SHARP HARDY SPACE ESTIMATES FOR MULTIPLIERS

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Abstract. We provide an improvement of Calderón and Torchinsky’s version [5] of the Hörmander multiplier theorem on Hardy spaces $H^p (0 < p < \infty)$, by replacing the Sobolev space $L^2_s(A_0)$ by the Lorentz-Sobolev space $L^r_{\tau(s,p),\min(1,p)}(A_0)$, where $\tau(s,p) = \frac{n}{s-\frac{n}{\min(1,p)} - n}$ and $A_0$ is the annulus $\{ \xi \in \mathbb{R}^n : 1/2 < |\xi| < 2 \}$. Our theorem also extends that of Grafakos and Slavíková [10] to the range $0 < p \leq 1$. Our result is sharp in the sense that the preceding Lorentz-Sobolev space cannot be replaced by a smaller Lorentz-Sobolev space $L^{r,q}_{\tau(s,p)}(A_0)$ with $r < \tau(s,p)$ or $q > \min(1,p)$.

1. Introduction

Let $S(\mathbb{R}^n)$ denote the Schwartz space and $S'(\mathbb{R}^n)$ the space of tempered distributions on $\mathbb{R}^n$. For the Fourier transform of $f \in S(\mathbb{R}^n)$ we use the definition $\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$ and denote by $f^\vee(\xi) := \hat{f}(-\xi)$ the inverse Fourier transform of $f$. We also extend these transforms to the space of tempered distributions.

Given a bounded function $\sigma$ on $\mathbb{R}^n$, the multiplier operator $T_\sigma$ is defined as

$$T_\sigma f(x) := \int_{\mathbb{R}^n} \sigma(\xi) \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi$$

for $f \in S(\mathbb{R}^n)$, where $\langle x, \xi \rangle$ is the dot product of $x$ and $\xi$ in $\mathbb{R}^n$. The classical Mikhlin multiplier theorem [15] states that if a function $\sigma$, defined on $\mathbb{R}^n$, satisfies

$$|\partial^\alpha \sigma(\xi)| \lesssim |\xi|^{-|\alpha|}, \quad |\alpha| \leq \left[ \frac{n}{2} \right] + 1,$$

then the operator $T_\sigma$ admits a bounded extension in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In [13] Hörmander sharpened Mikhlin’s result, using the weaker condition

$$\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^2_s(A_0)} < \infty$$

for $s > n/2$, where $L^2_s$ denotes the standard $L^2$-based Sobolev space on $\mathbb{R}^n$, $\Psi$ is a Schwartz function on $\mathbb{R}^n$ whose Fourier transform is supported in the annulus $A_0 = \{ \xi : 1/2 < |\xi| < 2 \}$ and satisfies $\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j} \xi) = 1$, $\xi \neq 0$. Calderón and Torchinsky [5] proved that if (1.1) holds for $s > n/p - n/2$, then $\sigma$ is a Fourier multiplier of Hardy space $H^p(\mathbb{R}^n)$ for $0 < p \leq 1$. A different proof was given by Taibleson and Weiss [22]. It turns out that the condition $s > n/\min(1,p) - n/2$ is optimal for boundedness to hold and it is natural to ask whether (1.1) can be weakened by replacing $L^2_s(A_0)$ by other spaces. Baernstein and Sawyer [1] obtained endpoint $H^p(\mathbb{R}^n)$ estimates by using

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Herz space conditions for \( (\sigma(2^j \cdot \hat{\Psi})^\vee) \). An endpoint \( H^1 - L^{1,2} \) estimate involving Besov space was given by Seeger [17, 18] and these estimates were improved and extended to Triebel-Lizorkin spaces by Seeger [19] and Park [16]. Grafakos, He, Honzík, and Nguyen [11] replaced \( L^2(\mathbb{R}^n), s > n/2 \) in (1.1) by \( L^r_s(\mathbb{R}^n), s > n/r \), while Grafakos and Slavíková [10] recently improved this, replacing (1.1) by
\[
\sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot \hat{\Psi}) \|_{L^{n/s,1}_s(A_0)} < \infty
\]
where \( L^{n/s,1}_s \) is a Lorentz-type Sobolev space (defined in (1.2)).

Before stating our results, we recall the definition of Lorentz spaces \( L^{p,q}(\mathbb{R}^n) \) and Lorentz-Sobolev spaces \( L^{p,q}_s(\mathbb{R}^n) \). For any measurable function \( f \) defined on \( \mathbb{R}^n \), the decreasing rearrangement of \( f \) is defined by
\[
f^*(t) := \inf \{ s > 0 : d_f(s) \leq t \}, \quad t > 0
\]
where \( d_f(s) := |\{ x \in \mathbb{R}^n : |f(x)| > s \}| \). Here we adopt the convention that the infimum of the empty set is \( \infty \). Then for \( 0 < p, q \leq \infty \) we define
\[
\| f \|_{L^{p,q}(\mathbb{R}^n)} := \begin{cases} \left( \int_0^\infty \left( t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q}, & q < \infty \\ \sup_{t > 0} t^{1/p} f^*(t), & q = \infty. \end{cases}
\]
The set of all \( f \) with \( \| f \|_{L^{p,q}(\mathbb{R}^n)} < \infty \) is called the Lorentz space and is denoted by \( L^{p,q}(\mathbb{R}^n) \). For \( s > 0 \) let \( (I - \Delta)^{s/2} \) be the inhomogeneous fractional Laplacian operator, defined by
\[
(I - \Delta)^{s/2} f := ((1 + 4\pi^2 \cdot |x|^{2})^{s/2} \hat{f})^\vee.
\]
Then for \( 0 < p, q \leq \infty \) and \( s > 0 \) let
\[
\| f \|_{L^{p,q}_s(\mathbb{R}^n)} := \| (I - \Delta)^{s/2} f \|_{L^{p,q}(\mathbb{R}^n)}.
\]

**Theorem A.** [10] Let \( 1 < p < \infty \) and \( 0 < s < n \) satisfy
\[
s > \left| n/p - n/2 \right|.
\]
Then there exists \( C > 0 \) such that
\[
\| T_{\sigma} f \|_{L^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot \hat{\Psi}) \|_{L^{n/s,1}_s(\mathbb{R}^n)} \| f \|_{L^p(\mathbb{R}^n)}.
\]

Moreover, a counterexample showing that condition (1.3) is optimal can be found in Slavíková [21]; this means that \( L^p \) boundedness could fail on the line \( |n/p - n/2| = s \).

The purpose of this paper is to extend Theorem A to Hardy spaces \( H^p(\mathbb{R}^n) \) for \( 0 < p < \infty \). Let \( \Phi \) be a Schwartz function satisfying \( \int_{\mathbb{R}^n} \Phi(x) dx = 1 \) and \( \text{Supp}(\Phi) \subset \{ x \in \mathbb{R}^n : |x| \leq 2 \} \), and \( \Phi_k := 2^{kn} \Phi(2^k \cdot) \). We define \( H^p \) to be the collection of all tempered distributions \( f \) satisfying
\[
\| f \|_{H^p(\mathbb{R}^n)} := \| \sup_{k \in \mathbb{Z}} |\Phi_k \ast f| \|_{L^p(\mathbb{R}^n)} < \infty.
\]
Let  \( \tau^{(s,p)} := \frac{n}{s - (n/ \min(1,p) - n)} \).

One of the main results is

**Theorem 1.1.** Let \( 0 < p < \infty \) and \( 0 < s < n/ \min(1,p) \) satisfy (1.3). Then there exists \( C > 0 \) such that

\[
\|T_{\sigma}f\|_{H^p(\mathbb{R}^n)} \leq C \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^{\tau^{(s,p)},\min(1,p)}(\mathbb{R}^n)} \|f\|_{H^p(\mathbb{R}^n)}.
\]

The above theorem coincides with Theorem A if \( 1 < p < \infty \) because \( H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \), and so we mainly deal with the case \( 0 < p \leq 1 \) in the paper. However, a complex interpolation argument between \( H^1 \) and \( L^2 \)-boundedness yields the result for \( 1 < p < 2 \); this recovers Theorem A by a duality argument, as our proof for \( 0 < p \leq 1 \) is in fact independent of that of Theorem A. We will give a sketch of this in the appendix. Actually the construction of analytic family of operators and interpolation techniques are very similar to those used in [10].

**Remark.** We point out that Theorem 1.1 could be obtained as a consequence of the results of Baernstein and Sawyer [1, Corollary 1 (Chapter 3)] combined with the recent generalization of the Franke-Jawerth embedding theorem for Triebel-Lizorkin-Lorentz spaces of Seeger and Trebels [20]. In the sequel we provide a self-contained proof based on the atomic decomposition of Hardy spaces.

