A SIMPLER PROOF OF TOROIDALIZATION OF MORPHISMS FROM 3-FOLDS TO SURFACES

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Abstract. We give a simpler and more conceptual proof of toroidalization of morphisms of 3-folds to surfaces, over an algebraically closed field of characteristic zero. A toroidalization is obtained by performing sequences of blow ups of nonsingular subvarieties above the domain and range, to make a morphism toroidal. The original proof of toroidalization of morphisms of 3-folds to surfaces, which appeared in Springer Lecture Notes in Math. in 2002 [15], is much more complicated.

1. Introduction

Let \( \mathfrak{k} \) be an algebraically closed field of characteristic zero. Toroidal varieties and morphisms of toroidal varieties over \( \mathfrak{k} \) are defined in [32], [4] and [5]. If \( X \) is nonsingular, then the choice of a SNC divisor on \( X \) makes \( X \) into a toroidal variety.

Suppose that \( \Phi : X \to Y \) is a dominant morphism of nonsingular \( \mathfrak{k} \)-varieties, and there is a SNC divisor \( D_Y \) on \( Y \) such that \( D_X = \Phi^{-1}(D_Y) \) is a SNC divisor on \( X \). Then \( \Phi \) is toroidal (with respect to \( D_Y \) and \( D_X \)) if and only if \( \Phi^*(\Omega^1_Y(\log D_Y)) \) is a subbundle of \( \Omega^1_X(\log D_X) \) (Lemma 1.5 [15]). A toroidal morphism can be expressed locally by monomials. All of the cases are written down for toroidal morphisms from a 3-fold to a surface in Lemma 19.3 [15].

The toroidalization problem is to determine, given a dominant morphism \( f : X \to Y \) of \( \mathfrak{k} \)-varieties, if there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are products of blow ups of nonsingular subvarieties, \( X_1 \) and \( Y_1 \) are nonsingular, and there exist SNC divisors \( D_{Y_1} \) on \( Y_1 \) and \( D_{X_1} = f^*(D_{Y_1}) \) on \( X_1 \) such that \( f_1 \) is toroidal (with respect to \( D_{X_1} \) and \( D_{Y_1} \)). This is stated in Problem 6.2.1 of [5]. Some papers where related problems are considered are [4] and [35].

The toroidalization problem does not have a positive answer in positive characteristic \( p \), even for maps of curves; \( t = x^p + x^{p+1} \) gives a simple example.

In characteristic zero, the toroidalization problem has an affirmative answer if \( Y \) is a curve and \( X \) has arbitrary dimension; this is really embedded resolution of hypersurface singularities, so follows from resolution of singularities ([27], and simplified proofs [7], [8], [18], [22], [23], [34] and [41]). There are several proofs for the case of maps of a surface to a surface (some references are [3], [20] and Corollary 6.2.3 [5]). The case of a morphism from a 3-fold to a surface is proven in [15], and the case of a morphism from a 3-fold to a 3-fold is proven in [16].

Partially supported by NSF.
The problem of toroidalization is a resolution of singularities type problem. When the dimension of the base is larger than one, the problem shares many of the complexities of resolution of vector fields ([38], [9], [36]) and of resolution of singularities in positive characteristic (some references are [1], [2], [28], [10], [11], [12], [17], [21], [24], [25], [26], [29], [30], [31], [33], [39], [40], [6]). In particular, natural invariants do not have a "hypersurface of maximal contact" and are sometimes not upper semicontinuous.

Toroidalization, locally along a fixed valuation, is proven in all dimensions and relative dimensions in [13] and [14].

The proof of toroidalization of a dominant morphism from a 3-fold to a surface given in [15] consists of 2 steps.

The first step is to prove "strong preparation". Suppose that $X$ is a nonsingular variety, $S$ is a nonsingular surface with a SNC divisor $D_S$, and $f : X \to S$ is a dominant morphism such that $D_X = f^{-1}(D_S)$ is a SNC divisor on $X$ which contains the locus where $f$ is not smooth. $f$ is strongly prepared if $f^*(\Omega^2_S(\log D_S)) = IM$ where $I \subset \mathcal{O}_X$ is an ideal sheaf, and $\mathcal{M}$ is a subbundle of $\Omega^2_X(\log D_X)$ (Lemma 1.7 [15]). A strongly prepared morphism has nice local forms which are close to being toroidal (page 7 of [15]).

Strong preparation is the construction of a commutative diagram

$$
\begin{array}{ccc}
X_1 & \rightarrow & S \\
\downarrow & \searrow & \\
X & \rightarrow & S
\end{array}
$$

where $S$ is a nonsingular surface with a SNC divisor $D_S$ such that $D_X = f^*(D_S)$ is a SNC divisor on the nonsingular variety $X$ which contains the locus where $f$ is not smooth, the vertical arrow is a product of blow ups of nonsingular subvarieties so that $X_1 \rightarrow S$ is strongly prepared. Strong preparation of morphisms from 3-folds to surfaces is proven in Theorem 17.3 of [15].

The second step is to prove that a strongly prepared morphism from a 3-fold to a surface can be toroidalized. This is proven in Sections 18 and 19 of [15].

This second step is generalized in [19] to prove that a strongly prepared morphism from an $n$-fold to a surface can be toroidalized. Thus to prove toroidalization of a morphism from an $n$-fold to a surface, it suffices to proof strong preparation.

The proof of strong preparation in [15] is extremely complicated, and does not readily generalize to higher dimensions. The proof of this result occupies 170 pages of [15]. We mention that that the main invariant considered in this paper, $\nu$, can be interpreted as the adopted order of Section 1.2 of [9] of the 2-form $du \wedge dv$.

In this paper, we give a significantly simpler and more conceptual proof of strong preparation of morphisms of 3-folds to surfaces. It is our hope that this proof can be extended to prove strong preparation for morphisms of $n$-folds to surfaces, for $n > 3$. The proof is built around a new upper semicontinuous invariant $\sigma_D$, whose value is a natural number or $\infty$. if $\sigma_D(p) = 0$ for all $p \in X$, then $X \rightarrow S$ is prepared (which is slightly stronger than being strongly prepared). A first step towards obtaining a reduction in $\sigma_D$ is to make $X$ 3-prepared, which is achieved in Section 3. This is a nicer local form, which is proved by making a local reduction to lower dimension. The proof proceeds by performing a toroidal morphism above $X$ to obtain that $X$ is 3-prepared at all points except for a finite number of 1-points. Then general curves through these points lying on $D_X$ are blown up to achieve 3-preparation everywhere on $X$. if $X$ is 3-prepared at a point $p$, then there exists an étale cover $U_p$ of an affine neighborhood of $p$ and a local toroidal structure $\overline{D}_p$ at $p$ (which contains $D_X$) such that there exists a projective toroidal morphism $\Psi : U' \rightarrow U_p$ such that
\( \sigma_D \) has dropped everywhere above \( p \) (Section 4). The final step of the proof is to make these local constructions algebraic, and to patch them. This is accomplished in Section 5. In Section 6 we state and prove strong preparation for morphisms of 3-folds to surfaces (Theorem 6.1) and toroidalization of morphisms from 3-folds to surfaces (Theorem 6.2).

2. The invariant \( \sigma_D \), 1-preparation and 2-preparation.

For the duration of the paper, \( k \) will be an algebraically closed field of characteristic zero. We will write curve (over \( k \)) to mean a 1-dimensional \( k \)-variety, and similarly for surfaces and 3-folds. We will assume that varieties are quasi-projective. This is not really a restriction, by the fact that after a sequence of blow ups of nonsingular subvarieties, all varieties satisfy this condition. By a general point of a \( k \)-variety \( Z \), we will mean a member of a nontrivial open subset of \( Z \) on which some specified good condition holds.

A reduced divisor \( D \) on a nonsingular variety \( Z \) of dimension \( n \) is a simple normal crossings divisor (SNC divisor) if all irreducible components of \( D \) are nonsingular, and if \( p \in Z \), then there exists a regular system of parameters \( x_1, \ldots, x_n \) in \( \mathcal{O}_{Z,p} \) such that \( x_1 x_2 \cdots x_r = 0 \) is a local equation of \( D \) at \( p \), where \( r \leq n \) is the number of irreducible components of \( D \) containing \( p \). Two nonsingular subvarieties \( X \) and \( Y \) intersect transversally at \( p \in X \cap Y \) if there exists a regular system of parameters \( x_1, \ldots, x_n \) in \( \mathcal{O}_{Z,p} \) and subsets \( I, J \subset \{1, \ldots, n\} \) such that \( \mathcal{I}_X, p = (x_i \mid i \in I) \) and \( \mathcal{I}_Y, p = (x_j \mid j \in J) \).

**Definition 2.1.** Let \( S \) be a nonsingular surface over \( k \) with a reduced SNC divisor \( D_S \). Suppose that \( X \) is a nonsingular 3-fold, and \( f : X \to S \) is a dominant morphism. \( X \) is 1-prepared (with respect to \( f \)) if \( D_X = f^{-1}(D_S)_{\text{red}} \) is a SNC divisor on \( X \) which contains the locus where \( f \) is not smooth, and if \( C_1, C_2 \) are the two components of \( D_S \) whose intersection is nonempty, \( T_1 \) is a component of \( X \) dominating \( C_1 \) and \( T_2 \) is a component of \( D_X \) which dominates \( C_2 \), then \( T_1 \) and \( T_2 \) are disjoint.

The following lemma is an easy consequence of the main theorem on resolution of singularities.

**Lemma 2.2.** Suppose that \( g : Y \to T \) is a dominant morphism of a 3-fold over \( k \) to a surface over \( k \) and \( D_T \) is a 1-cycle on \( T \) such that \( g^{-1}(D_R) \) contains the locus where \( g \) is not smooth. Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{g_1} & T_1 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
Y & \xrightarrow{g} & T
\end{array}
\]

such that the vertical arrows are products of blow ups of nonsingular subvarieties contained in the preimage of \( D_T \), \( Y_1 \) and \( T_1 \) are nonsingular and \( D_{T_1} = \pi_1^{-1}(D_T) \) is a SNC divisor on \( T_1 \) such that \( Y_1 \) is 1-prepared with respect to \( g_1 \).

For the duration of this paper, \( S \) will be a fixed nonsingular surface over \( k \), with a (reduced) SNC divisor \( D_S \). To simplify notation, we will often write \( D \) to denote \( D_X \), if \( f : X \to S \) is 1-prepared.

Suppose that \( X \) is 1-prepared with respect to \( f : X \to S \). A permissible blow up of \( X \) is the blow up \( \pi_1 : X_1 \to X \) of a point of \( D_X \) or a nonsingular curve contained in \( D_X \) which makes SNCs with \( D_X \). Then \( D_{X_1} = \pi_1^{-1}(D_X)_{\text{red}} = (f \circ \pi_1)^{-1}(X_S)_{\text{red}} \) is a SNC divisor on \( X_1 \) and \( X_1 \) is 1-prepared with respect to \( f \circ \pi_1 \).

Assume that \( X \) is 1-prepared with respect to \( D \). We will say that \( p \in X \) is an \( n \)-point (for \( D \)) if \( p \) is on exactly \( n \) components of \( D \). Suppose \( q \in D_S \) and \( u, v \) are regular parameters
in $\mathcal{O}_{S,q}$ such that either $u = 0$ is a local equation of $D_S$ at $q$ or $uv = 0$ is a local equation of $D_S$ at $q$. $u,v$ are called permissible parameters at $q$.

For $p \in f^{-1}(q)$, we have regular parameters $x,y,z$ in $\hat{\mathcal{O}}_{X,p}$ such that

1) If $p$ is a 1-point,

\[
(u = x^a, v = P(x) + x^bF)
\]

where $x = 0$ is a local equation of $D$, $x \nmid F$ and $x^bF$ has no terms which are a power of $x$.

2) If $p$ is a 2-point, after possibly interchanging $u$ and $v$,

\[
(u = (x^a y^b)^l, v = P(x^a y^b) + x^c y^dF)
\]

where $xy = 0$ is a local equation of $D$, $a,b > 0$, $\gcd(a,b) = 1$, $x,y \nmid F$ and $x^c y^dF$ has no terms which are a power of $x^a y^b$.

3) If $p$ is a 3-point, after possibly interchanging $u$ and $v$,

\[
(u = (x^a y^b z^c)^l, v = P(x^a y^b z^c) + x^d y^e z^fF)
\]

where $xyz = 0$ is a local equation of $D$, $a,b,c > 0$, $\gcd(a,b,c) = 1$, $x,y,z \nmid F$ and $x^d y^e z^fF$ has no terms which are a power of $x^a y^b z^c$.

regular parameters $x,y,z$ in $\hat{\mathcal{O}}_{X,p}$ giving forms (1), (2) or (3) are called permissible parameters at $p$ or $u,v$.

Suppose that $X$ is 1-prepared. We define an ideal sheaf

\[
\mathcal{I} = \text{fitting ideal sheaf of the image of } f^*: \Omega^2_S \to \Omega^2_X(\log(D))
\]
in $\mathcal{O}_X$. $\mathcal{I} = \mathcal{O}_X(-G)\mathcal{I}$ where $G$ is an effective divisor supported on $D$ and $\mathcal{I}$ has height $\geq 2$.

Suppose that $E_1, \ldots, E_n$ are the irreducible components of $D$. For $p \in X$, define

\[
\sigma_D(p) = \operatorname{order}_{\mathcal{O}_X,p/(\sum_{p \in E_i} \mathcal{I}_{E_i,p})} \mathcal{I}_p \left( \mathcal{O}_{X,p}/ \sum_{p \in E_i} \mathcal{I}_{E_i,p} \right) \in \mathbb{N} \cup \{\infty\}.
\]

**Lemma 2.3.** $\sigma_D$ is upper semicontinuous in the Zariski topology of the scheme $X$.

**Proof.** For a fixed subset $J \subset \{1, 2, \ldots, n\}$, we have that the function

\[
\operatorname{order}_{\mathcal{O}_X,p/(\sum_{i \in J} \mathcal{I}_{E_i,p})} \mathcal{I}_p \left( \mathcal{O}_{X,p}/ \sum_{i \in J} \mathcal{I}_{E_i,p} \right)
\]

is upper semicontinuous, and if $J \subset J' \subset \{1, 2, \ldots, n\}$. we have that

\[
\operatorname{order}_{\mathcal{O}_X,p/(\sum_{i \in J} \mathcal{I}_{E_i,p})} \mathcal{I}_p \left( \mathcal{O}_{X,p}/ \sum_{i \in J} \mathcal{I}_{E_i,p} \right) \leq \operatorname{order}_{\mathcal{O}_X,p/(\sum_{i \in J'} \mathcal{I}_{E_i,p})} \mathcal{I}_p \left( \mathcal{O}_{X,p}/ \sum_{i \in J'} \mathcal{I}_{E_i,p} \right).
\]

Thus for $r \in \mathbb{N} \cup \{\infty\}$,

\[
\operatorname{Sing}_r(X) = \{p \in X \mid \sigma_D(p) \geq r\}
\]
is a closed subset of $X$, which is supported on $D$ and has dimension $\leq 1$ if $r > 0$.

**Definition 2.4.** A point $p \in X$ is prepared if $\sigma_D(p) = 0$. 

We have that \( \sigma_D(p) = 0 \) if and only if \( I_p = O_{X,p} \). Further, 
\[ \text{Sing}_1(X) = \{ p \in X \mid I_p \neq O_{X,p} \} . \]

If \( p \in X \) is a 1-point with an expression (1) we have 
\[ (I_p + (x))\hat{O}_{X,p} = (x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) . \]

If \( p \in X \) is a 2-point with an expression (2) we have 
\[ (I_p + (x,y))\hat{O}_{X,p} = (x, y, (ad - bc)F, \frac{\partial F}{\partial z}) . \]

If \( p \in X \) is a 3-point with an expression (3) we have 
\[ (I_p + (x,y,z))\hat{O}_{X,p} = (x, y, z, (ae - bd)F, (af - cd)F, (bf - ce)F) . \]

If \( p \in X \) is a 1-point, then \( \sigma_D(p) = \text{ord} F(0, y, z) - 1 \). We have 
\[ 0 \leq \sigma_D(p) < \infty \] if \( p \) is a 1-point. If \( p \in X \) is a 2-point, we have 
\[ \sigma_D(p) = \begin{cases} 0 & \text{if ord } F(0,0,z) = 0 \text{ (in this case, } ad - bc \neq 0) \\ \text{ord } F(0,0,z) - 1 & \text{if } 1 \leq \text{ord } F(0,0,z) < \infty \\ \infty & \text{if ord } F(0,0,z) = \infty . \end{cases} \]

If \( p \in X \) is a 3-point, let 
\[ A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} . \]
we have 
\[ \sigma_D(p) = \begin{cases} 0 & \text{if ord } F(0,0,0) = 0 \text{ (in this case, rank}(A) = 2) \\ \infty & \text{if ord } F(0,0,0) = \infty . \end{cases} \]

**Lemma 2.5.** Suppose that \( X \) is 1-prepared and \( \pi_1 : X_1 \to X \) is a toroidal morphism with respect to \( D \). Then \( X_1 \) is 1-prepared and \( \sigma_D(p_1) \leq \sigma_D(p) \) for all \( p \in X \) and \( p_1 \in \pi_1^{-1}(p) \).

*Proof.* Suppose that \( p \in X \) is a 2-point and \( p_1 \in \pi_1^{-1}(p) \). Then there exist permissible parameters \( x, y, z \) at \( p \) giving an expression (2). In \( \hat{O}_{X_1,p_1} \), there are regular parameters \( x_{11}, y_{11}, z \) where 
\[ x = x_{11}^{\alpha_1}(y_1 + \alpha)^{\alpha_{12}}, \quad y = x_{11}^{\alpha_2}(y_1 + \alpha)^{\alpha_{22}} \]
with \( \alpha \in \mathfrak{k} \) and \( a_{11} \sigma_{22} - a_{12} \sigma_{22} = \pm 1 \). If \( \alpha = 0 \), so that \( p_1 \) is a 2-point, then \( x_{11}, y_{11}, z \) are permissible parameters at \( p_1 \) and substitution of (7) into (2) gives an expression of the form (2) at \( p_1 \), showing that \( \sigma_D(p_1) \leq \sigma_D(p) \). If \( \alpha \neq 0 \in \mathfrak{k} \), so that \( p_1 \) is a 1-point, set \( \lambda = \frac{a_{12} + b_{22}}{a_{11} + b_{21}} \) and \( \bar{x}_1 = x_1(y_1 + \alpha)^{\lambda} \). Then \( \bar{x}_1, y_1, z \) are permissible parameters at \( p_1 \). Substitution into (2) leads to a form (1) with \( \sigma_D(p_1) \leq \sigma_D(p) \).

