SHARP ESTIMATES FOR FOURIER MULTIPLIERS: LINEAR AND MULTILINEAR THEORY

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ABSTRACT. We present some recent results in the theory of Fourier multipliers. These concern sharp versions of the classical multiplier theorems of Hörmander and of Marcinkiewicz and their bilinear analogues. We also discuss optimal, and in some cases necessary and sufficient, criteria for certain bilinear Fourier multiplier operators to be bounded from $L^2 \times L^2 \rightarrow L^1$.

1. The Hörmander multiplier theorem - an introduction

Fourier multipliers are formed by composing three operators: the Fourier transform, multiplication, and the inverse Fourier transform. The Fourier transform of a Schwartz function $f$ on $\mathbb{R}^n$ is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

and its inverse Fourier transform is defined as

$$f^\vee(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} dx.$$ 

Using these definitions, a general Fourier multiplier operator has the form

$$T_\sigma(f)(x) = (\hat{f} \sigma)\vee(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) \sigma(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

where $\sigma$ is a bounded function on $\mathbb{R}^n$; the function $\sigma$ has to be in $L^\infty$ if $T_\sigma$ is going to be bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$. An old and important problem in harmonic analysis is to find optimal sufficient conditions on $\sigma$ so that the operator $T_\sigma$, initially defined on Schwartz functions, admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself. If this is the case for a given $p$, then $\sigma$ is called an $L^p$ Fourier multiplier. An easy application of Plancherel’s identity $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$ yields that $\sigma$ is an $L^2$ Fourier multiplier if and only if $\sigma$ is a bounded function. Also, duality gives that $\sigma$ is an $L^p$ Fourier multiplier if and only if it is an $L^{p'}$ Fourier multiplier for any $p \in (1, \infty)$. Then

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it follows by interpolation that $\sigma$ is also an $L^q$ Fourier multiplier for any $q$ between $p$ and $p' = \frac{p}{p-1}$.

The first nontrivial multiplier result was provided by Mikhlin [39] who proved that if the condition

\begin{equation}
|\partial^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0,
\end{equation}

holds for all multi-indices $\alpha$ with size $|\alpha| \leq [n/2]+1$, then $T_\sigma$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. Here $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, where $\alpha_j$ are nonnegative integers denoting the number of differentiations in the $j$th coordinate. This theorem is well suited for functions that are homogeneous of degree zero and are sufficiently differentiable on the unit sphere. The key point is that the $\alpha$th derivative of a homogeneous of degree zero function is homogeneous of degree $-|\alpha|$ and the constant $C_\alpha$ in (1.1) is the $L^\infty$ norm of $\partial^\alpha \sigma$ on $S^{n-1}$ in this case.

**Example:** The functions

$$
\sigma_1(\xi_1, \xi_2, \xi_3) = \frac{\xi_1 \xi_2 \xi_3}{(\xi_1^2 + \xi_2^2 + \xi_3^2)^{3/2}}
$$

$$
\sigma_2(\xi_1, \xi_2, \xi_3) = \frac{\xi_2 \xi_3}{i\xi_1^2 + (\xi_2^2 + \xi_3^2)}
$$

$$
\sigma_3(\xi_1, \xi_2, \xi_3) = \frac{\xi_1 \xi_2}{1 + i(\xi_1^2 + \xi_2^2 + \xi_3^2)}
$$

defined on $\mathbb{R}^3$ satisfy condition (1.1). To verify this assertion we use that $\sigma_1$, $\sigma_2$ are homogeneous of degree zero and smooth on $S^2$; this implies that their $\alpha$-th derivatives are homogeneous of degree $-|\alpha|$. For $\sigma_3$ we introduce the homogeneous of degree zero function $F(t, \xi_1, \xi_2, \xi_3) = \xi_1 \xi_2 (t^2 + i(\xi_1^2 + \xi_2^2 + \xi_3^2))^{-1}$ on $\mathbb{R}^4$ and we note that its $\alpha$-th derivative in the variables $\xi_1, \xi_2, \xi_3$ is homogeneous of degree $-|\alpha|$.

An extension of Mikhlin’s result was obtained by Hörmander [37] who showed that the conclusion of Mikhlin’s theorem still holds if condition (1.1) is replaced by

\begin{equation}
\sup_{k \in \mathbb{Z}} 2^{-kn+2k|\alpha|} \int_{2^k < |\xi| < 2^{k+1}} |\partial^\alpha \sigma(\xi)|^2 d\xi < \infty.
\end{equation}

To compare conditions (1.1) and (1.2), for every multiindex $\alpha$ we introduce the function

$$
M_\alpha(\xi) = |\partial^\alpha \sigma(\xi)||\xi|^{|\alpha|}.
$$

Then condition (1.1) requires $M_\alpha$ to be bounded on $\mathbb{R}^n \setminus \{0\}$ for all $\alpha$ with $|\alpha| \leq [n/2]+1$, while (1.2) relaxes this assumption to the weaker requirement that averages of $M_\alpha$ with respect to the $L^2$ norms over dyadic annuli of the form $\{2^k < |\xi| < 2^{k+1}\}$ are uniformly
bounded. Additionally, it is useful to observe that Hörmander’s condition (1.2) can be rewritten in the form

\[
\sup_{k \in \mathbb{Z}} \| \partial^\alpha [\sigma(2^k \cdot)] \|_{L^2(A)} < \infty,
\]

where \( A = \{ \xi \in \mathbb{R}^n : 1 < |\xi| < 2 \} \) denotes the unit annulus in \( \mathbb{R}^n \).

In order to obtain a sharp variant of the Hörmander multiplier theorem, we need to introduce derivatives of fractional order. Let \( \Delta \) be the Laplacian on \( \mathbb{R}^n \). We denote by \( (I-\Delta)^{s/2} \) the operator given by multiplication by \((1+4\pi^2|\xi|^2)^{s/2}\) on the Fourier transform. Informally speaking, when \( s \) is an even natural number, \( (I-\Delta)^{s/2} \) corresponds to taking all derivatives of a function up to and including order \( s \). For other \( s > 0 \), \( (I-\Delta)^{s/2} \) corresponds to taking all derivatives of a function up to and including the (potentially) fractional number \( s \). Even when \( s \) is an odd integer, \( (I-\Delta)^{s/2} \) contains all derivatives of a function up to and including order \( s \), at least when measured in the \( L^p \) sense, i.e., for \( 1 < p < \infty \) and \( s \in \mathbb{Z}^+ \) we have

\[
\| (I-\Delta)^{s/2} f \|_{L^p} \approx \sum_{|\alpha| \leq s} \| \partial^\alpha f \|_{L^p}.
\]

The quantity \( \| (I-\Delta)^{s/2} f \|_{L^p} \) is denoted by \( \| f \|_{L^p_s} \) and is referred to as the \( L^p \) Sobolev norm of order \( s \) of a function \( f \); if \( s \in \mathbb{Z}^+ \), this is equivalent with the sum of the \( L^p \) norms of all partial derivatives of \( f \) up to and including order \( s \), as indicated in (1.4).

A variant of Hörmander’s result involving fractional derivatives can be formulated as follows: let \( s > 0 \) and let \( \Psi \) be a Schwartz function whose Fourier transform is supported in the annulus \( \{ \xi : 1/2 < |\xi| < 2 \} \) and which satisfies \( \sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j} \xi) = 1 \) for all \( \xi \neq 0 \). If for some \( 1 \leq r \leq 2 \) and \( s > n/r \), \( \sigma \) satisfies

\[
\sup_{k \in \mathbb{Z}} \| (I-\Delta)^{s/2} [\hat{\Psi} \sigma(2^k \cdot)] \|_{L^r(\mathbb{R}^n)} < \infty,
\]

then \( T_\sigma \) admits a bounded extension from \( L^p(\mathbb{R}^n) \) to itself for all \( 1 < p < \infty \). We would like to point out that in the special case when \( s \) is a positive integer and \( r = 2 \), the present version of the Hörmander multiplier theorem is equivalent to the original one; this can be verified by making use of (1.3) and of the equivalence (1.4).

One may wonder if condition (1.5) still implies that \( \sigma \) is an \( L^p \) Fourier multiplier for some \( p \in (1, \infty) \) if \( s \leq \frac{n}{2} \). This is indeed true and, roughly speaking, the closer \( p \) is to 2, the fewer derivatives are needed in condition (1.5). More precisely, Calderón and Torchinsky [3, Theorem 4.7] showed, via an interpolation argument, that \( T_\sigma \) is bounded from \( L^p(\mathbb{R}^n) \) to itself whenever condition (1.5) holds for all \( p \) satisfying \( \frac{1}{p} - \frac{1}{2} \leq \frac{s}{n} \) and \( \frac{1}{p} - \frac{1}{2} = \frac{1}{r} \). It was observed in [22] that the assumption \( \frac{1}{p} - \frac{1}{2} = \frac{1}{r} \) can be
replaced by a weaker one, namely, by $\frac{1}{p} - \frac{1}{2} < \frac{s}{n}$. We observe that the latter condition is also necessary as it is dictated by the embedding of $L^s_r(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ (we recall that all $L^p$ Fourier multipliers need to be bounded functions). Recently, Grafakos and Slavíková have eliminated the index $r$ from the statement of this theorem, by showing that $T_\sigma$ is $L^p$ bounded if $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ and the $L^{2,1}(\mathbb{R}^n)$ quasinorm of $(I - \Delta)^{s/2}[\hat{\psi}\sigma(2^k \cdot)]$ is bounded by a finite constant uniformly over all $k \in \mathbb{Z}$. Here $L^{2,1}$ is the Lorentz space, defined in Section 2.

In addition, it is known that if $T_\sigma$ is bounded from $L^p(\mathbb{R}^n)$ to itself for every $\sigma$ satisfying (1.5), then $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$. This can be shown using the following classical example of Hirschman [36, comments after Theorem 3c], Wainger [53, Part II], and Miyachi [40, Theorem 3].

**Example:** Let $a > 0$, $a \neq 1$, $b > 0$, and assume that $\phi$ is a smooth function which vanishes in a neighborhood of 0 and is equal to 1 for large $\xi$ in $\mathbb{R}^n$. Let $\sigma_{a,b}$ be the bounded function defined as

$\sigma_{a,b}(\xi) = \phi(\xi)|\xi|^{-b}e^{i\xi a}$.

Then $\sigma_{a,b}$ satisfies condition (1.5) with $s = b/a$ and $r > n/s$ and $T_{\sigma_{a,b}}$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $|\frac{1}{p} - \frac{1}{2}| \leq \frac{b/a}{n}$.

Additionally, Slavíková [46] recently constructed an example to show that $L^p$ boundedness does not hold on the line $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$. This means that conditions $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ and $rs > n$ are optimal for assumption (1.5). Prior to this, positive endpoint results on $L^p$ and on $H^1$ involving Besov spaces were given by Seeger [43], [44], [45].

2. A sharp version of the Hörmander multiplier theorem

In this section, we discuss the aforementioned improvement to the Hörmander multiplier theorem in which the Lebesgue space $L^r(\mathbb{R}^n)$, $r > \frac{n}{s}$, in condition (1.5) is replaced by the locally larger Lorentz space $L^{2,1}(\mathbb{R}^n)$. This space is defined in terms of the nonincreasing rearrangement of the function $f$, namely, the unique nonincreasing left-continuous function on $(0, \infty)$ equimeasurable with $f$, defined as follows:

$f^*(t) = \inf \{ r \geq 0 : |\{ y \in \mathbb{R}^n : |f(y)| > r \}| < t \}$

We recall the definitions of Lorentz spaces. For any measurable function $f$ on $\mathbb{R}^n$, we define

$$\|f\|_{L^{p,1}(\mathbb{R}^n)} = \int_0^\infty f^*(t)t^{\frac{1}{p} - 1} dt$$
and
\[ \|f\|_{L^p,\infty(\mathbb{R}^n)} = \sup_{t>0} t^{\frac{s}{p}} f^*(t), \]
where \( 1 < p < \infty \). It can be shown that
\[ \|f\|_{L^p,1(\mathbb{R}^n)} = p \int_0^\infty \lambda \left| \{ x \in \mathbb{R}^n : |f(x)| > \lambda \} \right|^{\frac{1}{p}} d\lambda \]
and
\[ \|f\|_{L^p,\infty(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left| \{ x \in \mathbb{R}^n : |f(x)| > \lambda \} \right|^{\frac{1}{p'}}. \]
The space \( L^{p',\infty}(\mathbb{R}^n) \), where \( p' = \frac{p}{p-1} \), is a sort of a measure-theoretic dual of the space \( L^{n,1}(\mathbb{R}^n) \), in view of the following version of Hölder’s inequality
\[ (2.1) \quad \int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leq \|f\|_{L^{p,1}(\mathbb{R}^n)} \|g\|_{L^{p',\infty}(\mathbb{R}^n)}. \]

We now discuss the proof of the following theorem.

**Theorem 2.1.** [31] Let \( \Psi \) be a Schwartz function on \( \mathbb{R}^n \) whose Fourier transform is supported in the annulus \( 1/2 < |\xi| < 2 \) and satisfies \( \sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j} \xi) = 1, \xi \neq 0 \). Let \( p \in (1,\infty), n \in \mathbb{N} \), and let \( s \in (0,n) \) satisfy
\[ (2.2) \quad \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{s}{n}. \]
Then for all functions \( f \) in the Schwartz class of \( \mathbb{R}^n \) we have the a priori estimate
\[ (2.3) \quad \|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi} \sigma(2^j \cdot)] \|_{L^{\frac{n}{p},1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \]

The Lorentz space \( L^{\frac{n}{s},1}(\mathbb{R}^n) \) appears naturally in this context, since at least if derivatives of integer order are considered, this space is locally the largest rearrangement-invariant function space \( X \) such that if all partial derivatives of \( f \) of order up to and including order \( s \) lie in \( X \), then \( f \) is bounded, see [49, 7].

The strategy to proving Theorem 2.1 is as follows: We show that inequality (2.3) holds for any \( p \in (1,\infty) \), provided that \( s \in (n/2,n) \), in Proposition 2.3 stated below. We then interpolate between this estimate with \( p \) near 1 and the trivial \( L^2 \) bound which essentially holds with zero derivatives.

In what follows, \( B(x,r) \) denotes the ball centered at point \( x \) and having the radius \( r \). If a ball of radius \( r \) is centered at the origin, we shall denote it simply by \( B_r \).

We consider the centered Hardy-Littlewood maximal operator \( M \) defined by
\[ M(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy \]
for a measurable function $f$. An important property of this operator is that for any $p \in (1, \infty]$ there is a constant $C_{p,n}$ such that

$$
\|M(f)\|_{L^p(\mathbb{R}^n)} \leq C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}.
$$

Moreover, when $p = 1$ there is an analogous inequality with $L^{1,\infty}$ in place of $L^1$ on the left in (2.4). There is also a vector-valued version of the preceding inequality due to Fefferman and Stein [16] saying:

$$
\left\| \left( \sum_k M(f_k)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n} \left\| \left( \sum_k |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)},
$$

where $1 < p, q < \infty$.

We now consider a related maximal operator defined for $q \geq 1$ and $q < \infty$. We define a maximal operator $M_{L^q}$ by

$$
M_{L^q}(f)(x) = \sup_{r>0} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|^q \, dy \right)^{\frac{1}{q}}.
$$

Observe that

$$
M_{L^q}(f) = (M(|f|^q))^{\frac{1}{q}},
$$

where $M$ stands for the classical Hardy-Littlewood maximal operator, which coincides with $M_{L^1}$.

A crucial step in proving Proposition 2.3 is the next lemma, which in some sense sharpens the following estimate (see [18, Theorem 2.1.10])

$$
\sup_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|f(x + 2^{-j}y)|}{(1 + |y|)^s} \, dy \leq C_{n,s} M(f)(x), \quad s > n,
$$

valid for measurable functions $f$; here $C_{n,s}$ is a constant depending only on the dimension and on $s > n$.

**Lemma 2.2.** Assume that $n \in \mathbb{N}$, $s \in (0, n)$ and $q > \frac{n}{s}$. Then there is a positive constant $C$ depending on $n$, $s$ and $q$ such that for any $j \in \mathbb{Z}$ and any measurable function $f$ on $\mathbb{R}^n$,

$$
\left\| \frac{f(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n, dy)} \leq CM_{L^q}(f)(x), \quad x \in \mathbb{R}^n.
$$

**Proof.** Setting $g(y) = f(x + 2^{-j}y)$, we obtain

$$
\left\| \frac{f(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n, dy)} = \left\| \frac{g(y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n, dy)}
$$

for a measurable function $f$. An important property of this operator is that for any $p \in (1, \infty]$ there is a constant $C_{p,n}$ such that

$$
\|M(f)\|_{L^p(\mathbb{R}^n)} \leq C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}.
$$

Moreover, when $p = 1$ there is an analogous inequality with $L^{1,\infty}$ in place of $L^1$ on the left in (2.4). There is also a vector-valued version of the preceding inequality due to Fefferman and Stein [16] saying:

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\left\| \left( \sum_k M(f_k)^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \leq C_{p,q,n} \left\| \left( \sum_k |f_k|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)},
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where $1 < p, q < \infty$.