We now turn our attention to the sharpness of Theorem 1.1. We point out that the example of Slavíková [21] is still applicable to the case \( 0 < p \leq 1 \) with the dilation property \( \|f(e^j \cdot)\|_{H^p(\mathbb{R}^n)} = \epsilon^{-n/p}\|f\|_{H^p(\mathbb{R}^n)} \), and therefore (1.3) is sharp in Theorem 1.1. We now consider the optimality of different parameters. Note that for \( 0 < r_1 < r_2 < \infty \) and \( 0 < q_1, q_2 \leq \infty \)

\[
\|\sigma(2^j \cdot) \hat{\Psi}\|_{L^{r_1,q_1}(\mathbb{R}^n)} \lesssim \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^{r_2,q_2}(\mathbb{R}^n)} \text{ uniformly in } j,
\]

which follows from the Hölder inequality with even integers \( s \), complex interpolation technique, and a proper embedding theorem. Moreover, if \( q_1 \geq q_2 \), then the embedding \( L^{r_2,q_2}(\mathbb{R}^n) \hookrightarrow L^{r_1,q_1}(\mathbb{R}^n) \) yields that

\[
\|\sigma(2^j \cdot) \hat{\Psi}\|_{L^{r_1,q_1}(\mathbb{R}^n)} \lesssim \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^{r_2,q_2}(\mathbb{R}^n)} \text{ uniformly in } j.
\]

Consequently, we may replace \( L^{\tau^{(s,p)},\min(1,p)}(\mathbb{R}^n) \) in Theorem 1.1 by \( L^{r,q}(\mathbb{R}^n) \) for \( r > \tau^{(s,p)} \) and \( 0 < q \leq \infty \), or by \( L^{\tau^{(s,p)},q}(\mathbb{R}^n) \) for \( 0 < q < \min(1,p) \).

The second main result of this paper is the sharpness of the parameters \( \tau^{(s,p)} \) and \( \min(1,p) \). That is, Theorem 1.1 is sharp in the sense that \( \tau^{(s,p)} \) cannot be replaced by any smaller number \( r \), and if \( r = \tau^{(s,p)} \), then \( \min(1,p) \) cannot be replaced by any larger number \( q \).

**Theorem 1.2.** Let \( 0 < p < \infty \) and \( |n/p - n/2| < s < n/ \min(1,p) \).

1. For any \( 0 < r < \tau^{(s,p)} \) and \( 0 < q \leq \infty \), there exists a function \( \sigma \) that satisfies

\[
\sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^{r,q}(\mathbb{R}^n)} < \infty.
\]
such that $T_\sigma$ is unbounded on $H^p(\mathbb{R}^n)$.

(2) For any $q > \min(1, p)$, there exists a function $\sigma$ that satisfies

$$\sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \hat{\Psi} \right\|_{L^{(s,p),q}_s(\mathbb{R}^n)} < \infty$$

such that $T_\sigma$ is unbounded on $H^p(\mathbb{R}^n)$.

The paper is organized as follows. Section 2 is dedicated to preliminaries, mostly extensions of inequalities in Lebesgue spaces to Lorentz spaces thanks to a real interpolation technique. We address the case $0 < p \leq 1$ of Theorem 1.1 in Section 3 and the proof of Theorem 1.2 is given in Section 4. In the appendix, a complex interpolation method is discussed whose purpose is to establish the $L^p$-boundedness for $1 < p < 2$.

2. Preliminaries

The Lorentz spaces are generalization of Lebesgue spaces, which occur as intermediate spaces for the real interpolation, so called $K$-method. For $0 < p, p_0, p_1 < \infty$, $0 < r \leq \infty$, and $0 < \theta < 1$ satisfying $p_0 \neq p_1$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$,

$$(L^{p_0}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n))_{\theta,r} = L^{p,r}(\mathbb{R}^n).$$

This remains valid for vector-valued spaces. For $0 < p, p_0, p_1 < \infty$, $0 < q, r \leq \infty$, and $0 < \theta < 1$ satisfying $p_0 \neq p_1$ and $1/p = (1 - \theta)/p_0 + \theta/p_1$,

$$(L^{p_0}(\ell^q), L^{p_1}(\ell^q))_{\theta,r} = L^{p,r}(\ell^q), \quad (\ell^q(L^{p_0}), \ell^q(L^{p_1}))_{\theta,r} = \ell^q(L^{p,r}).$$

We remark that $((L^{p_0}(\ell^{q_0}), L^{p_1}(\ell^{q_1}))_{\theta,r} \neq L^{p,r}(\ell^q), \ell^q(L^{p_0}), \ell^q(L^{p_1}))_{\theta,r} \neq \ell^q(L^{p,r})$ for $q_0 \neq q_1$ with $1/q = (1 - \theta)/q_0 + \theta/q_1$. See [2, 3, 6, 7] for more details.

Then many inequalities in Lebesgue spaces can be extended to Lorentz spaces from the following real interpolation method, which appears in [2, 3, 7, 12].

**Proposition B.** Let $\mathcal{A}$ and $\mathcal{B}$ be two topological vector spaces. Suppose $(A_0, A_1)$ and $(B_0, B_1)$ be couples of quasi-normed spaces continuously embedded into $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $0 < \theta < 1$ and $0 < r \leq \infty$. If $T$ is a linear operator such that

$$T : A_0 \to B_0, \quad T : A_1 \to B_1,$$

with the quasi-norms $M_0$ and $M_1$, respectively, then

$$T : (A_0, A_1)_{\theta,r} \to (B_0, B_1)_{\theta,r}$$

is also continuous, and for its quasi-norm we have

$$\|T\|_{(A_0, A_1)_{\theta,r} \to (B_0, B_1)_{\theta,r}} \leq M_0^{1-\theta} M_1^\theta.$$

As applications of Proposition B, we shall extend Young inequality, Hausdorff-Young inequality, Minkowski inequality, and Kato-Ponce type inequality into Lorentz spaces.
Lemma 2.1. Let $1 < p \leq r < \infty$, $1 \leq q < r$, and $0 < t \leq \infty$ satisfy $1/r + 1 = 1/p + 1/q$. Then
\[ \|f \ast g\|_{L^{p',t}(\mathbb{R}^n)} \leq \|f\|_{L^{p,t}(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} \]
for all $f,g \in S(\mathbb{R}^n)$.

Proof. For a fixed $g \in S(\mathbb{R}^n)$, we define the linear operator $T_g$ by
\[ T_g f := f \ast g. \]
Choose $r_1$, $\theta$, and $p_1$ such that $r < r_1 < \infty$, $0 < \theta < 1$, $p < p_1 < \infty$, $1/r = (1-\theta)/q + \theta/r_1$, and $1/r_1 + 1 = 1/p_1 + 1/q$. Then note that $1/p = 1 - \theta + \theta/p_1$. By using Young inequality, we obtain that
\[ \|T_g f\|_{L^q(\mathbb{R}^n)} \leq \|g\|_{L^q} \|f\|_{L^1(\mathbb{R}^n)} \]
and
\[ \|T_g f\|_{L^{p_1}(\mathbb{R}^n)} \leq \|g\|_{L^{p_1}} \|f\|_{L^{p_1}(\mathbb{R}^n)}. \]
Then Proposition B with (2.1) completes the proof.

Lemma 2.2. Let $2 < p < \infty$ and $0 < r \leq \infty$. Then
\[ \|\hat{f}\|_{L^{p,r}(\mathbb{R}^n)} \leq \|f\|_{L^{p',r}(\mathbb{R}^n)} \]
where $1/p + 1/p' = 1$.

Proof. It follows immediately from Hausdorff-Young inequality and Proposition B with (2.1).

Lemma 2.3. Let $1 < p < \infty$, $0 < r \leq \infty$, and $s > 0$. For any $\vartheta \in S(\mathbb{R}^n)$, we have
\[ \|\vartheta \cdot f\|_{L^{p,r}(\mathbb{R}^n)} \lesssim_{n,s,p,r,\vartheta} \|f\|_{L^{p',r}(\mathbb{R}^n)}. \]

Proof. Pick $p_0$, $p_1$ satisfying $1 < p_0 < p < p_1 < \infty$ and let $T$ be the linear operator defined by
\[ Tf := (I - \Delta)^{s/2} (\vartheta \cdot (I - \Delta)^{-s/2} f). \]
Then we apply the Kato-Ponce inequality [14] to obtain
\[ \|Tf\|_{L^{p_j}} \lesssim \|f\|_{L^{p_j}} \quad \text{for} \quad j = 0, 1. \]
Then (2.3) follows from Proposition B and (2.1).

Lemma 2.4. Let $1 \leq q < p < \infty$ and $0 < r \leq \infty$. Then
\[ \left\| \left( \sum_{k \in \mathbb{Z}} |f_k|^q \right)^{1/q} \right\|_{L^{p,r}(\mathbb{R}^n)} \lesssim \left( \sum_{k \in \mathbb{Z}} \|f_k\|_{L^{p,r}(\mathbb{R}^n)}^q \right)^{1/q} \]

Proof. We select $p_1 > 0$ and $0 < \theta < 1$ so that $p < p_1 < \infty$ and $1/p = (1-\theta)/p_1 + \theta/q$. Using Minkowski inequality, $\| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^{p_1}(\mathbb{R}^n)} \lesssim \| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^{p_1}(\mathbb{R}^n)}$ and we interpolate this with $\| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^{q}(\mathbb{R}^n)} = \| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^{q}(\mathbb{R}^n)}$ to obtain
\[ \| \{ f_k \}_{k \in \mathbb{Z}} \|_{(L^{p_1}(\mathbb{R}^n),L^{q}(\mathbb{R}^n))_{\theta,r}} \lesssim \| \{ f_k \}_{k \in \mathbb{Z}} \|_{(L^{q}(\mathbb{R}^n))_{\theta,r}}. \]
Then the proof is completed in view of (2.2).
The next ingredient we need is Hölder’s inequality in Lorentz spaces, which is an immediate consequence of the Hardy-Littlewood inequality
\[
\int_{\mathbb{R}^n} |f(x)g(x)| \,dx \leq \int_0^\infty f^*(t)g^*(t) \,dt
\]
and Hölder’s inequality for Lebesgue spaces.

