RIEMANN-ROCH FOR DELIGNE-MUMFORD STACKS

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ABSTRACT. We give a simple proof of the Riemann-Roch theorem for Deligne-Mumford stacks using the equivariant Riemann-Roch theorem and the localization theorem in equivariant K-theory, together with some basic commutative algebra of Artin local rings.

1. INTRODUCTION

The Riemann-Roch theorem is one of the most important and deep results in mathematics. At its essence, the theorem gives a method to compute the dimension of the space of sections of a vector bundle on a compact analytic manifold in terms of topological invariants (Chern classes) of the bundle and manifold.

Beginning with Riemann's inequality for linear systems on curves, work on the Riemann-Roch problem spurred the development of fundamental ideas in many branches of mathematics. In algebraic geometry Grothendieck viewed the classical Riemann-Roch theorem as an example of a transformation between K-theory and Chow groups of a smooth projective variety. In differential geometry Atiyah and Singer saw the classical theorem as a special case of their celebrated index theorem which computes the index of an elliptic operator on a compact manifold in terms of topological invariants.

Recent work in moduli theory has employed the Riemann-Roch theorem on Deligne-Mumford stacks. A version of the theorem for complex V-manifolds was proved by Kawasaki [Kaw] using index-theoretic methods. Toen [Toe] also proved a version of Grothendieck-Riemann-Roch on Deligne-Mumford stacks using cohomology theories with coefficients in representations. Unfortunately, both the statements and proofs that appear in the literature are quite technical and as a result somewhat inaccessible to many working in the field.

The purpose of this article is to state and prove a version of the Riemann-Roch theorem for Deligne-Mumford stacks based on the equivariant Riemann-Roch theorem for schemes and the localization theorem in equivariant K-theory. Our motivation is the belief that equivariant methods give the simplest and least technical proof of the theorem. The proof here is based on the author's joint work with W. Graham [EG2, EG3, EG4] in equivariant intersection theory and equivariant K-theory. It requires

little more background than some familiarity with Fulton's intersection theory [Ful] and its equivariant analogue developed in [EG1].

The contents of this article are as follows. In Section 2 we review the algebraic development of the Riemann-Roch theorem from its original statement for curves to the version for arbitrary schemes proved by Baum, Fulton and MacPherson. Our main reference for this materia, with some slight notational changes, is Fulton's intersection theory book [Ful].

In Section 3 we explain how the equivariant Riemann-Roch theorem [EG2] easily yields a Grothendieck-Riemann-Roch theorem for representable morphisms of smooth Deligne-Mumford stacks.

Section 4 is the heart of the article. In it we prove the Hirzebruch-Riemann-Roch theorem for smooth, complete Deligne-Mumford stacks. Using the example of the weighted projective line stack $\mathbb{P}(1,2)$ as motivation, we first prove (Section 4.2) the result for quotient stacks of the form $[X/G]$ with G diagonalizable. This proof combines the equivariant Riemann-Roch theorem with the classical localization theorem in equivariant K-theory and originally appeared in [EG3]. In Section 4.3 we explain how the non-abelian localization theorem of [EG4] is used to obtain the general result. We also include several computations to illustrate how the theorem can be applied.

In Section 5 we briefly discuss the Grothendieck-Riemann-Roch theorem for Deligne-Mumford stacks and illustrate its use by computing the Todd class of a weighted projective space.

For the convenience of the reader we also include an Appendix with some basic definitions used in the theory.

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Dedication: It is a pleasure to dedicate this article to my teacher, Joe Harris.

2. The Riemann-Roch theorem for schemes

The material in Sections 2.1 - 2.3 is well known and further details can be found in the book [Ful].

2.1. **Riemann-Roch through Hirzebruch.** The original Riemann-Roch theorem is a statement about curves. If D is a divisor on a smooth complete curve C then the result can be stated as:

$$l(D) - l(K_C - D) = \deg D + 1 - g$$

where K_C is the canonical divisor and $l(D)$ indicates the dimension of the linear series of effective divisors equivalent to D. Using Serre duality we can rewrite this as

$$\chi(C, L(D)) = \deg D + 1 - g.$$

where $L(D)$ is the line bundle determined by D. Or, in slightly fancier notation

(1) $$\chi(C, L(D)) = \deg c_1(L(D)) + 1 - g.$$

The Hirzebruch-Riemann-Roch theorem extends (1) to arbitrary smooth complete varieties.

Theorem 2.1 (Hirzebruch-Riemann-Roch). *Let X be a smooth projective variety and let V be a vector bundle on X. Then*

(2) $$\chi(X, V) = \int_X \operatorname{ch}(E) \operatorname{Td}(X)$$

where $\operatorname{ch}(V)$ is the Chern character of V, $\operatorname{Td}(X)$ is the Todd class of the tangent bundle and $\int_X$ is refers to the degree of the 0-dimensional component in the product.

The Hirzebruch version of Riemann-Roch yields many useful formulas. For example, if X is a smooth algebraic surface then the arithmetic genus can be computed as

(3) $$\chi(X, \mathcal{O}_X) = \frac{1}{12} \int_X c_1^2 + c_2 = \frac{1}{12}(K^2 + \chi)$$

where χ is the topological genus.

2.2. The Grothendieck-Riemann-Roch theorem.

The Grothendieck-Riemann-Roch theorem extends the Hirzebruch-Riemann-Roch theorem to the relative setting. Rather than considering Euler characteristics of vector bundles on smooth, complete varieties we consider the relative Euler characteristic for proper morphisms of smooth varieties.

Let $f \colon X \to Y$ be a proper morphism of smooth varieties. The Chern character defines homomorphisms $\operatorname{ch} \colon K_0(X) \to \operatorname{Ch}^* X \otimes \mathbb{Q}$, and $\operatorname{ch} \colon K_0(Y) \to \operatorname{Ch}^* Y \otimes \mathbb{Q}$. Likewise, there are two pushforward maps: the relative Euler characteristic $f_* \colon K_0(X) \to K_0(Y)$ and proper pushforward $f_* \colon \operatorname{Ch}^*(X) \to \operatorname{Ch}^*(Y)$. Since we have 4 groups and 4 natural maps we obtain a diagram - which which does not commute!

(4)
$$\begin{array}{ccc} K_0(X) & \overset{\operatorname{ch}}{\to} & \operatorname{Ch}^*(X) \otimes \mathbb{Q} \\ f_* \downarrow & & f_* \downarrow \\ K_0(Y) & \overset{\operatorname{ch}}{\to} & \operatorname{Ch}^*(Y) \otimes \mathbb{Q} \end{array}$$

The Grothendieck-Riemann-Roch theorem supplies the correction that makes (4) commutative. If $\alpha \in K_0(X)$ then

(5) $$\operatorname{ch}(f_* \alpha) \operatorname{Td}(Y) = f_* \left(\operatorname{ch}(\alpha) \operatorname{Td}(X) \right) \in \operatorname{Ch}^*(Y) \otimes \mathbb{Q}.$$

In other words the following diagram commutes:

$$
\text{(6)} \qquad
\begin{array}{ccc}
K_0(X) & \overset{\operatorname{ch}\operatorname{Td}(X)}{\to} & \operatorname{Ch}^*(X) \otimes \mathbb{Q} \\
f_* \downarrow & & f_* \downarrow \\
K_0(Y) & \overset{\operatorname{ch}\operatorname{Td}(Y)}{\to} & \operatorname{Ch}^*(Y) \otimes \mathbb{Q}
\end{array}
$$

Since $\operatorname{Td}(Y)$ is invertible in $\operatorname{Ch}^*(Y)$ we can rewrite equation (5) as

$$
\text{(7)} \qquad\qquad \operatorname{ch}(f_*\alpha) = f_*\left(\operatorname{ch}(\alpha)\operatorname{Td}(T_f)\right)
$$

where $T_f = [TX] - [f^*TY] \in K_0(X)$ is the relative tangent bundle.

Example 2.2. Equation (7) can be viewed as a relative version of the Hirzebruch-Riemann-Roch formula, but it is also more general. For example, it can also be applied when $f \colon X \to Y$ is a regular embedding of codimension d. In this case a more refined statement holds. If N is the normal bundle of f and V is a vector bundle of rank r on X then the equation

$$
c(f_*V)) = 1 + f_*P(c_1(V),\ldots,c_r(V),c_1(N),\ldots c_d(N))
$$

holds in $\operatorname{Ch}^*(Y)$ where $P(T_1,\ldots,T_d,U_1,\ldots,U_d)$ is a universal power series with integer coefficients.

This result is known as Riemann-Roch without denominators and was conjectured by Grothendieck and proved by Grothendieck and Jouanolou.

2.3. **Riemann-Roch for singular schemes.** If $Z \subset X$ is a subvariety of codimension k then $\operatorname{ch}[\mathcal{O}_Z] = [Z] + \beta$ where β is an element of $\operatorname{Ch}^*(X)$ supported in codimension strictly greater than k. Since $\operatorname{Td}(X)$ is invertible in $\operatorname{Ch}^*(X)$ the Grothendieck-Riemann-Roch theorem can be restated as follows:

Theorem 2.3. *The map $\tau_X \colon K_0(X) \to \operatorname{Ch}^*(X) \otimes \mathbb{Q}$ defined by $[V] \mapsto \operatorname{ch}(V)\operatorname{Td}(X)$ is covariant for proper morphisms of smooth schemes*[1] *and becomes an isomorphism after tensoring $K_0(X)$ with $\mathbb{Q}$.*

The Riemann-Roch theorem of Baum, Fulton and MacPherson generalizes previous Riemmann-Roch theorems to maps of arbitrary schemes. However, the Grothendieck group of vector bundles $K_0(X)$ is replaced by the Grothendieck group of coherent sheaves $G_0(X)$.

Theorem 2.4. [Ful, Theorem 18.3, Corollary 18.3.2] *For all schemes X there is a homomorphism $\tau_X \colon G_0(X) \to \operatorname{Ch}^*(X) \otimes \mathbb{Q}$ satisfying the following properties:*

(a) τ_X is covariant for proper morphisms.

(b) If V is a vector bundle on X then $\tau_X([V]) = \operatorname{ch}(V)\tau_X(\mathcal{O}_X)$.

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[1]This means that if $f \colon X \to Y$ is a proper morphism of smooth schemes then $f_* \circ \tau_X = \tau_Y \circ f_*$ as maps $K_0(X) \to \operatorname{Ch}^*(Y) \otimes \mathbb{Q}$.

(c) If $f\colon X \to Y$ is an lci morphism with relative tangent bundle T_f then for every class $\alpha \in G_0(Y)$ $\tau_X f^* \alpha = \mathrm{Td}(T_f) \cap f^* \tau(\alpha)$.

(d) If $Z \subset X$ is a subvariety of codimension k then $\tau(\mathcal{O}_Z) = [Z] + \beta$ where $\beta \in \mathrm{Ch}^*(X)$ is supported in codimension strictly greater than k.

(e) The map τ_X induces an isomorphism $G_0(X) \otimes \mathbb{Q} \to \mathrm{Ch}^*(X) \otimes \mathbb{Q}$.

Remark 2.5. If X is smooth then $K_0(X) = G_0(X)$ and using (c) we see that $\tau_X(\mathcal{O}_X) = \mathrm{Td}(X)$ and thereby obtain the Hirzebruch and Grothendieck Riemann-Roch theorems. In [Ful] the Chow class $\tau_X(\mathcal{O}_X)$ is called the *Todd class* of X.

Remark 2.6. Theorem 2.4 is proved by a reduction to the (quasi)-projective case via Chow's lemma. Since Chow's lemma also holds for algebraic spaces, the Theorem immediately extends to the category of algebraic spaces.

3. Grothendieck Riemann-Roch for representable morphisms of quotient Deligne-Mumford stacks

The goal of this section explain how the equivariant Riemann-Roch theorem 3.1 yields a Grothendieck-Riemann-Roch theorem for *representable* morphisms of Deligne-Mumford quotient stacks.

3.1. **Equivariant Riemann-Roch.** If G is an algebraic group acting on a scheme X then there are equivariant versions of K-theory, Chow groups and Chern classes (see the appendix for definitions). Thus it is natural to expect an equivariant Riemann-Roch theorem relating equivariant K-theory with equivariant Chow groups. Such a theorem was proved in [EG2] for the arbitrary action of an algebraic group G on a separated algebraic space X. Before we state the equivariant Riemann-Roch theorem we introduce some further notation.

The equivariant Grothendieck group of coherent sheaves, $G_0(G, X)$, is a module for both $K_0(G, X)$, the Grothendieck ring of G-equivariant vector bundles, and $R(G) = K_0(G, \mathrm{pt})$, the Grothendieck ring of G-modules. Each of these rings has a distinguished ideal, the augmentation ideal, corresponding to virtual vector bundles (resp. representations) of rank 0. A result of [EG2] shows that the two augmentation ideals generate the same topology on $G_0(G, X)$ and we denote by $\widehat{G_0(G, X)}$ the completion of $G_0(G, X)_{\mathbb{Q}}$ with respect to this topology.

