ARITHMETICAL DEVELOPMENT: COMMENTARY ON CHAPTERS 9 THROUGH 15 AND FUTURE DIRECTIONS

David C. Geary
University of Missouri at Columbia

The second half of this book focuses on the important issues of arithmetical instruction and arithmetical learning in special populations. The former represents an area of considerable debate in the United States, and the outcome of this debate could affect the mathematics education of millions of American children (Hirsch, 1996; Loveless, 2001). At one of the end points of this debate are those who advocate an educational approach based on a constructivist philosophy, whereby teachers provide materials and problems to solve but allow students leeway to discover for themselves the associated concepts and problem-solving procedures. At the other end are those who advocate direct instruction, whereby teachers provide explicit instruction on basic skills and concepts and children are required to practice these basic skills. In the first section of my commentary, I briefly address this debate, as it relates to chapters 9, 10, and 11 (by Dowker; Fuson and Burghardt; and Ambrose; Baek, and Carpenter respectively). Chapters 12 to 15 (by Donlan; Jordan, Hanich, and Uberti; and Delazer; and Heavey respectively) focus on the important issue of arithmetical learning in special populations, a topic that has been largely neglected, at least in relation to research on reading competencies in special populations. I discuss these chapters and related issues in the second section of my commentary. In the third, I briefly consider future directions.

Instructional Research (Chapters 9 to 11)

Constructivism is an instructional approach based on the developmental theories of Piaget and Vygotsky (e.g., Cobb, Yackel, & Wood, 1992). The basic assumption is that children are active learners and should be provided a context within which they can discover mathematics principles and procedures or algorithms for themselves. The teacher provides appropriate materials and a social context within which the materials are explored and discussed but does not provide formal instruction (e.g., a lecture) or strongly guide students' exploration of the material. As with any educational philosophy, the use of these principles to guide instruction is more extreme for some than for others. At one extreme, Cobb et al. (1992) have argued that, with appropriate social-mathematical contexts, "it is possible for students to construct for themselves the mathematical practices that, historically, took several thousand years to evolve" (p. 28). Claims such as these are highly speculative and almost certainly untrue. It is very unlikely that most, or even a handful of, students will be capable of discovering all of the mathematical concepts and procedures that will enable them to succeed, for instance, in college-level mathematics courses (Geary, 1995).

Even the relatively simple domain of modern arithmetic emerged only with the sustained and focused efforts of earlier mathematicians, as they struggled—and continue to struggle—to develop better systems for representing numbers (e.g., the base-ten system) and manipulating them (e.g., multiplication algorithms). In other words, modern arithmetic represents the work of many mathematicians over many centuries (Al-Uqlidisi, 952/1978), and it may be naive to believe that elementary-school children, even with a well-designed curriculum, can duplicate this work. Still, I am willing to grant that this is a largely unresolved issue, and more moderate doses of constructivism may prove to be a useful feature of the elementary-school curriculum. The issue, however, can only be resolved through the use of well-designed empirical studies that test the educational utility of these approaches.

The research described by Fuson and Burghardt in chapter 10 and Ambrose et al. in chapter 11 represent a more moderate, and potentially useful, application of the principles of constructivist philosophy to mathematics education (see Geary, 1994, 1995, for discussion of more extreme applications of constructivist philosophy). Ambrose et al. provide a nice summary of children's early strategies for using multiplication and division to address issues related to measurement and the distribution of resources. From there, they describe a clinical-interview study in which third- to fifth-grade elementary school children were asked to solve a series of arithmetical word problems involving multiplication and division. The children's instruction did not involve the teaching of standard algorithms or an emphasis on memorizing basic facts. As a result, the children were required to invent their own algorithms for solving these problems. Ambrose et al. found that many, presumably most, of these children were able to rely on their knowledge of addition, subtraction, measurement, and related concepts to invent their own multiplication and division algorithms. The finding that children used related knowledge to construct these algorithms is not surprising, given that the tendency to "fall back" on related knowledge and procedures when solving novel problems is found in many, if not all, cognitive domains (Geary, 1994; Siegler, 1983).