**Lemma 2.5.** Let \(1 < p < \infty\) and \(1 \leq q \leq \infty\). Then
\[
\int_{\mathbb{R}^n} |f(x)g(x)| \,dx \leq \|f\|_{L^{p,q}(\mathbb{R}^n)} \|g\|_{L^{p',q'}(\mathbb{R}^n)}
\]
where \(1/p + 1/p' = 1/q + 1/q' = 1\).

The following Lorentz space variant of the Sobolev embedding theorem can be easily obtained from the classical Sobolev embedding theorem combined with the Marcinkiewicz interpolation theorem; the proof is omitted.

**Lemma 2.6.** Let \(s_0, s_1 \in \mathbb{R}\), \(1 < p_0, p_1 < \infty\), and \(0 < r_0, r_1 \leq \infty\). Then the embedding
\[L^{p_0,r_0}_{s_0}(\mathbb{R}^n) \hookrightarrow L^{p_1,r_1}_{s_1}(\mathbb{R}^n)\]
holds if \(p_0 = p_1\), \(s_0 \geq s_1\), \(r_0 \leq r_1\), or if \(s_0 - s_1 = n/p_0 - n/p_1 > 0\).

We remark that a generalization of the preceding lemma can be found in the recent work of Seeger and Trebels [20].

Finally, we describe the behavior of decreasing rearrangement of radial functions.

**Lemma 2.7.** Suppose \(f\) is a radial function with \(f(x) = g(|x|)\) for \(x \in \mathbb{R}^n\). Then
\[f^*(t) = g^*((t/\Omega_n)^{1/n})\]
where \(\Omega_n\) stands for the volume of the unit ball in \(\mathbb{R}^n\).

**Proof.** We observe that
\[
d_f(s) = \left| \{x \in \mathbb{R}^n : |f(x)| > s\} \right| = \left| \{r\theta \in \mathbb{R}^n : |g(r)| > s, \theta \in \mathbb{S}^{n-1}\} \right|
= \Omega_n \left| \{r > 0 : |g(r)| > s\} \right|^n
= \Omega_n (d_g(s))^n
\]
and this proves that
\[
f^*(t) = \inf \{s > 0 : d_f(s) \leq t\} = \inf \{s > 0 : \Omega_n (d_g(s))^n \leq t\}
= \inf \{s > 0 : d_g(s) \leq (t/\Omega_n)^{1/n}\}
= g^*((t/\Omega_n)^{1/n}).
\]
\(\Box\)
3. Proof of Theorem 1.1

The set of Schwartz functions whose Fourier transform is compactly supported away from the origin is dense in $H^p(\mathbb{R}^n)$; this is a consequence of Littlewood-Paley theory for $H^p$ as one can approximate $f \in H^p$ by

$$f^{(N)} := \sum_{k=-N}^{N} 2^{kn} \Psi(2^k \cdot) * f \to f \quad \text{in } H^p(\mathbb{R}^n) \quad \text{as } N \to \infty.$$ 

See [24] for more details. Thus we may work with such Schwartz functions. Let $f$ be a Schwartz function with compact support away from the origin in frequency space and suppose $\sigma \in L^\infty(\mathbb{R}^n)$ satisfies

$$\sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \widehat{\Psi} \|_{L^s(\mathbb{R}^n)} < \infty.$$ 

Let $\Lambda \in S(\mathbb{R}^n)$ have the properties that $\text{Supp}(\Lambda) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 1 \}$ and $\int_{\mathbb{R}^n} \Lambda(\xi) d\xi = 1$. For $0 < \epsilon < 1/100$, we introduce

$$\sigma^\epsilon(\xi) := \sum_{j \in \mathbb{Z}} (\sigma \widehat{\Psi}(\cdot / 2^j)) * \Lambda^j \epsilon(\xi)$$

where $\Lambda^j \epsilon := (2^j \epsilon)^{-n} \Lambda(\cdot / 2^j \epsilon)$. Then since $\widehat{f}$ has compact support away from the origin,

$$T_{\sigma^\epsilon} f = \sum_{j \in \mathbb{Z}} \left( (\sigma \widehat{\Psi}(\cdot / 2^j)) * \Lambda^j \epsilon \right) \widehat{f}$$

is a finite sum and thus, using the argument of approximation of identity, for each $k \in \mathbb{Z}$

$$\lim_{\epsilon \to 0} \Phi_k * (T_{\sigma^\epsilon} f)(x) = \Phi_k * (T_{\sigma} f)(x).$$

This proves that

$$\| T_{\sigma^\epsilon} f \|_{H^p(\mathbb{R}^n)} \leq \lim \inf_{\epsilon \to 0} \sup_{k \in \mathbb{Z}} \| \Phi_k * (T_{\sigma^\epsilon} f) \|_{L^p(\mathbb{R}^n)} \leq \lim \inf_{\epsilon \to 0} \| T_{\sigma^\epsilon} f \|_{H^p(\mathbb{R}^n)}$$

where we applied Fatou's lemma in the last inequality. Therefore, it suffices to show that

$$\| T_{\sigma^\epsilon} f \|_{H^p(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \widehat{\Psi} \|_{L^s(\mathbb{R}^n)} \| f \|_{H^p(\mathbb{R}^n)}, \quad \text{uniformly in } \epsilon.$$ 

Now there exist a sequence of $L^\infty$-atoms $\{a_l\}^\infty_{l=1}$ for $H^p(\mathbb{R}^n)$, and a sequence of scalars $\{\lambda_l\}^\infty_{l=1}$ so that

$$f = \sum_{l=1}^{\infty} \lambda_l a_l \quad \text{in } S'$$

and

$$\left( \sum_{l=1}^{\infty} |\lambda_l|^p \right)^{1/p} \approx \| f \|_{H^p(\mathbb{R}^n)},$$
where $L^\infty$-atom $a_l$ for $H^p(\mathbb{R}^n)$ means that there exists a cube $Q_l$ such that $a_l$ is supported in $Q_l$, $|a_l| \leq |Q_l|^{-1/p}$, and $\int_{\mathbb{R}^n} x^\gamma a_l(x) dx = 0$ for all multi-indices $\gamma$ with $|\gamma| \leq [n/p - n]$.

We note that $T_{\sigma^*}$ maps $\mathcal{S}(\mathbb{R}^n)$ to itself, which implies that $T_{\sigma^*}$ is well-defined on $\mathcal{S}'(\mathbb{R}^n)$ using duality argument, and actually, $T_{\sigma^*} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. This yields that

$$T_{\sigma^*}f = \sum_{l=1}^{\infty} \lambda_l(T_{\sigma^*}a_l)$$

in the sense of tempered distribution.

Hence we have

$$\|T_{\sigma^*}f\|_{H^p(\mathbb{R}^n)} \leq \left( \sum_{l=1}^{\infty} |\lambda_l|^p \|T_{\sigma^*}a_l\|_{H^p(\mathbb{R}^n)}^p \right)^{1/p},$$

using subadditive property of $\|\cdot\|_{H^p(\mathbb{R}^n)}^p$.

Moreover, due to support assumptions and dilations, for each $j \in \mathbb{Z}$, we have

$$\sigma^*(2^j \xi) \hat{\Psi}(\xi) = \sum_{l=j-2}^{j+2} (\sigma \hat{\Psi}(\cdot/2^l)) * \Lambda^l(2^j \xi) \hat{\Psi}(\xi) = \sum_{l=-2}^{2} (\sigma(2^j \cdot) \hat{\Psi}(\cdot/2^l)) * \Lambda^l(\xi) \hat{\Psi}(\xi),$$

from which it follows

$$\sup_{j \in \mathbb{Z}} \left\| (\sigma^*(2^j \cdot) \hat{\Psi}) \right\|_{L^s(\mathbb{R}^n),p(\mathbb{R}^n)} \lesssim \sum_{l=-2}^{2} \sup_{j \in \mathbb{Z}} \left\| (I - \Delta)^{s/2} \left((\sigma(2^j \cdot) \hat{\Psi}(\cdot/2^l)) * \Lambda^l\right) \right\|_{L^s(\mathbb{R}^n)}$$

$$\lesssim \sum_{l=-2}^{2} \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \hat{\Psi}(\cdot/2^l) \right\|_{L^s(\mathbb{R}^n),p(\mathbb{R}^n)} \lesssim \sum_{l=-2}^{2} C_l \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j+1 \cdot) \hat{\Psi} \right\|_{L^s(\mathbb{R}^n),p(\mathbb{R}^n)}$$

uniformly in $\epsilon$; here we applied Lemmas 2.3 and 2.1 combined with the fact that $\|\Lambda^l\|_{L^1(\mathbb{R}^n)} = \|\Lambda\|_{L^1(\mathbb{R}^n)}$.