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A crucial step in proving Proposition 2.3 is the next lemma, which in some sense sharpens the following estimate (see [18, Theorem 2.1.10])

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\sup_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|f(x + 2^{-j}y)|}{(1 + |y|)^s} \, dy \leq C_{n,s} M(f)(x), \quad s > n,
$$

valid for measurable functions $f$; here $C_{n,s}$ is a constant depending only on the dimension and on $s > n$.

**Lemma 2.2.** Assume that $n \in \mathbb{N}$, $s \in (0, n)$ and $q > \frac{n}{s}$. Then there is a positive constant $C$ depending on $n$, $s$ and $q$ such that for any $j \in \mathbb{Z}$ and any measurable function $f$ on $\mathbb{R}^n$,

$$
\left\| \frac{f(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n, dy)} \leq CM_{L^q}(f)(x), \quad x \in \mathbb{R}^n.
$$

**Proof.** Setting $g(y) = f(x + 2^{-j}y)$, we obtain

$$
\left\| \frac{f(x + 2^{-j}y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n, dy)} = \left\| \frac{g(y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s},\infty}(\mathbb{R}^n, dy)}
$$
and

\begin{align*}
M_{L^q}(f)(x) &= \sup_{r>0} \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)|^q \, dy \right)^{\frac{1}{q}} \\
&= \sup_{r>0} \left( \frac{1}{2^{jn}|B(x, r)|} \int_{B(0, 2^j r)} |f(x + 2^{-j} z)|^q \, dz \right)^{\frac{1}{q}} \\
&= \sup_{r'>0} \left( \frac{1}{|B(0, r')|} \int_{B(0, r')} |g(y)|^q \, dy \right)^{\frac{1}{q}} \\
&= M_{L^q}(g)(0).
\end{align*}

This says that we may assume, without loss of generality, that \( j = 0 \) and \( x = 0 \). Hence, it suffices to show that for any measurable function \( g \) on \( \mathbb{R}^n \),

\begin{equation}
\left\| \frac{g(y)}{(1 + |y|)^s} \right\|_{L^{\frac{q}{q-s, \infty}}(\mathbb{R}^n, dy)} \leq CM_{L^q}(g)(0).
\end{equation}

If \( M_{L^q}(g)(0) = \infty \), then inequality (2.10) holds trivially, so we can assume in what follows that \( M_{L^q}(g)(0) < \infty \). Since the case \( M_{L^q}(g)(0) = 0 \) is trivial as well (as \( g \) needs to vanish a.e. in this case), dividing the function \( g \) by the positive constant \( M_{L^q}(g)(0) \), we can in fact assume that \( M_{L^q}(g)(0) = 1 \).

Fix any \( a > 0 \) and \( k \in \mathbb{N}_0 \). Then

\[
\left| \left\{ y \in B_{2^{k+1}} \setminus B_{2^k} : |g(y)| > a \right\} \right| \leq \frac{1}{a^q} \int_{B_{2^{k+1}} \setminus B_{2^k}} |g(y)|^q \, dy \\
\leq \frac{|B_{2^{k+1}}|}{a^q} \cdot \frac{1}{|B_{2^{k+1}}|} \int_{B_{2^{k+1}}} |g(y)|^q \, dy \\
\leq \frac{\omega_n 2^{(k+1)n}}{a^q},
\]

where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \). Combining this with the trivial estimate

\[
\left| \left\{ y \in B_{2^{k+1}} \setminus B_{2^k} : |g(y)| > a \right\} \right| \leq \omega_n 2^{(k+1)n},
\]

we deduce that

\[
\left| \left\{ y \in \mathbb{R}^n : \frac{|g(y)|}{(1 + |y|)^s} > a \right\} \right| \\
= \left| \left\{ y \in B_1 : \frac{|g(y)|}{(1 + |y|)^s} > a \right\} \right| + \sum_{k=0}^{\infty} \left| \left\{ y \in B_{2^{k+1}} \setminus B_{2^k} : \frac{|g(y)|}{(1 + |y|)^s} > a \right\} \right| \\
\leq \left| \left\{ y \in B_1 : |g(y)| > a \right\} \right| + \sum_{k=0}^{\infty} \left| \left\{ y \in B_{2^{k+1}} \setminus B_{2^k} : |g(y)| > 2^k a \right\} \right|
\]
\[
\begin{align*}
\leq |\{y \in B_1 : |g(y)| > a\}| + \sum_{k=0}^{\infty} \omega_n 2^{(k+1)n} \min \left\{ \frac{1}{2^{ksq}a^q}, 1 \right\} \\
\leq |\{y \in B_1 : |g(y)| > a\}| + \sum_{k \in \mathbb{N}_0 \cap \left( \frac{1}{a^{1/s}} \right)} \omega_n 2^n \cdot 2^{kn} + \sum_{k \in \mathbb{N}_0 \cap \left( \frac{1}{a^{1/s}} \right)} \frac{\omega_n 2^n}{a^q} \cdot 2^{k(n-sq)} \\
\leq |\{y \in B_1 : |g(y)| > a\}| + \frac{C}{a^{\frac{n}{q}}}
\end{align*}
\]

Notice that in the last inequality we have used the fact that \(n - sq < 0\). Hence,

\[
\left\| \frac{g(y)}{(1 + |y|)^s} \right\|_{L^{\frac{n}{s}}(\mathbb{R}^n,dy)} = \sup_{a > 0} a \left\{ y \in \mathbb{R}^n : \frac{|g(y)|}{(1 + |y|)^s} > a \right\}^{\frac{n}{s}}
\]

\[
\leq \sup_{a > 0} a |\{y \in B_1 : |g(y)| > a\}|^{\frac{n}{s}} + C
\]

\[
= \|g\|_{L^{\frac{n}{s}}(B_1)} + C
\]

\[
\leq C'\|g\|_{L^q(B_1)} + C
\]

\[
\leq C'\omega_n^{\frac{1}{q}} M_{L^q}(g)(0) + C
\]

\[
\leq C'\omega_n^{\frac{1}{q}} M_{L^q}(g)(0) + C
\]

where \(C' > 0\) is the constant from the embedding \(L^q(B_1) \hookrightarrow L^{\frac{n}{s}}(B_1)\). As \(M_{L^s}(g)(0) = 1\), this proves (2.10), and thus (2.7).

Assume that \(\Psi\) is the function from the statement of Theorem 2.1. For any \(j \in \mathbb{Z}\) we define the Littlewood-Paley operator \(\Delta_j^\Psi\) as

\[
\Delta_j^\Psi(f)(x) = \int_{\mathbb{R}^n} f(x - y) 2^{jn} \Psi(2^j y) dy.
\]

The associated square function is given by

\[
f \mapsto \left( \sum_{j \in \mathbb{Z}} | \Delta_j^\Psi f |^2 \right)^{\frac{1}{2}},
\]

and the Littlewood-Paley theorem asserts that if \(1 < p < \infty\) then

(2.11) \[
\|f\|_{L^p(\mathbb{R}^n)} \approx \left\| \left( \sum_{j \in \mathbb{Z}} | \Delta_j^\Psi f |^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}
\]

for any \(f \in L^p(\mathbb{R}^n)\).

Let us also recall some properties of the Fourier transform that will be needed in the sequel. It is well known that the Fourier transform is an isometry on \(L^2\), and a bounded operator from \(L^1\) into \(L^\infty\). An interpolation argument then yields that

(2.12) \[
\|\hat{f}\|_{L^{p'}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}
\]
whenever \(1 \leq p \leq 2\). This result is called the Hausdorff-Young inequality, and can be extended to the setting of Lorentz spaces as well. In particular, one has
\[
\|\hat{f}\|_{L^{p',1}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,1}(\mathbb{R}^n)}
\]
for any \(1 < p < 2\).

**Proposition 2.3.** Let \(p \in (1, \infty), \ n \in \mathbb{N}, \ s \in (\frac{n}{2}, n)\). Let \(\Psi\) be as in Theorem 2.1. Then
\[
\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C\sup_{j \in \mathbb{Z}} \| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi}\sigma(2^j \cdot)] \|_{L^{\frac{p}{2},1}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.
\]

**Proof.** Let
\[
K = \sup_{j \in \mathbb{Z}} \| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi}\sigma(2^j \cdot)] \|_{L^{\frac{p}{2},1}(\mathbb{R}^n)} < \infty.
\]

Define a function \(\Theta\) in terms of \(\hat{\Theta}(\xi) = \hat{\Psi}(\xi/2) + \hat{\Psi}(\xi) + \hat{\Psi}(2\xi)\), and observe that \(\hat{\Theta}\) is equal to 1 on the support of the function \(\hat{\Psi}\).

Let us denote by \(\Delta^\Psi_j\) and \(\Delta^\Theta_j\) the Littlewood-Paley operators associated with \(\Psi\) and \(\Theta\), respectively. If \(f\) is a Schwartz function on \(\mathbb{R}^n\), then standard manipulations yield
\[
\Delta^\Psi_jT_\sigma(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{\Psi}(2^{-j} \xi)\sigma(\xi)e^{2\pi ix \cdot \xi} d\xi
\]
\[
= \int_{\mathbb{R}^n} (\Delta^\Psi_j f)^* (\xi)\hat{\Psi}(2^{-j} \xi)\sigma(\xi)e^{2\pi ix \cdot \xi} d\xi
\]
\[
= 2^jn \int_{\mathbb{R}^n} (\Delta^\Theta_j f)^* (2^j \xi')\hat{\Psi}(\xi')\sigma(2^j \xi')e^{2\pi ix \cdot 2^j \xi'} d\xi'
\]
\[
= \int_{\mathbb{R}^n} (\Delta^\Theta_j f)(x + 2^{-j} y)[\hat{\Psi}\sigma(2^j \cdot)]\hat{\Psi}(\xi')\sigma(2^j \xi')e^{2\pi ix \cdot 2^j \xi'} d\xi' dy
\]
\[
= \int_{\mathbb{R}^n} (\Delta^\Theta_j f)(x + 2^{-j} y)(1 + |y|^s)[\hat{\Psi}\sigma(2^j \cdot)]\hat{\Psi}(\xi')\sigma(2^j \xi')e^{2\pi ix \cdot 2^j \xi'} d\xi' dy.
\]

By the Hölder inequality in Lorentz spaces (2.1), we therefore obtain
\[
|\Delta^\Psi_jT_\sigma(f)(x)| \leq \left\| (\Delta^\Theta_j f)(x + 2^{-j} y) \right\|_{L^{\frac{p}{2},\infty}(\mathbb{R}^n, dy)} \left\| (1 + |y|^s)[\hat{\Psi}\sigma(2^j \cdot)]\hat{\Psi}(\xi')\sigma(2^j \xi')e^{2\pi ix \cdot 2^j \xi'} d\xi' dy \right\|_{L^{\frac{p}{2},1}(\mathbb{R}^n)}.
\]
Since \(\frac{n}{s} < 2\), we can find a real number \(q\) such that \(\frac{n}{s} < q < 2\). Lemma 2.2 now yields that
\[
\left\| (\Delta^\Theta_j f)(x + 2^{-j} y) \right\|_{L^{\frac{p}{2},\infty}(\mathbb{R}^n, dy)} \leq CM_{L^q}(\Delta^\Theta_j f)(x).
\]
Using inequality (2.13) with \(p = \frac{n}{s}\), we deduce that
\[
\left\| (1 + |y|^s)[\hat{\Psi}\sigma(2^j \cdot)]\hat{\Psi}(\xi')\sigma(2^j \xi')e^{2\pi ix \cdot 2^j \xi'} d\xi' dy \right\|_{L^{\frac{p}{2},1}(\mathbb{R}^n)} \leq C\left\| (1 + |y|^2)^{\frac{s}{2}}[\hat{\Psi}\sigma(2^j \cdot)]\hat{\Psi}(\xi')\sigma(2^j \xi')e^{2\pi ix \cdot 2^j \xi'} d\xi' dy \right\|_{L^{\frac{p}{2},1}(\mathbb{R}^n)}.
\]
\[ \leq C \left\| (I - \Delta)^{\frac{s}{2}} \left[ \hat{\Psi} \sigma(2^j \cdot) \right] \right\|_{L^{\frac{2 n}{n - 1}}(\mathbb{R}^n)} \]
\[ \leq CK. \]

Altogether, we obtain the estimate
\[ |\Delta_j T_\sigma(f)(x)| \leq CK_M L_{p(q)}(\Delta^\Theta_j f)(x). \]

Assume that \( p \geq 2 \). Then applying the Littlewood-Paley theorem (2.11) and the Fefferman-Stein inequality (2.5) (since \( \frac{p}{q} \geq \frac{2}{q} > 1 \)) we obtain the following sequence of inequalities:
\[
\left\| T_\sigma(f) \right\|_{L^p(\mathbb{R}^n)} \leq C \left( \sum_{j \in \mathbb{Z}} \left\| \Delta_j^\Theta T_\sigma(f) \right\|^2 \right)^{\frac{1}{2}} \left\| \sum_{j \in \mathbb{Z}} \left( M L^p_{p(q)}(\Delta^\Theta_j f)^{\frac{q}{2}} \right)^{\frac{q}{2}} \right\|_{L^q(\mathbb{R}^n)} \]
\[ \leq CK \left( \sum_{j \in \mathbb{Z}} \left\| \Delta_j^\Theta f \right\|^q \right)^{\frac{1}{q}} \left\| \sum_{j \in \mathbb{Z}} \left( M L^p_{p(q)}(\Delta^\Theta_j f)^{\frac{q}{2}} \right)^{\frac{q}{2}} \right\|_{L^q(\mathbb{R}^n)} \]
\[ \leq CK \left( \sum_{j \in \mathbb{Z}} \left\| \Delta_j^\Theta f \right\|^q \right)^{\frac{1}{q}} \left\| \sum_{j \in \mathbb{Z}} \left( M L^p_{p(q)}(\Delta^\Theta_j f)^{\frac{q}{2}} \right)^{\frac{q}{2}} \right\|_{L^q(\mathbb{R}^n)} \]
\[ \leq CK \| f \|_{L^p(\mathbb{R}^n)}. \]

If \( p \in (1, 2) \) then the result follows by duality. \( \square \)

To complete the proof, we need to properly interpolate between \( L^2 \) and \( L^{p_1} \), for \( p_1 \) near 1. This is achieved via the following result, whose proof is not included here but the reader is referred to [31] for a proof.

**Proposition 2.4.** Suppose that \( 1 < p_1 < \infty \) and \( 0 < s_1 < n \). If
\[
\| T_\sigma(f) \|_{L^{p_1}(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s_1}{2}} \left[ \hat{\Psi} \sigma(2^j \cdot) \right] \right\|^{\frac{1}{p_1}} \left\| f \right\|_{L^{p_1}(\mathbb{R}^n)},
\]
then
\[ \| T_\sigma(f) \|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{s}{2}} \left[ \hat{\Psi} \sigma(2^j \cdot) \right] \right\|^{\frac{1}{p}} \left\| f \right\|_{L^p(\mathbb{R}^n)} \]
for any \( 1 < p < \infty \) and \( 0 < s < s_1 \) satisfying
\[
\frac{1}{s} \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{s_1} \left| \frac{1}{p_1} - \frac{1}{2} \right|.
\]
Assuming Proposition 2.4, and using the estimate from Proposition 2.3 as the assumption (2.15), we complete the proof of Theorem 2.1 as follows:

**Proof of Theorem 2.1.** If \( s \in \left( \frac{n}{2}, n \right) \), then inequality (2.3) follows from Proposition 2.3. If \( s \leq \frac{n}{2} \), then we denote

\[
\alpha = \frac{1}{s} \left| \frac{1}{p} - \frac{1}{2} \right|
\]

Since \( \alpha \in (0, \frac{1}{n}) \), we can find \( p_1 \in (1, \infty) \) and \( s_1 \in \left( \frac{n}{2}, n \right) \) such that

\[
\alpha < \frac{1}{s_1} \left| \frac{1}{p_1} - \frac{1}{2} \right|
\]

A combination of Propositions 2.3 and 2.4 thus yields the desired assertion (2.3). □

### 3. An example

Unlike the Mikhlin multiplier theorem, the Hörmander multiplier theorem and its extension due to Calderón and Torchinsky (see Section 1 for more details) apply to multipliers whose derivatives have infinitely many singularities, such as the multiplier

\[
\sigma(x) = \sum_{k \in \mathbb{Z}} \phi(2^{-k}x) |2^{-k}x - a_k|^\beta,
\]

where \( \beta > 0 \), \( \phi \) is a smooth function supported in the set \( \{x \in \mathbb{R}^n : \frac{1}{2} < |x| < 2\} \) and, for every \( k \in \mathbb{Z} \), \( a_k \in \mathbb{R}^n \) belongs to the same set.

