THE HÖRMANDER MULTIPLIER THEOREM, I: THE LINEAR CASE REVISITED

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Abstract. We discuss $L^p(\mathbb{R}^n)$ boundedness for Fourier multiplier operators that satisfy the hypotheses of the Hörmander multiplier theorem in terms of an optimal condition that relates the distance $|\frac{1}{p} - \frac{1}{2}|$ to the smoothness $s$ of the associated multiplier measured in some Sobolev norm. We provide new counterexamples to justify the optimality of the condition $|\frac{1}{p} - \frac{1}{2}| < \frac{s}{n}$ and we discuss the endpoint case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$.

1. Introduction

To a bounded function $\sigma$ on $\mathbb{R}^n$ we associate a linear multiplier operator

$$T_\sigma(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\sigma(\xi)e^{2\pi i x \cdot \xi}d\xi,$$

where $f$ is a Schwartz function on $\mathbb{R}^n$ and $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi}dx$ is its Fourier transform. The classical theorem of Mikhlin [12] (see also Stein [20]) states that if condition

$$(1) \quad |\partial^\alpha \sigma(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

holds for all multi-indices $\alpha$ with size $|\alpha| \leq [n/2]+1$, then $T_\sigma$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$.

Mikhlin’s theorem was extended by Hörmander [10] to multipliers with fractional derivatives in some $L^r$ space. To precisely describe this extension, let $\Delta$ be the Laplacian, let $(I - \Delta)^{s/2}$ denote the operator given on the Fourier transform by multiplication by $(1 + 4\pi^2|\xi|^2)^{s/2}$ and for $s > 0$, and let $L^r_s$ be the standard Sobolev space of all functions $h$ on $\mathbb{R}^n$ with norm

$$\|h\|_{L^r_s} := \|(I - \Delta)^{s/2}h\|_{L^r} < \infty.$$ 

Let $\Psi$ be a Schwartz function whose Fourier transform is supported in the annulus of the form $\{\xi : 1/2 < |\xi| < 2\}$ which satisfies $\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1$ for all $\xi \neq 0$.

Hörmander’s extension of Mikhlin’s theorem says that if $1 < r \leq 2$ and $s > n/r$, a bounded function $\sigma$ satisfies

$$(2) \quad \sup_{k \in \mathbb{Z}} \|\hat{\Psi}\sigma(2^k \cdot)\|_{L^r_s} < \infty,$$

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i.e., $\sigma$ is uniformly (over all dyadic annuli) in the Sobolev space $L^p_s$, then $T_\sigma$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$, and is also of weak type $(1, 1)$. An endpoint result for this multiplier theorem involving a Besov space was given by Seeger [16]. The least number of derivatives imposed on the multiplier in Hörmander’s condition (2) is when $r = 2$. In this case, under the assumption of $n/2 + \varepsilon$ derivatives in $L^2$ uniformly (over all dyadic annuli), we obtain boundedness of $T_\sigma$ on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$. It is natural to ask whether $L^p$ boundedness holds for some $p$ if $s < n/2$.

Calderón and Torchinsky [2] used an interpolation technique to prove that if (2) holds, then the multiplier operator $T_\sigma$ is bounded from $L^p(\mathbb{R}^n)$ to itself whenever $p$ satisfies

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{s}{n}$$

and

$$\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{r}.$$  

It is not hard to verify that if $\sigma$ satisfies (2) and $T_\sigma$ is bounded from $L^p(\mathbb{R}^n)$ to itself, then we must necessarily have $rs \geq n$; see Proposition 2.1. Thus $\frac{1}{r} \leq \frac{s}{n}$ and comparing conditions (3) and (4) we notice that (4) restricts (3). On the other hand, if we only have conditions (2) and (3) for some $r, s$ with $rs > n$, $r \in (1, \infty)$, $s \in (0, \infty)$, then one can find an $r_o$ such that $\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{1}{r_o} < \frac{s}{n}$ and $r_o < r$. In view of standard embeddings between Sobolev spaces\(^1\) we obtain that

$$\sup_{k \in \mathbb{Z}} \left\| \hat{\Psi}_\sigma \psi^{(k)} \right\|_{L^p_s} \leq C \sup_{k \in \mathbb{Z}} \left\| \hat{\Psi}_\sigma \psi^{(k)} \right\|_{L^s} < \infty,$$

and thus we can deduce the boundedness of $T_\sigma$ on $L^p(\mathbb{R}^n)$ by the aforementioned Calderón and Torchinsky [2] result using the space $L^s_{r_o}$. So assumption (4) is not necessary.