Therefore, the proof of (3.1) is reduced to the following proposition.

**Proposition 3.1.** Let $0 < p \leq 1$ and $a$ be a $H^p$-atom, associated with a cube $Q$ in $\mathbb{R}^n$. Then we have

$$\|T_{\sigma^*}a\|_{H^p(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \hat{\Psi} \right\|_{L^s(\mathbb{R}^n),p(\mathbb{R}^n)},$$

where the constant in the inequality is independent of $\sigma$ and $a$.

**Proof.** Introducing the function $\Theta$ satisfying $\hat{\Theta}(\xi) := \hat{\Psi}(\xi/2) + \hat{\Psi}(\xi) + \hat{\Psi}(2\xi)$ so that $\hat{\Theta} = 1$ on the support of $\hat{\Psi}$, let $L_j$ and $L_j^0$ be the Littlewood-Paley operators associated with $\Psi$ and $\Theta$, respectively. Let $Q^*$ and $Q^{**}$ denote the concentric dilates of $Q$.
with side length $10l(Q)$ and $100l(Q)$, respectively. Then we write
\[ \| T_\sigma a \|_{H^p(\mathbb{R}^n)} \approx \left\| \left( \sum_{j \in \mathbb{Z}} |L_j T_\sigma a|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \]
\[ \lesssim_p \| \left( \sum_{j \in \mathbb{Z}} |L_j T_\sigma a|^2 \right)^{1/2} \|_{L^p(Q^{**})} + \| \left( \sum_{j \in \mathbb{Z}} |L_j T_\sigma a|^2 \right)^{1/2} \|_{L^p((Q^{**})^c)}. \]
In view of Hölder's inequality, the first part is controlled by
\[ |Q^{**}|^{1/p-1/2} \| \left( \sum_{j \in \mathbb{Z}} |L_j T_\sigma a|^2 \right)^{1/2} \|_{L^2(\mathbb{R}^n)} \lesssim_n |Q|^{1/p-1/2} \| T_\sigma a \|_{L^2(\mathbb{R}^n)} \]
and we see that
\[ \| T_\sigma a \|_{L^2(\mathbb{R}^n)} \leq \| \sigma \|_{L^\infty(\mathbb{R}^n)} \| a \|_{L^2(\mathbb{R}^n)} \leq \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \hat{\Psi} \|_{L^\infty(\mathbb{R}^n)} |Q|^{-(1/p-1/2)}. \]
Now using Lemma 2.5, 2.2, and 2.6 with $1 < \tau^{(s,p)} < 2$, we obtain
\[ \| \sigma(2^j \cdot) \hat{\Psi} \|_{L^\infty(\mathbb{R}^n)} \leq \| \sigma(2^j \cdot) \hat{\Psi} \|_{L^1(\mathbb{R}^n)} \]
\[ \lesssim \| (1 + 4 \pi^2 |\cdot|^2)^{(s-(n/p-n))/2} (\sigma(2^j \cdot) \hat{\Psi}) \|_{L^{(s,p)'}(\mathbb{R}^n)} \]
\[ \leq \| \sigma(2^j \cdot) \hat{\Psi} \|_{L^{(s,p),1}_{s-(n/p-n)}(\mathbb{R}^n)} \lesssim \| \sigma(2^j \cdot) \hat{\Psi} \|_{L^{s,p}_{s-(s,p)-}(\mathbb{R}^n)}, \]
which finishes the proof of
\[ \left\| \left( \sum_{j \in \mathbb{Z}} |L_j T_\sigma a|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \hat{\Psi} \|_{L^{s,p}_{s-(s,p)-}(\mathbb{R}^n)}. \]

To verify (3.2)
\[ \left\| \left( \sum_{j \in \mathbb{Z}} |L_j T_\sigma a|^2 \right)^{1/2} \right\|_{L^p((Q^{**})^c)} \lesssim \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \hat{\Psi} \|_{L^{s,p}_{s-(s,p)-}(\mathbb{R}^n)}, \]
we notice that $L_j T_\sigma a(x)$ can be written as $(\sigma \hat{\Psi}(\cdot/2^j))^\vee \ast (L_j^\theta a)(x)$. We decompose the left-hand side of (3.2) to
\[ I := \left\| \left( \sum_{j:2^j l(Q) < 1} |\sigma \hat{\Psi}(\cdot/2^j)|^\vee \ast (L_j^\theta a)|^2 \right)^{1/2} \right\|_{L^p((Q^{**})^c)} \]
and
\[ J := \left\| \left( \sum_{j:2^j l(Q) \geq 1} |\sigma \hat{\Psi}(\cdot/2^j)|^\vee \ast (L_j^\theta a)|^2 \right)^{1/2} \right\|_{L^p((Q^{**})^c)} \]
In view of the embedding $\ell^p \hookrightarrow \ell^2$
\[ I \leq \left( \sum_{j:2^j l(Q) < 1} \| (\sigma \hat{\Psi}(\cdot/2^j))^\vee \ast (L_j^\theta a) \|^p_{L^p(\mathbb{R}^n)} \right)^{1/p} \]
and Bernstein’s inequality, we obtain
\[ \| (\sigma \hat{\Psi}(\cdot/2^j))^\vee \ast (L_j^\theta a) \|_{L^p(\mathbb{R}^n)} \lesssim 2^{jn(1/p-1)} \| (\sigma \hat{\Psi}(\cdot/2^j))^\vee \|_{L^p(\mathbb{R}^n)} \| L_j^\theta a \|_{L^p(\mathbb{R}^n)}. \]
Using dilation, Lemma 2.5 and 2.2, we have
\[
2^{jn(1/p-1)} \|(\sigma \tilde{\Psi} \cdot /2^j)\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |(\sigma(2^j \cdot) \tilde{\Psi})^{\vee}(x)|^p dx \right)^{1/p} \leq \left\| \left(1 + 4\pi^2 |.|^2 \right)^{s/2} (\sigma(2^j \cdot) \tilde{\Psi})^{\vee} \right\|_{L^{(n/(sp))',\mathbb{R}^n}}^{1/p} = \left\| \left(1 + 4\pi^2 |.|^2 \right)^{s/2} (\sigma(2^j \cdot) \tilde{\Psi})^{\vee} \right\|_{L^{(n/(sp))',\mathbb{R}^n}}^{1/p}
\]
(3.3)

since \(2 < p(n/(sp))' < \infty\) and \(\tau^{(s,p)} = (p(n/(sp))')'\). Moreover, for any \(M > 0\)

\[
|\mathcal{L}^\Theta_j a(x)| \lesssim_M |Q|^{-1/p} (2^j l(Q))^{\frac{n}{(n-p) + 1}} \frac{2^{jn}}{(1 + 2^j |x - c_Q|)^M}
\]

using standard arguments in [9, Appendix B] with \(2^j l(Q) < 1\) and the fact that

\[
|a(x)| \lesssim_{n,M} |Q|^{-1/p} \frac{1}{(1 + |x - c_Q|/l(Q))^M}, \quad \int_{\mathbb{R}^n} x^n a(x) dx = 0 \quad \text{for} \ |\alpha| \leq [n/p-n],
\]

where \(c_Q\) denotes the center of \(Q\). Selecting \(M > n/p\), we have

\[
\|\mathcal{L}_j a\|_{L^p} \lesssim (2^j l(Q))^{\frac{n}{(n-p) + 1}}
\]

and thus

\[
\mathcal{I} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \tilde{\Psi}\|_{L^{(s,p),p(\mathbb{R}^n)}} \left( \sum_{j: 2^j l(Q) < 1} (2^j l(Q))^{p[n/p] + 1-n/p} \right)^{1/p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \tilde{\Psi}\|_{L^{(s,p),p(\mathbb{R}^n)}},
\]

since \([n/p] + 1 - n/p > 0\).

