and let $\delta_{2^{-j}}(t)$ be Dirac mass at the point $t = 2^{-j}$. Then there is a constant $C_{n,\delta}$ such that
\[
\int_{\mathbb{R}^{n+1}_+} |(\Psi_{2^{-j}} * b)(x)|^2 \, dx \delta_{2^{-j}}(t)
\]
is a Carleson measure on $\mathbb{R}^{n+1}_+$ with norm at most $C_{n,\delta}(A + B)^2 \|b\|^2_{\text{BMO}}$.

(b) Suppose that
\[
\sup_{\xi \in \mathbb{R}^n} \int_0^\infty |\hat{\Psi}(t\xi)|^2 \frac{dt}{t} \leq B^2 < \infty.
\]
Then the continuous version $d\nu(x,t)$ of $d\mu(x,t)$ defined by
\[
d\nu(x,t) = |(\Psi_t * b)(x)|^2 \, dx \, dt
\]
is a Carleson measure on $\mathbb{R}^{n+1}_+$ with norm at most $C_{n,\delta}(A + B)^2 \|b\|^2_{\text{BMO}}$ for some constant $C_{n,\delta}$.

(c) Let $\delta, A > 0$. Suppose that $\{K_t\}_{t > 0}$ are functions on $\mathbb{R}^n \times \mathbb{R}^n$ that satisfy
\[
|K_t(x,y)| \leq \frac{At^\delta}{(t + |x-y|)^{n+\delta}}
\]
for all $t > 0$ and all $x, y \in \mathbb{R}^n$. Let $R_t$ be the linear operator
\[
R_t(f)(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y) \, dy,
\]
which is well defined for all $f \in \bigcup_{1 \leq p \leq \infty} L^p(\mathbb{R}^n)$. Suppose that $R_t(1) = 0$ for all $t > 0$ and that there is a constant $B > 0$ such that
\[
\int_0^\infty \int_{\mathbb{R}^n} |R_t(f)(x)|^2 \frac{dx\,dt}{t} \leq B^2 \|f\|^2_{L^2(\mathbb{R}^n)}
\]
for all $f \in L^2(\mathbb{R}^n)$. Then for all $b$ in BMO the measure
\[
|R_t(b)(x)|^2 \frac{dx\,dt}{t}
\]
is Carleson with norm at most a constant multiple of $(A + B)^2 \|b\|^2_{\text{BMO}}$.

We note that if, in addition to (3.3.12), the function $\Psi$ has mean value zero and satisfies $|\nabla \Psi(x)| \leq A(1 + |x|)^{-n-\delta}$, then (3.3.13) and (3.3.14) hold and therefore conclusions (a) and (b) of Theorem 3.3.8 follow. (See for instance [156, Page 422]).

Proof. We prove (a). The measure $\mu$ is defined so that for every $\mu$-integrable function $F$ on $\mathbb{R}^{n+1}_+$ we have
\[
\int_{\mathbb{R}^{n+1}_+} F(x,t) \, d\mu(x,t) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |(\Psi_{2^{-j}} * b)(x)|^2 F(x,2^{-j}) \, dx.
\]
For a cube $Q$ in $\mathbb{R}^n$ we let $Q^\ast$ be the cube with the same center and orientation whose side length is $3\sqrt{n}\ell(Q)$, where $\ell(Q)$ is the side length of $Q$. Fix a cube $Q$ in $\mathbb{R}^n$, take $F$ to be the characteristic function of the tent of $Q$, and split $b$ as

$$b = (b - \text{Avg}_Q b) \chi_{Q^\ast} + (b - \text{Avg}_Q b) \chi_{Q^\ast} + \text{Avg}_Q b.$$ 

Since $\Psi$ has mean value zero, $\Psi_{2^{-j}} \ast \text{Avg}_Q b = 0$. Then (3.3.17) gives

$$\mu(T(Q)) = \sum_{2^{-j} \leq \ell(Q)} \int_Q |\Delta_j(b)(x)|^2 dx \leq 2\Sigma_1 + 2\Sigma_2,$$

where

$$\Sigma_1 = \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} |\Delta_j((b - \text{Avg}_Q b) \chi_{Q^\ast})(x)|^2 dx,$$

$$\Sigma_2 = \sum_{2^{-j} \leq \ell(Q)} \int_Q |\Delta_j((b - \text{Avg}_Q b) \chi_{Q^\ast})(x)|^2 dx.$$

Using Plancherel’s theorem and (3.3.13), we obtain

$$\Sigma_1 \leq \sup_{\xi} \sum_{j \in \mathbb{Z}} |\hat{\Psi}(2^{-j}\xi)|^2 \int_{\mathbb{R}^n} \left|((b - \text{Avg}_Q b) \chi_{Q^\ast})(\eta)\right|^2 d\eta$$

$$\leq B^2 \int_{Q^\ast} \left|b(x) - \text{Avg}_Q b\right|^2 dx$$

$$\leq 2B^2 \int_{Q^\ast} \left|b(x) - \text{Avg}_Q b\right|^2 dx + 2B^2 \left|Q^\ast\right| \left|\text{Avg}_Q b - \text{Avg}_Q b\right|^2$$

$$\leq B^2 \int_{Q^\ast} \left|b(x) - \text{Avg}_Q b\right|^2 dx + c_n 2B^2 \left\|b\right\|_{\text{BMO}}^2 |Q|$$

$$\leq C_n B^2 \left\|b\right\|_{\text{BMO}}^2 |Q|,$$

where the used the analogue of (3.1.4) for cubes and Corollary 3.1.8. To estimate $\Sigma_2$, we use the size estimate of the function $\Psi$. We obtain

$$\left|\Psi_{2^{-j}} \ast (b - \text{Avg}_Q b) \chi_{Q^\ast}(x)\right| \leq \int_{(Q^\ast)^c} \frac{A 2^{-j\delta} |b(y) - \text{Avg}_Q b|}{(2^{-j} + |x - y|)^{n+\delta}} dy. \quad (3.3.18)$$

But note that if $c_Q$ is the center of $Q$, then

$$2^{-j} + |x - y| \geq |y - x|$$

$$\geq |y - c_Q| - |c_Q - x|$$

$$\geq \frac{1}{2} |c_Q - y| + \frac{3\sqrt{n}}{4} \ell(Q) - |c_Q - x|$$