5.2 Singular Integrals and the Method of Rotations

Remark 5.2.9. It follows from the proof of Theorem 5.2.7 and from Theorems 5.1.7 and 5.1.12 that whenever $\Omega$ is an odd function on $S^{n-1}$, we have

$$
\|T_\Omega\|_{L^p \to L^p} \leq \|\Omega\|_{L^1} \begin{cases} 
ap \quad \text{when } p \geq 2, \\
ap \(p-1)^{-1} \quad \text{when } 1 < p \leq 2,
\end{cases}
$$

for some $a > 0$ independent of $p$ and the dimension.

5.2.4 Singular Integrals with Even Kernels

Since a general integrable function $\Omega$ on $S^{n-1}$ with mean value zero can be written as a sum of an odd and an even function, it suffices to study singular integral operators $T_\Omega$ with even kernels. For the rest of this section, fix an integrable even function $\Omega$ on $S^{n-1}$ with mean value zero. The following idea is fundamental in the study of such singular integrals. Proposition 5.1.16 implies that

$$
T_\Omega = - \sum_{j=1}^{n} R_j R_j T_\Omega.
$$

(5.2.23)

If $R_j T_\Omega$ were another singular integral operator of the form $T_{\Omega_j}$ for some odd $\Omega_j$, then the boundedness of $T_\Omega$ would follow from that of $T_{\Omega_j}$ via the identity (5.2.23) and Theorem 5.2.7. It turns out that $R_j T_\Omega$ does have an odd kernel, but it may not be integrable on $S^{n-1}$ unless $\Omega$ itself possesses an additional amount of integrability. The amount of extra integrability needed is logarithmic, more precisely of this sort:

$$
c_\Omega = \int_{S^{n-1}} |\Omega(\theta)| \log(e + |\Omega(\theta)|) d\theta < \infty.
$$

(5.2.24)

Observe that we always have

$$
\|\Omega\|_{L^1} \leq c_\Omega.
$$

The following theorem is the main result of this section.

Theorem 5.2.10. Let $n \geq 2$ and let $\Omega$ be an even integrable function on $S^{n-1}$ with mean value zero that satisfies (5.2.24). Then the corresponding singular integral $T_\Omega$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, with norm at most a dimensional constant multiple of the quantity $\max\left(\((p-1)^{-2}, p^2\right) c_\Omega$.

If the operator $T_\Omega$ in Theorem 5.2.10 is weak type $(1, 1)$, then the estimate on the $L^p$ operator norm of $T_\Omega$ can be improved to $\|T_\Omega\|_{L^p \to L^p} \leq C_n (p-1)^{-1} c_\Omega$ as $p \to 1$. This is indeed the case; see the historical comments at the end of this chapter.
Proof. Let $W_{\Omega}$ be the distributional kernel of $T_{\Omega}$. We have that $W_{\Omega}$ coincides with the function $\Omega(x/|x|)|x|^{-n}$ on $\mathbb{R}^n \setminus \{0\}$. Using Proposition 5.2.3 and the fact that $\Omega$ is an even function, we obtain the formula

$$\hat{W}_{\Omega}(\xi) = \int_{S^{n-1}} \Omega(\theta) \log \frac{1}{|\xi \cdot \theta|} d\theta,$$

which implies that $\hat{W}_{\Omega}$ is itself an even function. Now, using Exercise 5.2.3 and condition (5.2.24), we conclude that $\hat{W}_{\Omega}$ is a bounded function. Therefore, $T_{\Omega}$ is $L^2$ bounded. To obtain the $L^p$ boundedness of $T_{\Omega}$, we use the idea mentioned earlier involving the Riesz transforms. In view of (5.1.46), we have that

$$T_{\Omega} = -\sum_{j=1}^{n} R_j T_j,$$

where $T_j = R_j T_{\Omega}$. Equality (5.2.26) makes sense as an operator identity on $L^2(\mathbb{R}^n)$, since $T_{\Omega}$ and each $R_j$ are well defined and bounded on $L^2(\mathbb{R}^n)$.

The kernel of the operator $T_j$ is the inverse Fourier transform of the distribution $-i \frac{\xi_j}{|\xi|} \hat{W}_{\Omega}(\xi)$, which we denote by $K_j$. At this point we know only that $K_j$ is a tempered distribution whose Fourier transform is the function $-i \frac{\xi_j}{|\xi|} \hat{W}_{\Omega}(\xi)$. Our first goal is to show that $K_j$ coincides with an integrable function on an annulus. To prove this assertion we write

$$W_{\Omega} = W_{\Omega}^0 + W_{\Omega}^1 + W_{\Omega}^\infty,$$

where $W_{\Omega}^0$ is a distribution and $W_{\Omega}^1, W_{\Omega}^\infty$ are functions defined by

$$\langle W_{\Omega}^0, \varphi \rangle = \lim_{\epsilon \to 0} \int_{\epsilon < |x| \leq \frac{1}{2}} \frac{\Omega(x/|x|)}{|x|^n} \varphi(x) dx,$$

$$W_{\Omega}^1(x) = \frac{\Omega(x/|x|)}{|x|^n} \chi_{\frac{1}{2} \leq |x| \leq 2},$$

$$W_{\Omega}^\infty(x) = \frac{\Omega(x/|x|)}{|x|^n} \chi_{2 < |x|}.$$

We now fix a $j \in \{1, 2, \ldots, n\}$ and we write

$$K_j = K_j^0 + K_j^1 + K_j^\infty,$$

where

$$K_j^0 = (-i \frac{\xi_j}{|\xi|} \hat{W}_{\Omega}^0(\xi))^\vee,$$

$$K_j^1 = (-i \frac{\xi_j}{|\xi|} \hat{W}_{\Omega}^1(\xi))^\vee,$$

$$K_j^\infty = (-i \frac{\xi_j}{|\xi|} \hat{W}_{\Omega}^\infty(\xi))^\vee.$$

Notice that $K_j^0$ is well defined via Theorem 2.3.21.
5.2 Singular Integrals and the Method of Rotations

