the multiplier does not suffice to yield boundedness in the bilinear setting. The key argument of this article is to circumvent this obstacle and prove that the norms of the operators $T_j$ indeed decay exponentially. Our proof is new in this context and is based on a suitable wavelet expansion combined with combinatorial arguments.

3. BOUNDEDNESS: A GOOD POINT

In this section we prove the following result which is a special case of Theorem 1:

**Theorem 4.** Suppose $\Omega \in L^q(S^{2n-1})$ with $2 \leq q \leq \infty$, then for $f,g$ in $L^2(\mathbb{R}^n)$ we have

$$\|T\Omega(f,g)\|_{L^1(\mathbb{R}^n)} \leq C \|\Omega\|_{L^q} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$  

In view of Proposition 3, Theorem 4 will be a consequence of the following proposition.

**Proposition 5.** Given $2 \leq q \leq \infty$ and $0 < \delta < 1/(8q')$, then for any $j \geq 0$, the operator $T_j$ associated with the kernel $K_j$ maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ with norm at most $C \|\Omega\|_{L^q} 2^{-\delta j}$.

To obtain the proof of the proposition, we utilize wavelets with compact support. Their existence is due to Daubechies [12] and can also be found in Meyer’s book [25]. For our purposes we need product type smooth wavelets
with compact supports; the construction of such objects can be found in Triebel [32].

Lemma 6. For any fixed $k \in \mathbb{N}$ there exist real compactly supported functions $\psi_F, \psi_M \in C^k(\mathbb{R})$, which satisfy $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$, for $0 \leq \alpha \leq k$ we have $\int_{\mathbb{R}} x^\alpha \psi_M(x) dx = 0$, and, if $\Psi^G$ is defined by

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1) \cdots \psi_{G_{2n}}(x_{2n})$$

for $G = (G_1, \ldots, G_{2n})$ in the set

$$I := \{(G_1, \ldots, G_{2n}) : G_i \in \{F, M\}\},$$

then the family of functions

$$\bigcup_{\vec{\mu} \in \mathbb{Z}^{2n}} \left[ \{\Psi^{(F, \ldots, F)}(\vec{x} - \vec{\mu})\} \cup \bigcup_{\lambda = 0}^{\infty} \{2^\lambda n \Psi^G(2^\lambda \vec{x} - \vec{\mu}) : G \in I \setminus \{(F, \ldots, F)\}\} \right]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1, \ldots, x_{2n})$.

Proof of Proposition 5. To obtain the estimate, we first decompose the symbol into dyadic pieces, estimate them separately, and then use orthogonality arguments to put them back together. Let us take a look at the symbol $\hat{K}_j^0$ which we denote $m_{j,0}$. The classical estimates show that

$$\|m_{j,0}\|_{L^\infty} = \|\hat{K}_j^0\|_{L^\infty} \leq C\|\Omega\|_{L^q} 2^{-\delta j}, \quad 0 < \delta < 1/q',$$

while for $2 \leq q \leq \infty$

$$\|m_{j,0}\|_{L^2} = \|\beta_j(\hat{\psi}_0 K)\|_{L^2} \leq C\|\hat{\psi}_0 K\|_{L^2} \leq C\|\Omega\|_{L^2} \leq C\|\Omega\|_{L^q}.$$

We observe that for the case $i \neq 0$ we have the identity $m_{j,i} = \hat{K}_j^i = m_{j,0}(2^i \cdot)$ from the homogeneity of the symbol, and thus $m_{j,i}$ also lies in $L^2$.

We utilize a wavelet transform of $m_{j,0}$. We take the product wavelets described above, with compact supports and $M$ vanishing moments, where $M$ is a large number to be determined later. Here we choose generating functions with support diameter approximately 1. The wavelets with the same dilation factor $2^\lambda$ have some bounded overlap $N$ independent of $\lambda$.

Since the inverse Fourier transform of $m_{j,0}$ is essentially supported in the dyadic annulus of radius 1, the symbol is smooth and the wavelet transform has a nice decay. Precisely with

$$\Psi_{\vec{\mu}}^{\lambda, G}(\vec{x}) = 2^\lambda n \Psi^G(2^\lambda \vec{x} - \vec{\mu}), \quad \vec{x} \in \mathbb{R}^{2n},$$

we have the following result:
Lemma 7. Using the preceding notation, for any \(j \in \mathbb{Z}\) and \(\lambda \in \mathbb{N}_0\) we have
\[
\langle \Psi_{\vec{\mu}}^{\lambda,G}, m_{j,0} \rangle \leq C \|\Omega\|_{L^q} 2^{-\delta j} 2^{-(M+1+n)\lambda},
\]
where \(M\) is the number of vanishing moments of \(\psi\) and \(\delta\) is as in (4).

Proof. Let \(\lambda \geq 0\) and \(G \in \mathcal{I} \setminus \{(F,\ldots,F)\}\). We apply the smoothness-cancellation estimate in Appendix B.2 of [17] with \(\Psi_{\vec{\mu}}^{\lambda,G}\), \(L = M + 1\), and \(\Phi\) being the function \(m_{j,0}\). Then we have the properties

(i) \(\int_{\mathbb{R}^n} \Psi_{\vec{\mu}}^{\lambda,G}(\vec{x}) \vec{x}^\beta \, d\vec{x} = 0\) for \(|\beta| \leq L - 1\),
(ii) \(|\Psi_{\vec{\mu}}^{\lambda,G}(\vec{x})| \leq \frac{C^{2\lambda_k_n}}{(1+2^\lambda |\vec{x}^-2-\vec{\mu}|)^{q_1}}\),
(iii) For \(|\alpha| = L\), \(|\partial^\alpha (m_{j,0})(\vec{x})| \leq \frac{C\|\Omega\|_{L^q} 2^{-j\delta}}{(1+2^{\lambda-j\delta}|\vec{x}|)^{M_2}}\). To verify this property we notice that since \(\beta_0\) is a Schwartz function, we have
\[
|\partial^\alpha (\beta_j \tilde{K}^0)(\vec{x})| \leq \sum_{\gamma \leq \alpha} 2^{-j|\gamma|} |\partial^\gamma \beta_0(2^{-j\lambda} \vec{x})| \partial^\alpha - \gamma \tilde{K}^0(\vec{x})|
\]
\[
\leq C\|\Omega\|_{L^q} \sum_{\gamma \leq \alpha} 2^{-j|\gamma|} \frac{2^{-j\delta}}{(1+2^{-j\lambda}|\vec{x}|)^{M_2}}
\]
\[
\leq C\|\Omega\|_{L^q} \frac{2^{-j\delta}}{(1+2^{-j\lambda}|\vec{x}|)^{M_2}},
\]
where we used Lemma 2, i.e. the property that \(|\partial^\alpha \tilde{K}^0(\vec{x})| \leq C\|\Omega\|_{L^q} |\vec{x}|^{-\delta}\)
for all multiindices \(\alpha\).