In any case, there are some features of this instructional technique that appear to have promise, when used judiciously, for facilitating children's understanding multiplication and division concepts. For instance, many of the children used their understanding of and associated algorithms for solving multiplication problems to solve related division problems. As noted by Ambrose et al., making such conceptual links between multiplication and division is important and often not achieved if students are only taught the standard algorithms for solving multiplication and division problems. It is also likely that the invented algorithms provided children with experiences, such as decomposing and recombining sets of numbers, which could facilitate their understanding of numbers and how algorithms can be used to manipulate numbers.

However, there are other aspects of this instructional strategy that appeared to have been less effective. First, many of the students did not apply the commutative property of multiplication (that, e.g., $3 \times 4 = 4 \times 3$), nor did many invent a base-ten strategy for solving these problems. Second, it was not clear from the Ambrose et al. chapter how many children were successful and how many were not successful in inventing effective algorithms. Third, as Fuson and

Burghardt's work (described next) suggests, a discovery approach will be difficult to implement, at least in some areas, with many children. Fourth is the issue of teaching the standard algorithms and other facets of direct instruction. This last issue will be addressed after the discussion of Fuson and Burghardt's study.

Fuson and Burghardt provided groups of high-achieving second graders with base-ten blocks and with materials to represent quantities with written marks and asked them to solve relatively complex addition and subtraction problems (e.g., 2834 + 1963). As with the Ambrose et al. study, Fuson and Burghardt's approach was based on the general principles of constructivism and was designed to foster children's invention of addition and subtraction algorithms. An especially useful feature of this work is the detailed analysis of the children's progress during the learning sessions, as well as a detailed analysis of their successes and their failures. As for the successes, some of the invented algorithms, especially for addition, were clever and a few were better than the standard algorithm taught in U.S. schools; Method I, for instance, results in fewer demands on working memory than does the standard algorithm (see Fig. 10.3 in chap. 10). As with the Ambrose et al. study, there are features of this approach that are likely to facilitate children's conceptual understanding of numbers and addition and subtraction. For instance, the use of base-ten blocks to concretely represent quantities and the addition of these quantities is an especially attractive feature of this approach (Fuson, 1990; Geary, 1994), although it was less effective for subtraction. As Fuson and Burghardt note, their approach has the advantage of building on what children already understand about number and arithmetic and their associated strategies. In this way, children's new learning is strongly tied to what they already know, which is not always the case with direct instruction.

At the same time, there were difficulties with this approach. The social context was not always conducive to mathematics learning—for instance, being dominated by one child—and the general approach was time consuming and apparently not effective for all children. Even many of these high-achieving children invented incorrect algorithms, and many did not have the time to work on subtraction algorithms. It is likely that many lower achieving children will have similar difficulties, that is, a lack of time and invention of incorrect algorithms. On the basis of these difficulties, Fuson and Burghardt suggest that some balance between constructivist and direct instruction will be needed. I agree with this suggestion, with the caveat that the determination of this balance must be based on empirical research, that is, controlled studies that assess the most effective approach to teaching concepts and procedures. However, I disagree with Ambrose et al.'s suggestion that basic procedural and computational skills do not need to be taught in school and that the use of constructivist or direct instruction is a matter of values, not just empirical research.

In a recent empirical study, we found that cross-national differences in computational skills, including skill at retrieving facts and executing procedures, appeared to contribute to cross-national differences on tests of arithmetical reasoning, as measured by the ability to solve multistep word problems (Geary et al. 1997; Geary, Liu, Chen, Saults, & Hoard, 1999). Here, the advantage of Chinese students over their American peers on arithmetical reasoning tests was moderately to substantively reduced (depending on the sample), but not eliminated, when national differences in computational skills were statistically controlled. These findings are in keeping with the position that the automaticity of computational skills (e.g., fast retrieval of facts) reduces working memory demands associated with problem solving and thereby allows these resources to be allocated to more important aspects of problem solving, such as translating statements into quantitative representations (Mayer, 1985).