Define the annulus
\[ A = \{ x \in \mathbb{R}^n : 2/3 < |x| < 3/2 \}. \]

For a smooth function \( \phi \) supported in the annulus \( 2/3 < |x| < 3/2 \) we have
\[
\langle K_0^j, \phi \rangle = \langle \left( -i \frac{\xi_j}{|\xi|} \tilde{W}_0^j(\xi) \right) \hat{\mathcal{W}}_0^j \Omega(\xi), \phi \rangle \\
= \langle \tilde{W}_0^j(\xi) \left( -i \frac{\xi_j}{|\xi|} \hat{\mathcal{W}}_0^j \Omega(\xi) \right), \phi \rangle \\
= \langle \tilde{W}_0^j(\xi), \Omega \left( -i \frac{\xi_j}{|\xi|} \phi \right) \hat{\mathcal{W}}_0^j \Omega(\xi) \rangle \\
= -\langle \tilde{W}_0^j, R_j(\phi) \rangle \\
= -\lim_{\varepsilon \to 0} \int_{|y| < 1/2} \frac{\Omega(y/|y|)}{|y|^n} R_j(\phi)(-y) dy \quad (\Omega \text{ is even}) \\
= -\frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \to 0} \int_{|y| < 1/2} \frac{\Omega(y/|y|)}{|y|^n} \int_{\mathbb{R}^n} \frac{y_j - x_j}{|y - x|^{n+1}} \phi(x) dx dy,
\]

noticing that \( |y - x| \) stays away from zero when \( |y| < 1/2 \) and \( x \) lies in \( A \).

It follows that \( K_0^j \) coincides in \( A \) with the function inside the absolute value below:
\[
\left| \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{\frac{n+1}{2}}} \lim_{\varepsilon \to 0} \int_{|y| < 1/2} \frac{x_j - y_j}{|x - y|^{n+1}} \frac{\Omega(y/|y|)}{|y|^n} dy \right| \\
= \left| \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{\frac{n+1}{2}}} \int_{|y| < 1/2} \left( \frac{x_j - y_j}{|x - y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) \frac{\Omega(y/|y|)}{|y|^n} dy \right| \\
\leq \int_{|y| < 1/2} C_n |y| \frac{|\Omega(y/|y|)|}{|y|^n} dy \\
= C_n \| \Omega \|_{L^1},
\]

where we used the fact that \( \Omega(y/|y|)|y|^{-n} \) has integral zero over annuli of the form \( \varepsilon < |y| < \frac{1}{2} \), the mean value theorem applied to the function \( x_j/|x|^{-(n+1)} \), and the fact that \( |x - y| \geq 1/6 \) for \( x \) in the annulus \( A \). We conclude that on \( A \), \( K_0^j \) coincides with the bounded function inside the absolute value in (5.2.27).
Likewise, for \( x \in A \) we have

\[
\frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{n/2}} \left( \int_{|y| > 2} \frac{x_j - y_j}{|x-y|^{n+1}} \frac{\Omega(y/|y|)}{|y|^n} \, dy \right) \leq \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{n/2}} \int_{|y| > 2} \frac{1}{|x-y|^n} \frac{\Omega(y/|y|)}{|y|^n} \, dy \\
\leq \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi^{n/2}} \int_{|y| > 2} \frac{4^n}{|y|^{2n}} \frac{\Omega(y/|y|)}{|y|^n} \, dy \\
= C \| \Omega \|_\ell^1,
\]

from which it follows that on the annulus, \( K_j^\infty \) coincides with the bounded function inside the absolute value in (5.2.29) or in (5.2.28).

Now observe that condition (5.2.24) gives that the function \( W_j^1 \) satisfies

\[
\int_{|x| \leq 2} |W_j^1(x)| \log(e + |W_j^1(x)|) \, dx \\
\leq \int_{1/2}^2 \int_{S_{r^2}} \frac{\Omega(\theta)}{r^n} \log \left[ e + 2^n |\Omega(\theta)| \right] d\theta r^{n-1} \, dr \\
\leq (\log 4) \left[ n (\log 2) \| \Omega \|_\ell^1 + c_\Omega \right] \leq c'_n c_\Omega < \infty.
\]

Since the Riesz transform \( R_j \) is countably subadditive and maps \( L^p \) to \( L^p \) with norm at most \( 4(p - 1)^{-1} \) for \( 1 < p < 2 \), it follows from Exercise 1.3.7 that \( K_j^1 = R_j(W_j^1) \) is integrable over the ball \( |x| \leq 3/2 \) and moreover, it satisfies

\[
\int_{|x| \leq 2} |K_j^1(x)| \, dx \leq C_n \left[ \int_{|x| \leq 2} |W_j^1(x)| \log^+ \| W_j^1(x) \|_1^1 \, dx + 1 \right] \leq C'_n c_\Omega.
\]

Furthermore, since \( \tilde{K}_j \) is homogeneous of degree zero, \( K_j \) is a homogeneous distribution of degree \(-n\) (Exercise 2.3.9). This means that for all test functions \( \varphi \) and all \( \lambda > 0 \) we have

\[
\langle K_j, \delta^\lambda(\varphi) \rangle = \langle K_j, \varphi \rangle,
\]

where \( \delta^\lambda(\varphi)(x) = \varphi(\lambda x) \). But for \( \varphi \in C_0^\infty \) supported in the annulus \( 3/4 < |x| < 4/3 \) and for \( \lambda \) in \( (8/9, 9/8) \) we have that \( \delta^\lambda(\varphi) \) is supported in \( A \) and thus we can express (5.2.30) as convergent integrals as follows:

\[
\int_{\mathbb{R}^n} K_j(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} \tilde{K}_j(x) \varphi(\lambda^{-1} x) \, dx = \int_{\mathbb{R}^n} \lambda^n \tilde{K}_j(\lambda x) \varphi(x) \, dx.
\]