In this note we show that (3) is optimal in the sense that within the class of multipliers $\sigma$ for which (2) holds, if $T_\sigma$ maps $L^p$ to $L^p$, then we must necessarily have $\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{s}{n}$.

**Theorem 1.1.** Fix $1 < r < \infty$ and $0 < s \leq \frac{n}{2}$ such that $rs > n$. Assume that (2) holds. Then $T_\sigma$ maps $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$ such that $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{s}{n}$. Moreover, if $T_\sigma$ is bounded from $L^p(\mathbb{R}^n)$ to itself for all $\sigma$ such that (2) holds, then we must necessarily have $\left| \frac{1}{p} - \frac{1}{2} \right| \leq \frac{s}{n}$.

The proof of the positive direction in Theorem 1.1 is mostly folklore, and is omitted. We only mention that the theorem could be proved via the interpolation result of Connnett and Schwartz [5] or directly via the interpolation technique of Calderón and Torchinsky [2]; on this see also the presentation in Carbery, Gasper, and Trebels [3]. In this note we focus on certain counterexamples related to the optimality of the hypotheses $rs > n$ and $\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{s}{n}$ of Theorem 1.1; these are in the spirit of the presentation of chapter 4 in the book of Wolff [21].

\(^1\)This could be proved via the Kato-Ponce inequality $\left\| FG \right\|_{L^1} \leq C \left\| F \right\|_{L^2} \left\| G \right\|_{L^2}$, $1/q = 1/q_1 + 1/q_2$ with $q = r_o$ and $q_1 = r$; see [11], [8].
On the line $|\frac{1}{p} - \frac{1}{2}| = \frac{1}{2}$ there are positive results for $1 < p < 2$ (see Seeger [15]) and for $p = 1$ by Seeger [16]. In Section 3 we discuss a direct way to relate the results in the cases $p = 1$ and $1 < p < 2$ via direct interpolation that yields the following result as a consequence of the main theorem in [16].

**Proposition 1.2.** Given $0 \leq s \leq \frac{n}{2}$, $1 < p < 2$ satisfy $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$, then we have

$$\|T_\sigma\|_{L^p \to L^{p,2}} \leq C \sup_{k \in \mathbb{Z}} \|\sigma(2^k \cdot)\widehat{\Psi}\|_{B^{s,1}_{n,1}}.$$ 

Here $L^{p,2}$ denotes the Lorentz space of functions $f$ for which $t^{1/p} f(t)$ lies in $L^2((0, \infty), \frac{dt}{t})$, where $f^*$ is the nondecreasing rearrangement of $f$; for the definition of the Besov space $B^{s,1}_{n,1}$ see the last section. Other types of endpoint results involving $L^p$ norms as opposed to $L^{p,2}$ norms were provided by Seeger [17]. We are also aware of a direct proof of Proposition 1.2 based on weighted $L^2$ inequalities as in Christ [4].

2. **Necessary Conditions**

In this section we discuss examples that reinforce the minimality of the conditions on the indices in Theorem 1.1. One way to see this is to use the multiplier $m_{a,b}(\xi) = \psi(\xi)|\xi|^{-a}e^{ib|\xi|^2}$ where $a > 0$, $a \neq 1$, $b > 0$, and $\psi$ is a smooth function which vanishes in a neighborhood of the origin and is equal to 1 for large $\xi$. One can verify that $m_{a,b}$ satisfies (2) for $s = b/a$ and $r > n/s$. But it is known that $T_{m_{a,b}}$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $|\frac{1}{p} - \frac{1}{2}| \leq \frac{b/a}{n}$ (see Hirschman [9, comments after Theorem 3c], Wainger [19, Part II], and Miyachi [13, Theorem 3]). Alternative examples were given in Miyachi and Tomita [14, Section 7].