The equivariant Riemann-Roch theorem generalizes Theorem 2.4 as follows:

Theorem 3.1. *There is a homomorphism $\tau_X \colon G_0(G, X) \to \prod_{i=0}^{\infty} \mathrm{Ch}_G^i(X) \otimes \mathbb{Q}$ which factors through an isomorphism $\widehat{G_0(G, X)} \to \prod_{i=0}^{\infty} \mathrm{Ch}_G^i(X) \otimes \mathbb{Q}$. The map τ_X is covariant for proper equivariant morphisms and when X is a smooth scheme and V is a*

vector bundle then

$$(8) \qquad\qquad \tau_X(V) = \text{ch}(V)\,\text{Td}(TX - \mathfrak{g})$$

where $\mathfrak{g}$ is the adjoint representation of G.

Remark 3.2. The K-theory class $TX - \mathfrak{g}$ appearing (9) corresponds to the tangent bundle of the quotient stack $[X/G]$. If G is finite then $\mathfrak{g} = 0$ and if G is diagonalizable (or more generally solvable) then $\mathfrak{g}$ is a trivial representation of G and the formula $\tau_X(V) = \text{ch}(V)\,\text{Td}(TX)$ also holds.

Example 3.3. If $X = \text{pt}$ and $G = \mathbb{C}^*$ then $R(G)$ is the representation ring of G. Since G is diagonalizable the representation ring is generated by characters and $R(G) = \mathbb{Z}[\xi, \xi^{-1}]$ where ξ is the character of weight one. If we set $t = c_1(\xi)$ then the map τ_X is simply the exponential map $\mathbb{Z}[\xi, \xi^{-1}] \to \mathbb{Q}[[t]]$, $\xi \mapsto e^t$. The augmentation ideal of $R(G)$ is $\mathfrak{m} = (\xi - 1)$. If we tensor with $\mathbb{Q}$ and complete at the ideal $\mathfrak{m}$ then the completed ring $\widehat{R(G)}$ is isomorphic to the power series ring $\mathbb{Q}[[x]]$ where $x = \xi - 1$. The map τ_X is the isomorphism sending x to $e^t - 1 = t(1 + t/2 + t^2/3! + \ldots)$.

3.2. Quotient stacks and moduli spaces.

Definition 3.4. A quotient stack is a stack $\mathcal{X}$ equivalent to the quotient $[X/G]$ where $G \subset \text{GL}_n$ is a linear algebraic group and X is a scheme (or more generally an algebraic space[2]).

A quotient stack is Deligne-Mumford if the stabilizer of every point is finite and geometrically reduced. Note that in characteristic 0 the second condition is automatic.

A quotient stack $\mathcal{X} = [X/G]$ is *separated* if the action of G on X is proper - that is, the map $\sigma \colon G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ is proper. Since G is affine σ is proper if and only if it is finite. In characteristic 0 any separated quotient stack is automatically a Deligne-Mumford stack.

The hypothesis that a Deligne-Mumford stack is a quotient stack is not particularly restrictive. Indeed, the author does not know any example of a separated Deligne-Mumford stack which is not a quotient stack. Moreover, there are a number of general results which show that "most" Deligne-Mumford stacks are quotient stacks [EHKV, KV]. For example if $\mathcal{X}$ satisfies the resolution property - that is, every coherent sheaf is the quotient of a locally free sheaf then $\mathcal{X}$ is quotient stack.

It is important to distinguish two classes of morphisms of Deligne-Mumford stacks, *representable* and *non-representable* morphisms. Roughly speaking, a morphism of Deligne-Mumford stacks $\mathcal{X} \to \mathcal{Y}$ is representable if the fibers of f are schemes. Any morphism $X' \to \mathcal{X}$ from a scheme to a Deligne-Mumford stack is representable. If $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$ are quotient stacks and $f \colon \mathcal{X} \to \mathcal{Y}$ is representable then $\mathcal{X}$

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[2]The fact that X is an algebraic space as opposed to a scheme makes little difference in this theory.

is equivalent to a quotient $[Z/H]$ (where $Z = Y \times_{\mathcal{Y}} \mathcal{X}$)and the map of stacks $\mathcal{X} \to \mathcal{Y}$ is induced by an H-equivariant morphism $Z \to Y$. Thus, for quotient stacks we may think of representable morphisms as those corresponding to G-equivariant morphisms.

The non-representable morphisms that we will encounter are all morphisms from a Deligne-Mumford stack to a scheme or algebraic space. Specifically we consider the structure map from a Deligne-Mumford stack to a point and the map from a stack to its coarse moduli space.

Every Deligne-Mumford stack $\mathcal{X}$ is *finitely parametrized*. This means that there is finite surjective morphism $X' \to \mathcal{X}$ where X is a scheme. Thus we can say that a separated stack $\mathcal{X}$ is *complete* if it is finitely parametrized by a complete scheme.

A deep result of Keel and Mori [KM] implies that every separated Deligne-Mumford stack $\mathcal{X}$ has a *coarse moduli* space M in the category of algebraic spaces. Roughly speaking, this means that there is a proper surjective (but not representable) morphism $p \colon \mathcal{X} \to M$ which is a bijection on geometric points and satisfies the universal property that any morphism $\mathcal{X} \to M'$ with M' an algebraic space must factor through p. When $\mathcal{X} = [X/G]$ then the coarse moduli space M is the geometric quotient in the category of algebraic spaces. When $X = X^s$ is the set of stable points for the action of a reductive group G then M is the geometric invariant theory quotient of [MFK].

The map $\mathcal{X} \to M$ is not finite in the usual scheme-theoretic sense, because it is not representable, but it behaves like a finite morphism in the sense that if $f \colon X' \to \mathcal{X}$ is a finite parametrization then the composite morphism $X' \to M$ is finite. Note, however, that if we define $\deg p$ by requiring $\deg p \deg f = \deg X'/M$ then $\deg p$ may be fractional (see below).

Since p is a bijection on geometric points, some of the geometry of the stack $\mathcal{X}$ can be understood by studying the coarse space M. Note, however, that when $\mathcal{X}$ is smooth the space M will in general have finite quotient singularities.

3.2.1. *K-theory and Chow groups of quotient stacks.* If $\mathcal{X}$ is a stack then we use the notation $K_0(\mathcal{X})$ to denote the Grothendieck group of vector bundles on $\mathcal{X}$ and we denote by $G_0(\mathcal{X})$ the Grothendieck group of coherent sheaves on $\mathcal{X}$. If $\mathcal{X}$ is smooth and has the resolution property then the natural map $K_0(\mathcal{X}) \to G_0(\mathcal{X})$ is an isomorphism.

If $\mathcal{X} = [X/G]$ then $K_0(\mathcal{X})$ (resp. $G_0(\mathcal{X})$) is naturally identified with the equivariant Grothendieck ring $K_0(G, X)$ (resp. equivariant Grothendieck group $G_0(G, X)$.

Chow groups of Deligne-Mumford stacks were defined with rational coefficients by Gillet [Gil] and Vistoli [Vis] and with integral coefficients by Kresch [Kre]. When $\mathcal{X} = [X/G]$ Kresch's Chow groups agree integrally with the equivariant Chow groups $\mathrm{Ch}^*_G(X)$ defined in [EG1]. The proper pushforward of rational Chow groups $p \colon \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q} \to \mathrm{Ch}^*(M) \otimes \mathbb{Q}$ is an always an isomorphism [Vis, EG1]. In particular this means

that if $\mathcal{X} = [X/G]$ is a Deligne-Mumford stack then every equivariant Chow class can be represented by a G-invariant cycle on X (as opposed to $X \times \mathbf{V}$ where $\mathbf{V}$ is a representation of G). Consequently $\mathrm{Ch}^k(\mathcal{X}) \otimes \mathbb{Q} = 0$ for $k > \dim \mathcal{X}$.

The theory of Chern classes in equivariant intersection theory implies that a vector bundle V on $\mathcal{X} = [X/G]$ has Chern classes $c_i(V)$ which operate on $\mathrm{Ch}^*(\mathcal{X})$. If $\mathcal{X}$ is smooth then we may again view the Chern classes as elements of $\mathrm{Ch}^*(\mathcal{X})$. If $\mathcal{X}$ is smooth and Deligne-Mumford the Chern character and Todd class are again maps $K_0(\mathcal{X}) \to \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$.

Every smooth Deligne-Mumford stack has a tangent bundle. If $\mathcal{X} = [X/G]$ is a quotient stack then the map $X \to [X/G]$ is a G-torsor so the tangent bundle to $\mathcal{X}$ corresponds to the quotient $TX/\mathfrak{g}$ where $\mathfrak{g}$ is the adjoint representation of G. In particular under the identification of $\mathrm{Ch}^*(\mathcal{X}) = \mathrm{Ch}_G^*(X)$, $c(T\mathcal{X}) = c(TX)c(\mathfrak{g})^{-1}$. If G is finite or diagonalizable then $\mathfrak{g}$ is a trivial representation so $c_t(\mathfrak{g}) = 1$. Thus, the Chern classes of $T\mathcal{X}$ are just the equivariant Chern classes of TX in these cases.

3.2.2. *Restatement of the equivariant Riemann-Roch theorem for Deligne-Mumford quotient stacks.* As already noted, when G acts properly then $\mathrm{Ch}_G^i(X)_{\mathbb{Q}} = 0$ for $i > \dim[X/G]$ so the infinite direct product in Theorem 3.1 is just $\mathrm{Ch}^*(\mathcal{X})$ where $\mathcal{X} = [X/G]$. A more subtle fact proved in [EG2] is that if G acts with finite stabilizers (in particular if the action is proper) then $G_0(G, X) \otimes \mathbb{Q}$ is supported at a finite number of points of $\mathrm{Spec}(R(G) \otimes \mathbb{Q})$. It follows that $\widehat{G_0(G, X)}$ is the same as the localization of the $R(G) \otimes \mathbb{Q}$-module $G_0(G, X) \otimes \mathbb{Q}$ at the augmentation ideal in $R(G) \otimes \mathbb{Q}$. For reasons that will become clear in the next section we denote this localization by $G_0(G, X)_1$ (or $K_0(G, X)_1$). Identifying equivariant K-theory with the K-theory of the stack $\mathcal{X} = [X/G]$ we will also write $K_0(\mathcal{X})_1$ and $G_0(\mathcal{X})_1$ respectively. Theorem 3.1 implies the following result about smooth Deligne-Mumford quotient stacks.

Theorem 3.5. *There is a homomorphism $\tau_X : G_0(\mathcal{X}) \to \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$ which factors through an isomorphism $G_0(\mathcal{X})_1 \to \mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$. The map τ_X is covariant for proper representable morphisms and when $\mathcal{X}$ is a smooth and V is a vector bundle then*

$$(9) \qquad\qquad \tau_X(V) = \mathrm{ch}(V)\,\mathrm{Td}(\mathcal{X})$$

4. Hirzebruch Riemann-Roch for quotient Deligne-Mumford stacks

At first glance, Theorem 3.5 looks like the end of the Riemann-Roch story for Deligne-Mumford stacks, since it gives a stack-theoretic version of the Grothendieck-Riemann-Roch theorem for representable morphisms and also explains the relationship between K-theory and Chow groups of a quotient stack. Unfortunately, the theorem cannot be directly used to compute the Euler characteristic of vector bundles or coherent sheaves on complete Deligne-Mumford stacks.

The problem is that the Euler characteristic of a vector bundle V on $\mathcal{X}$ is the K-theoretic direct image $f_!V := \sum(-1)^i H^i(\mathcal{X}, V)$ under the projection map $f: K_0(\mathcal{X}) \to K_0(\mathrm{pt}) = \mathbb{Z}$. However, the projection map $\mathcal{X} \to \mathrm{pt}$ is not representable - since if it were then $\mathcal{X}$ would be a scheme or algebraic space.

A Hirzebruch-Riemann-Roch theorem for a smooth, complete, Deligne-Mumford stack $\mathcal{X}$ should be a formula for the Euler characteristic of a bundle in terms of degrees of Chern characters and Todd classes. In this section, which is the heart of the paper, we show how to use Theorem 3.5 and generalizations of the localization theorem in equivariant K-theory to obtain such a result. Henceforth, we will work exclusively over the complex numbers $\mathbb{C}$.

4.1. Euler characteristics and degrees of 0-cycles.
If V is a coherent sheaf on $\mathcal{X} = [X/G]$ then the cohomology groups of V are representations of G and we make the following definition.

Definition 4.1. If V is a G-equivariant vector bundle on X then Euler characteristic of V viewed as a bundle on $\mathcal{X} = [X/G]$ is $\sum_i(-1)^i \dim H^i(X, V)^G$ where $H^i(X, V)^G$ denotes the invariant subspace. We denote this by $\chi(\mathcal{X}, V)$.