As an application of Theorem 2.1 we show that the function \( \sigma \) in (3.1) continues to be an \( L^p \) Fourier multiplier for any \( p \in (1, \infty) \) if \( |2^{-k}x - a_k| \) is replaced by \( \left( \log \frac{e^4 n}{|2^{-k}x - a_k|^n} \right)^{-1} \). This conclusion cannot be reached via the multiplier theorems of Hörmander or Calderón and Torchinsky as the \( s \)-order derivative of the function \( \left( \log \frac{e^4 n}{|x|^n} \right)^{-\beta} \), with \( \beta > 0 \), does not belong locally to any Lebesgue space \( L^r(\mathbb{R}^n) \) with \( r > n/s \) (but it does belong locally to the Lorentz space \( L^{\frac{r}{s},1}(\mathbb{R}^n) \), as we will see below).

**Example 3.1.** Assume that \( n \in \mathbb{N} \), \( n \geq 2 \), and \( \beta < 0 \). Let \( \phi \) be a smooth function supported in the set \( A = \{x \in \mathbb{R}^n : 1/2 < |x| < 2\} \) and let \( a_k \in A \), \( k \in \mathbb{Z} \). Then the function

\[
\sigma(x) = \sum_{k \in \mathbb{Z}} \phi(2^{-k}x) \left( \log \frac{e^4 n}{|2^{-k}x - a_k|^n} \right)^\beta
\]

is an \( L^p \) Fourier multiplier for any \( p \in (1, \infty) \).
To verify the statement of Example 3.1, we fix a positive integer $s$ and observe that for any $j \in \mathbb{Z}$,

$$
\| (I - \Delta)^{\frac{s}{2}} [\hat{\Psi} \sigma(2^j \cdot)] \|_{L^\frac{n}{s+1} (\mathbb{R}^n)} \leq \left\| (I - \Delta)^{\frac{s}{2}} \left[ \hat{\Psi}(x) \phi(x) \left( \log \frac{e^{4n}}{x - a_j^n} \right)^{\beta} \right] \right\|_{L^\frac{n}{s+1} (\mathbb{R}^n)}
$$

\[ + \left\| (I - \Delta)^{\frac{s}{2}} \left[ \hat{\Psi}(x) \phi(2x) \left( \log \frac{e^{4n}}{2x - a_{j-1}^n} \right)^{\beta} \right] \right\|_{L^\frac{n}{s+1} (\mathbb{R}^n)}
\]

\[ + \left\| (I - \Delta)^{\frac{s}{2}} \left[ \hat{\Psi}(x) \phi(x/2) \left( \log \frac{e^{4n}}{\frac{x}{2} - a_{j+1}^n} \right)^{\beta} \right] \right\|_{L^\frac{n}{s+1} (\mathbb{R}^n)}.
\]

In what follows, let us deal with the first term only, since the remaining two terms can be estimated in a similar way.

Fix $j \in \mathbb{Z}$ and denote

$$f_j(x) = \hat{\Psi}(x) \phi(x) \left( \log \frac{e^{4n}}{x - a_j^n} \right)^{\beta}.$$

For any multiindex $\alpha$ satisfying $|\alpha| \geq 1$ we have

$$|\partial^\alpha f_j(x)| \leq C \chi_A(x) \left( \log \frac{e^{4n}}{|x - a_j^n|} \right)^{\beta - 1} |x - a_j|^{-|\alpha|}.$$

Since $|A| \leq 2^n \omega_n$, where $\omega_n$ stands for the volume of the unit ball in $\mathbb{R}^n$, the previous estimate implies

$$(\partial^\alpha f_j)^* (t) \leq C \chi_{(0,2^n \omega_n)}(t) \left( \log \frac{e^{4n} \omega_n}{t} \right)^{\beta - 1} t^{-\frac{|\alpha|}{n}},$$

where the constant $C$ is independent of $j$. Therefore, if $s$ is a positive integer and $\alpha$ is a multiindex with $1 \leq |\alpha| \leq s$, then

$$(\partial^\alpha f_j)^* (t) \leq C \chi_{(0,2^n \omega_n)}(t) \left( \log \frac{e^{4n} \omega_n}{t} \right)^{\beta - 1} t^{-\frac{s}{n}}.$$

Consequently,

$$(3.3) \quad \sup_{1 \leq |\alpha| \leq s} \| \partial^\alpha f_j \|_{L^\frac{n}{s+1} (\mathbb{R}^n)} \leq C \int_0^{2^n \omega_n} \left( \log \frac{e^{4n} \omega_n}{t} \right)^{\beta - 1} t^{-1} dt < \infty.$$

Since each $|f_j|$ is bounded by a constant independent of $j$ and compactly supported in the set $A$, we also have

$$\|f_j\|_{L^\frac{n}{s} (\mathbb{R}^n)} \leq C < \infty.$$

It remains to observe that the quantity $\| (I - \Delta)^{\frac{s}{2}} f_j \|_{L^\frac{n}{s+1} (\mathbb{R}^n)}$ is equivalent to

$$\sum_{|\alpha| \leq s} \| \partial^\alpha f_j \|_{L^\frac{n}{s+1} (\mathbb{R}^n)}.$$
This can be proved in exactly the same way as the corresponding result for the Lebesgue spaces (1.4), see, e.g., [48, Theorem 3, Chapter 5]. Therefore, we deduce that
\[
\sup_{j \in \mathbb{Z}} \| (I - \Delta)^{s/2} \hat{\Psi} \sigma(2^j \cdot) \|_{L^{2^s,1}(\mathbb{R}^n)} < \infty
\]
for any positive integer \( s \). Theorem 2.1 now yields that \( \sigma \) is an \( L^p \) Fourier multiplier for any \( p \in (1, \infty) \).

4. The Marcinkiewicz multiplier theorem

We recall the classical version of the Marcinkiewicz multiplier theorem:

**Theorem 4.1.** Suppose that \( \sigma(\xi_1, \ldots, \xi_n) \) is a function on \( \mathbb{R}^n \) such that
\[
\left| \partial_{\xi_1}^{\beta_1} \cdots \partial_{\xi_n}^{\beta_n} \sigma(\xi_1, \ldots, \xi_n) \right| \leq A_{\beta_1, \ldots, \beta_n} |\xi_1|^{-\beta_1} \cdots |\xi_n|^{-\beta_n}
\]
for all \( \beta_j \in \{0, 1\}, j = 1, \ldots, n \). Then \( \sigma \) is an \( L^p \) Fourier multiplier for all \( 1 < p < \infty \) with bound
\[
\| T_\sigma \|_{L^p \to L^p} \leq C_{n,p} \sup_{\beta_j \in \{0,1\}} A_{\beta_1, \ldots, \beta_n}.
\]

**Example:** The following functions satisfy conditions (4.1) for all \( \beta_j \in \mathbb{Z}^+ \cup \{0\} \):
\[
m_1(\xi) = \frac{\xi_1}{\xi_1 + i(\xi_2^2 + \cdots + \xi_n^2)},
m_2(\xi) = \frac{|\xi_1|^{\alpha_1} \cdots |\xi_n|^{\alpha_n}}{(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2)^{\alpha/2}},
\]
where \( \alpha_1 + \alpha_2 + \cdots + \alpha_n = \alpha, \alpha_j > 0 \),
\[
m_3(\xi) = \frac{\xi_2 \xi_3^2}{i\xi_1 + \xi_2^2 + \xi_3^2}.
\]
The functions \( m_1 \) and \( m_2 \) are defined on \( \mathbb{R}^n \setminus \{0\} \) and \( m_3 \) on \( \mathbb{R}^3 \setminus \{0\} \).

These examples and many other examples that satisfy conditions (4.1) are invariant under a set of dilations in the following sense: suppose that there exist \( k_1, \ldots, k_n \in \mathbb{R}^+ \) and \( s \in \mathbb{R} \) such that the smooth function \( m \) on \( \mathbb{R}^n \setminus \{0\} \) satisfies
\[
m(\lambda^{k_1} \xi_1, \ldots, \lambda^{k_n} \xi_n) = \lambda^{is} m(\xi_1, \ldots, \xi_n)
\]
for all \( \xi_1, \ldots, \xi_n \in \mathbb{R} \) and \( \lambda > 0 \). Then \( m \) satisfies condition (4.1). Indeed, differentiation gives
\[
\lambda^{\alpha_1 k_1 + \cdots + \alpha_n k_n} \partial^{\alpha} m(\lambda^{k_1} \xi_1, \ldots, \lambda^{k_n} \xi_n) = \lambda^{is} \partial^{\alpha} m(\xi_1, \ldots, \xi_n)
\]
Theorem 4.2. Let $L$ be a function on the line whose Fourier transform is supported in $|\xi| < \pi$. Then $\lambda_{\xi}^{\alpha, \alpha_j} \leq |\xi_j|^{-\alpha_j}$, and it follows that

$$|\partial^\alpha m(\xi_1, \ldots, \xi_n)| \leq \left( \sup_{\xi \in \mathbb{R}^n} |\partial^\alpha m\right) \lambda_{\xi}^{\alpha_1 k_1 + \cdots + \alpha_n k_n} \leq C_\alpha |\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n}.

**Example:** Let $\kappa \in \mathbb{R}$. Consider the function

$$m(\xi, \eta) = \left( \frac{1 + |\xi + \eta|^2}{(1 + |\xi|^2)(1 + |\eta|^2)} \right)^\kappa$$
defined on $\mathbb{R}^{2n}$. Define the function

$$M(\xi, t, \eta, s) = \left( \frac{t^2 s^2 + |s \xi + t \eta|^2}{(t^2 + |\xi|^2)(s^2 + |\eta|^2)} \right)^\kappa$$
on $\mathbb{R}^{2n+2}$. Notice that

$$M(\lambda \xi, \lambda t, \mu \eta, \mu s) = M(\xi, t, \eta, s)$$
for any $\lambda, \mu > 0$. This says that for fixed $(\eta, s)$, the function $(\xi, t) \mapsto M(\xi, t, \eta, s)$ is homogeneous of degree zero, hence $\partial^\alpha m(\xi, t, \eta, s)$ is homogeneous of degree $-|\alpha|$ in $(\xi, t)$. By the same argument the function $(\eta, s) \mapsto \partial^\beta \partial^\alpha M(\xi, t, \eta, s)$ is homogeneous of degree $-|\beta|$ in $(\eta, s)$. From these observations we conclude that

$$|\partial^\alpha \partial^\beta m(\xi, \eta)| = |\partial^\alpha \partial^\beta M(\xi, 1, \eta, 1)| \leq \frac{C_{\alpha, \beta, s}}{(1 + |\xi|)^{|\alpha|} (1 + |\eta|)^{|\beta|}}.$$ 

Here $\alpha$ and $\beta$ are multiindices of $n$ entries. It follows that

$$|\partial^\alpha \partial^\beta m(\xi, \eta)| \leq C_{\alpha, \beta, s} |\xi_1|^{-\alpha_1} \cdots |\xi_n|^{-\alpha_n} |\eta_1|^{-\beta_1} \cdots |\eta_n|^{-\beta_n},$$
that is, condition (4.1) holds for all $\beta_j \in \mathbb{Z}^+ \cup \{0\}$.

Let us now study a product-type Sobolev space version of the Marcinkiewicz multiplier theorem. We define $(I - \partial^2_\xi)^{-\gamma} f$ as the linear operator $((1 + 4\pi^2 |\eta|^2)^{\gamma} \hat{f}(\eta))^\gamma$ associated with the multiplier $(1 + 4\pi^2 |\eta|^2)^{-\gamma}$. We present here a proof of the Marcinkiewicz multiplier theorem in which only $|1/p - 1/2| + \varepsilon$ derivatives per variable are required to guarantee $L^p$ boundedness of $T_\alpha$, instead of a full derivative as in (4.1).

**Theorem 4.2.** Let $n \in \mathbb{N}$, $n \geq 2$. Suppose that $1 \leq r < \infty$ and $\psi$ is a Schwartz function on the line whose Fourier transform is supported in $[-2, -1/2] \cup [1/2, 2]$ and which satisfies $\sum_{j \in \mathbb{Z}} \hat{\psi}(2^{-j} \xi) = 1$ for all $\xi \neq 0$. Let $\gamma_\ell > 1/r$, $\ell = 1, \ldots, n$. If a function $\sigma$ on $\mathbb{R}^n$ satisfies

$$\sup_{j_1, \ldots, j_n \in \mathbb{Z}} \left\| (I - \partial^2_1)^{\gamma_1} \cdots (I - \partial^2_n)^{\gamma_n} \left( \hat{\psi}(\xi_1) \cdots \hat{\psi}(\xi_n) \sigma(2^{j_1} \xi_1, \ldots, 2^{j_n} \xi_n) \right) \right\|_{L^r} < \infty,$$
then $T_\sigma$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$ with
\begin{equation}
\left| \frac{1}{p} - \frac{1}{2} \right| < \min(\gamma_1, \ldots, \gamma_n).
\end{equation}

Moreover, (4.3) is optimal in the sense that if $T_\sigma$ is $L^p$-bounded for every $\sigma$ satisfying (4.2), then the strict inequality in (4.3) must necessarily hold.

Carbery [4] first formulated a version of Theorem 4.2 in which the multiplier lies in a product-type $L^2$-based Sobolev space. Carbery and Seeger [5, Remark after Prop. 6.1] obtained Theorem 4.2 in the case when $\gamma_1 = \cdots = \gamma_n > \left| \frac{1}{p} - \frac{1}{2} \right| = \frac{1}{r}$. The positive direction of their result also appeared in [6, Condition (1.4)] but this time the range of $p$ is $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{r}$. The present variant of Theorem 4.2 appeared in [32].

Let us now focus on the proof of Theorem 4.2. We use $\psi$ to denote the bump from Theorem 4.2; further, $\theta$ will stand for the function on the line satisfying $\hat{\theta}(\eta) = \hat{\psi}(\eta/2) + \hat{\psi}(\eta) + \hat{\psi}(2\eta)$.

One can observe that $\hat{\theta}$ is supported in $\{ \frac{1}{4} \leq |\xi| \leq 4 \}$ and $\hat{\theta} = 1$ on the support of $\hat{\psi}$.