If \( p \in X \) is a 3-point and \( \sigma_D(p) \neq \infty \), then \( \sigma_D(p) = 0 \) so that \( p \) is prepared. Thus there exist permissible parameters \( x, y, z \) at \( p \) giving an expression (3) with \( F = 1 \). Suppose that \( p_1 \in \pi_1^{-1}(p) \). In \( \hat{O}_{X_1,p_1} \) there are regular parameters \( x_{11}, y_{11}, z_1 \) such that
\[ x = (x_1 + \alpha)^{\alpha_1}(y_1 + \beta)^{\alpha_{12}}(z_1 + \gamma)^{\alpha_{13}} \]
\[ y = (x_1 + \alpha)^{\alpha_2}(y_1 + \beta)^{\alpha_{22}}(z_1 + \gamma)^{\alpha_{23}} \]
\[ z = (x_1 + \alpha)^{\alpha_3}(y_1 + \beta)^{\alpha_{32}}(z_1 + \gamma)^{\alpha_{33}} \]
where at least one of \( \alpha, \beta, \gamma \in \mathfrak{k} \) is zero. Substituting into (3), we find permissible parameters at \( p_1 \) giving a prepared form. \( \square \)
Suppose that $X$ is 1-prepared with respect to $f : X → S$. Define

$$Γ_D(X) = \max{\{σ_D(p) | p ∈ X\}}.$$

**Lemma 2.6.** Suppose that $X$ is 1-prepared and $C$ is a 2-curve of $D$ and there exists $p ∈ C$ such that $σ_D(p) < ∞$. Then $σ_D(q) = 0$ at the generic point $q$ of $C$.

**Proof.** If $p$ is a 3-point then $σ_D(p) = 0$ and the lemma follows from upper semicontinuity of $σ_D$.

Suppose that $p$ is a 2-point. If $σ_D(p) = 0$ then the lemma follows from upper semicontinuity of $σ_D$, so suppose that $0 < σ_D(p) < ∞$. There exist permissible parameters $x, y, z$ at $p$ giving a form (2), such that $x, y, z$ are uniformizing parameters on an étale cover $U$ of an affine neighborhood of $p$. Thus for $α$ in a Zariski open subset of $\mathfrak{f}$, $x, y, z = z − α$ are permissible parameters at a 2-point $\bar{p}$ of $C$. After possibly replacing $U$ with a smaller neighborhood of $p$, we have

$$\frac{∂F}{∂z} = \frac{1}{x^cy^d} \frac{∂v}{∂z} ∈ Γ(U, O_X)$$

and $\frac{∂F}{∂z}(0, 0, z) ≠ 0$. Thus there exists a 2-point $\bar{p} ∈ C$ with permissible parameters $x, y, z = z − α$ such that $\frac{∂F}{∂z}(0, 0, α) ≠ 0$, and thus there is an expression (2) at $\bar{p}$

$$\begin{align*}
u &= (x^ay^b)^t \\
v &= P_1(x^ay^b) + x^cy^dF_1(x, y, z)
\end{align*}$$

with $F_1(0, 0, z) = 0$ or 1, so that $σ_D(\bar{p}) = 0$. By upper semicontinuity of $σ_D$, $σ_D(q) = 0$.

**Proposition 2.7.** Suppose that $X$ is 1-prepared with respect to $f : X → S$. Then there exists a toroidal morphism $π_1 : X_1 → X$ with respect to $D$, such that $π_1$ is a sequence of blow ups of 2-curves and 3-points, and

1) $σ_D(p) < ∞$ for all $p ∈ D_{X_1}$.

2) $X_1$ is prepared (with respect to $f_1 = f ◦ π_1 : X_1 → S$) at all 3-points and the generic point of all 2-curves of $D_{X_1}$.

**Proof.** By upper semicontinuity of $σ_D$, Lemma 2.6 and Lemma 2.5, we must show that if $p ∈ X$ is a 3-point with $σ_D(p) = ∞$ then there exists a toroidal morphism $π_1 : X_1 → X$ such that $σ_D(p_1) = 0$ for all 3-points $p_1 ∈ π_1^{-1}(p)$ and if $p ∈ X$ is a 2-point with $σ_D(p) = ∞$ then there exists a toroidal morphism $π_1 : X_1 → X$ such that $σ_D(p_1) < ∞$ for all 2-points $p_1 ∈ π_1^{-1}(p)$.

First suppose that $p$ is a 3-point with $σ_D(p) = ∞$. Let $x, y, z$ be permissible parameters at $p$ giving a form (3). There exist regular parameters $\bar{x}, \bar{y}, \bar{z}$ in $O_{X,p}$ and unit series $α, β, γ ∈ O_{X,p}$ such that $x = α\bar{x}$, $y = β\bar{y}$, $z = γ\bar{z}$. Write $F = ∑ b_{ijk}x^iy^jz^k$ with $b_{ijk} ∈ \mathfrak{f}$. Let $I = (\bar{x}i\bar{y}j\bar{z}k | b_{ijk} ≠ 0)$, an ideal in $O_{X,p}$. Since $\bar{x}\bar{y}\bar{z} = 0$ is a local equation of $D$ at $p$, there exists a toroidal morphism $π_1 : X_1 → X$ with respect to $D$ such that $IO_{X_1,p_1}$ is principal for all $p_1 ∈ π_1^{-1}(p)$. At a 3-point $p_1 ∈ π_1^{-1}(p)$, there exist permissible parameters $x_1, y_1, z_1$ such that

$$\begin{align*}
x &= x_1^{α_{11}}y_1^{α_{12}}z_1^{α_{13}} \\
y &= x_1^{α_{21}}y_1^{α_{22}}z_1^{α_{23}} \\
z &= x_1^{α_{31}}y_1^{α_{32}}z_1^{α_{33}}
\end{align*}$$
with $\text{Det}(a_{ij}) = \pm 1$. Substituting into (3), we obtain an expression (3) at $p_1$, where

\[
\begin{align*}
    u &= (x_1^{a_1} y_1^{b_1} z_1^{c_1})^l \\
    v &= P_1(x_1^{a_1} y_1^{b_1} z_1^{c_1}) + x_1^{d_1} y_1^{e_1} z_1^{f_1} F_1
\end{align*}
\]

where $P_1(x_1^{a_1} y_1^{b_1} z_1^{c_1}) = P(x^a y^b z^c)$ and

\[
F(x, y, z) = x_1^{\pi} y_1^{\beta} z_1^{\gamma} F_1(x_1, y_1, z_1).
\]

with $x_1^{\pi} y_1^{\beta} z_1^{\gamma}$ a generator of $I\mathcal{O}_{X_1, p_1}$ and $F_1(0, 0, 0) \neq 0$. Thus $\sigma_D(p_1) = 0$.

Now suppose that $p$ is a 2-point and $\sigma_D(p) = \infty$. There exist permissible parameters $x, y, z$ at $p$ giving a form (2). Write $F = \sum a_i(x, y) z^i$, with $a_i(x, y) \in \mathfrak{t}[[x, y]]$ for all $i$. We necessarily have that no $a_i(x, y)$ is a unit series.

Let $I$ be the ideal $I = (a_i(x, y) \mid i \in \mathbb{N})$ in $\mathfrak{t}[[x, y]]$. There exists a sequence of blow ups of 2-curves $\pi_1 : X_1 \rightarrow X$ such that $\mathcal{O}_{X_1, p_1}$ is principal at all 2-points $p_1 \in \pi_1^{-1}(p)$. There exist $x_1, y_1 \in \mathcal{O}_{X_1, p_1}$ so that $x_1, y_1, z$ are permissible parameters at $p_1$, and

\[
x = x_1^{a_{11}} y_1^{a_{12}}, \quad y = x_1^{a_{21}} y_1^{a_{22}}
\]

with $a_{11} a_{22} - a_{12} a_{21} = \pm 1$. Let $x_1^{\pi} y_1^{\beta}$ be a generator of $I\mathcal{O}_{T_1, q_1}$. Then $F = x_1^{\pi} y_1^{\beta} F_1(x_1, y_1, z)$ where $F_1(0, 0, z) \neq 0$, and we have an expression (2) at $p_1$, where

\[
\begin{align*}
    u &= (x_1^{a_1} y_1^{b_1})^l \\
    v &= P_1(x_1^{a_1} y_1^{b_1}) + x_1^{d_1} y_1^{e_1} F_1
\end{align*}
\]

where $P_1(x_1^{a_1} y_1^{b_1}) = P(x^a y^b)$. Thus $\sigma_D(p_1) < \infty$ and $\sigma_D(q) < \infty$ if $q$ is the generic point of the 2-curve of $D_{X_1}$ containing $p_1$.

We will say that $X$ is 2-prepared (with respect to $f : X \rightarrow S$) if it satisfies the conclusions of Proposition 2.7. We then have that $\Gamma_D(X) < \infty$.

If $X$ is 2-prepared, we have that $\text{Sing}_2(X)$ is a union of (closed) curves whose generic point is a 1-point and isolated 1-points and 2-points. Further, $\text{Sing}_1(X)$ contains no 3-points.

3. 3-PREPARATION

Lemma 3.1. Suppose that $X$ is 2-prepared. Suppose that $p \in X$ is such that $\sigma_D(p) > 0$. Let $m = \sigma_D(p) + 1$. Then there exist permissible parameters $x, y, z$ at $p$ such that there exist $\tilde{x}, y, z \in \mathcal{O}_{X, p}$, an étale cover $U$ of an affine neighborhood of $p$, such that $x, z \in \Gamma(U, \mathcal{O}_X)$ and $x, y, z$ are uniformizing parameters on $U$, and $x = \gamma \tilde{x}$ for some unit series $\gamma \in \mathcal{O}_{X, p}$. We have an expression (1) or (2), if $p$ is respectively a 1-point or a 2-point, with

\[
F = \tau x^m + a_2(x, y) z^{m-2} + \cdots + a_{m-1}(x, y) z + a_m(x, y)
\]

where $m \geq 2$ and $\tau \in \mathcal{O}_{X_1, p} = \mathfrak{t}[[x, y, z]]$ is a unit, and $a_i(x, y) \neq 0$ for $i = m-1$ or $i = m$. Further, if $p$ is a 1-point, then we can choose $x, y, z$ so that $x = y = 0$ is a local equation of a generic curve through $p$ on $D$.

For all but finitely many points $p$ in the set of 1-points of $X$, there is an expression (9) where

\[
a_i \text{ is either zero or has an expression } a_i = \overline{a}_i x^{r_i} \text{ where } \overline{a}_i \text{ is a unit and } r_i > 0 \text{ for } 2 \leq i \leq m, \text{ and } a_m = 0 \text{ or } a_m = x^m \overline{a}_m \text{ where } r_m > 0 \text{ and } \text{ord}(\overline{a}_m(0, y)) = 1.
\]

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Proof. There exist regular parameters \( \bar{x}, y, \bar{z} \) in \( \mathcal{O}_{X,p} \) and a unit \( \gamma \in \mathcal{O}_{X,p} \) such that \( x = \gamma \bar{x}, y, \bar{z} \) are permissible parameters at \( p \), with \( \text{ord}(F(0,0,\bar{z})) = m \). Thus there exists an affine neighborhood \( \text{Spec}(A) \) of \( p \) such that \( V = \text{Spec}(R) \), where \( R = A[\gamma^\frac{1}{m}] \) is an étale cover of \( \text{Spec}(A) \), \( x, y, \bar{z} \) are uniformizing parameters on \( V \), and \( u, v \in \Gamma(V, \mathcal{O}_{X}) \). Differentiating with respect to the uniformizing parameters \( x, y, \bar{z} \) in \( R \), set

\[
\bar{z} = \frac{\partial^{m-1} F}{\partial \bar{z}^{m-1}} = \omega(\bar{z} - \varphi(x, y))
\]

where \( \omega \in \mathcal{O}_{X,p} \) is a unit series, and \( \varphi(x, y) \in \mathfrak{f}[[x, y]] \) is a nonunit series, by the formal implicit function theorem. Set \( z = \bar{z} - \varphi(x, y) \). Since \( R \) is normal, after possibly replacing \( \text{Spec}(A) \) with a smaller affine neighborhood of \( p \),

\[
\bar{z} = \frac{1}{x^b} \frac{\partial^{m-1} v}{\partial \bar{z}^{m-1}} \in R.
\]

By Weierstrass preparation for Henselian local rings (Proposition 6.1 [37]), \( \varphi(x, y) \) is integral over the local ring \( \mathfrak{f}[[x, y]]/(x, y) \). Thus after possibly replacing \( A \) with a smaller affine neighborhood of \( p \), there exists an étale cover \( U \) of \( V \) such that \( \varphi(x, y) \in \Gamma(U, \mathcal{O}_{X}) \), and thus \( z \in \Gamma(U, \mathcal{O}_{X}) \).

Let \( G(x, y, z) = F(x, y, \bar{z}) \). We have that

\[
G = G(x, y, 0) + \frac{\partial G}{\partial z}(x, y, 0)z + \cdots + \frac{1}{(m-1)!} \frac{\partial^{m-1} G}{\partial z^{m-1}}(x, y, 0)z^{m-1} + \frac{1}{m!} \frac{\partial^m G}{\partial z^m}(x, y, 0)z^m + \cdots
\]

We have

\[
\frac{\partial^{m-1} G}{\partial z^{m-1}}(x, y, 0) = \frac{\partial^{m-1} F}{\partial \bar{z}^{m-1}}(x, y, \varphi(x, y)) = 0
\]

and

\[
\frac{\partial^m G}{\partial z^m}(x, y, 0) = \frac{\partial^m F}{\partial \bar{z}^m}(x, y, \varphi(x, y))
\]

is a unit in \( \mathcal{O}_{X,p} \). Thus we have the desired form (9), but we must still show that \( a_m \neq 0 \) or \( a_{m-1} \neq 0 \). If \( a_i(x, y) = 0 \) for \( i = m \) and \( i = m - 1 \), we have that \( z^2 \mid F \) in \( \mathcal{O}_{X,p} \), since \( m \geq 2 \). This implies that the ideal of \( 2 \times 2 \) minors

\[
I_2 \left( \begin{array}{cccc}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial \bar{x}} & \frac{\partial u}{\partial \bar{y}} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial \bar{x}} & \frac{\partial v}{\partial \bar{y}}
\end{array} \right) \subset (z),
\]

which implies that \( z = 0 \) is a component of \( D \) which is impossible. Thus either \( a_{m-1} \neq 0 \) or \( a_m \neq 0 \).

Suppose that \( C \) is a curve in \( \text{Sing}_1(X) \) (containing a 1-point) and \( p \in C \) is a general point. Let \( r = \sigma_D(p) \). Set \( m = r + 1 \). Let \( x, y, \bar{z} \) be permissible parameters at \( p \) with \( y, \bar{z} \in \mathcal{O}_{X,p} \), which are uniformizing parameters on an étale cover \( U \) of an affine neighborhood of \( p \) such that \( x = \bar{z} = 0 \) are local equations of \( C \) and we have a form (1) at \( p \) with

\[
F = \tau \bar{z}^m + a_1(x, y)\bar{z}^{m-1} + \cdots + a_m(x, y).
\]

For \( \alpha \) in a Zariski open subset of \( \mathfrak{f} \), \( x, \bar{y} = y - \alpha, \bar{z} \) are permissible parameters at a point \( q \in C \cap U \). For most points \( q \) on the curve \( C \cap U \), we have that \( a_i(x, y) = x^n \bar{a}_i(x, y) \) where \( \bar{a}_i(x, y) \) is a unit or zero for \( 1 \leq i \leq m - 1 \) in \( \mathcal{O}_{X,q} \). Since \( \sigma_D(p) = r \) at this point,
we have that $1 \leq r_i$ for all $i$. We further have that if $a_m \neq 0$, then $a_m = x^{r_m} a'$ where
\[ a' = f(y) + x \Omega \] where $f(y)$ is non constant. Thus
\[ 0 \neq \frac{\partial a_m}{\partial y}(0, y) = \frac{\partial F}{\partial y}(0, y, 0). \]
After possibly replacing $U$ with a smaller neighborhood of $p$, we have
\[ \frac{\partial F}{\partial y} = \frac{1}{x^b} \frac{\partial \nu}{\partial y} \in \Gamma(U, \mathcal{O}_X). \]
Thus $\frac{\partial a_m}{\partial y}(0, \alpha) \neq 0$ for most $\alpha \in \mathfrak{t}$. Since $r > 0$, we have that $r_m > 0$, and thus $r_i > 0$ for all $i$ in (12). We have
\[ \frac{\partial^{m-1} F}{\partial \Omega^{m-1}} = \xi \Omega + a_1(x, y), \]
where $\xi$ is a unit series. Comparing the above equation with (11), we observe that $\varphi(x, y)$ is a unit series in $x$ and $y$ times $a_1(x, y)$. Thus $x$ divides $\varphi(x, y)$. Setting $z = \Omega - \varphi(x, y)$, we obtain an expression (9) such that $x$ divides $a_i$ for all $i$. Now argue as in the analysis of (12), after substituting $z = \Omega - \varphi(x, y)$, to conclude that there is an expression (9), where (10) holds at most points $q \in C \cap U$. Thus a form (9) and (10) holds at all but finitely many 1-points of $X$.

\[ \square \]

**Lemma 3.2.** Suppose that $X$ is 2-prepared, $C$ is a curve in $\text{Sing}_1(X)$ containing a 1-point and $p$ is a general point of $C$. Let $m = \sigma_D(p) + 1$. Suppose that $\tilde{x}, y \in \mathcal{O}_{X, p}$ are such that $\tilde{x} = 0$ is a local equation of $D$ at $p$ and the germ $\tilde{x} = y = 0$ intersects $C$ transversally at $p$. Then there exists an étale cover $U$ of an affine neighborhood of $p$ and $z \in \Gamma(U, \mathcal{O}_X)$ such that $\tilde{x}, y, z$ give a form (9) at $p$.

