Thus
\[ \max |f_n(z)g(z) - f(z)g(z)| \leq M \max |f_n(z) - f(z)| \to 0, \]
because \( f_n \) converges uniformly to \( f \) on \( E \). Thus \( f_n g \to fg \) uniformly on \( E \). To prove \((ii)\), apply \((i)\) to the sequence of partial sums \( u_1 + \cdots + u_n \) which converges to \( u \).

**Proof (Theorem 4.1.10).** (i) The function \( f \) is continuous by Theorem 4.1.3\((i)\). To prove that \( f \) is analytic, we apply Morera's theorem (Theorem 3.8.10). Let \( \gamma \) be an arbitrary closed triangular path lying in a closed disk in \( \Omega \). It is enough to show that \( \int_\gamma f(z)\,dz = 0 \). We have \( \int_\gamma f_n(z)\,dz = 0 \) for all \( n \), by Cauchy's theorem (Theorem 3.5.4), because \( f_n \) is analytic inside and on \( \gamma \); and by Theorem 4.1.5, \( \int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz \) as \( n \to \infty \). So \( \int_\gamma f(z)\,dz = 0 \) and \((i)\) follows.

(ii) Let \( z_0 \in \Omega \) and let \( B_R(z_0) \) be a closed disk contained in \( \Omega \), centered at \( z_0 \) with radius \( R > 0 \), with positively oriented boundary \( C_R(z_0) \). Since \( f_n \to f \) uniformly on \( C_R(z_0) \) and \( \frac{1}{(z-z_0)^k} \) is continuous on \( C_R(z_0) \), it follows from Lemma 4.1.11\((i)\) that
\[
\frac{f_n(z)}{(z-z_0)^{k+1}} \to \frac{f(z)}{(z-z_0)^{k+1}}
\]
uniformly for all \( z \) on \( C_R(z_0) \). Applying Theorem 4.1.5 and using the generalized Cauchy integral formula, we deduce
\[
f_n^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{C_R(z_0)} \frac{f_n(z)}{(z-z_0)^{k+1}}\,dz \to \frac{k!}{2\pi i} \int_{C_R(z_0)} \frac{f(z)}{(z-z_0)^{k+1}}\,dz = f^{(k)}(z_0),
\]
which proves \((ii)\).

Theorem 4.1.10 may fail if we replace analytic functions by differentiable functions of a real variable. That is, if \( E \) is a subset of the real line and \( f_n(x) \to f(x) \) uniformly on \( E \), it does not follow in general that \( f'_n \) converges to \( f' \), as the next example shows.

**Example 4.1.12. (Failure of termwise differentiation)** For \( 0 \leq x \leq 2\pi \) and \( n = 1, 2, \ldots \), define \( f_n(x) = \frac{e^{nx}}{n^2} \). It is clear that \( f_n \to 0 \) uniformly on \([0,2\pi]\). But \( f'_n(x) = e^{nx} \) and this sequence does not converge except at \( x = 0 \) or \( x = 2\pi \). (See Example 1.5.9) Consequently, \( f'_n \) does not converge to \( 0 \). Can we understand how this occurs within the larger framework of complex functions? Replace \( x \) by \( z \) and consider the sequence functions \( f_n(z) = \frac{e^{nz}}{n^2} \). We cannot find a complex neighborhood of the real interval \([0,2\pi]\) where \( f_n \) converges, as such a neighborhood would contain \( z \) with \( \text{Im} \, z < 0 \). Thus Theorem 4.1.10 does not apply.

**Corollary 4.1.13.** Suppose that \( \{u_n\}_{n=1}^{\infty} \) is a sequence of analytic functions on a region \( \Omega \) and that \( u = \sum_{n=1}^{\infty} u_n \) converges uniformly on every closed disk in \( \Omega \). Then \( u \) is analytic on \( \Omega \). Moreover, for all integers \( k \geq 1 \), the series may be differentiated term by term \( k \) times to yield