To simplify the notation, if $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$, we shall write
\begin{equation*}
2^j \xi = (2^{j_1} \xi_1, \ldots, 2^{j_n} \xi_n)
\end{equation*}
and
\begin{equation*}
\hat{\psi}(\xi) = \prod_{\ell=1}^n \hat{\psi}(\xi_\ell), \quad \hat{\theta}(\xi) = \prod_{\ell=1}^n \hat{\theta}(\xi_\ell).
\end{equation*}

Let $k \in \{1, \ldots, n\}$. For $j \in \mathbb{Z}$ we define the Littlewood-Paley operators associated to the bumps $\psi$ and $\theta$ by
\begin{equation*}
\Delta^{\psi,k}_j(f)(x) = \int_{\mathbb{R}} f(x_1, \ldots, x_{k-1}, x_k - y, x_{k+1}, \ldots, x_n) 2^j \psi(2^j y) dy
\end{equation*}
and
\begin{equation*}
\Delta^{\theta,k}_j(f)(x) = \int_{\mathbb{R}} f(x_1, \ldots, x_{k-1}, x_k - y, x_{k+1}, \ldots, x_n) 2^j \theta(2^j y) dy.
\end{equation*}

We begin with the following lemma:

**Lemma 4.3.** Let $1 \leq r < \infty$, let $1 \leq \rho < 2$ satisfy $1 \leq \rho \leq r$ and let $\gamma_1, \ldots, \gamma_n$ be real numbers such that $\gamma_\ell \rho > 1$, $\ell = 1, \ldots, n$. Then, for any function $f$ on $\mathbb{R}^n$ and for all integers $j_1, \ldots, j_n$, we have
\begin{equation}
|\Delta^{\psi,1}_{j_1} \cdots \Delta^{\psi,n}_{j_n} T_\sigma(f)| \leq C K \left[ M^{(1)} \cdots M^{(n)}( |\Delta^{\theta,1}_{j_1} \cdots \Delta^{\theta,n}_{j_n} f|^\rho \right]^{1/\rho},
\end{equation}
where \( M^{(\ell)} \) denotes the one-dimensional Hardy-Littlewood maximal operator in the \( \ell \)-th coordinate and
\[
K = \sup_{j_1,\ldots,j_n \in \mathbb{Z}} \left\| (I - \partial_{j_1}^2)^{\frac{1}{2}} \cdots (I - \partial_{j_n}^2)^{\frac{1}{2}} \left[ \sigma(2^{j_1} \xi_1, \ldots, 2^{j_n} \xi_n) \hat{\psi}(\xi_1) \cdots \hat{\psi}(\xi_n) \right] \right\|_{L^r}.
\]

**Proof.** Throughout the proof we shall use the notation introduced above and, whenever \( J = (j_1, \ldots, j_n) \), we write
\[
\Delta^\psi f = \Delta^\psi_{j_1} \cdots \Delta^\psi_{j_n} f, \quad \Delta^\theta f = \Delta^\theta_{j_1} \cdots \Delta^\theta_{j_n} f.
\]

Since \( \hat{\theta} \) is equal to 1 on the support of \( \hat{\psi} \), we have
\[
\Delta^\psi T_\sigma(f)(x_1, \ldots, x_n)
\]
\[
= \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{\psi}(2^{-j} \xi) \sigma(\xi) e^{2\pi i x \cdot \xi} d\xi
\]
\[
= \int_{\mathbb{R}^n} \hat{f}(\xi) \hat{\theta}(2^{-j} \xi) \hat{\psi}(2^{-j} \xi) \sigma(\xi) e^{2\pi i x \cdot \xi} d\xi
\]
\[
= \int_{\mathbb{R}^n} (\Delta^\theta f) \hat{\psi}(2^{-j} \xi) \sigma(\xi) e^{2\pi i x \cdot \xi} d\xi
\]
\[
= \int_{\mathbb{R}^n} 2^{j_1 + \cdots + j_n} (\Delta^\theta f) \hat{\psi}(2^{-j} \xi) \sigma(2^j \xi) e^{2\pi i (2^j x \cdot \xi')} d\xi'
\]
\[
= 2^{j_1 + \cdots + j_n} \int_{\mathbb{R}^n} (\Delta^\theta f)(y) \left[ \hat{\psi}(\xi') \sigma(2^j \xi') \right] \hat{\psi}(x-y) dy
\]
\[
= \int_{\mathbb{R}^n} \frac{2^{j_1 + \cdots + j_n} (\Delta^\theta f)(y)}{\prod_{\ell=1}^n (1 + 2^{j_\ell} |x_\ell - y_\ell|)^{\gamma_\ell}} \cdot \prod_{\ell=1}^n (1 + 2^{j_\ell} |x_\ell - y_\ell|)^{\gamma_\ell} \left[ \hat{\psi}(\xi) \sigma(2^j \xi) \right] (2^j (x-y)) dy.
\]

Hölder’s inequality thus yields that \( |\Delta^\theta T_\sigma(f)(x)| \) is bounded by
\[
\left( \int_{\mathbb{R}^n} 2^{j_1 + \cdots + j_n} \frac{|(\Delta^\theta f)(y)|^\rho}{\prod_{\ell=1}^n (1 + 2^{j_\ell} |x_\ell - y_\ell|)^{\gamma_\ell \rho}} dy \right)^{\frac{1}{\rho}} \cdot \left( \int_{\mathbb{R}^n} 2^{j_1 + \cdots + j_n} \left| \prod_{\ell=1}^n (1 + 2^{j_\ell} |x_\ell - y_\ell|)^{\gamma_\ell} \left[ \hat{\psi}(\xi) \sigma(2^j \xi) \right] (2^j (x-y)) \right|^{\rho'} dy \right)^{\frac{1}{\rho'}}
\]
where, when \( \rho = 1 \), the second term in the product is to be interpreted as
\[
\left\| \prod_{\ell=1}^n (1 + 2^{j_\ell} |x_\ell - y_\ell|)^{\gamma_\ell} \left[ \hat{\psi}(\xi) \sigma(2^j \xi) \right] (2^j (x-y)) \right\|_{L^\infty}.
\]

Since \( \gamma_\ell \rho > 1 \) for all \( \ell = 1, \ldots, n \), \( n \) consecutive applications of (2.6) yield the estimate
\[
\left( \int_{\mathbb{R}^n} 2^{j_1 + \cdots + j_n} \frac{|(\Delta^\theta f)(y)|^\rho}{\prod_{\ell=1}^n (1 + 2^{j_\ell} |x_\ell - y_\ell|)^{\gamma_\ell \rho}} dy \right)^{\frac{1}{\rho}} \leq C \left[ M^{(1)} \cdots M^{(n)} \left( |\Delta^\theta f|^\rho \right)(x) \right]^{\frac{1}{\rho}}.
\]
We now write
\[
\left( \int_{\mathbb{R}^n} \left| \prod_{\ell=1}^{n} (1 + 2^{j\ell} |x_\ell - y_\ell|)^{\gamma_\ell} \left( \hat{\psi}(\xi) \sigma(2^j \xi) \right) \left( 2^J(x - y) \right) \right|^{\rho'} \, dy \right)^{\frac{1}{\rho'}}
\]
\[
\leq \left( \int_{\mathbb{R}^n} \left| \prod_{\ell=1}^{n} (1 + |y_\ell|^2)^{\frac{\gamma_\ell}{2}} \left( \hat{\psi}(\xi) \sigma(2^j \xi) \right) \left( 2^J(x - y) \right) \right|^{\rho'} \, dy \right)^{\frac{1}{\rho'}}
\]
\[
\leq \left\| (I - \partial_1^{2j})^{\frac{\gamma_1}{2}} \cdots (I - \partial_n^{2j})^{\frac{\gamma_n}{2}} \left[ \sigma(2^j \xi) \hat{\psi}(\xi) \right] \right\|_{L^p}
\]
\[
\leq C \left\| (I - \partial_1^{2j})^{\frac{\gamma_1}{2}} \cdots (I - \partial_n^{2j})^{\frac{\gamma_n}{2}} \left[ \sigma(2^j \xi) \hat{\psi}(\xi) \right] \right\|_{L^r}
\]
\[
\leq CK.
\]
Notice that the second inequality is the Hausdorff-Young inequality (2.12) while (4.5) is a consequence of the Kato-Ponce inequality [29] (if \(\rho < r\)). A combination of the preceding estimates yields (4.4).

**Proposition 4.4.** Let \(1 \leq r < \infty\) and let \(\gamma_\ell > \max\{1/2, 1/r\}, \ell = 1, \ldots, n\). If a function \(\sigma\) on \(\mathbb{R}^n\) satisfies (4.2), then \(T_\sigma\) admits a bounded extension from \(L^p(\mathbb{R}^n)\) to itself for all \(1 < p < \infty\).

**Proof.** Suppose first that \(p > 2\). Since \(\gamma_\ell > \max\{1/2, 1/r\}, \ell = 1, \ldots, n\), we can find \(\rho \in [1, 2)\) such that \(\rho \leq r\) and \(\rho \gamma_\ell > 1, \ell = 1, \ldots, n\). Then
\[
\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C_p(n) \left\| \left( \sum_{j_1, \ldots, j_n \in \mathbb{Z}} |\Delta_1^{\psi,j_1} \cdots \Delta_n^{\psi,j_n} T_\sigma(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^p}
\]
\[
\leq C_p(n)K \left\| \left( \sum_{j_1, \ldots, j_n \in \mathbb{Z}} \left[ M^{(1)} \cdots M^{(n)}(|\Delta_1^{\sigma,j_1} \cdots \Delta_n^{\sigma,j_n} f|^{\rho}) \right]^{\frac{1}{2}} \right)^{\frac{1}{2}} \right\|_{L^p}
\]
\[
\leq C_p(n)K \left\| \left( \sum_{j_1, \ldots, j_n \in \mathbb{Z}} |\Delta_1^{\sigma,j_1} \cdots \Delta_n^{\sigma,j_n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p}
\]
\[
\leq C_p(n)K\|f\|_{L^p}.
\]
Notice that the second inequality follows from Lemma 4.3 and the third inequality is obtained by applying the Fefferman-Stein inequality (2.5) on the Lebesgue space \(L^{\frac{r}{2}}\) in each of the variables \(y_1, \ldots, y_n\). Observe that the Fefferman-Stein inequality makes use of the assumptions \(2/\rho > 1\) and \(p/2 > 1\). The first and last inequality follow from Proposition 4.5 below.

The case \(1 < p < 2\) follows by a duality argument, while the case \(p = 2\) is a consequence of Plancherel’s theorem and of a Sobolev embedding into \(L^\infty\).
Proposition 4.5. If $\psi$ satisfies $\sum_{j \in \mathbb{Z}} \hat{\psi}(2^{-j} \xi) = 1$ for all $\xi \neq 0$, then we have

$$
(4.6) \quad \left\| \left( \sum_{j_1, \ldots, j_n \in \mathbb{Z}} |\Delta_{j_1}^{\theta_1} \cdots \Delta_{j_n}^{\theta_n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \approx \|f\|_{L^p(\mathbb{R}^n)},
$$

where $\hat{\theta}(\xi) = \hat{\psi}(2\xi) + \hat{\psi}(\xi) + \hat{\psi}(2^{-1}\xi)$.

Before we discuss the proofs we recall the Rademacher functions, which we plan to use. The Rademacher functions are defined on $[0, 1]$ as follows: $r_0(t) = 1$; $r_1(t) = 1$ for $0 \leq t \leq 1/2$ and $r_1(t) = -1$ for $1/2 < t \leq 1$; $r_2(t) = 1$ for $0 \leq t \leq 1/4$, $r_2(t) = -1$ for $1/4 < t \leq 1/2$, $r_2(t) = 1$ for $1/2 < t \leq 3/4$, and $r_2(t) = -1$ for $3/4 < t \leq 1$; and so on. According to this definition, we have that $r_j(t) = \text{sgn}(\sin(2^j \pi t))$ for $j = 0, 1, 2, \ldots$. They are mutually independent random variables on $[0, 1]$ that satisfy Khintchine’s inequalities:

For any $0 < p < \infty$ and for any complex-valued square summable sequences $\{z_j\}$ we have

$$
(4.7) \quad B_p \left( \sum_j |z_j|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_j z_j r_j \right\|_{L^p([0, 1])} \leq A_p \left( \sum_j |z_j|^2 \right)^{\frac{1}{2}}
$$

for some constants $0 < A_p, B_p < \infty$ that depend only on $p$.

These inequalities also extend to several variables. Set

$$
F_n(t_1, \ldots, t_n) = \sum_{j_1} \cdots \sum_{j_n} c_{j_1, \ldots, j_n} r_{j_1}(t_1) \cdots r_{j_n}(t_n),
$$

for $t_j \in [0, 1]$, where $c_{j_1, \ldots, j_n}$ is a sequence of complex numbers.

For any $0 < p < \infty$ and for any complex-valued square summable sequence of $n$ variables $\{c_{j_1, \ldots, j_n}\}_{j_1, \ldots, j_n}$, we have the following inequalities for $F_n$:

$$
B^*_p \left( \sum_{j_1} \cdots \sum_{j_n} |c_{j_1, \ldots, j_n}|^2 \right)^{\frac{1}{2}} \leq \left\| F_n \right\|_{L^p([0, 1]^n)} \leq A^*_p \left( \sum_{j_1} \cdots \sum_{j_n} |c_{j_1, \ldots, j_n}|^2 \right)^{\frac{1}{2}},
$$

where $A_p, B_p$ are the constants in (4.7).

Using the Rademacher functions we can now prove Proposition 4.5.

Proof. We begin by noting that (4.6) holds when $n = 1$, as $\sum_{j \in \mathbb{Z}} \hat{\theta}(2^{-j} \xi) = 3$ when $\xi \neq 0$; see [18, Theorem 6.1.6, Corollary 6.1.7] So it suffices to prove (4.6) in higher dimensions. Using the preceding inequalities we write:

$$
\left\| \left( \sum_{j_1, \ldots, j_n \in \mathbb{Z}} |\Delta_{j_1}^{\theta_1} \cdots \Delta_{j_n}^{\theta_n} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} \left( \sum_{j_1, \ldots, j_n \in \mathbb{Z}} |\Delta_{j_1}^{\theta_1} \cdots \Delta_{j_n}^{\theta_n} f|^2 \right)^{\frac{p}{2}} dx
$$
Theorem 5.1. Fix

The following result will be the key interpolation estimate:

Then there is a constant $\|w\|_{L^p}$

(5.1)

$$\int_{\mathbb{R}^n} \int_{[0,1]^n} \left| \sum_{j_1, \ldots, j_n \in \mathbb{Z}} r_{j_1}(t_1) \cdots r_{j_n}(t_n) \Delta_{j_1}^{\theta,1} \cdots \Delta_{j_n}^{\theta,n} f(x) \right|^p dt_1 \cdots dt_n \, dx$$

$$\approx \int_{\mathbb{R}^n} \int_{[0,1]^n} \left| \sum_{j_1 \in \mathbb{Z}} \Delta_{j_1}^{\theta,1} \left( \sum_{j_n \in \mathbb{Z}} r_{j_n}(t_n) \Delta_{j_n}^{\theta,n} f(x) \right) \right|^p dt_1 \cdots dt_n \, dx$$

$$\approx \int_{\mathbb{R}^{n-1}} \int_{[0,1]^{n-1}} \int_{\mathbb{R}} \left( \sum_{j_1 \in \mathbb{Z}} \Delta_{j_1}^{\theta,1} \left\{ \prod_{i=2}^{n-1} \left( \sum_{j_i \in \mathbb{Z}} r_{j_i}(t_i) \Delta_{j_i}^{\theta,i} f \right) \right\} (x_1, x') \right)^2 \, dx_1 \, dx_2 \cdots dx_n \, dx'$$

where $x' = (x_2, \ldots, x_n)$. We now apply the Littlewood-Paley theorem (2.11) in the first variable $x_1$ to eliminate the square function in $j_1$ and replace the inner integral by the $p$-th power of the function in the curly brackets. We then continue the same reasoning to the remaining variables $x_2, \ldots, x_n$ to conclude the proof of (4.6).

\[ \square \]

5. The interpolation argument needed in the proof of Theorem 4.2

When $p = 2$ no derivatives are required of $\sigma$ for $T_\sigma$ to be bounded. To mitigate the effect of the requirement of the derivatives of $\sigma$ for $T_\sigma$ to be bounded on $L^p$ for $p \neq 2$, we apply an interpolation argument between $p = 2$ and $p$ near 1.