To estimate \(\mathcal{J}\) we further separate into two terms

\[
\mathcal{J}_1 := \left\| \left( \sum_{j: 2^j l(Q) \geq 1} |(\sigma \tilde{\Psi} \cdot /2^j)\|_{L^p((Q^*)^c)}\right)^{1/2} \right\|_{L^p}\]
and

\[
\mathcal{J}_2 := \left\| \left( \sum_{j: 2^j l(Q) \geq 1} |(\sigma \tilde{\Psi} \cdot /2^j)\|_{L^p((Q^*)^c)}\right)^{1/2} \right\|_{L^p((Q^*)^c)}.
\]

Using the embedding \(\ell^p \hookrightarrow \ell^2\), Bernstein inequality with

\[
(\sigma \tilde{\Psi} \cdot /2^j)\|_{L^p((Q^*)^c)}\right)^{1/2} \right\|_{L^p((Q^*)^c)}.
\]

Using the embedding \(\ell^p \hookrightarrow \ell^2\), Bernstein inequality with

\[
(\sigma \tilde{\Psi} \cdot /2^j)\|_{L^p((Q^*)^c)}\right)^{1/2} \right\|_{L^p((Q^*)^c)}.
\]

and the inequality (3.3), we have

\[
\mathcal{J}_1 \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot) \tilde{\Psi}\|_{L^{(s,p),p(\mathbb{R}^n)}} \left( \sum_{j: 2^j l(Q) \geq 1} \|\mathcal{L}^\Theta_j (\chi_{(Q^*)^c} \mathcal{L}^\Theta_j a)\|_{L^p(\mathbb{R}^n)}^{p} \right)^{1/p}.
\]
We see that for $x \in (Q^*)^c$ and $M > n/p$

$$|L_j^\Theta a(x)| \lesssim_M |Q|^{-1/p} \int_{y \in Q} \frac{2^{jn}}{(1 + 2^j|x-y|)^{2M}} dy \lesssim_M |Q|^{-1/p} \frac{1}{(2^j|x-c_Q|)^M}$$

$$\lesssim_M |Q|^{-1/p}(2^j l(Q))^{-M} \frac{1}{(1 + |x-c_Q|/l(Q))^M}$$

since $|x-y| \geq \frac{9}{10} |x-c_Q|$. Then

$$\|L_j^\Theta (\chi_{(Q^*)^c}L_j^\Theta a)\|_{L^p(\mathbb{R}^n)} \lesssim \|Q|^{-1/p}(2^j l(Q))^{-M} \left[ \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |2^{jn}\Theta(2^j(x-y))| \frac{1}{(1 + |x-c_Q|/l(Q))^M} dy \right)^p dx \right]^{1/p}.$$ 

Standard manipulations with $2^j l(Q) \geq 1$ in [9, Appendix B] yield that the last expression is less than a constant times

$$|Q|^{-1/p}(2^j l(Q))^{-M} \left( \int_{\mathbb{R}^n} \frac{1}{(1 + |x-c_Q|/l(Q))^{Mp}} dx \right)^{1/p} \lesssim (2^j l(Q))^{-M}.$$ 

Accordingly,

$$J_1 \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \hat{\Psi})\|_{L^p_s(x,p,\mathbb{R}^n)} \left( \sum_{k:2^k l(Q) \geq 1} (2^k l(Q))^{-Mp} \right)^{1/p} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma(2^j \cdot \hat{\Psi})\|_{L^p_s(x,p,\mathbb{R}^n)}.$$ 

We now consider $J_2$. Choose $n/p - n/2 < N < s$ so that $n/2 < Np < sp < n$ and $2 < p(n/(Np))' < \infty$. For notational convenience we write

$$E_j^N \sigma(x) := (1 + 4\pi^2 (2^j |x|)^2)^{N/2} (\sigma \hat{\Psi}((\cdot)/2^j))^\vee(x).$$

Observe that if $x \in (Q^*)^c$ and $y \in Q^*$, then $|x-c_Q| \lesssim |x-y|$ and thus

$$|x-c_Q|^N (\sigma \hat{\Psi}((\cdot)/2^j))^\vee (\chi_Q \cdot L_j^\Theta a) (x) \lesssim 2^{-jN} |E_j^N \sigma| \ast \chi_Q \cdot L_j^\Theta a (x).$$

This proves that $J_2$ is less than a constant times

$$\left\| \frac{1}{|x-c_Q|^N} \left( \sum_{j:2^j l(Q) \geq 1} 2^{-2jN} \left( |E_j^N \sigma| \ast \chi_Q \cdot L_j^\Theta a \right)^2 \right)^{1/2} \right\|_{L^p((Q^*)^c)}$$

$$\lesssim \left\| \left( \sum_{j:2^j l(Q) \geq 1} 2^{-2jN} \left( |E_j^N \sigma| \ast \chi_Q \cdot L_j^\Theta a \right)^2 \right)^{p/2} \right\|_{L^{(n/(Np))'}(\mathbb{R}^n)}^{1/p}$$

$$= \left\| \left( \sum_{j:2^j l(Q) \geq 1} 2^{-2jN} \left( |E_j^N \sigma| \ast \chi_Q \cdot L_j^\Theta a \right)^2 \right)^{1/2} \right\|_{L^p(n/(Np))'(\mathbb{R}^n)},$$

where we made use of Lemma 2.5 with $n/(Np) > 1$. Now using Lemma 2.4 with $p(n/(Np))' > 2$, the preceding expression is dominated by a constant multiple of

$$\left( \sum_{j:2^j l(Q) \geq 1} 2^{-2jN} \left\| |E_j^N \sigma| \ast \chi_Q \cdot L_j^\Theta a \right\|_{L^{p(n/(Np))'}(\mathbb{R}^n)}^2 \right)^{1/2}.$$
and Lemma 2.1 yields that
\[ \left\| \mathcal{E}_j^N \sigma \ast \chi Q J_j^O a \right\|_{L^p(n/(Np),p(\mathbb{R}^n))} \lesssim \left\| \mathcal{E}_j^N \sigma \right\|_{L^p(n/(Np),p(\mathbb{R}^n))} \left\| J_j^O a \right\|_{L^1(Q^*)}. \]
We see that
\[ \left\| \mathcal{E}_j^N \sigma \right\|_{L^p(n/(Np),p(\mathbb{R}^n))} \lesssim 2^{-j(n/p-n)} 2^{jN} \left\| \sigma \right\|_{L^p_n(n/(Np),p(\mathbb{R}^n))} \lesssim 2^{-j(n/p-n)} 2^{jN} \left\| \sigma \right\|_{L^p_n(n/(Np),p(\mathbb{R}^n))} \]
by applying dilation, Lemma 2.2 with \((p(n/(Np)))' = \tau(Np)\), and Lemma 2.6 with \(s > N\). Combining with the estimate \(\left\| \mathcal{L}_j^O a \right\|_{L^1(Q^*)} \lesssim |Q|^{1/2} \left\| \mathcal{L}_j^O a \right\|_{L^2(\mathbb{R}^n)}\), we finally obtain
\[ J_2 \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \hat{\Psi} \right\|_{L^p_x(s,p,p(\mathbb{R}^n))} |Q|^{1/2} \left( \sum_{j:2^j l(Q) \geq 1} 2^{-2j(n/p-n)} \left\| \mathcal{L}_j^O a \right\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} \]
\[ \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \hat{\Psi} \right\|_{L^p_x(s,p,p(\mathbb{R}^n))} |Q|^{1/2} \left\{ \left\| \mathcal{L}_j^O a \right\|_{L^2(\mathbb{R}^n)} \right\}_{j \in \mathbb{Z}} \]
\[ \lesssim \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \hat{\Psi} \right\|_{L^p_x(s,p,p(\mathbb{R}^n))} \]
because \(\left\{ \left\| \mathcal{L}_j^O a \right\|_{L^2(\mathbb{R}^n)} \right\}_{j \in \mathbb{Z}} \approx \left\| a \right\|_{L^2(\mathbb{R}^n)} \lesssim |Q|^{-1/p+1/2}.\)

This concludes the proof of the proposition. \(\square\)

4. PROOF OF THEOREM 1.2

The construction of our counterexamples is based on the idea in [16] and the following lemma is crucial in the proof.

**Lemma 4.1.** Let \(0 < s, \gamma < \infty\) and define the function on \(\mathbb{R}^n\)
\[ \mathcal{H}^{(s,\gamma)}(x) := \frac{1}{(1 + 4\pi^2|x|^2)^{s/2}} \frac{1}{(1 + \ln(1 + 4\pi^2|x|^2))^{\gamma/2}}. \]

Then \(\mathcal{H}^{(s,\gamma)}\) is a positive radial function and there exist \(c_{s,\gamma,n}, d_{s,\gamma,n} > 0\) such that
\[ \mathcal{H}^{(s,\gamma)}(\xi) \leq c_{s,\gamma,n} e^{-|\xi|/2} \quad \text{when} \quad |\xi| \geq 1 \]
and
\[ \frac{1}{d_{s,\gamma,n}} \leq \frac{\mathcal{H}^{(s,\gamma)}(\xi)}{\mathcal{H}^{(s,\gamma)}(\xi)} \leq d_{s,\gamma,n} \quad \text{when} \quad |\xi| \leq 1 \]
where
\[ \mathcal{H}^{(s,\gamma)}(\xi) := \begin{cases} |\xi|^{-(n-s)}(1 + 2 \ln |\xi|^{-1})^{-\gamma/2} & \text{for} \quad 0 < s < n \\ 1 & \text{for} \quad s \geq n. \end{cases} \]

**Proof.** It is known that the Fourier transform of \((1 + 4\pi^2|x|^2)^{-s/2}\) is the Bessel potential \(G_s(\xi)\). Recall that \(G_s\) is a positive radial function, \(\|G_s\|_{L^1(\mathbb{R}^n)} = 1\), and there exist \(C_{s,n}, D_{s,n} > 0\) such that
\[ G_s(\xi) \leq C_{s,n} e^{-|\xi|/2} \quad \text{for} \quad |\xi| \geq 1, \]
and
\begin{equation}
\frac{1}{D_{(s,n)}} \leq \frac{G_s(\xi)}{\mathcal{G}_s(\xi)} \leq D_{(s,n)} \quad \text{for} \ |\xi| \leq 1
\end{equation}
where
\[ \mathcal{G}_s(\xi) := \begin{cases} 
|\xi|^{-(n-s)} & \text{for } 0 < s < n \\
\ln (2|\xi|^{-1}) & \text{for } s = n \\
1 & \text{for } s > n.
\end{cases} \]

Here we note that for any \( \epsilon > 0 \)
\begin{equation}
C_{(s,n)}, D_{(s,n)} \lesssim \epsilon, n e^{\epsilon|s-n|}.
\end{equation}

We refer to [9, Ch. 1.2.2] for more details.

