with the interpretation that $2^{-1}I_u = \emptyset$. $(2^k I_u$ has the same center as $I_u$ but $2^k$ times its length.) It follows that for all $u$ in $U_{\text{max}}$ there exists an integer $k \geq 0$ such that

$$|E| \frac{\mu}{8} |I_u| 2^{-k} < \int_{E \cap N^{-1}[\omega_u] \cap (2^k I_u \setminus 2^{k-1} I_u)} \frac{dx}{(1 + \frac{|x-c(I_u)|}{|I_u|})^{10}} \leq \frac{|E \cap N^{-1}[\omega_u] \cap 2^k I_u|}{(\frac{2}{3})^{10}(1 + 2^{k-2})^{10}}.$$

We therefore conclude that

$$U_{\text{max}} = \bigcup_{k=0}^{\infty} U_k,$$

where

$$U_k = \{ u \in U_{\text{max}} : |I_u| \leq 8 \cdot 5^{10} 2^{-9k} \mu^{-1} |E|^{-1} |E \cap N^{-1}[\omega_u] \cap 2^k I_u| \}.$$

The required estimate (6.1.42) will be a consequence of the sequence of estimates

$$\sum_{u \in U_k} |I_u| \leq C 2^{-8k} \mu^{-1}, \quad k \geq 0. \quad (6.1.43)$$

We now fix a $k \geq 0$ and we concentrate on (6.1.43). Select an element $v_0 \in U_k$ such that $|I_{v_0}|$ is the largest possible among elements of $U_k$. Then select an element $v_1 \in U_k \setminus \{v_0\}$ such that the enlarged rectangle $(2^k I_{v_1}) \times \omega_{v_1}$ is disjoint from the enlarged rectangle $(2^k I_{v_0}) \times \omega_{v_0}$ and $|I_{v_1}|$ is the largest possible. Continue this process by induction. At the $j$th step select an element of

$$U_k \setminus \{v_0, \ldots, v_{j-1}\}$$

such that the enlarged rectangle $(2^k I_{v_j}) \times \omega_{v_j}$ is disjoint from all the enlarged rectangles of the previously selected tiles and the length $|I_{v_j}|$ is the largest possible. This process will terminate after a finite number of steps. We denote by $V_k$ the set of all selected tiles in $U_k$.

We make a few observations. Recall that all elements of $U_k$ are maximal rectangles in $U$ and therefore disjoint. For any $u \in U_k$ there exists a selected $v \in V_k$ with $|I_u| \leq |I_v|$ such that the enlarged rectangles corresponding to $u$ and $v$ intersect. Let us associate this $u$ to the selected $v$. Observe that if $u$ and $u'$ are associated with the same selected $v$, they are disjoint, and since both $\omega_u$ and $\omega_{u'}$ contain $\omega_v$, the intervals $I_u$ and $I_{u'}$ must be disjoint. Thus, tiles $u \in U_k$ associated with a fixed $v \in V_k$ have disjoint $I_u$'s and satisfy

$$I_u \subseteq 2^{k+2} I_v.$$

Consequently,

$$\sum_{u \in U_k} |I_u| \leq |2^{k+2} I_v| = 2^{k+2} |I_v|.$$