We shall use the notation introduced at the beginning of the previous section, and we shall denote

$$\Gamma(\{s_\ell\}_{\ell=1}^n) = \Gamma(s_1, \ldots, s_n) = (I - \partial_1^{s_1})^{\frac{\theta}{s_1}} \cdots (I - \partial_n^{s_n})^{\frac{\theta}{s_n}}.$$ 

The following result will be the key interpolation estimate:

**Theorem 5.1.** Fix $1 < p_0, p_1, r_0, r_1 < \infty$, $0 < s_1^0, \ldots, s_n^0, s_1, \ldots, s_n^1 < \infty$. Suppose that $r_0 s_\ell^0 > 1$ and $r_1 s_\ell^1 > 1$ for all $\ell = 1, \ldots, n$. Let $\psi$ be as before. Assume that for $k \in \{0, 1\}$ we have

(5.1) $$\|T_\sigma(f)\|_{L^{p_k}} \leq K_k \sup_{j_1, \ldots, j_n \in \mathbb{Z}} \left\| \Gamma(s_1^k, \ldots, s_n^k) \left[ \sigma(2^J \xi) \prod_{\ell=1}^n \hat{\psi}(\xi_\ell) \right] \right\|_{L^{p_k}} \|f\|_{L^{p_k}}$$

for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. For $0 < \theta < 1$ and $\ell = 1, \ldots, n$ define

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}, \quad s_\ell = (1 - \theta)s_\ell^0 + \theta s_\ell^1.$$

Then there is a constant $C_*$ such that for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ we have

(5.2) $$\|T_\sigma(f)\|_{L^p} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{j_1, \ldots, j_n \in \mathbb{Z}} \left\| \Gamma(s_1, \ldots, s_n) \left[ \sigma(2^J \xi) \prod_{\ell=1}^n \hat{\psi}(\xi_\ell) \right] \right\|_{L^r} \|f\|_{L^p}.$$

Assuming Theorem 5.1, we complete the proof of Theorem 4.2 as follows:
Indeed, since \( \min_{\ell} \gamma_{\ell} > 1/\ell, \ell = 1, \ldots, n \), fix \( p \in (1, \infty) \) satisfying (4.3). In fact, we can assume that \( p \in (1, 2) \), since the case \( p \in (2, \infty) \) follows by duality and the case \( p = 2 \) is a consequence of Plancherel’s theorem and of a Sobolev embedding into \( L^\infty \). In addition, assume first that \( \min_{\ell} \gamma_{\ell} \leq \frac{1}{2} \). In view of (4.3), there is \( \tau \in (0,1) \) such that

\[
\frac{1}{p} - \frac{1}{2} < \tau \min_{\ell} \gamma_{\ell}.
\]

Set \( p_1 = \frac{2}{\tau + 1} \), \( r_1 = 2 r \min_{\ell} \gamma_{\ell} \) and \( \gamma_{\ell} = \frac{1}{2} + \varepsilon, \ell = 1, \ldots, n \), where \( \varepsilon > 0 \) is a real number whose exact value will be specified later. Since \( p_1 > 1 \) and \( r_1 \gamma_{\ell} > 2 \gamma_{\ell} > 1, \ell = 1, \ldots, n \), Proposition 4.4 yields that

\[
\|T_\sigma(f)\|_{L^{p_1}} \leq C_1 \sup_{J_1, \ldots, J_n \in \mathbb{Z}} \left\| \Gamma(\gamma_{1}, \ldots, \gamma_{n}) \left[ \sigma(2^J \xi) \prod_{\ell=1}^{n} \hat{\psi}(\xi_{\ell}) \right] \right\|_{L^{p_1}} \|f\|_{L^{p_1}}.
\]

Pick \( p_0 = 2 \). Let \( \theta \) be the real number satisfying

\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},
\]

namely, \( \theta = \frac{2}{\tau} \left( \frac{1}{p} - \frac{1}{2} \right) \). Observe that, by (5.3), \( 0 < \theta < 2 \min_{\ell} \gamma_{\ell} \leq 1 \). Finally, choose real numbers \( r_0 \) and \( \gamma_{\ell} = (1 - \theta) \gamma_{\ell}^0 + \theta \gamma_{\ell}^1 \).

We claim that, for a suitable choice of \( \varepsilon > 0 \), one has \( r_0 > 2 \) and \( r_0 \gamma_{\ell} > 1, \ell = 1, \ldots, n \). Indeed, since \( \min_{\ell} \gamma_{\ell} \leq \frac{1}{2} \), we have \( r_1 \leq r \), and thus, by (5.5), \( r_0 \geq r \geq r_1 > 2 \). Further,

\[
r_0 \gamma_{\ell} = \frac{r_1 r (\gamma_{\ell} - \gamma_{\ell}^1)}{r_1 - \theta r} = \frac{r \min_{k} \gamma_{k}(\gamma_{\ell} - \frac{\theta}{2} - \theta \varepsilon)}{\min_{k} \gamma_{k} - \frac{\theta}{2}} \geq \frac{r \min_{k} \gamma_{k}(\min_{k} \gamma_{k} - \frac{\theta}{2} - \theta \varepsilon)}{\min_{k} \gamma_{k} - \frac{\theta}{2}}.
\]

Since \( r \min_{k} \gamma_{k} > 1 \) and \( \min_{k} \gamma_{k} - \frac{\theta}{2} > 0 \), one gets \( r_0 \gamma_{\ell} > 1 \) if \( \varepsilon > 0 \) is small enough. Consequently, the space \( \{ g : \Gamma(\gamma_{1}, \ldots, \gamma_{n}) \in L^{p_0} \} \) embeds in \( L^\infty \), and we thus have

\[
\|T_\sigma(f)\|_{L^2} \leq C_1 \sup_{J_1, \ldots, J_n \in \mathbb{Z}} \left\| \Gamma(\gamma_{1}, \ldots, \gamma_{n}) \left[ \sigma(2^J \xi) \prod_{\ell=1}^{n} \hat{\psi}(\xi_{\ell}) \right] \right\|_{L^{p_0}} \|f\|_{L^2}.
\]

The boundedness of \( T_\sigma \) on \( L^p(\mathbb{R}^n) \) for any \( \sigma \) satisfying (4.2) thus follows from Theorem 5.1. Finally, if \( \min_{\ell} \gamma_{\ell} > \frac{1}{2} \), then the required assertion follows directly from Proposition 4.4.

**Proof of Theorem 5.1.** The proof of Theorem 5.1 follows closely that of [3, Theorem 4.7] and for this reason we only provide an outline of its proof with few details. Throughout
the proof we shall use the notation introduced at the beginning of the previous section. Also, whenever $J \in \mathbb{Z}^n$, we denote

$$
\varphi_J = \Gamma(s_1, \ldots, s_n) \left[ \sigma(2^J \xi) \hat{\psi}(\xi) \right],
$$

and for $z$ with real part in $[0, 1]$ we define

$$
\sigma_z(\xi) = \sum_{J \in \mathbb{Z}^n} \Gamma \left( \left\{ -s_1^0(1 - z) - s_1^1 z \right\}_{\ell=1}^n \right) \left[ |\varphi_J|^{r \left( \frac{1 - z}{r_0} + \frac{zt}{r_1} \right)} e^{i \text{Arg}(\varphi_J)} \right] (2^{-J} \xi) \widehat{\theta}(2^{-J} \xi).
$$

For any given $\xi \in \mathbb{R}^n$, this sum has only finitely many terms and one can show that

$$
\|\sigma_{t + it}\|_{L^\infty} \lesssim (1 + |t|)^{\frac{3p}{2}} \left( \sup_{J \in \mathbb{Z}^n} \|\Gamma(s_1, \ldots, s_n) [\sigma(2^J \xi) \hat{\psi}(\xi)]\|_{L^r} \right)^{\frac{1}{r}},
$$

where $r_\tau$ is the real number satisfying $\frac{1}{r_\tau} = \frac{1 - \tau}{r_0} + \frac{\tau}{r_1}$.

Let $T_z$ be the family of operators associated to the multipliers $\sigma_z$. Fix $f, g \in C_0^\infty$ and $1 < p_0 < p < p_1 < \infty$. Given $\varepsilon > 0$ there exist functions $f^\varepsilon_z$ and $g^\varepsilon_z$ such that

$$
\|f^\varepsilon_z - f\|_{L^p} < \varepsilon, \quad \|g^\varepsilon_z - g\|_{L^{p_1}} < \varepsilon,
$$

and that

$$
\|f^\varepsilon_z\|_{L^{p_0}} \leq \left( \|f\|_{L^p} + \varepsilon \right)^{\frac{1}{p_0}}, \quad \|f^\varepsilon_z\|_{L^{p_1}} \leq \left( \|f\|_{L^p} + \varepsilon \right)^{\frac{1}{p_1}},
$$

$$
\|g^\varepsilon_z\|_{L^{p_0}} \leq \left( \|g\|_{L^{p_1}} + \varepsilon \right)^{\frac{1}{p_0}}, \quad \|g^\varepsilon_z\|_{L^{p_1}} \leq \left( \|g\|_{L^{p_1}} + \varepsilon \right)^{\frac{1}{p_1}}.
$$

The existence of $f^\varepsilon_z$ and $g^\varepsilon_z$ is folklore and is omitted; for a similar construction see [3, Theorem 3.3]. Let $F(z) = \int T_{\sigma_z}(f^\varepsilon_z)g^\varepsilon_z \, dx$. Then $F(z)$ is equal to

$$
\int_{\mathbb{R}^n} \sigma_z(\xi) \hat{f}^\varepsilon_z(\xi) \hat{g}^\varepsilon_z(\xi) \, d\xi
$$

and

$$
= \sum_{J \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \Gamma \left( \left\{ -s_1^0(1 - z) - s_1^1 z \right\}_{\ell=1}^n \right) \left[ |\varphi_J|^{r \left( \frac{1 - z}{r_0} + \frac{zt}{r_1} \right)} e^{i \text{Arg}(\varphi_J)} \right] (2^{-J} \xi) \widehat{\theta}(2^{-J} \xi) \hat{f}^\varepsilon_z(\xi) \hat{g}^\varepsilon_z(\xi) \, d\xi
$$

$$
= \sum_{J \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \left[ |\varphi_J|^{r \left( \frac{1 - z}{r_0} + \frac{zt}{r_1} \right)} e^{i \text{Arg}(\varphi_J)} \right] (2^{-J} \xi) \Gamma \left( \left\{ -s_1^0(1 - z) - s_1^1 z \right\}_{\ell=1}^n \right) \left[ \widehat{\theta}(2^{-J} \cdot) \hat{f}^\varepsilon_z(\xi) \hat{g}^\varepsilon_z(\xi) \right] \, d\xi.
$$

The function $F(z)$ is analytic on the strip $0 < \Re(z) < 1$ and continuous up to the boundary. Notice that $\sigma_{it}(2^K \cdot) \hat{\psi}$ picks up only the terms of (5.7) for which $J$ differs from $K$ in some coordinate by at most one unit. For simplicity we may therefore take $K = J$ in the calculation below. Using the Kato-Ponce inequality we may “remove” the factor $\widehat{\theta}$ and write

$$
\|T_{\sigma_{it}}(f^\varepsilon_{it})\|_{L^{p_0}} \leq K_0 \sup_{K \in \mathbb{Z}^n} \left\| \Gamma(s_1^0, \ldots, s_n^0) \left[ \sigma_{it}(2^K \cdot) \hat{\psi} \right] \right\|_{L^{p_0}} \|f^\varepsilon_{it}\|_{L^{p_0}}
$$

$$
\leq K_0 \sup_{K \in \mathbb{Z}^n} \left\| \Gamma \left( \left\{ s_1^0 - s_1^0(1 - it) - s_1^1 it \right\}_{\ell=1}^n \right) \left[ |\varphi_K|^{r \left( \frac{1 - it}{r_0} + \frac{it}{r_1} \right)} e^{i \text{Arg}(\varphi_K)} \right] \right\|_{L^{p_0}} \|f^\varepsilon_{it}\|_{L^{p_0}}
$$
\[ \leq (1 + |t|)^\frac{3^\alpha}{2} K_0 \sup_{K \in \mathbb{Z}^N} \| \varphi_K \|_{L^\infty} \left( \| f \|_{L^p} + \varepsilon \right)^{\frac{1}{\rho}}. \]

Using Hölder’s inequality \( |F(it)| \leq \| T_{\sigma(it)} (f_{it}) \|_{L^{p_0}} \| g_{it}^\varepsilon \|_{L^{p_0}'} \), we may therefore write
\[
|F(it)| \leq C(1 + |t|)^\frac{3^\alpha}{2} K_0 \sup_{J \in \mathbb{Z}^n} \left\| \Gamma(\{ s_\ell \}_{\ell=1}^n) \sigma(2^{j_\ell} \cdot) \widehat{\psi} \right\|_{L^\infty} \left( \| f \|_{L^p} + \varepsilon \right)^{\frac{1}{\rho}} \left( \| g \|_{L^{p_0}'} + \varepsilon \right)^{\frac{1}{\rho}}
\]
for some constant \( C = C(n, r_0, s^0_\ell, s^1_\ell) \). Similarly, for some constant \( C = C(n, r_1, s^0_\ell, s^1_\ell) \) we obtain
\[
|F(1 + it)| \leq C(1 + |t|)^\frac{3^\alpha}{2} K_1 \sup_{J \in \mathbb{Z}^n} \left\| \Gamma(\{ s_\ell \}_{\ell=1}^n) \sigma(2^{j_\ell} \cdot) \widehat{\psi} \right\|_{L^\infty} \left( \| f \|_{L^p} + \varepsilon \right)^{\frac{1}{\rho}} \left( \| g \|_{L^{p_0}'} + \varepsilon \right)^{\frac{1}{\rho}}
\]
Thus for \( z = \tau + it, t \in \mathbb{R} \) and \( 0 \leq \tau \leq 1 \) it follows from (5.8) and from the definition of \( F(z) \) that
\[
|F(z)| \leq C''(1 + |t|)^\frac{3^\alpha}{2} \left( \sup_{J \in \mathbb{Z}^n} \left\| \Gamma(s_1, \ldots, s_n) \sigma(2^{j_\ell} \cdot) \widehat{\psi} \right\|_{L^\infty} \right)^{\frac{1}{\rho}} \left\| f \right\|_{L^2} \left\| g \right\|_{L^2} = A_\tau(t),
\]
noting that \( \left\| f \right\|_{L^2} \left\| g \right\|_{L^2} \) is bounded above by constants independent of \( t \) and \( \tau \). Since \( A_\tau(t) \leq \exp(Ae^{a|t|}) \), the hypotheses of three lines lemma are valid. It follows that
\[
|F(\theta)| \leq C K_0^{1-\theta} K_1^\theta \sup_{J \in \mathbb{Z}^n} \left\| \Gamma(\{ s_\ell \}_{\ell=1}^n) \sigma(2^{j_\ell} \cdot) \widehat{\psi} \right\|_{L^\infty} \left( \| f \|_{L^p} + \varepsilon \right)^{\frac{1}{\rho}} \left( \| g \|_{L^{p_0}'} + \varepsilon \right)^{\frac{1}{\rho}}.
\]
Taking the supremum over all functions \( g \in L^{p_0'} \) with \( \| g \|_{L^{p_0}'} \leq 1 \), a simple density argument yields for some \( C_* = C_*(n, r_1, r_2, s^0_\ell, s^1_\ell) \)
\[
\| T_\sigma(f) \|_{L^p} \leq C_* K_0^{1-\theta} K_1^\theta \sup_{J \in \mathbb{Z}^n} \left\| \Gamma(s_1, \ldots, s_n) \sigma(2^{j_\ell} \cdot) \widehat{\psi} \right\|_{L^\infty} \| f \|_{L^p}.
\]
This completes the proof of the sufficiency part of Theorem 4.2. The proof of the necessity part is postponed to section 6. \( \square \)

6. The sharpness of the Marcinkiewicz multiplier theorem

In this section we discuss examples indicating the sharpness of Theorem 4.2. We first consider the multiplier
\[
\sigma(\xi) = e^{i|\xi|^2} \prod_{\ell=1}^n \phi(\xi_\ell)|\xi_\ell|^{-2\gamma_\ell},
\]
where \( \phi \) is a smooth function on \( \mathbb{R} \) which vanishes in a neighborhood of the origin and is equal to 1 near infinity, and \( \gamma > 0 \). This multiplier can be obtained by taking tensor products of the functions \( \sigma_{a,b} \) introduced at the end of Section 1, with \( a = 2 \) and \( b = 2\gamma_\ell \). Then \( \sigma \) satisfies (4.2) when \( r > 1 \) and \( \min_{1 \leq \ell \leq n} \gamma_\ell > \frac{1}{r} \). In addition, if boundedness holds for \( T_\sigma \) from \( L^p(\mathbb{R}^n) \) to itself, then by testing on tensor type functions, we must necessarily have that each \( m_{2,2\gamma_\ell}(\xi_\ell) \) is bounded from \( L^p(\mathbb{R}) \) to itself and thus we must have \( \frac{1}{2} - \frac{1}{p} \leq \gamma_\ell \) for all \( \ell = 1, \ldots, n \).
Next we discuss an example indicating that Theorem 4.2 does not hold in the limiting case when $|\frac{1}{2} - \frac{1}{p}| = \gamma_\ell$ for some $\ell = 1, \ldots, n$. Such an example appeared, at least in the two-dimensional case with both smoothness parameters equal, in Carbery and Seeger [5, remark after Proposition 6.1]. We provide an example in the spirit of theirs, given by an explicit closed-form expression and valid in all dimensions $n \geq 2$.

**Example 6.1** ([32]). Given $\alpha \in (0, 1)$, consider the function

$$
\sigma(\xi, \eta) = \varphi(|\eta|) e^{-\frac{\xi^2}{2}|\eta|^\alpha (\log |\eta|)^{-\alpha}}, \quad (\xi, \eta) \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n,
$$

where $\varphi$ is a smooth function on the line such that $0 \leq \varphi \leq 1$, $\varphi = 0$ on $(-\infty, 8]$ and $\varphi = 1$ on $[9, \infty)$. Then

(i) $\sigma$ satisfies (4.2), with $r$ large enough, whenever $\gamma_1 = \alpha$ and $\gamma_2, \ldots, \gamma_n$ are arbitrary positive real numbers;

(ii) $\sigma$ is an $L^p$ Fourier multiplier for a given $1 < p < \infty$ if and only if $\alpha > |\frac{1}{p} - \frac{1}{2}|$.