**Proof.** There exists $\Omega \in \mathcal{O}_{X, p}$ such that $\tilde{x}, y, \Omega$ are regular parameters in $\mathcal{O}_{X, p}$ and $x = \Omega = 0$ is a local equation of $C$ at $p$. There exists a unit $\gamma \in \mathcal{O}_{X, p}$ such that $x = \gamma \tilde{x}, y, \Omega$ are permissible parameters at $p$. We have an expression of the form (1),
\[ u = x^a, v = P(x) + x^b F \]
at $p$. Write $F = f(y, \Omega) + x \Omega$ in $\mathcal{O}_{X, p}$. Let $I$ be the ideal in $\mathcal{O}_{X, p}$ generated by $x$ and
\[ \{ \frac{\partial^{i+j} f}{\partial y^i\partial \Omega^j} \mid 1 \leq i + j \leq m - 1 \}. \]
The radical of $I$ is the ideal $(x, \Omega)$, as $x = \Omega = 0$ is a local equation of $\text{Sing}_{m-1}(X)$ at $p$. Thus $\Omega$ divides $\frac{\partial^{i+j} f}{\partial y^i\partial \Omega^j}$ for $1 \leq i + j \leq m - 1$ (with $m \geq 2$). Expanding
\[ f = \sum_{i=0}^{\infty} b_i(y) \Omega^i \]
(where $b_0(0) = 0$) we see that $\frac{\partial b_0}{\partial y} = 0$ (so that $b_0(y) = 0$) and $b_1(y) = 0$ for $1 \leq i \leq m - 1$. Thus $\Omega^m$ divides $f(y, \Omega)$. Since $\sigma_D(p) = m - 1$, we have that $f = \tau \Omega^m$ where $\tau$ is a unit series. Thus $x, y, \Omega$ gives a form (1) with $\text{ord}(F(0, 0, \Omega)) = m$. Now the proof of Lemma 3.1 gives the desired conclusion. \[ \square \]

Let $\omega(m, r_2, \ldots, r_{m-1})$ be a function which associates a positive integer to a positive integer $m$, natural numbers $r_2, \ldots, r_{m-2}$ and a positive integer $r_{m-1}$. We will give a precise form of $\omega$ after Theorem 4.1.
**Definition 3.3.** $X$ is 3-prepared (with respect to $f : X \to S$) at a point $p \in D$ if $\sigma_D(p) = 0$ or if $\sigma_D(p) > 0$, $f$ is 2-prepared with respect to $D$ at $p$ and there are permissible parameters $x,y,z$ at $p$ such that $x,y,z$ are uniformizing parameters on an étale cover of an affine neighborhood of $p$ and we have one of the following forms, with $m = \sigma_D(p) + 1$:

1) $p$ is a 2-point, and we have an expression (2) with

$$F = \tau_0 z^m + \tau_2 x^{r_2} y^{s_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} y^{s_{m-1}} z + \tau_m x^r y^s$$

where $\tau_0 \in \hat{O}_{X,p}$ is a unit, $\tau_i \in \hat{O}_{X,p}$ are units (or zero), $r_i + s_i > 0$ whenever $\tau_i \neq 0$ and $(r_m + c)b - (s_m + d)a \neq 0$. Further, $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

2) $p$ is a 1-point, and we have an expression (1) with

$$F = \tau_0 z^m + \tau_2 x^{r_2} y^{s_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} y^{s_{m-1}} z + \tau_m x^r y^s$$

where $\tau_0 \in \hat{O}_{X,p}$ is a unit, $\tau_i \in \hat{O}_{X,p}$ are units (or zero), and $\ord(\tau_m(0,0)) = 1$ (or $\tau_m = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$, and $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

3) $p$ is a 1-point, and we have an expression (1) with

$$F = \tau_0 z^m + \tau_2 x^{r_2} y^{s_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} y^{s_{m-1}} z + x^t \Omega$$

where $\tau_0 \in \hat{O}_{X,p}$ is a unit, $\tau_i \in \hat{O}_{X,p}$ are units (or zero) for $2 \leq i \leq m - 1$, $\Omega \in \hat{O}_{X,p}$, $\tau_{m-1} \neq 0$ and $t > \omega(m, r_2, \ldots, r_{m-1})$ (where we set $r_i = 0$ if $\tau_i = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$.

$X$ is 3-prepared if $X$ is 3-prepared for all $p \in X$.

**Lemma 3.4.** Suppose that $X$ is 2-prepared with respect to $f : X \to S$. Then there exists a sequence of blow ups of 2-curves $\pi_1 : X \to X_1$ such that $X_1$ is 3-prepared with respect to $f \circ \pi_1$, except possibly at a finite number of 1-points.

**Proof.** The conclusions follow from Lemmas 3.1, 2.6 and 2.5, and the method of analysis above 2-points of the proof of 2.7. \qed

**Lemma 3.5.** Suppose that $u, v \in k[[x,y]]$. Let $T_0 = \Spec(k[[x,y]])$. Suppose that $u = x^a$ for some $a \in \mathbb{Z}_+$, or $u = (x^a y^b)\ell$ where $\gcd(a, b, \ell) = 1$ for some $a, b, \ell \in \mathbb{Z}_+$. Let $p \in T_0$ be the maximal ideal $(x,y)$. Suppose that $v \in (x,y)k[[x,y]]$. Then either $v \in k[[x]]$ or there exists a sequence of blow ups of points $\lambda : T_1 \to T_0$ such that for all $p_1 \in \lambda^{-1}(p)$, we have regular parameters $x_1, y_1$ in $\hat{O}_{T_1,p_1}$, regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\hat{O}_{T_1,p_1}$ and a unit $\gamma_1 \in \hat{O}_{T_1,p_1}$ such that $x_1 = \gamma_1 \tilde{x}_1$, and one of the following holds:

1) $u = x_1^{a_1}, v = P(x_1) + x_1^{b_1}y_1^c$

with $c > 0$ or

2) There exists a unit $\gamma_2 \in \hat{O}_{T_1,p_1}$ such that $y_1 = \gamma_2 \tilde{y}_1$ and

$$u = (x_1^{a_1} y_1^{b_1})\ell_1, v = P(x_1^{a_1} y_1^{b_1}) + x_1^{c_1} y_1^{d_1}$$

with $\gcd(a_1, b_1) = 1$ and $a_1d_1 - b_1c_1 \neq 0$.

**Proof.** Let

$$J = \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$
First suppose that \( J = 0 \). Expand \( v = \sum \gamma_{ij}x^iy^j \) with \( \gamma_{ij} \in \mathfrak{t} \). If \( u = x^a \), then 
\[
\sum j\gamma_{ij}x^iy^j = 1 \implies \gamma_{ij} = 0 \text{ if } j > 0.
\]

Thus \( v = P(x) \in \mathfrak{t}[[x]] \). If \( u = (x^ay^b)^l \), then
\[
0 = J = lx^{la-1}y^{lb-1}(\sum \gamma_{ij}x^iy^j)
\]

implies \( \gamma_{ij} = 0 \) if \( ja - ib \neq 0 \), which implies that \( v \in \mathfrak{t}[x^ay^b] \).

Now suppose that \( J \neq 0 \). Let \( E \) be the divisor \( uJ = 0 \) on \( T_0 \). There exists a sequence of blow ups of points \( \lambda : T_1 \to T_0 \) such that \( \lambda^{-1}(E) \) is a SNC divisor on \( T_1 \). Suppose that \( p_1 \in \lambda^{-1}(p) \). There exist regular parameters \( \tilde{x}_1, \tilde{y}_1 \) in \( \mathcal{O}_{T_1, p_1} \) such that

\[
J_1 = \det \left( \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial y_1} \right) ,
\]

then

\[
u = \tilde{x}_1^{a_1}, \quad J_1 = \delta \tilde{x}_1 \tilde{y}_1^{c_1}
\]

where \( a_1 > 0 \) and \( \delta \) is a unit in \( \mathcal{O}_{T_1, p_1} \), or

\[
u = (\tilde{x}_1^{a_1} \tilde{y}_1^{b_1})^{l_1}, \quad J_1 = \delta \tilde{x}_1 \tilde{y}_1^{d_1}
\]

where \( a_1, b_1 > 0 \), \( \gcd(a_1, b_1) = 1 \) and \( \delta \) is a unit in \( \mathcal{O}_{T_1, p_1} \). Expand \( v = \sum \gamma_{ij} \tilde{x}_1^i \tilde{y}_1^j \) with \( \gamma_{ij} \in \mathfrak{t} \).

First suppose (16) holds. Then

\[
a_1x_1^{a_1-1} \left( \sum \gamma_{ij}x_1^i \tilde{y}_1^{j-1} \right) = \delta \tilde{x}_1 \tilde{y}_1^{c_1}.
\]

Thus \( v = P(\tilde{x}_1) + \varepsilon \tilde{x}_1^i \tilde{y}_1^j \) where \( P(\tilde{x}_1) \in \mathfrak{t}[[\tilde{x}_1]] \), \( e = b_1 - a_1 + a \), \( f = c_1 + 1 \) and \( \varepsilon \) is a unit series. Since \( f > 0 \), we can make a formal change of variables, multiplying \( \tilde{x}_1 \) by an appropriate unit series to get the form 1) of the conclusions of the lemma.

Now suppose that (17) holds. Then

\[
x_1^{a_1l_1-1} \tilde{y}_1^{b_1l_1-1} \left( \sum \gamma_{ij}x_1^i \tilde{y}_1^{j-1} \right) = \delta \tilde{x}_1 \tilde{y}_1^{d_1}.
\]

Thus \( v = P(\tilde{x}_1^{a_1} \tilde{y}_1^{b_1}) + \varepsilon \tilde{x}_1^i \tilde{y}_1^j \), where \( P \) is a series in \( \tilde{x}_1^{a_1} \tilde{y}_1^{b_1} \), \( \varepsilon \) is a unit series, \( e = c_1 + 1 - a_1l_1 \), \( f = d_1 + 1 - b_1l_1 \). Since \( a_1l_1f - b_1l_1e \neq 0 \), we can make a formal change of variables to reach 2) of the conclusions of the lemma. \( \square \)

**Lemma 3.6.** Suppose that \( X \) is 2-prepared with respect to \( f : X \to S \). Suppose that \( p \in D \) is a 1-point with \( m = \sigma_D(p) + 1 > 1 \). Let \( u, v \) be permissible parameters for \( f(p) \) and \( x, y, z \) be permissible parameters for \( D \) at \( p \) such that a form (9) holds at \( p \). Let \( U \) be an étale cover of an affine neighborhood of \( p \) such that \( x, y, z \) are uniformizing parameters on \( U \). Let \( C \) be the curve in \( U \) which has local equations \( x = y = 0 \) at \( p \).

Let \( T_0 = \text{Spec}(\mathfrak{t}[x, y]) \), \( \Lambda_0 : U \to T_0 \). Then there exists a sequence of quadratic transforms \( T_1 \to T_0 \) such that if \( U_1 = U \times_{T_0} T_1 \) and \( \psi_1 : U_1 \to U \) is the induced sequence of blow ups of sections over \( C \), \( \Lambda_1 : U_1 \to T_1 \) is the projection, then \( U_1 \) is 2-prepared with respect to \( f \circ \psi_1 \) at all \( p_1 \in \psi_1^{-1}(p) \). Further, for every point \( p_1 \in \psi_1^{-1}(p) \), there exist regular parameters \( x_1, y_1 \) in \( \mathcal{O}_{T_1, \Lambda_1(p_1)} \) such that \( x_1, y_1, z \) are permissible parameters at \( p_1 \), and there exist regular parameters \( \tilde{x}_1, \tilde{y}_1 \) in \( \mathcal{O}_{T_1, \Lambda_1(p_1)} \) such that if \( p_1 \) is a 1-point,
\[ x_1 = \alpha(\tilde{x}_1, \tilde{y}_1) \tilde{x}_1 \] where \( \alpha(\tilde{x}_1, \tilde{y}_1) \in \tilde{O}_{T_1, \Lambda_1(p_1)} \) is a unit series and \( y_1 = \beta(\tilde{x}_1, \tilde{y}_1) \) with \( \beta(\tilde{x}_1, \tilde{y}_1) \in \tilde{O}_{T_1, \Lambda_1(p_1)} \), and if \( p_1 \) is a 2-point, then \( x_1 = \alpha(\tilde{x}_1, \tilde{y}_1) \tilde{x}_1 \) and \( y_1 = \beta(\tilde{x}_1, \tilde{y}_1) \tilde{y}_1 \), where \( \alpha(\tilde{x}_1, \tilde{y}_1), \beta(\tilde{x}_1, \tilde{y}_1) \in \tilde{O}_{T_1, \Lambda_1(p_1)} \) are unit series. We have one of the following forms:

1) \( p_1 \) is a 2-point, and we have an expression (2) with

\[ F = \tau z^m + \tilde{a}(x_1, y_1)x_1^r_1 y_1^s z^{m-2} + \cdots + \tilde{a}_{m-1}(x_1, y_1)x_1^r_{m-1} y_1^s z + \tilde{a}_m x_1^r y_1^s \]

where \( \tau \in \tilde{O}_{U_1, p_1} \) is a unit, \( \tilde{a}_i(x_1, y_1) \in \mathfrak{t}[[x_1, y_1]] \) are units (or zero) for \( 2 \leq i \leq m-1 \), \( \tilde{a}_m = 0 \) or 1 and if \( \tilde{a}_m = 0 \), then \( \tilde{a}_{m-1} \neq 0 \). Further, \( r_i + s_i > 0 \) whenever \( \tilde{a}_i \neq 0 \) and \( a(r_m + c)b - (s_m + d)a \neq 0 \).

2) \( p_1 \) is a 1-point, and we have an expression (1) with

\[ F = \tau z^m + \tilde{a}_2(x_1, y_1)x_1^r_2 z^{m-2} + \cdots + \tilde{a}_{m-1}(x_1, y_1)x_1^r_{m-1} z + \tilde{a}_m x_1^r y_1^s \]

where \( \tau \in \tilde{O}_{U_1, p_1} \) is a unit, \( \tilde{a}_i(x_1, y_1) \in \mathfrak{t}[[x_1, y_1]] \) are units (or zero) for \( 2 \leq i \leq m-1 \). Further, \( r_i > 0 \) (whenever \( \tilde{a}_i \neq 0 \)).

3) \( p_1 \) is a 1-point, and we have an expression (1) with

\[ F = \tau z^m + \tilde{a}_2(x_1, y_1)x_1^r_2 z^{m-2} + \cdots + \tilde{a}_{m-1}(x_1, y_1)x_1^r_{m-1} z + \tilde{a}_m x_1^r y_1 \]

where \( \tau \in \tilde{O}_{U_1, p_1} \) is a unit, \( \tilde{a}_i(x_1, y_1) \in \mathfrak{t}[[x_1, y_1]] \) are units (or zero) for \( 2 \leq i \leq m-1 \) and \( r_i > 0 \) whenever \( \tilde{a}_i \neq 0 \). We also have \( t > \omega(m, r_2, \ldots, r_{m-1}) \). Further, \( \tilde{a}_{m-1} \neq 0 \) and \( \Omega \in \tilde{O}_{U_1, p_1} \).

Proof. Let \( \bar{p} = \Lambda_0(p) \). Let \( T = \{ i \mid a_i(x, y) \neq 0 \text{ and } 2 \leq i \leq m \} \). There exists a sequence of blow ups \( \varphi_1 : T_1 \to T_0 \) of points over \( \bar{p} \) such that at all points \( q \in \psi_1^{-1}(p) \), we have permissible parameters \( x_1, y_1, z \) such that \( x_1, y_1 \) are regular parameters in \( \tilde{O}_{T_1, \Lambda_1(q)} \) and \( u \) has a monomial in \( x_1 \) and \( y_1 \) times a unit in \( \tilde{O}_{T_1, \Lambda_1(q)} \), where \( q = \prod_{i \in T} a_i(x, y) \).

Suppose that \( a_m(x, y) \neq 0 \). Let \( \bar{v} = x^b a_m(x, y) \) if (1) holds and \( \bar{v} = x^c y^d a_m(x, y) \) if (2) holds. We have \( \bar{v} \notin \mathfrak{t}[[x]] \) (respectively \( \bar{v} \notin \mathfrak{t}[[x^a y^b]] \)). Then by Theorem 3.5 applied to \( u, \bar{v} \), we have that there exists a further sequence of blow ups \( \varphi_2 : T_2 \to T_1 \) of points over \( \bar{p} \) such that at all points \( q \in (\psi_1 \circ \psi_2)^{-1}(p) \), we have permissible parameters \( x_2, y_2, z \) such that \( x_2, y_2 \) are regular parameters in \( \tilde{O}_{T_2, \Lambda_2(q)} \) such that \( u = 0 \) is a SNC divisor and either

\[ u = x_2^{\bar{v}} \bar{v} = \bar{P}(x_2) + x_2^{\xi_2} \bar{y}_2 \]

with \( \bar{v} > 0 \) or

\[ u = (x_2^{\xi_2} \bar{y}_2)^t \bar{v} = \bar{P}(x_2^{\xi_2} \bar{y}_2) + x_2^{\xi_2} \bar{y}_2 \]

where \( \bar{a}a - \bar{b} \bar{c} \neq 0 \).

If \( q \) is a 2-point, we have thus achieved the conclusions of the lemma. Further, there are only finitely many 1-points \( q \) above \( p \) on \( U_2 \) where the conclusions of the lemma do not hold. At such a 1-point \( q \), \( F \) has an expression

\[ F = \tau z^m + \tilde{a}_2(x_2, y_2)x_2^r_2 y_2^s z^{m-2} + \cdots + \tilde{a}_{m-1}(x_2, y_2)x_2^r_{m-1} y_2^s z + \tilde{a}_m x_2^r y_2^s \]

where \( \tilde{a}_m = 0 \) or 1, \( \tilde{a}_i \) are units (or zero) for \( 2 \leq i \leq m \).

Let

\[ J = I_2 \begin{pmatrix} \frac{\partial u}{\partial x_2} & \frac{\partial u}{\partial y_2} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x_2} & \frac{\partial v}{\partial y_2} & \frac{\partial v}{\partial z} \\ \end{pmatrix} = x^n \begin{pmatrix} \frac{\partial F}{\partial y_2} \\ \frac{\partial F}{\partial z} \\ \end{pmatrix} \]

for some positive integer \( n \). Since \( D \) contains the locus where \( f \) is not smooth, we have that the localization \( J_p = (\tilde{O}_{U_2, q})_p \), where \( p \) is the prime ideal \( (y_2, z_2) \) in \( \tilde{O}_{U_2, q} \).
We compute
\[ \frac{\partial F}{\partial z} = \bar{u} x_m z^{r_m-1} y_2^{s_m-1} + \Lambda_1 z \]
and
\[ \frac{\partial F}{\partial y_2} = s_m \bar{u} y_2^{s_m-1} z^{r_m} + \Lambda_2 z \]
for some $\Lambda_1, \Lambda_2 \in \hat{O}_{U_2,q}$, to see that either $\bar{u}x_m \neq 0$ and $s_m = 1$, or $\bar{u}x_m \neq 0$ and $s_m = 1$.