In this section we provide yet new examples to indicate the necessity of the indices in Theorem 1.1. We are not sure as to whether boundedness into $L^p$, or even weak $L^p$, is valid in general under assumption (2) exactly on the critical line $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$.

**Proposition 2.1.** If for all $\sigma \in L^\infty(\mathbb{R}^n)$ such that $\sup_k \|\sigma(2^k \cdot)\widehat{\Psi}\|_{L^\infty(\mathbb{R}^n)} < \infty$ we have

$$\|T_\sigma\|_{L^p(\mathbb{R}^n) \to L^{p,2}(\mathbb{R}^n)} \leq C_p \sup_k \|\sigma(2^k \cdot)\widehat{\Psi}\|_{L^\infty(\mathbb{R}^n)} < \infty,$$

then we must necessarily have $rs > n$ and $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$.

**Proof.** First we prove the necessary condition $rs \geq n$. Let $\widehat{\chi}$ be a smooth function supported in the ball $B(0,1/10)$ in $\mathbb{R}^n$ and let $\widehat{\phi}$ be supported in the ball $B(0,1/2)$ equal to 1 on $B(0,1/5)$. Define $\widehat{f}(\xi) = \widehat{\chi}(N(\xi - a))$ with $|a| = 1$, and $\sigma(\xi) = \widehat{\phi}(N(\xi - a))$. A direct calculation gives $\|f\|_{L^p(\mathbb{R}^n)} \approx N^{-n+np}$ and $\|\sigma\|_{L^1(\mathbb{R}^n)} \leq CNsN^{-n/r}$; for the last estimate see Lemma 2.2. Moreover, $T_\sigma(f)(x) = N^{-n}\zeta(x/N)e^{2\pi i x \cdot a}$. We thus obtain that $\|T_\sigma(f)\|_{L^p(\mathbb{R}^n)} \approx N^{-n+np}$. Then (6) yields the inequality $N^{-n+np} \leq CNsN^{-n/r}N^{-n+np}$, which forces $s - n/r \geq 0$ by letting $N$ go to infinity. We note that the strict inequality $rs > n$ follows from the fact that there exist unbounded functions $\sigma$ for which the second inequality in (6) holds when $s = n/r$.

We now turn to the other necessary condition $|\frac{1}{p} - \frac{1}{2}| \leq \frac{s}{n}$. By duality it suffices to prove the case when $1 < p \leq 2$. We will prove our result by constructing an example.
We consider the case \( n = 1 \) first while the higher dimensional case will be an easy generalization.

Let \( \hat{\psi}, \hat{\varphi} \in C_0^\infty(\mathbb{R}) \) such that \( 0 \leq \hat{\varphi} \leq \chi_{[-1/100,1/100]} \) and \( \chi_{[-1/10,1/10]} \leq \hat{\psi} \leq \chi_{[-1/2,1/2]} \). Therefore \( \hat{\psi} \hat{\varphi} = \hat{\varphi} \). For a fixed large positive integer \( N \), we define

\[
(7) \quad \hat{f}_N(\xi) = \sum_{j=-N}^{N} \hat{\varphi}(N\xi - j), \quad \sigma_{N,t}(\xi) = \sum_{j \in J_N} a_j(t) \hat{\psi}(N\xi - j),
\]

where \( J_N = \{ j \in \mathbb{Z} : \frac{N}{2} \leq |j| \leq 2N \} \) and \( t \in [0,1] \). Here \( \{a_j\}_{j=-\infty}^{\infty} \) is the sequence of Rademacher functions indexed by all integers.