Thus \(\Psi_{\vec{\mu}}^{\lambda,G}\) has cancellation and \(m_{j,0}\) has appropriate smoothness and so it follows that
\[
|\langle \Psi_{\vec{\mu}}^{\lambda,G}, m_{j,0} \rangle| \leq C\|\Omega\|_{L^q} \frac{2^{-j\delta} 2^{\lambda n} 2^{-\lambda (L+2n)}}{(1+2^{-j\lambda}|\vec{\mu}|)^{M_2}} \leq C\|\Omega\|_{L^q} 2^{-j\delta} 2^{-\lambda (M+1+n)},
\]
thus (6) holds. Notice that the constant \(C\) is independent of \(\vec{\mu}\).

Next we consider the case \(\lambda = 0\) and \(G = (F,\ldots,F)\). In this case we have \(|\Psi_{\vec{\mu}}^{\lambda,G}(\vec{x})| \leq \frac{C}{(1+|\vec{x}^-2-\vec{\mu}|)^{q_1}}\) and \(|m_{j,0}(\vec{x})| \leq \frac{C\|\Omega\|_{L^q} 2^{-j\delta}}{(1+2^{-j\lambda}|\vec{x}|)^{M_2}}\). Using the result in Appendix B1 in [17] we deduce that
\[
|\langle \Psi_{\vec{\mu}}^{\lambda,G}, m_{j,0} \rangle| \leq C\|\Omega\|_{L^q} \frac{2^{-j\delta}}{(1+2^{-j\lambda}|\vec{\mu}|)^{M_2}} \leq C\|\Omega\|_{L^q} 2^{-j\delta}
\]
and thus (6) follows in this case as well. \(\square\)

The wavelets sharing the same generation index may be organized into \(C_{n,M,N}\) groups so that members of the same group have disjoint supports and are of the same product type, i.e., they have the same index \(G \in \mathcal{I}\).
For $1 \leq \kappa \leq C_{n,M,N}$ we denote by $D_{\lambda, \kappa}$ one of these groups consisting of wavelets whose supports have diameters about $2^{-\lambda}$. We now have that the wavelet expansion

$$m_{j,0} = \sum_{\lambda \geq 0} \sum_{\omega \in D_{\lambda, \kappa}} a_{\omega} \omega$$

and $\omega$ all have disjoint supports within the group $D_{\lambda, \kappa}$. For the sequence $a = \{a_{\omega}\}$ we get $\|a\|_{l^2} \leq C$, in view of (5), because $\{\omega\}$ is an orthonormal basis. Since the $\omega$ are continuous functions and and bounded by $2^{\lambda n}$, if we set $b_{\omega} = \|a_{\omega}\|_{L^\infty}$, we have

$$\|\{b_{\omega}\}_{\omega \in D_{\lambda, \kappa}}\|_{l^2} \leq 2^{\lambda n} \left( \sum_{\omega \in D_{\lambda, \kappa}} |a_{\omega}|^2 \right)^{1/2} \leq C \|\Omega\|_{L^2} 2^{\lambda n}.$$ 

Clearly we also have

$$\|\{b_{\omega}\}_{\omega \in D_{\lambda, \kappa}}\|_{l^\infty} \leq \|\{a_{\omega}\}_{\omega \in D_{\lambda, \kappa}}\|_{l^\infty} 2^{\lambda n} \leq C \|\Omega\|_{L^q} 2^{-\delta j - (M+1)\lambda}. \tag{7}$$

Now, we split the group $D_{\lambda, \kappa}$ into three parts. Recall the fixed integer $j$ in the statement of Proposition 5. We define sets

$$D_{\lambda, \kappa}^1 = \{ \omega \in D_{\lambda, \kappa} : a_{\omega} \neq 0, \text{ supp} \omega \subset \{ (\xi_1, \xi_2) : 2^{-j} |\xi_1| \leq |\xi_2| \leq 2^j |\xi_1| \} \},$$

$$D_{\lambda, \kappa}^2 = \{ \omega \in D_{\lambda, \kappa} : a_{\omega} \neq 0, \text{ supp} \omega \cap \{ (\xi_1, \xi_2) : 2^{-j} |\xi_1| \geq |\xi_2| \} \neq \emptyset \},$$

and

$$D_{\lambda, \kappa}^3 = \{ \omega \in D_{\lambda, \kappa} : a_{\omega} \neq 0, \text{ supp} \omega \cap \{ (\xi_1, \xi_2) : 2^{-j} |\xi_2| \geq |\xi_1| \} \neq \emptyset \}.$$ 

These groups are disjoint for large $j$. Notice that $D_{\lambda, \kappa}^1 \cap D_{\lambda, \kappa}^2 = \emptyset$ is obvious. For $D_{\lambda, \kappa}^2$ and $D_{\lambda, \kappa}^3$, the worst case is $\lambda = 0$ when we have balls of radius 1 centered at integers, and $D_{\lambda, \kappa}^2 \cap D_{\lambda, \kappa}^3 = \emptyset$ if $j$ is sufficiently large, for instance $j \geq 100\sqrt{n}$ works, since if $a_{\omega} \neq 0$, then $\omega$ is supported in an annulus centered at the origin of size about $2^j$. We are assuming here that $j \geq 100\sqrt{n}$ but notice that for $j < 100\sqrt{n}$, Proposition 5 is an easy consequence of Proposition 3.

We denote, for $t = 1, 2, 3$,

$$m_{j,0}^t = \sum_{\lambda, \kappa} \sum_{\omega \in D_{\lambda, \kappa}^t} a_{\omega} \omega,$$

and define

$$m_j^t = \sum_{k=-\infty}^{\infty} m_{j,k}^t.$$
with $m_{j,k}(\xi) = m_{j,0}(2^k\xi)$. We prove boundedness for each piece $m_j^1, m_j^2, m_j^3$. We call $m_j^1$ the diagonal part of $m_j$ and $m_j^2, m_j^3$ the off-diagonal parts of $m_j = \widehat{K}_j$.

4. The diagonal part

We first deal with the first group $D^1_{\lambda,k}$. Each $\omega \in D^1_{\lambda,k}$ is of tensor product type $\omega = \omega_1 \omega_2$, therefore, we may index the sequences by two indices $k,l \in \mathbb{Z}^n$ according to the first and second variables. Thus $\omega_{k,l} = \omega_{1,k} \omega_{2,l}$. Likewise, we index the sequence $b = \{b_{(k,l)}\}_{k,l}$. Now for $r \geq 0$ we define sets

$$U_r = \{(k,l) \in \mathbb{Z}^{2n} : 2^{-r-1} \|b\|_\infty < |b_{(k,l)}| \leq 2^{-r+1} \|b\|_\infty\}.$$ 

From the $l^2$ norm of $b$, we find that the cardinality of this set is at most $C\|\Omega\|_2^2 2^{2n} 2^{2r} \|b\|_{l^\infty}^{-2}$. Indeed, we have

$$|U_r| \leq 4 \sum_{(k,l) \in U_r} |b_{(k,l)}|^2 (\|b\|_{l^2} 2^{-r})^{-2} \leq 4 \|b\|_{l^2}^2 \|b\|_{l^\infty}^{-2} 2^{2r} \leq C \|\Omega\|_2^2 2^{2n} 2^{2r} \|b\|_{l^\infty}^{-2}.$$ 

We split each $U_r = U_r^1 \cup U_r^2 \cup U_r^3$, where

$$U_r^1 = \{(k,l) \in U_r : \text{card}\{s : (k,s) \in U_r\} \geq 2^{(r+\delta)j+M^{\lambda}/4}\},$$

$$U_r^2 = \{(k,l) \in U_r \setminus U_r^1 : \text{card}\{s : (s,l) \in U_r \setminus U_r^1\} \geq 2^{(r+\delta)j+M^{\lambda}/4}\},$$

and the third set is the remainder. These three sets are disjoint. We notice that if the index $k$ satisfies $\text{card}\{s : (k,s) \in U_r\} \geq 2^{(r+\delta)j+M^{\lambda}/4}$, then the pair $(k,l)$ lies in $U_r^1$ for all $l \in \mathbb{Z}^n$ such that $(k,l) \in U_r$.