From this it would be ideal to be able to directly obtain that \( K_j(x) = \lambda^n K_j(\lambda x) \) for all \( 8/9 < |x| < 9/8 \) and \( 8/9 < \lambda < 9/8 \), in particular when \( \lambda = |x|^{-1} \). But unfortunately, we can only deduce that for every \( \lambda \in (8/9, 9/8) \), \( K_j(x) = \lambda^n K_j(\lambda x) \) holds for all \( x \) in the annulus except a set of measure zero that depends on \( \lambda \). To be able to define the restriction of \( K_j \) on \( S^{n-1} \), we employ a more delicate argument.
For any \( J \) subinterval of \([8/9, 9/8]\) we obtain from (5.2.31) that

\[
\int_{\mathbb{R}^n} K_{J}(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} \int_{J} \lambda^n K_{J}(\lambda x) \, d\lambda \, \varphi(x) \, dx,
\]

where integral with the slashed integral denotes the average of a function over the set \( J \). Since \( \varphi \) was an arbitrary \( C^\infty \) function supported in the annulus \( 3/4 < |x| < 4/3 \), it follows that for every \( J \) subinterval of \([8/9, 9/8]\), there is a null\(^1\) subset \( E_J \) of the annulus \( A' = \{ x : 27/32 < |x| < 32/27 \} \) such that

\[
K_{J}(x) = \int_{J} \lambda^n K_{J}(\lambda x) \, d\lambda \tag{5.2.32}
\]

for all \( x \in A' \setminus E_J \).

Let \( J_0 = [\sqrt{8/9}, \sqrt{9/8}] \). We claim that there is a null subset \( E \) of \( A' \) such that for all \( x \in A' \setminus E \) we have

\[
\int_{J_0} \lambda^n K_{J}(\lambda x) \, d\lambda = \int_{J_0} \lambda^n K_{J}(\lambda x) \, d\lambda \tag{5.2.33}
\]

for every \( r \) in \( J_0 \). Indeed, let \( E \) be the union of \( E_{rJ_0} \) over all \( r \) in \( J_0 \cap Q \). Then in view of (5.2.32), identity (5.2.33) holds for \( x \in A' \setminus E \) and \( J_0 \cap Q \). But for a fixed \( x \) in \( A' \setminus E \), the function of \( r \) on the right hand side of (5.2.33) is constant on the rationals and is also continuous (in \( r \)), hence it must be constant for all \( r \in J_0 \). Thus the claim follows since both sides of (5.2.33) are equal to (5.2.32).

Writing \( x = \delta \theta \), where \( 27/32 < \delta < 32/27 \) and \( \theta \in S^{n-1} \), it follows by Fubini’s theorem that there is a \( \delta \in (27/32, 32/27) \) (in fact almost all \( \delta \) have this property) such that

\[
\int_{J_0} \lambda^n K_{J}(\lambda \delta \theta) \, d\lambda = \int_{J_0} \lambda^n K_{J}(\lambda \delta \theta) \, d\lambda \tag{5.2.34}
\]

for almost all \( \theta \in S^{n-1} \) and all \( r \in J_0 \). We fix such a \( \delta \), which we denote \( \delta_0 \).

We now define a function \( \Omega_j \) on \( S^{n-1} \) by setting

\[
\Omega_j(\theta) = \int_{J_0} \delta_0^n \lambda^n K_{J}(\lambda \delta_0 \theta) \, d\lambda = \int_{J_0} \delta_0^n \lambda^n K_{J}(\lambda \delta_0 \theta) \, d\lambda
\]

for all \( r \in J_0 \). The function \( \Omega_j \) is defined almost everywhere and is integrable over \( S^{n-1} \), since \( K_j \) is integrable over the annulus \( A \).

Let \( e_1 = (1, 0, \ldots, 0) \). Let \( \Psi \) be a \( C^\infty \) function supported in the annulus \( 32/(27\sqrt{2}) < |x| < 27\sqrt{2}/32 \) around \( S^{n-1} \). We start with

\[
\Omega_j(\theta) = \int_{J_0} \delta_0^n \lambda^n K_{J}(\lambda \delta_0 \theta) \, d\lambda = \int_{J_0} \delta_0^n \lambda^n K_{J}(r \lambda \delta_0 \theta) \, d\lambda,
\]

which holds for all \( r \in J_0 \), we multiply by \( \Psi(re_1) \), and we integrate over \( S^{n-1} \) and over \( (0, \infty) \) with respect to the measure \( dr/r \). We obtain

\(^1\) here we are making use of the following version of du Bois-Reymond’s lemma: if \( U \) is an open subset of \( \mathbb{R}^n \) and \( g \) is an integrable function on \( U \) such that \( \int_U g(x) \psi(x) \, dx = 0 \) for all \( \psi \) smooth functions with compact support contained in \( U \), then \( g = 0 \) a.e. on \( U \).
\[
\int_0^\infty \Psi(re_1) \frac{dr}{r} \int_{S^{n-1}} \Omega_j(\theta) \, d\theta = \int_{j_0}^\infty \int_{S^{n-1}} \delta_0^r \lambda^n K_j(\lambda \delta_0 r \theta) \Psi(re_1) r^n \, d\theta \frac{dr}{r} \, d\lambda \\
= \int_{j_0}^\infty \int_{R^n} \delta_0^r \lambda^n K_j(\lambda \delta_0 x) \Psi(x) \, dx \, d\lambda \\
= \int_{j_0}^\infty \int_{R^n} K_j(x) \Psi((\lambda \delta_0)^{-1} x) \, dx \, d\lambda \\
= \int_{j_0} \langle K_j, \Psi \rangle \, d\lambda , \\
= \langle K_j, \Psi \rangle
\]
in view of the homogeneity of \( K_j \). But, as \( \hat{\Psi} = \Psi^{\vee} \), for some constant \( c_\Psi \) we have

\[
\langle K_j, \Psi \rangle = \langle \hat{K}_j, \Psi^{\vee} \rangle = \int_{R^n} \frac{-i\xi}{|\xi|} \hat{\Omega}_j(\xi) \hat{\Psi}(\xi) \, d\xi = c_\Psi \int_{S^{n-1}} \frac{-i\theta}{|\theta|} \hat{\Omega}_j(\theta) \, d\theta = 0,
\]
since by (5.2.25), \( \frac{-i\xi}{|\xi|} \hat{\Omega}_j(\xi) \) is an odd function. We conclude that \( \Omega_j \) has mean value zero over \( S^{n-1} \).

