The Chow class of the hyperelliptic Weierstrass divisor

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1 Introduction

The purpose of this note is to compute the rational Chow class of the universal hyperelliptic Weierstrass divisor \( \mathcal{H}_{g,w} \subset \mathcal{H}_{g,1} \) where \( \mathcal{H}_{g,1} \) is the universal stable hyperelliptic curve of genus \( g \).

Our result is expressed in terms of a basis for \( \text{Cl}(\mathcal{H}_{g,1})_\mathbb{Q} = \text{Pic}(\mathcal{H}_{g,1})_\mathbb{Q} \) obtained by Scavia in [4] and is stated as follows.

Theorem 1.1.

\[
[\mathcal{H}_{g,w}] = \left( \frac{g+1}{g-1} \right) \psi - \frac{1}{2(2g+1)(g-1)} \eta_{irr} + \sum_{i=1}^{\lfloor (g-1)/2 \rfloor} \left[ -\frac{i(2i+3)}{(2g+1)(g-1)} \eta_{i,0} - \frac{(g-i-1)(2g-i+1)}{(2g+1)(g-1)} \eta_{i,1} \right] + \sum_{i=1}^{\lfloor g/2 \rfloor} \left[ -\frac{2i(2i+1)}{(2g+1)(g-1)} \delta_{i,0} - \frac{2(g-i)(2g-i+1)}{(2g+1)(g-1)} \delta_{i,1} \right]
\]

where \( \psi \) is the \( \psi \)-class associated to the section and the other classes are boundary divisors which we describe below. When \( g = 2 \), our formula agrees with the formula proved by Eisenbud and Harris in [2] using Porteous’s formula. Unfortunately, we cannot apply their method in higher genus, because not every curve is hyperelliptic and the corresponding degeneracy locus has expected codimension higher than 1. Instead we use the method of test curves, and the main challenge is to find sufficiently interesting families of branched double covers of rational nodal curves. The key construction to do this is done in Section 4.2.

2 Stacks of hyperelliptic curves

Let \( \mathcal{H}_g \) be the stack of smooth hyperelliptic curves. This a locally closed smooth substack of the moduli space \( \mathcal{M}_g \). We denote by \( \overline{\mathcal{H}}_g \) the closure of \( \mathcal{H}_g \) in \( \overline{\mathcal{M}}_g \). This is a smooth substack of \( \overline{\mathcal{M}}_g \) followed from [1] Chapter XI, Lemma (6.15)]. The theory of admissible covers allows us to describe the boundary divisors of \( \overline{\mathcal{H}}_g \); i.e. the irreducible components of \( \overline{\mathcal{H}}_g \setminus \mathcal{H}_g \).

Likewise, let \( \overline{\mathcal{H}}_{g,n} \) denote the stack of \( n \)-pointed hyperelliptic curves. It is defined as the inverse image of \( \overline{\mathcal{H}}_g \) under the forgetful map \( \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g \). The stack \( \overline{\mathcal{H}}_{g,1} \) is smooth but it is not known if \( n > 1 \) [4]. However, \( \overline{\mathcal{H}}_{g,n} \) always contains a smooth open substack \( \overline{\mathcal{H}}_{g,n}^o \) whose complement has codimension more than two. The stack \( \overline{\mathcal{H}}_{g,n}^o \) parametrizes stable hyperelliptic curves without rational tails.

2.1 The divisor class group of the stack of stable hyperelliptic curves

We begin by describing the boundary divisors of \( \overline{\mathcal{H}}_g \).

- \( \eta_{irr} \): it parametrizes stable hyperelliptic curves \( C \) with at least one nonseparating node such that its partial normalization at such point is connected. A general curve in \( \eta_{irr} \) has a single node. Its normalization is a smooth hyperelliptic curve of genus \( g - 1 \). If \( \{P, Q\} \) is the inverse image of then node, then points \( P \) and \( Q \) conjugated by the hyperelliptic involution.
• \( \delta_i \) for \( 1 \leq i \leq \lfloor \frac{g}{2} \rfloor \): it parametrizes curves \( C \) admitting a node such that its partial normaliza-
tion at such point is the disjoint union of two curves with genera \( i \) and \( g - i \). Moreover, such
a node is fixed by the hyperelliptic involution of \( C \) which means that each point in the inverse
image of the node is a Weierstrass point on its corresponding component.

• \( \eta_i \) for \( 1 \leq i \leq \lfloor \frac{g-1}{2} \rfloor \): it parametrizes curves \( C \) having two nodes that are conjugated by the
hyperelliptic involution such that the partial normalization is the disjoint union of two curves
of genera \( i \) and \( g - 1 - i \).

**Theorem 2.1.** [Chapter XIII, Theorem (8.4)] For any \( g \geq 2 \), \( \text{Pic}(\overline{\mathcal{M}}_{g,1})/\mathbb{Q} \) is a vector space of
dimension \( g \), freely generated by \( \eta_{irr} \), \( \{ \delta_i \}_{1 \leq i \leq \lfloor g/2 \rfloor} \), and \( \{ \eta_i \}_{1 \leq i \leq \lfloor (g-1)/2 \rfloor} \).

### 2.2 The divisor class group \( \overline{\mathcal{H}}_{g,1} \)

We first describe the inverse images of these boundary divisors under the projection \( \pi: \overline{\mathcal{H}}_{g,1} \to \overline{\mathcal{H}}_g \).

• \( \eta_{irr} \): The inverse image of \( \eta_{irr} \) is an irreducible component of \( \overline{\mathcal{H}}_{g,1} \setminus \overline{\mathcal{H}}_g \) and we again denote
it by \( \eta_{irr} \).

• \( \delta_i \): If \( i < g/2 \) then the inverse image of \( \delta_i \) consists of two irreducible components \( \delta_{i,1} \) and \( \delta_{i,0} \)
where the second index is one if the marked point is on the component of genus \( i \) and zero if
the marked point is on the component of genus \( g - i \). If \( i = g/2 \) then the inverse image of \( \delta_i \)
is irreducible.

• \( \eta_i \): If \( i < \frac{g-1}{2} \) then the inverse image of \( \eta_i \) again has two irreducible components \( \eta_{i,0} \) and \( \eta_{i,1} \)
corresponding to whether the marked point is on the component of \( g - i \) or the component
of genus \( i \). When \( i = \frac{g-1}{2} \) then \( \eta_{i,0} = \eta_{i,1} \).