Note that, if $\dim G > 0$ then X will never be complete, so $H^i(X, V)$ need not be finite dimensional. Nevertheless, if $\mathcal{X}$ is complete then $H^i(X, V)^G$ is finite dimensional as it can be identified with the cohomology of the coherent sheaf $H^i(M, p_*E)$ under the proper morphism $p: \mathcal{X} \to M$ from $\mathcal{X}$ to its coarse moduli space.

If G is linearly reductive (for example if G is diagonalizable) then the cohomology group $H^i(X, V)$ decomposes as direct sum of G-modules and $H^i(X, V)^G$ is the trivial summand. In this case it easily follows that the assignment $V \mapsto \sum_i(-1)^i \dim H^i(X, V)^G$ defines an Euler characteristic homomorphism $K_0(G, X) \to \mathbb{Z}$. The identification of vector bundles on $\mathcal{X}$ with G-equivariant bundles on X yields an Euler characteristic map $\chi: K_0(\mathcal{X}) \to \mathbb{Z}$. When the action of G is free and $\mathcal{X}$ is represented by a scheme, this is the usual Euler characteristic.

However, even if G is not reductive but acts properly on X then the assignment $V \mapsto \sum_i(-1)^i \dim H^i(X, V)^G$ still defines an Euler characteristic map $\chi: K_0(\mathcal{X}) \to \mathbb{Z}$. This follows from Keel and Mori's description of the finite map $[X/G] \to M = X/G$ as being étale locally in M a quotient $[V/H] \to V/H$ where V is affine and H is finite (and hence reductive because we work in characteristic 0).

The above reasoning also applies to G-linearized coherent sheaves on X and we also obtain an Euler characteristic map $\chi: G_0(\mathcal{X}) \to \mathbb{Z}$. These maps can be extended by linearity to maps $\chi: K_0(\mathcal{X}) \otimes F \to F$ (resp. $G_0(\mathcal{X}) \otimes F \to F$) where F is any coefficient ring.

Example 4.2. If G is a finite group let $BG = [\mathrm{pt}/G]$ be the classifying stack parametrizing algebraic G coverings. The identity morphism $pt \to pt$ factors as $pt \to BG \to pt$

where the first map is the universal G-covering and which associates to any scheme T the trivial covering $G \times T \to T$. The map $BG \to pt$ is the coarse moduli space map and associates to any G-torsor $Z \to T$ to the ground scheme T.

The map $pt \to BG$ is representable and the pushforward in map $K_0(pt) \to K_0(BG)$ is the map $\mathbb{Z} \to R(G)$ which sends the a vector space V to the representation $V \otimes \mathbb{C}[G]$ where $\mathbb{C}[G]$ is the regular representation of G.

Since the $\mathbb{C}[G]$ contains a copy of the trivial representation with multiplicity one, it follows that, with our definition, the composition of pushforwards $\mathbb{Z} = K_0(pt) \to R(G) = K_0(BG) \to \mathbb{Z} = K_0(pt)$ is the identity - as expected.

4.1.1. *The degree of a 0-cycle.* Some care is required in understanding 0-cycles on a Deligne-Mumford stack. The reason is that a closed 0-dimensional integral substack η is not in general a closed point but rather a *gerbe*. That is, it is isomorphic after étale base change to BG for some finite group G. Assuming that the ground field is algebraically closed then the degree of $[\eta]$ is defined to be $1/|G|$.

If $\mathcal{X} = [X/G]$ is a complete Deligne-Mumford quotient stack then 0-dimensional integral substacks correspond to G-orbits of closed points and we can define for a closed point $x \in X$ $\deg[Gx/G] = 1/|G_x|$ where G_x is the stabilizer of x.

Example 4.3. The necessity of dividing by the order the stabilizer can be seen by again looking at the factorization of the morphism $pt \to BG \to pt$ when G is a finite group. The map $pt \to BG$ has degree $|G|$ so the map $BG \to pt$ must have degree $\frac{1}{|G|}$.

4.2. **Hirzebruch Riemann-Roch for quotients by diagonalizable groups.** The goal of this section is to understand the Riemann-Roch theorem in an important special case: separated Deligne-Mumford stacks of the form $\mathcal{X} = [X/G]$ where X is a smooth variety and $G \subset (\mathbb{C}^*)^n$ is a diagonalizable group. We will develop the theory using a very simple example - the weighted projective line stack $\mathbb{P}(1, 2)$.

4.2.1. *Example: The weighted projective line stack $\mathbb{P}(1, 2)$, Part I.* Consider the weighted projective line stack $\mathbb{P}(1, 2)$. This stack is defined as the quotient of $[\mathbb{A}^2 \smallsetminus \{0\}/\mathbb{C}^*]$ where $\mathbb{C}^*$ acts with weights $(1, 2)$; i.e., $\lambda(v_0, v_1) = (\lambda v_0, \lambda^2 v_1)$. Because $X = \mathbb{A}^2 \smallsetminus \{0\}$ is an open set in a two-dimensional representation, every equivariant vector bundle on X is of the form $X \times V$ where V is a representation of $\mathbb{C}^*$. In this example we consider two line bundles on $\mathbb{P}(1, 2)$ - the line bundle L associated to the weight one character ξ of $\mathbb{C}^*$ and the line bundle $\mathcal{O}$ associated to the trivial character.

Direct calculation of $\chi(\mathbb{P}(1, 2), \mathcal{O})$ and $\chi(\mathbb{P}(1, 2), L)$: It is easy to compute $\chi(\mathbb{P}(1, 2), L)$ and $\chi(\mathbb{P}(1, 2), \mathcal{O})$ directly. The coarse moduli space of $\mathbb{P}(1, 2)$ is the geometric quotient $(\mathbb{A}^2 \smallsetminus \{0\})/\mathbb{C}^*$. Even though $\mathbb{C}^*$ no longer acts freely the quotient is still $\mathbb{P}^1$ since it has a covering by two affines $\operatorname{Spec} \mathbb{C}[x_0^2/x_1]$ and $\operatorname{Spec} \mathbb{C}[x_1/x_0^2]$, where x_0 and x_1 are the coordinate functions on $\mathbb{A}^2$. The Euler characteristic pushforward

$K_0(\mathbb{P}(1,2)) \to K_0(\text{pt}) = \mathbb{Z}$ factors through the proper pushforward $K_0(\mathbb{P}(1,2)) \to K_0(\mathbb{P}^1)$. Consequently, we can compute $\chi(\mathbb{P}(1,2), L)$ and $\chi(\mathbb{P}(1,2), \mathcal{O})$ by identifying the images of these bundles on $\mathbb{P}^1$. A direct computation using the standard covering of $\mathbb{A}^2 \smallsetminus \{0\}$ by $\mathbb{C}^*$ invariant affines shows that both L and $\mathcal{O}$ pushforward to the trivial bundle on $\mathbb{P}^1$. Hence

$$\chi(\mathbb{P}(1,2), L) = \chi(\mathbb{P}(1,2), \mathcal{O}) = 1$$

An attempt to calculate $\chi(\mathbb{P}(1,2), \mathcal{O})$ and $\chi(\mathbb{P}(1,2), L)$ using Riemann-Roch methods: Following Hirzebruch-Riemann-Roch for smooth varieties we might expect to compute $\chi(\mathbb{P}(1,2), L)$ as $\int_{\mathbb{P}(1,2)} \text{ch}(L) \, \text{Td}(\mathbb{P}(1,2))$. To do that we will use the presentation of $\mathbb{P}(1,2)$ as a quotient by $\mathbb{C}^*$. The line bundle L corresponds to the pullback to $\mathbb{A}^2$ of the standard character ξ of $\mathbb{C}^*$ and the tangent bundle to the stack $\mathbb{P}(1,2)$ fits into a weighted Euler sequence

$$0 \to \mathbf{1} \to \xi + \xi^2 \to T\mathbb{P}(1,2) \to 0$$

where $\mathbf{1}$ denotes the trivial character of $\mathbb{C}^*$ and again ξ is the character of $\mathbb{C}^*$ of weight 1. If we let $t = c_1(\xi)$ then

$$\text{ch}(L) \, \text{Td}(\mathbb{P}(1,2)) = (1+t)(1+3t/2) = 1 + 5t/2$$

Now the Chow class t is represented by the invariant cycle $[x = 0]$ on $\mathbb{A}^2$ and the corresponding point of $\mathbb{P}(1,2)$ has stabilizer of order 2. Thus

$$\int_{\mathbb{P}(1,2)} \text{ch}(L) \, \text{Td}(\mathbb{P}(1,2)) = 1/2(5/2) = 5/4$$

which is $1/4$ too big. On the other a hand then again $\chi(\mathbb{P}(1,2), \mathcal{O}) = 1$ but

$$\int_{\mathbb{P}(1,2)} \text{ch}(\mathcal{O}) \, \text{Td}(\mathbb{P}(1,2)) = 3/4$$

is too small by $1/4$. In particular

(10)
$$\int_{\mathbb{P}(1,2)} \text{ch}(\mathcal{O} + L) \, \text{Td}(\mathbb{P}(1,2)) = 2$$

which is indeed equal to $\chi(\mathbb{P}(1,2), \mathcal{O} + L)$.

Equation (10) may seem unremarkable but is in fact a hint as to how to obtain a Riemann-Roch formula that works for all bundles on $\mathbb{P}(1,2)$.

4.2.2. *The support of equivariant K-theory.* To understand why (10) holds we need to study $K_0(\mathbb{P}(1,2))$ as an $R(\mathbb{C}^*)$-module. Precisely,

$$K_0(\mathbb{P}(1,2)) = K_0(\mathbb{C}^*, \mathbb{A}^2 \smallsetminus \{0\}) = \mathbb{Z}[\xi, \xi^{-1}]/(\xi^2 - 1)(\xi - 1).$$

This follows from the fact that $\mathbb{A}^2$ is a representation of $\mathbb{C}^*$ so $K_0(\mathbb{C}^*, \mathbb{A}^2) = R(\mathbb{C}^*) = \mathbb{Z}[\xi, \xi^{-1}]$ where again ξ denotes the weight one character of $\mathbb{C}^*$. Because we delete the origin we must quotient by the ideal generated by the K-theoretic Euler class of the

tangent space to the origin. With our action, $\mathbb{A}^2$ is the representation $\xi + \xi^2$ so the tangent space of the origin is also $\xi + \xi^2$. The Euler class of this representation is $(1 - \xi^{-1})(1 - \xi^{-2})$ which generates the ideal $(\xi^2 - 1)(\xi - 1)$.

From the above description we see that $K_0(\mathbb{C}^*, \mathbb{A}^2 \smallsetminus \{0\}) \otimes \mathbb{C}$ is an Artin ring supported at the points 1 and -1 of $\operatorname{Spec} R(G) \otimes \mathbb{C} = \mathbb{C}^*$. The vector bundle $\mathcal{O} + L$ on $\mathbb{P}(1,2)$ corresponding to the element $1 + \xi \in R(\mathbb{C}^*)$ is supported at $1 \in \mathbb{C}^*$ and the formula

$$\chi(\mathbb{P}(1,2), \mathcal{O} + L) = \int_{\mathbb{P}(1,2)} (\operatorname{ch}(\mathcal{O} + L) \operatorname{Td}(\mathbb{P}(1,2)))$$

is correct. On the other hand the class of the bundle $\mathcal{O}$ decomposes as $[\mathcal{O}]_1 + [\mathcal{O}]_{-1}$ where $[\mathcal{O}]_1 = 1/2(1 + \xi)$ is supported at 1 and $[\mathcal{O}]_{-1} = 1/2(1 - \xi)$ is supported at -1. In this case the integral $\int_{\mathbb{P}(1,2)} \operatorname{ch}(\mathcal{O}) \operatorname{Td}(\mathbb{P}(1,2))$ computes $\chi(\mathbb{P}(1,2), [\mathcal{O}]_1)$.

This phenomenon is general. If $\alpha \in K_0(G, X) \otimes \mathbb{Q}$, denote by α_1 the component supported at the augmentation ideal of $R(G)$.

Corollary 4.4. [EG4, cf. Proof of Theorem 6.8] *Let G be a linear algebraic group (not necessarily diagonalizable) acting properly on smooth variety X. Then if $\alpha \in K_0(\mathcal{X}) \otimes \mathbb{Q}$*

$$(11) \qquad \int_{\mathcal{X}} \operatorname{ch}(\alpha) \operatorname{Td}(\mathcal{X}) = \chi(\mathcal{X}, \alpha_1).$$

Proof. Since the equivariant Chern character map factors through $K_0(G, X)_1$ it suffices to prove that

$$(12) \qquad \int_{\mathcal{X}} \operatorname{ch}(\alpha) \operatorname{Td}(\mathcal{X}) = \chi(\mathcal{X}, \alpha)$$

for $\alpha \in K_0(G, X)_1$. To prove our result we use the fact that every Deligne-Mumford stack $\mathcal{X}$ is finitely parametrizable. Translated in terms of group actions this means that there is a finite, surjective G-equivariant morphism $X' \to X$ such that G acts freely on X' and the quotient $\mathcal{X}' = [X'/G]$ is represented by a scheme. (This result was first proved by Seshadri in [Ses] and is the basis for the finite parametrization theorem for stacks proved in [EHKV].) The scheme X' is in general singular[3], but the equivariant Riemann-Roch theorem implies the following proposition.