Let $q$ be one of these points, and let $\varphi_3 : T_3 \to T_2$ be the blow up of $\Lambda_2(q)$. We then have that the conclusions of the lemma hold in the form (18) at the 2-point which has permissible parameters $x_3, y_3, z$ defined by $x_2 = x_3 y_3$ and $y_2 = y_3$. At a 1-point which has permissible parameters $x_3, y_3, z$ defined by $x_2 = x_3 y_3$, we have that a form (19) holds. Thus the only case where we may possibly have not achieved the conclusions of the lemma is at the 1-point which has permissible parameters $x_3, y_3, z$ defined by $x_2 = x_3$ and $y_2 = x_3 y_3$. We continue to blow up, so that there is at most one point where the conclusions of the lemma do not hold. This point is a 1-point, which has permissible parameters $x_3, y_3, z$ where $x_2 = x_3$ and $y_2 = x_3 y_3$ where we can take $n$ as large as we like. We thus have a form
\begin{equation}
(22) \quad u = x_3^a, v = P(x_3) + x_3^b F_3
\end{equation}
with $F_3 = \tau z^m + \bar{b}_m x_3^r z^{m-2} + \cdots + \bar{b}_{m-1} x_3^{r_m-2} z + x_3^{r_m} \Omega$, where either $\bar{b}_i(x_3, y_3)$ is a unit or is zero, $\bar{b}_{m-1} \neq 0$, and $t > \omega(m, r_2, \ldots, r_m-1)$ if $\bar{u}x_m \neq 0$ and $s_m = 1$ which is of the form of (20), or we have a form (19) (after replacing $y_3$ with $y_3$ times a unit series in $x_3$ and $y_3$) if $\bar{u}x_m \neq 0$ and $s_m = 1$.

\[ \Box \]

**Lemma 3.7.** Suppose that $X$ is 2-prepared with respect to $f : X \to S$. Suppose that $p \in D$ is a 1-point with $\sigma_D(p) > 0$. Let $m = \sigma_D(p) + 1$. Let $x, y, z$ be permissible parameters for $D$ at $p$ such that a form (9) holds at $p$.

Let notation be as in Lemma 3.6. For $p_1 \in \psi_i^{-1}(p)$ let $\bar{\sigma}(p_1) = m + 1 + r_m$, if a form (19) holds at $p_1$, and
\[ \bar{\sigma}(p_1) = \begin{cases} \max\{m + 1 + r_m, m + 1 + s_m\} & \text{if } \bar{u}x_m = 1 \\ \max\{m + 1 + r_m - 1, m + 1 + s_m + 1\} & \text{if } \bar{u}x_m = 0 \end{cases} \]
if a form (18) holds at $p_1$. Let $\bar{\sigma}(p_1) = m + 1 + r_m - 1$ if a form (20) holds at $p_1$.

Let $r' = \max\{\bar{\sigma}(p_1) | p_1 \in \psi_i^{-1}(p)\}$. Let
\[ r = r(p) = m + 1 + r' \]
if $\sigma(p) \equiv 1 \mod m_i^\ast \hat{O}_{X,p}$ is such that $x = \gamma x^*$ for some unit $\gamma \in \hat{O}_{X,p}$ with $\gamma \equiv 1 \mod m_i^\ast \hat{O}_{X,p}$.

Let $V$ be an affine neighborhood of $p$ such that $x^*,y \in \Gamma(V, \mathcal{O}_X)$, and let $C^*$ be the curve in $V$ which has local equations $x^* = y = 0$ at $p$.

Let $T_0^* = \text{Spec}(k[x^*, y])$. Then there exists a sequence of blow ups of points $T_1^* \to T_0^*$ above $(x^*, y)$ such that if $V_1 = V \times_{T_0^*} T_1^*$ and $\psi_1^* : V_1 \to V$ is the induced sequence of blow ups of sections over $C^*$, $\Lambda_1^* : V_1 \to T_1^*$ is the projection, then $V_1$ is 2-prepared at all $p_i^* \in (\psi_1^*)^{-1}(p)$. Further, for every point $p_i^* \in (\psi_1^*)^{-1}(p)$, there exist $\bar{x}_1, \bar{y}_1 \in \mathcal{O}_{V_1, p_i^*}$ such that $\bar{x}_1, \bar{y}_1, z$ are permissible parameters at $p_i^*$ and we have one of the following forms:

1. $p_i^*$ is a 2-point, and we have an expression (2) with
\begin{equation}
(24) \quad F = \bar{\tau}_0 z^m + \bar{\tau}_2 \bar{x}_1 \bar{y}_1 z^{m-2} + \cdots + \bar{\tau}_{m-1} \bar{x}_1^{m-1} \bar{y}_1^{s_m-1} z + \bar{\tau}_m \bar{x}_1^{m} \bar{y}_1^{s_m}
\end{equation}
where \( \tau_0 \in \mathcal{O}_{V_1, p_1^*} \) is a unit, \( \tau_i \in \mathcal{O}_{V_1, p_1^*} \) are units (or zero) for \( 0 \leq i \leq m - 1 \), \( \tau_m \) is zero or 1, \( \tau_{m-1} \neq 0 \) if \( \tau_m = 0 \), \( r_i + s_i > 0 \) if \( \tau_i \neq 0 \), and
\[
(r_m + c)b - (s_m + d)a \neq 0.
\]

2) \( p_1^* \) is a 1-point, and we have an expression (1) with
\[
F = \tau_0 z^m + \tau_2 z^{m-2} z + \cdots + \tau_m z^{m-1} + x \tau_1
\]
where \( \tau_0 \in \mathcal{O}_{V_1, p_1^*} \) is a unit, \( \tau_i \in \mathcal{O}_{V_1, p_1^*} \) are units (or zero), and \( \text{ord}(\tau_m, y_1, 0) = 1 \). Further, \( r_i > 0 \) if \( \tau_i \neq 0 \).

3) \( p_1^* \) is a 1-point, and we have an expression (1) with
\[
F = \tau_0 z^m + \tau_2 z^{m-2} z + \cdots + \tau_m z^{m-1} + x \tau_1
\]
where \( \tau_0 \in \mathcal{O}_{V_1, p_1^*} \) is a unit, \( \tau_i \in \mathcal{O}_{V_1, p_1^*} \) are units (or zero), \( \tau \in \mathcal{O}_{V_1, p_1^*} \), \( \tau_{m-1} \neq 0 \) and \( t > \omega(m, r_2, \ldots, r_{m-1}) \). Further, \( r_i > 0 \) if \( \tau_i \neq 0 \).

Proof. The isomorphism \( T_0^* \rightarrow T_0 \) obtained by substitution of \( x^* \) for \( x \) and subsequent base change by the morphism \( T_1 \rightarrow T_0 \) of Lemma 3.6, induces a sequence of blow ups of points \( T_1 \rightarrow T_0 \). The base change \( \psi_1 : V_1 = V \times_{T_0^*} T_1 \rightarrow V \cong V \times_{T_0^*} T_0 \) factors as a sequence of blow ups of sections over \( C^* \). Let \( \Lambda_1 : V_1 \rightarrow T_1 \) be the natural projection.

Let \( p_1^* \in (\psi_1^*)^{-1}(p) \), and let \( p_1 \in \psi_1^{-1}(p) \subset U_1 \) be the corresponding point.

First suppose that \( p_1 \) has a form (19). With the notation of Lemma 3.6, we have polynomials \( \varphi, \psi \) such that
\[
x = \varphi(\bar{x}, \bar{y}), y = \psi(\bar{x}, \bar{y})
\]
determines the birational extension \( \mathcal{O}_{T_0, p_0} \rightarrow \mathcal{O}_{T_1, \Lambda_1(p_1)} \), and we have a formal change of variables
\[
x_1 = \alpha(\bar{x}, \bar{y}) \bar{x}, y_1 = \beta(\bar{x}, \bar{y})
\]
for some unit series \( \alpha \) and series \( \beta \). We further have expansions
\[
a_i(x, y) = x_i \bar{a}_i(x, y_1)
\]
for \( 2 \leq i \leq m - 1 \) where \( \bar{a}_i(x, y_1) \) are unit series or zero, and
\[
am(x, y) = x_1 y_1.
\]

We have \( x = \bar{\gamma} x^* \) with \( \bar{\gamma} \equiv 1 \mod m^*_p \mathcal{O}_{X, p} \). Set \( y^* = y \). At \( p_1 \), we have regular parameters \( x_1^*, y_1^* \) in \( \mathcal{O}_{T_1, \Lambda_1(p_1)} \) such that
\[
x^* = \varphi(x_1^*, y_1^*), y^* = \psi(x_1^*, y_1^*)
\]
and \( x_1^*, y_1^*, \bar{x} \) are regular parameters in \( \mathcal{O}_{V_1, p_1^*} \) (recall that \( z = \sigma \bar{x} \) in Lemma 3.1). We have regular parameters \( \bar{x}_1, \bar{y}_1, \in \mathcal{O}_{T_1, \Lambda_1(p_1)} \) defined by
\[
\bar{x}_1 = \alpha(x_1^*, y_1^*) x_1^*, \bar{y}_1 = \beta(x_1^*, y_1^*)
\]
We use \( u = x^d = x_1^{a_1} \) where \( a_1 = ad \) for some \( d \in \mathbb{Z}_+ \). Since \( \alpha(\bar{x}_1, \bar{y}_1) \bar{x}_1 \) we have that \( \alpha(x_1^*, y_1^*) x_1^* \) is a unit in \( \mathcal{O}_{V_1, p_1^*} \), and \( \bar{x}_1 = \bar{x}_1^d \). Thus \( x = \bar{x}_1 \) (with an appropriate choice of root \( \bar{\gamma} \)). We have \( u = x_1^{a_1} \), so that \( \bar{x}_1, \bar{y}_1, \bar{z} \) are permissible parameters at \( p_1^* \).

For \( 2 \leq i \leq m - 1 \), we have
\[
a_i(x, y) = a_i(\bar{\gamma} x^*, y^*) \equiv a_i(x^*, y^*) \mod m^*_p \mathcal{O}_{V, p}.
\]
and
\[ a_i(x^*, y^*) = a_i(\varphi(x_1^*, y_1^*), \psi(x_1^*, y_1^*)) \]
\[ = x_1^r \tau \psi(x_1, y_1) \equiv x_1^r \psi(x_1, y_1) \mod m_p^r \mathcal{O}_{V_1, p_1^*}. \]

We further have
\[ a_m(x^*, y^*) \equiv x_1^{r_m} y_1 \mod m_p^r \mathcal{O}_{V_1, p_1^*}. \]

Thus we have expressions
\begin{equation}
\begin{aligned}
 u & = x_1^{da} \\
v & = P(x_1^d) + x_1^{bd} P_1(x_1) + x_1^{bd}(\tau z^m + x_1^{s_2(1, y_1)} z^{m-2} + \cdots + x_1^{r_m} y_1 + h)
\end{aligned}
\end{equation}

where \( \tau \in \mathcal{O}_{V_1, p_1^*} \) is a unit series and
\[ h \in m_p^r \mathcal{O}_{V_1, p_1^*} \subset (x_1, z)^r. \]

Set \( s = r - m \), and write
\[ h = z^m \lambda_0(x_1, y_1, z) + z^{m-1} x_1^{1+s} \lambda_1(x_1, y_1) + z^{m-2} x_1^{2+s} \lambda_2(x_1, y_1) + \cdots + z x_1^{(m-1)+s} \lambda_{m-1}(x_1, y_1) + x_1^{m+s} \lambda_m(x_1, y_1) \]
with \( \lambda_0 \in m_{p_1^*} \mathcal{O}_{V_1, p_1^*} \) and \( \Lambda_i \in \mathfrak{t}[x_1, y_1] \) for \( 1 \leq i \leq m \).

Substituting into (27), we obtain an expression
\begin{equation}
\begin{aligned}
 u & = x_1^{da} \\
v & = P(x_1^d) + x_1^{bd} P_1(x_1) + x_1^{bd}(\tau_0 z^m + x_1^{s_2(1, y_1)} z^{m-2} + \cdots + x_1^{r_m} \tau_{m-1} z + x_1^{r_m} \tau_m)
\end{aligned}
\end{equation}

where \( \tau_0 \in \mathcal{O}_{V_1, p_1^*} \) is a unit, \( \tau_i \in \mathcal{O}_{V_1, p_1^*} \) are units (or zero), for \( 1 \leq i \leq m-1 \) and \( \tau_m \in \mathfrak{t}[x_1, y_1] \) with \( \text{ord}(\tau_m(0, y_1)) = 1 \).

We have \( \tau_0 = \tau + \Lambda_0, \tau_i = \Omega_i(x_1, y_1) \) for \( 2 \leq i \leq m-1 \), and
\[ \tau_m = y_1 + z^{m-1} x_1^{1+s-r_m} \Lambda_1(x_1, y_1) + \cdots + x_1^{m+s-r_m} \Lambda_m(x_1, y_1). \]

We thus have the desired form (25).

In the case when \( p_1 \) has a form (20), a similar argument to the analysis of (19) shows that \( p_1^* \) has a form (26).

Now suppose that \( p_1 \) has a form (18). We then have
\begin{equation}
m_p \mathcal{O}_{U_1, p_1} \subset (x_1, y_1, z) \mathcal{O}_{U_1, p_1},
\end{equation}
unless there exist regular parameters \( x_1^*, y_1^* \in \mathcal{O}_{T_1, \Lambda_1(p_1)} \) such that \( x_1^*, y_1^* \) are regular parameters in \( \mathcal{O}_{U_1, p_1} \) and
\begin{equation}
x = x_1^*, y = (x_1^*)^n y_1^*
\end{equation}
or
\begin{equation}
x = x_1^*(y_1^*)^n, y = y_1^*
\end{equation}
for some \( n \in \mathbb{N} \). If (29) or (30) holds, then \( \mathcal{O}_{V_1, p_1^*} = \mathcal{O}_{U_1, p_1} \), and (taking \( \hat{x}_1 = x_1, \hat{y}_1 = y_1 \)) we have that a form (24) holds at \( p_1^* \). We may thus assume that (28) holds.

With the notation of Lemma 3.6, we have polynomials \( \varphi, \psi \) such that
\[ x = \varphi(\hat{x}_1, \hat{y}_1), y = \psi(\hat{x}_1, \hat{y}_1) \]
determines the birational extension \( \mathcal{O}_{T_0, p_0} \to \mathcal{O}_{T_1, \Lambda_1(p_1)} \), and we have a formal change of variables
\[ x_1 = \alpha(\hat{x}_1, \hat{y}_1) \hat{x}_1, y_1 = \beta(\hat{x}_1, \hat{y}_1) \hat{y}_1 \]
for some unit series $\alpha$ and $\beta$. We further have expansions
\[ a_i(x, y) = x_1^i y_1^i \alpha_i(x_1, y_1) \]
for $2 \leq i \leq m - 1$ where $\alpha_i(x_1, y_1)$ are unit series or zero, and
\[ a_m(x, y) = x_1^m y_1^m \alpha_m, \]
where $\alpha_m = 0$ or 1. We have $x = \overline{\tau} x^* \mod \overline{\tau} \equiv 1 \mod m_p^r \tilde{O}_{X, p}$. Set $y^* = y$. At $p_1^r$, we have regular parameters $x_1^*, y_1^*$ in $O_{T_1^*, A_1^*}$ such that
\[ x^* = \varphi(x_1^*, y_1^*), \quad y^* = \psi(x_1^*, y_1^*), \]
and $x_1^*, y_1^*$ are regular parameters in $O_{V, p_1^r}$ (recall that $z = \sigma \tilde{z}$ in Lemma 3.1). We have regular parameters $\overline{x_1}, \overline{y_1} \in \tilde{O}_{T_1^*, A_1^*}$ defined by
\[ \overline{x_1} = \alpha(x_1^*, y_1^*) x_1^*, \quad \overline{y_1} = \beta(x_1^*, y_1^*) y_1^*. \]
We calculate
\[ u = x^a = (x_1^a y_1^b)^t_1 = [\alpha(\hat{x}_1, \hat{y}_1)]^{a_1 t_1} [\beta(\hat{x}_1, \hat{y}_1)]^{b_1 t_1} \]
which implies
\[ (x^*)^a = [\alpha(x_1^*, y_1^*) x_1^a y_1^b]^{a_1 t_1} [\beta(x_1^*, y_1^*) y_1^b]^{b_1 t_1} = \overline{x}_1^{a_1 t_1} \overline{y}_1^{b_1 t_1}. \]
Set $\hat{x}_1 = \overline{\overline{x}_1} \overline{x}_1$ to get $u = (\hat{x}_1^a \overline{y}_1^b)^t_1$, so that $\hat{x}_1, \overline{y}_1, \tilde{z}$ are permissible parameters at $p_1^r$.