Using the identity
\[ A^{-\gamma/2} = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-tA}t^{\gamma/2} dt, \]
which is valid for \( A > 0 \), we write
\[ \left(1 + \log(1 + 4\pi^2|x|^2)\right)^{-\gamma/2} = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t}e^{-t\log(1+4\pi^2|x|^2)t^{\gamma/2}} dt = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t} \frac{1}{(1+4\pi^2|x|^2)} t^{\gamma/2} dt. \]

We obtain from this that the Fourier transform of \( \left(1 + \log(1 + 4\pi^2|x|^2)\right)^{-\gamma/2} \) is
\[ \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t}G_{2t}(\xi)t^{\gamma/2} dt \]
and consequently,
\[ \mathcal{H}^{(s,\gamma)}(\xi) = G_s * \left( \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t}G_{2t}(\cdot)t^{\gamma/2} dt \right)(\xi) = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t}G_{2t+s}(\xi)t^{\gamma/2} dt. \]

Clearly, \( \mathcal{H}^{(s,\gamma)} \) is a positive radial function since so is \( G_{2t+s} \).

Suppose \( |\xi| \geq 1 \). Then using (4.3) and (4.5) with \( 0 < \epsilon < 1/100 \),
\[ |\mathcal{H}^{(s,\gamma)}(\xi)| \lesssim \epsilon, n \frac{1}{\Gamma(\gamma/2)} \left( \int_0^\infty e^{-t}e^{t[2t+s-n]t^{\gamma/2}} dt \right) e^{-|\xi|/2} \lesssim_{s,n,\gamma} e^{-|\xi|/2}, \]
which proves (4.2).

Now we assume that \( |\xi| \leq 1 \). When \( 0 < s < n \)
\[ \mathcal{H}^{(s,\gamma)}(\xi) = \frac{1}{\Gamma(\gamma/2)} \int_0^{n-s} e^{-t}G_{2t+s}(\xi)t^{\gamma/2} dt + \frac{1}{\Gamma(\gamma/2)} \int_{n-s}^\infty e^{-t}G_{2t+s}(\xi)t^{\gamma/2} dt. \]
Then using (4.4), (4.5), and change of variables,
\[
\frac{1}{\Gamma(\gamma/2)} \int_0^{\frac{n-s}{2}} e^{-t} G_{2t+s}(\xi) t^{\gamma/2} dt
\]
\[
\lesssim_{n,\epsilon} |\xi|^{-(n-s)} \frac{1}{\Gamma(\gamma/2)} \int_0^{\frac{n-s}{2}} e^{-t} |\xi|^{2t} e^{-\epsilon(n-2t-s)} t^{\gamma/2} dt
\]
\[
\leq e^{\epsilon(n-s)} |\xi|^{-(n-s)} \frac{1}{\Gamma(\gamma/2)} \int_0^{\frac{n-s}{2}} e^{-t(1+2 \ln(|\xi|^{-1}))} dt
\]
\[
\leq e^{\epsilon(n-s)} |\xi|^{-(n-s)} (1 + 2 \ln(|\xi|^{-1}))^{-\gamma/2} \frac{1}{\Gamma(\gamma/2)} \int_0^{\infty} e^{-\frac{t}{2}} dt
\]
\[
\lesssim_{s,n,\gamma} |\xi|^{-(n-s)} (1 + 2 \ln(|\xi|^{-1}))^{-\gamma/2}
\]

and
\[
\frac{1}{\Gamma(\gamma/2)} \int_0^{\frac{n-s}{2}} e^{-t} G_{2t+s}(\xi) t^{\gamma/2} dt
\]
\[
\gtrsim_{n,\epsilon} |\xi|^{-(n-s)} \frac{1}{\Gamma(\gamma/2)} \int_0^{\frac{n-s}{2}} e^{-t} |\xi|^{2t} e^{-\epsilon(n-2t-s)} t^{\gamma/2} dt
\]
\[
\geq e^{-\epsilon(n-s)} |\xi|^{-(n-s)} \frac{1}{\Gamma(\gamma/2)} \int_0^{\frac{n-s}{2}} e^{-t(1+2 \ln(|\xi|^{-1}))} dt
\]
\[
\geq e^{-\epsilon(n-s)} |\xi|^{-(n-s)} (1 + 2 \ln(|\xi|^{-1}))^{-\gamma/2} \frac{1}{\Gamma(\gamma/2)} \int_0^{\frac{n-s}{2}} e^{-\frac{t}{2}} dt
\]
\[
\gtrsim_{s,n,\gamma} |\xi|^{-(n-s)} (1 + 2 \ln(|\xi|^{-1}))^{-\gamma/2}
\]

Similarly, we can also prove that
\[
\frac{1}{\Gamma(\gamma/2)} \int_0^{\infty} e^{-t} G_{2t+s}(\xi) t^{\gamma/2} dt \approx_{s,n,\gamma} 1.
\]

A similar computation, together with (4.4) and (4.5), will lead to an estimate for $s \geq n$, in which $\mathcal{H}^{(s,\gamma)} \approx_{s,\gamma,n} 1$ for $|\xi| \leq 1$. We leave this to the reader to avoid unnecessary repetition. \hfill \Box

In what follows let $\eta, \widehat{\eta}$ denote Schwartz functions so that $\eta \geq 0$, $\eta(x) \geq c$ on $\{x \in \mathbb{R}^n : |x| \leq 1/100\}$ for some $c > 0$, $\text{Supp}(\widehat{\eta}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1/1000\}$, $\widehat{\eta}(\xi) = 1$ for $|\xi| \leq 1/1000$, and $\text{Supp}(\widehat{\eta}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 1/100\}$. Let $e_1 := (1, 0, \ldots, 0) \in \mathbb{Z}^n$ and $0 < t, \gamma < \infty$. Define $\mathcal{H}^{(t,\gamma)}$ as in (4.1),
\[
K^{(t,\gamma)}(x) := \mathcal{H}^{(t,\gamma)} * \widehat{\eta}(x) e^{2\pi i(x,e_1)},
\]
and
\[
\sigma^{(t,\gamma)}(\xi) := \widehat{K^{(t,\gamma)}}(\xi).
\]

We investigate an upper bound of $\sup_{j \in \mathbb{Z}} \|\sigma^{(t,\gamma)}(2^j \cdot)\widehat{\Psi}\|_{L^p_n(\mathbb{R}^n)}$ and a lower bound of $\|T^{(t,\gamma)}\|_{H^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)}$ when $t - n < s$. 

4.1. **Upper bound of** $\sup_{j \in \mathbb{Z}} \| \sigma^{(t, \gamma)}(2^j \cdot) \hat{\Psi} \|_{L^r_s(\mathbb{R}^n)}$. Note that, due to the supports of $\sigma^{(t, \gamma)}$ and $\hat{\Psi}$, we have

$$\sigma^{(t, \gamma)}(2^j \xi) \hat{\Psi}(\xi) = \begin{cases} K(t, \gamma)(2^j \xi) \hat{\Psi}(\xi), & -2 \leq j \leq 2 \\
0, & \text{otherwise.} \end{cases}$$

For $-2 \leq j \leq 2$ and $t - n < s$,

$$\| \sigma^{(t, \gamma)}(2^j \cdot) \hat{\Psi} \|_{L^r_s(\mathbb{R}^n)} \lesssim \| \sigma^{(t, \gamma)} \|_{L^s_r(\mathbb{R}^n)} \lesssim \| \hat{H}^{(t, \gamma)} \|_{L^s_r(\mathbb{R}^n)} = \| \hat{H}^{(t-s, \gamma)} \|_{L^r_q(\mathbb{R}^n)}$$

where Lemma 2.3 is applied.