**Theorem 2.2.** [Theorem 1.1] \( \text{Pic}(\overline{\mathcal{H}}_{g,1})/\mathbb{Q} = \text{Cl}(\overline{\mathcal{H}}_{g,1})/\mathbb{Q} \) is freely generated by the classes \( \psi \), \( \eta_{irred} \),
\( \delta_{i,0}, \delta_{i,1}, \eta_{i,0}, \eta_{i,1} \), where \( \psi \) is the pullback to \( \overline{\mathcal{H}}_{g,1} \) of the \( \psi \)-class\(^1\) on \( \overline{\mathcal{M}}_{g,1} \).

By Theorem 2.2 we can write
\[
[\overline{\mathcal{H}}_{g,w}] = \sum_{i=1}^{\lfloor g/2 \rfloor} [a_{i,0}\delta_{i,0} + a_{i,1}\delta_{i,1}] + \sum_{j=1}^{\lfloor (g-1)/2 \rfloor} [b_{j,0}\eta_{j,0} + b_{j,1}\eta_{j,1}] + c\eta_{irr,1} + d\psi. \tag{2.1}
\]

Our goal is to determine the \( \mathbb{Q} \)-coefficients \( a_{i,0}, a_{i,1}, b_{j,0}, b_{j,1} \) and \( c, d \) of these generators. We will
use the method of test curves, as explained by Harris and Morrison in [M].

### 3 Test curves with trivial moduli

We begin by considering three test curves \( (C \to B, \sigma) \) where the section but not the moduli of
the curves varies. In this case the composite map \( B \to \overline{\mathcal{H}}_{g,1} \to \overline{\mathcal{H}}_g \) is constant. Using these test curves
we can compute the coefficient \( d \) and obtain linear relations between \( a_{i,0} \) and \( a_{i,1} \), and \( b_{j,0} \) and \( b_{j,1} \)
respectively. It is not strictly necessary to compute these relations, but they serve as a way to check
the correctness of some of the more difficult calculations.

#### 3.1 The first test curve - calculating the coefficient \( d \)

**Proposition 3.1.** The coefficient \( d \) of \( \psi \) equals \( \left( \frac{g+1}{g-1} \right) \).

**Proof.** Let \( X \) be a fixed smooth hyperelliptic curve and consider the family of pointed curves over
\( X \), defined by \((X \times X \xrightarrow{\psi} X, \Delta)\) where \( \Delta \) denotes the diagonal section. Since all fibers of this family
are smooth we know that all boundary divisors vanish on \( X \). Also \( \deg_X([\overline{\mathcal{H}}_{g,w}]) = 2g + 2 \) since \( X \)
has \( 2g + 2 \) Weierstrass points. Since the section is the diagonal, \( \deg_X \psi = -(\Delta)^2 = 2g - 2 \). Hence
\( 2g + 2 = (2g - 2)d \) or equivalently, \( d = \left( \frac{g+1}{g-1} \right) \). \( \square \)

\(^1\)Recall that the \( \psi \)-class is line bundle on the stack \( \overline{\mathcal{M}}_{g,1} \) whose pullback to a pointed family of stable curves
\( (C \xrightarrow{\psi} B, \sigma) \) is the line bundle \( \sigma^*(\omega_{C/B}) \).
3.2 The second test curve - calculating a linear relation between $a_{i,0}$ and $a_{i,1}$

The second test curve is described as below. Let $(X, A)$ be a hyperelliptic curve of genus $i$ with a fixed Weierstrass point $A$; let $(Y, B, p)$ be another hyperelliptic curve of genus $g - i$ with a fixed Weierstrass point $B$ together with a moving point $p \in Y$. By identifying the fixed point $A$ on $X$ with the point $B$ on $Y$, we obtained our second test curve. Here the base is defined to be $Y$. And obviously, $\deg_Y(\eta_{j,0}) = \deg_Y(\eta_{j,1}) = \deg_Y(\eta_{irr,1}) = 0$, for $1 \leq j \leq \lfloor \frac{g-1}{2} \rfloor$, and also $\deg_Y(\delta_{j,1}) = 0$ for all $1 \leq j \leq \lfloor g/2 \rfloor, j \neq i$ together with $\deg_Y(\delta_{j,0}) = 0$ for $j \neq i$. So it remains to compute $\deg_Y(\delta_{i,0})$, $\deg_Y(\delta_{i,1})$, $\deg_Y(\psi)$ and $\deg_Y([\overline{\mathcal{M}}_{g,w}])$.

First it is easy to see that the first test curve defined as above is a degeneration of curves in $\overline{\mathcal{M}}_{g-i,2}$. Thus we can consider the family of smooth hyperelliptic curves of genus $g - i$ with two sections $\sigma_1$ and $\sigma_2$ passing through fixed $B$ and moving marked point $p$ respectively. The special fiber appears when the moving point $p$ meets with the fixed point $B$. By blowing up the point $(B, B)$ in $Y \times Y$, we obtain the following diagram and finally we identify the section $Y \times \{B\}$ with the section $Y \times \{A\}$:

![Diagram](image)

Figure 1: The second test curve

By observation, if we look at the forgetful morphism $\pi_g : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$, we will have the relation $\pi_g^*(D_i) = D_{i,1} + D_{i,0}$. Since the stable hyperelliptic curve defined by $X \cup Y/A \sim B$ has no moduli in $\overline{\mathcal{M}}_g$ due to the fact that everything is fixed, we have the following important relation between $\deg_Y(\delta_{i,0})$ and $\deg_Y(\delta_{i,1})$:

$$\deg_Y(\delta_{i,0}) + \deg_Y(\delta_{i,1}) = 0.$$ 

Indeed, it’s easy to see that the curve of the following form contributes to both $\delta_{i,0}$ and $\delta_{i,1}$.

Thus the $\mathbb{Q}$-linear relation we are going to find here is reduced to

$$\deg_Y([\overline{\mathcal{M}}_{g,w}]) = (a_{i,0} - a_{i,1}) \deg_Y(\delta_{i,0}) + d \deg_Y(\psi).$$

Since both $A$ and $B$ are Weierstrass points for $X$ and $Y$ respectively, the admissible double cover of the stable hyperelliptic curve of above form is clearly ramified over the two nodes. Thus the point $p$ on the exceptional divisor $E \cong \mathbb{P}^1$ cannot be the ramification point of the double cover since we have used up all the $2g + 2$ ramification points. This give us:

$$\deg_Y([\overline{\mathcal{M}}_{g,w}]) = 2(g - i) + 1.$$ 

Now we want to find $Y \cdot \delta_{i,0}$. In fact, since our surface is obtained by gluing two smooth surfaces (not sure this quite the right statement, but our surface certainly isn’t smooth since it’s reducible),
by Lemma (3.94) in [3], the value on $Y$ of the divisor class $\delta_{i,0}$ on the moduli is the tensor product of the normal bundles

$$N_{Y \times \{A\}/Y \times X} \otimes N_{Y \times \{B\}/Bl(B,B)(Y \times Y)}$$

where $Y \times \{B\}$ is the proper transform of $Y \times \{B\}$ on the blowup of $Y \times Y$ at $(B,B)$. The first factor here is trivial, and the second factor has degree equal to the self-intersection of $Y \times \{B\}$. On $Y \times Y$, we have $(Y \times \{B\})^2 = 0$. Since $Y \times \{B\} + E = Y \times \{B\}$, and $E^2 = -1$, then the self-intersection will drop by one after blowing up, i.e.