We observe that in the first set $U_r^1$, we have

$$N_1 := \text{card}\{k : \text{there is } l \text{ s.t. } (k,l) \in U_r^1\} \leq C \|\Omega\|_2^2 2^{(n-M^{\lambda}/4)j + \frac{2r}{4} - \frac{\delta_0}{4}},$$

since $N_1 2^{(r+\delta)j+M^{\lambda}/4} \leq C \|\Omega\|_2^2 2^{2n} 2^{2r} \|b\|_{l^\infty}^{-2}$. We now write

$$m_{j,1}^r = \sum_{(k,l) \in U_r^1} a_{(k,l)} \omega_{1,k} \omega_{2,l}$$

and estimate the norm of $m_{j,1}^r$ as a bilinear multiplier as follows:

$$\|T_{m_{j,1}^r}(f,g)\|_{L^1} \leq \left\| \sum_{(k,l) \in U_r^1} a_{(k,l)} F^{-1}(\omega_{1,k}) \mathcal{F}^{-1}(\omega_{2,l} \hat{g}) \right\|_{L^1} \leq \sum_{k \in E} \|\omega_{1,k} \hat{f}\|_{L^2} \sum_{(k,l) \in U_r^1} a_{(k,l)} \|\omega_{2,l} \hat{g}\|_{L^2}.$$
For fixed $k$, by the choice of $D_{\lambda,k}$, the supports of $\omega_{k,l} = \omega_{1,k} \omega_{2,l}$ are disjoint, in particular, the supports of $\omega_{2,l}$ are disjoint. Since $\|\omega_{1,k}\|_{L^\infty} \approx 2^{\lambda n/2}$, we have the estimate

$$\left\| \sum_{(k,l) \in U_1^j} a_{(k,l)} \omega_{2,l} \right\|_{L^\infty} \leq C \left\| \sum_{(k,l) \in U_1^j} |b_{(k,l)}| 2^{-\lambda n/2} \chi_{E_l} \right\|_{L^\infty} \leq C \|b\|_\infty 2^{-r_2-\lambda n/2},$$

where $E_l \subset \mathbb{R}^n$ is the support of $\omega_{2,l}$. As a result,

$$\left\| \sum_{(k,l) \in U_1^j} a_{(k,l)} \omega_{2,l} \hat{g} \right\|_{L^2} \leq C \|b\|_\infty 2^{-r_2-\lambda n/2} \|g\|_{L^2}.$$

Now let $E = \{k : \exists l \text{ s.t. } (k,l) \in U_1^j \}$ and note that $|E| = N_1$.

Notice that the $\omega_{k,l}$ in $U_1^j$ have the following property. If $(k,l) \neq (k',l')$, then the supports of $\omega_{1,k}$ and $\omega_{1,k'}$ are disjoint. Since the $\omega_{1,k}$ satisfy $\|\omega_{1,k}\|_{L^\infty} \approx 2^{\lambda n/2}$ and have disjoint supports, we have

$$\|T_{m_j^{r_1}}(f,g)\|_{L^1} \leq \sum_{k \in E} \|\omega_{1,k} \hat{f} \|_{L^2} 2^{-\lambda n/2} 2^{-r} \|b\|_{\infty} \|g\|_{L^2} \leq C \|\Omega\|_{L^2} 2^{(n-M/4)\lambda+7r/4-\delta_{0,j/4}} \|b\|_{\infty} 2^{-\lambda n/2} \|f\|_{L^2} 2^{-\lambda n/2} 2^{-r} \|b\|_{\infty} \|g\|_{L^2} \leq C \|\Omega\|_{L^2} 2^{(n-M/8)\lambda-r/8-\delta_{0,j/4}} \|f\|_{L^2} \|g\|_{L^2},$$

where we used (8) and (7). This gives sufficient decay in $j$, $r$ and $\lambda$ if $M \geq 16n$. The set $U_2^2$ is handled the same way.

To estimate the set $U_3^3$, we further decompose it into at most $2^{(r+\delta_{0,j}+M\lambda)/2}$ disjoint sets $V_s$, such that if $(k,l), (k',l') \in V_s$ then $(k,l) \neq (k',l')$ implies $k \neq k'$ and $l \neq l'$. Indeed, by the definition of $U_3^3$, for each $(k,l)$ in it with $k$ fixed there exist at most $N_2$ pairs $(k',l')$ in $U_3^3$ with $N_2 = 2^{(r+\delta_{0,j}+M\lambda)/4}$. Otherwise, it is in $U_1^1$ and therefore a contradiction. Similarly for each $(k,l)$ in $U_3^3$ with $l$ fixed we have at most $N_2$ pairs $(k',l)$ in $U_3^3$. Therefore we have at most $N_2^2 = C2^{(r+\delta_{0,j}+M\lambda)/2}$ sets $V_s$ satisfying the claimed property.

For each of these sets, since $|a_{\omega}| = C|b_{\omega}| 2^{-\lambda n}$, for the multiplier

