AN ALTERNATIVE TO PLANCHEREL’S CRITERION FOR BILINEAR OPERATORS

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Abstract. We prove that bilinear operators associated with $L^q$ multipliers with sufficiently many derivatives in $L^\infty$ are bounded from $L^2 \times L^2$ to $L^1$ when $q < 4$. In the absence of Plancherel’s identity on $L^1$, the range $q < 4$ in the bilinear case should be compared to $q = \infty$ in the classical $L^2 \to L^2$ boundedness for linear multiplier operators.

1. Introduction

Function spaces provide quantitative ways to measure integrability, smoothness, and to certain extent, cancellation properties of functions. A space of central importance is $L^2(\mathbb{R}^n)$ as it appears at the crossroads of many echelons of function spaces. An important feature of $L^2(\mathbb{R}^n)$ is Plancherel’s identity which says that the Fourier transform

$$\widehat{f}(\xi) = \lim_{N \to \infty} \int_{|x| \leq N} f(x)e^{-2\pi ix \cdot \xi} dx \quad \text{(limit in } L^2)$$

of a square-integrable function $f$ satisfies

$$\|f\|_{L^2} = \|\widehat{f}\|_{L^2}.$$  

(Here $x \cdot y$ is the dot product on $\mathbb{R}^n$.) This simply identity provides an alternative way to calculate $L^2$ norms. It also trivializes the characterization of the $L^2$-boundedness of convolution operators $\varphi \mapsto \varphi * K$, where $K$ is a tempered distribution. Plancherel’s identity yields that such a convolution operator is bounded on $L^2(\mathbb{R}^n)$ if and only if the distributional Fourier transform of $K$ is a bounded function. Convolution operators can also be expressed as multiplier operators. A multiplier operator has the form

$$S_m(\varphi)(x) = \int_{\mathbb{R}^n} m(\xi)\widehat{\varphi}(\xi)e^{2\pi ix \cdot \xi} d\xi$$

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where \( m \) is a bounded function on \( \mathbb{R}^n \) and \( \varphi \) is a Schwartz function. We note that \( S_m(\varphi) = \varphi * K \) whenever \( \hat{K} = m \). In view of Plancherel’s identity we have

\[
\left\| S_m(f) \right\|_{L^2} = \left\| \hat{S_m}(f) \right\|_{L^2} = \left\| \hat{m \hat{f}} \right\|_{L^2}
\]

and it follows from this that \( S_m \) is \( L^2 \) bounded if and only if \( m \) is an \( L^\infty \) function. Moreover, the norm of \( S_m \) from \( L^2 \) to itself is equal to \( \|m\|_{L^\infty} \). This simple characterization of the \( L^2 \rightarrow L^2 \) boundedness of multiplier operators is a direct consequence of Plancherel’s identity, and for this reason we simply referred to it as Plancherel’s criterion.

In this note we ask whether there exist boundedness criteria for bilinear translation-invariant operators analogous to Plancherel’s criterion. Bilinear translation-invariant operators have the form

\[
T(f, g)(x) = \text{p.v.} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x - y, x - z) f(y) g(z) \, dy \, dz, \quad x \in \mathbb{R}^n,
\]

where \( f, g \) are Schwartz functions and \( K \) is a distribution on \( \mathbb{R}^{2n} \) that coincides with a suitable function on \( \mathbb{R}^{2n} \setminus \{(0,0)\} \). These operators can also be expressed as bilinear multiplier operators given by

\[
T_m(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{2\pi i x \cdot (\xi + \eta)} \, d\xi d\eta,
\]

where \( f, g \) are Schwartz functions and \( m \) is a bounded function on \( \mathbb{R}^{2n} \) which coincides with the distributional Fourier transform of \( K \).

We refer to [4, Section 6] for general material related to the bilinear translation-invariant operators. These operators may map the product \( L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) when \( 1/p_1 + 1/p_2 = 1/p \) but in this note, we only focus on the \( L^2 \times L^2 \rightarrow L^1 \) boundedness of such operators. Such estimates are central and play the same role in in bilinear theory as the \( L^2 \) boundedness plays in linear multiplier theory. As Plancherel’s identity (1) does not hold on \( L^1 \), there does not seem to be a straightforward way to characterize the boundedness of bilinear multiplier operators from \( L^2 \times L^2 \rightarrow L^1 \). But if the functions \( m \) have bounded derivatives up to a certain order, we show that such a characterization is possible.