$$(Y \times \{B\})^2 + 2 - 1 = (Y \times \{B\})^2.$$

Thus we obtain the following:

$$\text{deg}_Y(\delta_{i,0}) = -\text{deg}_Y(\delta_{i,1}) = -1.$$

In order to find $Y \cdot \psi$, note that the degree relates to the self-intersection of $\widetilde{\Delta}$ where $\widetilde{\Delta}$ is the proper transform of the diagonal $\Delta$. Again, on $Y \times Y$, we have $\widetilde{\Delta}^2 = \chi_{\text{top}}(Y) = 2 - 2(g - i)$. And the self-intersection will drop by one by the similar argument as above, thus $\Delta^2 = 1 - 2(g - i)$.

$$\text{deg}_Y(\psi) = -\Delta^2 = 2(g - i) - 1,$$

according to the self-intersection formula $\pi_* (\sigma_2^2) = -\psi \cap [Y]$ in terms of divisors where we use $\pi$ as the family of curves over the base $Y$. Thus, we obtain the relation

$$2(g - i) + 1 = (a_{i,0} - a_{i,1})(-1) + (2(g - i) - 1)d.$$

Substituting $d = \left(\frac{g+1}{g-1}\right)$, we conclude that

$$a_{i,0} - a_{i,1} = \frac{2(g - 2i)}{g - 1} \quad (3.1)$$

By symmetry, we could also choose the test curve similar to what we defined as above except the moving point $p$ lying on the genus $i$ component $X$. Thus we could get a pair of relations:

$$2(g - i) + 1 = (a_{i,0} - a_{i,1})(-1) + (2(g - i) - 1)d$$  
$$2i + 1 = (a_{i,1} - a_{i,0})(-1) + (2i - 1)d$$

which imply that in addition

$$d = \frac{g + 1}{g - 1}$$

which gives us a consistency check with our previous computation of $d$. 

Figure 2: The special fiber in the second test curve
3.3 The third test curve - calculating a linear relation between $b_{i,0}$ and $b_{i,1}$

Take two smooth hyperelliptic curves $X$ and $Y$ of genus $i$ and $g - i - 1$ respectively. Let $A, B$ (resp. $A', B'$) be two points of $X$ (resp $Y$) which are conjugate under the hyperelliptic involution. Add one moving point $p \in X$ and then blow up $X \times X$ at two points $(A, A)$ and $(B, B)$. Here we denote the proper transforms of $\Delta, X \times \{A\}$ and $X \times \{B\}$ by $\tilde{\Delta}, \tilde{X \times \{A\}}$ and $\tilde{X \times \{B\}}$ respectively. By identifying sections $X \times \{A\}$ with $X \times \{A'\}$ and $X \times \{B\}$ with $X \times \{B'\}$, one would get a smooth family of curve in $\mathcal{H}_g$ with section $\Delta$.

![Figure 3: A general curve](image)

Now we need to determine that whether the moving point $p$ is ramified in the admissible double cover of this stable model of hyperelliptic curve. First notice that the admissible double cover is not ramified at the nodes, so by counting the number of ramification points, it is easy to see that $p$ cannot be a ramification point. This will give us the following fact:

$$\deg_X([\mathcal{H}_{g,w}]) = 2i + 2.$$

Now consider the special fiber as below

![Figure 4: Special fiber in the third test curve](image)

We can see that it contributes to both classes $\eta_{i,0}$ and $\eta_{i,1}$, so we will get nonzero degrees for both $\eta_{i,0}$ and $\eta_{i,1}$ on this test family.

By the same argument for the second test curve, since $X \cup Y/(A \sim A', B \sim B')$ without the moving point has no moduli in $\mathcal{H}_g$. If we pullback the point via the forgetful morphism (after base change if necessary), we would get the trivial restriction on the class $\eta_{i,0} + \eta_{i,1}$ which will give us the relation:

$$\deg_X(\eta_{i,0}) + \deg_X(\eta_{i,1}) = 0.$$

Using the Lemma (3.94) in [3] again, we have

$$\deg_X(\eta_{i,1}) = \deg(X \times \{A\})^2 + \deg(X \times \{B\})^2 = -2.$$

And the self-intersection of the proper transform of the diagonal section gives rise to the degree of the psi class:

$$\deg_X(\psi) = -\deg(\tilde{\Delta}) = -(2 - 2i - 2) = 2i,$$
since the section $\Delta$ passes through two points we blown up. Again by the exactly same argument as for the first test curve, we get for any $1 \leq i \leq \lfloor \frac{g-1}{2} \rfloor$,
\[
2i + 2 = (b_{i,1} - b_{i,0})(-2) + 2td \\
2(g - 1 - i) + 2 = (b_{i,0} - b_{i,1})(-2) + 2(g - 1 - i)d.
\]
It gives us a relation between $b_{i,0}$ and $b_{i,1}$:
\[
b_{i,0} - b_{i,1} = \frac{g - 2i - 1}{g - 1}.
\]
which again gives another consistency check with $d$. Now we need to find some extra test curves of nontrivial moduli in $\mathcal{H}_g$ in order to compute $b_{i,0}$ and $b_{i,1}$.

4 Families of pointed hyperelliptic curves with non-trivial moduli

4.1 The fourth test curve - calculating the coefficient $c$

Similar to the idea of varying an elliptic curve in an elliptic surface, we can also vary a hyperelliptic curve in some special pencil of hyperelliptic curves to obtain a family of pointed hyperelliptic curves with at worst a single node.

Let $X$ be a smooth hyperelliptic curve of genus $g$, then there exists a double cover $\pi_1 : X \to \mathbb{P}^1$ ramified over $2g + 2$ points. Moreover, we can also associate a degree $g + 1$ line bundle to the divisor which is the formal sum of $g + 1$ distinct Weierstrass points. By Riemann-Roch Theorem, we have $h^0(X, \mathcal{L}) = 2$, which means that the complete linear series $|\mathcal{L}|$ forms a degree $g + 1$ map $\pi_2$ to $\mathbb{P}^1$.

And the product morphism $X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ is actually an embedding.