Even without the cognitive advantages of automaticity and the associated need for some degree of direct instruction and practice, the constructivist approach used by Ambrose et al. and Fuson and Burghardt appears to have considerable opportunity costs. Even if the constructivist approach was fully effective with all children (which has not been proven), the time required to construct invented algorithms is time that cannot be spent on other mathematical topics. Of course this is why early mathematicians invested so much time in developing procedures and representational systems (e.g., the base-ten system) for number and arithmetic. These procedures, such as the standard algorithm taught in U.S. schools for solving complex multiplication problems, are simply an efficient way to solve problems, although not the only or the most efficient ones. The goal of teaching these algorithms is to circumvent the centuries it took to develop these procedures (e.g., Al-Uqlidisi, 952/1978) in favor of more important aspects of arithmetic, such as conceptually understanding the base-ten system.

These arguments are not to say that students should be taught procedures without understanding the associated concepts; students should be taught effective computational procedures—and practice them to the point of automaticity—and should know the associated concepts (see Geary, 1995, 2001). It is now clear that the learning of conceptual and procedural skills are interrelated. A solid conceptual understanding of the domain (e.g., base-ten system) is important for avoiding and correcting procedural errors (e.g., Sophian, 1997), and the practice of procedures provides a context for children to learn associated concepts and problem-solving strategies (Siegler & Stern, 1998). Nor does this mean that constructivist techniques cannot sometimes be useful in mathematics education. As noted earlier, there are features of this approach that appear to be very useful, as illustrated by some children's linking of multiplication and division found in the Ambrose et al. study. I am simply arguing for a judicious use of this approach and for research—as nicely demonstrated by Fuson and Burghardt—aimed at determining when this approach is useful and when it is not.

Although Dowker's study (chap. 9) is not directly related to the issues associated with direct instruction versus constructivism, it has clear educational implications and speaks to the importance of computational skills. Dowker begins by providing a nice review of the relation between computational abilities and estimation skills in children and the dissociation between these skills that is sometimes found with certain forms of brain injury. In short, computational and estimation skills interact in many contexts but appear to be supported by distinct cognitive and neural systems (see also Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999). An innovative and important aspect of Dowker's work is the focus on how these distinct competencies interact. Specifically, she proposes that arithmetical estimation derives from and, in a sense, extends computational skills. For problems that have been mastered (e.g., facts have been memorized), there is no reason to use estimation. Estimation is, however, very useful in contexts in which an exact answer cannot or does not need to be calculated, but its usefulness is largely confined to a zone of partial knowledge and understanding. The proposal that estimation is only useful within this zone—a zone determined by the level of computational skill and number knowledge—is especially interesting and potentially important. The zone provides an important framework for studying the relation between computational skills and estimation and for determining when use of an estimation strategy is likely to be effective (i.e., providing a reasonable close approximation of the actual answer).

Her large-scale study of the development of arithmetical estimation skills in elementary-school children demonstrates the utility of this approach. Here, 215 children participated in a 3-year study of the relation between computational and estimation skills. The children were classified based on their computational skills and administered estimation problems of varying difficulty. As found by others (e.g., Baroody, 1992), even when 5 years old, these children were able to make reasonable estimates for simple problems. The most interesting finding was that the development of children's estimation skills was tied to the development of computational skills. As computational skills improved, so did their ability to make good estimates. More important, the children's estimation skills appeared to be best at the level just beyond their ability to do accurate computations. As predicted by Dowker's model, children could estimate on problems that were somewhat more difficult than the problems on which they could easily compute exact answers, but could not estimate for problems several levels of difficulty beyond this. Finally, Dowker's finding that children used a variety of approaches to estimate is interesting and extends the work of Siegler and his colleagues, who have demonstrated that variability in problem-solving approaches is found in many—probably all—cognitive and academic domains (Siegler, 1996)