The previous example indicates that condition (4.2) does not guarantee that $T_{\sigma}$ is $L^p$ bounded unless all indices $\gamma_1, \ldots, \gamma_n$ in (4.2) are larger than $|\frac{1}{p} - \frac{1}{2}|$. In particular, for a given $i \in \{1, \ldots, n\}$, one does not have boundedness on the critical line $\gamma_i = |\frac{1}{p} - \frac{1}{2}|$, no matter how large the remaining parameters are.

Let us now verify the statement of part (i) of Example 6.1. We shall first prove that

$$
(6.1) \quad \sup_{k, \ell \in \mathbb{Z}} \| (I - \partial_\xi^{\alpha_2})(I - \Delta_\eta)^{\alpha_2} \hat{\psi}(\xi) \hat{\Phi}(\eta) \sigma(2^k \xi, 2^\ell \eta) \|_{L^r} < \infty
$$

for any $s > 0$ and $r > 1$. Here, $\Phi$ denotes a Schwartz function on $\mathbb{R}^{n-1}$ whose Fourier transform is supported in the set $\{\eta \in \mathbb{R}^{n-1} : \frac{1}{2} \leq |\eta| \leq 2\}$ and which satisfies $\sum_{\ell \in \mathbb{Z}} \hat{\Phi}(2^\ell \eta) = 1$ for all $\eta \neq 0$. Indeed, for any $k, \ell \in \mathbb{Z}$, $\ell \geq 3$, and for any given nonnegative integer $m$, we have

$$
\| \hat{\psi}(\xi) \hat{\Phi}(\eta) \sigma(2^k \xi, 2^\ell \eta) \|_{L^r} \leq C \ell^{-\alpha}
$$

and

$$
\| (I - \partial_\xi)^{\frac{1}{2}} (I - \Delta_\eta)^{\alpha_2} \hat{\psi}(\xi) \hat{\Phi}(\eta) \sigma(2^k \xi, 2^\ell \eta) \|_{L^r} \leq C \ell \cdot \ell^{-\alpha},
$$

where the constant $C$ is independent of $k$ and $\ell$. Interpolating between these two estimates, we obtain

$$
\| (I - \partial_\xi)^{\frac{1}{2}} (I - \Delta_\eta)^{\alpha_2} \hat{\psi}(\xi) \hat{\Phi}(\eta) \sigma(2^k \xi, 2^\ell \eta) \|_{L^r} \leq C.
$$
Notice also that the last inequality in fact holds for all integers \( k, \ell \), since the function \( \widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi,2^\ell\eta) \) is identically equal to 0 if \( \ell \leq 2 \). Hence, we have

\[
\sup_{k,\ell \in \mathbb{Z}} \| (I - \partial_\xi)^{\frac{\sigma}{2}} (I - \Delta_\eta)^{\frac{m}{2}} [\widehat{\psi}(\xi)\widehat{\Phi}(\eta)\sigma(2^k\xi,2^\ell\eta)] \|_{L^s_p} \leq C,
\]

and interpolating between variants of this estimate corresponding to different values of \( m \), we obtain (6.1) for any \( s > 0 \). Now, part (i) of Example 6.1 follows by an application of Theorem 7.1, which will be stated and proved in the next section, in the variable \( \eta \).

Let us finally focus on part (ii) of Example 6.1. If \( \alpha > |\frac{1}{p} - \frac{1}{2}| \), then \( \sigma \) is an \( L^p \) Fourier multiplier thanks to (i) and Theorem 4.2. Let us now prove that \( T_\sigma \) is not \( L^p \) bounded if \( \alpha \leq |\frac{1}{p} - \frac{1}{2}| \). By duality, it suffices to discuss only the case when \( 1 < p < 2 \). Furthermore, one can make use of the result of Herz and Rivière [34] which asserts that if \( T_\sigma \) is \( L^p \) bounded then it is necessarily bounded also on the mixed norm space \( L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1})) \), defined as

\[
\|f\|_{L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1}))} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^{n-1}} |f(\xi,\eta)|^2 \, d\eta \right)^\frac{p}{2} \, d\xi \right)^{\frac{1}{2}}.
\]

Thus, the proof will be complete if we show that \( T_\sigma \) is not bounded on \( L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1})) \).

Let \( f \) be the function on \( \mathbb{R}^n \) whose Fourier transform satisfies

\[
\widehat{f}(\xi,\eta) = e^{-\frac{\xi^2}{2}} \varphi(|\eta|)|\eta|^{\frac{1-n}{2}} (\log |\eta|)^{-\frac{1}{2}} (\log \log |\eta|)^{-\beta}, \quad (\xi,\eta) \in \mathbb{R} \times \mathbb{R}^{n-1}.
\]

Using Plancherel’s theorem in the variable \( \eta \), it is easy to check that \( f \in L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1})) \) whenever \( \beta > \frac{1}{2} \). Our next goal is to prove that \( T_\sigma f = \mathcal{F}^{-1}(\sigma \hat{f}) \) does not belong to \( L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1})) \) if \( \beta \in (\frac{1}{2}, \frac{1}{p}) \). Using Plancherel’s theorem in the variable \( \eta \) again, this is equivalent to showing that \( \mathcal{F}_\xi^{-1}(\sigma \hat{f}) \) is not in \( L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1})) \), where \( \mathcal{F}_\xi^{-1} \) stands for the inverse Fourier transform in the \( \xi \) variable.

Observe that

\[
\mathcal{F}_\xi^{-1}(\sigma \hat{f})(x,\eta) = C e^{-\frac{1}{2}(2\pi x + \log |\eta|)^2} \varphi^2(|\eta|)|\eta|^{\frac{1-n}{2}} (\log |\eta|)^{-\alpha} (\log \log |\eta|)^{-\beta} \\
\geq C \chi((x,\eta): x < -2, e^{-2\pi x - 1} < |\eta| < e^{-2\pi} \sigma(x,\eta) e^{2\pi x n-1} (x,\eta) e^{2\pi x n-rac{1}{2}} (x) - \frac{1}{2} (\log(-x))^{-\beta}.
\]

Therefore,

\[
\| \mathcal{F}_\xi^{-1}(\sigma \hat{f}) \|_{L^p(\mathbb{R}; L^2(\mathbb{R}^{n-1}))} \geq C \left( \int_{-\infty}^{-2} (-x)^{(-\alpha - \frac{1}{2})p} (\log(-x))^{-\beta p} \, dx \right)^{\frac{1}{p}} = \infty,
\]

which yields the desired conclusion.
7. Comparison between the Hörmander and Marcinkiewicz multiplier theorems

In this section we show that the class of multipliers which satisfies the assumptions of Theorem 4.2 is strictly larger than the set of multipliers treated by the version of the Hörmander multiplier theorem due to Calderón and Torchinsky [3, Theorem 4.6]; see Section 1 for the statement of this theorem. Before we come to the proof we would like to emphasize that such a comparison is not possible for the classical versions of these two theorems (see (1.2) for the Hörmander condition and (4.1) for the Marcinkiewicz condition). Indeed, while condition (1.2) requires the multiplier to have more than $n/2$ derivatives in each variable, condition (4.1) assumes $n$ derivatives in total, but only one in each variable. Therefore, there are multipliers satisfying (1.2) but not (4.1), and also multipliers satisfying (4.1) but not (1.2).

To compare the fractional versions of the Hörmander and Marcinkiewicz multiplier theorems, we first notice that Theorem 4.2 assumes the multiplier $\sigma$ to have $1/r + \varepsilon$ derivatives in each variable, while the Hörmander multiplier theorem requires more than $n/r$ derivatives in all variables, and so there are multipliers which can be treated by Theorem 4.2, but not by [3, Theorem 4.6]. On the other hand, it is an easy consequence of the following theorem that every multiplier satisfying the assumptions of the Hörmander multiplier theorem falls under the scope of Theorem 4.2 as well.

**Theorem 7.1.** Let $\psi$ be a Schwartz function on the line whose Fourier transform is supported in the set $\{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and which satisfies $\sum_{k \in \mathbb{Z}} \hat{\psi}(2^k \xi) = 1$ for every $\xi \neq 0$. Also, let $\Phi$ be a Schwartz function on $\mathbb{R}^n$ having analogous properties. If $1 < r < \infty$ and $\gamma_1, \ldots, \gamma_n$ are real numbers larger than $\frac{1}{r}$, then

$$
\sup_{j_1, \ldots, j_n \in \mathbb{Z}} \left\| (I - \partial^2_1)^{\frac{j_1}{2}} \cdots (I - \partial^2_n)^{\frac{j_n}{2}} \left[ \sigma(2^{j_1} \xi_1, \ldots, 2^{j_n} \xi_n) \prod_{\ell=1}^{n} \hat{\psi}(\xi_\ell) \right] \right\|_{L^r} 
\leq C \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{\frac{\gamma_1 + \cdots + \gamma_n}{2}} \left[ \sigma(2^j \xi) \hat{\Phi}(\xi) \right] \right\|_{L^r}.
$$

Crucial ingredients needed for the proof of Theorem 7.1 are two one-dimensional inequalities contained in the following lemma.

**Lemma 7.2.** Let $\psi$ be as in Theorem 7.1. If $k \in \mathbb{Z}$, $\gamma > 0$ and $1 < r < \infty$ are such that $\gamma r > 1$, then

$$
\left\| f(2^k \cdot \hat{\psi}) \right\|_{L^r(\mathbb{R})} \leq C \left\| (I - \partial^2)^{\frac{j}{2}} f \right\|_{L^r(\mathbb{R})}
$$

(7.2)
and
\[(7.3) \qquad \| (\partial^2)^{\frac{3}{2}} [f(2^k \cdot \hat{\psi})] \|_{L^r(\mathbb{R})} \leq C(1 + 2^{k(\gamma - \frac{1}{2})}) \| (I - \partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})}. \]

Proof. Since $\gamma r > 1$, the Sobolev embedding theorem yields
\[(7.4) \qquad |f(2^k x)| \leq \| f \|_{L^{\infty}(\mathbb{R})} \leq C \| (I - \partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})} \quad \text{for a.e. } x \in \mathbb{R}. \]

Therefore,
\[
\| f(2^k \cdot \hat{\psi}) \|_{L^r(\mathbb{R})} \leq C \| (I - \partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})} \| \hat{\psi} \|_{L^r(\mathbb{R})} = C' \| (I - \partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})},
\]

This proves (7.2).

Further, using the Kato-Ponce inequality [29], the estimate (7.4) and the fact that $\hat{\psi}$ is smooth and with compact support, we obtain
\[
\| (-\partial^2)^{\frac{3}{2}} [f(2^k \cdot \hat{\psi})] \|_{L^r(\mathbb{R})} \\
\leq C \left( \| (-\partial^2)^{\frac{3}{2}} [f(2^k \cdot \hat{\psi})] \|_{L^r(\mathbb{R})} \| \hat{\psi} \|_{L^r(\mathbb{R})} + \| f(2^k \cdot \hat{\psi}) \|_{L^{\infty}(\mathbb{R})} \| (-\partial^2)^{\frac{3}{2}} \hat{\psi} \|_{L^r(\mathbb{R})} \right) \\
\leq C \left( \| (-\partial^2)^{\frac{3}{2}} [f(2^k \cdot \hat{\psi})] \|_{L^r(\mathbb{R})} + \| (I - \partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})} \right) \\
= C' \left( 2^{k(\gamma - \frac{1}{2})} \| (-\partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})} + \| (I - \partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})} \right) \\
\leq C(2^{k(\gamma - \frac{1}{2})} + 1) \| (I - \partial^2)^{\frac{3}{2}} f \|_{L^r(\mathbb{R})},
\]

namely, (7.3). \hfill $\Box$

Proof of Theorem 7.1. Set $F(\xi) = \sum_{k=-n}^{n} \hat{\Phi}(2^k \xi)$, $\xi \in \mathbb{R}^n$. Then $F(\xi) = 1$ for any $\xi$ satisfying $\frac{1}{2^n} \leq |\xi| \leq 2^n$. Therefore, if $j_1, \ldots, j_n$ are integers and $j := \max\{j_1, \ldots, j_n\}$, then $F(2^{j-i} \xi_1, \ldots, 2^{j-n} \xi_n) = 1$ on $\{(\xi_1, \ldots, \xi_n) : \frac{1}{2} \leq |\xi_1| \leq 2, \ldots, \frac{1}{2} \leq |\xi_n| \leq 2\}$. Consequently,
\[
\prod_{\ell=1}^{n} \hat{\psi}(\xi_\ell) = F(2^{j-i} \xi_1, \ldots, 2^{j-n} \xi_n) \prod_{\ell=1}^{n} \hat{\psi}(\xi_\ell).
\]

Using this, we can write
\[
\| (I - \partial_1^2)^{\frac{3}{2}} \cdots (I - \partial_n^2)^{\frac{3}{2}} \left[ \sigma(2^{j_1} \xi_1, \ldots, 2^{j_n} \xi_n) \prod_{\ell=1}^{n} \hat{\psi}(\xi_\ell) \right] \|_{L^r} \\
= \| (I - \partial_1^2)^{\frac{3}{2}} \cdots (I - \partial_n^2)^{\frac{3}{2}} \left[ \sigma(2^{j_1} \xi_1, \ldots, 2^{j_n} \xi_n) F(2^{j_1-i} \xi_1, \ldots, 2^{j-n} \xi_n) \prod_{\ell=1}^{n} \hat{\psi}(\xi_\ell) \right] \|_{L^r} \\
\leq C \sum_{a=-n}^{n} \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}} \sum_{\ell=1}^{n} \hat{\psi}(\xi_\ell),
\]
\[ \left\| \left( -\partial_{i_1}^{\gamma_{i_1}} \right) \cdots \left( -\partial_{i_k}^{\gamma_{i_k}} \right) \right\|_{L^r} \cdot \left( \sigma(2^{j_1} \xi_1, \ldots, 2^{j_n} \xi_n) \hat{\Phi}(2^{j_1-j+a} \xi_1, \ldots, 2^{j_n-j+a} \xi_n) \prod_{\ell=1}^{n} \hat{\psi}(\xi_{\ell}) \right) \leq C \left( 1 + 2^n \max_{s=1}^{n} (\gamma_{is} - \frac{1}{2}) \right)^{n} \left\| (I - \Delta)^{\gamma_1 + \gamma_2 + \cdots + \gamma_n} \right\|_{L^r} \left\| (I - \Delta)^{\gamma_1 + \gamma_2 + \cdots + \gamma_n} \right\|_{L^r} \leq C \sup_{m \in \mathbb{Z}} \left\| (I - \Delta)^{\gamma_1 + \gamma_2 + \cdots + \gamma_n} \right\|_{L^r} \left\| (I - \Delta)^{\gamma_1 + \gamma_2 + \cdots + \gamma_n} \right\|_{L^r} . \]

This implies (7.1).

8. A boundedness criterion for bilinear Fourier multiplier operators

As physical and natural phenomena depend on numerous inputs, it natural to develop theories that model dependencies on many variables. Multilinear Fourier Analysis provides a framework to study operations that depend linearly on several input functions. Multilinear multiplier operators are special kinds of multilinear operators in which the product of frequencies is jointly altered by a common symbol. Based on this definition, bilinear multiplier operators are given by

\[ T_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} d\xi d\eta , \]

where \( f, g \) are Schwartz functions and \( m \) is a bounded function on \( \mathbb{R}^{2n} \). These are exactly the bilinear operators that commute with simultaneous translations of functions. The study of general bilinear operators was initiated by Coifman and Meyer [9], [10] but since the turn of the present century this area has been enjoying a resurgence of activity. We refer to [11], [33], and [19] for general material related to the multilinear operators. A classical by now criterion for boundedness of bilinear multiplier operators says that if \( m \) satisfies

\[ |\partial^\alpha \partial^\beta \sigma(\xi, \eta)| \leq C_{\alpha, \beta} (|\xi| + |\eta|)^{1-|\alpha|-|\beta|} \]

for sufficiently large multiindices \( \alpha, \beta \), then the associated bilinear operator \( T_m \) admits a bounded extension from \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) when \( 1/p_1 + 1/p_2 = 1/p, 1 < p_1, p_2 \leq \infty \) and \( 1/2 < p < \infty \). This was proved by Coifman and Meyer [10] in the case when \( p > 1 \) and was extended to the case \( p \leq 1 \) by Grafakos and Torres [33] and
independently by Kenig and Stein [38]. This theorem is essentially saying that linear Mikhlin multipliers on \( \mathbb{R}^{2n} \) are bounded bilinear multipliers on \( \mathbb{R}^n \times \mathbb{R}^n \). It should be noted that the analogue to the Marcinkiewicz condition
\[
|\partial^\alpha \partial^\beta \sigma(\xi, \eta)| \leq C_{\alpha, \beta} |\xi|^{-|\alpha}|\eta|^{-|\beta|}
\]
does not suffice to guarantee boundedness of \( T_m \) on any product of Lebesgue spaces; a counterexample to indicate this fact was constructed in Grafakos and Kalton [25].