As we restrict attention to multipliers all of whose derivatives are bounded, we introduce the space

\[
L^\infty(\mathbb{R}^{2n}) = \{ m : \mathbb{R}^{2n} \to \mathbb{C} : \partial^\alpha m \text{ exist for all } \alpha \text{ and } \|\partial^\alpha m\|_{L^\infty} < \infty \}.
\]

In the linear setting we have \( m \in L^\infty \) if and only if the corresponding linear operator is bounded on \( L^2 \). So one may guess that a bilinear operator \( T_m \) is bounded from \( L^2 \times L^2 \) to \( L^1 \) when \( m \) lies in \( L^\infty \). However
Bényi and Torres [1] provided an example of a function $m \in L^\infty$ for which the associated bilinear operator $T_m$ is unbounded from $L^{p_1} \times L^{p_2}$ to $L^p$ for any $1 \leq p_1, p_2 < \infty$ satisfying $1/p = 1/p_1 + 1/p_2$. The counterexample of Bényi and Torres is also complemented by a subsequent positive result of He, Honzík, and the author [2, Corollary 8], who showed that the mere $L^2$ integrability of functions in $L^\infty$ suffices to yield the $L^2 \times L^2 \to L^1$ boundedness of $T_m$.

It turns out that the magnitude of integrability of a function $m$ in $L^\infty$ characterizes the boundedness of the bilinear multiplier operator $T_m$ from $L^2 \times L^2 \to L^1$. We provide a proof of the main direction of this equivalence, the one that yields the boundedness of the operator.

**Theorem 1.1.** [3] Let $1 \leq q < 4$ and set $M_q = \left\lfloor \frac{2n}{4-q} \right\rfloor + 1$. Let $m$ be a function in $L^q(\mathbb{R}^{2n}) \cap C^{M_q}(\mathbb{R}^{2n})$ satisfying

\begin{equation}
\|\partial^\alpha m\|_{L^\infty} \leq C_0 < \infty \quad \text{for all multiindices } \alpha \text{ with } |\alpha| \leq M_q.
\end{equation}

Then there is a constant $C$ depending on $n$ and $q$ such that the bilinear operator $T_m$ with multiplier $m$ satisfies

\begin{equation}
\|T_m\|_{L^2 \times L^2 \to L^1} \leq C \left| \frac{q}{4} \right| \|m\|_{L^q}^4.
\end{equation}

Additionally, we are aware of examples indicating that for any $q \geq 4$ there exist functions $m \in L^q(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ such that the associated operator $T_m$ does not map $L^2 \times L^2$ to $L^1$; see [3] for $q > 4$ and [5] for $q = 4$. These counterexamples complement Theorem 1.1 and indicate its sharpness; as this note is based on the lecture of the author at the Function Spaces XII conference, we do not describe these counterexamples here.

### 2. Product-type wavelets

We plan to outline the proof of Theorem 1.1. This is based on the product-type wavelet method initiated by He, Honzík and the author in [2]. Our approach here incorporates several crucial combinatorial improvements. For the sake of a simple and clear presentation, we prove Theorem 1.1 only in the case where $n = 1$.

We recall some facts related to product-type wavelets that will be crucial in our approach of proving Theorem 1.1. For a fixed $M \in \mathbb{N}$ there exist real-valued compactly supported functions $\psi_F, \psi_M$ in $C^k(\mathbb{R})$, called father wavelet and mother wavelet, respectively, that satisfy

\begin{equation}
\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1.
\end{equation}
and
\[ \int_{\mathbb{R}} x^k \psi_M(x) \, dx = 0 \quad \text{for all } 0 \leq k \leq M. \]