One can verify that \( T_{N,t}(f_N) = (\sigma_{N,t}f_N)^\vee = (\sum_{j \in J_N} a_j(t) \hat{\varphi}(N\xi - j))^{\vee} \). Recall that Rademacher functions satisfy for any \( p \in (0, \infty) \)

\[
c_p \left\| \sum_j a_j(t) A_j \right\|_{L^p([0,1])} \leq \left( \sum_j |A_j|^2 \right)^{1/2} \leq C_p \left\| \sum_j a_j(t) A_j \right\|_{L^p([0,1])},
\]

where \( c_p \) and \( C_p \) are constants. Therefore

\[
\left( \int_0^1 \| T_{N,t}(f_N) \|^p_{L^p(\mathbb{R})} \, dt \right)^{1/p} = \left( \int_0^1 \left( \int_\mathbb{R} \left| \sum_{j \in J_N} a_j(t) N^{-1} \varphi(N^{-1}x)e^{2\pi ij/N} \right|^p \, dx \right) \, dt \right)^{1/p}
\]

\[
\approx \left( \int_\mathbb{R} \left( \sum_{j \in J_N} \left| N^{-1} \varphi(N^{-1}x) e^{2\pi ij/N} \right|^2 \right)^{p/2} \, dx \right)^{1/p}
\]

\[
\approx N^{-1} \left( \int_\mathbb{R} \left| N^{1/2} \varphi(N^{-1}x) \right|^p \, dx \right)^{1/p}
\]

\[
\approx N^{1/p-1/2}.
\]

The Sobolev norm of \( \sigma_{N,t} \) is given by the following lemma, proved in all dimensions. For \( n \geq 1 \) and \( \vec{t} = (t_1, \ldots, t_n) \in [0,1]^n \) we define a function on \( \mathbb{R}^n \) by

\[
\sigma_{N,\vec{t}}(\xi_1, \ldots, \xi_n) = \sum_{j \in J_N} a_{j_1}(t_1) \cdots a_{j_n}(t_n) \hat{\varphi}(N\xi_1 - j_1) \cdots \hat{\varphi}(N\xi_n - j_n),
\]

where \( J_N = \{ \vec{j} = (j_1, \ldots, j_n) \in \mathbb{Z}^n : N \geq j_k \leq 2N, 1 \leq k \leq N \} \). This \( \sigma_{N,\vec{t}} \) coincides with \( \sigma_{N,t} \) when \( n = 1 \).

**Lemma 2.2.** We have that \( \| \sigma_{N,\vec{t}} \|_{L^4(\mathbb{R}^n)} \leq CN^4 \).

**Proof.** It is easy to verify that \( \| \sigma_{N,\vec{t}} \|_{L^4} \leq C \) and \( \| \sigma_{N,\vec{t}} \|_{L^2} \leq CN^2 \). The rest follows by interpolation between Sobolev spaces \([1]\). \( \square \)

We continue with the proof of Proposition 2.1 when \( n = 1 \). We note that \( \hat{f}_N \) has \( L^q \) norm bounded by a constant independent of \( N \), which implies by the Young’s inequality that \( \| f_N \|_{L^q} \leq C \) with \( C \) independent of \( N \) when \( 2 \leq q \leq \infty \). We show in the following lemma that this property is valid for all \( q \in (1, \infty] \).

**Lemma 2.3.** Let \( f_N \) be as in (7) and let \( p \in (1, \infty] \). Then there is a constant \( C_p \) independent of \( N \) such that \( \| f_N \|_{L^p} \leq C_p \).
Proof. We note that $f_N = \sum_{j=-N}^{N} \frac{1}{N} \varphi(x/N) e^{2\pi ijx/N} = \frac{1}{N} \varphi(x/N) D_N(x/N)$, where $D_N$ is the Dirichlet kernel, whose $L^p$-norm over $[0, 1]$ is comparable to $N^{1/p'}$ when $p > 1$; see for example [6, Exercise 3.1.6]. Using this fact and that $\varphi$ is a Schwartz function we obtain

$$
\|f_N\|_{L^p(\mathbb{R})} = \left\| \frac{1}{N} \varphi(x/N) D_N(x/N) \right\|_{L^p(\mathbb{R})}
= \frac{1}{N} N^{1/p} \|\varphi D_N\|_{L^p(\mathbb{R})}
= N^{-1/p'} \left( \sum_{j=-\infty}^{\infty} \int_{j-1}^{j} |\varphi(x) D_N(x)|^p dx \right)^{1/p}
\leq C N^{-1/p'} \left( \sum_{j=-\infty}^{\infty} \frac{1}{(1+|j|)^M} \int_{j-1}^{j} |D_N(x)|^p dx \right)^{1/p}
\leq C_p N^{-1/p'} N^{1/p'} = C_p.
$$