Proposition 4.5. *Let G act properly on X and let $f: X' \to X$ be a finite surjective G-equivariant map. Then the proper pushforward $f_*: G_0(G, X') \to G_0(G, X)$ induces a surjection $G_0(G, X')_1 \to G_0(G, X)_1$, where $G_0(G, X)_1$ (resp. $G_0(G, X)_1$) denotes the localization of $G_0(G, X) \otimes \mathbb{Q}$ (resp. $G_0(G, X') \otimes \mathbb{Q}$) at the augmentation ideal of $R(G) \otimes \mathbb{Q}$.*

---

[3]If the quotient X/G is quasi-projective then a result of Kresch and Vistoli [KV] shows that we can take X' to be smooth, but this is not necessary for our purposes.

Proof of Proposition 4.5. Because G acts properly on X and $X' \to X$ is finite (hence proper) it follows that G acts properly on X'. Thus $\mathrm{Ch}^*_G(X') \otimes \mathbb{Q}$ and $\mathrm{Ch}^*_G(X) \otimes \mathbb{Q}$ are generated by G-invariant cycles. Since f is finite and surjective any G-invariant cycle on X is the direct image of some rational G-invariant cycle on X'; i.e., the pushforward of Chow groups $f_*\colon \mathrm{Ch}^*_G(X') \to \mathrm{Ch}^*_G(X)$ is surjective after tensoring with $\mathbb{Q}$. Hence by Theorem 3.5 the corresponding map $f_*\colon G_0(G, X')_1 \to G_0(G, X)_1$ is also surjective. $\square$

Now G acts freely on X' so $G_0(G, X') \otimes \mathbb{Q}$ is supported entirely at the augmentation ideal of $R(G) \otimes \mathbb{Q}$. Therefore we have a surjection $G_0(G, X') \otimes \mathbb{Q} \to G_0(G, X)_1$. Since X is smooth, we can also identify $K_0(G, X)_1$ with $G_0(G, X)_1$ and express the class $\alpha \in K_0(G, X)_1$ as $\alpha = f_*\beta$. Since f is finite we see that $\chi(\mathcal{X}', \alpha) = \chi(\mathcal{X}, \beta)$. Since $\mathcal{X}'$ is a scheme, we know by the Riemann-Roch theorem for the singular schemes that $\chi(\mathcal{X}', \beta) = \int_{\mathcal{X}'} \tau_{\mathcal{X}'}(\beta)$. Applying the covariance of the equivariant Riemann-Roch map for proper equivariant morphisms we conclude that

$$\int_{\mathcal{X}} \mathrm{ch}(\alpha) \, \mathrm{Td}(\mathcal{X}) = \int_{\mathcal{X}'} \tau_{\mathcal{X}'}(\beta) = \chi(\mathcal{X}, \beta) = \chi(\mathcal{X}, \alpha).$$

$\square$

4.2.3. *The localization theorem in equivariant K-theory.* Corollary 4.4 tells us how to deal with the component of $G_0(G, X)$ supported at the augmentation ideal. We now turn to the problem of understanding what to do with the rest of equivariant K-theory. The key tool is the *localization theorem.*

The correspondence between diagonalizable groups and finitely generated abelian groups implies that if G is a complex diagonalizable group then $R(G) \otimes \mathbb{C}$ is the coordinate ring of G. Since the $R(G) \otimes \mathbb{Q}$-module $G_0(G, X) \otimes \mathbb{Q}$ is supported at a finite number of closed points of $\mathrm{Spec}\, R(G) \otimes \mathbb{Q}$ it follows that $G_0(G, X) \otimes \mathbb{C}$ is also supported at a finite number of closed points of $G = \mathrm{Spec}\, R(G) \otimes \mathbb{C}$. If $h \in G$ then we denote by $G_0(G, X)_h$ the localization of $G_0(G, X) \otimes \mathbb{C}$ at the corresponding maximal ideal of $R(G) \otimes \mathbb{C}$. In the course of the proof of [Tho3, Theorem 2.1] Thomason showed that $G_0(G, X)_h = 0$ if h acts without fixed point on X. Hence $h \in \mathrm{Supp}\, G_0(G, X)$ implies that $X^h \neq \emptyset$. Since G is assumed to act with finite stabilizers (because it acts properly) it follows that h must be of finite order if $h \in \mathrm{Supp}\, G_0(G, X)$.

If X is a smooth scheme then we can identify $G_0(G, X) = K_0(G, X)$ and the discussion of the above paragraph applies to the Grothendieck ring of vector bundles.

Let X^h be the fixed locus of $h \in G$. If X is smooth then X^h is a smooth closed subvariety of X so the inclusion $i_h\colon X^h \to X$ is a regular embedding. Since the map i_h is G-invariant the normal bundle N_h of $X^h \to X$ comes with a natural G-action. The key to understanding what happens to the summand $G_0(G, X)_h$ is the localization theorem:

Theorem 4.6. *Let G be a diagonalizable group acting on a smooth variety X. The pullback $i_h^*\colon G_0(G,X) \to G_0(G,X^h)$ is an isomorphism after tensoring with $\mathbb{C}$ and localizing at h. Moreover, the Euler class of the normal bundle, $\lambda_{-1}(N_h^*)$, is invertible in $G_0(G,X^h)_h$ and if $\alpha \in G_0(G,X)$ then*

$$\alpha = (i_h)_* \left(\frac{i_h^*\alpha}{\lambda_{-1}(N_h^*)} \right)$$

Remark 4.7. The localization theorem in equivariant K-theory was originally proved by Segal in [Seg]. The version stated above is essentially [Tho3, Lemma 3.2].

4.2.4. *Hirzebruch-Riemann-Roch for diagonalizable group actions.* The localization theorem implies that if $\alpha \in G_0(G,X)_h$ then

$$\chi(\mathcal{X}, \alpha) = \chi([X^h/G], \frac{i_h^*\alpha}{\lambda_{-1}N_h^*}).$$

Thus if $\alpha \in G_0(G,X)_h$ then we can compute $\chi([X/G],\alpha)$ by restricting to the fixed locus X^h. This is advantageous because there is an automorphism of $G_0(G,X^h)$ which moves the component of a K-theory class supported at h to the component supported at 1 without changing the Euler characteristic.

Definition 4.8. Let V be a G-equivariant vector bundle on a space Y and suppose that an element $h \in G$ of finite order acts trivially on Y. Let H be the cyclic group generated by h and let $X(H)$ be its character group. Then V decomposes into a sum of h-eigenbundles $\oplus_{\xi \in X(H)} V_\xi$ for the action of H on the fibres of $V \to Y$. Because the action of H commutes with the action of G (since G is abelian) each eigenbundle is a G-equivariant vector bundle. Define $t_h([V]) \in K_0(G,Y) \otimes \mathbb{C}$ to be the class of the virtual bundle $\sum_{\xi \in X(H)} \xi(h)V_\xi$. A similar construction for G-linearized coherent sheaves defines an automorphism $t_h\colon G_0(G,Y) \otimes \mathbb{C} \to G_0(G,Y) \otimes \mathbb{C}$.

The map t_h is compatible with the automorphism of $R(G) \otimes \mathbb{C}$ induced by the translation map $G \to G$, $k \mapsto kh$ and maps the localization $K_0(G,Y)_h$ to the localization $K_0(G,Y)_1$. The analogous statement also holds for the corresponding localizations of $G_0(G,Y) \otimes \mathbb{C}$.

The crucial property of t_h is that it preserves invariants.

Proposition 4.9. *If G acts properly on Y and Y/G is complete then $\chi([Y/G],\beta) = \chi([Y/G], t_h(\beta))$.*

Proof. Observe that if $V = \oplus_{\xi \in X(H)} V_\xi$ then the invariant subbundle V^G is contained in the H-weight 0 submodule of V. Since $t_h(E)$ fixes the 0 weight submodule we see that the invariants are preserved. $\square$

Combining the localization theorem with Proposition 4.9 we obtain the Hirzebruch-Riemann-Roch theorem for actions of diagonalizable groups.

Theorem 4.10. [EG3, cf. Theorem 3.1] *Let G be a diagonalizable group acting properly on smooth variety X such that the quotient stack $\mathcal{X} = [X/G]$ is complete. Then if V is an equivariant vector bundle on X*

$$(13) \qquad \chi(\mathcal{X}, V) = \sum_{h \in \text{Supp } K_0(G,X)} \int_{[X^h/G]} \text{ch}\left(t_h\left(\frac{i_h^* V}{\lambda_{-1}(N_h^*)}\right)\right) \text{Td}([X^h/G]).$$

4.2.5. *Conclusion of the $\mathbb{P}(1,2)$ example.* Since $K_0(\mathbb{P}(1,2)) = \mathbb{Z}[\xi]/(\xi^2 - 1)(\xi - 1)$, we see that K-theory is additively generated by the class $1, \xi, \xi^2$. We use Theorem 4.10 to compute $\chi(\mathbb{P}(1,2), \xi^l)$. First

$$\chi(\mathbb{P}(1,2), \xi_1^l) = \int_{\mathbb{P}(1,2)} \text{ch}(\xi^2)\,\text{Td}(\mathbb{P}(1,2)) = \int_{\mathbb{P}(1,2)} (1+lt)(1+3/2t) = \int_{\mathbb{P}(1,2)} (l+3/2)t = \frac{(2l+3)}{4}.$$

Now we must calculate the contribution from the component supported at -1. If we let $X = \mathbb{A}^2 \smallsetminus \{0\}$ then X^{-1} is the linear subspace $\{(0,a)|a \neq 0\}$. Because $\mathbb{C}^*$ acts with weight 2 on X^{-1} the stack $[X^{-1}/\mathbb{C}^*]$ is isomorphic to the classifying stack $B\mathbb{Z}_2$ and $K_{\mathbb{C}^*}(X^{-1}) = \mathbb{Z}[\xi]/(\xi^2 - 1)$ while $\text{Ch}^*_{\mathbb{C}^*}(X^{-1}) = \mathbb{Z}[t]/2t$ where again $t = c_1(\xi)$ and $\int_{[X^{-1}/\mathbb{C}^*]} 1 = 1/2$. Using our formula we see that

$$\chi(\mathbb{P}(1,2), \xi_{-1}^l) = \int_{[X^{-1}/\mathbb{C}^*]} \text{ch}\left(\frac{(-1)^l \xi^l}{1+\xi^{-1}}\right) \text{Td}([X^{-1}/\mathbb{C}^*]).$$

Since $c_1(\xi)$ is torsion, the only contribution to the integral on the 0-dimensional stack $[X^{-1}/\mathbb{C}^*]$ is from the class 1 and we see that $\chi(\mathbb{P}(1,2), \xi_{-1}^l) = (-1)^l/4$, so we conclude that

$$\chi(\mathbb{P}(1,2), \xi^l) = \frac{2l + 3 + (-1)^l}{4}.$$

In particular, $\chi(\mathbb{P}(1,2), \mathcal{O}) = \chi(\mathbb{P}(1,2), L) = 1$. Note however that $\chi(\mathbb{P}(1,2), L^2) = 2$.

Exercise 4.11. You should be able to work things out for arbitrary weighted projective stacks. The stack $\mathbb{P}(4,6)$ is known to be isomorphic to the stack of elliptic curve $\overline{\mathcal{M}}_{1,1}$ and so $K_0(\overline{\mathcal{M}}_{1,1}) = \mathbb{Z}[\xi]/(\xi^4 - 1)(\xi^6 - 1)$. Hence $K_0(\overline{\mathcal{M}}_{1,1})$ is supported at $\pm 1, \pm i, \omega, \omega^{-1}, \eta, \eta^{-1}$ where $\omega = e^{2\pi i/3}$ and $\eta = e^{2\pi i/6}$. Use Theorem 4.10 to compute $\chi(\overline{\mathcal{M}}_{1,1}, \xi^k)$. This computes the dimension of the space of level one weight k-modular forms. The terms in the sum will be complex numbers but the total sum is of course integral.

4.2.6. *Example: The quotient stack $[(\mathbb{P}^2)^3/\mathbb{Z}_3]$.* To further illustrate Theorem 4.10 we consider Hirzebruch-Riemann-Roch on the quotient stack $\mathcal{X} = [(\mathbb{P}^2)^3/\mathbb{Z}_3]$ where $\mathbb{Z}_3$ acts on $(\mathbb{P}^2)^3$ by cyclic permutation. This example will serve as a warm-up for Section 4.3.1 where we consider the stack $[(\mathbb{P}^2)^3/S_3]$.