Then the family of functions
\[ \bigcup_{\mu_1, \mu_2 \in \mathbb{Z}} \left\{ \psi_F(x_1 - \mu_1) \psi_F(x_2 - \mu_2) \right\} \perp \bigcup_{\mu_1, \mu_2 \in \mathbb{Z}} \left\{ 2^{\frac{\lambda}{2}} \psi_F(2^\lambda x_1 - \mu_1) 2^{\frac{\lambda}{2}} \psi_M(2^\lambda x_2 - \mu_2) \right\} \]
\[ \bigcup_{\mu_1, \mu_2 \in \mathbb{Z}} \left\{ 2^{\frac{\lambda}{2}} \psi_M(2^\lambda x_1 - \mu_1) 2^{\frac{\lambda}{2}} \psi_F(2^\lambda x_2 - \mu_2) \right\} \bigcup_{\mu_1, \mu_2 \in \mathbb{Z}} \left\{ 2^{\frac{\lambda}{2}} \psi_M(2^\lambda x_1 - \mu_1) 2^{\frac{\lambda}{2}} \psi_M(2^\lambda x_2 - \mu_2) \right\} \]
forms an orthonormal basis of \(L^2(\mathbb{R}^2)\). This result is due to Triebel\(^1\) and its proof can be found in Triebel [6].

We denote by \(J\) the set of all pairs \((\lambda, G)\) such that either \(\lambda = 0\) and \(G = (F, F)\), or \(\lambda\) is a nonnegative integer and \(G\) has the form \((F, M)\), \((M, F)\), or \((M, M)\). For \((\lambda, G) \in J\) and \((\mu_1, \mu_2) \in \mathbb{Z}^2\) we set
\[ \Psi_{\mu_1, \mu_2}^G(x_1, x_2) = 2^{\frac{\lambda}{2}} \psi_{G_1}(2^\lambda x_1 - \mu_1) 2^{\frac{\lambda}{2}} \psi_{G_2}(2^\lambda x_2 - \mu_2). \]

for \((x_1, x_2) \in \mathbb{R}^2\), where \(G = (G_1, G_2)\) and \((\lambda, G) \in J\).

The cancellation of wavelets is manifested in the following result.

**Lemma 2.1.** Let \(M\) be a positive integer. Assume that \(m \in C^{M+1}\) is a function on \(\mathbb{R}^2\) such that
\[ \sup_{|\alpha| \leq M+1} \|\partial^\alpha m\|_{L^\infty} \leq C_0 < \infty. \]

Then for \((\lambda, G) \in J\) and \((\mu_1, \mu_2) \in \mathbb{Z}^2\) we have
\[ |\langle \Psi_{\mu_1, \mu_2}^G, m \rangle| \leq C C_0 2^{-(M+2)\lambda}, \]
provided that \(\psi_M\) has \(M\) vanishing moments.

This lemma can be easily proved and is essentially a restatement of Lemma 7 in [2]. Note that if \(G = (F, F)\) there is no cancellation, however, there is no decay claimed in (4), as \(\lambda = 0\) in this case.

\(^1\)as confirmed by him during the Function Spaces XII conference
3. Proof of Theorem 1.1

Proof. To prove the theorem we use the product type wavelets introduced. We begin by fixing a large number $M$ to be determined later, which denotes the number of vanishing moments of the mother wavelet.

For $(\lambda, G) \in J$ and $\mu \in \mathbb{Z}^2$ we denote the wavelet coefficient by

$$b_{\lambda,G}^\mu = \langle \Psi_{\lambda,G}^\mu, m \rangle.$$ 

By [7, Theorem 1.64] and by the fact that $L^q = F_{q,2}$, we obtain

$$\|m\|_{L^q(\mathbb{R}^2)} \approx \left\| \left( \sum_{(\lambda, G) \in J} \sum_{\mu \in \mathbb{Z}^2} |b_{\lambda,G}^\mu 2^\lambda \chi_{Q_{\lambda\mu}}|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^2)},$$

where $Q_{\lambda\mu}$ is the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{1-\lambda}$.