Now, according to the above statement, every smooth hyperelliptic curve of genus $g$ can be viewed as a bidegree $(2, g + 1)$ curve on a smooth quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. Let $C$ be a general pencil of curves of bidegree $(2, g + 1)$ on $Q$ in $\mathbb{P}^3$ over the base $B = \mathbb{P}^1$. Namely,

\[C = \{\lambda G(x, y, u, v) + \mu H(x, y, u, v) \mid (\lambda, \mu) \in \mathbb{P}^1\},\]

where both $G$ and $H$ are some fixed general bihomogeneous polynomials of bidegree $(2, g + 1)$. Then there are $4(g + 1)$ base points in this pencil. Choose one of them, say $P$, as the marked point. And there is only one section of linear series $g^2_2$ that contains $P$, so

\[
\deg_B(\mathcal{H}_{g,w}) = B \cdot [\mathcal{H}_{g,w}] = 1
\]

Blow up these $4(g + 1)$ base points, we get a surface $S \subset Q \times \mathbb{P}^1$ which is a one-parameter family of genus $g$ hyperelliptic curves over the base $B$.

Lemma 4.1. The number of nodal fibers in the pencil is $4(2g + 1)$.

Proof. Since $S$ is the blow-up of $Q$ at $(4g + 1)$ points, we have

\[
\chi_{\top}(S) = \chi_{\top}(Q) + 4(g + 1)
\]

where $\chi_{\top}(Q) = \chi_{\top}(\mathbb{P}^1) = 4$. Moreover, we know that a general fiber $C_\eta$ of the pencil is a smooth hyperelliptic curve of genus $g$, thus $\chi_{\top}(C_\eta) = 2 - 2g$. Note that each singular fiber $C$ appearing in a general pencil of plane curves has a single node as singularity, so the normalization $\overline{C}$ of $C$ is of genus $g - 1$ and hence of Euler characteristic $2 - 2(g - 1)$. Since $C$ is obtained from $\overline{C}$ by identifying two points, thus

\[
\chi_{\top}(C) = 1 - 2(g - 1).
\]

It means that the each singular fiber of $S \to \mathbb{P}^1$ contributes one to the Hurwitz formula, so that we can compute the number of singular fibers as following:

\[
\text{number of singular fibers} = \chi_{\top}(S) - \chi_{\top}(\mathbb{P}^1)\chi_{\top}(C_\eta) \\
= 4 + 4(g + 1) - 2(2 - 2g) \\
= 4(2g + 1).
\]

\[\square\]
Therefore we have the following intersection multiplicity with the boundary class $\eta_{irr}$

$$\deg_B(\eta_{irr}) = 4(2g+1).$$

Furthermore, since the marked point gets blown up once, thus

$$\deg_B(\psi) = -(-1) = 1.$$

Together with the fact that this family intersects other boundary divisors trivially, we have the following linear relation:

$$1 = 4(2g+1)c + d, \quad \text{where} \quad d = \frac{g+1}{g-1}$$

which implies

$$c = \frac{1}{2(2g+1)(g-1)}.$$

### 4.2 A general construction of pencils of pointed hyperelliptic curves with at worst nodes as singularities

The following construction is a variant on the construction of the pencil in Section 4.1 but it allows more flexibility for creating families with sections which have certain properties.

Instead of either compactifying $z^2 = f_{2g+2}(x, y)$ in $\mathbb{P}^2$ in which case we obtain a complicated singularity at infinity, or embedding a smooth hyperelliptic curve in a rational ruled surface as we did in the previous section, we view $z^2 = f_{2g+2}(x, y)$ as a defining equation for a smooth hyperelliptic curve embedded in the weighted projective surface $\mathbb{P}(1, 1, g+1)$.

Note that any smooth hyperelliptic curve $C$ over some algebraically closed field, say $\mathbb{C}$, can be embedded in the weighted projective space $\mathbb{P}(1, 1, g+1)$ and expressed by an equation

$$z^2 = f(x, y) = f_0x^{2g+2} + f_1x^{2g+1}y + \cdots + f_{2g+2}y^{2g+2},$$

where $f_0, \cdots, f_{2g+2}$ are some coefficients and $f$ factors into distinct linear factors over $\mathbb{C}$.

For the weighted projective space $\mathbb{P}(1, 1, g+1)$, either it can be viewed as

$$\mathbb{P}(1, 1, g+1) = \mathbb{C}^3 \setminus \{0\} / \sim, \quad (x, y, z) \sim (\lambda x, \lambda y, \lambda^{g+1} z) \text{ where } \lambda \in \mathbb{C}^*,$$

or it can also be embedded in some higher dimensional projective space, say $\mathbb{P}^{g+1}$ via

$$\mathbb{P}(1, 1, g+1) \longrightarrow \mathbb{P}^{g+2}$$

$$(x : y : z) \longrightarrow (x^{g+1} : x^g y : \cdots : y^{g+1} : z).$$

So that the image of above embedding is a cone over the rational normal curve of degree $g+1$ in $\mathbb{P}^{g+1}$ which has a unique singular point at the vertex $(0 : \cdots : 0 : 1) \in \mathbb{P}^{g+2}$ when $g \geq 1$. Then $z^2 = f(x, y)$ defines a curve of genus $g$ and defines a hypersurface of degree $2g+2$ in $\mathbb{P}(1, 1, g+1)$. Notice that the curve defined by $z^2 = f(x, y)$ is also the intersection of this cone with a quadratic hypersurface that doesn’t pass through the vertex. So a pencil of hyperelliptic curves can also be viewed as a pencil of quadratic sections on the weighted projective space $\mathbb{P}(1, 1, g+1)$.

Since weighted projective spaces are also toric varieties, if we blow up $\mathbb{P}(1, 1, g+1)$ at the vertex $(0 : 0 : 1)$, it is easy to see that we get the Hirzebruch surface $\mathbb{F}_{g+1}$ with the exceptional divisor $E \cong \mathbb{P}^1$ satisfying $E \cdot E = -g - 1$. Thus this construction of embedding a smooth hyperelliptic curve in $\mathbb{P}(1, 1, g+1)$ is the same as embedding in a corresponding Hirzebruch surface $\mathbb{F}_{g+1}$.

**Lemma 4.2.** A pencil of binary forms in $x, y$ of homogeneous degree $2g+2$ over some algebraically closed field $k$ defined as

$$\prod_{i=1}^{2} ((a_i \lambda + b_i \mu) x + (c_i \lambda + d_i \mu) y) \prod_{j=1}^{2g} (x + \alpha_j y)$$

for some general coefficients $a_i, b_i, c_i, d_i, \alpha_j \in k$ intersects its discriminant locus at $4g + 2$ points, each of multiplicity 2.
We will use \( \tilde{\pi} \) and \( \tilde{\sigma} \) where the \( \sigma \)'s of multiplicity \( 2 \) \( \Delta \) branched along \( \rho \) able to take its smooth double cover which gives us a pencil of double covers branched exactly at \( \Delta \). The pencil total surface is singular.