Research in Special Populations (Chapters 12 to 15)

As noted earlier, research on the arithmetical abilities and learning of individuals composing special populations, such as children with learning disabilities, has been scant in relation to research on reading abilities in these same populations. Given this, the work of by Donlan, Jordan et al., Delazer, and Heavey chaps. 12, 13, 14, and 15, respectively) is a needed contribution to the literature. Donlan and Jordan et al's work focuses on the arithmetical development of children with learning disabilities and will be discussed first. Delazer's chapter adds to our understanding of the relation between brain injury and arithmetical competencies, which, in turn, also contributes to the understanding of learning disabilities (Geary, 1993). Heavey's chapter is a fascinating and unique description of arithmetical savants and is discussed following the commentary on Delazer's work.

Jordan et al. address several important topics. The first concerns their nonverbal calculation task and the accompanying results. Here, a child views an array of disks, which is then hidden. Next, disks are either added to the hidden array or removed from it. The child is then asked to represent the resulting number of hidden disks using other disks. The task is important in that it allows children to demonstrate their ability to add and subtract without the use of language (e.g., verbally counting the array). The finding that many children can perform basic addition and subtraction using these disks before they can solve more formal arithmetic problems is also important, and in keeping with other studies that have used a nonverbal procedure (Starkey, 1992). An unresolved issue is the source of this ability. Jordan et al. argue that the ability to perform these nonverbal calculations is the result of the development of general intellectual abilities, and not the operation of a specific arithmetical module (see also Huttenlocher, Jordan, & Levine, 1994). Other researchers, including myself, argue that the ability to perform these nonverbal calculations reflects the operation of an evolved system of number-counting-arithmetic abilities (Gallistel & Gelman, 1992; Geary, 1995). In any event, the discovery and ability to assess these preverbal arithmetical competencies is an important contribution, whatever the source of the abilities.

Jordan et al. and I are in closer agreement with respect to the deficits that define a mathematical disability (MD). Jordan et al.'s research, our own work, and that of others have consistently revealed that children with MD demonstrate grade-appropriate arithmetical and mathematical skills in some areas but persistent deficits in others (e.g., Bull & Johnston, 1997; Geary, Hamson, & Hoard, 2000; Geary, Hoard, & Hamson, 1999; Jordan & Montani, 1997). The most consistently found deficit in children with MD is in the ability to quickly and accurately retrieve basic arithmetic facts from long-term memory (LTM). These children retrieve fewer facts than their academically normal peers do, and they quickly forget many of the facts that they do learn. When they can retrieve facts from LTM, children with MD make many more retrieval errors and sometimes show unusual reaction time patterns, in relation to their academically normal peers (Geary, 1993). For many of these children, the retrieval deficit appears to reflect a persistent, perhaps lifelong, disability (e.g., Ostad, 1997). The source of this deficit is currently unknown, although there appear to be two contributing factors. Some of these children appear to have a fundamental deficit in the ability to represent and retrieve information from semantic memory, as discussed next. Other children appear to have difficulties inhibiting irrelevant associations during the retrieval process (Barrouillet, Fayol, & Lathulière, 1997; Geary et al., 2000). For instance, when asked to determine the sum of 3 + 6, many of these children appear to recall 4, 7, and 9; 4 and 7 are the numbers following 3 and 6 in the counting string. These three "answers" then compete for expression, which, in turn, disrupts reaction times and results in more retrieval errors (Geary et al., 2000).