Now, let us fix $(\lambda, G) \in J$. For notational simplicity, we write $b_\mu$ instead of $b_{\lambda,G}^\mu$ in what follows. We also denote by $\tilde{Q}_{\lambda\mu}$ the cube centered at $2^{-\lambda}\mu$ with sidelength $2^{-\lambda}$. Noting that these cubes are pairwise disjoint in $\mu$ (for the fixed value of $\lambda$), the equivalence (5) yields

$$\|m\|_{L^q(\mathbb{R}^2)} \geq 2^\lambda \left\| \left( \sum_{\mu \in \mathbb{Z}^2} |b_\mu| \chi_{\tilde{Q}_{\lambda\mu}} \right)^{1/2} \right\|_{L^q(\mathbb{R}^2)}$$

$$\geq 2^\lambda \left\| \left( \sum_{\mu \in \mathbb{Z}^2} |b_\mu| \chi_{\tilde{Q}_{\lambda\mu}} \right) \right\|_{L^q(\mathbb{R}^2)}$$

$$= 2^\lambda \|b_\mu \chi_{\tilde{Q}_{\lambda\mu}}\|_{L^q(\mathbb{R}^2)}$$

$$= 2^{\lambda(1-\frac{q}{2})} \left( \sum_{\mu \in \mathbb{Z}^2} |b_\mu|^q \right)^{\frac{1}{q}}.$$

Setting $b = (b_\mu)_{\mu \in \mathbb{Z}^2}$, the preceding sequence of inequalities yields

$$\|b\|_{L^q} \leq C 2^{-\lambda(M+2)} \|m\|_{L^q}$$

Also, Lemma 2.1 implies that

$$\|b\|_{L^\infty} \leq C 2^{-\lambda(M+2)},$$

where $M$ is the number of vanishing moments of $\psi_M$.

We have an infinite × infinite matrix of wavelet coefficients indexed by $\mathbb{Z}^2$. To better organize these coefficients, define

$$U_r = \{(k, l) \in \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 : 2^{-r-1}\|b\|_{L^\infty} < |b_{(k,l)}| \leq 2^{-r}\|b\|_{L^\infty}\}.$$
where \( r \) is a nonnegative integer. Also, we write \( U_r \) as a union of the following two disjoint sets:

\[
U^1_r = \{(k, l) \in U_r : \text{card}\{s : (k, s) \in U_r\} \geq K\}; \\
U^2_r = \{(k, l) \in U_r : \text{card}\{s : (k, s) \in U_r\} < K\},
\]

where \( K \) is a positive number to be determined. Thinking of \( U_r \) an infinite \( \times \) infinite matrix with integers entries, in this splitting, we placed in \( U^1_r \) all columns of \( U_r \) that have size greater than or equal to \( K \) and in \( U^2_r \) the remaining ones. We call \( U^1_r \) the long columns of \( U_r \) and \( U^2_r \) the short columns. Let us denote

\[
E = \{k \in \mathbb{Z} : (k, l) \in U^1_r \text{ for some } l \in \mathbb{Z}\}.
\]

This set is exactly the set of projections of all long columns. Then

\[
\frac{1}{2} \left\{ 2^{-(r+1)} \|b\|_{\ell^\infty} \right\}^q \leq \sum_{(k, l) \in U^1_r} |b(k, l)|^q \leq \|b\|_{\ell^q}^q,
\]

and therefore

\[
(\text{card}E) K \left\{ 2^{-(r+1)} \|b\|_{\ell^\infty} \right\}^q \leq \sum_{(k, l) \in U^1_r} |b(k, l)|^q \leq \|b\|_{\ell^q}^q.
\]

Having broken down the wavelet coefficients in groups we proceed with the analysis of the sums of the decomposition associated these groups. Given \( (k, l) \in \mathbb{Z} \times \mathbb{Z} \), it follows from the definition of \( \Psi^\lambda_G \) that \( \Psi^\lambda_G \) can be written in the tensor product form

\[
\Psi^\lambda_G(x_1, x_2) = \omega_{1, k}(x_1) \omega_{2, l}(x_2)
\]

and

\[
\|\omega_{1, k}\|_{L^\infty} \approx \|\omega_{2, l}\|_{L^\infty} = 2^{\frac{\lambda}{2}}.
\]

Define

\[
m_{\text{r.1}} = \sum_{(k, l) \in U^1_r} b(k, l) \Psi^\lambda_G = \sum_{(k, l) \in U^1_r} b(k, l) \omega_{1, k} \omega_{2, l}.
\]