Thus \( \Omega_j \in L^1(S^{n-1}) \) has mean value zero and the distribution \( \hat{W}_{\Omega_j} \) is well defined.

We claim that

\[
K_j = W_{\Omega_j}.
\]
(5.2.35)

To establish (5.2.35), we show first that \( \langle K_j, \varphi \rangle = \langle W_{\Omega_j}, \varphi \rangle \) whenever \( \varphi \) is supported in the annulus \( 8/9 < |x| < 9/8 \). Using (5.2.32) we have

\[
\int_{R^n} K_j(x) \varphi(x) \, dx = \int_{R^n} \int_{R^n} K_j(\delta_0 \lambda x) \delta_0^n \lambda^n \, d\lambda \, \varphi(x) \, dx \\
= \int_0^\infty \int_{S^{n-1}} \int_{j_0}^\infty K_j(\delta_0 \lambda x) \delta_0^n \lambda^n r^n \, d\lambda \, \varphi(r \theta) \, d\theta \frac{dr}{r} \\
= \int_0^\infty \int_{S^{n-1}} \int_{j_0}^\infty K_j(\delta_0 \lambda' x) \delta_0^n (\lambda')^n \, d\lambda' \, \varphi(r \theta) \, d\theta \frac{dr}{r} \\
= \int_0^\infty \int_{S^{n-1}} \Omega_j(\theta) \varphi(r \theta) \, d\theta \frac{dr}{r} \\
= \langle W_{\Omega_j}, \varphi \rangle ,
\]
having used (5.2.34) in the second to last equality.

Given a general \( \mathcal{C}_0^\infty \) function \( \varphi \) whose support is contained in an annulus of the form \( M^{-1} < |x| < M \), for some \( M > 0 \), via a smooth partition of unity, we write \( \varphi \) as a finite sum of smooth functions \( \varphi_k \) whose supports are contained in annuli of the form \( 8s/9 < |x| < 9s/8 \) for some \( s > 0 \). These annuli can be brought inside the annulus \( 8/9 < |x| < 9/8 \) by a dilation. Since both \( K_j \) and \( W_{\Omega_j} \) are homogeneous distributions of degree \( -n \) and agree on the annulus \( 8/9 < |x| < 9/8 \) they must agree on annuli \( 8s/9 < |x| < 9s/8 \). Consequently, \( \langle K_j, \varphi \rangle = \langle W_{\Omega_j}, \varphi \rangle \) for all \( \varphi \in \mathcal{C}_0^\infty(R^n \setminus \{0\}) \).

Therefore, \( K_j - W_{\Omega_j} \) is supported at the origin, and since it is homogeneous of degree \( -n \), it must be equal to \( b\delta_0 \), a constant multiple of the Dirac mass. But \( \hat{K}_j \) is an
odd function and hence \( K_j \) is also odd. It follows that \( W_{\Omega_j} \) is an odd function on \( \mathbb{R}^n \setminus \{0\} \), which implies that \( \Omega_j \) is an odd function. We say that \( u \in \mathcal{S}'(\mathbb{R}^n) \) is odd if \( \tilde{u} = -u \), where \( \tilde{u} \) is defined by \( (\tilde{u}, \psi) = (u, \tilde{\psi}) \) for all \( \psi \in \mathcal{S}(\mathbb{R}^n) \) and \( \tilde{\psi}(x) = \psi(-x) \). We have that \( K_j - W_{\Omega_j} \) is an odd distribution, and thus \( b\delta_0 \) must be an odd distribution. But if \( b\delta_0 \) is odd, then \( b = 0 \). We conclude that for each \( j \) there exists an odd integrable function \( \Omega_j \) on \( S^{n-1} \) with \( \|\Omega_j\|_{L^1} \) controlled by a constant multiple of \( c_\Omega \) such that (5.2.35) holds.

Then we use (5.2.26) and (5.2.35) to write

\[
T_\Omega = -\sum_{j=1}^n R_j T_{\Omega_j},
\]

and appealing to the boundedness of each \( T_{\Omega_j} \) (Theorem 5.2.7) and to that of the Riesz transforms, we obtain the required \( L^p \) boundedness for \( T_\Omega \).

We note that Theorem 5.2.10 holds for all \( \Omega \in L^1(S^{n-1}) \) that satisfy (5.2.24), not necessarily even \( \Omega \). Simply write \( \Omega = \Omega_e + \Omega_o \), where \( \Omega_e \) is even and \( \Omega_o \) is odd, and check that condition (5.2.24) holds for \( \Omega_o \).

### 5.2.5 Maximal Singular Integrals with Even Kernels

We have the corresponding theorem for maximal singular integrals.

**Theorem 5.2.11.** Let \( n \geq 2 \) and let \( \Omega \) be an even integrable function on \( S^{n-1} \) with mean value zero that satisfies (5.2.24). Then the corresponding maximal singular integral \( T^{(\ast \ast)}_\Omega \), defined in (5.2.4), is bounded on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \) with norm at most a dimensional constant multiple of \( \max(p^2, (p-1)^{-1}) c_\Omega \).

**Proof.** For \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), we define the maximal function of \( f \) in the direction \( \theta \) by setting

\[
M_\theta(f)(x) = \sup_{a>0} \frac{1}{2a} \int_{|r| \leq a} |f(x-r\theta)| \, dr.
\]

(5.2.36)

In view of Exercise 5.2.5 we have that \( M_\theta \) is bounded on \( L^p(\mathbb{R}^n) \) with norm at most \( 3p(p-1)^{-1} \).