For $u > 0$ define

$$\mathcal{T}^{(t-s, \gamma)}(u) := \begin{cases} u^{-(n-t+s)}(1 + 2 \ln u^{-1})^{-\gamma/2} & \text{for } u \leq 1 \\
e^{-u/2+1/2} & \text{for } u > 1. \end{cases}$$

Then $\mathcal{T}^{(t-s, \gamma)}$ is a positive decreasing function and this implies that

$$\left( \mathcal{T}^{(t-s, \gamma)} \right)^*(u) = \mathcal{T}^{(t-s, \gamma)}(u).$$

We first assume $0 < q < \infty$. By using Lemma 4.1, we have

$$\| \hat{H}^{(t-s, \gamma)}(\xi) \|_{s,t,\gamma,n} \lesssim \mathcal{T}^{(t-s, \gamma)}(\| \xi \|),$$

from which

$$\| \hat{H}^{(t-s, \gamma)} \|_{L^r_q(\mathbb{R}^n)} \lesssim s,t,\gamma,n \| \mathcal{T}^{(t-s, \gamma)}(\| \cdot \|) \|_{L^r_q(\mathbb{R}^n)}$$

$$= \left( \int_0^\infty \left( \mathcal{T}^{(t-s, \gamma)} \left( (u/\Omega_n)^{1/n} \right) u^{1/r} \right)^q du \right)^{1/q}$$

$$= \Omega_n^{1/r} n^{1/q} \left( \int_0^\infty \left( \mathcal{T}^{(t-s, \gamma)}(u) \right)^q u^{nq/r} du \right)^{1/q}$$

where Lemma 2.7 is applied with (4.6). Furthermore,

$$\left( \int_0^1 \left( \mathcal{T}^{(t-s, \gamma)}(u) \right)^q u^{nq/r} du \right)^{1/q} = \left( \int_0^1 \frac{1}{u^{n-t+s-n/r}} \frac{1}{(1 + 2 \ln u^{-1})^{\gamma/2}} du \right)^{1/q}$$

$$= \left( \int_1^\infty u^{n-t+s-n/r-q} \frac{1}{(1 + 2 \ln u)^{\gamma/2}} du \right)^{1/q}$$

and

$$\left( \int_1^\infty \left( \mathcal{T}^{(t-s, \gamma)}(u) \right)^q u^{nq/r} du \right)^{1/q} = e^{1/2} \left( \int_1^\infty e^{-u^{q/2}} u^{nq/r} du \right)^{1/q} \lesssim_{q,r,n} 1$$

Finally, we conclude that

$$\sup_{j \in \mathbb{Z}} \| \sigma^{(t, \gamma)}(2^j \cdot) \hat{\Psi} \|_{L^r_s(\mathbb{R}^n)} \lesssim_{s,\gamma,n,q,r} 1 + \left( \int_1^\infty u^{(n-t+s-n/r-q)} \frac{1}{(1 + 2 \ln u)^{\gamma/2}} du \right)^{1/q}$$

and with the usual modification if $q = \infty$ we may also obtain

$$\sup_{j \in \mathbb{Z}} \| \sigma^{(t, \gamma)}(2^j \cdot) \hat{\Psi} \|_{L^r_s(\mathbb{R}^n)} \lesssim_{s,\gamma,n,r} 1 + \sup_{u > 1} \frac{u^{n-t+s-n/r}}{(1 + 2 \ln u)^{\gamma/2}}.$$
4.2. Lower bound of $\|T_{\sigma(t,\gamma)}\|_{H^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)}$. If $1 \leq p < \infty$, then

$$
\|T_{\sigma(t,\gamma)}\|_{H^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)} \geq \|\sigma(t,\gamma)\|_{L^\infty(\mathbb{R}^n)} \geq |\sigma(t,\gamma)(e_1)| \gtrsim \|H^{(t,\gamma)}\|_{L^1(\mathbb{R}^n)}.
$$

Moreover, for $0 < p < 1$, define $f(x) := \eta(x)e^{2\pi i \langle x, e_1 \rangle}$. Observe that $|T_{\sigma(t,\gamma)}f(x)| = |H^{(t,\gamma)}*\eta(x)|$ and thus

$$
\|T_{\sigma(t,\gamma)}\|_{H^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)} \gtrsim \|T_{\sigma(t,\gamma)}f\|_{H^p(\mathbb{R}^n)} \geq \|T_{\sigma(t,\gamma)}f\|_{L^p(\mathbb{R}^n)}
$$

$$
= \|H^{(t,\gamma)}*\eta\|_{L^p(\mathbb{R}^n)} \gtrsim \|H^{(t,\gamma)}\|_{L^p(\mathbb{R}^n)},
$$

where the last inequality follows from the fact that $H^{(t,\gamma)}\eta \geq 0$ and $H^{(t,\gamma)}(x-y) \geq H^{(t,\gamma)}(x) H^{(t,\gamma)}(y)$.

Consequently, for any $0 < p < \infty$,

$$
\|T_{\sigma(t,\gamma)}\|_{H^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)} \gtrsim \|H^{(t,\gamma)}\|_{L^{\min(1,p)}(\mathbb{R}^n)}
$$

(4.9) 

$$
= \left( \int_{\mathbb{R}^n} \frac{1}{(1 + 4\pi^2|x|^2)^{t\min(1,p)/2}} \frac{1}{(1 + \ln(1 + 4\pi^2|x|^2))^{\min(1,p)\gamma/2}} \,dx \right)^{1/\min(1,p)}.
$$

4.3. Completion of the proof of Theorem 1.2. We are only concerned with the case $0 < p \leq 2$ as the other cases follow by a duality argument. Suppose $n/p - n/2 < s < n/\min(1,p)$.

We first assume $r < \tau(s,p)$ and $0 < q \leq \infty$. Then we can choose $t < \frac{n}{\min(1,p)}$ so that

$$
r < \frac{n}{s - (t - n)} < \frac{n}{s - (n/\min(1,p) - n)} = \tau(s,p).
$$

Note that $t - n < s$ and $n - t + s - n/r < 0$, from which

$$
\sup_{j \in \mathbb{Z}} \|\sigma(t,\gamma)(2^j \Psi)\|_{L^{s,q}_r(\mathbb{R}^n)} \lesssim_{s,\gamma,n,q,r} 1
$$

due to (4.7) and (4.8). Moreover, since $t \min(1,p) < n$

$$
\|T_{\sigma(t,\gamma)}\|_{H^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)} = \infty,
$$

using (4.9).

Now suppose $r = \tau(s,p)$ and $\min(1,p) < q < \infty$. Choose

(4.10) 

$$
2/q < \gamma \leq 2/\min(1,p)
$$

and let $t = \frac{n}{\min(1,p)}$ such that $n - t + s - n/r = 0$. Then

$$
\sup_{j \in \mathbb{Z}} \|\sigma(t,\gamma)(2^j \Psi)\|_{L^{s,q}_r(\mathbb{R}^n)} \lesssim_{s,\gamma,n,q} 1 + \left( \int_{1}^{\infty} \frac{1}{(1 + 2 \ln u)^{\gamma q/2}} \,du \right)^{1/q} \lesssim 1
$$

because of (4.10) for $0 < q < \infty$, and similarly, $\sup_{j \in \mathbb{Z}} \|\sigma(t,\gamma)(2^j \Psi)\|_{L^{s,q}_r(\mathbb{R}^n)} \lesssim_{s,\gamma,n} 1$ for $q = \infty$. On the other hand, $\|T_{\sigma(t,\gamma)}\|_{H^p(\mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)}$ is bounded below by

$$
\left( \int_{\mathbb{R}^n} \frac{1}{(1 + 4\pi^2|x|^2)^{n/2}} \frac{1}{(1 + \ln(1 + 4\pi^2|x|^2))^{\min(1,p)\gamma/2}} \,dx \right)^{1/\min(1,p)},
$$

which diverges for the choice of $\gamma$ in (4.10).
Appendix A. Complex Interpolation of $H^1$- and $L^2$-boundedness

In this section, we review the complex interpolation method of Calderón-Torchinsky [5] and Triebel [23], which is a generalization of the well-known method of Calderón [4] and Fefferman and Stein [8].

Let $A := \{ z \in \mathbb{C} : 0 < \text{Re}(z) < 1 \}$ be a strip in the complex plane $\mathbb{C}$ and $\overline{A}$ denote its closure. We say that the mapping $z \mapsto f_z \in S'(\mathbb{R}^n)$ is a $S'$-analytic function on $A$ if the following properties are satisfied:

1. For any $\varphi \in S(\mathbb{R}^n)$ with compact support, $g(x, z) := (\varphi \hat{f}_z)(x)$ is a uniformly continuous and bounded function on $\mathbb{R}^n \times \overline{A}$.
2. For any $\varphi \in S(\mathbb{R}^n)$ with compact support and any fixed $x \in \mathbb{R}^n$, $h_x := (\varphi \hat{f}_z)^\vee$ is an analytic function on $A$.

Let $0 < p_0, p_1 < \infty$. Then we define $F(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))$ to be the collection of all $S'$-analytic functions $f_z$ on $A$ such that

$$f_{it} \in H^{p_0}(\mathbb{R}^n), \quad f_{1+it} \in H^{p_1}(\mathbb{R}^n) \quad \text{for any} \ t \in \mathbb{R}$$

and

$$\sup_{t \in \mathbb{R}} \| f_{1+it} \|_{H^{p_1}(\mathbb{R}^n)} < \infty \quad \text{for each} \ l = 1, 2.$$

Moreover,

$$\| f_z \|_{F(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))} := \max \left( \sup_{t \in \mathbb{R}} \| f_{it} \|_{H^{p_0}(\mathbb{R}^n)}, \sup_{t \in \mathbb{R}} \| f_{1+it} \|_{H^{p_1}(\mathbb{R}^n)} \right).$$

For $0 < \theta < 1$ the intermediate space $(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))_\theta$ is defined by

$$(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))_\theta := \{ g : \exists f_z \in F(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)) \text{ so that } g = f_\theta \}$$

and the (quasi-)norm in the space is

$$\| g \|_{(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))_\theta} := \inf_{f_z \in F(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n)) : g = f_\theta} \| f_z \|_{F(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))}$$

where the infimum is taken over all admissible functions $f_z$ in the sense that $f_z \in F(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))$ and $g = f_\theta$. It is known in [5, 23] that for any $0 < p_0, p_1 < \infty$ and $0 < \theta < 1$

$$(H^{p_0}(\mathbb{R}^n), H^{p_1}(\mathbb{R}^n))_\theta = H^p(\mathbb{R}^n) \quad \text{when} \ 1/p = (1 - \theta)/p_0 + \theta/p_1. \ (A.1)$$

We now use this method to interpolate $H^1$- and $L^2$-boundedness of the multiplier operator $T_\alpha$ to obtain $L^p$ estimates for $1 < p < 2$. Note that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Since most arguments are very similar to that used in the proof of [10, Theorem 3.1], we shall provide only the outline of the proof, omitting the details.