The work of Jordan et al.'s also represents an essential next step in MD research, the identification and study of MD subtypes. The finding of different patterns of functions and deficits in groups of children with different patterns of reading and mathematics achievement scores mirrors our recent work and speaks to the usefulness of this approach (e.g., Geary, Hoard, et al., 1999, 2000). Our results compliment those of Jordan et al. and suggest that children with low mathematics achievement scores but average reading achievement scores have difficulty holding information in working memory while counting and show the retrieval-inhibition deficit described earlier. Although definitive conclusions cannot be reach at this time, the pattern suggests that these children may have a developmental delay or a more fundamental deficit in the prefrontal cortex and the associated executive functions (see also Barrouillet et al., 1997; Geary & Hoard, 2001). Children with poor reading scores but average mathematics scores also show the retrieval-inhibition deficit. However, they do not appear to have working-memory problems while counting. In keeping with Jordan et al.'s results, we have found that children with low achievement in both reading and math show more pervasive deficits than do other children with learning disabilities, even after controlling for IQ, although they have grade-appropriate skills in some areas (e.g., number recognition).

In his chapter, Donlan addresses the important, but not fully understood, link between language development and arithmetical development. The approach taken by Donlan is innovative and complements the work of Jordan et al. In this approach, children with specific language impairment (SLI) are contrasted with language-matched and age-matched children on a variety of number and arithmetic tasks. The basic question is whether SLI will be associated with delays in the acquisition of number and arithmetic skills generally or in language-dependent number and arithmetic skills only. These questions are important in their own right, but they also are relevant to the issue of the modularity of the early number-counting-arithmetic system. Although Donlan's study is not a definitive test, a correlation between SLI and pervasive number and arithmetic deficits would be consistent with the position that early quantitative skills emerge from general cognitive abilities (e.g., Huttenlocher et al., 1994). Although not the only interpretation (see, e.g., Jordan et al., chap. 13, this volume), an association between SLI and a more circumscribed number and arithmetic deficits would be consistent with the position that certain early quantitative abilities are modular (Gallistel & Gelman, 1992; Geary, 1995).

Generally in keeping with the latter position, Donlan found that young children with SLI had age-appropriate number and arithmetic skills in some areas (e.g., conceptual understanding of counting) and deficits in others. Moreover, the deficits largely involved numerical processes, such as the use of verbal counting procedures to count objects, that are dependent on the language system, in general, and the phonological loop, in particular. These results are consistent with Jordan et al.'s findings that children with learning disabilities show both strengths and weaknesses in basic number and arithmetic skills and are in keeping with the position that disruptions in the phonological system underlie language and reading disorders and certain forms of MD (Geary, 1993). Because children use counting procedures to initially solve verbal arithmetic problems and because the articulation of number words involves basic phonetic and language systems, the associations in LTM between problems and answers should be represented, at least in part, in the same phonetic and semantic memory systems that support word processing and word retrieval. Any neurodevelopmental disruptions in the functioning of these systems might then place the individual at risk for SLI and for difficulties in the arithmetical processes that are supported by the same systems, such as arithmetic-fact retrieval.

On the basis of these theoretical considerations and on the comorbidity of MD and dyslexia, I argued that these disorders co-occur "because of a common underlying neuropsychological deficit, perhaps involving the posterior regions of the left hemisphere. At the cognitive level, this deficit manifests itself as difficulties in the representation and retrieval of semantic information from LTM. This would include fact-retrieval difficulties in simple arithmetic and, for instance, word-recognition and phonological-awareness difficulties in reading" (Geary, 1993, p. 356). Although it now appears that fact-retrieval deficits are more varied than originally believed, as described earlier, recent results confirm that children with comorbid MD and dyslexia are slower at accessing number and word names from LTM than are their normal peers and show arithmetic fact retrieval deficits (Geary et al., 2000). In short, there is reason to believe that SLI and certain forms of MD will be related and the approach used by Donlan to address this relation represents a very useful methodological adjunct to the approach used by Jordan et al. and in our research.