Extensions of the Coifman-Meyer–Kenig-Stein–Grafakos-Torres result for bilinear multipliers that satisfy Hörmander’s [37] classical Sobolev space weakening of Mikhlin’s condition for linear operators are available in the literature as well. They were initiated by Tomita [50] and subsequently further investigated by Grafakos, Fujita, Miyachi, Nguyen, Si, and Tomita among others; see [30] [17], [26], [41], [42], [28], [27].

As we have seen, bilinear Fourier multiplier operators may map \( L^{p_1} \times L^{p_2} \) to \( L^p \) when \( 1/p = 1/p_1 + 1/p_2 \) for a variety of parameters \( p_1 \) and \( p_2 \) but in this note, we only focus on the \( L^2 \times L^2 \) to \( L^1 \) boundedness of such operators. Such estimates are central and play the same role in bilinear theory as the \( L^2 \) boundedness plays in linear multiplier theory.

As Plancherel’s identity \( \|f\|_{L^2} = \|\hat{f}\|_{L^2} \) does not hold on \( L^1 \), there does not seem to be a straightforward way to characterize the boundedness of bilinear multiplier operators from \( L^2 \times L^2 \to L^1 \); however, different types of sharp sufficient conditions are available.

For instance, a bilinear variant of the Hörmander multiplier theorem asserts that if the functions \( m(2^k \cdot) \phi \) have \( s \) derivatives in \( L^r(\mathbb{R}^{2n}) \) \((1 < r < \infty)\) uniformly in \( k \in \mathbb{Z} \), with \( \phi \) being a suitable smooth bump supported in \( 1/2 < |\xi| < 2 \), then \( T_m \) is bounded from \( L^2 \times L^2 \) to \( L^1 \) when \( s > s_0 = \max(n/2, 2n/r) \) and \( s_0 \) cannot be replaced by any smaller number; see [21]. Thus more than \( n/2 \) derivatives of \( m(2^k \cdot) \phi \) in \( L^4(\mathbb{R}^{2n}) \) uniformly in \( k \) are required of a generic multiplier \( m \) for \( T_m \) to map \( L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \) to \( L^1(\mathbb{R}^n) \).

In this note we restrict our attention to multipliers whose derivatives are merely bounded. We introduce the space
\[
\mathcal{L}^\infty(\mathbb{R}^{2n}) = \{ m : \mathbb{R}^{2n} \to \mathbb{C} : \partial^\alpha m \text{ exist for all } \alpha \text{ and } \|\partial^\alpha m\|_{L^\infty} < \infty \}.
\]

In the linear setting we have \( m \in L^\infty \) if and only if the corresponding linear operator is bounded on \( L^2 \). So one may guess that a bilinear operator \( T_m \) is bounded from \( L^2 \times L^2 \) to \( L^1 \) when \( m \) lies in \( \mathcal{L}^\infty \). However Bényi and Torres [2] provided an example of a function \( m \in \mathcal{L}^\infty \) for which the associated bilinear operator \( T_m \) is unbounded from \( L^{p_1} \times L^{p_2} \) to \( L^p \) for any \( 1 \leq p_1, p_2 < \infty \) satisfying \( 1/p = 1/p_1 + 1/p_2 \). The counterexample of Bényi and Torres is also complemented by a positive result of the same authors [2] involving
mixed norm spaces and a subsequent positive result of Grafakos, He and Honzík [20, Corollary 8], who showed that the mere $L^2$ integrability of functions in $L^\infty$ suffices to yield the $L^2 \times L^2 \to L^1$ boundedness of $T_m$.

It turns out that if $m \in L^\infty$ then a sharp criterion for the boundedness of the bilinear multiplier operator $T_m$ from $L^2 \times L^2 \to L^1$ can be formulated in terms of the magnitude of integrability of the function $m$. We provide a proof of the main direction of this result, the one that yields the boundedness of the operator. In what follows, $C^M(\mathbb{R}^{2n})$ denotes the class of all functions on $\mathbb{R}^{2n}$ whose partial derivatives of order up to and including order $M$ are continuous.

**Theorem 8.1.** [24] Let $1 \leq q < 4$ and set $M_q = \left\lfloor \frac{2n}{4-q} \right\rfloor + 1$. Let $m$ be a function in $L^q(\mathbb{R}^{2n}) \cap C^{M_q}(\mathbb{R}^{2n})$ satisfying

$$
\left\| \partial^\alpha m \right\|_{L^\infty} \leq C_0 < \infty \quad \text{for all multiindices } \alpha \text{ with } |\alpha| \leq M_q.
$$

Then there is a constant $C$ depending on $n$ and $q$ such that the bilinear operator $T_m$ with multiplier $m$ satisfies

$$
\left\| T_m \right\|_{L^2 \times L^2 \to L^1} \leq C C_0^{1-\frac{q}{4}} \|m\|_{L^q}^{\frac{q}{3}}.
$$

Additionally, we are aware of examples indicating that for any $q \geq 4$ there exist functions $m \in L^q(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ such that the associated operator $T_m$ does not map $L^2 \times L^2$ to $L^1$; see [24] for $q > 4$ and [47] for $q = 4$. These counterexamples are discussed in Section 11.

9. **Bumps centered at lattice points**

Before we prove Theorem 8.1 we present the idea of its proof in a simpler context. We examine the situation where the bilinear operator is given as a finite sum of products of smooth bumps supported in small discs centered at some lattice points in $\mathbb{R}^2$. As the set of lattice points may not be a product of subsets of $\mathbb{Z}$, the associated bilinear operator cannot be written as a product of linear operators and an alternative approach needs to be employed for its boundedness.

Let us define a linear operator as follows:

$$
S_k(f)(x) = \int_{\mathbb{R}} \hat{f}(\xi) \phi(\xi - k) e^{2\pi i \xi x} d\xi,
$$

where $\phi$ is a smooth function on the line supported in the interval $(-1/10, 1/10)$.

Let $E$ be a subset of $\mathbb{Z}^2$ of size $N$. We denote by $E_1$ the set of all $k \in \mathbb{Z}$ with the property that there exists an $l \in \mathbb{Z}$ such that the point $(k, l) \in E$. That is $E_1$ is the set
of all first coordinates of elements of $E$. We think of the set $E$ as a union of columns $Col_k$ indexed by $k \in E_1$ and we write
$$E = \bigcup_{k \in E_1} Col_k$$
and
$$T_{\sigma_N}(f, g) = \sum_{k \in E_1} S_k(f) \sum_{l : (k,l) \in Col_k} S_l(g).$$
We split the columns in large and small. Precisely, we write
$$E = E_{\text{large}} \cup E_{\text{small}},$$
where $E_{\text{large}}$ contains all columns of size $\geq K$ and $E_{\text{small}}$ contains all columns of size $< K$, for some $K$ to be chosen later. Analogously we split
$$E_1 = E_{1\text{large}} \cup E_{1\text{small}},$$
where $E_{1\text{large}}$ and $E_{1\text{small}}$ is the set of all first coordinates of columns in $E_{\text{large}}$ and $E_{\text{small}}$, respectively. Correspondingly we define:
$$T_{\sigma_N}^{\text{large}}(f, g) = \sum_{k \in E_{1\text{large}}} S_k(f) \sum_{l : (k,l) \in Col_k} S_l(g)$$
and
$$T_{\sigma_N}^{\text{small}}(f, g) = \sum_{k \in E_{1\text{small}}} S_k(f) \sum_{l : (k,l) \in Col_k} S_l(g) = \sum_{l : \exists (k,l) \in E_{1\text{small}}} S_l(g) \sum_{k : (k,l) \in E_{1\text{small}}} S_k(f)$$
so that
$$T_{\sigma_N}(f, g) = T_{\sigma_N}^{\text{large}}(f, g) + T_{\sigma_N}^{\text{small}}(f, g).$$
We start with $T_{\sigma_N}^{\text{large}}$. We have
$$\|T_{\sigma_N}^{\text{large}}(f, g)\|_{L^1} \leq \sum_{k \in E_{1\text{large}}} \|S_k(f)\|_{L^2} \sum_{l : (k,l) \in Col_k} \|S_l(g)\|_{L^2} \leq \sum_{k \in E_{1\text{large}}} \|S_k(f)\|_{L^2}^2 \|\sum_{l : (k,l) \in Col_k} S_l(g)\|_{L^2}^2 \leq \left( \sum_{k \in E_{1\text{large}}} \|S_k(f)\|_{L^2}^4 \right)^{\frac{1}{2}} \left( \sum_{k \in E_{1\text{large}}} \|\sum_{l : (k,l) \in Col_k} S_l(g)\|_{L^2}^4 \right)^{\frac{1}{2}} \leq \|\phi\|_{L^\infty} \|f\|_{L^2} (\# E_{1\text{large}})^{\frac{1}{2}} \|\phi\|_{L^\infty} \|g\|_{L^2},$$
exploiting the orthogonality of $S_k$’s on $L^2$. 
Notice that as there are $N$ points in $E$ and each column in $E^{large}$ has at least $K$ elements, this means that there are at most $N/K$ columns in $E^{large}$. We conclude that

\[ \left\| T^{\text{large}}_{\sigma_N}(f, g) \right\|_{L^1} \leq (N/K)^{1/2} \| \phi \|_{L^\infty}^2 \| f \|_{L^2} \| g \|_{L^2}. \]

We continue with $T^{\text{small}}_{\sigma_N}$. We have

\[
\left\| T^{\text{small}}_{\sigma_N}(f, g) \right\|_{L^1} = \left\| \sum_{l: \exists k, (k, l) \in E^{\text{small}}} S_l(g) \sum_{k: (k, l) \in E^{\text{small}}} S_k(f) \right\|_{L^1} \\
\leq \sum_{l: \exists k, (k, l) \in E^{\text{small}}} \left\| S_l(g) \right\|_{L^2} \left\| \sum_{k: (k, l) \in E^{\text{small}}} S_k(f) \right\|_{L^2} \\
\leq \left( \sum_{l: \exists k, (k, l) \in E^{\text{small}}} \left\| S_l(g) \right\|_{L^2}^2 \right)^{1/2} \left( \sum_{l: \exists k, (k, l) \in E^{\text{small}}} \left\| \sum_{k: (k, l) \in E^{\text{small}}} S_k(f) \right\|_{L^2}^2 \right)^{1/2} \\
\leq \| \phi \|_{L^\infty} \| g \|_{L^2} \left( \sum_{l: \exists k, (k, l) \in E^{\text{small}}} \sum_{k: (k, l) \in E^{\text{small}}} \left\| S_k(f) \right\|_{L^2}^2 \right)^{1/2} \\
= \| \phi \|_{L^\infty} \| g \|_{L^2} \left( \sum_{k: (k, l) \in \text{Col}_k} \left\| S_k(f) \right\|_{L^2}^2 \right)^{1/2} \\
\leq \| \phi \|_{L^\infty} \| g \|_{L^2} K^{1/2} \left( \sum_{k \in E^{\text{small}}_1} \left\| S_k(f) \right\|_{L^2}^2 \right)^{1/2} \\
\leq \| \phi \|_{L^\infty} \| g \|_{L^2} K^{1/2} \| \phi \|_{L^\infty} \| f \|_{L^2},
\]

as all columns in $E^{\text{small}}$ have size at most $K$. This yields

\[ \left\| T^{\text{small}}_{\sigma_N}(f, g) \right\|_{L^1} \leq K^{1/2} \| \phi \|_{L^\infty} \| f \|_{L^2} \| g \|_{L^2}. \]

In view of (9.2) and (9.3), the optimal choice of $K = N^{1/2}$. This proves

\[ \left\| T_{\sigma_N}(f, g) \right\|_{L^1} \leq N^{1/2} \| \phi \|_{L^\infty} \| f \|_{L^2} \| g \|_{L^2}. \]

10. PROOF OF THEOREM 8.1

We plan to outline the proof of Theorem 8.1. This is based on the product-type wavelet method initiated in [20]. Our approach here incorporates several crucial combinatorial improvements compared to [20].
We recall some facts related to product-type wavelets that will be crucial in our approach of proving Theorem 8.1. For a fixed $M \in \mathbb{N}$ there exist real-valued compactly supported functions $\psi_F, \psi_M$ in $\mathcal{C}^k(\mathbb{R})$, called father wavelet and mother wavelet, respectively, that satisfy
\[
\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1
\]
and
\[
\int_{\mathbb{R}} x^k \psi_M(x) dx = 0 \quad \text{for all } 0 \leq k \leq M.
\]
Then the family of functions of the variables $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$
\[
\bigcup_{\mu_1, \mu_2 \in \mathbb{Z}^n} \left\{ \psi_F(x_1 - \mu_1) \psi_F(x_2 - \mu_2) \right\}
\]
\[
\bigcup_{\mu_1, \mu_2 \in \mathbb{Z}^n} \bigcup_{\lambda=0}^{\infty} \left\{ 2^{n \lambda} \psi_F(2^\lambda x_1 - \mu_1) 2^{n \lambda} \psi_M(2^\lambda x_2 - \mu_2) \right\}
\]
\[
\bigcup_{\mu_1, \mu_2 \in \mathbb{Z}^n} \bigcup_{\lambda=0}^{\infty} \left\{ 2^{n \lambda} \psi_M(2^\lambda x_1 - \mu_1) 2^{n \lambda} \psi_F(2^\lambda x_2 - \mu_2) \right\}
\]
\[
\bigcup_{\mu_1, \mu_2 \in \mathbb{Z}^n} \bigcup_{\lambda=0}^{\infty} \left\{ 2^{n \lambda} \psi_M(2^\lambda x_1 - \mu_1) 2^{n \lambda} \psi_M(2^\lambda x_2 - \mu_2) \right\}
\]
forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$. This result is due to Triebel [52].

We denote by $\mathcal{J}$ the set of all pairs $(\lambda, G)$ such that either $\lambda = 0$ and $G = (F, F)$, or $\lambda$ is a nonnegative integer and $G$ has the form $(F, M)$, $(M, F)$, or $(M, M)$. For $(\lambda, G) \in \mathcal{J}$ and $(\mu_1, \mu_2) \in \mathbb{Z}^{2n}$ we set
\[
\Psi_{\mu_1, \mu_2}^{\lambda, G}(x_1, x_2) = 2^{n \lambda} \psi_{G_1}(2^\lambda x_1 - \mu_1) 2^{n \lambda} \psi_{G_2}(2^\lambda x_2 - \mu_2).
\]
for $(x_1, x_2) \in \mathbb{R}^{2n}$, where $G = (G_1, G_2)$ and $(\lambda, G) \in \mathcal{J}$.

The cancellation of wavelets is manifested in the following result.

**Lemma 10.1.** Let $M$ be a positive integer. Assume that $m \in \mathcal{C}^{M+1}$ is a function on $\mathbb{R}^{2n}$ such that
\[
\sup_{|\alpha| \leq M+1} \|\partial^\alpha m\|_{L^\infty} \leq C_0 < \infty.
\]
Then for $(\lambda, G) \in \mathcal{J}$ and $(\mu_1, \mu_2) \in \mathbb{Z}^{2n}$ we have
\[
|\langle \Psi_{\mu_1, \mu_2}^{\lambda, G}, m \rangle| \leq CC_0 2^{-(M+n+1)\lambda},
\]
provided that $\psi_M$ has $M$ vanishing moments.
This lemma can be easily proved and is essentially a restatement of Lemma 7 in [20]. Note that if \( G = (F, F) \) there is no cancellation, however, there is no decay claimed in (10.1), as \( \lambda = 0 \) in this case.

**Proof of Theorem 8.1.** To prove the theorem we use the product type wavelets introduced. We begin by fixing a large number \( M \) to be determined later, which denotes the number of vanishing moments of the mother wavelet.