We may consider a Schwartz function $f$ whose Fourier transform is compactly supported via a density argument. Suppose that $1 < p < 2$ and $n/p - n/2 < s < n$. Let $0 < \theta < 1$ satisfy $1/p = (1 - \theta)/1 + \theta/2$. Then we have $s > n/p - n/2 = (1 - \theta)n/2$. Pick $s_0 > n/2$ so that

$$s > (1 - \theta)s_0 > (1 - \theta)n/2$$

and let $s_1 := \frac{s - (1 - \theta)s_0}{\theta} > 0$ which implies

$$s = (1 - \theta)s_0 + \theta s_1.$$
Since \( f \in L^p(\mathbb{R}^n) = H^p(\mathbb{R}^n) = (H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))_\theta \), by definition, for any \( \epsilon > 0 \), there exists \( f^\epsilon_z \in F(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n)) \) such that \( f = f^\epsilon_\theta \) and
\[
\|f^\epsilon_z\|_{F(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))} < \|f\|_{(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))_\theta} + \epsilon. \tag{A.2}
\]

Now let \( \hat{\Theta}(\xi) := \hat{\Psi}(\xi/2) + \hat{\Psi}(\xi) + \hat{\Psi}(2\xi) \) as before, and \( \sigma^{j,s} := (I - \Delta)^{s/2}(\sigma(2^j \cdot) \hat{\Psi}) \) for each \( j \in \mathbb{Z} \). We define, as in [10, (3.18)],
\[
\sigma_z(\xi) := \frac{(1 + \theta)^{n+1}}{(1 + z)^{n+1}} \sum_{j \in \mathbb{Z}} (I - \Delta)^{-\frac{s(1-z)}{2} + \frac{s_z - s_{z+1}}{n}} \left( \sigma^{j,s} h_{j,s} \right) (\xi/2^j) \hat{\Theta}(\xi/2^j)
\]
where \( h_{j,s} : \mathbb{R}^n \to (0, \infty) \) is a measure preserving transformation so that \( |\sigma^{j,s}| = (\sigma^{j,s})^* \circ h_{j,s} \). Then we note that \( \sigma_\theta = \sigma \) and \( F_z := T_{\sigma_z} f^\epsilon_z \) is a \( \mathcal{S}' \)-analytic function on \( A \). Moreover,
\[
\|T_{\sigma} f\|_{H^p(\mathbb{R}^n)} \approx \|T_{\sigma_\theta} f^\epsilon\|_{(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))_\theta} = \|f\|_{(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))_\theta} \leq \|F_z\|_{F(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))} = \max \left( \sup_{t \in \mathbb{R}} \|F_{it}\|_{H^1(\mathbb{R}^n)}, \sup_{t \in \mathbb{R}} \|F_{1+it}\|_{H^2(\mathbb{R}^n)} \right).
\]

By using Theorem 1.1 for \( p = 1 \), we have
\[
\|F_{it}\|_{H^1(\mathbb{R}^n)} = \|T_{\sigma_{it}} f^\epsilon_{it}\|_{H^1(\mathbb{R}^n)} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma_{it}(2^j \cdot) \hat{\Psi}\|_{L^0_{\nu/0.1}}(\|f^\epsilon_{it}\|_{H^1(\mathbb{R}^n)}) \lesssim \sup_{j \in \mathbb{Z}} \|\sigma_{it}(2^j \cdot) \hat{\Psi}\|_{L^0_{\nu/0.1}}\left( \|f\|_{(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))_\theta} + \epsilon \right),
\]
where (A.2) is applied in the last inequality. Similarly, with \( L^2 \)-boundedness,
\[
\|F_{1+it}\|_{H^2(\mathbb{R}^n)} = \|T_{\sigma_{1+it}} f^\epsilon_{1+it}\|_{H^2(\mathbb{R}^n)} \lesssim \|\sigma_{1+it}(2^j \cdot) \hat{\Psi}\|_{L^\infty}(\|f^\epsilon_{1+it}\|_{H^2(\mathbb{R}^n)}) \lesssim \sup_{j \in \mathbb{Z}} \|\sigma_{1+it}(2^j \cdot) \hat{\Psi}\|_{L^\infty}\left( \|f\|_{(H^1(\mathbb{R}^n), H^2(\mathbb{R}^n))_\theta} + \epsilon \right).
\]

Therefore, once we prove
\[
\|\sigma_{it}(2^j \cdot) \hat{\Psi}\|_{L^0_{\nu/0.1}}(\|\sigma_{1+it}(2^j \cdot) \hat{\Psi}\|_{L^\infty}) \lesssim \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^2_{\nu/1}} \tag{A.3}
\]
uniformly in \( j \), then we are done by using (A.1) and taking \( \epsilon \to 0 \).

Let us prove (A.3). We first observe that
\[
\sigma_z(2^j \xi) \hat{\Psi}(\xi) = \frac{(1 + \theta)^{n+1}}{(1 + z)^{n+1}} \sum_{k \in \mathbb{Z}} (I - \Delta)^{-\frac{s(1-z)}{2} + \frac{s_z - s_{z+1}}{n}} \left( \sigma^{k,s} h_{k,s} \right) (\xi/2^j) \hat{\Theta}(\xi/2^j) \hat{\Psi}(\xi)
\]
is actually finite sum over \( k \) near \( j \) due to the supports of \( \hat{\Theta} \) and \( \hat{\Psi} \), and for simplicity, we may therefore take \( k = j \) in the calculation below.

Using Lemma 2.3, we have
\[
\|\sigma_{it}(2^j \cdot) \hat{\Psi}\|_{L^0_{\nu/0.1}} \lesssim \frac{1}{(1 + |t|^2)^{(n+1)/2}} \|I - \Delta\|^{(\nu_{s-1})/2} \left( \sigma^{j,s} h_{j,s} \right) \|\sigma(2^j \cdot) \hat{\Psi}\|_{L^2_{\nu/1}}.
\]
Then we apply [10, Lemma 3.5, 3.7] to bound this by

\[
\left\| \sigma^{j,s} h_{j,s}^{s-s_0+(s_0-s_1)t/n} \right\|_{L^{n/s_0,1}(\mathbb{R}^n)} \lesssim \left\| (\sigma^{j,s})^*(r) (s-s_0)/n \right\|_{L^{n/s_0,1}(0,\infty)} \\
\lesssim \left\| (\sigma^{j,s})^* \right\|_{L^{n/s_1,1}(\mathbb{R}^n)} \lesssim \left\| \sigma^{j,s} \right\|_{L^{n/s_0,1}(\mathbb{R}^n)} = \left\| \sigma (2^j \cdot) \hat{\Psi} \right\|_{L^{n/s_1,1}(\mathbb{R}^n)}.
\]

On the other hand, using [10, Lemma 3.4, 3.5, 3.7],

\[
\left\| \sigma_{1+it} (2^j \cdot) \hat{\Psi} \right\|_{L^{\infty}(\mathbb{R}^n)} \\
\lesssim \frac{1}{(1 + |it|^2)^{(n+1)/2}} \left\| (I - \Delta)^{-s_1/2} (I - \Delta)^{(s_0-s_1)it/2} \left( \sigma^{j,s} h_{j,s}^{s-s_0+(s_0-s_1)t/2} \right) \right\|_{L^{\infty}(\mathbb{R}^n)} \\
\lesssim \frac{1}{(1 + |it|^2)^{(n+1)/2}} \left\| (I - \Delta)^{(s_0-s_1)it/2} \left( \sigma^{j,s} h_{j,s}^{s-s_0+(s_0-s_1)t} \right) \right\|_{L^{n/s_1,1}(\mathbb{R}^n)} \\
\lesssim \left\| \sigma^{j,s} h_{j,s}^{s-s_0+(s_0-s_1)t/n} \right\|_{L^{n/s_1,1}(\mathbb{R}^n)} \lesssim \left\| (\sigma^{j,s})^* \right\|_{L^{n/s_1,1}(\mathbb{R}^n)} \lesssim \left\| (\sigma^{j,s})^* (r) (s-s_0)/n \right\|_{L^{n/s_1,1}(0,\infty)} \\
\lesssim \left\| \sigma^{j,s} \right\|_{L^{n/s_0,1}(\mathbb{R}^n)} = \left\| \sigma (2^j \cdot) \hat{\Psi} \right\|_{L^{n/s_1,1}(\mathbb{R}^n)},
\]

which finishes the proof of (A.3).

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