This proves the claim. \( \square \)

In view of Lemma 2.3 we obtain the following inequalities

$$
N^{\frac{1}{p'} - \frac{1}{2}} \leq C \left( \int_0^1 \| T_{N,t}(f_N) \|_{L^p(L^p(\mathbb{R}))}^p dt \right)^{\frac{1}{p}} \leq C A \| f_N \|_p \left( \int_0^1 \| \sigma_{N,t} \|_{L^s_{\infty}}^{p} dt \right)^{\frac{1}{p}} \leq C C_p A N^s.
$$

Letting $N$ go to infinity forces $1/p - 1/2 \leq s$.

We now consider the higher dimensional case. Let $F_N(x) = f_N(x_1) \cdots f_N(x_n)$, where $f_N$ is as in (7). It follows from Lemma 2.2 and 2.3 that $\| F_N \|_{L^p} \leq C$ and $\| \sigma_N \|_{L^s_{\infty}} \leq C N^s$. A calculation similar to the one dimensional case shows that $\| T_N(F_N) \|_{L^p} \approx N^{(1/p-1/2)n}$, thus letting $N \to \infty$ we obtain that $|1/p - 1/2| \leq s/n$. \( \square \)

Related examples were given by Olevskii [18] who showed among other things that $L^2_{n/2} \cap L^\infty$ is not contained in the space of $L^p$ Fourier multipliers for $p \neq 2$.

3. The endpoint case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$

In this section we discuss an interpolation theorem applicable in the critical case $|\frac{1}{p} - \frac{1}{2}| = \frac{s}{n}$. We introduce the Besov space norm

$$
\| h \|_{B^s_{p,q}} := \left( \sum_{j \geq 1} \| 2^{js} \Delta_j h \|_{L^p}^q \right)^{\frac{1}{q}} + \| S_0 h \|_{L^p}
$$

where $\Delta_j$ are the Littlewood-Paley operators and $S_0$ is an averaging operator that satisfy $S_0 + \sum_{j=1}^{\infty} \Delta_j = I$. We assume that for $j \geq 1$, $\Delta_j$ have spectra supported in the annuli $2^j \leq |\xi| \leq 2^{j+2}$, while $S_0$ has spectrum inside the ball $B(0, 2)$.

We recall the following result of Seeger [16]

$$
\| T_\sigma \|_{H^{1/2} \to L^{1,2}} \leq C \sup_{k \in \mathbb{Z}} \| \sigma(2^k \cdot) \widehat{\Psi} \|_{B^{-\frac{n}{2},q}}
$$

(8)
concerning the endpoint case \( p = 1 \). We also have the trivial estimate
\[
\| T_\sigma \|_{L^2 \to L^2} = \| T_\sigma \|_{L^2 \to L^2} \leq C \sup_{k \in \mathbb{Z}} \| \sigma(2^k \cdot) \hat{\Psi} \|_{B^{0,1}_k}.
\]

(9)

In this section, we derive the intermediate estimate contained in Seeger [15]:
\[
\| T_\sigma \|_{L^p \to L^p} \leq C \sup_{k \in \mathbb{Z}} \| \sigma(2^k \cdot) \hat{\Psi} \|_{B^{0,1}_k}
\]

for \( |\frac{1}{p} - \frac{1}{2}| = \frac{\varepsilon}{n}, 1 < p < 2, \) and \( 0 \leq s \leq \frac{\varepsilon}{2} \). We deduce (10) from the following theorem.