Fix \( \Phi \) a smooth radial function such that \( \Phi(x) = 0 \) for \( |x| \leq 1/4 \), \( \Phi(x) = 1 \) for \( |x| \geq 3/4 \), and \( 0 \leq \Phi(x) \leq 1 \) for all \( x \) in \( \mathbb{R}^n \). For \( f \in L^p(\mathbb{R}^n) \) and \( 0 < \varepsilon < N < \infty \) we introduce the smoothly truncated singular integral

\[
\overline{T}^{(\varepsilon,N)}_\Omega(f)(x) = \int_{\mathbb{R}^n} \frac{\Omega(\frac{x}{|y|})}{|y|^n} \left( \Phi\left( \frac{x}{\varepsilon} \right) - \Phi\left( \frac{x}{N} \right) \right) f(x-y) \, dy
\]

and the corresponding maximal singular integral operator

\[
\overline{T}^{(\ast \ast)}_\Omega(f) = \sup_{0 < N < \infty} \sup_{0 < \varepsilon < N} |\overline{T}^{(\varepsilon,N)}_\Omega(f)|.
\]

(5.2.37)

It suffices to work with \( \overline{T}^{(\ast \ast)}_\Omega \) instead of \( T^{(\ast \ast)}_\Omega \) in view of the following argument.
For $f$ in $L^p(\mathbb{R}^n)$ (for some $1 < p < \infty$), we have
\[
\left| \bar{T}_{\Omega}^{(e,N)}(f)(x) - \bar{T}_{\Omega}^{(e,N)}(f)(x) \right|
\leq \left| \bar{T}_{\Omega}^{(e,N)}(f)(x) - \bar{T}_{\Omega}^{(e,N)}(f)(x) \right|
\leq \left[ \int_{|y| \geq \frac{1}{|x|}} \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \Phi \left( \frac{y}{|x|} \right) f(y) \, dy \right] - \left[ \int_{|y| \geq \frac{1}{|x|}} \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \Phi \left( \frac{y}{|x|} \right) f(y) \, dy \right]
\leq \int_{|y| \geq \frac{1}{|x|}} \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \Phi \left( \frac{y}{|x|} \right) f(y) \, dy \, dy
\leq \left( \int_{|y| \geq \frac{1}{|x|}} \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \Phi \left( \frac{y}{|x|} \right) f(y) \, dy \right) = \left( \int_{|y| \geq \frac{1}{|x|}} \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \Phi \left( \frac{y}{|x|} \right) f(y) \, dy \right)
\leq 16 \int_{|y| \geq \frac{1}{|x|}} \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \Phi \left( \frac{y}{|x|} \right) f(y) \, dy.
\]

Taking the supremum over $N > \varepsilon > 0$ and using the result of Exercise 5.2.5 we conclude that
\[
\left\| \bar{T}_{\Omega}^{(e,N)}(f) - \bar{T}_{\Omega}^{(e,N)}(f) \right\|_{L^p} \leq 100 \left\| \Omega \right\|_{L^1} \max \left( 1, (p-1)^{-1} \right) \left\| f \right\|_{L^p}.
\]

This implies that it suffices to obtain the required $L^p$ bound for the smoothly truncated maximal singular integral operator $\bar{T}_{\Omega}^{(e,N)}$.

In proving the required estimate, we may assume that the even function $\Omega$ is bounded. For, if we know that
\[
\left\| \bar{T}_{\Omega}^{(e,N)} \right\|_{L^p \rightarrow L^p} \leq C_n \max (p^2, (p-1)^{-3}) c_{\Omega}
\]
for $\Omega$ even and bounded, then we write a general even function $\Omega$ in $L \log L$ with vanishing integral as $\Omega = \Omega^0 + \sum_{m=1}^{\infty} \Omega^m$, where $\Omega^0 = \Omega \chi_{\Omega \leq 2} - \kappa^0$, $\Omega^m = \Omega \chi_{2^m \leq |\Omega| < 2^{m+1}} - \kappa^m$, and the $\kappa^m$ are constants chosen so that $\int_{S^{n-1}} \Omega^m \, d\sigma = 0$ for all $m \geq 0$. Then $\Omega^m$ are even and bounded and we obtain
\[
\left\| \bar{T}_{\Omega}^{(e,N)} \right\|_{L^p \rightarrow L^p} \leq \sum_{m=0}^{\infty} \left\| \bar{T}_{\Omega^m}^{(e,N)} \right\|_{L^p \rightarrow L^p} \leq C_n c_{\Omega^0} \sum_{m=0}^{\infty} c_{\Omega^m} \leq C_n c_{\Omega} c(p) c_{\Omega}.
\]

where $C(p) = \max (p^2, (p-1)^{-3})$, for a general even function $\Omega$ in $L \log L(S^{n-1})$.

So we fix a bounded function $\Omega$ on $S^{n-1}$ with integral zero. Let $K_\kappa$, $\Omega_j$, and $T_j$ be as in the previous theorem, and let $F_j$ be the Riesz transform of the function $\Omega(x/|x|) \Phi(x)|x|^{-\sigma}$. Let $f \in L^p(\mathbb{R}^n)$. A calculation yields the identity
\[
\bar{T}_{\Omega}^{(e,N)}(f)(x) = \int_{\mathbb{R}^n} \left[ \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} - \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \right] \Phi \left( \frac{y}{|x|} \right) - \frac{1}{N} \left[ \frac{\Omega \left( \frac{y}{|x|} \right)}{|y|^n} \right] \Phi \left( \frac{y}{|x|} \right) f(y) \, dy
\leq \left( \sum_{j=1}^{n} \left[ \frac{1}{N} F_j \left( \frac{y}{|x|} \right) - \frac{1}{N} F_j \left( \frac{y}{|x|} \right) \right] * R_j(f) \right)(x),
\]
where in the last step we used Proposition 5.1.16. Therefore we may write