To start with, we take our branch divisor in \( Q \) take the double cover of the quadric surface \( B \) the branch divisor is reducible, in order to get a smooth double cover, we have to blow up the \( g \) in order to get two sections adding up to \( g \), since each \( \sigma \)'s and \( \Delta \) intersecting each other and with the \( \sigma \)'s.

To be precise, we have to blow up all \( 4g+2 \) points to separate the sections, and then we will be able to take its smooth double cover which gives us a pencil of double covers branched exactly at \( 2g+2 \) points. When generating sections, we will get a family of stable curves. For the notations, let \( \tilde{\pi} \) \( \tilde{\sigma} \) \( \tilde{\Delta} \) denote the Weierstrass points on each fiber. To start with, we take our branch divisor in \( \tilde{\Pi}_g \) where the general fibers are nonsingular and the total surface is singular.

**Lemma 4.3.** The pencil \( \mathcal{X} \to B \) constructed above intersects the divisor \( \eta_{irr} \) at \( 4g+2 \) points, each of multiplicity \( 2 \).

To construct a pencil in \( \tilde{\Pi}_{g,1} \), we have to construct a pencil of curves with section. Since the branch divisor is reducible, in order to get a smooth double cover, we have to blow up the intersection points between the irreducible components in the branch locus. In the branch divisor \( B \), all \( \sigma \) mutually have trivial intersection; the only intersection appears when the moving diagonal \( \Delta_1 \) and \( \Delta_2 \) intersecting each other and with the \( \sigma \)'s.

Here are some intersection multiplicities that we will use quite often in the future computation:

\[
\begin{align*}
(\sigma_i \cdot \sigma_j)_Q &= 0, \quad (\Delta_i \cdot \Delta_j)_Q = 2,
\end{align*}
\]

since each \( \sigma_i \) gets blown up twice; each \( \Delta_j \) gets blown up \( 2g+2 \) times, we have

\[
(\tilde{\sigma}_i \cdot \tilde{\sigma}_j)_Q = -2, \quad (\tilde{\Delta}_i \cdot \tilde{\Delta}_j)_Q = -2g.
\]

After taking the double cover \( \tilde{\mathcal{X}} \), since all \( \tilde{\sigma}_i \)'s and \( \tilde{\Delta}_j \)'s are contained in the branch locus \( \tilde{B}_0 \), then

\[
(C_i \cdot C_i)_{\tilde{\mathcal{X}}} = -1, \quad (D_j \cdot D_j)_{\tilde{\mathcal{X}}} = -g,
\]

where \( C_i = \pi^{-1}(\tilde{\sigma}_i) \) and \( D_j = \pi^{-1}(\tilde{\Delta}_j) \), with \( \pi^*(\tilde{\sigma}_i) = 2C_i \) and \( \pi^*(\tilde{\Delta}_j) = 2D_j \).

But we still need sections of the double cover \( f : \mathcal{X} \to B \), either for attachment or for making it as the marking point. We will be using the following four ways to get sections.

In order to get a Weierstrass section either for marking, so that the family is contained entirely in \( \tilde{\Pi}_{g,w} \); or for gluing, so that the family is contained entirely in some \( \delta_i \), we could

1. start with a horizontal ruling, say \( \sigma_1 \), which is contained in the branch locus \( B \), or
2. start with a diagonal section, say \( \Delta_1 \), which lies in the branch locus \( B \).

In order to get two sections adding up to \( g^1_{2} \) on each fiber for gluing, we could

3. start with a general ruling \( \sim (1,0) \). Since it intersects the branch locus at two distinct points, after taking the double cover, it becomes a double cover of \( \mathbb{P}^1 \) branched along two points. Or
4. start with a horizontal ruling which intersects \( \sigma_1 \) and \( \Delta_1 \) at a common point.
Proposition 4.4. For $1 \leq i \leq \lfloor g/2 \rfloor$, the coefficients $a_{i,0}$ and $a_{i,1}$ of $\delta_{i,0}$ and $\delta_{i,1}$ respectively, are

$$a_{i,0} = -\frac{2i(2i+1)}{(2g+1)(g-1)}$$

$$a_{i,1} = -\frac{2(g-i)[2(g-i) + 1]}{(2g+1)(g-1)}$$

For $1 \leq i \leq \lfloor (g-1)/2 \rfloor$, the coefficients $b_{i,0}$ and $b_{i,1}$ of $\eta_{i,0}$ and $\eta_{i,1}$ respectively, are

$$b_{i,0} = -\frac{i(2i+3)}{(2g+1)(g-1)}$$

$$b_{i,1} = -\frac{(g-i-1)[2(g-i-1) + 3]}{(2g+1)(g-1)}$$

Proof. We will take either $\sigma_j$’s or $\Delta_j$’s to be the Weierstrass section for attaching another fixed nonsingular hyperelliptic curve along a fixed Weierstrass point with another non-Weierstrass marked point. To be more explicit, we start with a family of hyperelliptic curves of genus $i$ constructed similarly as above.

(1) First let the horizontal ruling $\sigma_1$ be the section for attachment. We already have $(C_1 \cdot C_1)_X = -1$. After gluing a fixed genus $g-i$ hyperelliptic curve with a non-Weierstrass marking, we obtain a family of marked hyperelliptic curves contained entirely in $\delta_{i,0}$. We have the following types of fibers:

Note that the fibers of the middle type in Figure 6 are contained in classes $\eta_{irr}, \delta_{i,0}$ and also in $\eta_{i-1,0}$. Moreover, the family has nontrivial intersections only with $\eta_{irr}, \delta_{i,0}$ and $\eta_{i-1,0}$.

$$\deg_B(\eta_{irr}) = 2(4i+2)$$

$$\deg_B(\delta_{i,0}) = C^2_1 = -1$$

$$\deg_B(\eta_{i-1,0}) = 2$$

where the multiple 2 in equation (4.6) is due to Lemma 4.3. Thus

$$0 = -a_{i,0} + 2(4i+2)c + 2b_{i-1,0}.$$
We can also switch the genera of these two components. By symmetry, we should get the following result:

\[ 0 = -a_{i,1} + 2(4(g - i) + 2)c + 2b_{i,1}. \]  

(4.9)

