Example 4.5.3. (Order of zeros) Find the order $m$ of the zero of $\sin z$ at $z_0 = 0$; then express $\sin z = z^m \lambda(z)$, where $\lambda$ is analytic and satisfies $\lambda(0) \neq 0$.

**Solution.** Clearly, 0 is a zero of $\sin z$. The order of the zero is equal to the order of the first nonvanishing derivative of $f(z) = \sin z$ at 0. Since $f''(z) = \cos z$ and $\cos 0 = 1 \neq 0$, we conclude that the order of the zero at 0 is 1. We have for all $z$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \right) = z \lambda(z),$$

where $\lambda(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$. The function $\lambda(z)$ is entire (because it is a convergent power series for all $z$), and $\lambda(0) = 1$. Also, for $z \neq 0$, $\lambda(z) = \frac{\sin z}{z}$, which is entire by Example 4.3.10.

We now address the following question: Suppose that a function is analytic and not identically zero on a region. Can it have zeros that are not isolated? The answer is no, and this depends on the fact that the region is connected.

**Theorem 4.5.4. All zeros of a nonconstant analytic function on a region are isolated.**

**Proof.** Let $f$ be an analytic function defined on a region $\Omega$. Define the subsets

$$\Omega_0 = \{ z \in \Omega : f(z) = 0 \text{ and } z \text{ is not isolated} \}$$

$$\Omega_1 = \{ z \in \Omega : f(z) \neq 0 \} \cup \{ z \in \Omega : f(z) = 0 \text{ and } z \text{ is isolated} \}.$$

Then $\Omega_0$ and $\Omega_1$ are disjoint sets whose union is $\Omega$. We show that $\Omega_0$ and $\Omega_1$ are open; then by the connectedness of $\Omega$ (Proposition 2.1.7) we have either $\Omega = \Omega_0$ or $\Omega = \Omega_1$. If $\Omega = \Omega_0$, then every point in $\Omega$ is a zero of $f$; in this case $f$ vanishes everywhere on $\Omega$. If $\Omega = \Omega_1$ then every point of $\Omega$ is either not a zero or an isolated zero of $f$. In this case all zeros of $f$ are isolated.

If $w \in \Omega_0$, then assertion (ii) in Theorem 4.5.2 does not hold, hence assertion (i) must hold and thus $f$ vanishes identically in some neighborhood $B_r(w)$. Clearly, all the points in $B_r(w)$ are not isolated zeros of $f$, so $B_r(w)$ is contained in $\Omega_0$. This shows that $\Omega_0$ is open in this case. Now let $w \in \Omega_1$. If $f(w) \neq 0$, then there is a neighborhood of $w$ on which $f$ is nonvanishing; this neighborhood is contained in $\Omega_1$. If $f(w) = 0$ and $w$ is an isolated zero, then Theorem 4.5.2 guarantees the existence of a neighborhood $B_\delta(w)$ of $w$ on which $f$ has no zeros except the isolated one at $w$. This neighborhood is also contained in $\Omega_1$, since $B_\delta'(w) = B_\delta(w) \setminus \{w\}$ is contained in $\{ z \in \Omega : f(z) \neq 0 \}$. Thus $\Omega_1$ is open.

**Theorem 4.5.5. (Identity Principle)** Suppose that $f, g$ are analytic functions on a region $\Omega$. If $\{z_n\}_{n=1}^\infty$ is an sequence of distinct points in $\Omega$ with $f(z_n) = g(z_n)$ for all $n$ and $z_n \to z_0 \in \Omega$, then $f(z) = g(z)$ for all $z \in \Omega$.

**Proof.** If $z_n$ is a zero of $f - g$ in $\Omega$ and $z_n \to z_0$ as $n \to \infty$, then $f(z_n) = g(z_n)$ for all $n$ and by continuity it follows that