\[-\mathcal{F}^{(e,N)}(f)(x) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left[ \frac{1}{\epsilon^{n}} F_j \left( \frac{x-y}{\epsilon} \right) - \frac{1}{\epsilon^{n}} F_j \left( \frac{z-y}{\epsilon} \right) \right] R_j(f)(y) \, dy \]

\[= A_1^{(e,N)}(f)(x) + A_2^{(e,N)}(f)(x) + A_3^{(e,N)}(f)(x), \tag{5.2.38} \]

where

\[A_1^{(e,N)}(f)(x) = \sum_{j=1}^{n} \frac{1}{\epsilon^n} \int_{|x-y| \leq \epsilon} F_j \left( \frac{x-y}{\epsilon} \right) R_j(f)(y) \, dy \]

\[\quad - \sum_{j=1}^{n} \frac{1}{\epsilon^n} \int_{|x-y| \leq \epsilon} F_j \left( \frac{z-y}{\epsilon} \right) R_j(f)(y) \, dy, \]

\[A_2^{(e,N)}(f)(x) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left[ \frac{1}{\epsilon^n} \chi_{|x-y| > \epsilon} \left\{ F_j \left( \frac{x-y}{\epsilon} \right) - K_j \left( \frac{z-y}{\epsilon} \right) \right\} \right] R_j(f)(y) \, dy, \]

\[A_3^{(e,N)}(f)(x) = \sum_{j=1}^{n} \int_{\mathbb{R}^n} \left[ \frac{1}{\epsilon^n} \chi_{|x-y| > \epsilon} K_j \left( \frac{x-y}{\epsilon} \right) - \frac{1}{\epsilon^n} \chi_{|x-y| > \epsilon} K_j \left( \frac{z-y}{\epsilon} \right) \right] R_j(f)(y) \, dy. \]

It follows from the definitions of $F_j$ and $K_j$ that

\[F_j(z) - K_j(z) = \frac{\Gamma(n+1)}{\pi^n} \lim_{\epsilon \to 0} \int_{|y| < \epsilon} \frac{\Omega(y/|y|)}{|y|^{n+1}} (\Phi(y) - 1) \frac{z_j - y_j}{|z_j - y_j|^{n+1}} \, dy \]

\[= \frac{\Gamma(n+1)}{\pi^n} \int_{|y| \leq \frac{1}{2}} \frac{\Omega(y/|y|)}{|y|^{n+1}} (\Phi(y) - 1) \left\{ \frac{z_j - y_j}{|z_j - y_j|^{n+1}} - \frac{z_j}{|z_j|^{n+1}} \right\} \, dy \]

whenever $|z| \geq 1$. But using the mean value theorem, the last expression is easily seen to be bounded by

\[C_n \int_{|y| \leq \frac{1}{2}} \frac{|\Omega(y/|y|)|}{|y|^{n}|z_j|^{n+1}} \, dy = C_n \|\Omega\|_{L^1} |z|^{-(n+1)}, \]

whenever $|z| \geq 1$. Using this estimate, we obtain that the $j$th term in $A_2^{(e,N)}(f)(x)$ is bounded by

\[C_n \|\Omega\|_{L^1} \epsilon^n \int_{|x-y| > \epsilon} \frac{|R_j(f)(y)| \, dy}{(|x-y|/\epsilon)^{n+1}} \leq C_n \frac{2 \|\Omega\|_{L^1}}{2^{n-\epsilon^n}} \int_{\mathbb{R}^n} \frac{|R_j(f)(y)| \, dy}{(1 + |x-y|/\epsilon)^{n+1}}, \]

It follows that for functions $f$ in $L^p$ we have

\[\sup_{0 < \epsilon < N < \infty} |A_2^{(e,N)}(f)| \leq C_n \|\Omega\|_{L^1} M(R_j(f)), \]
in view of Theorem 2.1.10. \((M\) here is the Hardy–Littlewood maximal operator.) By Theorem 2.1.6, \(M\) maps \(L^p(\mathbb{R}^n)\) to itself with norm bounded by a dimensional constant multiple of \(\max(1, (p-1)^{-1})\). Since by Remark 5.2.9 the norm \(\|R_j\|_{L^p} \to L^p\) is controlled by a dimensional constant multiple of \(\max(p, (p-1)^{-1})\), it follows that

\[
\left\| \sup_{0 < \varepsilon < N < \infty} |A_2^{(\varepsilon,N)}(f)| \right\|_{L^p} \leq C_n \|\Omega\|_{L^1} \max(p, (p-1)^{-2}) \|f\|_{L^p}.
\] (5.2.39)

Next, recall that in the proof of Theorem 5.2.10 we showed that

\[
K_j(x) = \frac{\Omega_j(x/|x|)}{|x|^n},
\]

where \(\Omega_j\) are integrable functions on \(S^{n-1}\) that satisfy

\[
\left\| \Omega_j \right\|_{L^1} \leq C_n c_{\Omega}.
\] (5.2.40)

Consequently, for functions \(f\) in \(L^p(\mathbb{R}^n)\) we have

\[
\sup_{0 < \varepsilon < N < \infty} |A_3^{(\varepsilon,N)}(f)| \leq 2 \sum_{j=1}^n \mathcal{T}_{\Omega_j}^{(\varepsilon)}(R_j(f)),
\]

and by Remark 5.2.9 this last expression has \(L^p\) norm at most a dimensional constant multiple of \(\|\Omega\|_{L^1} \max(p, (p-1)^{-2}) \|R_j(f)\|_{L^p}\). It follows that

\[
\left\| \sup_{0 < \varepsilon < N < \infty} |A_3^{(\varepsilon,N)}(f)| \right\|_{L^p} \leq C_n \max(p, (p-1)^{-2})(c_{\Omega} + 1) \|f\|_{L^p}.
\] (5.2.41)