Let \( F^{-1} \) denote the inverse Fourier transform. Then

\[
\left\| T_{m_{\text{r.1}}} (f, g) \right\|_{L^1} \leq \left\| \sum_{(k, l) \in U^1_r} b(k, l) F^{-1}(\omega_{1, k} \hat{f}) F^{-1}(\omega_{2, l} \hat{g}) \right\|_{L^1} \leq \sum_{k \in E} \left\| \omega_{1, k} \hat{f} \right\|_{L^2} \left\| \sum_{l : (k, l) \in U^1_r} b(k, l) \omega_{2, l} \hat{g} \right\|_{L^2} \leq C \sum_{k \in E} \left\| \omega_{1, k} \hat{f} \right\|_{L^2} 2^{\frac{\lambda}{2} - r} \|b\|_{\ell^\infty} \|g\|_{L^2}.
\]
the estimate $K$ are equal. The optimal choice of $K$ is

$$M \left( \sum_{k \in E} 1 \right)^{1/2} \left( \sum_{k \in E} \left\| \omega_{1,k} \hat{f} \right\|_{L^2}^2 \right)^{1/2} 2^{3/2} 2^{-r} \left\| b \right\|_{L^2} \left\| g \right\|_{L^2}$$

$$\leq C \left\{ K^{-\frac{1}{2}} \left[ 2^{-(r+1)} \left\| b \right\|_{L^2} \right] \left\| b \right\|_{L^2}^{\frac{2}{3}} \right\} \left\{ 2^{3/2} 2^{-r} \left\| b \right\|_{L^2} \right\} 2^{3/2} \left\| f \right\|_{L^2} \left\| g \right\|_{L^2},$$

where we used estimate (8) and the property that the supports of the functions $\omega_{1,k}$ and $\omega_{2,l}$ have finite overlap.

Now define

$$m^{r,2} = \sum_{(k,l) \in U_r^2} b_{(k,l)} \omega_{1,k} \omega_{2,l}.$$

Then

$$\left\| T_{m^{r,2}} (f, g) \right\|_{L^1} \leq \left\{ \sum_{(k,l) \in U_r^2} b_{(k,l)} \mathcal{F}^{-1}(\omega_{1,k} \hat{f}) \mathcal{F}^{-1}(\omega_{2,l} \hat{g}) \right\}_{L^1} \leq \left\{ \sum_{l \in Z} \left\| \omega_{2,l} \hat{g} \right\|_{L^2} \right\} \left\{ \sum_{k \in U_r^2} b_{(k,l)} \omega_{1,k} \hat{f} \right\}_{L^2} \frac{1}{2} \left( \sum_{l \in Z} \left\| \omega_{2,l} \hat{g} \right\|_{L^2}^2 \right)^{1/2} \left( \sum_{k \in U_r^2} \left\| \omega_{1,k} \hat{f} \right\|_{L^2}^2 \left\| b_{(k,l)} \right\|_{L^2}^2 \right)^{1/2} \leq C 2^{3/2} \left\| g \right\|_{L^2} \left( \sum_{k \in U_r^2} \left\| \omega_{1,k} \hat{f} \right\|_{L^2}^2 \left\| b_{(k,l)} \right\|_{L^2} \right)^{1/2} \leq C 2^{3/2} \left\| g \right\|_{L^2} 2^{-r} \left\| b \right\|_{L^2} \left\| f \right\|_{L^2} \left\| g \right\|_{L^2}.$$
Using (6) and (7) we obtain
\[ \| T_{m'} \|_{L^2 \times L^2 \to L^1} \leq C C_0^{1-\frac{q}{4}} 2^{\lambda - \lambda (1-\frac{q}{4}) (M+2) + (\frac{q}{4} - 1) \frac{3}{4} \lambda} 2^{r/2 - (1-\frac{q}{4})} \| m \|_{L^q}^{\frac{q}{4}}. \]

But
\[ 2^{\lambda - \lambda (1-\frac{q}{4}) (M+2) + (\frac{q}{4} - 1) \frac{3}{4} \lambda} = 2^{\lambda [\frac{1}{4} - \frac{1}{4} (M+1)]}, \]
and the exponent is negative only when \( M + 1 > \frac{2}{4-q} \). Thus, if we choose \( M = \lfloor \frac{2}{4-q} \rfloor \), we can sum first over \( r \) and then over \((\lambda, G)\) in \( J \), obtaining (3). This completes the proof of Theorem 1.1. \( \square \)

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References

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