(2) In the same manner, we can also choose \( \Delta_1 \) as the Weierstrass section for attachment. Now we have \( (D_1 \cdot D_1)_X = -i \). And

\[
\begin{align*}
\deg_B(\eta_{ir}) &= 2(4i + 2) \\
\deg_B(\delta_{i,0}) &= D_i^2 = -i \\
\deg_B(\eta_{i-1,0}) &= 2i + 2 
\end{align*}
\]

give us the following relation:

\[ 0 = 2(4i + 2)c - ia_{i,0} + (2i + 2)b_{i-1,0}. \]  

(4.10)

Similarly by symmetry,

\[ 0 = 2(4(g - i) + 2)c - (g - i)a_{i,1} + (2(g - i) + 2)b_{i,1} \]  

(4.11)

According to these relations (4.8), (4.9), (4.10) and (4.11) together with (3.1) and (3.2), we would get the formula for the coefficients of \( \delta_{i,0}, \delta_{i,1} \) and \( \eta_{i,0}, \eta_{i,1} \):

\[
\begin{align*}
a_{i,0} &= -\frac{2i(2i + 1)}{(2g + 1)(g - 1)} \\
a_{i,1} &= -\frac{2(g - i)(2(g - i) + 1)}{(2g + 1)(g - 1)} \\
b_{i,0} &= -\frac{i(2i + 3)}{(2g + 1)(g - 1)} \\
b_{i,1} &= -\frac{(g - i - 1)(2(g - i - 1) + 3)}{(2g + 1)(g - 1)}
\end{align*}
\]

which prove the proposition.

Therefore, together with the results from the previous section, we have proved Theorem 1.1.

4.3 Other variants of test curves

The main purpose of this section is to give more variants of family of hyperelliptic curves with sections satisfying certain properties.

Let’s start with a basic construction of a family of genus \( g \) hyperelliptic curves without gluing. We will do the following two cases: taking a general horizontal ruling \( \sigma_0 \) or a diagonal \( \Delta_0 \) section on \( Q \), to start with for the section corresponding to the marking point.

![Figure 7: Construction of section](image-url)
(1) Let $\sigma_0 \sim (1,0)$ be any general ruling on $Q$ distinct from all $\sigma_i$’s. Then it will hit the branching $B_0$ at two points with two $\Delta_j$’s. Since $\sigma_0$ is away from the points we have blown up, then the strict transform $\widetilde{\sigma}_0$ is isomorphism with $\sigma_0$ under the blowup morphism with $(\sigma_0 \cdot \sigma_0)_Q = (\widetilde{\sigma}_0 \cdot \widetilde{\sigma}_0)_{\widetilde{Q}} = 0$. Moreover, $\widetilde{\sigma}_0$ intersects with the branch divisor $B_0$ transversally at two distinct points. Therefore, $C_0 := \pi^*(\widetilde{\sigma}_0)$ is a smooth irreducible curve on the smooth surface $\mathcal{X}$, such that $\pi|_{C_0} : C_0 \to \widetilde{\sigma}_0$ is a double cover branched along the 2 points of intersection of $\widetilde{\sigma}_0$ with $B_0$. Moreover, $(C_0 \cdot C_0)_X = 2(\widetilde{\sigma}_0 \cdot \widetilde{\sigma}_0)_{\widetilde{Q}} = 0$. Now via a base change $\beta$ to $C_0$ defined by the above double cover of $\widetilde{\sigma}_0 \cong \mathbb{P}^1$, the pullback of $C_0 \subset \mathcal{X}$ consists of two components, namely $C'_0$ and $C''_0$, intersecting at 2 points transversally. Thus
\[
(C'_0 + C''_0)^2 = 2C'_0 + 2C''_0 = 0
\]
which yields $C'^2_0 = C''^2_0 = -2$. If we choose either $C'_0$ or $C''_0$ as the section for marking points on fibers, we have

\[
\begin{align*}
\deg_{C_0}(\overline{\mathcal{F}_{g,w}}) &= 2 \\
\deg_{C_0}(\psi) &= 2 \\
\deg_{C_0}(\eta_{\text{irr}}) &= 2 \cdot 2(4g + 2)
\end{align*}
\]

where in the degree of $\Delta_{\text{irr}}$ the first multiple 2 comes from the base change, and the second multiple 2 comes from the order of the tangency of the divisor $\Delta_{\text{irr}}$ intersecting with the discriminant locus by Lemma 4.3. Then we get the same formula as in (4.1).

Notice that from how we construct these two sections $C'_0$ and $C''_0$, away from the points of intersection, the corresponding two points on each fiber add up to a $g^1_2$. If we blow up the two intersection points, we obtain two sections on this family adding up to a $g^1_2$. By observation, we can easily see that the following type of singular curve is actually in the locus $\eta_{i,0}$ or $\eta_{i,1}$.

![Figure 8: Special fiber](image)

(2) Let $\Delta_0 \sim (1,1)$ be any general diagonal on $Q$, which will hit the horizontal ruling $\sigma_i$ each at 1 point, and hit the other diagonal $\Delta_j$ each at two distinct points. Note that the diagonal $\Delta_0$ can be chosen to be intersecting with $\Delta_1$ and $\Delta_2$ at 4 distinct points. We will use the similar set of notations as above. Then
\[
(\Delta_0 \cdot \Delta_0)_Q = 2, \quad (\widetilde{\Delta}_0 \cdot \widetilde{\Delta}_0)_{\widetilde{Q}} = 2, \quad (D_0 \cdot D_0)_X = 4
\]
where $D_0 := \pi^{-1}(\widetilde{\Delta}_0) \to \widetilde{\Delta}_0$ is a smooth irreducible curve on $\mathcal{X}$ which is also a double cover of $\widetilde{\Delta}_0$ branched over $2g + 4$ points. By similar argument,
\[
(D'_0 + D''_0)^2 = 2D'_0 + 2D''_0 = 8, \quad D'_0 \cdot D''_0 = 2g + 4,
\]
implies that $D'_0^2 = D''_0^2 = -2g$. Therefore

\[
\begin{align*}
\deg_{D_0}(\overline{\mathcal{F}_{g,w}}) &= 2g + 4 \\
\deg_{D_0}(\psi) &= 2g \\
\deg_{D_0}(\eta_{\text{irr}}) &= 2 \cdot 2(4g + 2)
\end{align*}
\]

from which we get the same formula for the coefficient $c$ of $\eta_{\text{irr}}$ in (4.1).
Figure 9: Construction of section

Now we do a little variation to make the extra section passing through a common point.