Finally, we turn our attention to the term \(A_4^{(\varepsilon,N)}(f)\). To prove the required estimate, we first show that there exist nonnegative homogeneous of degree zero functions \(G_j\) on \(\mathbb{R}^n\) that satisfy

\[
|F_j(x)| \leq G_j(x) \quad \text{when}\ |x| \leq 1
\] (5.2.42)

and

\[
\int_{S^{n-1}} |G_j(\theta)| d\theta \leq C_n c_{\Omega}.
\] (5.2.43)

To prove (5.2.42), first note that if \(|x| \leq 1/8\), then

\[
|F_j(x)| = \frac{\Gamma \left( \frac{n+1}{2} \right)}{\pi \frac{n+1}{2}} \left| \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} \Phi(y) \frac{x_j - y_j}{|x - y|^{n+1}} dy \right|
\leq C_n \int_{|y| \geq \frac{1}{8}} \frac{\Omega(y/|y|)}{|y|^{2n}} dy
\leq C'_n \left\| \Omega \right\|_{L^1}.
\]
5.2 Singular Integrals and the Method of Rotations 351

We now fix an $x$ satisfying $1/8 \leq |x| \leq 1$ and we write

$$|F_j(x)| \leq \Phi(x)|K_j(x)| + |F_j(x) - \Phi(x)K_j(x)|$$

$$\leq |K_j(x)| + \frac{\Gamma(n+1)}{\pi^{n+2}} \lim_{\epsilon \to 0} \int_{|y| > \epsilon} \frac{x_j - y_j}{|x-y|^{n+1}} (\Phi(y) - \Phi(x)) \frac{\Omega(y/|y|)}{|y|^n} dy$$

$$= |K_j(x)| + \frac{\Gamma(n+1)}{\pi^{n+2}} (P_1(x) + P_2(x) + P_3(x)),$$

where

$$P_1(x) = \int_{|y| \leq \frac{1}{8}} \left( \frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) (\Phi(y) - \Phi(x)) \frac{\Omega(y/|y|)}{|y|^n} dy,$$

$$P_2(x) = \int_{\frac{1}{8} \leq |y| \leq 2} \frac{x_j - y_j}{|x-y|^{n+1}} (\Phi(y) - \Phi(x)) \frac{\Omega(y/|y|)}{|y|^n} dy,$$

$$P_3(x) = \int_{|y| \geq 2} \frac{x_j - y_j}{|x-y|^{n+1}} (\Phi(y) - \Phi(x)) \frac{\Omega(y/|y|)}{|y|^n} dy.$$

But since $1/8 \leq |x| \leq 1$, we see that

$$P_1(x) \leq C_n \int_{|y| \leq \frac{1}{8}} \frac{|y|}{|x|^{n+1}} \frac{|\Omega(y/|y|)|}{|y|^n} dy \leq C_n \|\Omega\|_{L^1}$$

and that

$$P_3(x) \leq C_n \int_{|y| \geq 2} \frac{|\Omega(y/|y|)|}{|y|^{2n}} dy \leq C_n \|\Omega\|_{L^1}.$$

For $P_2(x)$ we use the estimate $|\Phi(y) - \Phi(x)| \leq C|x-y|$ to obtain

$$P_2(x) \leq \int_{\frac{1}{8} \leq |y| \leq 2} \frac{C}{|x-y|} \frac{|\Omega(y/|y|)|}{|y|^{2n}} dy$$

$$\leq 4C \int_{\frac{1}{8} \leq |y| \leq 2} \frac{|\Omega(y/|y|)|}{|x-y|^{n-1} |y|^{n-\frac{1}{2}}} dy$$

$$\leq 4C \int_{\mathbb{R}^n} \frac{|\Omega(y/|y|)|}{|x-y|^{n-1} |y|^{n-\frac{1}{2}}} dy.$$

Recall that $K_j(x) = \Omega_j(x/|x|)|x|^{-n}$. We now set

$$G_j(x) = C_n \left( \|\Omega\|_{L^1} + \Omega_j \left( \frac{x}{|x|} \right) \right) + |x|^{n-\frac{3}{2}} \int_{\mathbb{R}^n} \frac{|\Omega(y/|y|)|}{|x-y|^{n-1} |y|^{n-\frac{1}{2}}} dy \right)$$

and we observe that $G_j$ is a homogeneous of degree zero function, it satisfies (5.2.42), and it is integrable over the annulus $\frac{1}{2} \leq |x| \leq 2$. To verify the last assertion, we split up the double integral
\[ I = \int_{\frac{1}{2} \leq |x| \leq 2} \int_{\mathbb{R}^n} \frac{|\Omega(y/|y|)| dy}{|x-y|^{n-\frac{3}{2}}} \, dx \]

into the pieces \(1/4 \leq |y| \leq 4\), \(|y| > 4\), and \(|y| < 1/4\). The part of \(I\) where \(1/4 \leq |y| \leq 4\) is pointwise bounded by a constant multiple of

\[ \int_{\frac{1}{2} \leq |x| \leq 2} \int_{\frac{1}{2} \leq |y| \leq 4} \frac{dx}{|x-y|^{n-1}} \, dy \]

which is pointwise controlled by a constant multiple of \(\|\Omega\|_{L^1}\). In the part of \(I\) where \(|y| > 4\) we use that \(|x-y|^{-n+1} \leq (|y|/2)^{-n+1}\) to obtain rapid decay in \(y\) and hence a bound by a constant multiple of \(\|\Omega\|_{L^1}\). Finally, in the part of \(I\) where \(|y| < 1/4\) we use that \(|x-y|^{-n+1} \leq (1/4)^{-n+1}\), and then we also obtain a similar bound. It follows from (5.2.44) and (5.2.40) that

\[ \int_{\frac{1}{2} \leq |x| \leq 2} |G_j(x)| \, dx \leq C_n \left( \|\Omega\|_{L^1} + \|\Omega\|_{L^1} + \|\Omega\|_{L^1} \right) \leq C_n \epsilon \Omega. \]

Since \(G_j\) is homogeneous of degree zero, we deduce (5.2.43).