Delazer provides an excellent overview of prominent neuropsychological models of number and arithmetic processing and makes an important contribution to this literature. The basic strategy in this area is to examine the performance of individuals with dyscalculia—number and arithmetic deficits associated with overt brain injury—on an array of number, counting, and arithmetic tasks and note systematic deficits as well as intact functions. As noted by Delazer, most of these studies have focused on arithmetic-fact retrieval and the ability to execute arithmetical and other procedures (e.g., borrowing). Many of these studies show a double dissociation between these two competencies, with some individual showing fact-retrieval deficits and age-appropriate procedural competencies and others showing the opposite pattern (e.g., Temple, 1991). Delazer contributes to this literature, and to our general understanding of arithmetical and mathematical development, by demonstrating dissociations between conceptual knowledge in arithmetic and algebra and retrieval and procedural competencies. In other words, conceptual knowledge appears to be represented in different cognitive and neural systems than are arithmetic facts and procedural knowledge. Caution about this conclusion is warranted, because the interpretation of the dissociative data is confounded by potential schooling effects (see Baroody, chap. 1, this volume; Geary, 2000).

Delazer's research and that of other neuropsychologists also has the potential to inform research on MD and vice versa. Many of the cognitive deficits associated with dyscalculia appear to be the same as those found with children with MD (Geary, 1993; Geary & Hoard, 2001). Although these patterns do not necessarily indicate that children with MD have suffered from some form of overt brain injury, they do suggest subtle neurodevelopment abnormalities. Neuropsychological and neuroimaging research on individuals with dyscalculia thus provides a framework for future neuroimaging research with individuals with MD. For instance, neuropsychological studies suggest that the difficulty that some children with MD have in retrieving arithmetic facts and words from semantic memory might be the result of abnormalities in the posterior regions of the left hemisphere and some subcortical structures, such as the basal ganglia (Dehaene & Cohen, 1991). Difficulties in inhibiting irrelevant associations, in contrast, might result from abnormalities in the prefrontal cortex (Geary & Hoard, 2001). Neuroimaging studies of individual with MD will enable a test of these hypotheses. At the same time, the tasks used to study normal arithmetical development and children with MD, such as the nonverbal calculation task described by Jordan et al., might be fruitfully used in future studies of dyscalculia.

Heavey's chapter provides an excellent overview of arithmetical savants, that is, individuals who show moderate to severe general cognitive or social deficits but show exceptional abilities in one or several areas of arithmetic or mathematics. These studies again demonstrate that many quantitative abilities are modular, that is, distinct from social (e.g., language) and general cognitive abilities. The studies are also consistent with the neuropsychological research in that dissociations between different arithmetical and mathematical competencies are common among savants, that is, they may show exceptional abilities in one area, such as algebra, and poor abilities in another, such as geometry. In other words, even if quantitative abilities are distinct from language and other abilities, submodules, so to speak, of quantitative competencies would be expected (Geary, 1995). In any case, the studies of arithmetical savants speak to the importance of memorizing many basic facts and procedures, as well as understanding numerical relations. They also speak to then using this knowledge as a platform for solving related but novel problems, in much the same way that Dowker described for the relation between computational abilities and estimation.

The most intriguing of Heavey's findings is that many savants show an early obsession with counting and number. Of course, such an obsession is probably needed to memorize the facts and procedures and to understand the conceptual nature of numbers that support savant abilities. More important, this developmental pattern suggests an inherent interest in early quantitative features of the environment that is separate from language and other social interests. An early interest in quantitative features of the environment, in turn, is in keeping with the position that very basic number, counting, and arithmetic skills represent an evolved domain of mind (see Geary, 1995, 2001; Gelman, 1990). Some scholars propose that evolved cognitive abilities are initially skeletal in structure. That is, the basic cognitive and neural systems are inherent but these systems are fleshed out, as children play, explore the environment, and interact socially. In theory, these early experiences flesh out skeletal competencies so that they are adapted to the local ecology and social group. If there is a rudimentary but inherent number, counting, and arithmetic system, then children should naturally engage in counting and other quantitative activities, which, in turn, would flesh out this system. Perhaps this inherent system is intact in many arithmetical savants and provides, along with an obsession with numbers, the foundation for acquiring their exceptional but nonevolved (e.g., an understanding prime numbers) arithmetical competencies.