**Theorem 3.1.** Fix \( 1 < r_0, r_1 \leq \infty, 1 < p_0, p_1 < \infty, 0 \leq s_0, s_1 < \infty \). Let \( \hat{\Psi} \) be supported in the annulus \( 1/2 \leq |\xi| \leq 2 \) on \( \mathbb{R}^n \) and satisfy
\[
\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j} \xi) = 1, \quad \xi \neq 0.
\]

Assume that for \( k \in \{0, 1\} \) we have
\[
\| T_\sigma(f) \|_{L^{p_k}, 2} \leq K_k \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \hat{\Psi} \|_{B^{0,1}_{p_k}} \| f \|_{L^{p_k}}
\]

(11)

for all \( f \in C_c^\infty(\mathbb{R}^n) \) and \( \sigma \) which make the right hand side finite. For \( 0 < \theta < 1 \) define
\[
\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}, \quad s = (1 - \theta)s_0 + \theta s_1.
\]

Then there is a constant \( C_* = C_*(r_0, r_1, s_0, s_1, p_0, p_1, p, n) \) such that for all \( f \in C_c^\infty(\mathbb{R}^n) \) we have
\[
\| T_\sigma(f) \|_{L^{p(\mathbb{R}^n)}} \leq C_* K_0^{1 - \theta} K_1^\theta \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \hat{\Psi} \|_{B^{0,1}_r} \| f \|_{L^p(\mathbb{R}^n)}.
\]

Moreover, conclusion (12) also holds under the assumption that \( p_0 = 1 \) and (11) is substituted (only for \( k = 0 \)) by
\[
\| T_\sigma(f) \|_{L^{1, 2}} \leq K_0 \sup_{j \in \mathbb{Z}} \| \sigma(2^j \cdot) \hat{\Psi} \|_{B^{0,1}_{p_0}} \| f \|_{H^1}
\]

(13)

for all \( f \in C_c^\infty(\mathbb{R}^n) \) with vanishing integral.

**Proof.** Let \( \hat{\Phi}(\xi) = \sum_{j \leq 0} \hat{\Psi}(2^{-j} \xi) \) and \( \hat{\Phi}(0) = 1 \); then \( \hat{\Phi} \) is supported in \( |\xi| \leq 2 \). Fix a bounded function \( \sigma \). For an integer \( k \) define the dilation of \( \sigma^k \) by setting \( \sigma^k(\xi) = \sigma(2^k \xi) \).

For \( z \) in the closed unit strip we introduce linear functions
\[
L(z) = \frac{r}{r_0} (1 - z) + \frac{r}{r_1} z, \quad M(z) = s - (1 - z)s_0 - zs_1
\]

and when \( j \geq 1 \) introduce Littlewood-Paley operators \( \Delta_j(g) = g \ast \hat{\Psi}_{2^{-j}}, \tilde{\Delta}_j(g) = g \ast \hat{\Psi}_{2^{-j}}, \) where \( \hat{\Psi} \) is a Schwartz function whose Fourier transform is supported in an annulus only slightly larger than \( 1/2 \leq |\xi| \leq 2 \) and equals 1 on the support of \( \hat{\Psi} \). We also define \( \Delta_0(g) = g \ast \hat{\Phi} \) and \( \tilde{\Delta}_0(g) = g \ast \hat{\Phi} \), where the Fourier transform of \( \hat{\Phi} \) is supported in \( |\xi| \leq 4 \) and equals 1 on the support of \( \hat{\Phi} \). Then define:
\[
\sigma_z = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{M(z)} (e_j^k)^{1-L(z)} \tilde{\Delta}_j \left( \left| \Delta_j (\sigma^k \hat{\Psi}) \right|^{L(z)} e^{i \text{Arg} \left( \Delta_j (\sigma^k \hat{\Psi}) \right)} \right) (2^{-k}) \hat{\Psi}(2^{-k})
\]
where
\[
e_j^k = \left\| \Delta_j (\sigma^k \hat{\Psi}) \right\|_{L^p} \left( \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls} \left\| \Delta_i (\sigma^\mu \hat{\Psi}) \right\|_{L^p} \right)^{-1}.
\]