For $2 \leq i \leq m$, we have
\[ a_i(x, y) = a_i(\overline{\tau} x^*, y^*) \equiv a_i(x^*, y^*) \mod m_p^r \tilde{O}_{V, p} \]
and
\[ a_i(x^*, y^*) = a_i(\varphi(x_1^*, y_1^*), \psi(x_1^*, y_1^*)) = \overline{x}_1^i \overline{y}_1^i \alpha_i(\overline{x}_1, \overline{y}_1) \equiv \hat{x}_1^i \overline{y}_1^i \alpha_i(\overline{x}_1, \overline{y}_1) \mod m_p^r \tilde{O}_{V, p_1^r}. \]

Thus we have expressions
\[
\begin{align*}
\overline{u} &= (\hat{x}_1^a \overline{y}_1^b)^t_1, \\
\overline{v} &= P((\hat{x}_1^a \overline{y}_1^b)^t_1) + (\hat{x}_1^a \overline{y}_1^b)^t_1 P_1(\hat{x}_1^a \overline{y}_1^b) + (\hat{x}_1^a \overline{y}_1^b)^t_1 \overline{P}_1(\overline{\tau} z^m) \\
&\quad + \hat{x}_1^a \overline{y}_1^b \overline{\alpha}(\hat{x}_1, \hat{y}_1) z^{m-2} + \cdots + \hat{x}_1^{x_m^r} \overline{y}_1^{y_m^r} \overline{\alpha}(\hat{x}_1, \hat{y}_1) + h)
\end{align*}
\]
where $\overline{P} \in \tilde{O}_{V_1, p_1^r}$ is a unit series and
\[ h \in m_p^r \tilde{O}_{V_1, p_1^r} \subset (\hat{x}_1^a \overline{y}_1^b)^r. \]

Set $s = r - m$, and write
\[
\begin{align*}
h &= z^m \Lambda_0(x_1, y_1, z) + z^{m-1}(\hat{x}_1 \overline{y}_1) \Lambda_1(\hat{x}_1, \overline{y}_1) + z^{m-2}(\hat{x}_1 \overline{y}_1)^2 \Lambda_2(\hat{x}_1, \overline{y}_1) + \cdots \\
&\quad + z(\hat{x}_1 \overline{y}_1)^{(m-1)+s} \Lambda_{m-1}(\hat{x}_1, \overline{y}_1) + (\hat{x}_1 \overline{y}_1)^{m+s} \Lambda_m(\hat{x}_1, \overline{y}_1)
\end{align*}
\]
with $\Lambda_0 \in m_p^r \tilde{O}_{V_1, p_1^r}$ and $\Lambda_i \in \mathfrak{t}[[\hat{x}_1, \overline{y}_1]]$ for $1 \leq i \leq m$.

First suppose that $\overline{\alpha}_m = 1$. Substituting into (31), we obtain an expression
\[
\begin{align*}
\overline{u} &= (\hat{x}_1^a \overline{y}_1^b)^t_1, \\
\overline{v} &= P((\hat{x}_1^a \overline{y}_1^b)^t_1) + (\hat{x}_1^a \overline{y}_1^b)^t_1 P_1(\hat{x}_1^a \overline{y}_1^b) \\
&\quad + (\hat{x}_1^a \overline{y}_1^b)^t_1 \overline{P}_1(\overline{\tau} z^m) \\
&\quad + \hat{x}_1^a \overline{y}_1^b \overline{\alpha}(\hat{x}_1, \hat{y}_1) \overline{y}_1^{m-2} + \cdots + \hat{x}_1^{x_m^r} \overline{y}_1^{y_m^r} \overline{\alpha}(\hat{x}_1, \hat{y}_1)
\end{align*}
\]
where $\overline{\tau}_0, \overline{\tau}_m \in \tilde{O}_{V_1, p_1^r}$ are units, $\overline{\tau}_i \in \tilde{O}_{V_1, p_1^r}$ are units (or zero) for $2 \leq i \leq m - 1$.
We have $\overline{\tau}_0 = \overline{\tau} + \Lambda_0, \overline{\tau}_i = \overline{\alpha}_i(\hat{x}_1, \overline{y}_1)$ for $2 \leq i \leq m - 1$, and
\[ \overline{\tau}_m = \overline{\alpha}_m + \hat{x}_1^{m-1} \overline{y}_1^{m-s} \Lambda_1(\hat{x}_1, \overline{y}_1) + \cdots + \hat{x}_1^{m+s} \overline{y}_1^{m-s} \Lambda_m(\hat{x}_1, \overline{y}_1). \]
We thus have the desired form (24).

Now suppose that \( \overline{a}_m = 0 \). Then \( \overline{a}_{m-1} \neq 0 \), and \( z \) divides \( h \) in (31), so that \( \Lambda_m = 0 \) in (32). Substituting into (31), we obtain an expression

\[
\begin{align*}
  u &= (x_1^{a_1} \bar{y}_1^{b_1})^{t_1} \\
  v &= P((x_1^{a_1} \bar{y}_1^{b_1} + x_1^{a_1} \bar{y}_1^{b_1}) 1^{(a_1 + b_1) + x_1^{a_1} \bar{y}_1^{b_1}} + x_1^{a_1} y_1^{b_1}) 1^{(1 + b_1) + x_1^{a_1} \bar{y}_1^{b_1} + y_1^{b_1}} \\
  &\quad + x_1^{a_1} y_1^{b_1}) 1^{(\tau_0 z^m + x_1^{a_1} \bar{y}_1^{b_1} \tau_2 z^{m-2} + \cdots + x_1^{a_1} \bar{y}_1^{b_1} \tau_{m-1} z^{m-1})}
\end{align*}
\]

where \( \tau_0, \tau_{m-1} \in \bar{O}_{V_1, p_1^0} \) are units, \( \tau_i \in \bar{O}_{V_1, p_1^i} \) are units (or zero) for \( 2 \leq i \leq m - 2 \).

We have \( \tau_0 = \tau = \Lambda_0, \tau_i = \tau_i(\hat{x}_1, \hat{y}_1) \) for \( 2 \leq i \leq m - 2 \), and

\[
\tau_{m-1} = \overline{a}_{m-1} + z^{m-1} \hat{x}_1^{1 + s - r_m - 1} \hat{y}_1^{1 + s - s_m - 1} \Lambda_1(\hat{x}_1, \hat{y}_1) + \cdots + \hat{x}_1^{m-1 + s - r_m - 1} \hat{y}_1^{m-1 + s - s_m - 1} \Lambda_{m-1}(\hat{x}_1, \hat{y}_1).
\]

We thus have the form (24).

\[\square\]

**Lemma 3.8.** Suppose that \( X \) is \( 2 \)-prepared. Suppose that \( p \in X \) is a 1-point with \( \sigma_p(p) > 0 \) and \( E \) is the component of \( D \) containing \( p \). Suppose that \( Y \) is a finite set of points in \( X \) (not containing \( p \)). Then there exists an affine neighborhood \( U \) of \( p \) in \( X \) such that

1. \( Y \cap U = \emptyset \).
2. \( [E - U \cap E] \cap \text{Sing}_1(X) \) is a finite set of points.
3. \( U \cap D = U \cap E \) and there exists \( \bar{\pi} \in \Gamma(U, \mathcal{O}_X) \) such that \( \bar{\pi} = 0 \) is a local equation of \( E \) in \( U \).
4. There exists an étale map \( \pi : U \to \mathbb{A}_k^3 = \text{Spec}(\mathcal{O}[\bar{\pi}, \bar{y}, \bar{z}]) \).
5. The Zariski closure \( C \) in \( X \) of the curve in \( U \) with local equations \( \bar{\pi} = \bar{y} = 0 \) satisfies the following:
   i) \( C \) is a nonsingular curve through \( p \).
   ii) \( C \) contains no 3-points of \( D \).
   iii) \( C \) intersects 2-curves of \( D \) transversally at prepared points.
   iv) \( C \cap \text{Sing}_1(X) \cap (X - U) = \emptyset \).
   v) \( C \cap Y = \emptyset \).
   vi) \( C \) intersects \( \text{Sing}_1(X) - \{p\} \) transversally at general points of curves in \( \text{Sing}_1(X) \).
   vii) There exist permissible parameters \( x, y, z \) at \( p \), with \( \hat{x} = \bar{\pi}, \hat{y} = \bar{y} = 0 \), which satisfy the hypotheses of lemma 3.1.

**Proof.** Let \( H \) be an effective, very ample divisor on \( X \) such that \( H \) contains \( Y \) and \( D - E \), but \( H \) does not contain \( p \) and does not contain any one dimensional components of \( \text{Sing}_1(X, D) \cap E \). There exists \( n > 0 \) such that \( E + nH \) is ample, \( \mathcal{O}_X(E + nH) \) is generated by global sections and a general member \( H' \) of the linear system \( |E + nH| \) does not contain any one dimensional components of \( \text{Sing}_1(X, D) \cap E \), and does not contain \( p \). \( H + H' \) is ample, so \( V = X - (H + H') \) is affine. Further, there exists \( f \in \mathcal{O}(X) \), the function field of \( X \), such that \( f = H' - (E + nH) \). Thus \( \bar{\pi} = \frac{1}{f} \in \Gamma(V, \mathcal{O}_X) \) as \( X \) is normal and \( \bar{\pi} \) has no poles on \( V \). \( \bar{\pi} = 0 \) is a local equation of \( E \) on \( V \). We have that \( V \) satisfies the conclusions 1), 2) and 3) of the lemma.

Let \( R = \Gamma(V, \mathcal{O}_X) \). \( R = \bigcup_{s \in \mathbb{Z}_{\geq 1}} \Gamma(X, \mathcal{O}_X(s(H + H'))) \) is a finitely generated \( \mathcal{K} \)-algebra. Thus for \( s \gg 0 \), \( R \) is generated by \( \Gamma(X, \mathcal{O}_X(s(H + H'))) \) as a \( \mathcal{K} \)-algebra.

From the exact sequences

\[
0 \to \Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p) \to \Gamma(X, \mathcal{O}_X(s(H + H'))) \to \mathcal{O}_{X,p}/m_p \cong k
\]

and the fact that \( 1 \in \Gamma(X, \mathcal{O}_X(s(H + H'))) \), we have that \( R \) is generated by \( \Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p) \) as a \( \mathcal{K} \)-algebra for all \( s \gg 0 \).
For $s \gg 0$, and a general member $\sigma$ of $\Gamma(X, \mathcal{O}_X(s(H + H'))) \otimes \mathcal{I}_p$ we have that the curve $C = B \cdot E$, where $B$ is the divisor $B = (\sigma) + s(H + H')$, satisfies the conclusions of 5 of the lemma; since each of the conditions 5i) through 5vii) is an open condition on $\Gamma(X, \mathcal{O}_X(s(H + H') \otimes \mathcal{I}_p))$, we need only establish that each condition holds on a nonempty subset. This follows from the fact that $H + H'$ is ample, Bertini’s theorem applied to the base point free linear system $|\varphi^* (s(H + H')) - A|$, where $\varphi : W \to X$ is the blow up of $p$ with exceptional divisor $A$, and the fact that

$$\varphi^* (\mathcal{O}_W (\varphi^* (s(H + H') - A)) = \mathcal{O}_X (s(H + H')) \otimes \mathcal{I}_p.$$ 

For $s \gg 0$, let $\mathfrak{x}, \mathfrak{y}_1, \ldots, \mathfrak{y}_n$ be a $\mathfrak{t}$-basis of $\Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p)$, so that $R = \mathfrak{t}[\mathfrak{x}, \mathfrak{y}_1, \ldots, \mathfrak{y}_n]$. We have shown that there exists a Zariski open set $Z$ of $k^n$ such that for $(b_1, \ldots, b_n) \in Z$, the curve $C$ in $X$ which is the Zariski closure of the curve with local equation $\mathfrak{x} = b_1 \mathfrak{y}_1 + \cdots + b_n \mathfrak{y}_n = 0$ in $V$ satisfies 5 of the conclusions of the lemma.

Let $C_1, \ldots, C_t$ be the curves in $\text{Sing}(X) \cap V$, and let $p_i \in C_i$ be closed points such that $p, p_1, \ldots, p_t$ are distinct. Let $Q_0$ be the maximal ideal of $p$ in $R$, and $Q_i$ be the maximal ideal in $R$ of $p_i$ for $1 \leq i \leq t$. We have that $\mathfrak{x}$ is nonzero in $Q_i/Q_i^2$ for all $i$. For a matrix $A = (a_{ij}) \in \mathfrak{t}^{2n}$, and $1 \leq i \leq 2$, let

$$L^A_i (\mathfrak{y}_1, \ldots, \mathfrak{y}_n) = \sum_{j=1}^n a_{ij} \mathfrak{y}_j.$$ 

There exist $\alpha_{jk} \in \mathfrak{t}$ such that $Q_k = (\mathfrak{y}_1 - \alpha_{1,k}, \ldots, \mathfrak{y}_n - \alpha_{n,k})$ for $0 \leq k \leq t$. By our construction, we have $\alpha_{1,0} = \cdots = \alpha_{n,0} = 0$. For each $0 \leq k \leq t$, there exists a non empty Zariski open subset $Z_k$ of $k^{2n}$ such that

$$\mathfrak{x}, L^A_1 (\mathfrak{y}_1, \ldots, \mathfrak{y}_n) - L^A_1 (\alpha_{1,1}, \ldots, \alpha_{n,1}), L^A_2 (\mathfrak{y}_1, \ldots, \mathfrak{y}_n) - L^A_2 (\alpha_{1,2}, \ldots, \alpha_{n,2})$$

is a $\mathfrak{t}$-basis of $Q_k/Q_{k+1}^2$. Suppose $(a_{1,1}, \ldots, a_{1,n}) \in Z$ and $A \in Z_0 \cap \cdots \cap Z_t$.

We will show that $\mathfrak{x}, L^A_1, L^A_2$ are algebraically independent over $\mathfrak{t}$. Suppose not. Then there exists a nonzero polynomial $h \in \mathfrak{t}[t_1, t_2, t_3]$ such that $h(\mathfrak{x}, L^A_1, L^A_2) = 0$. Write $h = H + h'$ where $H$ is the leading form polynomial of $h$, and $h' = h - H$ is a polynomial of larger order than the degree of $H$. Now $H(\mathfrak{x}, L^A_1, L^A_2) = -h'(\mathfrak{x}, L^A_1, L^A_2)$, so that $H(\mathfrak{x}, L^A_1, L^A_2) = 0$ in $Q_0^{2}/Q_0^{r+1}$. Thus $H = 0$, since $R_{Q_0}$ is a regular local ring, which is a contradiction. Thus $\mathfrak{x}, L^A_1, L^A_2$ are algebraically independent. Without loss of generality, we may assume that $L^A_i = \mathfrak{y}_i$ for $1 \leq i \leq 2$.

Let $S = \mathfrak{t}[\mathfrak{x}, \mathfrak{y}_1, \mathfrak{y}_2]$; a polynomial ring in 3 variables over $\mathfrak{t}$. $S \to R$ is unramified at $Q_i$ for $0 \leq i \leq t$ since

$$(\mathfrak{x}, \mathfrak{y}_1 - \alpha_{1,i}, \mathfrak{y}_2 - \alpha_{2,i}) R_{Q_i} = Q_i R_{Q_i}$$ 

for $0 \leq i \leq t$.

Let $W$ be the closed locus in $V$ where $V \to \text{Spec}(S)$ is not étale. We have that $p, p_1, \ldots, p_t \not\in W$, so there exists an ample effective divisor $H'$ on $X$ such that $W \subset H'$ and $p, p_1, \ldots, p_t \not\subset H'$. Let $U = V - H'$. $U$ is affine, and $U \to \text{Spec}(S) \cong A^3$ is étale, so satisfies 4 of the conclusions of the lemma.

□

**Lemma 3.9.** Suppose $X$ is 2-prepared with respect to $f : X \to S$, $p \in D$ is a prepared point, and $\pi_1 : X_1 \to X$ is the blow up of $p$. Then all points of $\pi^{-1}_1(p)$ are prepared.

**Proof.** The conclusions follow from substitution of local equations of the blow up of a point into a prepared form (1), (2) or (3). □
Lemma 3.10. Suppose that $X$ is 2-prepared with respect to $f : X \to S$, and that $C$ is a permissible curve for $D$, which is not a 2-curve. Suppose that $p \in C$ satisfies $\sigma_D(p) = 0$. Then there exist permissible parameters $x,y,z$ at $p$ such that one of the following forms hold:

1) $p$ is a 1-point of $D$ of the form of (1), $F = z$ and $x = y = 0$ are formal local equations of $C$ at $p$.
2) $p$ is a 1-point of $D$ of the form of (1), $F = z$ and $x = z = 0$ are formal local equations of $C$ at $p$.
3) $p$ is a 1-point of $D$ of the form of (1), $F = z$, $x = z + y^r \sigma(y) = 0$ are formal local equations of $C$ at $p$, where $r > 1$ and $\sigma$ is a unit series.
4) $p$ is a 2-point of $D$ of the form of (2), $F = z$, $x = z = 0$ are formal local equations of $C$ at $p$.
5) $p$ is a 2-point of $D$ of the form of (2), $F = z$, $x = f(y,z) = 0$ are formal local equations of $C$ at $p$, where $f(y,z)$ is not divisible by $z$.
6) $p$ is a 2-point of $D$ of the form of (2), $F = 1$ (so that $ad - bc \neq 0$) and $x = z = 0$ are formal local equations of $C$ at $p$.

Further, there are at most a finite number of 1-points on $C$ satisfying condition 3) (and not satisfying condition 1) or 2)).

Proof. Suppose that $p$ is a 1-point. We have permissible parameters $x,y,z$ at $p$ such that a form (1) holds at $p$ with $F = z$. There exists a series $f(y,z)$ such that $x = f = 0$ are formal local equations of $C$ at $p$. By the formal implicit function theorem, we get one of the forms 1), 2) or 3). A similar argument shows that one of the forms 4), 5) or 6) must hold if $p$ is a 2-point.

Now suppose that $p \in C$ is a 1-point, $\sigma_D(p) = 0$ and a form 3) holds at $p$. There exist permissible parameters $x,y,z$ at $p$, with an expression (1), such that $x = z = 0$ are formal local equations of $C$ at $p$ and $x,y,z$ are uniformizing parameters on an étale cover $U$ of an neighborhood of $p$, where we can choose $U$ so that

$$\frac{\partial F}{\partial y} = \frac{1}{y^r} \frac{\partial v}{\partial y} \in \Gamma(U, \mathcal{O}_X).$$

Since there is not a form 2) at $p$, we have that $z$ does not divide $F(0,y,z)$, so that $F(0,y,0) \neq 0$. Since $F$ has no constant term, we have that $\frac{\partial F}{\partial y}(0,y,0) \neq 0$. There exists a Zariski open subset of $\mathfrak{t}$ such that $\alpha \in \mathfrak{t}$ implies $x,y - \alpha,z$ are regular parameters at a point $q \in U$. There exists a Zariski open subset of $\mathfrak{t}$ of such $\alpha$ so that $\frac{\partial F}{\partial y}(0,\alpha,0) \neq 0$. Thus $x,y - \alpha,z$ are permissible parameters at $q$ giving a form 1) at $q \in C$.