Our goal is to compute $\chi(\mathcal{X}, L)$ where $L = \mathcal{O}(m) \boxtimes \mathcal{O}(m) \boxtimes \mathcal{O}(m)$ viewed as a $\mathbb{Z}_3$-equivariant line bundle on $(\mathbb{P}^2)^3$. To make this computation we observe that $\text{Ch}^*(\mathcal{X}) =$

$\mathrm{Ch}^*_{\mathbb{Z}_3}((\mathbb{P}^2)^3)$ is generated by $\mathbb{Z}_3$ invariant cycles. It follows that every element $\mathrm{Ch}^*(\mathcal{X}) \otimes \mathbb{Q}$ is represented by a symmetric polynomial (of degree at most 6) in the variables H_1, H_2, H_3, where H_i is the hyperplane class on the i-th copy of $\mathbb{P}^2$.

As before we have that

$$\chi(\mathcal{X}, L_1) = \int_{\mathcal{X}} \mathrm{ch}(L) \, \mathrm{Td}(\mathcal{X}). \tag{14}$$

Since $\mathcal{X} \to (\mathbb{P}^2)^3$ is a $\mathbb{Z}_3$ covering we can identify $T\mathcal{X}$ with $T((\mathbb{P}^2)^3)$ viewed as $\mathbb{Z}_3$-equivariant vector bundle. Using the standard formula for the Todd class of projective space we can rewrite equation (14) as

$$\chi(\mathcal{X}, L_1) = \int_{\mathcal{X}} \prod_{i=1}^{3} (1 + mH_i + m^2 H_i^2/2)(1 + 3H_i/2 + H_i^2). \tag{15}$$

The only term which contributes to the integral on the right-hand side of (15) is $(H_1 H_2 H_3)^2$. Now if $P \in \mathbb{P}^3$ is any point then $(H_1 H_2 H_3)^2$ is represented by the invariant cycle $[P \times P \times P]$. Since $\mathbb{Z}_3$ fixes this cycle we see that $\int_{\mathcal{X}} [P \times P \times P] = 1/3$ and conclude that

$$\chi(X, L_1) = 1/3 \left(\text{coefficient of } (H_1 H_2 H_3)^2 \right). \tag{16}$$

Expanding the product in (15) shows that

$$\chi(\mathcal{X}, L_1) = 1/3 \left(1 + 9m/2 + 33m^2/4 + 63m^3/8 + 33m^4/8 + 9m^5/8 + m^6/8 \right). \tag{17}$$

Since $R(\mathbb{Z}_3) \otimes \mathbb{C} = \mathbb{C}[\xi]/(\xi^3 - 1)$ we may identify $\mathrm{Spec}\, R(\mathbb{Z}_3) \otimes \mathbb{C}$ as the subgroup $\mu_3 \subset \mathbb{C}^*$ and compute the contributions to $\chi(\mathcal{X}, L)$ from the components of L supported at $\omega = e^{2\pi i/3}$ and ω^2.

For both ω and ω^2 the fixed locus of the corresponding element of $\mathbb{Z}_3$ is the diagonal $\Delta_{(\mathbb{P}^2)^3} \xrightarrow{\Delta} (\mathbb{P}^2)^3$. The group $\mathbb{Z}_3$ acts trivially on the diagonal so $K_{\mathbb{Z}_3}(\Delta_{(\mathbb{P}^2)^3}) = K_0(\mathbb{P}^2) \otimes R(\mathbb{Z}_3)$. Under this identification, the pullback of the tangent bundle of $(\mathbb{P}^2)^3$ is $T\mathbb{P}^2 \otimes V$ where V is the regular representation of $\mathbb{Z}_3$ corresponding to the action of $\mathbb{Z}_3$ on a 3-dimensional vector space by cyclic permutation. Hence

$$\Delta^*(T(\mathbb{P}^2)^3)) = T\mathbb{P}^2 \otimes \mathbf{1} + T\mathbb{P}^2 \otimes \xi + T\mathbb{P}^2 \otimes \xi^2$$

where ξ is the character of $\mathbb{Z}_3$ with weight $\omega = e^{2\pi i/3}$. The $\mathbb{Z}_3$-fixed component of this $\mathbb{Z}_3$ equivariant bundle is the tangent bundle to fixed locus $\Delta_{(\mathbb{P}^2)^3}$ and its complement is the normal bundle. Thus $T\Delta_{(\mathbb{P}^2)^3} = T\mathbb{P}^2 \otimes \mathbf{1}$ and $N_\Delta = (T\mathbb{P}^2 \otimes \xi) + (T\mathbb{P}^2 \otimes \xi^2)$. Computing the K-theoretic Euler characteristic gives:

$$\begin{aligned}
\lambda_{-1}(N_\Delta^*) &= \lambda_{-1}(T^*\mathbb{P}^2 \otimes \xi^2) \lambda_{-1}(T^*\mathbb{P}^2 \otimes \xi) \\
&= (1 - T^*_{\mathbb{P}^2} \otimes \xi^2 + K_{\mathbb{P}^2} \otimes \xi)(1 - T^*_{\mathbb{P}^2} \otimes \xi + K_{\mathbb{P}^2} \otimes \xi^2).
\end{aligned}$$

(Here we use the fact that $\xi^* = \xi^{-1} = \xi^2$ in $R(\mathbb{Z}_3)$.) Because the above expression is symmetric in ξ and ξ^2, applying the twisting operator for either ω or ω^2 yields

$$t(\lambda_{-1}(N_\Delta^*)) = (1 - \omega^2 T^*\mathbb{P}^2 \otimes \xi^2 + \omega K_{\mathbb{P}^2} \otimes \xi)(1 - \omega T^*\mathbb{P}^2 \otimes \xi + \omega^2 K_{\mathbb{P}^2}).$$

Expanding the product in K-theory gives:

(18) $\quad t(\lambda_{-1})(N_\Delta^*) = 1 + K_{\mathbb{P}^2}^2 + (T^*\mathbb{P}^2)^2 - (T^*\mathbb{P}^2 - K_{\mathbb{P}^2} + T^*\mathbb{P}^2 K_{\mathbb{P}^2}) \otimes (\omega\xi + \omega^2\xi).$

Expression (18) simplifies after taking the Chern character because the Chern classes of any representation are torsion. Precisely,

$$\mathrm{ch}(t(\lambda_{-1}(N_\Delta^*))) = 9 - 27H + 99H^2/2.$$

where H is the hyperplane class on $\Delta_{(\mathbb{P}^2)^3}$. Also note that $\Delta^*L = \mathcal{O}(3m) \otimes \mathbf{1}$ where $\mathbf{1}$ denotes the trivial representation of $\mathbb{Z}_3$. Hence $t(\Delta^*L) = \Delta^*L$ and

(19)
$$\begin{aligned}
\chi(\mathcal{X}, L_\omega) &= \int_{[\Delta_{(\mathbb{P}^2)^3}/\mathbb{Z}_3]} \mathrm{ch}(\mathcal{O}(3m)) \, \mathrm{ch}(t(\lambda_{-1}(N_\Delta^*)^{-1} \, \mathrm{Td}(\mathbb{P}^2) \\
&= 1/3 \, (\text{ coefficient of } H^2) \\
&= 1/3(1 + 3m/2 + m^2/2)
\end{aligned}$$

with the same answer for $\chi(\mathcal{X}, L_{\omega^2})$. Putting the pieces together we see that

(20) $\quad \chi(\mathcal{X}, L) = 1 + 5m/2 + 37m^2/12 + 21m^3/8 + 11m^4/8 + 3m^5/8 + m^6/24.$

Remark 4.12. Note that we have quick consistency check for our computation - namely that $\chi(\mathcal{X}, L)$ is an integer-valued polynomial in m. The values of $\chi(\mathcal{X}, L)$ for $m = 0, 1, 2, 3$ are $1, 11, 76, 340$.

4.3. **Hirzebruch Riemann-Roch for arbitrary quotient stacks.** We now turn to the general case of quotient stacks $\mathcal{X} = [X/G]$ with X smooth and G an arbitrary linear algebraic group acting properly[4] on X. Again $G_0(\mathcal{X}) \otimes \mathbb{C}$ is a module supported a finite number of closed points of $\mathrm{Spec}\, R(G) \otimes \mathbb{C}$. For a general group G, $R(G) \otimes \mathbb{C}$ is the coordinate ring of the quotient of G by its conjugation action. As a result, points of $\mathrm{Spec}\, R(G) \otimes \mathbb{C}$ are in bijective correspondence with conjugacy classes of semi-simple (i.e. diagonalizable) elements in G. An element $\alpha \in G_0(G, X)$ decomposes as $\alpha = \alpha_1 + \alpha_{\Psi_2} + \ldots + \alpha_{\Psi_r}$ where α_{Ψ_r} is the component supported at the maximal ideal corresponding to the semi-simple conjugacy class $\Psi_r \subset G$. Moreover, if a conjugacy class Ψ is in $\mathrm{Supp}\, G_0(\mathcal{X}) \otimes \mathbb{C}$ then Ψ consists of elements of finite order.

By Corollary 4.4 if $\mathcal{X} = [X/G]$ is complete then $\chi(\mathcal{X}, \alpha_1) = \int_\mathcal{X} \mathrm{ch}(\alpha) \, \mathrm{Td}(\mathcal{X})$. To understand what happens away from the identity we use a non-abelian version of the localization theorem proved in [EG4]. Before we state the theorem we need to introduce some notation. If Ψ is a semi-simple conjugacy class let $S_\Psi = \{(g, x) | gx = x, g \in \Psi\}$.

---

[4]Because we work in characteristic 0, the hypothesis that G acts properly implies that the stabilizers are linearly reductive since they are finite. In addition every linear algebraic group over $\mathbb{C}$ has a Levy decomposition $G = LU$ with L reductive and U unipotent and normal. If G acts properly then U necessarily acts freely because a complex unipotent group has no non-trivial finite subgroups. Thus, if we want, we can quotient by the free action of U and reduce to the case that G is reductive.

The condition the G acts properly on X implies that S_Ψ is empty for all but finitely many Ψ and the elements of these Ψ have finite order. In addition, if S_Ψ is non-empty then the projection $S_\Psi \to X$ is a finite unramified morphism.

Remark 4.13. Note that the map $S_\Psi \to X$ need not be an embedding. For example if $G = S_3$ acts on $X = \mathbb{A}^3$ by permuting coordinates and Ψ is the conjugacy class of two-cycles, then S_Ψ is the disjoint union of the three planes $x = y$, $y = z$, $x = z$.

If we fix an element $h \in \Psi$ then the map $G \times X^h \to S_\Psi$, $(g, x) \mapsto (ghg^{-1}, gx)$ identifies S_Ψ as the quotient $G \times_Z X^h$ where $Z = \mathcal{Z}_G(h)$ is the centralizer of the semi-simple element $h \in G$. In particular $G_0(G, S_\Psi)$ can be identified with $G_0(Z, X^h)$. The element h is central in Z and if $\beta \in G_0(G, S_\Psi)$ we denote by β_{c_Ψ} the component of β supported at $h \in \operatorname{Spec} Z$ under the identification described above. It is relatively straightforward [EG4, Lemma 4.6] to show that β_{c_Ψ} is in fact independent of the choice of representative element $h \in \psi$, and thus we obtain a distinguished "central" summand $G_0(G, S_\Psi)_{c_\Psi}$ in $G_0(G, S_\Psi)$.

Theorem 4.14 (Non-abelian localization theorem). [EG4, Theorem 5.1] *The pullback map $f_\Psi^* \colon G_0(G, X) \to G_0(G, S_\Psi)$ induces an isomorphism between the localization of $G_0(G, X)$ at the maximal ideal $m_\Psi \in \operatorname{Spec} R(G) \otimes \mathbb{C}$ corresponding to the conjugacy class Ψ and the summand $G_0(G, S_\Psi)_{c_\Psi}$ in $G_0(G, S_\Psi)$. Moreover, the Euler class of the normal bundle, $\lambda_{-1}(N_{f_\Psi^*})$ is invertible in $G_0(G, S_\Psi)_{c_\Psi}$ and if $\alpha \in G_0(G, X)_{m_\Psi}$ then*

$$\alpha = f_{\Psi*} \left(\frac{f^* \alpha_{c_\Psi}}{\lambda_{-1}(N_f^*)} \right). \tag{21}$$

The theorem can be restated in way that is sometimes more useful for calculations. Fix an element $h \in \Psi$ and again let $Z = \mathcal{Z}_G(h)$ be the centralizer of h in G. Let $\iota^! \colon G_0(G, X) \to G_0(Z, X^h)$ be the composition of the restriction of groups map $G_0(G, X) \to G_0(Z, X)$ with the pullback $G_0(Z, X) \xrightarrow{i_h^*} G_0(Z, X^h)$. Let β_h denote the component of $\beta \in G_0(Z, X^h)$ in the summand $G_0(Z, X^h)_{m_h}$. Let $\mathfrak{g}$ (resp. $\mathfrak{z}$) be the adjoint representation of G (resp. Z). The restriction of the adjoint representation to the subgroup Z makes $\mathfrak{g}$ a Z-module, so $\mathfrak{g}/\mathfrak{z}$ is a Z-module. Since $S_\Psi = G \times_Z X^h$, under the identification $G_0(G, S_\Psi) = G_0(Z, X^h)$ the class of the conormal bundle of the map f_Ψ is identified with $N_{i_h}^* - \mathfrak{g}/\mathfrak{z}^*$. Thus we can restate the non-abelian localization theorem as follows:

Corollary 4.15. *Let $\iota_!$ be the composite of $f_{\Psi*}$ with the isomorphism $G_0(Z, X^h) \to G_0(G, S_\Psi)$. Then for $\alpha \in G_0(G, X)_{m_\Psi}$*

$$\alpha = \iota_! \left(\frac{(\iota^! \alpha)_h \cdot \lambda_{-1}(\mathfrak{g}/\mathfrak{z}^*)}{\lambda_{-1}(N_{i_h}^*)} \right). \tag{22}$$

The element $h \in Z(h)$ is central, and as in the abelian case we obtain a twisting map $t_h \colon G_0(Z, X^h) \to G_0(Z, X^h)$ which maps the summand $G_0(Z, X^h)_h$ to the summand $G_0(Z, X^h)_1$ and also preserves invariants.

