We now write \( f_L = f^1_L + f^2_L \), where
\[
\begin{align*}
f^1_L &= \sum_{j=-L}^{L} \Delta^0_\gamma(f) * \eta_{2^{-j}} * \eta_{2^{-j}}, \\
f^2_L &= \sum_{j=0}^{L} \Delta^0_\gamma(f) * \eta_{2^{-j}} * \eta_{2^{-j}}.
\end{align*}
\]

It follows from (1.4.13) that with \( C_0 = (2^\gamma + 1 + 2^{-\gamma})C_0 \) we have
\[
||\Delta^0_\gamma(f) * \eta_{2^{-j}} * \eta_{2^{-j}}||_{L^\infty} \leq ||\Delta^0_\gamma(f)||_{L^\infty} ||\eta_{2^{-j}} * \eta_{2^{-j}}||_{L^1} \leq C_0 ||\eta * \eta||_{L^2} 2^{-j\gamma};
\]

thus, \( f^2_L \) converges uniformly to a continuous and bounded function \( g_2 \) as \( L \to \infty \).

Also, \( \partial^\beta f^2_L \) converges uniformly for all \( |\beta| < \gamma \) as \( L \to \infty \). Using Lemma 1.4.7 we conclude that \( g_2 \) is in \( C^{[\gamma]} \) and that \( \partial^\beta f^2_L \) converges uniformly to \( \partial^\beta g_2 \) as \( L \to \infty \) for all \( |\beta| < \gamma \).

We now turn our attention to \( f^1_L \). Obviously, \( f^1_L \) is in \( C^{[\alpha]} \) and
\[
\partial^\alpha f^1_L = \sum_{j=-L}^{1} \Delta^0_\gamma(f) * 2^{j|\alpha|} (\partial^\alpha \eta)_{2^{-j}} * \eta_{2^{-j}}.
\]

Thus for all multi-indices \( \alpha \) with \( |\alpha| \geq |\gamma| + 1 \) we have
\[
\sup_{L \in \mathbb{Z}^+} ||\partial^\alpha f^1_L||_{L^\infty} \leq \sum_{j=-\infty}^{1} C_0 2^{-j(\gamma+1)} ||\partial^\alpha \eta * \eta||_{L^1} = c_{\alpha,\gamma} C_0 < \infty. \tag{1.4.19}
\]

Let \( P^d_L \) be the Taylor polynomial of \( f^1_L \) of degree \( d \). By Taylor’s theorem we have
\[
f^1_L(x) - P^d_L(x) = (|\gamma| + 1) \sum_{|\alpha|=|\gamma|+1} \frac{\alpha!}{\alpha!} \int_0^1 (1-t)^{\gamma}(\partial^\alpha f^1_L)(tx) \, dt. \tag{1.4.20}
\]

Using (1.4.19), with \( |\alpha| \in \{[|\gamma|] + 1, \ldots, [|\gamma|] + |\beta| + 2 \} \), we obtain that the sequence \( \{\nabla(\partial^\beta (f^1_L - P^d_L))\} \}_{L=1}^{\infty} \) is uniformly bounded on every ball \( B(0,K) \); thus, the sequence \( \{\partial^\beta (f^1_L - P^d_L)\} \}_{L=1}^{\infty} \) is equicontinuous on every such ball. By the Arzelà–Ascoli theorem, for every \( K = 1, 2, \ldots \) and for every \( |\beta| < \gamma \) there is a subsequence of \( \{\partial^\beta (f^1_L - P^d_L)\} \}_{L=1}^{\infty} \) that converges uniformly on \( B(0,K) \). The diagonal subsequence of these subsequences converges uniformly on every compact subset of \( \mathbb{R}^n \) for all \( |\beta| < \gamma \). Hence, there is a continuous function \( g_1 \) on \( \mathbb{R}^n \) and a subsequence \( L_m \) of \( \mathbb{Z}^+ \) such that \( f^1_{L_m} - P_{L_m}^d \to g_1 \) uniformly on compact sets as \( m \to \infty \) and \( \partial^\beta (f^1_{L_m} - P_{L_m}^d) \) converges uniformly on compact sets for all \( |\beta| \geq |\gamma| \). Using Lemma 1.4.7, stated at the end of this proof, we deduce that \( g_1 \) is \( C^{[\gamma]} \) and that \( \partial^\beta (f^1_{L_m} - P_{L_m}^d) \to \partial^\beta g_1 \) as \( m \to \infty \) for all \( |\beta| \leq |\gamma| \).

Set \( g = g_1 + g_2 \). It follows from (1.4.20) and from \( \sup_{L \in \mathbb{Z}^+} ||f^2_L||_{L^\infty} < \infty \) that \( |g(x)| \leq C_{n,\gamma} C_0(1 + |x|)^{[|\gamma|] + 1} \) for all \( x \in \mathbb{R}^n \). Thus, \( g \) can be viewed as an element of \( C^{[|\gamma|]}(\mathbb{R}^n) \). Since both \( g_1 \) and \( g_2 \) are in \( C^{[|\gamma|]} \), it follows that so is \( g \).