$$m_j^{V_s} = \sum_{(k,l) \in V_s} a_{(k,l)} \omega_{1,k} \omega_{2,l}$$
we have the following estimate
\[
\|T_{m_j}^V(f,g)\|_{L^1} \leq \sum_{(k,l) \in V_s} |b_{(k,l)}| 2^{-\lambda n} \left\| \mathcal{F}^{-1}(\omega_1 \hat{f}) \mathcal{F}^{-1}(\omega_2 \hat{g}) \right\|_{L^1} \\
\leq C 2^{-r} \|b\|_{\ell^\infty} 2^{-\lambda n} \left[ \sum_{(k,l) \in V_s} \|\omega_1 \hat{f}\|_{L^2} \right]^{1/2} \left[ \sum_{(k,l) \in V_s} \|\omega_2 \hat{g}\|_{L^2} \right]^{1/2} \\
\leq C 2^{-r} \|b\|_{\ell^\infty} \|f\|_{L^2} \|g\|_{L^2}.
\]
Summing over \(s\) and using estimate (7) and the fact that \(N_2^2 = C 2^{r^2/2} \|b\|_{\ell^\infty}^{1/2}\) yields
\[
\|T_{m_j}^V(f,g)\|_{L^1} \leq N_2^2 2^{-r} \|b\|_{\ell^\infty} \|f\|_{L^2} \|g\|_{L^2} \\
\leq C 2^{(r+\delta_0 j+M\lambda)/2} 2^{-r} \|b\|_{\ell^\infty} \|f\|_{L^2} \|g\|_{L^2} \\
\leq C \|\Omega\|_{L^q} 2^{(-r-(2\delta-\delta_0) j-M\lambda)/2} \|f\|_{L^2} \|g\|_{L^2},
\]
which is also a good decay. Take \(\delta_0 = 8 \delta / 5\). Then we have
\[
\|T_{m_j}^V(f,g)\|_{L^1} \leq \left[ \|T_{m_j}^V(f,g)\|_{L^1} + \|T_{m_j}^V(f,g)\|_{L^1} \right] + \|T_{m_j}^V(f,g)\|_{L^1} \\
\leq C \|\Omega\|_{L^q} 2^{(n-M/8)\lambda} 2^{-r/8} 2^{-\delta_0 j/8} \|f\|_{L^2} \|g\|_{L^2} \\
+ C \|\Omega\|_{L^q} 2^{(r-(2\delta-\delta_0) j-M\lambda)/2} \|f\|_{L^2} \|g\|_{L^2} \\
\leq C \|\Omega\|_{L^q} 2^{-\delta j/2} 2^{(-M\lambda-r)/16} \|f\|_{L^2} \|g\|_{L^2}.
\]
Set \(f^j = \mathcal{F}^{-1}(\hat{f} \chi_{c_1 |\xi_1| \leq c_2 2^{j+1}})\) and \(g^j = \mathcal{F}^{-1}(\hat{g} \chi_{c_1 |\xi_1| \leq c_2 2^{j+1}})\) for some suitable constants \(c_1, c_2 > 0\). In view of the preceding estimate for the piece \(m_{j,0} = \sum_{\lambda} \sum_{\omega \in D_{\lambda,k}^1} a_\omega \omega \Omega\), we have
\[
\|T_{m_{j,0}}^V(f,g)\|_{L^1} = \|T_{m_{j,0}}^V(f^j,g^j)\|_{L^1} \\
\leq C \|\Omega\|_{L^q} \sum_{\lambda=1}^{C_0 M} \sum_{\lambda \geq 0} \sum_{r \geq 0} 2^{(2\delta j-M\lambda-r)/16} \|f^j\|_{L^2} \|g^j\|_{L^2} \\
\leq C \|\Omega\|_{L^q} 2^{-\delta j/5} \|f^j\|_{L^2} \|g^j\|_{L^2}.
\]
The first equality was obtained from the support properties of \(m_{j,0}\), which comes from the observation that \(m_{j,0}(\xi) \neq 0\) only if \(|\xi| \approx 2^j\), and that \(2^{-j} |\xi_1| \leq |\xi_2| \leq 2^j |\xi_1|\). Now recall that \(m_{j,k}(\xi) = m_{j,0}(2^k \xi)\), so
\[
T_{m_{j,k}}(f,g)(x) \\
= \int_{\mathbb{R}^n} m_{j,0}(2^k \xi) \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2
\]
Then there is a dimensional constant \( C \) such that the bilinear operator \( T \) with multiplier \( m \) satisfies:

\[
\| T_1 \|_{L^2 \times L^2 \to L^1} \leq C \| m \|_{L^\infty}^{\frac{1}{4}}.
\]

**Corollary 8.** Suppose that \( m(\xi, \eta) \) is a function in \( L^2(\mathbb{R}^{2n}) \) such that

\[
\sup_{|\alpha| \leq 4n+1} \| \partial^\alpha m \|_{L^\infty} \leq C_0 < \infty.
\]

Then there is a dimensional constant \( C \) such that the bilinear operator \( T_m \) satisfies:

\[
\| T_m \|_{L^2 \times L^2 \to L^1} \leq C \| m \|_{L^2}^{\frac{4}{5}}.
\]

**Remark 1.** Bényi and Torres [1] constructed a counterexample of a function \( m \) all of whose derivatives are bounded such that the corresponding bilinear
operator \(T_m\) is unbounded. So the assumption \(m \in L^2\) here in some sense is necessary.

5. The Off-Diagonal Parts

We now estimate the off-diagonal parts of the operator, namely \(T_{m^j}\) and \(T_{m^j}^1\). To control these two operators, we need the following inequality,

\[
\|T_{m^j}(f, g) + T_{m^j}(f, g)\|_{L^1} \leq C \|\Omega\|_{L^8} 2^{-j\delta} \|f\|_{L^2} \|g\|_{L^2},
\]

which will be discussed in Lemma 9.

Let us select a group \(D^2_{\lambda, \kappa}\) for some \(\kappa\). For \(\omega \in D^2_{\lambda, \kappa}\) we have the estimate \(\|b_\omega\|_{L^\infty} \leq C\|\Omega\|_{L^8} 2^{-j\delta} 2^{-M\lambda}\). We further divide the group \(D^2_{\lambda, \kappa}\) into columns \(D^2_{\lambda, \kappa}^a\) such that all wavelets in a given column have the form \(\omega = \omega_1 \omega_2^a\) with the same \(\omega_2^a\), where \(a = (\mu_{n+1}, \ldots, \mu_{2n}) \in \mathbb{Z}^n\). Notice that \(\omega \in D^2_{\lambda, \kappa}\) implies that \(|\xi_2| \leq 2\), and each \(\omega_2^a\) is supported in the cube

\[
Q = [2^{-\lambda}(\mu_{n+1} - c), 2^{-\lambda}(\mu_{n+1} + c)] \times \cdots \times [2^{-\lambda}(\mu_{2n} - c), 2^{-\lambda}(\mu_{2n} + c)]
\]

for some \(c \approx 1\). Therefore, we have at most \(C 2^{\lambda n}\) choices of \((\mu_{n+1}, \ldots, \mu_{2n})\), i.e. there exist at most \(C 2^{\lambda n}\) different \(\omega_2^a\) and \(C 2^{\lambda n}\) different columns.

For the multiplier \(m_{\lambda, \kappa}^2a\) related to the column of \(\omega_2^a\), we then get

\[
\int_{\mathbb{R}^n} \sum_{\omega_1} a_{\omega} \omega_1(\xi_1) \omega_2^a(\xi_2) \hat{f}(\xi_1) \hat{g}(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2
\]

\[
= \left[ \sum_{\omega_1} a_{\omega} \omega_1 f(x) \right] \left[ T \omega_2^a g(x) \right]
\]

with \(\omega_2^a(\xi_2) = 2^{\lambda n/2} \omega_2(2^{\lambda} \xi_2 - a)\). Notice that

\[
|T \omega_2^a g(x)| = \left| 2^{-\lambda n/2} \int_{\mathbb{R}^n} \mathcal{F}^{-1}(\omega_2)(2^{-\lambda} (x - y)) g(y) e^{2\pi i 2^{-\lambda} (x - y) a} dy \right|
\]

\[
\leq 2^{-\lambda n/2} \int_{\mathbb{R}^n} \frac{g(y)}{(1 + 2^{-\lambda} |x - y|)^M} dy
\]

\[
\leq 2^{\lambda n/2} \mathcal{M}(g)(x),
\]

where \(\mathcal{M}\) is the Hardy-Littlewood maximal function. We define

\[
m_{\alpha, \lambda}(\xi_1) = \sum_{\omega_1} a_{\omega} \omega_1(\xi_1) \chi_{2^{-j-1} |\xi_1| \leq 2^{j+1}} 2^{-j\delta} 2^{-(M+\frac{\lambda}{2})\xi_1},
\]