Lemma 3.11. Suppose that $X$ is 2-prepared. Suppose that $C$ is a permissible curve on $X$ which is not a 2-curve and $p \in C$ satisfies $\sigma_D(p) = 0$. Further suppose that either a form 3) or 5) of the conclusions of Lemma 3.10 hold at $p$. Then there exists a sequence of blow ups of points $\pi_1 : X_1 \to X$ above $p$ such that $X_1$ is 2-prepared and $\sigma_D(p_1) = 0$ for all $p_1 \in \pi_1^{-1}(p)$, and the strict transform of $C$ on $X_1$ is permissible, and has the form 4) or 6) of Lemma 3.10 at the point above $p$.

Proof. If $p$ is a 1-point, let $\pi' : X' \to X$ be the blow ups of $p$, and let $C'$ be the strict transform of $C$ on $X'$. Let $p'$ be the point on $C'$ above $p$. Then $p'$ is a 2-point and $\sigma_D(p') = 0$. We may thus assume that $p$ is a 2-point and a form 5) holds at $p$. For $r \in \mathbb{Z}_+$, let

$$X_r \to X_{r-1} \to \cdots \to X_1 \to X.$$
be the sequence of blow ups of the point $p_i$ which is the intersection of the strict transform $C_i$ of $C$ on $X_i$ with the preimage of $p$.

There exist permissible parameters $x, y, z$ at $p$ such that $x = z = 0$ are formal local equations of $C$ at $p$, and a form (2) holds at $p$ with $F = x \Omega + f(y, z)$. We have that ord $f(y, z) = 1$, ord $\Omega(0, y, z) \geq 1$, $y$ does not divide $f(y, z)$ and $z$ does not divide $f(y, z)$.

At $p_r$, we have permissible parameters $x_r, y_r, z_r$ such that

$x = x_r y_r^r, \quad y = y_r, \quad z = z_r y_r^r. \quad x_r = z_r = 0$ are local equations of $C_r$ at $p_r$. We have a form (2) at $p_r$ with

$u = (x_r y_r^{ar+b})^l, \quad v = P(x_r y_r^{ar+b}) + x_r x_r^{cr+d+r} F'$

where

$F' = x_r \Omega + \frac{f(y_r, z_r y_r^r)}{y_r^l},$ 

if $f(y_r - 1, x_r^{a-1} y_r^{r-1})$ is not a unit series. Thus for $r$ sufficiently large, we have that $F'$ is a unit, so that a form 6) holds at $p_r$.

**Lemma 3.12.** Suppose that $X$ is 2-prepared and that $C_1$ is a permissible curve on $X$. Suppose that $q \in C$ is a point with $\sigma_D(q) = 0$ which has a form 1), 4) or 6) of Lemma 3.10. Let $\pi_1 : X_1 \to X$ be the blow up of $C$. Then $X_1$ is 3-prepared in a neighborhood of $\pi_1^{-1}(q)$. Further, $\sigma_{D_1}(q_1) = 0$ for all $q_1 \in \pi_1^{-1}(q)$.

**Proof.** The conclusions follow from substitution of local equations of the blow up of $C$ into the forms 1), 4) and 6) of Lemma 3.10.

**Proposition 3.13.** Suppose that $X$ is 2-prepared. Then there exists a sequence of permissible blow ups $\pi_1 : X_1 \to X$, such that $X_1$ is 3-prepared. We further have that $\sigma_D(p_1) \leq \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.

**Proof.** Let $T$ be the points $p \in X$ such that $X$ is not 3-prepared at $p$. By Lemmas 3.4 and 2.5, after we perform a sequence of blow ups of 2-curves, we may assume that $T$ is a finite set consisting of 1-points of $D$.

Suppose that $p \in T$. Let $T' = T \setminus \{p\}$. Let $U = \text{Spec}(R)$ be the affine neighborhood of $p$ in $X$ and let $C$ be the curve in $X$ such that $C$ is 3-prepared in a neighborhood of $\pi_1^{-1}(q)$. Then $X_1$ is 3-prepared in a neighborhood of $\pi_1^{-1}(q)$.

Let $\Sigma_1 = C \cap \text{Sing}_1(X)$. $\Sigma_1 = \{p = p_0, \ldots, p_r\}$ is the union of curves in $\text{Sing}_1(X)$, which must be 1-points. We have that $\Sigma_1 \subset U$. Let

$\Sigma_2 = \{q \in C \cap U \mid \sigma_D(q) = 0 \text{ and a form 2) of Lemma 3.10 holds at } q\}.$

$\Sigma_2$ is a finite set by Lemma 3.10. Let $\Sigma_3 = C \setminus U$, a finite set of 1-points and 2-points which are prepared.

Set $U' = U \setminus \Sigma_2$. There exists a unit $\tau \in R$ and $a \in \mathbb{Z}_+$ such that $u = \tau x^a$.

By 5 vi), 5 vii) of Lemma 3.8 and Lemma 3.2, there exist $z_i \in \hat{O}_{X, p_i}$ such that for all $p_i \in \Sigma_1$, $x = \tau^{\frac{1}{2}} \pi, \gamma, z_i$ are permissible parameters at $p_i$ giving a form (9).

Let $t = \max\{r(p_i) \mid 0 \leq i \leq r\}$, where $r(p_i)$ are calculated from (23)) of Lemma 3.7. There exists $\lambda \in R$ such that $\lambda \equiv \tau^{\frac{1}{2}} \pi \mod m_{p_i} \hat{O}_{X, p_i}$ for $0 \leq i \leq r$. Let $x^* = \lambda^{-1} \pi, \gamma = \tau^{\frac{1}{2}} \lambda$. Then $x = \tau^{\frac{1}{2}} \pi = \gamma x^*$ with $\gamma = 1 \mod m_{p_i} \hat{O}_{X, p_i}$ for $0 \leq i \leq r$. Let $U' = U \setminus \Sigma_2$.
Let $T^*_0 = \text{Spec}(k[x^*, \overline{y}])$, and let $T^*_1 \to T^*_0$ be a sequence of blow ups of points above $(x^*, \overline{y})$ such that the conclusions of Lemma 3.7 hold on $U_1' = U' \times_{T^*_0} T^*_1$ above all $p_i$ with $0 \leq i \leq r$. The projection $\lambda_1 : U'_1 \to U'$ is a sequence of blow ups of sections over $C$. $\lambda_1$ is permissible and $\lambda_1^{-1}(C \cap (U' \setminus \Sigma_1))$ is prepared by Lemma 3.12.

All points of $\Sigma_2 \cup \Sigma_3$ are prepared. Thus by Lemma 3.9, Lemmas 3.11 and Lemma 3.12, by interchanging some blowups of points above $\Sigma_2 \cup \Sigma_4$ between blow ups of sections over $C$, we may extend $\lambda_1$ to a sequence of permissible blow ups over $X$ to obtain the desired sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that $X_1$ is 2-prepared. $\pi_1$ is an isomorphism over $T'$, $X_1$ is 3-prepared over $\pi_1^{-1}(X_1 \setminus T')$, and $\sigma_D(p_1) \leq \sigma_D(p)$ for all $p \in X_1 \setminus T'$.

By induction on $|T|$, we may iterate this procedure a finite number of times to obtain the conclusions of Proposition 3.13.

\[ \square \]

The following proposition is proven in a similar way.

**Proposition 3.14.** Suppose that $X$ is 1-prepared and $D'$ is a union of irreducible components of $D$. Suppose that there exists a neighborhood $V$ of $D'$ such that $V$ is 2-prepared and $V$ is 3-prepared at all 2-points and 3-points of $V$.

Let $A$ be a finite set of 1-points of $D'$, such that $A$ is contained in $\text{Sing}_1(X)$ and $A$ contains the points where $V$ is not 3-prepared, and let $B$ be a finite set of 2-points of $D'$. Then there exists a sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that

1) $X_1$ is 3-prepared in a neighborhood of $\pi_1^{-1}(D')$.
2) $\pi_1$ is an isomorphism over $X_1 \setminus D'$.
3) $\pi_1$ is an isomorphism in a neighborhood of $B$.
4) $\pi_1$ is an isomorphism over generic points of 2-curves on $D'$ and over 3-points of $D'$.
5) Points on the intersection of the strict transform of $D'$ on $X_1$ with $\pi_1^{-1}(A)$ are 2-points of $D_{X_1}$.
6) $\sigma_D(p_1) \leq \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.

4. Reduction of $\sigma_D$ above a 3-prepared point.

**Theorem 4.1.** Suppose that $p \in X$ is a 1-point such that $X$ is 3-prepared at $p$, and $\sigma_D(p) > 0$. Let $x, y, z$ be permissible parameters at $p$ giving a form (14) at $p$. Let $U$ be an étale cover of an affine neighborhood of $p$ in which $x, y, z$ are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\overline{D}$ on $U$. Let $I$ be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by $z^m, x^r$ if $\tau_m \neq 0$, and by $\{x^i z^{m-i} \mid 2 \leq i \leq m - 1 \text{ and } \tau_i \neq 0\}$.

Suppose that $\psi : U' \to U$ is a toroidal morphism with respect to $\overline{D}$ such that $U'$ is nonsingular and $I\mathcal{O}_{U'}$ is locally principal. Then (after possibly replacing $U$ with a smaller neighborhood of $p$) $U'$ is 2-prepared and $\sigma_D(q) < \sigma_D(p)$ for all $q \in U'$.

There is (after possibly replacing $U$ with a smaller neighborhood of $p$) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\overline{D}$ with the property that $U'$ is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map $\psi$ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections $C_i$ over the two curve $C$ of $\overline{D}$. $U_i$ is 1-prepared for $U_i \to S$. We have that the curve $C_i$ blown up in $U_{i+1} \to U_i$ is in $\text{Sing}_{\sigma_D(p)}(U_i)$ if $C_i$ is not a 2-curve of $D_{U_i}$, and that $C_i$ is in $\text{Sing}_1(U_i)$ if $C_i$ is a 2-curve of $D_{U_i}$.
Proof. Suppose that $\psi : U' \to U$ is toroidal for $\overline{D}$ and $U'$ is nonsingular. Let $\overline{D}' = \psi^{-1}(\overline{D})$.

The set of 2-curves of $\overline{D}'$ is the disjoint union of the 2-curves of $D_{U'}$ and the 2-curve which is the intersection of the strict transform of the surface $z = 0$ on $U'$ with $D_{U'}$. $\psi$ factors as a sequence of blow ups of 2-curves of (the preimage of) $\overline{D}$. We will verify the following three statements, from which the conclusions of the theorem follow.

If $q \in \psi^{-1}(p)$ and $IO_{U',q}$ is principal, then $\sigma_D(q) < \sigma_D(p)$.
In particular, $\sigma_D(q) < \sigma_D(p)$ if $q$ is a 1-point of $\overline{D}'$.

If $C'$ is a 2-curve of $D_{U'}$, then $U'$ is prepared at $q = C' \cap \psi^{-1}(p)$
if and only if $\sigma_D(q) < \infty$
if and only if $IO_{U',q}$ is principal
if and only if $U'$ is prepared at all $q' \in C'$ in a neighborhood of $q$.

If $C'$ is the 2-curve of $\overline{D}'$ which is the intersection of $D_{U'}$ with the strict transform of $\tilde{z} = 0$ in $U'$,
then $\sigma_D(q) \leq \sigma_D(p)$ if $q = C' \cap \psi^{-1}(p)$, and $\sigma_D(q') = \sigma_D(q)$
for $q' \in C'$ in a neighborhood of $q$.

Suppose that $q \in \psi^{-1}(p)$ is a 1-point for $\overline{D}'$. Then $IO_{U',q}$ is principal. At $q$, we have permissible parameters $x_1, y, z_1$ defined by

$$x = x_1^{a_1}, z = x_1^{b_1}(z_1 + \alpha)$$

for some $a_1, b_1 \in \mathbb{Z}_+$ and $0 \neq \alpha \in \mathfrak{f}$. Substituting into (14), we have

$$u = x_1^{aa_1}, v = P(x_1^{a_1}) + x_1^{ba_1}G$$

where

$$G = r_0x_1^{b_1m}(z_1 + \alpha)^m + r_2x_1^{a_1r_2+b_1(m-2)}(z_1)^{m-2} + \cdots + r_mx_1^{a_1r_m+b_1}(z_1+\alpha) + r_mx_1^{a_1r_m}.$$ 

Let $x_1^{a}$ be a local generator of $IO_{U',q}$. Let $G' = \frac{G}{x_1^a}$.

If $z^m$ is a local generator of $IO_{U',q}$, then $G'$ has an expansion

$$G' = \tau'(z_1 + \alpha)^m + g_2(z_1 + \alpha)^{m-2} + \cdots + g_m(z_1 + \alpha) + g_m + x_1\Omega_1 + y\Omega_2$$

where $0 \neq \tau' = \tau(0, 0, 0) \in \mathfrak{f}$, $g_2, \ldots, g_m \in \mathfrak{f}$ and $\Omega_1, \Omega_2 \in \hat{O}_{U',q}$. We have ord($G'(0, 0, z_1)$) $\leq m - 1$. Setting $F' = G' - G'(x_1, 0, 0)$ and $P'(x_1) = P(x_1^{a_1}) + x_1^{ba_1+b_1m}G'(x_1, 0, 0)$, we have an expression

$$u = x_1^{aa_1}, v = P'(x_1) + x_1^{ba_1+b_1m}F'$$

of the form of (1). Thus $U'$ is 2-prepared at $q$ with $\sigma_{D'}(q) < m - 1 = \sigma_D(p)$.

Suppose that $z^m$ is not a local generator of $IO_{U',q}$, but there exists some $i$ with $2 \leq i \leq m - 1$ such that $x_i^{a_1}z^{m-i}$ is a local generator of $IO_{U',q}$. Let $h$ be the smallest $i$ with this property. Then $G'$ has an expression

$$G' = g_h(z_1 + \alpha)^{m-h} + \cdots + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{f}$ with $g_h \neq 0$ and $\Omega_1, \Omega_2 \in \hat{O}_{U',q}$. As in the previous case, we have that $U'$
is 2-prepared at $q$ with $\sigma_D(q) < m - h - 1 < m - 1 = \sigma_D(p)$.
Suppose that \( z^m \) is not a local generator of \( \hat{\mathcal{O}}_{U',q} \) and \( x^{r_i} z^{m-i} \) is not a local generator of \( \hat{\mathcal{O}}_{U',q} \) for \( 2 \leq i \leq m - 1 \). Then \( x_1^{r_m} \) is a local generator of \( \hat{\mathcal{O}}_{U',q} \), and we have an expression

\[
G' = \Lambda + x_1 \Omega_1,
\]

where \( \Lambda(x_1, y, z_1) = \tau_m(x_1^{a_1}, y, x_1^{b_1}(z_1 + \alpha)) \) and \( \Omega_1 \in \hat{\mathcal{O}}_{U',q} \). Then

\[
\text{ord} \Lambda(0, y, 0) = \text{ord} \tau_m(0, y, 0) = 1,
\]

and we have that \( U' \) is prepared at \( q \).

Now suppose that \( q \in \psi^{-1}(p) \) is a 2-point for \( D_{U'} \). We have permissible parameters \( x_1, y, z_1 \) in \( \hat{\mathcal{O}}_{U',q} \) such that

\[
x = x_1^{a_1} z_1^{b_1}, \quad z = x_1^{c_1} z_1^{d_1}
\]

with \( a_1, b_1 > 0 \) and \( a_1 d_1 - b_1 c_1 = \pm 1 \). Substituting into (14), we have

\[
u = x_1^{a_1} z_1^{b_1}, \quad v = P(x_1^{a_1} z_1^{b_1}) + x_1^{a_1 b} z_1^{b_1 b} G'
\]

where

\[
G = \tau_0 x_1^{c_1 m} z_1^{d_1 m} + \tau_2 x_1^{r_2 a_1 + c_1 (m - 2)} z_1^{r_2 b_1 + d_1 (m - 2)} + \cdots + \tau_m x_1^{a_1 r_m - 1 + c_1} z_1^{b_1 r_m - 1 + d_1} + \tau_m x_1^{a_1 r_m - 1} z_1^{b_1 r_m - 1}.
\]

Let \( C' \) be the 2-curve of \( D_{U'} \) containing \( q \). Since \( \text{ord} \tau_m(0, y, 0) = 1 \) (if \( \tau_m \neq 0 \)) we see that the three statements \( \sigma_D(q) < \infty \), \( \text{ord} \sigma_D(q) = 0 \) and \( \hat{\mathcal{O}}_{U',q} \) is principal are equivalent. Further, we have that \( \sigma_D(q') = \sigma_D(q) \) for \( q' \in C' \) in a neighborhood of \( q \).

Suppose that \( \hat{\mathcal{O}}_{U',q} \) is principal and let \( x_1^a z_1^b \) be a local generator of \( \hat{\mathcal{O}}_{U',q} \). Let \( G' = G/x_1^a z_1^b \). We have that

\[
u = (x_1^{a_1} z_1^{b_1})^a, \quad v = P(x_1^{a_1} z_1^{b_1}) + x_1^{a_1 b + s} z_1^{b_1 + t} G'
\]

has the form (2), since we have made a monomial substitution in \( x \) and \( z \). If \( z^m \) or \( x^{r_i} z^{m-i} \) for some \( i < m \) is a local generator of \( \hat{\mathcal{O}}_{U',q} \), then \( G' \) is a unit in \( \hat{\mathcal{O}}_{U',q} \). If none of \( z^m \), \( x^{r_i} z^{m-i} \) for \( i < m \) are local generators of \( \hat{\mathcal{O}}_{U',q} \), then

\[
G' = \Lambda + x_1 \Omega_1 + z_1 \Omega_2,
\]

where

\[
\Lambda(x_1, y, z_1) = \tau_m(x_1^{a_1} z_1^{b_1}, y, x_1^{c_1} z_1^{d_1})
\]

and \( \Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U',q} \). Thus

\[
\text{ord} \Lambda(0, y, 0) = \text{ord} \tau_m(0, y, 0) = 1.
\]

We thus have that \( U' \) is prepared at \( q \).