Future Directions

The previously described chapters highlight both the progress that has been made in recent years in understanding children's arithmetical development and the sources of variation in this development, and point to future directions for research in mathematics education and mathematical cognition (see also Ashcraft, 1995). Issues centered on children's mathematics education are especially controversial these days and will likely require many years of further research on children's mathematical learning and constructive dialog before we fully understand how to best teach mathematics to children. The Ambrose et al.

and Fuson and Burghardt chapters approach this issue from a constructivist position, whereas others, myself included, have advocated a greater emphasis on direct instruction and practice of basic skills (Geary, 1995, 2001). The emphasis on direct instruction and practice is based on an evolutionary approach to cognition and cognitive development. Specifically, the argument is that the mind and brain were not designed to learn much of modern arithmetic and mathematics. Thus, social discourse and other activities sufficient for the acquisition of evolved competencies, such as language, are not likely to be sufficient for learning non-evolved competencies, such the base-ten system. In any case, the resolution of these conflicting views will require theoretical advances in conceptualizing children's cognitive and academic development—as illustrated by Dowker's chapter—and empirical research to determine the most effective combination of approaches for teaching different aspects of mathematics.

Another area of future direction, and an area illustrated by all of the remaining chapters, involves research on individual and group differences in arithmetic development and competencies. One example is research on learning disabilities in mathematics, as described by Jordan et al., which have been largely ignored, in relation to the extensive research efforts devoted to reading disabilities (Lyon, Alexander, & Yaffe, 1997). The other chapters also touched on important topics for future research as well, including the relation between language disorders and MD (Donlan), the identification of the cognitive and neural systems underlying MD and dyscalculia (Delazer), and the development of arithmetic competencies in other special populations, such as autism (Heavey). Other directions for future work include the development of assessment techniques that go beyond identifying children with low achievement and provide information about the source of their poor achievement. That is, we need assessment tools that are sensitive to the specific deficits (e.g., fact retrieval) of children with MD. Furthermore, we need to develop remedial techniques for use with these children and individuals with dyscalculia.

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References

Al-Uqlidisi, A. (1978). *The arithmetic of Al-Uqlidisi* (S. A. Saidan, Trans.). Boston, MA: D. Reidel. (Original work published 952 A.D.).

Ashcraft, M. H. (1995). Cognitive psychology and simple arithmetic: A review and summary of new directions. *Mathematical Cognition, 1*, 3–34.

Baroody, A. J. (1992). The development of kindergartners' mental-addition strategies. *Learning and Individual Differences, 4*, 215–235.

Barrouillet, P., Fayol, M., & Lathulière, E. (1997). Selecting between competitors in multiplication tasks: An explanation of the errors produced by adolescents with learning disabilities. *International Journal of Behavioral Development, 21*, 253–275.

Bull, R., & Johnston, R. S. (1997). Children's arithmetical difficulties: Contributions from processing speed, item identification, and short-term memory. *Journal of Experimental Child Psychology, 65*, 1–24.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*, 2–33.

Dehaene, S., & Cohen, L. (1991). Two mental calculation systems: A case study of severe acalculia with preserved approximation. *Neuropsychologia, 29*, 1045–1074.

Dehaene, S., Spelke, E., Pinel, P., Stanescu, R., & Tsivkin, S. (1999). Sources of mathematical thinking: Behavioral and brain-imaging evidence. *Science, 284*, 970–974.

Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction, 7*, 343–403.

Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44*, 43–74.

Geary, D. C. (1993). Mathematical disabilities: Cognitive, neuropsychological, and genetic components. *Psychological Bulletin, 114*, 345–362.

Geary, D. C. (1994). Children's mathematical development: Research and practical applications. Washington, DC: American Psychological Association.