Next, we estimate
\[
(14) \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls_0} \left\| \Delta_i (\sigma^\mu \hat{\Psi}) \right\|_{L^{p_0}}.
\]

We notice that for a given \( \mu \in \mathbb{Z} \), in the sum defining \( \sigma^\mu \), only finitely many terms in \( k \) appear, the ones with \( k = \mu, \mu + 1, \mu - 1 \). For simplicity we only consider the term with \( k = \mu \), since the other ones are similar. This part of (14) is estimated by
\[
(15) \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0}^{\infty} 2^{ls_0} 2^{l(s-s_0)} |c_j^l|^{1-\frac{\mu}{2 \mu_0}} \left\| \tilde{\Delta}_l \left( \Delta_j (\sigma^\mu \hat{\Psi}) \right|^{L(it)} e^{i \text{Arg} \left( \Delta_j (\sigma^\mu \hat{\Psi}) \right)} \right\|_{L^{p_0}}.
\]

Using Lemma 3.2 (stated and proved below) we obtain that (15) is bounded by
\[
(16) \sum_{\mu \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{js_0} 2^{j(s-s_0)} |c_j^l|^{1-\frac{\mu}{2 \mu_0}} C_M 2^{-2|l| \max(j,l) \mu} \left\| \Delta_j (\sigma^\mu \hat{\Psi}) \right\|_{L^{p_0}}.
\]

But the sum over \( l \) in (16) is bounded by \( C_M 2^{j s_0} 2^{-2|j| \max(j,l) \mu} \leq C_M 2^{js_0} \) for \( M \) sufficiently large, and consequently (16) is bounded by
\[
(17) \sum_{\mu \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{j(s-s_0)} 2^{js_0} |c_j^l|^{1-\frac{\mu}{2 \mu_0}} \left\| \Delta_j (\sigma^\mu \hat{\Psi}) \right\|_{L^{p_0}} \leq C_M \left( \sup_{\mu \in \mathbb{Z}} \sum_{j=0}^{\infty} 2^{js} \left\| \Delta_j (\sigma^\mu \hat{\Psi}) \right\|_{L^{p_0}} \right)^{\frac{\mu}{\mu_0}}
\]
by the choice of \( c_j^l \). Likewise we obtain a similar estimate for the point \( 1 + it \). We summarize these two estimates as follows: for \( m = 0, 1 \) and \( \text{Re} \ z = m \) we have
\[
(18) \sup_{\mu \in \mathbb{Z}} \sum_{l \geq 0} 2^{ls_m} \left\| \Delta_i (\sigma^\mu \hat{\Psi}) \right\|_{L^{p_m}} \leq C_M \left( \sup_{\mu \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \left\| \Delta_j (\sigma^\mu \hat{\Psi}) \right\|_{L^{p_1}} \right)^{\frac{\mu}{\mu_0}}, \quad m = 0, 1.
\]

Now consider the analytic family of operators \( T_z \) associated with \( \sigma_z \) defined by \( f \mapsto T_{\sigma_z}(f) \). When \( \text{Re} \ z = 0 \), \( T_z \) maps \( L^{p_0,2} \) to \( L^{p_0} \) if \( p_0 > 1 \) and \( H^1 \) to \( L^{1,2} \) if \( p_0 = 1 \) with constant \( B_0 \) and when \( \text{Re} \ z = 1 \), \( T_z \) maps \( L^{p_1,2} \) to \( L^{p_1} \) with constant \( B_1 \), where
\[
B_m = C_M K \left( \sup_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^{js} \left\| \Delta_j (\sigma^k \hat{\Psi}) \right\|_{L^{p_1}} \right)^{\frac{\mu}{\mu_0}}, \quad m = 0, 1.
\]

We now interpolate using Theorem 1.1 (with \( m = 1 \)) in [7]. We obtain
\[
\| T_{\sigma_1} (f) \|_{(L^{p_0,2})^{1-\theta} (L^{p_1,2})^\theta} \leq C(p_0, p_1, p) B_0^{1-\theta} B_1^{\theta} \| f \|_{(L^{p_0,2} (L^{p_1,2}))^\theta}.
\]