and then we have

\[
\sum_{\omega_1} a_{\omega} \omega_1 f(x) = 2^{-j\delta} 2^{-(M+\frac{\lambda}{2})\xi_1} \int_{\mathbb{R}^n} m_{\alpha, \lambda}(\xi_1) \hat{f}(\xi_1) e^{2\pi i \xi_1} d\xi_1,
\]
since the supports of $\omega_1$’s are disjoint and are all contained in the annulus \( \{ \xi_1 : 2^{j-1} \leq |\xi_1| \leq 2^j \} \). In view of (6) in Lemma 7 we have \( |a_\omega| \leq C_M \Omega ||L^2 Q_1 \sum_{\Omega} \sum_{j} \sum_{m_\omega, \lambda} 2^{-M \lambda} \Omega ||L^2 \sum_{m_\omega, \lambda} \Omega ||L^2 \sum_{m_\omega, \lambda} \Omega ||L^2 \sum_{m_\omega, \lambda} \Omega ||L^2 \). This combined with \( ||\Omega||L^\infty \leq C 2^{2n/2} \) implies that \( m_\omega, \lambda \leq C \Omega ||L^2 \sum_{\Omega} \sum_{j} \sum_{m_\omega, \lambda} 2^{-M \lambda} \Omega ||L^2 \sum_{m_\omega, \lambda} \Omega ||L^2 \sum_{m_\omega, \lambda} \Omega ||L^2 \sum_{m_\omega, \lambda} \Omega ||L^2 \). Therefore for the multiplier \( m = \sum_{\lambda} 2^{-M \lambda} \sum_{a} m_{a, \lambda} \) we have

\[
||T_m(f)||_{L^2} \leq C \Omega ||L^2 ||f||_{L^2} ||\hat{f}(\xi_{2j-1} \leq |\xi_1| \leq 2^{j+1} ||L^2 \),
\]

since for each fixed \( \lambda \) there exist at most \( C 2^{\lambda n} \) indices \( a \).

We now can control \( T_{m_{j,0}}^2 (f,g)(x) \) by \( C 2^{-\delta_2^2} M(g)(x) T_m(f)(x) \). Recall that \( m_{j,k}(\xi) = m_{j,0}^2 (2^k \xi) \), then if \( f_k \) is the function whose Fourier transform is \( \hat{f}(2^{-k} \xi) \), we have

\[
|T_{m_{j,k}}^2 (f,g)(x)| \leq C 2^{-\delta_2^2} 2^{-kn} M(g_k)(2^{-k} x) |T_m(f_k)(2^{-k} x)|.
\]

As a result

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |T_{m_{j,k}}^2 (f,g)|^2 \right)^{1/2} \right\|_{L^1} \leq \int_{\mathbb{R}^n} \left( \sum_{k \in \mathbb{Z}} |2^{-\delta_2^2} 2^{-kn} M(g)(x) T_m(f_k)(2^{-k} x)|^2 \right)^{1/2} dx
\]

\[
\leq C 2^{-\delta_2^2} \| M(g) \|_{L^2} \left( \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} |2^{-kn} T_m(f_k)(2^{-k} x)|^2 dx \right)^{1/2}
\]

\[
\leq C \| \Omega \|_{L^2} 2^{-\delta_2^2} \| g \|_{L^2} \left( \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \chi_{2^{j+k-1} \leq |\xi_1| \leq 2^{j+k+1}} |\hat{f}(\xi_1)|^2 d\xi_1 \right)^{1/2}
\]

\[
\leq C \| \Omega \|_{L^2} 2^{-\delta_2^2} \| f \|_{L^2} \| g \|_{L^2}.
\]

The estimate for \( T_{m_{j,k}}^3 \) is similar. Thus the proof of Proposition 5 will be finished once we establish (9). The preceding estimate implies that for \( f, g \) in \( L^2 \) we have

\[
\left\| \left( \sum_{k \in \mathbb{Z}} |T_{m_{j,k}}^2 (f,g)|^2 \right)^{1/2} \right\|_{L^1} < \infty
\]

a fact that will be useful in the sequel.

**Lemma 9.** There is a constant \( C \) such that (9) holds for all \( f, g \) in \( L^2(\mathbb{R}^n) \).

**Proof.** We first show that there exists a polynomial \( Q_1 \) of \( n \) variables such that \( T_{m_{j,k}}^2 (f,g) - Q_1 \in L^1 \).

Let \( \psi \in C_c(\mathbb{R}^n) \) such that \( \hat{\psi} \geq 0 \) with \( \text{supp} \hat{\psi} \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \) and \( \sum_{j=-\infty}^{\infty} \hat{\psi}(2^{-j} \xi) = 1 \) for \( \xi \neq 0 \). Set \( \Phi = \sum_{j=-\infty}^{\infty} \hat{\psi}(2^{-j} \xi) \) and define \( \Delta_k(f) = \mathcal{F}^{-1} (\Phi(2^{-k}) \hat{f}) \).
For \( r = 0, 1, 2, 3, 4 \), define \( m^{(r)}_j = \sum_{k \in \mathbb{Z}^d} m^{2}_{j,k} \). We will show that there exists a polynomial \( Q'_j \) such that

\[
\|T^{(r)}_{m^{(r)}_j}(f, g) - Q'_j\|_{L^1} \leq \left( \sum_{k \in \mathbb{Z}^d + r} |T^{2}_{m^{2}_{j,k}}(f, g)|^2 \right)^{1/2} \|L^1\|.
\]

We prove this assertion only in the case \( r = 0 \) as the remaining cases are similar. By Corollary 2.2.10 in [17] there is a polynomial \( Q^0_1 \) such that

\[
\|T^{(0)}_{m^{(0)}_j}(f, g) - Q^0_1\|_{L^1} \leq C \left( \sum_{k \in \mathbb{Z}^d} |\Delta_k(T^{(0)}_{m^{(0)}_j}(f, g))|^2 \right)^{1/2} \|L^1\|.
\]

Notice that

\[
\Delta_k(T^{(0)}_{m^{(0)}_j}(f, g))(x) = \sum_{l \in \mathbb{Z}^d} \int_{\mathbb{R}^{2n}} \widehat{\Phi}(2^{-k}(\xi_1 + \xi_2)) m^{2}_{j,0}(2^l \xi) \widehat{f}(\xi_1) \widehat{g}(\xi_2) e^{2\pi i \xi \cdot (\xi_1 + \xi_2)} d\xi_1 d\xi_2.
\]