Geary, D. C. (1995). Reflections of evolution and culture in children's cognition: Implications for mathematical development and instruction. *American Psychologist, 50*, 24–37.

Geary, D. C. (2000). From infancy to adulthood: The development of numerical abilities. *European Child & Adolescent Psychiatry, 9*, II11–II16.

Geary, D. C. (2001). Darwinism, schooling, and mathematics: How an understanding of evolution can inform instructional practices. In T. Loveless (Ed.), *Curriculum wars: Alternative approaches to reading and mathematics*. *The great curriculum debate: How should we teach reading and math?* (pp. 85–107). Cambridge, MA: Harvard University Press. Washington, DC: Brookings Institute.

Geary, D. C., Hamson, C. O., Chen, G.-P., Liu, F., Hoard, M. K., & Salthouse, T. A. (1997). Computational and reasoning abilities in arithmetic: Cross-generational change in China and the United States. *Psychonomic Bulletin & Review, 4*, 425–430.

Geary, D. C., Hamson, C. O., & Hoard, M. K. (2000). Numerical and arithmetical cognition: A longitudinal study of process and concept deficits in learning disabled children. *Journal of Experimental Child Psychology*, *77*, 236–263.

Geary, D. C., & Hoard, M. K. (2001). Numerical and arithmetical deficits in learning disabled children: Relation to dyscalculia and dyslexia. *Aphasiology*, *15,* 635–647.

Geary, D. C., Hoard, M. K., & Hamson, C. O. (1999). Numerical and arithmetical cognition: Patterns of functions and deficits in children at risk for a mathematical disability. *Journal of Experimental Child Psychology, 74,* 213–239.

Geary, D. C., Liu, F., Chen, G.-P., Saults, S. J., & Hoard, M. K. (1999). Contributions of computational fluency to cross-national differences in arithmetical reasoning. *Journal of Educational Psychology, 91,* 716–719.

Gelman, R. (1990). First principles organize attention to and learning about relevant data: Number and animate-inanimate distinction as examples. *Cognitive Science, 14*, 79–106.

Hirsch, E. D., Jr. (1996). *The schools we need: Why we don't have them*. New York: Doubleday.

Huttenlocher, J., Jordan, N. C., & Levine, S. C. (1994). A mental model for early arithmetic. *Journal of Experimental Psychology: General, 123*, 284–296.

Jordan, N. C., & Montani, T. O. (1997). Cognitive arithmetic and problem solving: A comparison of children with specific and general mathematics difficulties. *Journal of Learning Disabilities, 30*, 624–634.

Loveless, T. (Ed.). (2001). *Curriculum wars: Alternative approaches to reading and mathematics*. Cambridge, MA: Harvard University Press. *The great curriculum debate: How should we teach reading and math?* Washington, DC: Brookings Institute.

Lyon, G. R., Alexander, D., & Yaffe, S. (1997). Progress and promise in research in learning disabilities. *Learning Disabilities, 8*, 1–6.
Mayer, R. E. (1985). Mathematical ability. In R. J. Sternberg (Ed.), *Human abilities: An information processing approach* (pp. 127–150). San Francisco: Freeman.
Ostad, S. A. (1997). Developmental differences in addition strategies: A comparison of mathematically disabled and mathematically normal children. *British Journal of Educational Psychology, 67*, 345–357.
Siegler, R. S. (1983). Five generalizations about cognitive development. *American Psychologist, 38*, 263–277.
Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking*. New York: Oxford University Press.
Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: A microgenetic analysis. *Journal of Experimental Psychology: General, 127*, 377–397.
Sophian, C. (1997). Beyond competence: The significance of performance for conceptual development. *Cognitive Development, 12*, 281–303.
Starkey, P. (1992). The early development of numerical reasoning. *Cognition, 43*, 93–126.
Temple, C. M. (1991). Procedural dyscalculia and number fact dyscalculia: Double dissociation in developmental dyscalculia. *Cognitive Neuropsychology, 8*, 155–176.