To complete the proof, we argue as follows:

\[ \sup_{0 < \epsilon < N < \infty} |A_{1}^{(\epsilon,N)}(f)(x)| \]

\[ \leq 2 \sup_{\epsilon > 0} \sum_{j=1}^{n} \frac{1}{\epsilon^n} \int_{|z| \leq \epsilon} |F_j \left( \frac{z}{\epsilon} \right)| |R_j(f)(x-z)| \, dz \]

\[ \leq 2 \sup_{\epsilon > 0} \sum_{j=1}^{n} \frac{1}{\epsilon^n} \int_{r=0}^{\epsilon} \int_{S^{n-1}} |F_j \left( \frac{r \theta}{\epsilon} \right)| |R_j(f)(x-r \theta)| \, r^{n-1} \, d\theta \, dr \]

\[ \leq 2 \sum_{j=1}^{n} \int_{S^{n-1}} |G_j(\theta)| \left\{ \sup_{\epsilon > 0} \frac{1}{\epsilon^n} \int_{r=0}^{\epsilon} |R_j(f)(x-r \theta)| \, r^{n-1} \, dr \right\} \, d\theta \]

\[ \leq 4 \sum_{j=1}^{n} \int_{S^{n-1}} |G_j(\theta)| |M_\theta(R_j(f))(x)| \, d\theta. \]

Using (5.2.43) together with the \(L^p\) boundedness of the Riesz transforms and of \(M_\theta\) we obtain

\[ \left\| \sup_{0 < \epsilon < N < \infty} |A_{1}^{(\epsilon,N)}(f)| \right\|_{L^p} \leq C_n \max(p^2, (p-1)^{-3}) (c_\Omega + 1) \|f\|_{L^p}. \quad (5.2.45) \]

Combining (5.2.45), (5.2.39), and (5.2.41), we obtain the required conclusion. \(\square\)

The following corollary is a consequence of Theorem 5.2.11.

Corollary 5.2.12. Let \(n \geq 2\) and \(\Omega\) be as in Theorem 5.2.11. Then for \(1 < p < \infty\) and \(f \in L^p(\mathbb{R}^n)\) the functions \(T_{\Omega}^{(\epsilon,N)}(f)\) converge to \(T_{\Omega}(f)\) in \(L^p\) and almost everywhere as \(\epsilon \to 0\) and \(N \to \infty\).
Proof. The a.e. convergence is a consequence of Theorem 2.1.14. The $L^p$ convergence is a consequence of the Lebesgue dominated convergence theorem since for $f \in L^p(\mathbb{R}^n)$ we have that $|T^{(\xi,N)}_{\Omega}(f)| \leq T^{(\ast)}_{\Omega}(f)$ and $T^{(\ast)}_{\Omega}(f)$ is in $L^p(\mathbb{R}^n)$. □

Exercises

5.2.1. Show that the directional Hilbert transform $\mathcal{H}_\theta$ is given by convolution with the distribution $w_\theta$ in $\mathscr{S}'(\mathbb{R}^n)$ defined by

$$\langle w_\theta, \phi \rangle = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{\phi(t)}{t} \, dt.$$ 

Compute the Fourier transform of $w_\theta$ and prove that $\mathcal{H}_\theta$ maps $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. [Hint: Use that $H$ maps $L^1(\mathbb{R})$ to $L^{1,\infty}(\mathbb{R})$, which follows from Theorem 5.3.3.]

5.2.2. Extend the definitions of $W_{\Omega}$ and $T_{\Omega}$ to $\Omega = d\mu$ a finite signed Borel measure on $S^{n-1}$ with mean value zero. Compute the Fourier transform of $W_{d\mu}$ and find a necessary and sufficient condition on measures $d\mu$ so that $T_{d\mu}$ is $L^2$ bounded. Notice that the directional Hilbert transform $\mathcal{H}_\theta$ is a special case of such an operator $T_{d\mu}$.

5.2.3. Use the inequality $AB \leq A \log A + e^B$ for $A \geq 1$ and $B > 0$ to prove that if $\Omega$ satisfies (5.2.24) then it must satisfy (5.2.16). Conclude that if $|\Omega| \log^+|\Omega|$ is in $L^1(S^{n-1})$, then $T_{\Omega}$ is $L^2$ bounded.

[Hint: Use that $\int_{S^{n-1}} |\xi - \theta|^{-\alpha} \, d\theta$ converges when $\alpha < 1$. See Appendix D.3.]

5.2.4. Let $\Omega$ be a nonzero integrable function on $S^{n-1}$ with mean value zero. Let $f$ be integrable over $\mathbb{R}^n$ with nonzero integral. Prove that $T_{\Omega}(f)$ is not in $L^1(\mathbb{R}^n)$.

[Hint: Show that $T_{\Omega}(f)$ cannot be continuous at zero.]

5.2.5. Let $\theta \in S^{n-1}$. Use an identity similar to (5.2.18) to show that the maximal operators

$$\sup_{a > 0} \frac{1}{a} \int_{0}^{a} |f(x - r\theta)| \, dr, \quad \sup_{a > 0} \frac{1}{2a} \int_{-a}^{a} |f(x - r\theta)| \, dr$$

are $L^p(\mathbb{R}^n)$ bounded for $1 < p < \infty$ with norm at most $3p(p - 1)^{-1}$.

5.2.6. For $\Omega \in L^1(S^{n-1})$ and $f$ locally integrable on $\mathbb{R}^n$, define

$$M_{\Omega}(f)(x) = \sup_{R>0} \frac{1}{v_{n} R^n} \int_{|y| \leq R} |\Omega(y/|y|)| |f(x - y)| \, dy.$$ 

Apply the method of rotations to prove that $M_{\Omega}$ maps $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$.

5.2.7. Let $\Omega(x, \theta)$ be a function on $\mathbb{R}^n \times S^{n-1}$ satisfying

(a) $\Omega(x, -\theta) = -\Omega(x, \theta)$ for all $x$ and $\theta$.

(b) $\sup_r |\Omega(x, \theta)|$ is in $L^1(S^{n-1})$.