(1) Let \( \sigma_0 \sim (1, 0) \) be a horizontal ruling which intersects at the point \( q \) where \( \Delta_1 \) and \( \Delta_2 \) intersect. Now the problematic point is the common point \( q \) where \( \sigma_0, \Delta_1 \) and \( \Delta_2 \) intersect. Again, we will use the same set of notation as before.

\[
(\sigma_0 \cdot \sigma_0)_Q = 0, \quad (\tilde{\sigma}_0 \cdot \tilde{\sigma}_0)_{\tilde{Q}} = -1
\]

since \( \tilde{\sigma}_0 \) is obtained by \( \sigma_0 \) getting blown up once. Now the strict transforms \( \tilde{\sigma}_0, \tilde{\Delta}_1 \) and \( \tilde{\Delta}_2 \) are getting separated on the exceptional \( \mathbb{P}^1 \). After taking the double cover branched along \( \tilde{B}_0 \), since the exceptional divisor is not branched, the preimage \( C_0 := \pi^{-1}(\tilde{\sigma}_0) = E_1 + E_2 \) is a multi-section, which is reducible, with \( \pi|_{E_i} : E_i \to \tilde{\sigma}_0 \) isomorphisms, and \( E^2_i = \tilde{\sigma}_0^2 = -1 \).

Figure 10: Section after taking double cover

Then we can choose either \( E_1 \) or \( E_2 \) as the section for marking point on the pencil. Notice that even if the section \( E_1 \) doesn’t hit the Weierstrass point on any fibers in the pencil, the special fiber containing the common point \( Q \) still has nontrivial contribution to the divisor \( [\mathcal{H}_{g,w}] \). Thus we have the followings:

\[
\deg_B([\mathcal{H}_{g,w}]) = 1 \\
\deg_B(\psi) = 1 \\
\deg_B(\eta_{irr}) = 2(4g + 2)
\]

in which case we still get the same result for \( c \) as in (4.1). We are using this test curve to test and verify that the contribution of the special fiber is counted as the same multiplicity as other nodal fibers (see (4.14)). The multiplicities are all compatible with the countings in (4.12) and (4.13).

(2) Now let \( \Delta_0 \sim (1, 1) \) be the diagonal which intersects with one of the common point \( Q \) of \( \Delta_1 \) and \( \Delta_2 \). In the same sense, we have

\[
(\Delta_0 \cdot \Delta_0)_Q = 2, \quad (\tilde{\Delta}_0 \cdot \tilde{\Delta}_0)_{\tilde{Q}} = 1.
\]
Thus $D_0 := \pi^{-1}(\Delta_0)$ is a double cover of $\Delta_0$ branched along $2g + 2$ points with $D_0^2 = 2$. Now we could take a base change to $D_0$ which is a degree 2 cover, therefore the pullback of $D_0 \subset X$ under the base change will consist two components, denoted by $D'_0$ and $D''_0$ satisfying:

$$(D'_0 + D''_0)^2 = 2D_0^2, \quad D'_0 \cdot D''_0 = 2g + 2$$

which imply $D_0^2 = D'_0^2 = -2g$. Thus

$$\deg_{D_0}(\mathcal{H}_{g,w}) = 2 + (2g + 2) = 2g + 4 \quad (4.15)$$

The first summand 2 in equation (4.15) comes from the contribution of the fiber corresponding to $Q$ after base change. It is consistent with what we’ve obtained using other sections.

Next we want to take one of $\sigma_i$’s or $\Delta_j$’s to be the Weierstrass section and take another horizontal $\sigma_0 \sim (1,0)$ for the marking point. We will need the Weierstrass section for attaching another hyperelliptic curve in order to construct a family of curves contained entirely in the class $\delta_{i,0}$ or $\delta_{i,1}$.

![Figure 11: Construction of section](image)

(1) First let the horizontal ruling $\sigma_1$ be the section for marking. Then we will have that $(C_1 \cdot C_1)_X = -1$. Since this family is contained entirely in $\mathcal{H}_{g,w}$, computing the degree of $\mathcal{H}_{g,w}$ on this family is computing the degree of the pullback of the whole branch divisor to this family. Since the pullback of the the branch divisor $B$ via the blowup morphism contains all exceptional divisors, we would have some positive components adding to $C^2_1$ which come from the intersection of the pullback of the exceptional divisors with the base of the family. Again, before attaching a fixed hyperelliptic curve along $C_1$, we have

$$\deg_B(\mathcal{H}_{g,w}) = C_1^2 + 2 = 1$$
$$\deg_B(\psi) = -(-1) = 1$$
$$\deg_B(\eta_{irr}) = 2(4g + 2).$$

which yield the relation we have already obtained.

If we start with the branch divisor defined by $2i$ horizontal rulings and $2$ other diagonal sections, in the same way constructed as above, we would arrive at a pencil of hyperelliptic curves with genus $i$. If we choose, say $C_1$, as the Weierstrass section for attaching another fixed hyperelliptic curve of genus $g - i$ along a fixed Weierstrass section, and choose a general horizontal $\sigma_0$ to start with for the marking point, we get a family of singular hyperelliptic curves of genus $g$ with one marked point. Since $\sigma_0$ is away from the locus where we’ve blown up, it is isomorphic to $\tilde{\sigma}_0$ and thus after the double cover, the preimage $C_0$ of $\tilde{\sigma}_0$ forms a double cover of $\tilde{\sigma}_0$ branched at two points coming from the intersections with the branch divisor. Now base change to $C_0$, the pullback of the original family $X$ under the base change $C_0 \to \mathbb{P}^1$ is now the double cover of $X$ branched along two fibers. Moreover, the pullback of $C_0$ on the new family consists of two components $C'_0, C''_0$ intersecting transversally at two points:

$$(C'_0 + C''_0)^2 = 2C_0^2 = 0 \implies C'_0 = C''_0 = -2.$$
Accordingly, $C_1$ is also a double of $\tilde{\sigma}_0$ branched along two points, thus after taking the base change, $C'_1$ is isomorphic to $C'_0$ together with the fact that $C'_1^2 = -2$. For the family we’ve obtained in this way, the general and special fibers are as follows.

![Figure 12: Fiber types](image)

And we have the following facts:

\[
\begin{align*}
\deg_{C_0}(\psi) &= -C'_0^2 = 2 \\
\deg_{C_0}(\bar{H}_{g,w}) &= 2 \\
\deg_{C_0}(\eta_{irr}) &= 4(4i + 2) \\
\deg_{C_0}(\delta_{i,1}) &= C'_1^2 = -2 \\
\deg_{C_0}(\eta_{i-1,1}) &= 4
\end{align*}
\]

which yield the following relation:

\[
4(4i + 2)c - 2a_{i,1} + 4b_{i-1,1} + 2d = 2 \quad (4.16)
\]

By symmetry, we could also start with a pencil of hyperelliptic curves of genus $g - i$ constructed similarly as above, then it will give us the relation:

\[
4(4(g - i) + 2)c - 2a_{i,0} + 4b_{i,0} + 2d = 2 \quad (4.17)
\]

We have checked by plugging in what we get from (4.2), (4.3) and (4.4), (4.5) that the above equations (4.16) and (4.17) hold.