We can then obtain the Riemann-Roch theorem in the general case. Let $1_G = \Psi_1, \ldots, \Psi_n$ be conjugacy classes corresponding to the support of $G_0(G, X)$ as an $R(G)$ module. Choose a representative element $h_r \in \Psi_r$ for each r. Let Z_r be the centralizer of h in G and let $\mathfrak{z}_r$ be its Lie algebra.

Theorem 4.16. *Let $\mathcal{X} = [X/G]$ be a smooth, complete Deligne-Mumford quotient stack. Then for any vector bundle V on $\mathcal{X}$*

$$(23) \qquad \chi(\mathcal{X}, V) = \sum_{r=1}^{n} \int_{[X^{h_r}/Z_r)]} \mathrm{ch}\left(t_{h_r}\big(\frac{[i_r^* V] \cdot \lambda_{-1}(\mathfrak{g}^*/\mathfrak{z}_r^*)}{\lambda_{-1}(N_{i_r}^*)}\big)\right) \mathrm{Td}([X^{h_r}/Z_r])$$

where $i_r \colon X^{h_r} \to X$ is the inclusion map.

4.3.1. *A computation using Theorem 4.16: The quotient stack $[(\mathbb{P}^2)^3/S_3]$.* We now generalize the calculation of Section 4.2.6 to the quotient $\mathcal{Y} = [Y/S_3]$ where the symmetric group S_3 acts on $Y = (\mathbb{P}^2)^3$ by permutation. Again we will compute $\chi(\mathcal{Y}, L)$ where $L = \mathcal{O}(m) \boxtimes \mathcal{O}(m) \boxtimes \mathcal{O}(m)$ viewed as an S_3-equivariant line bundle on $(\mathbb{P}^2)^3$. As was the case for the $\mathbb{Z}_3$ action the S_3-equivariant rational Chow group is generated by symmetric polynomials in H_1, H_2, H_3 where H_i is the hyperplane class on the i-th copy of $\mathbb{P}^2$. The calculation of $\chi(\mathcal{Y}, L_1)$ is identical to the one we did for the stack $\mathcal{X} = [(\mathbb{P}^2)/\mathbb{Z}_3]$ except that the cycle $[P \times P \times P]$ has stabilizer S_3 which has order 6. Thus,

$$(24) \quad \chi(\mathcal{X}, L_1) = 1/6 \left(1 + 9m/2 + 33m^2/4 + 63m^3/8 + 33m^4/8 + 9m^5)/8 + m^6/8\right).$$

Now $\mathrm{Spec}\, R(S_3) \otimes \mathbb{C}$ consists of 3 points, corresponding to the conjugacy classes of $\{1\}$, $\Psi_2 = \{(12), (13), (23)\}$ and $\Psi_3 = \{(123), (132)\}$. We denote the components of L at the maximal ideal corresponding to Ψ_2 and Ψ_3 by L_2 and L_3 respectively, so that $L = L_1 + L_2 + L_3$.

The computation of $\chi(\mathcal{Y}, L_3)$ is identical to the computation of $\chi(\mathcal{X}, L_\omega)$ in Section 4.2.6. If we choose the representative element $\omega = (123)$ in Ψ_3 then $\mathcal{Z}_{S_3}(\omega) = \langle \omega \rangle = \mathbb{Z}_3$. Again $Y^\omega = \Delta_{(\mathbb{P}^2)^3}$ and the tangent bundle to $(\mathbb{P}^2)^3$ restricts to the $\mathbb{Z}_3$-equivariant bundle $T\mathbb{P}^2 \otimes V$ where V is the regular representation. Hence (see (19))

$$(25) \qquad\qquad \chi(\mathcal{Y}, L_3) = 1/3(1 + 3m/2 + m^2/2)$$

To compute $\chi(\mathcal{Y}, L_2)$ choose the representative element $\tau = (12)$ in the conjugacy class $\Psi = (12)$. Then $\mathcal{Z}_{S_3}(\tau) = \langle \tau \rangle = \mathbb{Z}_2$ and the fixed locus of τ is $Y^\tau = \Delta_{(\mathbb{P}^2)^2} \times \mathbb{P}^2 \overset{\Delta_{12}}{\hookrightarrow} (\mathbb{P}^2)^3$ where $\Delta_{(\mathbb{P}^2)^2} \subset (\mathbb{P}^2)^2$ is the diagonal. The action of $\mathbb{Z}_2$ is trivial and the tangent bundle to $(\mathbb{P}^2)^3$ restricts to $(T_{\mathbb{P}^2} \otimes V) \boxtimes T\mathbb{P}^2$ where V is now the regular representation

of $\mathbb{Z}_2$ so $N_{\Delta_{12}} = (T\mathbb{P}^2 \otimes \xi) \boxtimes T\mathbb{P}^2$ where ξ is the non-trivial character of $\mathbb{Z}_2$. Since ξ is self-dual as a character of $\mathbb{Z}_2$ we see that

(26) $$\lambda_{-1}(N^*_{\Delta_{12}}) = (1 - (T^*\mathbb{P}^2 \otimes \xi) + K_{\mathbb{P}^2})$$

Applying the twisting operator yields

(27) $$t(\lambda_{-1}(N^*_{\Delta_{12}}) = (1 + T^*\mathbb{P}^2 \otimes \xi + K_{\mathbb{P}^2})$$

Taking the Chern character we have

$$\mathrm{ch}(t(\lambda_{-1}(N^*_{\Delta_{12}}))) = 4 - 6H + 6H^2$$

where H is the hyperplane class on the diagonal $\mathbb{P}^2$. The restriction of L to Y^τ is the line bundle $(\mathcal{O}(2m) \otimes \mathbf{1}) \boxtimes \mathcal{O}(m)$. Thus,

$$
\begin{aligned}
\chi(\mathcal{X}, L_2) &= \int_{[X^\tau/\mathbb{Z}_2]} \mathrm{ch}(\mathcal{O}(2m) \boxtimes \mathcal{O}(m)) \, \mathrm{ch}(t(\lambda_{-1}(N^*_\Delta)^{-1} \, \mathrm{Td}(Y^\tau) \\
&= 1/2 \left(\text{ coefficient of } H^2 H_3^2 \right) \\
&= 1/2(1 + 3m + 13m^2/4 + 3m^3/2 + m^4/4)
\end{aligned}
$$

Adding the Euler characteristics of L_1, L_2, L_3 gives

$$\chi(\mathcal{Y}, L) = 1 + 11m/4 + 19m^2/6 + 33m^3/16 + 13m^4/16 + 3m^5/16 + m^6/48$$

which is again an integer-valued polynomial in m.

4.3.2. *Statement of the theorem in terms of the inertia stack.* The computation of $\chi(\mathcal{X}, \alpha)$ does not depend on the choice of the representatives of elements in the conjugacy classes and Theorem 4.16 can be restated in terms of the S_Ψ and correspondingly in terms of the inertia stack.

Definition 4.17. Let $IX = \{(g, x) | gx = x\} \subset G \times X$ be the inertia scheme. The projection $IX \to X$ makes IX into a group scheme over X. If the stack $[X/G]$ is separated then IX is finite over X.

The group G acts on IX by $g(h, x) = (ghg^{-1}, gx)$ and the projection $IX \to X$ is G-equivariant with respect to this action. The quotient stack $I\mathcal{X} := [IX/G]$ is called the *inertia stack* of the stack $\mathcal{X} = [X/G]$ and there is an induced morphism of stacks $I\mathcal{X} \to \mathcal{X}$. Since G acts properly on X then the map $I\mathcal{X} \to \mathcal{X}$ is finite and unramified.

Since G acts with finite stabilizers a necessary condition for (g, x) to be in IX is for g to be of finite order.

Proposition 4.18. *If Ψ is a conjugacy class of finite order then S_Ψ is closed and open in IX and consequently there is a finite G-equivariant decomposition $IX = \coprod_\Psi S_\Psi$.*

Since IX has a G-equivariant decomposition into a finite disjoint sum of the S_Ψ we can define a twisting automorphism $t\colon G_0(G, IX) \otimes \mathbb{C} \to G_0(G, IX) \otimes \mathbb{C}$ and thus a corresponding twisting action on $G_0(I\mathcal{X})$. If V is a G-equivariant vector bundle on IX then its fiber at a point (h, x) is $\mathcal{Z}_G(h)$-module $V_{h,x}$ and $t(V)$ is the class in $G_0(G, IX) \otimes \mathbb{C}$ whose "fiber" at the point (h, x) is the virtual $\mathcal{Z}_G(h)$-module $\oplus_{\xi \in X(H)} \xi(h)(V_{h,x})_\xi$ where H is the cyclic group generated by h.

The Hirzebruch-Riemann-Roch theorem can then be stated very concisely as:

Theorem 4.19. *Let $\mathcal{X} = [X/G]$ be a smooth, complete Deligne-Mumford quotient stack and let $f\colon I\mathcal{X} \to \mathcal{X}$ be the inertia map. If V is a vector bundle on $\mathcal{X}$ then*

$$\chi(\mathcal{X}, V) = \int_{I\mathcal{X}} \mathrm{ch}\left(t(\frac{f^*V}{\lambda_{-1}(N_f^*)})\right) \mathrm{Td}(I\mathcal{X})$$

5. Grothendieck Riemann-Roch for proper morphisms of Deligne-Mumford quotient stacks

In the final section we state the Grothendieck-Riemann-Roch theorem for arbitrary proper morphisms of quotient Deligne-Mumford stacks.

5.1. Grothendieck-Riemann-Roch for proper morphisms to schemes and algebraic spaces.

The techniques used to prove the Hirzebruch-Riemann-Roch for proper Deligne-Mumford stacks actually yield a Grothendieck-Riemann-Roch result for arbitrary separated Deligne-Mumford stacks relative to map $\mathcal{X} \to M$ where M is the moduli space of the quotient stack $\mathcal{X} = [X/G]$.

Theorem 5.1. [EG4, Theorem 6.8] *Let $\mathcal{X} = [X/G]$ be a smooth quotient stack with coarse moduli space $p\colon \mathcal{X} \to M$. Then the following diagram commutes:*

$$\begin{array}{ccc} G_0(\mathcal{X}) & \overset{I\tau_\mathcal{X}}{\to} & \mathrm{Ch}^*(I\mathcal{X}) \otimes \mathbb{C} \\ p_* \downarrow & & p_* \downarrow \\ G_0(M) & \overset{\tau_M}{\to} & \mathrm{Ch}^*(M) \otimes \mathbb{C} \end{array}.$$

Here $I\tau_\mathcal{X}$ is the isomorphism that sends the class in $G_0(\mathcal{X})$ of a vector bundle V to $\mathrm{ch}\left(t(\frac{f^*V}{\lambda_{-1}(N_f^*)})\right) \mathrm{Td}(I\mathcal{X})$ and τ_M is the Fulton-MacPherson Riemann-Roch isomorphism.

Remark 5.2. If $\mathcal{X}$ is satisfies the resolution property then every coherent sheaf on $\mathcal{X}$ can be expressed as a linear combination of classes of vector bundles.

Using the universal property of the coarse moduli space and the covariance of the Riemann-Roch map for schemes and algebraic spaces we obtain the following Corollary.