Noting that \( (L^{p_0,2})^{1-\theta} (L^{p_1,2})^\theta = L^p \) and \( (L^{p_0}, L^{p_1})^\theta = L^p \) (even when \( p_0 = 1 \), in which case \( L^{p_0} \) is replaced by \( H^1 \)), we obtain the claimed assertion. \( \square \)
Lemma 3.2. Using the notation of Theorem 3.1, for any $M > 0$ there is a constant $C_M$ (also depending on the dimension $n$, on $\Psi$, and $\hat{\Psi}$) such that for any $1 \leq q \leq \infty$ we have
\[
\left\| \tilde{\Delta}_l(\Delta_j(g)\hat{\Psi}\hat{\Psi}) \right\|_{L^q} \leq C_M 2^{-2(1-\frac{l}{q})\max(j,l)M}\|g\|_{L^q}
\]
for all $l, j > 0$. We also have that for any $M > n$ there is a constant $C_M$ such that
\[
\left\| \tilde{\Delta}_l(\Delta_j(g)\hat{\Psi}\hat{\Psi}) \right\|_{L^1} \leq C_M 2^{-\max(j,l)(M-n)}\|g\|_{H^1}.
\]
Proof. The claimed estimate is obviously true when $q = 1$. So we prove it for $q = 2$ and derive (19) as a consequence of classical Riesz-Thorin interpolation theorem. Examining the Fourier transform of the operator in (19), matters reduce to computing the $L^\infty$ norm of the function
\[
\hat{\Psi}(2^{-j}\xi) \int_{\mathbb{R}^n} \hat{\Psi}(2^{l-j}(\xi - \eta))\phi(\eta)d\eta
\]
where $\phi(\eta) = \Psi \ast \hat{\Psi}$ is a Schwartz function. Since the integral is over the set $|\xi - \eta| \approx 2^l$, we estimate the absolute value of (21) by $C\|\hat{\Psi}\|_{L^\infty}$.

We now turn our attention to (20). Using Fourier inversion, we write
\[
\tilde{\Delta}_l(\Delta_j(g)\hat{\phi})(x) = \int_{\mathbb{R}^n} \hat{g}(\eta)\hat{\Psi}(2^{l-j}\eta) \int_{\mathbb{R}^n} \hat{\Psi}(2^{-j}\xi)\phi(\xi - \eta)e^{2\pi i x \cdot \xi} d\xi d\eta.
\]
We integrate by parts in the inner integral with respect to the operator $(I - \Delta_\xi)^N$ to obtain that the preceding expression is equal to
\[
\sum_{\beta + \gamma = 2N} \frac{C_{\beta,\gamma}}{(1 + 4\pi^2 |x|^2)^N} \int_{\mathbb{R}^n} \hat{g}(\eta)\hat{\Psi}(2^{l-j}\eta) \int_{\mathbb{R}^n} 2^{-j|\beta|}(\partial^\beta \hat{\Psi})(2^{-j}\xi)(\partial^\gamma \phi)(\xi - \eta)e^{2\pi i x \cdot \xi} d\xi d\eta.
\]
Since for $g \in H^1$ we have $|\hat{g}(\xi)| \leq c\|g\|_{H^1}$ for all $\xi$ and we deduce the estimate
\[
|\tilde{\Delta}_l(\Delta_j(g)\hat{\phi})(x)| \leq \frac{C_M\|g\|_{H^1}}{(1 + 4\pi^2 |x|^2)^N} 2^{ln} \sup_{|\eta| \approx 2^l} \int_{|\xi| \approx 2^j} \frac{d\xi}{(1 + |\xi - \eta|^2)^{2M}}
\]
for $M > n$. We easily derive from this estimate the validity of (20). Note that in the case $j = 0$ the notation $|\xi| \approx 2^j$ should be interpreted as $|\xi| \lesssim 2$; likewise when $l = 0$.

Proposition 1.2 is a consequence of Theorem 3.1 with initial estimates (8) and (9).

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