(2) Next take $D_1$ from the diagonal section of type $(1, 1)$ on $Q$ be the Weierstrass section. Likewise, before attaching along $D_1$, we obtain a family contained in $\bar{H}_{g,w}$ and moreover,

\[
\begin{align*}
\deg_B(\bar{H}_{g,w}) &= D'_1 + (2g + 2) = g + 2 \\
\deg_B(\psi) &= -(-g) = g \\
\deg_B(\eta_{irr}) &= 2(4g + 2)
\end{align*}
\]

give us the relation we already have. Now consider using $D_1$ for attachment, and choosing a general horizontal $\sigma_0$ to start with for the marking point. Again, we start with a branch divisor defined by $2i$ horizontal rulings and $2$ diagonals to construct a base family of pointed hyperelliptic curves of genus $i$. We will use almost the same method as above. The only difference is in this case, $\sigma_0$ has nontrivial intersection with $\Delta_1$. Thus once we get $C'_0$ and $D'_1$ via the base change, these two sections intersect transversally at 1 point. By blowing up this point, we have

\[
(\tilde{C}'_0 \cdot \tilde{C}'_0)_{\tilde{\mathbb{X}}} = -3, \quad (\tilde{D}'_1 \cdot \tilde{D}'_1)_{\tilde{\mathbb{X}}} = -2i - 1,
\]

and accordingly, after attaching a fixed pointed genus $g - i$ hyperelliptic curve, the final family which is contained entirely in the class $\delta_{i,1}$ has the following five types of fibers.
Therefore, we have the nontrivial degrees for the following divisor classes restricted on this pencil:

\[
\begin{align*}
\deg_{\tilde{\mathcal{C}}_0}(\psi) &= -\tilde{C}_0^2 = 3 \\
\deg_{\tilde{\mathcal{C}}_0}(\mathcal{H}_{g,w}) &= 1 \\
\deg_{\tilde{\mathcal{C}}_0}(\delta_{i,1}) &= -\tilde{D}_1^2 = -2i - 1 \\
\deg_{\tilde{\mathcal{C}}_0}(\delta_{i,0}) &= 1 \\
\deg_{\tilde{\mathcal{C}}_0}(\eta_{irr}) &= 4(4i + 2) \\
\deg_{\tilde{\mathcal{C}}_0}(\eta_{i-1,1}) &= 2(2i + 2)
\end{align*}
\]

In the further computation, we can see that the final formula for the coefficients of \( \delta_i \)'s do not depend on the degree of \( \delta_{i,0} \) on this family. Thus, we have the following relation:

\[
8(2i + 1)c - (2i + 1)a_{i,1} + a_{i,0} + 2(2i + 2)b_{i-1,1} + 3d = 2 \tag{4.18}
\]

Likewise by symmetry,

\[
8[2(g - i) + 1]c - [2(g - i) + 1]}a_{i,0} + a_{i,1} + 2[2(g - i) + 2]b_{i,0} + 3d = 2 \tag{4.19}
\]

The relations (4.18) and (4.19) are both consistent with the formulas we get from (4.2), (4.3) and (4.4), (4.5).

Furthermore, we could also use a horizontal ruling \( \sigma_0 \) which intersects with two diagonals \( \Delta_1 \) and \( \Delta_2 \) at a common point, for attaching another fixed smooth genus \( g - i - 1 \) hyperelliptic curve with two fixed points adding up to a \( g_1^1 \). And choose either a horizontal branched ruling, say \( \sigma_1 \), or a branched diagonal, say \( \Delta_1 \), for the marking point.

Then after taking the double cover, we obtain a family of singular hyperelliptic curves of arithmetic genus \( i \). Since after blowup of the branch locus, the strict transform of the horizontal ruling \( \tilde{\sigma}_0 \) does not hit the branch locus, thus once we take the double cover to get the family, the preimage of \( \tilde{\sigma}_0 \) will be reducible with two components of no intersection.

(1) First we choose \( \sigma_1 \) as the section for the marking. Then the family will contain the following types of fibers after attaching:

Note that the whole family is contained entirely in the divisor class \( \eta_{i,1} \). The fourth fiber in the Figure 13 is not contained in \( \eta_{irr} \) anymore, but it is in \( \eta_{i-1,1} \) since it can be regarded as a special fiber in the degeneration of a family of genus \( g - i \) hyperelliptic curves. Thus

\[
\begin{align*}
\deg_B(\psi) &= 1 \\
\deg_B(\eta_{irr}) &= 2(4i + 1) \\
\deg_B(\mathcal{H}_{g,w}) &= \tilde{C}_1^2 + 2 = 1 \\
\deg_B(\eta_{i,1}) &= -1 \\
\deg_B(\eta_{i-1,1}) &= 1
\end{align*}
\]

![Figure 13: Fiber types](image)

- non-WP
- WP
- g-i
- g-i
- g-i
- 4i fibers
- 2(2i+2) fibers
- 1 fiber

general

1 fiber

4i fibers

2(2i+2) fibers

1 fiber
yield the relation
\[ 2(4i + 1)c - b_{i,1} + b_{i-1,1} + d = 1, \]
which is consistent with formulas we get in \([4.5], [4.1]\) and for \(d\).

(2) If we choose \(\Delta_1\) as the section for the marking, the family consists of the following types of fibers: The family is contained entirely in \(\eta_{i,1}\) and the last fiber in Figure 16 is not in \(\eta_{i,0}\) but it is in \(\eta_{i-1,0}\). It cannot be contained in \(\eta_{i,0}\) since otherwise it is the special fiber in some 2-parameter family. Thus

\[
\begin{align*}
\deg_B(\psi) &= i \\
\deg_B(\eta_{i,1}) &= 2(4i + 1) \\
\deg_B(\eta_{i,0}) &= 0 \\
\deg_B([\mathcal{I}_{g,w}]) &= D_i^2 + (2i + 1) = i + 1 \\
\deg_B(\eta_{i-1,0}) &= -1 \\
\deg_B(\eta_{i-1,0}) &= 1
\end{align*}
\]

give us the relation
\[ 2(4i + 1)c - b_{i,1} + b_{i-1,1} + id = i + 1 \]
which also gives us another consistency check of the formulas in (4.5), (4.5), (4.1) and for $d$.

References