The final case is when \( q \in \psi^{-1}(p) \) is on the 2-curve \( C' \) of \( \overline{D} \) which is the intersection of \( D_{U'} \) with the strict transform of \( z = 0 \) in \( U' \). Then there exist permissible parameters \( x_1, y, z_1 \) at \( q \) such that

\[
x = x_1, \quad z = x_1^{b_1} z_1
\]

for some \( b_1 \in \mathbb{Z}_+ \). The equations \( x_1 = z_1 = 0 \) are local equations of \( C' \) at \( q \). Let

\[
s = \min\{b_1 m, r_i + b_1 (m - i) \text{ with } \tau_i \neq 0 \text{ for } 2 \leq i \leq m - 1, r_m \text{ if } \tau_m \neq 0 \}.
\]

We have an expression of the form (1) at \( q \),

\[
\begin{align*}
u &= x_1^2 \\
v &= P(x_1^q) + x_1^{ab + s} G'
\end{align*}
\]
with
\[ G' = \tau_0 x_1^{b_1 m-s} z_1^m + \tau_2 x_1^{r_2 + b_1 (m-2)-s} z_1^{m-2} + \cdots + \tau_{m-1} x_1^{r_{m-1} + b_1 - s} z_1 + \tau_m x_1^{r_m - s}. \]

We see that \( \sigma_D(q) \leq \sigma_D(p) \) (with \( \sigma_D(q) < \sigma_D(p) \) if \( s = r_i + b_1 (m-i) \) for some \( i \) with \( 2 \leq i \leq m-1 \) or \( s = r_m \)) and \( \sigma_D(q') = \sigma_D(q) \) for \( q' \) in a neighborhood of \( q \) on \( C' \).

Suppose that \( IO_{U',q} \) is principal. Then \( x'^m \) generates \( \hat{I} O_{U',q} \). We have that \( G' = x_1^{r_m} \Omega \) where \( \Omega \in \hat{O}_{U',q} \) satisfies \( \Omega(0,y,0) = 1 \). Thus \( U' \) is prepared at \( q \).

\[
\]

We will now construct the function \( \omega(m,r_2,\ldots,r_{m-1}) \) where \( m > 1, r_i \in \mathbb{N} \) for \( 2 \leq i \leq m-1 \) and \( r_{m-1} > 0 \).

Let \( I \) be the ideal in the polynomial ring \( \mathfrak{f}[x,z] \) generated by \( z^m \) and \( x^i z^{m-i} \) for all \( i \) such that \( 2 \leq i \leq m-1 \) and \( r_i > 0 \). Let \( m = (x,z) \) be the maximal ideal of \( k[x,z] \). Let \( \Phi : V_1 \to V = \text{Spec}(\mathfrak{f}[x,z]) \) be the toroidal morphism with respect to the divisor \( xz = 0 \) on \( V \) such that \( V_1 \) is the minimal nonsingular surface such that

1) \( IO_{V_1,q} \) is principal if \( q \in \Phi^{-1}(m) \) is not on the strict transform of \( z = 0 \).

2) If \( q \) is the intersection point of the strict transform of \( z = 0 \) and \( \Phi^{-1}(m) \), so that \( q \) has regular parameters \( x_1, z_1 \), with \( x = x_1, z = x_1^{b_1} z_1 \) for some \( b \in \mathbb{Z} \), then \( r_i + b_1 (m-i) < b_1 m \) for some \( 2 \leq i \leq m-1 \) with \( r_i > 0 \).

Every \( q \in \Phi^{-1}(m) \) which is not on the strict transform of \( z = 0 \) has regular parameters \( x_1, z_1 \) at \( q \) which are related to \( x, z \) by one of the following expressions:

\begin{equation}
(39) \quad x = x_1^{a_1}, \quad z = x_1^{b_1} (z_1 + \alpha)
\end{equation}

for some \( 0 \neq \alpha \in \mathfrak{f} \) and \( a_1, b_1 > 0 \), or

\begin{equation}
(40) \quad x = x_1^{a_1} z_1^{b_1}, \quad z = x_1^{c_1} z_1^{d_1}
\end{equation}

with \( a_1, b_1 > 0 \) and \( a_1 d_1 - b_1 c_1 = \pm 1 \). There are only finitely many values of \( a_1, b_1 \) occurring in expressions (39), and \( a_1, b_1, c_1, d_1 \) occurring in expressions (40).

The point \( q \) on the intersection of the strict transform of \( z = 0 \) and \( \Phi^{-1}(m) \) has regular parameters \( x_1, z_1 \) defined by

\begin{equation}
(41) \quad x = x_1, \quad z = x_1^{b_1} z_1
\end{equation}

for some \( b_1 > 0 \).

Now we define \( \omega = \omega(m,r_2,\ldots,r_{m-1}) \) to be a number such that

\[ \omega > \max \{ \frac{b_1}{a_1} m, r_i + \frac{b_1}{a_1} (m-i) \text{ for } 2 \leq i \leq m-1 \text{ such that } r_i > 0 \}. \]

For all expressions (39),

\[ \omega > \max \{ \frac{c_1}{a_1} m, r_i + \frac{c_1}{a_1} (m-i), r_i + \frac{d_1}{b_1} (m-i) \text{ for } 2 \leq i \leq m-1 \text{ such that } r_i > 0 \} \]

for all expressions (40), and

\[ \omega > \max \{ b_1 m, r_i + b_1 (m-i) \text{ for } 2 \leq i \leq m-1 \text{ such that } r_i > 0 \} \]

in (41).

**Theorem 4.2.** Suppose that \( p \in \text{Sing}_f(X) \) is a 1-point and \( X \) is 3-prepared at \( p \). Let \( x, y, z \) be permissible parameters at \( p \) giving a form (15) at \( p \). Let \( U \) be an étale cover of an affine neighborhood of \( p \) in which \( x, y, z \) are uniformizing parameters. Then \( xz = 0 \) gives a toroidal structure \( \hat{D} \) on \( U \).
There is (after possibly replacing $U$ with a smaller neighborhood of $p$) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\overline{D}$ with has the property that $U'$ is nonsingular, 2-prepared and $\Gamma_D(U') < \sigma_D(p)$. This map $\psi$ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections $C_i$ over the two curve $C$ of $\overline{D}$. $U_i$ is 1-prepared for $U_i \to S$. We have that the curve $C_i$ blown up in $U_{i+1} \to U_i$ is in $\Sing_{\sigma_D(p)}(U_i)$ if $C_i$ is not a 2-curve of $D_{U_i}$, and that $C_i$ is in $\Sing(U_i)$ if $C_i$ is a 2-curve of $D_{U_i}$.

Proof. The proof is similar to that of Theorem 4.1, using the fact that $t > \omega(m, r_2, \ldots, r_{m-1})$ as defined above. \hfill \square

**Theorem 4.3.** Suppose that $p \in X$ is a 2-point and $X$ is 3-prepared at $p$ with $\sigma_D(p) > 0$. Let $x, y, z$ be permissible parameters at $p$ giving a form (13) at $p$. Let $U$ be an étale cover of an affine neighborhood of $p$ in which $x, y, z$ are uniformizing parameters on $U$. Then $xyz = 0$ gives a toroidal structure $\overline{D}$ on $U$. Let $I$ be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by $z^m, x^my^n$ if $\tau_m \neq 0$ and

$$\{x^ny^mz^{-i} \mid 2 \leq i \leq m - 1 \text{ and } \tau_i \neq 0\}.$$

Suppose that $\psi : U_1 \to U$ is a toroidal morphism with respect to $\overline{D}$ such that $U_1$ is nonsingular and $I\mathcal{O}_{U_1}$ is locally principal. Then (after possibly replacing $U$ with a smaller neighborhood of $p$) $U_1$ is 2-prepared for $U_1 \to S$, with $\sigma_D(q) < \sigma_D(p)$ for all $q \in U_1$.

Proof. Suppose that $q \in \psi^{-1}(p)$ is a 1-point for $\psi^{-1}(\overline{D})$. Then $q$ is also a 1-point for $D_{U_1}$. Since $\psi$ is toroidal with respect to $\overline{D}$, there exist regular parameters $\hat{x}_1, \hat{y}_1, \hat{z}_1$ in $\mathcal{O}_{X_1, q}$ and a matrix $A = (a_{ij})$ with nonnegative integers as coefficients such that $\Det A = \pm 1$, and we have an expression

$$x = \hat{x}_1^{a_{11}}(\hat{y}_1 + \alpha)^{a_{12}}(\hat{z}_1 + \beta)^{a_{13}},$$
$$y = \hat{x}_1^{a_{21}}(\hat{y}_1 + \alpha)^{a_{22}}(\hat{z}_1 + \beta)^{a_{23}},$$
$$z = \hat{x}_1^{a_{31}}(\hat{y}_1 + \alpha)^{a_{32}}(\hat{z}_1 + \beta)^{a_{33}}$$

with $a_{11}, a_{21}, a_{31} \neq 0$ and $0 \neq \alpha, \beta \in \mathfrak{k}$. Set

$$\overline{\alpha} = \hat{x}_1^{a_{11}}(\hat{y}_1 + \alpha)^{a_{12}}(\hat{z}_1 + \beta)^{a_{13}} \in \mathcal{O}_{X_1, q}.$$

Substituting into (42), we have

$$x = \overline{\alpha}^{a_{11}},$$
$$y = \overline{\alpha}^{a_{21}},$$
$$z = \overline{\alpha}^{a_{31}}.$$

Let $B = (b_{ij})$ be the adjoint matrix of $A$. Let $\overline{\alpha} = \alpha^{a_{11}} \beta^{a_{12}} \alpha^{a_{13}}, \overline{\beta} = \alpha^{-1} \beta^{a_{11}} \alpha^{a_{12}} \beta^{a_{13}}$. Set

$$\overline{y}_1 = \frac{y}{\hat{x}_1^{a_{21}}}, \overline{z}_1 = \frac{z}{\hat{x}_1^{a_{31}}}, \overline{x}_1 = \frac{x}{\hat{x}_1}. $$

We will show that $\overline{y}_1, \overline{x}_1, \overline{z}_1$ are regular parameters in $\mathcal{O}_{X_1, q}$. We have that

$$ (\hat{y}_1 + \alpha)^{a_{22}} \frac{a_{21}a_{12}}{a_{11}} (\hat{z}_1 + \beta)^{a_{23}} \frac{a_{21}a_{13}}{a_{11}} = \overline{\alpha}^\frac{a_{21}a_{12}}{a_{11}} \beta^\frac{a_{21}a_{13}}{a_{11}},$$
$$ (\hat{y}_1 + \alpha)^{a_{32}} \frac{a_{31}a_{12}}{a_{11}} (\hat{z}_1 + \beta)^{a_{33}} \frac{a_{31}a_{13}}{a_{11}} = \overline{\beta}^\frac{a_{31}a_{12}}{a_{11}} \alpha^\frac{a_{31}a_{13}}{a_{11}}.$$

Let

$$C = \begin{pmatrix}
\frac{b_{12}a_{11} - b_{12}}{a_{11}} & \frac{b_{13}a_{11} - b_{13}}{a_{11}} & \frac{b_{14}a_{11} - b_{14}}{a_{11}} \\
\frac{b_{22}a_{11} - b_{22}}{a_{11}} & \frac{b_{23}a_{11} - b_{23}}{a_{11}} & \frac{b_{24}a_{11} - b_{24}}{a_{11}} \\
\frac{b_{32}a_{11} - b_{32}}{a_{11}} & \frac{b_{33}a_{11} - b_{33}}{a_{11}} & \frac{b_{34}a_{11} - b_{34}}{a_{11}}
\end{pmatrix}.$$
We must show that $C$ has rank 2. $C$ has the same rank as
\[
\begin{pmatrix}
b_{33}\beta & -b_{33}\alpha \\
b_{32}\beta & -b_{32}\alpha
\end{pmatrix}
= \begin{pmatrix}
b_{33} & b_{32} \\
b_{32} & b_{31}
\end{pmatrix}
\begin{pmatrix}
\beta & 0 \\
0 & -\alpha
\end{pmatrix}.
\]
Since $\alpha, \beta \neq 0$, $C$ has the same rank as
\[
B' = \begin{pmatrix}
b_{33} & b_{32} \\
b_{32} & b_{31}
\end{pmatrix}.
\]
Since $B$ has rank 3,
\[
\begin{pmatrix}
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{pmatrix}
\]
has rank 2. Since
\[
\begin{pmatrix}
b_{21} \\
b_{31}
\end{pmatrix}
= -\frac{a_{21}}{a_{11}} \begin{pmatrix}
b_{22} \\
b_{32}
\end{pmatrix}
+ \frac{a_{31}}{a_{11}} \begin{pmatrix}
b_{23} \\
b_{33}
\end{pmatrix},
\]
we have that $B'$ has rank 2, and hence $C$ has rank 2. Thus $\overline{x}_1, \overline{y}_1, \overline{z}_1$ are regular parameters in $\mathcal{O}_{X_1, q}$. We have
\[
x = \overline{x}_1^{a_1}, y = \overline{x}_1^{a_2}(\overline{y}_1 + \overline{\alpha}), z = \overline{x}_1^{a_3}(\overline{z}_1 + \overline{\beta}).
\]
We have that $u = (x^a y^b)^t$. Let
\[
t = -\frac{b}{a_{11}a + a_{21}b},
\]
and set $x_1 = x_1(y_1 + \alpha)^t$. Define $y_1 = y_1, \alpha = \overline{\alpha}, \beta = \overline{\alpha}t_{a_1}t_{a_2}$ and $z_1 = (y_1 + \alpha)^t_{a_3}(z_1 + \beta) - \beta$. Then $x_1, y_1, z_1$ are permissible parameters at $q$, with $u = x_1^{(a_{11}a + a_{21}b)}t$,
\[
x = x_1^{a_1}(y_1 + \alpha)^{a_2}, y = x_1^{a_2}(y_1 + \alpha)^{a_3}, z = x_1^{a_3}(z_1 + \beta).
\]
Thus we have shown that there exist (formal) permissible parameters $x_1, y_1, z_1$ at $q$ such that
\[
x = x_1^{e_1}(y_1 + \alpha)^{\lambda_1}, y = x_1^{e_2}(y_1 + \alpha)^{\lambda_2}, z = x_1^{e_3}(z_1 + \beta)
\]
where $e_1, e_2, e_3 \in \mathbb{Z}^+$, $\alpha, \beta \in \mathfrak{p}$ are nonzero, $\lambda_1, \lambda_2 \in \mathbb{Q}$ are both nonzero, and $u = x_1^{b_1t}$, where $b_1 = ae_1 + be_2, a\lambda_1 + b\lambda_2 = 0$. We then have an expression
\[
v = P(x_1^{a_1e_1+be_2}) + x_1^{c_1e_1+de_2}G,
\]
where
\[
G = (y_1 + \alpha)^{c_1+1+2} [\tau_0x_1^{c_1}m(z_1 + \beta)^m]
+ \tau_2 x_1^{r_1e_1+s_2e_2+(m-2)e_3} (y_1 + \alpha)^{r_1} + s_2^{r_2}(z_1 + \beta)^{m-2} + \cdots
+ \tau_m x_1^{r_1m-1e_1+s_m+e_2(e_3)} (y_1 + \alpha)^{r_m-1} + s_m^{r_m-1}(z_1 + \beta)
+ \tau_m x_1^{r_1m+1+s_m+e_2(y_1)^{r_m}+s_m^{r_m}(z_1 + \beta)}.
\]
Let $\tau' = \tau_0(0, 0, 0)$. Let $x_1 \in \mathfrak{O}$ be a generator of $I\mathcal{O}_{U_1, q}$. Let $G' = \frac{F}{x_1^t}$. If $x^m$ is a local generator of $I\mathcal{O}_{U_1, q}$, then $G'$ has an expression
\[
G' = \tau' \alpha^m (z_1 + \beta)^m + g_2(z_1 + \beta)^{m-2} + \cdots + g_{m-1}(z + \beta) + g_m + x_1\Omega_1 + y_1\Omega_2
\]
for some $g_i \in \mathfrak{p}$ and $\Omega_1, \Omega_2 \in \mathcal{O}_{U_1, q}$, where $\varphi = c\lambda_1 + d\lambda_2$. Setting $F' = G' - G'(x_1, 0, 0)$, and $P'(x_1) = P(x_1^{c_1+be_2}) + x_1^{c_1e_1+de_2+e}G'(x_1, 0, 0)$, we have that
\[
u = x_1^{b_1}, v = P'(x_1) + x_1^{c_1+de_2+e}F'
\]
has the form (1) and $\sigma_D(q) \leq \text{ord } F'(0, 0, z_1) - 1 \leq m - 2 < m - 1 = \sigma_D(p)$ since $0 \neq \beta$. 26
Suppose that $z^m$ is not a local generator of $I\hat{O}_{U_1,q}$, but there exists some $i$ with $2 \leq i \leq m - 1$ such that $\tau_i x^{r_i} y^{s_i} z^{m-i}$ is a local generator of $I\hat{O}_{U_1,q}$. Let $h$ be the smallest $i$ with this property. Then $G'$ has an expression
\[ G' = g_h(z_1 + \hat{\beta})^{m-h} + \cdots + g_{m-1}(z_1 + \hat{\beta}) + g_m + x_1\Omega_1 + y_2\Omega_2 \]
for some $g_i \in \k$ with $g_h \neq 0$. As in the previous case, we have
\[ \sigma_D(q) \leq m - h - 1 < m - 1 = \sigma_D(p). \]

Suppose that $z^m$ is not a local generator of $I\hat{O}_{U_1,q}$, and $\tau_i x^{r_i} y^{s_i} z^{m-i}$ is not a local generator of $I\hat{O}_{U_1,q}$ for $2 \leq i \leq m$. Then $x^{r_s} y^{s_s}$ is a local generator of $I\hat{O}_{U_1,q}$, and $G'$ has an expression
\[ G' = \tau'_m(y_1 + \hat{\alpha})^{p+r_m\lambda_1+s_m\lambda_2} + x_1\Omega \]
where $\tau'_m = \tau_m(0,0,0)$ for some $\Omega \in \hat{O}_{U_1,q}$. Suppose, if possible, that $\varphi + r_m\lambda_1 + s_m\lambda_2 = 0$. Since $\varphi + r_m\lambda_1 + s_m\lambda_2 = (c + r_m)\lambda_1 + (d + s_m)\lambda_2$, we then have that the nonzero vector $(\lambda_1, \lambda_2)$ satisfies $a\lambda_1 + b\lambda_2 = (c + r_m)\lambda_1 + (d + s_m)\lambda_2 = 0$. Thus the determinant $a(d + s_m) - b(c + r_m) = 0$, a contradiction to our assumption that $F$ satisfies (2).

Now since $\varphi + r_m\lambda_1 + s_m\lambda_2 \neq 0$ and $\hat{\alpha} \neq 0$, we have $1 = \text{ord} G'(0, y_1, 0) < m$, so that $\sigma_D(q) = 0 < m - 1 = \sigma_D(p)$.