Observe that \( m^{2}_{j,0}(2^l \xi) \) is supported in the set

\[
\{(\xi_1, \xi_2) : 2^{j-1} \leq |(\xi_1, \xi_2)| \leq 2^{j+1}, |\xi_1| \geq 2^j |\xi_2|\}
\]

which is a subset of

\[
\{(\xi_1, \xi_2) : 2^{j-2} \leq |\xi_1 + \xi_2| \leq 2^{j+2}\},
\]

so \( m^{2}_{j,0}(2^l \xi) \) is supported in \( \{(\xi_1, \xi_2) : 2^{j-l-2} \leq |\xi_1 + \xi_2| \leq 2^{j-l+2}\} \). The integrand in (12) is nonzero only if \( k = j - l \), when \( \Phi(2^{-k}\xi) m^{2}_{j,0}(2^l \xi) = m^{2}_{j,0}(2^l \xi) \), otherwise the product is 0. In summary we obtained

\[
\sum_{k \in \mathbb{Z}^d} |\Delta_k(T^{(0)}_{m^{(0)}_j}(f, g))|^2 = \sum_{k \in \mathbb{Z}^d} |T^{2}_{m^{2}_{j,k}}(f, g)|^2.
\]

Now (11) is a consequence of (10) and (13). Thus, there exist polynomials \( Q_1, Q_2 \) such that \( T^{(2)}_{m^{2}_{j}}(f, g) - Q_1, T^{(2)}_{m^{2}_{j}}(f, g) - Q_2 \in L^1 \). Given \( f, g \) in \( L^2(\mathbb{R}^n) \), we have already shown that \( T^{(2)}_{m^{2}_{j}}(f, g) \) lies in \( L^1 \). Moreover, we showed in Proposition 3 that \( T_j(f, g) \) lies in \( L^1 \). These facts imply that \( T^{(2)}_{m^{2}_{j}}(f, g) + T^{(2)}_{m^{2}_{j}}(f, g) \) lies in \( L^1 \), and therefore \( Q_1 + Q_2 = 0 \). Hence

\[
\|T^{(2)}_{m^{2}_{j}}(f, g) + T^{(2)}_{m^{2}_{j}}(f, g)\|_{L^1} \leq \|T^{(2)}_{m^{2}_{j}}(f, g) - Q_1\|_{L^1} + \|T^{(2)}_{m^{2}_{j}}(f, g) - Q_2\|_{L^1}
\]

\[
\leq C \|\Omega\|_{L^q} 2^{-j\delta} \|f\|_{L^2} \|g\|_{L^2}.
\]

This completes the proof of Proposition 5.
6. BOUNDEDNESS EVERYWHERE WHEN $q = \infty$

**Proposition 10.** Let $\Omega \in L^\infty(\mathbb{S}^{2n-1})$, $1 < p_1, p_2 < \infty$, $1/p = 1/p_1 + 1/p_2$. Then for any given $0 < \varepsilon < 1$ there is a constant $C_{n,\varepsilon}$ such that

$$
\|T_j\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C_{n,\varepsilon}\|\Omega\|_{L^\infty} 2^{j\varepsilon}
$$

for all $j \geq 0$.

To prove Proposition 10 we use Theorem 3 of [19] and Proposition 5. To apply the result in [19] we need to know that the kernel of $T_j$ is of bilinear Calderón-Zygmund type with bound $A \leq C_{n,\varepsilon}\|\Omega\|_{L^\infty} 2^{j\varepsilon}$ for any $\varepsilon \in (0, 1)$. This is proved in Lemma 11 below. Assuming this lemma, it follows that

$$
\|T_j\|_{L^{p_1} \times L^{p_2} \to L^p} \leq C(A + \|T_j\|_{L^2 \times L^2 \to L^1}) \leq C_{n,\varepsilon}\|\Omega\|_{L^\infty} 2^{j\varepsilon},
$$

which yields the claim in Proposition 10.

Recall that a bilinear Calderón-Zygmund kernel is a function $K$ defined away from the diagonal in $\mathbb{R}^{3n}$ which, for some $A > 0$, satisfies the size estimate

$$
|L(u, v, w)| \leq \frac{A}{(|u| + |v| |w|)}^{2n}
$$

and the smoothness estimate

$$
|L(u, v, w) - L(u', v, w)| \leq \frac{A|u - u'|^\varepsilon}{(|u| + |v| |w|)}^{2n+\varepsilon}
$$

when

$$
|u - u'| \leq \frac{1}{3}(|u| + |w|)
$$

(with analogous conditions in $v$ and $w$). Such a kernel is associated with the bilinear operator

$$(f, g) \mapsto T_L(f, g)(u) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v)g(w)L(u, v, w) \, dv \, dw.
$$

For the theory of such class of operators we refer to [19]. Thus we need to prove the following:

**Lemma 11.** Given $\Omega \in L^\infty(\mathbb{S}^{2n-1})$ and any $j \in \mathbb{Z}$, for any $0 < \varepsilon < 1$ there is a constant $C_{n,\varepsilon}$ such that

$$
L(u, v, w) = K_j(u - v, u - w) = \sum_{i \in \mathbb{Z}} K_j^i(u - v, u - w)
$$

is a bilinear Calderón-Zygmund kernel with constant $A \leq C_{n,\varepsilon}\|\Omega\|_{L^\infty} 2^{j|\varepsilon|}$.

**Proof.** We begin by showing that for given $x, y \in \mathbb{R}^{2n}$ with $|x| \geq 3|y|/2$ we have

$$
\sum_{i \in \mathbb{Z}} |K_j^i(x - y) - K_j^i(x)| \leq C_{n,\varepsilon}\|\Omega\|_{L^\infty} 2^{j|\varepsilon|}|y|^\varepsilon/|x|^{2n+\varepsilon}
$$

(14)
Assuming (14), we deduce the smoothness of $K_j(u - v, u - w)$ as follows:

(a) For $u, v, v', w \in \mathbb{R}^n$ satisfying $|v - v'| \leq \frac{1}{3}(|u - v| + |u - w|)$ we obtain

$$|K_j(u - v, u - w) - K_j(u - v', u - w)| \leq \sum_{i \in \mathbb{Z}} |K_j^i(u - v', u - w) - K_j^i(u - v, u - w)|$$

$$\leq C_n, [\Omega]|\Omega||L^m \frac{2^{\left\lfloor j \varepsilon \right\rfloor} |v - v'|^\varepsilon}{(|u - v| + |u - w|)^{2n+\varepsilon}}$$

$$\leq C_n, [\Omega]|\Omega||L^m \frac{2^{\left\lfloor j \varepsilon \right\rfloor} |v - v'|^\varepsilon}{(|u - v| + |u - w| + |v - w|)^{2n+\varepsilon}}$$

since $|u - v| + |u - w| + |v - w| \leq 2(|u - v| + |u - w|)$.

(b) For $u, u', v, w \in \mathbb{R}^n$ satisfying $|u - u'| \leq \frac{1}{3}(|u - v| + |u - w|)$ we take $x = (u - v, u - w)$ and $y = (u - u', u - u')$ in (14) to deduce the claimed smoothness.

(c) For $u, v, w, w' \in \mathbb{R}^n$ satisfying $|w - w'| \leq \frac{1}{3}(|u - v| + |u - w|)$ we take $x = (u - v, u - w)$ and $y = (0, w' - w)$.

We may therefore focus on (14). This will be a consequence of the following estimate

$$|K_j^i(x - y) - K_j^i(x)| \leq C_n, [\Omega]|\Omega||L^m \min\left(1, \frac{|y|}{2^{i-j}} \right) \frac{1}{2^{-j \varepsilon} 2^{2n+\varepsilon} |x|^{2n+\varepsilon}}$$

when $|x| \geq 3|y|/2$. Assuming (15) we prove (14) as follows: We pick an integer $N_3$ such that $(\log_2 |y|) + j \leq N_3 < (\log_2 |y|) + j + 1$.