Corollary 5.3. *Let $\mathcal{X} = [X/G]$ be a smooth quotient stack and let $\mathcal{X} \to Z$ be a proper morphism to a scheme or algebraic space. Then the following diagram commutes:*

$$
\begin{array}{ccc}
G_0(\mathcal{X}) & \overset{I\tau_{\mathcal{X}}}{\to} & \mathrm{Ch}^*(I\mathcal{X}) \otimes \mathbb{C} \\
p_* \downarrow & & p_* \downarrow \\
G_0(Z) & \overset{\tau_Z}{\to} & \mathrm{Ch}^*(Z) \otimes \mathbb{C}.
\end{array}
$$

5.1.1. *Example: The Todd class of a weighted projective space.* If X is an arbitrary scheme we define the *Todd class*, $\mathrm{td}(X)$, of X to be $\tau_X(\mathcal{O}_X)$ where τ_X is the Riemann-Roch map of Theorem 2.4. If X is smooth, then $\mathrm{td}(X) = \mathrm{Td}(TX)$, and for arbitrary complete schemes $\chi(X, V) = \int_X \mathrm{ch}(V)\, \mathrm{td}(X)$ for any vector bundle V on X.

In this section we explain how to use Theorem 5.1 to give a formula for the Todd class of the singular weighted projective space $\mathbf{P}(1, 1, 2)$. The method can be extended to any simplicial toric variety, complete or not, [EG3]. (See also [BV] for a computation of the equivariant Todd class of complete toric varieties using other methods.)

The singular variety $\mathbf{P}(1, 1, 2)$ is the quotient of $X = \mathbb{A}^3 \smallsetminus \{0\}$ where $\mathbb{C}^*$ acts with weights $(1, 1, 2)$. This variety is the coarse moduli space of the corresponding smooth stack $\mathbb{P}(1, 1, 2)$. A calculation similar to that of Section 4.2.2 shows that $K_0(\mathbb{P}(1, 1, 2)) = \mathbb{Z}[\xi]/(\xi - 1)^2(\xi^2 - 1)$ and $\mathrm{Ch}^*(\mathbb{P}(1, 1, 2)) = \mathbb{Z}[t]/2t^3$ where $t = c_1(\xi)$.

The stack $\mathbb{P}(1, 1, 2)$ is a toric Deligne-Mumford stack (in the sense of [BCS]) and the weighted projective space $\mathbf{P}(1, 1, 2)$ is the toric variety $X(\Sigma)$ where Σ is the complete 2-dimensional fan with rays by $\rho_0 = (-1, -2), \rho_1 = (1, 0), \rho_2 = (0, 1)$. This toric variety has an isolated singular point P_0 corresponding to the cone σ_{01} spanned by ρ_0 and ρ_1.
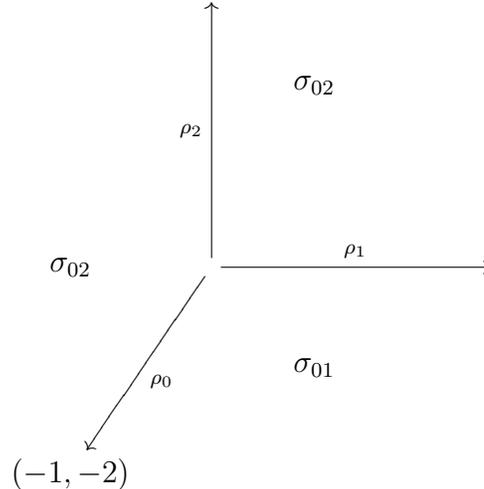
Each ray determines a Weil divisor D_{ρ_i} which is the image of the fundamental class of the hyperplane $x_i = 0$. With the given action, $[x_0 = 0] = [x_1 = 0] = t$ and $[x_2 = 0] = 2t$. Since the action of $\mathbb{C}^*$ on $\mathbb{A}^3$ is free on the complement of a set

of codimension 2, the pushforward defines an isomorphism of integral Chow groups $\mathrm{Ch}^1(\mathbb{P}(1,1,2)) = \mathrm{Ch}^1(\mathbf{P}(1,1,2))$. Thus, $\mathrm{Ch}^1(X(\Sigma) = \mathbb{Z}$ and $D_{\rho_0} \equiv D_{\rho_1}$ while $D_{\rho_2} \equiv 2D_{\rho_0}$. Also, $\mathrm{Ch}^2(X(\Sigma)) = \mathbb{Z}$ is generated by the class of the singular point P_0 and $[P_0] = 2[P]$ for any non-singular point P.

The tangent bundle to $\mathbb{P}(1,1,2)$ fits into the Euler sequence

$$0 \to \mathbf{1} \to 2\xi + \xi^2 \to T\mathbb{P}(1,1,2) \to 0$$

so $c_1(T\mathbb{P}(1,1,2) = 4t$ and $c_2(T\mathbb{P}(1,1,2) = 15t^2$. Thus

$$\mathrm{Td}(\mathbb{P}(1,1,2)) = 1 + 2t + 21/12t^2.$$

Pushing forward to $\mathbf{P}(1,1,2)$ gives a contribution of $1 + 2D_{\rho_0} + 21/24P_0$ to $\mathrm{td}(\mathbf{P}(1,1,2))$.

Now we must also consider the contribution coming from the fixed locus of (-1) acting on $\mathbb{A}^3 \setminus \{0\}$. The fixed locus is the line $x_0 = x_1 = 0$ and the normal bundle has K-theory class 2ξ. After twisting by -1 we obtain a contribution of

$$(28) \qquad p_* \left[\mathrm{ch}\left(\frac{1}{(1 + \xi^{-1})^2} \right) \mathrm{Td}([X^{-1}/\mathbb{C}^*]) \right]$$

Since $[X^{-1}/\mathbb{C}^*]$ is 0-dimensional and has a generic stabilizer of order 2 we obtain an additional contribution of $1/2 \, \mathrm{rk}(1/(1 + \xi^{-1})^2)[P_0] = (1/2 \times 1/4)[P_0] = 1/8[P_0]$ to $\mathrm{td}(\mathbf{P}(1,1,2))$. Combining the two contributions we conclude that:

$$\mathrm{td}(\mathbf{P}(1,1,2)) = 1 + 2D_{\rho_0} + [P_0]$$

in $\mathrm{Ch}^*(\mathbf{P}(1,1,2))$.

5.2. **Grothendieck-Riemann-Roch for Deligne-Mumford quotient stacks.** Suppose that $\mathcal{X} = [X/G]$ and $\mathcal{Y} = [Y/H]$ are smooth Deligne-Mumford quotient stacks and $f\colon \mathcal{X} \to \mathcal{Y}$ is a proper, but not-necessarily representable morphism. The most general Grothendieck-Riemann-Roch result we can write down is the following:

Theorem 5.4. [EK] *The following diagram of Grothendieck groups and Chow groups commutes:*

$$\begin{array}{ccc} G_0(\mathcal{X}) & \overset{I\tau_\mathcal{X}}{\to} & \mathrm{Ch}^*(I\mathcal{X}) \otimes \mathbb{C} \\ f_* \downarrow & & f_* \downarrow \\ G_0(\mathcal{Y}) & \overset{I\tau_\mathcal{Y}}{\to} & \mathrm{Ch}^*(I\mathcal{Y}) \otimes \mathbb{C} \end{array}$$

Remark 5.5. A proof of this result using the localization methods of [EG3, EG4] will appear in [EK]. A version of this Theorem (which also holds in some prime characteristics) was proved by Bertrand Toen in [Toe]. However, in that paper the target of the Riemann-Roch map is not the Chow groups but rather a "cohomology with coefficients in representations." Toen does not explicitly work with quotient stacks, but his hypothesis that the stack has the resolution property for coherent sheaves implies that the stack is a quotient stack.

In [EK] we will also give a version of Grothendieck-Riemann-Roch for proper morphisms of arbitrary quotient stacks.

6. Appendix on K-theory and Chow groups

In this section we recall some basic facts about K-theory and Chow groups both in the non-equivariant and equivariant settings. For more detailed references see [Ful, FL, Tho1, EG1].

6.1. K-theory of schemes and algebraic spaces.

Definition 6.1. Let X be an algebraic scheme. We denote by $G_0(X)$ the Grothendieck group of coherent sheaves on X and $K_0(X)$ the Grothendieck group of locally free sheaves; i.e vector bundles.

There is a natural map $K_0(X) \to G_0(X)$ which is an isomorphism when X is a smooth scheme. The reason is that if X is smooth every coherent sheaf has a finite resolution by locally free sheaves. For a proof see [Ful, Appendix B8.3].

Definition 6.2. If $X \to Y$ is a proper morphism then there is a pushforward map $f_* \colon G_0(X) \to G_0(Y)$ defined by $f_*[\mathcal{F}] = \sum_i (-1)^i [R^i f_* \mathcal{F}]$. When $Y = \mathrm{pt}$, then $G_0(Y) = \mathbb{Z}$ and $f_*(\mathcal{F}) = \chi(X, \mathcal{F})$.

The Grothendieck group $K_0(X)$ is a ring under tensor product and the map $K_0(X) \otimes G_0(X) \to G_0(X)$, $([V], \mathcal{F}) \mapsto \mathcal{F} \otimes V$ makes $G_0(X)$ into a $K_0(X)$-module. If $f \colon X \to Y$ is an arbitrary morphism of schemes then pullback of vector bundles defines a ring homomorphism $f^* \colon K_0(Y) \to K_0(X)$.

When $f \colon X \to Y$ is proper, the pullback for vector bundles and the pushforward for coherent sheaves are related by the projection formula. Precisely, if $\alpha \in K_0(Y)$ and $\beta \in G_0(X)$ then

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta$$

in $G_0(Y)$.

There is large class of morphisms $X \xrightarrow{f} Y$, for which there are pullbacks $f^* \colon G_0(Y) \to G_0(X)$ and pushforwards $f_* \colon K_0(X) \to K_0(Y)$. For example, if f is flat, the assignment $[\mathcal{F}] \mapsto [f^*\mathcal{F}]$ defines a pullback $f^* \colon G_0(Y) \to G_0(X)$.

Suppose that every coherent sheaf on Y is the quotient of a locally free sheaf (for example if Y embeds into a smooth scheme). If $f \colon X \to Y$ is a regular embedding then the direct image f_*V of a locally free sheaf has a finite resolution W by locally free sheaves. Thus we may define a pushforward $f_* \colon K_0(X) \to K_0(Y)$ by $f_*[V] = \sum_i (-1)^i [W_i]$ in this case. Also, if X and Y are smooth then there is a pushforward $f_* \colon K_0(X) \to K_0(Y)$. When X and Y admit ample line bundles then there are pushforwards $f_* \colon K_0(X) \to K_0(Y)$ for any proper morphism of finite Tor-dimension.

Definition 6.3. The Grothendieck ring $K_0(X)$ has an additional structure as a λ-ring. If V is a vector bundle of rank r set $\lambda^k[V] = [\Lambda^k V]$. If t is parameter define $\lambda_t(V) = \sum_{k=0}^{r} \lambda^k[V]t^k \in K_0(X)[t]$ where t is a parameter. The class $\lambda_{-1}(V^*) = 1 - [V^*] + [\Lambda^2 V^*] + \ldots + (-1)^r[\Lambda^r V^*]$ is called the K-*theoretic Euler class* of V.

Although, $K_0(X)$ is simpler to define and is functorial for arbitrary morphisms, it is actually much easier to prove results about the Grothendieck group $G_0(X)$. The reason is that G-functor behaves well with respect to localization. If $U \subset X$ is open with complement Z then there is an exact sequence

$$G_0(Z) \to G_0(X) \to G_0(U) \to 0.$$

The definitions of $G_0(X)$ and $K_0(X)$ also extend to algebraic spaces as does the basic functoriality of these groups. However, even if X is a smooth algebraic space there is no result guaranteeing that X satisfies the *resolution property* meaning that every coherent sheaf is the quotient of a locally free sheaf. Thus it is not possible to prove that the natural map $K_0(X) \to G_0(X)$ is actually an isomorphism. (Note however, that there no known examples of smooth separated algebraic spaces where the resolution property provably fails, c.f. [Tot].) In this case one can either replace $K_0(X)$ with the Grothendieck group of perfect complexes or work exclusively with $G_0(X)$.

6.2. **Chow groups of schemes and algebraic spaces.**

Definition 6.4. If X is a scheme (which for simplicity we assume to be equi-dimensional) we denote by $\mathrm{Ch}^i(X)$ the Chow group of codimension i-dimensional cycles modulo rational equivalence as in [Ful] and we set $\mathrm{Ch}^*(X) = \oplus_{i=0}^{\dim X} \mathrm{Ch}^i(X)$.

As was the case for the Grothendieck group $G_0(X)$, if $f \colon X \to Y$ is proper then there is a pushforward $f_* \colon \mathrm{Ch}^*(X) \to \mathrm{Ch}^*(Y)$. The map is defined as follows:

Definition 6.5. If $Z \subset X$ is a closed subvariety let $W = f(Z)$ with its reduced scheme structure

$$f_*[Z] = \left\{ \begin{array}{cc} [K(Z) : K(W)][W] & \text{if } \dim W = \dim Z \\ 0 & \text{otherwise} \end{array} \right\}$$

where $K(Z)$ (resp. $K(W)$) is the function field of Z (resp. W).

If X is complete then we denote the pushforward map $\mathrm{Ch}^* X \to \mathrm{Ch}^*(\mathrm{pt}) = \mathbb{Z}$ by $\int_X$.

