integral when \( g \) is a general element of \( H^1(\mathbb{R}^n) \) and \( \tilde{b} \) remains the same if \( b \) is replaced by \( b + c \), where \( c \) is an additive constant. Additionally, we observe that (3.2.1) is also an absolutely convergent integral. Observe that the integral in (3.2.1) and thus the functionals

\[
L_b(g) = \int_{\mathbb{R}^n} g(x)b(x)\,dx
\]  

as an absolutely convergent integral. Observe that the integral in (3.2.1) and thus the definition of \( L_b \) on \( H_0^1 \) remain the same if \( b \) is replaced by \( b + c \), where \( c \) is an additive constant. Additionally, we observe that (3.2.1) is also an absolutely convergent integral when \( g \) is a general element of \( H^1(\mathbb{R}^n) \) and the \( BMO \) function \( b \) is bounded.

To extend the definition of \( L_b \) on the entire \( H^1 \) for all functions \( b \) in \( BMO \) we need to know that

\[
\|L_b\|_{H^1 \to C} \leq C_n \|b\|_{BMO}, \quad \text{whenever } b \text{ is bounded},
\]  

(3.2.2)

a fact that will be proved momentarily. Assuming (3.2.2), take \( b \in BMO \) and let \( b_M(x) = b1_{|b| \leq M} \) for \( M = 1, 2, 3, \ldots \). Since \( \|b_M\|_{BMO} \leq \frac{M}{C_n} \|b\|_{BMO} \) (Exercise 3.1.4), the sequence of linear functionals \( \{L_{b_M}\}_M \) lies in a multiple of the unit ball of \( (H^1)^* \) and by the Banach–Alaoglu theorem there is a subsequence \( M_j \to \infty \) as \( j \to \infty \) such that \( L_{b_M} \) converges weakly to a bounded linear functional \( \tilde{L}_b \) on \( H^1 \). This means that for all \( f \) in \( H^1(\mathbb{R}^n) \) we have

\[
L_{b_{M_j}}(f) \to \tilde{L}_b(f)
\]

as \( j \to \infty \).

If \( a_Q \) is a fixed \( L^2 \) atom for \( H^1 \), the difference \( |L_{b_M}(a_Q) - L_b(a_Q)| \) is bounded by \( \|a_Q\|_{L^2} (\|b_M - \text{Avg}_Q b_M - b + \text{Avg}_Q b\|_{L^2(Q)}) \) which is in turn bounded by \( \|a_Q\|_{L^2} (\|b_M - b\|_{L^2(Q)} + |Q|^{1/2} \|\text{Avg}_Q(b_M - b)\|_{L^2(Q)}) \), and this expression tends to zero as \( j \to \infty \) by the Lebesgue dominated convergence theorem. The same conclusion holds for any finite linear combination of \( a_Q \)'s. Thus for all \( g \in H_0^1 \) we have

\[
L_{b_{M_j}}(g) \to L_b(g),
\]

and consequently, \( L_b(g) = \tilde{L}_b(g) \) for all \( g \in H_0^1 \). Since \( H_0^1 \) is dense in \( H^1 \) and \( L_b \) and \( \tilde{L}_b \) coincide on \( H_0^1 \), it follows that \( \tilde{L}_b \) is the unique bounded extension of \( L_b \) on \( H^1 \).

We have therefore defined \( L_b \) on the entire \( H^1 \) as a weak limit of bounded linear functionals.

Having set the definition of \( L_b \), we proceed by showing the validity of (3.2.2). Let \( b \) be a bounded \( BMO \) function. Given \( f \) in \( H^1 \), find a sequence \( a_k \) of \( L^2 \) atoms for \( H^1 \) supported in cubes \( Q_k \) such that

\[
f = \sum_{k=1}^{\infty} \lambda_k a_k
\]  

(3.2.3)