If $j \geq 0$, then for $i$ such that $2^{i-j} \leq |y|$, i.e., $i \leq N_3$, we have

$$\sum_{i \leq N_3} |K_j^i(x - y) - K_j^i(x)| \leq C_n, [\Omega]|\Omega||L^m \sum_{i \leq N_3} \frac{1}{2^{-i \varepsilon} |x|^{2n+\varepsilon}}$$

$$= C_n, [\Omega]|\Omega||L^m \frac{2^{2(2n+\varepsilon)} |y|^{\varepsilon}}{|x|^{2n+\varepsilon}}$$

$$= C_n, [\Omega]|\Omega||L^m \frac{2^{2(2n+\varepsilon)} |y|^{\varepsilon}}{|x|^{2n+\varepsilon}}.$$
\[
\sum_{i \leq N_3} |K_j^i(x-y) - K_j^i(x)| \leq C_{n, \varepsilon} 2^{2(n + \varepsilon)} \|\Omega\|_{L^\infty} \sum_{i \leq N_3} 2^{-i\varepsilon} \frac{1}{2^{i\varepsilon} |x|^{2n+\varepsilon}}
\]

If \( j < 0 \), then for \( i \leq N_3 \),

\[
\sum_{i \leq N_3} |K_j^i(x-y) - K_j^i(x)| \leq C_{n, \varepsilon} 2^{2(n + \varepsilon)} \|\Omega\|_{L^\infty} \sum_{i \leq N_3} \frac{|y|}{2^{i-j} 2^{-i\varepsilon} |2^j|^{2n+\varepsilon}} \sum_{i \leq N_3} 2^{i\varepsilon}
\]

\[
\leq C_{n, \varepsilon} 2^{2(n + \varepsilon)} \|\Omega\|_{L^\infty} \frac{|y|}{|x|^{2n+\varepsilon}} 2^{j(1-\varepsilon)} \sum_{i > N_3} 2^{i\varepsilon}
\]

\[
= C_{n, \varepsilon} 2^{2(n + \varepsilon)} \|\Omega\|_{L^\infty} \frac{|y|\varepsilon}{|x|^{2n+\varepsilon}}.
\]

And for \( j < 0 \) we have

\[
\frac{|y|\varepsilon}{|x|^{2n+\varepsilon}} \leq 2^{i\varepsilon}\frac{|y|\varepsilon}{|x|^{2n+\varepsilon}}.
\]

This concludes the proof of (14) assuming (15). Finally we prove (15).

We have a decreasing estimate of \( K^i(x) \), i.e. for \( \varepsilon \in (0, 1) \) and \( i \in \mathbb{Z} \)

\[
|K^i(x)| \leq \|\Omega\|_{L^\infty} 2^{-i\varepsilon} \chi_{\frac{1}{2} |x| \leq 2} (x)
\]

\[
\leq \|\Omega\|_{L^\infty} 2^{2n+\varepsilon} \frac{2^{-i\varepsilon}}{\left(1 + 2^{-i|x|}\right)^{2n+\varepsilon}} \chi_{\frac{1}{2} |x| \leq 2} (x)
\]

\[
\leq 2^{2n+\varepsilon} \|\Omega\|_{L^\infty} \frac{2^{-i\varepsilon}}{\left(1 + 2^{-i|x|}\right)^{2n+\varepsilon}}.
\]

Then recall the lemma from Appendix B1 of [17], by defining \( \Psi(x) = \frac{1}{(1 + |x|)^{2n+\varepsilon}} \), we have that for \( t \in [0, 1] \)

\[
(|K^i| * \Psi_{t-j})(x-ty)
\]
which gives the first part of (15) by taking $t = 0$ and 1 since

$$|\mathcal{F}^{-1}(\beta_j)(x)| \leq C\beta 2^{2jn} (1 + 2^{j} |x|)^{-2n-1} = C\beta \Psi_j(x).$$

The other part follows from the previous estimate in the following way

$$|K_j^i(x - y) - K_j^i(x)|
\leq \left| \int_{\mathbb{R}^{2n}} K_i^j(z)(\mathcal{F}^{-1}(\beta_\iota^{-j}))(x - y - z) - \mathcal{F}^{-1}(\beta_\iota^{-j})(x - z)dz \right|
\leq \left| \int_{\mathbb{R}^{2n}} K_i^j(z) \int_0^1 2^{-(i-j)n} 2^{-(i-j)} (\nabla (\mathcal{F}^{-1}(\beta)))(\frac{x - ty - z}{2^{i-j}}) \cdot y dtdz \right|
\leq \frac{C_{n,\varepsilon} |y|}{2^{i-j}} \int_0^1 \int_{\mathbb{R}^{2n}} |K_i^j(z)| 2^{-(i-j)n} (1 + 2^{j-i} |x - ty - z|)^{2n+1} dzdt
\leq C_{n,\varepsilon} \frac{|y|}{2^{i-j}} \int_0^1 (|K_i^j| * \Psi_{t^{-j}})(x - ty) dt
\leq C_{n,\varepsilon} 2^{(2n+\varepsilon)} ||\Omega||_{L^\infty} \frac{|y|}{2^{i-j}} \frac{C_{n,\varepsilon}}{2^{-i\varepsilon} 2^{\min(j,0)\varepsilon} |x|^{2n+\varepsilon}}.$$

To prove the size condition, notice that by the decreasing estimate (16) we have

$$\sum_{i \in \mathbb{Z}} |K_j^i(v, w)| \leq \sum_{i} |\int K_i^j(v_1, w_1) \beta_{i-j}(v - v_1, w - w_1) dv_1 dw_1|
\leq \sum_{i} C_{n,\varepsilon} 2^{-2in} \frac{(1 + c_k 2^{-i} |(v, w)|)^{2n+\varepsilon}}{(1 + 2^{-i} |x - t y - z|)^{2n+\varepsilon}}
\leq C_{n,\varepsilon} \sum_{i > N^*} 2^{-2in} + C(c_j |(v, w)|)^{-(2n+\varepsilon)} \sum_{i \leq N^*} 2^{i\varepsilon}
\leq C_{n,\varepsilon} |(v, w)|^{-2n}$$

where $c_j = 2^{\min(0,j)}$ and $N^*$ is the number such that $2^{N^*} \approx c_j |(v, w)|$. Hence

$$|K_j(u - v, u - w)| \leq \frac{C_{n,\varepsilon}}{(|u - v| + |u - w|)^{2n}} \leq \frac{C_{n,\varepsilon}}{(|u - v| + |u - w| + |v - w|)^{2n}}.$$
We improve Proposition 10 by giving a necessary decay via interpolation. Once this is proved, Theorem 1 follows trivially.