Because we index our Chow groups by codimension, the map f_* shifts degrees. If $f \colon X \to Y$ has (pure) relative dimension d then $f_*(\mathrm{Ch}^k(X)) \subset \mathrm{Ch}^{k+d}(Y)$.

There is again a large class of morphisms $X \xrightarrow{f} Y$ for which there are pullbacks $f^* \colon \mathrm{Ch}^*(Y) \to \mathrm{Ch}^*(X)$. Some of the most important examples are flat morphisms

where the pullback is defined by $f^*[Z] = [f^{-1}(Z)]$, regular embeddings and, more generally, local complete intersection morphisms.

We again have a localization exact sequence which can be used for computation. If $U \subset X$ is open with complement Z then there is a short exact sequence

$$\mathrm{Ch}^*(Z) \to \mathrm{Ch}^*(X) \to \mathrm{Ch}^*(U) \to 0$$

Definition 6.6. If X is smooth (and separated) then the diagonal $\Delta \colon X \to X \times X$ is a regular embedding. Pullback along the diagonal allows us to define an intersection product on $\mathrm{Ch}^*(X)$ making it into a graded ring, called the Chow ring. If $[Z] \subset \mathrm{Ch}^k(X)$ and $[W] \subset \mathrm{Ch}^l(X)$ then we define $[Z] \cdot [W] = \Delta^*([Z \times W]) \in \mathrm{Ch}^{k+l}(X)$.

Any morphism of smooth varieties is a local complete intersection morphism, so if $f \colon X \to Y$ is a morphism of smooth varieties then we have a pullback $f^* \colon \mathrm{Ch}^* Y \to \mathrm{Ch}^* X$ which is a homomorphism of Chow rings.

The theory of Chow groups carries through completely to algebraic spaces [EG1, Section 6.1].

6.3. **Chern classes and operations.** Associated to any vector bundle V on a scheme X are Chern classes $c_i(V)$, $0 \leq i \leq \mathrm{rk}\, V$. Chern classes are defined as operations on Chow groups. Specifically $c_i(V)$ defines a homomorphism $\mathrm{Ch}^k X \to \mathrm{Ch}^{k+i} X$, $\alpha \mapsto c_i(V)\alpha$, with c_0 taken to be the identity map and denoted by 1. Chern classes are compatible with pullback in the following sense: If $f \colon X \to Y$ is a morphism for which there is a pullback of Chow groups then $c_i(f^*V)f^*\alpha = f^*(c_i(V)\alpha)$.

Chern classes of a vector bundle V may be viewed as elements of the *operational Chow ring* $A^*X = \oplus_{i=0} A^i X$ defined in [Ful, Definition 17.3]. An element of $c \in A^i X$ is a collection of homomorphisms $c \colon \mathrm{Ch}^*(X') \to \mathrm{Ch}^{*+k}(X')$ defined for any morphism of schemes $X' \to X$. These homomorphisms should be compatible with pullbacks of Chow groups and should also satisfy the projection formula $f_*(c\alpha) = cf_*\alpha$ for any proper morphism of X-schemes $f \colon X'' \to X'$ and class $\alpha \in \mathrm{Ch}^*(X'')$. Composition of morphisms makes A^*X into a graded ring and it can be shown that $A^k X = 0$ for $k > \dim X$.

If X is smooth, then the map $A^*X \to \mathrm{Ch}^* X$, $c \mapsto c([X])$ is an isomorphism of rings where the product on $\mathrm{Ch}^* X$ is the intersection product. In particular, if X is smooth then the Chern class $c_i(V)$ is completely determined by $c_i(V)[X] \in \mathrm{Ch}^i(X)$ so in this way we may view $c_i(V)$ as an element of $\mathrm{Ch}^i(X)$.

The total Chern class $c(V)$ of a vector bundle is the sum $\sum_{i=0}^{\mathrm{rk}\, V} c_i(V)$. Since $c_0 = 1$ and $c_i(V)$ is nilpotent for $i > 0$ the total Chern class $c(V)$ is invertible in A^*X. Also, if $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of vector bundles then $c(V) = c(V')c(V'')$, so the assignment $[V] \mapsto c(V)$ defines a homomorphism from the Grothendieck group $K_0(X)$ to the multiplicative group of units in A^*X.

6.3.1. *Splitting, Chern characters and Todd classes.* If V is a vector bundle on a scheme X, then the *splitting construction* ensures that there is a scheme X' and a smooth, proper morphism $f\colon X' \to X$ such that $f^*\colon \mathrm{Ch}^* X \to \mathrm{Ch}^* X'$ is injective and f^*V has a filtration $0 = E_0 \subset E_1 \subset \ldots E_r = f^*V$ such that the quotients $L_i = E_i/E_{i-i}$ are line bundles. Thus $c(f^*V)$ factors as $\prod_{i=1}^r (1 + c_1(L_i))$. The classes $\alpha_i = c_1(L_i)$ are Chern roots of V and any symmetric expression in the α_i is the pullback from $\mathrm{Ch}^* X$ of a unique expression in the Chern classes of V.

Definition 6.7. If V is a vector bundle on X with Chern roots $\alpha_1, \ldots \alpha_r \in A^* X'$ for some $X' \to X$ then the *Chern character* of V is the unique class $\mathrm{ch}(V) \in A^* X \otimes \mathbb{Q}$ which pulls back to $\sum_{i=0}^r \exp(\alpha_i)$ in $A^*(X') \otimes \mathbb{Q}$. (Here exp is the exponential series.)

Likewise the *Todd class* of V is the unique class $\mathrm{Td}(V) \in A^* X \otimes \mathbb{Q}$ which pulls back to $\prod_{i=0}^r \frac{\alpha_i}{1 - \exp(-\alpha_i)}$ in $A^*(X') \otimes \mathbb{Q}$.

The Chern character can be expressed in terms of the Chern classes of V as

$$(29) \qquad \mathrm{ch}(V) = \mathrm{rk}\, V + c_1 + (c_1^2 - c_2)/2 + \ldots$$

and the Todd class as

$$(30) \qquad \mathrm{Td}(V) = 1 + c_1/2 + (c_1^2 + c_2)/12 + \ldots$$

Because $A^k(X) = 0$ for $k > \dim X$ the series for $\mathrm{ch}(V)$ and $\mathrm{Td}(X)$ terminate for any given scheme X and vector bundle V.

If V and W are vector bundles on X then $\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$ and $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\,\mathrm{ch}(W)$ so the Chern character defines a homomorphism of rings $\mathrm{ch}\colon K_0(X) \to A^* X \otimes \mathbb{Q}$. We also have that $\mathrm{Td}(V \oplus W) = \mathrm{Td}(V)\,\mathrm{Td}(W)$ so we obtain a homomorphism $\mathrm{Td}\colon K_0(X) \to (A^* X \otimes \mathbb{Q})^\star$ from the additive Grothendieck group to the multiplicative group of units in $A^* X \otimes \mathbb{Q}$.

When X is smooth we interpret the target of the Chern character and Todd class to be $\mathrm{Ch}^* X$.

6.4. **Equivariant K-theory and equivariant Chow groups.** We now turn to the equivariant analogues of Grothendieck and Chow groups.

6.4.1. *Equivariant K-theory.* Most of the material on equivariant K-theory can be found in [Tho1] while the material on equivariant Chow groups is in [EG1].

Definition 6.8. Let X be a scheme (or algebraic space) with the action of an algebraic group G. In this case we define $K_0(G, X)$ to be the Grothendieck group of G-equivariant vector bundles and $G_0(G, X)$ to be the Grothendieck group of G-linearized coherent sheaves.

As in the non-equivariant case there is pushforward of Grothendieck groups $G_0(G, X) \to$ $G_0(G, Y)$ for any proper G-equivariant morphism. Similarly, there is a pullback $K_0(G, Y) \to$ $K_0(G, X)$ for any G-equivariant morphism $X \to Y$. There are also pullbacks in G-theory for equivariant regular embeddings and equivariant lci morphisms. There is also a localization exact sequence associated to a G-invariant open set U with complement Z.

The Grothendieck group $K_0(G, X)$ is a ring under tensor product and $G_0(G, X)$ is a module for this ring. The equivariant Grothendieck ring $K_0(G, \mathrm{pt})$ is the representation ring $R(G)$ of G. Since every scheme maps to a point, $R(G)$ acts on both $G_0(G, X)$ and $K_0(G, X)$ for any G-scheme X. The $R(G)$-module structure on $G_0(G, X)$ plays a crucial role in the Riemann-Roch theorem for Deligne-Mumford stacks.

If V is a G-equivariant vector bundle then $\Lambda^k V$ has a natural G-equivariant structure. This means that the wedge product defines a λ-ring structure on $K_0(G, X)$. In particular we define the equivariant Euler class of a rank r bundle V by the formula

$$\lambda_{-1}(V^*) = 1 - [V^*] + [\Lambda^2 V^*] - \ldots + (-1)^r [\Lambda^r V^*].$$

Results of Thomason [Tho2, Lemmas 2.6, 2.10, 2.14] imply that if X is normal and quasi-projective or regular and separated over the ground field (both of which implies that X has the resolution property) and G acts on X then X has the G-equivariant resolution property. It follows that if X is a smooth G-variety then every G-linearized coherent sheaf has a finite resolution by G-equivariant vector bundles. Hence $K_0(G, X)$ and $G_0(G, X)$ may be identified if X is a smooth scheme.

The Grothendieck groups $G_0(G, X)$ and $K_0(G, X)$ are naturally identified with the corresponding Grothendieck groups of the categories of locally free and coherent sheaves on the quotient stack $\mathcal{X} = [X/G]$.

Remark 6.9 (Warning). If X is complete then there are pushforward maps $K_0(G, X) \to$ $K_0(G, \mathrm{pt}) = R(G)$ and $G_0(G, X) \to K_0(G, \mathrm{pt}) = R(G)$ that associate to a vector bundle V (resp. coherent sheaf $\mathcal{F}$) the virtual representation $\sum (-1)^i H^i(X, V)$ (resp. $\sum (-1)^i H^i(X, \mathcal{F})$.). Although V may be viewed as a vector bundle on the quotient stack $\mathcal{X} = [X/G]$ the virtual representation $\sum (-1)^i H^i(X, V)$ *is not* the Euler characteristic of V as a vector bundle on $\mathcal{X}$.

6.4.2. *Equivariant Chow groups.* The definition of equivariant Chow groups requires more care and is modeled on the Borel construction in equivariant cohomology. If G acts on X then the i-th equivariant Chow group is defined as $\mathrm{Ch}^i(X_G)$ where X_G is any quotient of the form $(X \times U)/G$ where U is an open set in a representation $\mathbf{V}$ of G such that G acts freely on U and $\mathbf{V} \smallsetminus U$ has codimension more than i. In [EG1] it is shown that such pairs $(U, \mathbf{V})$ exist for any algebraic group and that the definition of $\mathrm{Ch}^i_G(X)$ is independent of the choice of U and $\mathbf{V}$.

Because equivariant Chow groups are defined as Chow groups of certain schemes, they enjoy all of the functoriality of ordinary Chow groups. In particular, if X is smooth then pullback along the diagonal defines an intersection product on $\mathrm{Ch}_G^*(X)$.

Remark 6.10. Intuitively an equivariant cycle may be viewed as a G-invariant cycle on $X \times \mathbf{V}$ where $\mathbf{V}$ is some representation of G. Because representations can have arbitrarily large dimension $\mathrm{Ch}^i(X)$ can be non-zero for all i.

If G acts freely then a quotient X/G exists as an algebraic space and $\mathrm{Ch}_G^i(X) = \mathrm{Ch}^i(X/G)$. More generally, if G acts with finite stabilizers then elements of $\mathrm{Ch}_G^i(X) \otimes \mathbb{Q}$ are represented by G-invariant cycles on X and consequently $\mathrm{Ch}_G^i(X) = 0$ for $i > \dim X - \dim G$.

As in the non-equivariant case, an equivariant vector bundle V on a G-scheme defines Chern class operations $c_i(V)$ on $\mathrm{Ch}_G^*(X)$. The Chern class naturally live in the equivariant operational Chow ring $A_G^*(X)$ and as in the non-equivariant case the map $A_G^*(X) \to \mathrm{Ch}_G^*(X)$, $c \mapsto c[X]$ is a ring isomorphism if X is smooth.

We can again define the Chern character and Todd class of a vector bundle V. However, because $\mathrm{Ch}_G^i(X)$ can be non-zero for all i, the target of the Chern character and Todd class is the infinite direct product $\Pi_{i=0}^\infty \mathrm{Ch}_G^i(X) \otimes \mathbb{Q}$.

When G acts on X with finite stabilizers then $\mathrm{Ch}_G^i(X) \otimes \mathbb{Q}$ is 0 for $i > \dim X - \dim G$ so in this case the target of the Chern character and Todd class map is $\mathrm{Ch}_G^*(X)$.

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