For \((\lambda, G) \in \mathcal{J} \) and \( \mu \in \mathbb{Z}^{2n} \) we denote the wavelet coefficient by

\[
b_{\lambda, G}^\mu = \langle \Psi_{\lambda, G}^\mu, m \rangle.
\]

By [51, Theorem 1.64] and by the fact that \( L^q \) coincides with the Triebel-Lizorkin space \( F_{q, 2}^0 \), we obtain

\[
\| m \|_{L^q(\mathbb{R}^{2n})} \approx \left\| \left( \sum_{(\lambda, G) \in \mathcal{J}} \sum_{\mu \in \mathbb{Z}^{2n}} |b_{\mu}^{\lambda, G} 2^\lambda \chi_{Q_{\lambda \mu}}|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^{2n})},
\]

where \( Q_{\lambda \mu} \) is the cube centered at \( 2^{-\lambda} \mu \) with sidelength \( 2^{1-\lambda} \).

Now, let us fix \((\lambda, G) \in \mathcal{J} \). For notational simplicity, we write \( b_{\mu} \) instead of \( b_{\mu}^{\lambda, G} \) in what follows. We also denote by \( \tilde{Q}_{\lambda \mu} \) the cube centered at \( 2^{-\lambda} \mu \) with sidelength \( 2^{-\lambda} \). Noting that these cubes are pairwise disjoint in \( \mu \) (for the fixed value of \( \lambda \)), the equivalence (10.2) yields

\[
\| m \|_{L^q(\mathbb{R}^{2n})} \geq 2^{n\lambda} \left\| \left( \sum_{\mu \in \mathbb{Z}^{2n}} |b_{\mu}|^2 \chi_{\tilde{Q}_{\lambda \mu}} \right)^{1/2} \right\|_{L^q(\mathbb{R}^{2n})}
\]

\[
\geq 2^{n\lambda} \left\| \left( \sum_{\mu \in \mathbb{Z}^{2n}} |b_{\mu}|^2 \chi_{\tilde{Q}_{\lambda \mu}} \right)^{1/2} \right\|_{L^q(\mathbb{R}^{2n})}
\]

\[
= 2^{n\lambda} \left\| \sum_{\mu \in \mathbb{Z}^{2n}} |b_{\mu}| \chi_{\tilde{Q}_{\lambda \mu}} \right\|_{L^q(\mathbb{R}^{2n})}
\]

\[
= 2^{n\lambda(1-\frac{2}{q})} \left( \sum_{\mu \in \mathbb{Z}^{2n}} |b_{\mu}|^q \right)^{\frac{1}{q}}.
\]

Setting \( b = (b_{\mu})_{\mu \in \mathbb{Z}^{2n}} \), the preceding sequence of inequalities yields

\[
\| b \|_{\ell^q} \leq C 2^{-n\lambda(1-\frac{2}{q})} \| m \|_{L^q}.
\]

Also, Lemma 10.1 implies that

\[
\| b \|_{\ell^\infty} \leq CC_0 2^{-\lambda(M+n+1)},
\]

where \( M \) is the number of vanishing moments of \( \psi_M \).
We have an infinite × infinite matrix of wavelet coefficients indexed by \( \mathbb{Z}^{2n} \). To better organize these coefficients, define
\[
U_r = \{(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^n = \mathbb{Z}^{2n} : 2^{-r-1} \|b\|_\ell^\infty < |b_{(k,l)}| \leq 2^{-r} \|b\|_\ell^\infty \},
\]
where \( r \) is a nonnegative integer. Also, we write \( U_r \) as a union of the following two disjoint sets:
\[
U^1_r = \{(k, l) \in U_r : \text{card}\{s : (k, s) \in U_r\} \geq K\};
\]
\[
U^2_r = \{(k, l) \in U_r : \text{card}\{s : (k, s) \in U_r\} < K\},
\]
where \( K \) is a positive number to be determined. Thinking of \( U_r \) as an infinite × infinite matrix with integer entries, in this splitting, we placed in \( U^1_r \) all columns of \( U_r \) that have size greater than or equal to \( K \) and in \( U^2_r \) the remaining ones. We call \( U^1_r \) the long columns of \( U_r \) and \( U^2_r \) the short columns. Let us denote
\[
E = \{k \in \mathbb{Z}^n : (k, l) \in U^1_r \text{ for some } l \in \mathbb{Z}^n\}.
\]
This set is exactly the set of projections of all long columns. Then
\[
(\#E) K \left[2^{-(r+1)} \|b\|_\ell^\infty\right]^q \leq \sum_{(k,l) \in U^1_r} |b_{(k,l)}|^q \leq \|b\|^q_{\ell^q},
\]
and therefore
\[
(10.5) \quad \#E \leq K^{-1} \left[2^{-(r+1)} \|b\|_\ell^\infty\right]^{-q} \|b\|^q_{\ell^q}.
\]

Having broken down the wavelet coefficients in groups we proceed with the analysis of the sums of the decomposition associated with these groups. Given \((k, l) \in \mathbb{Z}^n \times \mathbb{Z}^n\), it follows from the definition of \( \Psi_{(k,l)}^{\lambda,G} \) that \( \Psi_{(k,l)}^{\lambda,G} \) can be written in the tensor product form
\[
\Psi_{(k,l)}^{\lambda,G}(x_1, x_2) = \omega_{1,k}(x_1) \omega_{2,l}(x_2)
\]
and
\[
\|\omega_{1,k}\|_{L^\infty} \approx \|\omega_{2,l}\|_{L^\infty} \approx 2^{\frac{n\lambda}{2}}.
\]
Define
\[
m_r^1 = \sum_{(k,l) \in U^1_r} b_{(k,l)} \Psi_{(k,l)}^{\lambda,G} = \sum_{(k,l) \in U^1_r} b_{(k,l)} \omega_{1,k} \omega_{2,l}.
\]
Let \( \mathcal{F}^{-1} \) denote the inverse Fourier transform. Then
\[
\|T_{m_r^1}(f, g)\|_{L^1} \leq \left\| \sum_{(k,l) \in U^1_r} b_{(k,l)} \mathcal{F}^{-1}(\omega_{1,k}\hat{f}) \mathcal{F}^{-1}(\omega_{2,l}\hat{g}) \right\|_{L^1} \leq \sum_{k \in E} \left\|\omega_{1,k}\hat{f}\right\|_{L^2} \sum_{l:(k,l) \in U^1_r} b_{(k,l)} \left\|\omega_{2,l}\hat{g}\right\|_{L^2}.
\[ \leq C \sum_{k \in E} \left\| \omega_{1,k} \hat{f} \right\|_{L^2} 2^n \lambda^2 2^{-r} \| b \|_{L^2} \| g \|_{L^2} \]

\[ \leq C \left( \sum_{k \in E} 1 \right)^{1/2} \left( \sum_{k \in E} \left\| \omega_{1,k} \hat{f} \right\|_{L^2}^2 \right)^{1/2} \left( \sum_{k \in E} 2^n \lambda^2 2^{-r} \| b \|_{L^2} \| g \|_{L^2} \right)^{1/2} \]

\[ \leq C \left\{ K^{-\frac{1}{2}} \left[ 2^{-(r+1)} \| b \|_{L^2} \right]^{-\frac{1}{2}} \| b \|_{L^2} \left( 2^n \lambda^2 2^{-r} \| b \|_{L^2} \right)^{1/2} \right\} \| f \|_{L^2} \| g \|_{L^2}, \]

where we used estimate (10.5) and the property that the supports of the functions \( \omega_{1,k} \) and \( \omega_{2,l} \) have finite overlap.

Now define

\[ m^{r,2} = \sum_{(k,l) \in U^2} b_{(k,l)} \omega_{1,k} \omega_{2,l}. \]

Then

\[ \| T_{m^{r,2}} (f, g) \|_{L^1} \leq \left\| \sum_{(k,l) \in U^2} b_{(k,l)} F^{-1} (\omega_{1,k} \hat{f}) F^{-1} (\omega_{2,l} \hat{g}) \right\|_{L^1} \]

\[ \leq \sum_{l : \exists k (k,l) \in U^2} \left\| \omega_{2,l} \hat{g} \right\|_{L^2} \left\| \sum_{k : (k,l) \in U^2} b_{(k,l)} \hat{\omega}_{1,k} \hat{f} \right\|_{L^2} \]

\[ \leq \left( \sum_{l \in \mathbb{Z}^n} \left\| \omega_{2,l} \hat{g} \right\|_{L^2}^2 \right)^{1/2} \left( \sum_{l : \exists k (k,l) \in U^2} \left\| \omega_{1,k} \hat{f} \right\|_{L^2}^2 \sum_{l : (k,l) \in U^2} \left| b_{(k,l)} \right|^2 \right)^{1/2} \]

\[ \leq C 2^{\frac{n}{2}} \| g \|_{L^2} 2^{-r} \| b \|_{L^2} \left( \sum_{k \in \mathbb{Z}} \left\| \omega_{1,k} \hat{f} \right\|_{L^2}^2 \right)^{1/2} \]

\[ \leq C 2^{\frac{n}{2}} 2^{-r} \| b \|_{L^2} \left\| \omega_{1,k} \hat{f} \right\|_{L^2} \left\| f \right\|_{L^2} \| g \|_{L^2}. \]

We have now obtained the estimates for an unknown quantity \( K \):

\[ \| T_{m^{r,1}} (f, g) \|_{L^1} \leq C K^{-\frac{1}{2}} \left[ 2^{-(r+1)} \| b \|_{L^2} \right]^{-\frac{1}{2}} \left( \frac{\| b \|_{L^2}^2 2^n \lambda^2 2^{-r} \| b \|_{L^2} \left\| f \right\|_{L^2} \| g \|_{L^2} \right)^{1/2} \]

\[ \| T_{m^{r,2}} (f, g) \|_{L^1} \leq C 2^n \lambda^2 2^{-r} \| b \|_{L^2} \| f \|_{L^2} \| g \|_{L^2}. \]

We choose \( K \) optimally so that the two quantities on the right in (10.6) and (10.7) are equal. The optimal choice of \( K \) is

\[ K = \left( \frac{2^r \| b \|_{L^2}}{\| b \|_{L^2} \left( \frac{\| b \|_{L^2}}{\| b \|_{L^2}} \right)^{1/2}} \right)^{1/2}. \]

This choice of \( K \) yields for

\[ m^r = \sum_{(k,l) \in U_r} b_{(k,l)} \omega_{1,k} \omega_{2,l} = m^{r,1} + m^{r,2}. \]
the estimate
\begin{equation}
\|T_m\|_{L^2 \times L^2 \to L^1} \leq C 2^{n\lambda} 2^{-(1-\frac{q}{4})} \|b\|_{L^\infty}^{1-\frac{q}{4}} \|b\|_{L^q}^{\frac{q}{4}}.
\end{equation}

Using (10.3) and (10.4) in (10.8) we obtain
\begin{equation}
\|T_m\|_{L^2 \times L^2 \to L^1} \leq C C_0^{1-\frac{q}{4}} 2^{n\lambda - \lambda (1-\frac{q}{4})} (M+n+1) + n(\frac{2}{q} - 1)^2 \lambda 2^{-(1-\frac{q}{4})} \|m\|_{L^q}^{\frac{q}{4}}.
\end{equation}
Notice that when \( q < 4 \) we have \( 1 - \frac{q}{4} < 0 \), hence we can sum in \( r \in \mathbb{Z}^+ \). Also,
\begin{equation}
2^{n\lambda - \lambda (1-\frac{q}{4})} (M+n+1) + n(\frac{2}{q} - 1)^2 \lambda = 2^{\lambda \left( \frac{n}{2} - \frac{1}{4} \right) (M+1)}
\end{equation}
and the exponent is negative only when \( M + 1 > \frac{2n}{4-q} \). Thus, if we choose \( M = \lfloor \frac{2n}{4-q} \rfloor \), we can sum first over \( \lambda \in \mathbb{Z}^+ \cup \{0\} \) when \( G \in \{(F,M), (M,M), (M,F)\} \). For \( G = (F,F) \) there is no need to sum over \( \lambda \). This yields (8.2) for any \( G \) and completes the proof of Theorem 8.1. \( \square \)

11. The sharpness of the condition \( q < 4 \)

In this section we discuss optimality of the assumption \( q < 4 \) of Theorem 8.1. The main result is the following.

**Theorem 11.1.** Suppose that \( q \geq 4 \). Then there exists a function \( m \in L^q(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n}) \) such that the associated operator \( T_m \) does not map \( L^2 \times L^2 \) to \( L^1 \).

Theorem 11.1 was proved in [24] in the case \( q > 4 \). The limiting case \( q = 4 \) is discussed in [47]. We do not include the full proof here as it is somewhat technical but we will describe the main ideas needed to reach the conclusion. To further simplify the presentation, we will assume that \( n = 1 \).

We start by studying a randomized variant of the operator \( T_{\sigma N} \) from Section 9. Namely, we fix a (large) positive integer \( K \) and consider the set \( E = \{(j,k) \in \mathbb{N}^2 : j + k \leq K\} \). We observe that \( N := \#E = \frac{K(K-1)}{2} \).

Let \( (r_j(t))_{j=0}^\infty \) denote the sequence of Rademacher functions; see Section 4 for the definition. Assume that \( \phi \) is a smooth function on \( \mathbb{R} \) supported in the interval \([-1/10, 1/10]\) assuming value 1 in \([-1/20, 1/20]\). Let \( S_k \) be the operator given by (9.1). We fix \( t \in [0,1] \) and define the operator \( T_{K,t} \) as
\begin{equation}
T_{K,t}(f,g) = \sum_{(j,k) \in E} r_{j+k}(t) S_j(f) S_k(g) = \sum_{j=1}^{K-1} \sum_{k=1}^{K-j} r_{j+k}(t) S_j(f) S_k(g).
\end{equation}
This operator is associated with the multiplier
\[ m_{K,t}(\xi, \eta) = \sum_{j=1}^{K-1} \sum_{k=1}^{K-j} r_{j+k}(t) \phi(\xi-j)\phi(\eta-k). \]

Let \( \varphi \) be a Schwartz function on \( \mathbb{R} \) whose Fourier transform is supported in the interval \([-1/100, 1/100]\). We define functions \( f \) and \( g \) on \( \mathbb{R} \) in terms of their Fourier transform by
\begin{align*}
\hat{f}(\xi) &= K^{-\frac{1}{2}} \sum_{j=1}^{K} \hat{\varphi}(\xi-j) \\
\hat{g}(\eta) &= K^{-\frac{1}{2}} \sum_{k=1}^{K} \hat{\varphi}(\eta-k).
\end{align*}

Then both \( f \) and \( g \) are Schwartz functions whose \( L^2 \)-norms are bounded by a constant independent of \( K \). We observe that
\[
\begin{align*}
T_{m_{K,t}}(f, g)(x) &= K^{-1} \sum_{j=1}^{K-1} \sum_{k=1}^{K-j} r_{j+k}(t)(\varphi(x))^2 e^{2\pi i x(j+k)} \\
&= K^{-1} \sum_{l=2}^{K} (l-1)r_l(t)e^{2\pi i lx}(\varphi(x))^2.
\end{align*}
\]

Using Fubini’s theorem and Khintchine’s inequality (4.7), we obtain
\[
\begin{align*}
\int_0^1 \|T_{m_{K,t}}(f, g)\|_{L^1} \, dt &= K^{-1} \int_\mathbb{R} \int_0^1 \left| \sum_{l=2}^{K} (l-1)r_l(t)e^{2\pi i lx}(\varphi(x))^2 \right| \, dt \, dx \\
&\approx K^{-1} \int_\mathbb{R} \left( \sum_{l=2}^{K} (l-1)^2 |\varphi(x)|^4 \right)^{\frac{1}{2}} \, dx \\
&\approx K^{\frac{1}{2}} \approx N^{\frac{1}{4}}.
\end{align*}
\]

This indicates the sharpness of the estimate in Section 9. Let us now observe that this calculation is also relevant for proving Theorem 11.1. We fix \( q \in [1, \infty) \) and consider the function
\[ M_{K,t} = K^{-\frac{2}{q}} m_{K,t}. \]

Then the \( L^q \) norm of \( M_{K,t} \) is bounded uniformly in \( K \) and \( t \), and the same is true for the \( L^\infty \) norms of all partial derivatives of \( M_{K,t} \). The calculation above yields
\[
\int_0^1 \|T_{M_{K,t}}(f, g)\|_{L^1} \, dt \approx K^{\frac{1}{2}} - \frac{2}{q}.
\]
We see that the right-hand side approaches infinity as $K \to \infty$ if $q > 4$, which implies that $T_m$ is not bounded from $L^2 \times L^2$ to $L^1$ with a constant depending only on the $L^q$ norm of $m$ and on the $L^\infty$ norms of its derivatives in this case. A modification of this example can then be used to construct a single function $m \in L^q(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ for which the associated operator $T_m$ does not map $L^2 \times L^2$ to $L^1$.

References