**Lemma 12.** Let $\Omega \in L^\infty(S^{2n-1})$, $1 < p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$, then there exist constants $\epsilon_0 > 0$ and $C_{n, \epsilon_0}$ such that for all $j \geq 0$ we have

$$\|T_j\|_{L^p_1 \times L^p_2 \to L^p} \leq C_{n, \epsilon_0} \|\Omega\|_{L^\infty} 2^{-j\epsilon_0}.$$  

**Proof.** For any triple $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ with $1/p = 1/p_1 + 1/p_2$, we can choose two triples $\vec{P}_1 = (\frac{1}{p_{1,1}}, \frac{1}{p_{1,2}}, \frac{1}{q_1})$ and $\vec{P}_2 = (\frac{1}{p_{2,1}}, \frac{1}{p_{2,2}}, \frac{1}{q_2})$ such that $\vec{P}_1, \vec{P}_2$ and $(\frac{1}{2}, \frac{1}{2}, 1)$ are not collinear and the point $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ is in the convex hull of them. By Proposition 10 and Proposition 5, $T_j$ is bounded at $\vec{P}_1, \vec{P}_2$ with bound $C_{n, \epsilon} \|\Omega\|_{L^\infty} 2^{j\epsilon}$ for any $\epsilon \in (0, 1)$ and at $(\frac{1}{2}, \frac{1}{2}, 1)$ with bound $C_n \|\Omega\|_{L^\infty} 2^{-j\delta}$ for some fixed $\delta < 1/8$. Applying Theorem 7.2.2 in [17] we obtain that

$$\|T_j(f, g)\|_{L^p} \leq C_{n, \epsilon_0} \|\Omega\|_{L^\infty} 2^{-j\epsilon_0} \|f\|_{L^p_1} \|g\|_{L^p_2}$$

for some constant $\epsilon_0$ depending on $p_1, p_2, p$.

As an application of Theorem 1 we derive the boundedness of the Calderón commutator in the full range of exponents $1 < p_1, p_2 < \infty$, a fact proved in [5]. The Calderón commutator is defined in [2] as

$$C(a, f)(x) = p.v. \int_{\mathbb{R}} \frac{A(x) - A(y)}{(x-y)^2} f(y) dy,$$

where $a$ is the derivative of $A$. It was shown in [2] that $C(a, f)(x)$ can be written as

$$p.v. \int_{\mathbb{R}} \int_{\mathbb{R}} K(x-y, x-z) f(y) a(z) dy dz$$

with $K(y, z) = e^{e(z)-e(z-y)} = \frac{\Omega((y,z)|/(y,z))}{(|y-z|^2)}$, where $e(t) = 1$ if $t > 0$ and $e(t) = 0$ if $t < 0$. $K(y, z)$ is odd and homogeneous of degree $-2$ whose restriction on $S^1$ is $\Omega(y, z)$. It is easy to check that $\Omega$ is odd, bounded and thus it satisfies the hypothesis of Theorem 1. As a consequence we obtain the following.

**Corollary 13.** Given $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$ there is a constant $C$ such that

$$\|C(a, f)\|_{L^p} \leq C \|a\|_{L^p_1} \|f\|_{L^p_2}$$

is valid for all functions $f$ and $a$ on the line.
7. BOUNDEDNESS OF $T_\Omega$ WHEN $\Omega \in L^q(S^{2n-1})$ WITH $2 \leq q < \infty$

Let $R$ be the rhombus of all points $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ with $1 \leq p_1, p_2 \leq \infty$ and $1/p = 1/p_1 + 1/p_2$. We let $B$ be the set of all points $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ such that either $p_1$ or $p_2$ are equal to $1$ or $\infty$, i.e., $B$ is the boundary of $R$.

**Theorem 14.** Given any dimension $n \geq 1$, there is a constant $C_n$ and there exists a neighborhood $S$ of the point $(\frac{1}{2}, \frac{1}{2}, 1)$ in $R$, whose size is at least $C_n(q)^{-2}$, such that if $\Omega$ lies in $L^q(S^{2n-1})$ with $2 \leq q \leq \infty$, then

$$\|T_\Omega\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} < \infty$$

for $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p}) \in S$.

**Proof.** In Proposition 5 we showed that $\|T_j\|_{L^2 \times L^2 \rightarrow L^1} \leq C\|\Omega\|_{L^q}^{2-\delta}$ with $\delta \approx \frac{1}{q}$. Consider the point $(\frac{1}{2}, \frac{1}{2}, 1)$. Find two other points $(\frac{1}{p_{11}}, \frac{1}{p_{12}}, \frac{1}{q_1})$ and $(\frac{1}{p_{21}}, \frac{1}{p_{22}}, \frac{1}{q_2})$ in the interior of $R$ such that these three points are not collinear.

Then if $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ lies in the open convex hull of these three points, precisely, if $\frac{1}{p_i} = \frac{1}{p_{1i}} \eta_1 + \frac{1}{p_{2i}} \eta_2 + \frac{1}{q_2} \eta_3$ for $i = 1, 2$, and $\eta_1 + \eta_2 + \eta_3 = 1$, then multilinear interpolation (Theorem 7.2.2 in [17]) yields that

$$\|T_j\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq C\|\Omega\|_{L^q}^{2j(2n(\eta_1 + \eta_2) - \delta \eta_3)}.$$  

Moreover, if $2n(\eta_1 + \eta_2) - \delta \eta_3 < 0$, then $\sum_{j \geq 0} \|T_j\|_{L^{p_1} \times L^{p_2} \rightarrow L^p} \leq C\|\Omega\|_{L^q}$. 

If $(\frac{1}{p_{11}}, \frac{1}{p_{12}}, \frac{1}{q_1})$ and $(\frac{1}{p_{21}}, \frac{1}{p_{22}}, \frac{1}{q_2})$ are close and let $\eta_1 = \eta_2 = \eta$, we roughly have $4n\eta - \delta(1 - 2\eta) < 0$, from which we get $\eta < \frac{\delta}{4n + 2\delta}$. In particular, all points $\vec{p} = (\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ in the set

$$\left\{\vec{p} = (1-t)(\frac{1}{2}, \frac{1}{2}, 1) + t\vec{B} : 0 \leq t \leq \delta/16n, \vec{B} \in B\right\}$$

are contained in the claimed neighborhood, whose size is comparable to $(q')^{-2}$. \hfill \Box

**Remark 2.** Theorem 14 is sharp in the following sense. Let $\vec{A} = (\frac{1}{2}, \frac{1}{2}, 1)$ and $\vec{B}_0 = (1, 1, 2)$. By Theorem 14, the smallest $p$ such that $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ lies in $S$ satisfies

$$\frac{1}{p} = 2 \cdot \frac{2\delta}{16n} + \frac{2\delta}{16n} = 1 + \frac{\delta}{8n},$$

from which $\frac{1}{p} - 1 = \frac{\delta}{8n} \approx \frac{1}{q'}$. For the case $n = 1$, by the example in [13], we have the requirement $\frac{1}{p} + \frac{1}{q} \leq 2$, which implies that $\frac{1}{p} - 1 \leq \frac{1}{q'}$.

We end this paper by stating two related open problems:

(a) Given $\Omega \in L^q(S^{2n-1})$ with $2 \leq q < \infty$, find the full range of $p_1, p_2, p$ such that $T_\Omega$ maps $L^{p_1} \times L^{p_2} \rightarrow L^p$. 


(b) Is $T_\Omega$ bounded when $\Omega \in L^q(S^{2n-1})$ for $q < 2$?

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