Suppose that $q \in \psi^{-1}(p)$ is a 2-point of $\psi^{-1}(D)$. Then there exist (normal) permissible parameters $\hat{x}_1, \hat{y}_1, \hat{z}_1$ at $q$ such that
\[ x = \hat{x}_1^{e_{11}} y_1^{e_{12}} (\hat{z}_1 + \hat{\alpha})^{e_{13}}, \]
\[ y = \hat{x}_1^{e_{21}} y_1^{e_{22}} (\hat{z}_1 + \hat{\alpha})^{e_{23}}, \]
\[ z = \hat{x}_1^{e_{31}} y_1^{e_{32}} (\hat{z}_1 + \hat{\alpha})^{e_{33}}. \]
where $e_{ij} \in \mathbb{N}$, with $\text{Det}(e_{ij}) = \pm 1$, and $\hat{\alpha} \in \k$ is nonzero. We further have
\[ e_{11} + e_{12} > 0, e_{21} + e_{22} > 0 \text{ and } e_{31} + e_{32} > 0. \]

First suppose that $e_{11} e_{22} - e_{12} e_{21} \neq 0$. Then $q$ is a 2-point of $D_{U_1}$.

There exist $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that upon setting
\[ \hat{x}_1 = x_1(z_1 + \hat{\alpha})^{\lambda_1} \text{ and } \hat{y}_1 = y_1(z_1 + \hat{\alpha})^{\lambda_2}, \]
we have
\[ x = x_1^{e_{11}} y_1^{e_{12}}; \]
\[ y = x_1^{e_{21}} y_1^{e_{22}}, \]
\[ z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \hat{\alpha})^r, \]
where
\[ \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}. \]

By Cramer’s rule,
\[ r = \pm \frac{1}{e_{11} e_{22} - e_{12} e_{21}} \neq 0. \]
Now set $z_1 = (z_1 + \hat{\alpha})^r - \hat{\alpha}^r$ and $\alpha = \hat{\alpha}^r$ to obtain permissible parameters $x_1, y_1, z_1$ at $q$ with
\[ x = x_1^{e_{11}} y_1^{e_{12}}; \]
\[ y = x_1^{e_{21}} y_1^{e_{22}}, \]
\[ z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \alpha). \]

We have an expression
\[ u = ((x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b) \ell_i = \ell_1 \ell_2 \ell_3, \]
where $t_1, t_2, \ell_1 \in \mathbb{Z}_+$ and $\text{gcd}(t_1, t_2) = 1$.

We then have an expression
\[ v = P((x_1^{t_1} y_1^{t_2} \ell_1^{\ell_1})^{\ell_1} + x_1^{ce_{11} + de_{21}} y_1^{ce_{12} + de_{22}} G, \]
where $\ell_1 \in \mathbb{Z}_+$ and $\text{gcd}(t_1, t_2) = 1$. 
Let $\tau' = 0(0, 0, 0)$. Let $x_1^iy_1^j$ be a generator of $I\hat{O}_{U_1,q}$. Let $G' = \frac{G}{x_1^iy_1^j}$.

If $z^m$ is a local generator of $I\hat{O}_{U_1,q}$, then $G'$ has an expression

$$G' = \tau'(z_1 + \alpha)^m + g_2(z_1 + \alpha)^{m-2} + \cdots + g_{m-1}(z - \alpha) + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{t}$ and $\Omega_1, \Omega_2 \in \hat{O}_{U_1,q}$. Let

$$P(x_1^{i_1}y_1^{j_1}) = \sum_{i_2, j_2} \frac{1}{i_2!j_2!} \partial^2 \frac{\partial G(x_1^{i_1}y_1^{j_1}, x_2^{i_2}y_2^{j_2}, x_3^{i_3}y_3^{j_3}, \ldots)}{\partial x_1^{i_1} \partial y_1^{j_1}} (0, 0, 0)$$

and $F' = G' - \frac{P(x_1^{i_1}y_1^{j_1})}{x_1^{i_1}y_1^{j_1}}$. Set $P'(x_1^{i_1}y_1^{j_1}) = P((x_1^{i_1}y_1^{j_1})^{\tau} + \hat{P}(x_1^{i_1}y_1^{j_1}))$. We have that

$$u = (x_1^{i_1}y_1^{j_1})^{\tau}, v = P'(x_1^{i_1}y_1^{j_1}) + x_1^{i_1}y_1^{j_1} + \partial x_1^{i_1}y_1^{j_1} + x_1^{i_1}y_1^{j_1}$$

has the form (2), and $\sigma_D(q) = \text{ord} F'(0, 0, z_1) - 1 \leq m - 2 < m - 1 = \sigma_D(p)$ since $0 \neq \alpha$.

Suppose that $z^m$ is not a local generator of $I\hat{O}_{U_1,q}$, but there exists some $i$ with $2 \leq i \leq m - 1$ such that $\tau_i x^iy^iz^{m-i}$ is a local generator of $I\hat{O}_{U_1,q}$. Let $h$ be the smallest $i$ with this property. Then $G'$ has an expression

$$G' = g_i(z_1 + \beta)^{m-h} + \cdots + g_m + x_1\Omega_1 + y_1\Omega_2$$

for some $g_i \in \mathfrak{t}$ with $g_i \neq 0$. As in the previous case, we have $\sigma_D(q) \leq m - h - 1 < m - 1 = \sigma_D(p)$.

Suppose that $z^m$ is not a local generator of $I\hat{O}_{U_1,q}$, and $\tau_i x^iy^iz^{m-i}$ is not a local generator of $I\hat{O}_{U_1,q}$ for $2 \leq i \leq m - 1$. Then $x^iy^iz^{m-i}$ is a local generator of $I\hat{O}_{U_1,q}$, and then $G'$ has an expression

$$G' = 1 + x_1\Omega_1 + y_1\Omega_2$$

for some $\Omega_1, \Omega_2 \in \hat{O}_{U_1,q}$.

We now claim that after replacing $G'$ with $F' = \frac{P(x_1^{i_1}y_1^{j_1})}{x_1^{i_1}y_1^{j_1}}$, where $\hat{P}$ is defined by (45), we have that $F'(0, 0, 0) \neq 0$. If this were not the case, we would have

$$0 = \left( \begin{array}{c} c + r_m e_{11} + (d + s_m)e_{21} \\ a e_{11} + b e_{21} \end{array} \right) \left( \begin{array}{c} c + r_m e_{12} + (d + s_m)e_{22} \\ a e_{12} + b e_{22} \end{array} \right)$$

Since $e_{11}e_{22} - e_{21}e_{12} \neq 0$ (by our assumption), we get

$$0 = \det \begin{pmatrix} c + r_m & d + s_m \\ a & b \end{pmatrix}$$

which is a contradiction to our assumption that $F$ satisfies (2). Since $F'(0, 0, 0) \neq 0$, we have that $\sigma_D(q) = 0 < m - 1 = \sigma_D(p)$.

Now suppose that $q$ is a 2-point of $\psi^{-1}(\bar{D})$ with $e_{11}e_{22} - e_{21}e_{12} = 0$ in (44). We make a substitution

$$\tilde{x}_1 = x_1(z_1 + \alpha)^{\psi_1}, \tilde{y}_1 = y_1(z_1 + \alpha)^{\psi_2}, \tilde{z}_1 = z_1$$
where \( \alpha = \hat{\alpha} \) and \( \varphi_1, \varphi_2 \in \mathbb{Q} \) satisfy

\[
0 = a(\varphi_1 e_{11} + \varphi_2 e_{12} + e_{13}) + b(\varphi_1 e_{21} + \varphi_2 e_{22} + e_{23}) = \varphi_1(ae_{11} + be_{21}) + \varphi_2(ae_{12} + be_{22}) + ae_{13} + be_{23}.
\]

We have \( ae_{11} + be_{21} > 0 \) and \( ae_{12} + be_{22} > 0 \) since \( a, b > 0 \) and by the condition satisfied by the \( e_{ij} \) stated after (44).

Let

\[
\lambda_1 = \varphi_1 e_{11} + \varphi_2 e_{12} + e_{13}, \quad \lambda_2 = \varphi_1 e_{21} + \varphi_2 e_{22} + e_{23}, \quad \lambda_3 = \varphi_1 e_{31} + \varphi_2 e_{32} + e_{33}.
\]

Then \( x_1, y_1, z_1 \) are permissible parameters at \( q \) such that

\[
(46) \quad x = x_1^{e_{11}} y_1^{e_{12}} (z_1 + \alpha)^{\lambda_1}, \quad y = x_1^{e_{21}} y_1^{e_{22}} (z_1 + \alpha)^{\lambda_2}, \quad z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \alpha)^{\lambda_3}
\]

with \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q} \), and \( a \lambda_1 + b \lambda_2 = 0 \).

Now suppose that \( e_{11} > 0 \) and \( e_{12} > 0 \), which is the case where \( q \) is a 2-point of \( D_{U_1} \).

Write

\[
u = \left( (x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b \right)^\ell = \left( x_1^{t_1} y_1^{t_2} \right)^{\ell_1}
\]

where \( t_1, t_2, \ell_1 \in \mathbb{Z}_+ \) and gcd\((t_1, t_2) = 1\).

We then have an expression

\[
v = P\left( (x_1^{t_1} y_1^{t_2})^{\ell_1} \right) + x_1^{c_{e_{11}+de_{21}} e_{12}+de_{22}} G,
\]

where

\[
G = (z_1 + \alpha)^{c_{\lambda_1} + d\lambda_2} \left[ \tau_0 x_1^{m_{c_{21}}} y_1^{m_{c_{22}}} (z_1 + \alpha)^{m_{\lambda_3}} + \tau_2 x_1^{r_{e_{11}} e_{21} + e_{22} + (m-2) e_{31}} y_1^{r_{e_{12}} e_{22} + e_{32} + (m-2) e_{32}} (z_1 + \alpha)^{r_{2} \lambda_1 + 2 \lambda_2 + (m-2) \lambda_3} + \ldots + \tau_m x_1^{r_{m-1} e_{11} + s_{m-1} e_{21} + e_{31} + r_{m-1} e_{12} + s_{m-1} e_{22} + e_{32}} (z_1 + \alpha)^{r_{m-1} \lambda_1 + s_{m-1} \lambda_2 + (m-1) \lambda_3} + \tau_m x_1^{r_{m} e_{11} + s_{m} e_{21} + e_{31} + r_{m} e_{12} + s_{m} e_{22} + e_{32}} (z_1 + \alpha)^{r_{m} \lambda_1 + s_{m} \lambda_2} \right].
\]

Let \( x_1^a y_1^b \) be a generator of \( I \hat{O}_{U_1,q} \). Let \( G' = \frac{F}{x_1^a y_1^b} \).

We will now establish that, with our assumptions, there is a unique element of the set \( S \) consisting of \( z^m \), and

\[
\{ x^n y^i z^{m-i} \mid 2 \leq i \leq m \text{ and } \tau_i \neq 0 \}
\]

which is a generator of \( I \hat{O}_{U_1,q} \); that is, is equal to \( x_1^a y_1^b \) times a unit in \( \hat{O}_{U_1,q} \). Let \( r_0 = 0 \) and \( s_0 = 0 \). Suppose that \( x^n y^i z^{m-i} \) (with \( 0 \leq i \leq m \) is a generator of \( I \hat{O}_{U_1,q} \). We have

\[
x^n y^i z^{m-i} = x_1^a y_1^b (z_1 + \alpha)^{\gamma_i}
\]

where

\[
\begin{align*}
r_i e_{11} + s_i e_{21} + (m - i) e_{31} &= s \\
r_i e_{12} + s_i e_{22} + (m - i) e_{32} &= t \\
r_i \lambda_1 + s_i \lambda_2 + (m - i) \lambda_3 &= \gamma_i.
\end{align*}
\]

Let

\[
(47) \quad A = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.
\]

We have

\[
(48) \quad A \begin{pmatrix} r_i \\ s_i \\ m - i \end{pmatrix} = \begin{pmatrix} s \\ t \\ \gamma_i \end{pmatrix}.
\]
Let $\omega = \text{Det}(A)$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_1 & \varphi_2 & 1 \end{pmatrix} \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$$

implies $\omega = \text{Det}(A) = \pm 1$.

By Cramer’s rule, we have

$$\omega(m - i) = \text{Det} \left( \begin{array}{ccc} e_{11} & e_{21} & s \\ e_{12} & e_{22} & t \\ \lambda_1 & \lambda_2 & \gamma_i \end{array} \right) = s \text{Det} \left( \begin{array}{cc} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{array} \right) - t \text{Det} \left( \begin{array}{cc} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{array} \right) + \gamma_i \text{Det} \left( \begin{array}{cc} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{array} \right).$$

Since $e_{11}e_{21} - e_{12}e_{22} = 0$ by assumption, we have that

$$i = m - \frac{1}{\omega} \left( s \text{Det} \left( \begin{array}{cc} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{array} \right) - t \text{Det} \left( \begin{array}{cc} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{array} \right) \right).$$

In particular, there is a unique element $x^iy^jz^{m-i} \in S$ which is a generator of $I\hat{O}_{U,q}$. We have $x^iy^jz^{m-i} = x_1^{t_1}(z_1 + \alpha)^{\gamma_i}$.

We thus have that $G = x_1^{t_1}y_1^i [g(z_1 + \alpha)^{\gamma_1} + c\lambda_1 + d\lambda_2 + x_1\lambda_1 + y_1\lambda_2]$ for some $\Omega_1, \Omega_2 \in \hat{O}_{U,q}$ and $0 \neq g \in \mathfrak{t}$.

We now establish that we cannot have that $\gamma_i + c\lambda_1 + d\lambda_2 = 0$ and $x_1^{e_{11} + d\gamma_1} y_1^{e_{12} + e\gamma_1} x_1^{t_1}$ is a power of $x_1^{t_1}y_1^2$. We will suppose that both of these conditions do hold, and derive a contradiction. Now we know that $x_1^{a_i}y_1^{b_i} = x_1^{a_{e_{11} + d\gamma_1}} y_1^{e_{12} + e\gamma_1}$ a power of $x_1^{t_1}y_1^2$. By (47), (48) and our assumptions, we have that

$$A \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

and

$$A \begin{pmatrix} c + r_i \\ d + s_i \\ m - i \end{pmatrix}$$

are rational multiples of

$$\begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix}.$$
has the form (2) and \( \sigma_D(q) = 0 \leq m - 2 = \sigma_D(p) \).

Now suppose that \( q \in \psi^{-1}(p) \) is a 2-point of \( \psi^{-1}(D) \), \( e_{11} e_{22} - e_{12} e_{21} = 0 \) in (44), and \( e_{11} = 0 \) or \( e_{12} = 0 \). Without loss of generality, we may assume that \( e_{12} = 0 \). \( q \) is a 1-point of \( D_{U_1} \), and we have permissible parameters (46) at \( q \). Since \( \text{Det}(e_{ij}) = \pm 1 \), we have that \( e_{32} = 1 \), and \( e_{11} e_{23} - e_{21} e_{13} = \pm 1 \). Replacing \( y_1 \) with \( y_1(z_1 + \alpha)^{\lambda_3} \) in (46), we find permissible parameters \( x_1, y_1, z_1 \) at \( q \) such that

\[
(49) \quad x = x_{e_{11}}^{e_{11}}(z_1 + \alpha)^{\lambda_1}, \quad y = x_{e_{21}}^{e_{21}}(z_1 + \alpha)^{\lambda_2}, \quad z = x_{e_{31}}^{e_{31}} y_1,
\]

with \( e_{11}, e_{21} > 0 \) and \( a \lambda_1 + b \lambda_2 = 0 \). We have

\[
\begin{align*}
 u &= x_1^{(a e_{11} + b e_{21})l} = x_1^{l_1} \\
v &= P(x_1^{a e_{11} + b e_{21}}) + x_1^{e_{11} + e_{21}} G
\end{align*}
\]

where

\[
G = \left( z_1 + \alpha \right)^{c \lambda_1 + d \lambda_2} \left[ \tau_0 x_1^{m e_{31}} y_1^m + \tau_2 x_1^{r e_{11} + s e_{21} + (m - 2) e_{31}} y_1^{m-2} (z_1 + \alpha)^{r \lambda_1 + s \lambda_2} + \ldots \\
+ \tau_m x_1^{r \lambda_1 + s \lambda_2} (z_1 + \alpha)^{r \lambda_1 + s \lambda_2} \right] + \tau_m x_1^{r \lambda_1 + s \lambda_2} (z_1 + \alpha)^{r \lambda_1 + s \lambda_2}.
\]

Since \( IO_{U_1,q} \) is principal and \( \tau_m \) or \( \tau_{m-1} \neq 0 \), we have that \( x_1^{r \lambda_1 + s \lambda_2} \) is a generator of \( IO_{U_1,q} \) if \( \tau_m \neq 0 \), and \( x_1^{r \lambda_1 + s \lambda_2} \) is a generator of \( IO_{U_1,q} \) if \( \tau_m = 0 \) and \( \tau_{m-1} \neq 0 \).

First suppose that \( \tau_m \neq 0 \) so that

\[
G = x_1^{r \lambda_1 + s \lambda_2} \left[ \left( z_1 + \alpha \right)^{c + r \lambda_m} y_1^l + \left( z_1 + \alpha \right)^{c + r \lambda_m} y_1^l + \left( z_1 + \alpha \right)^{c + r \lambda_m} y_1^l + \ldots \\
+ x_1^{l_1} \Omega + y_1 \Omega_2 \right]
\]

with \( 0 \neq g_m \in \mathfrak{t} \), \( \Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1,q} \). Since \( \lambda_1, \lambda_2 \) are not both zero, \( a \lambda_1 + b \lambda_2 = 0 \) and \( a(d + s_m) - b(c + r_m) \neq 0 \), we have that \( (c + r_m) \lambda_1 + (d + s_m) \lambda_2 \neq 0 \). Let \( \overline{P}(x_1) = G(x_1, 0, 0) \), and \( P'(x_1) = P(x_1^{a e_{11} + b e_{21}}) + \overline{P}(x_1) \).

Let

\[
F' = \frac{1}{x_1^{e_{11} + e_{21}} (G - \overline{P}(x_1))}.
\]

Then

\[
\begin{align*}
 u &= x_1^{l_1} \\
v &= P'(x_1)^{a e_{11} + b e_{21}} F'
\end{align*}
\]

is of the form (1) with \( \text{ord} F'(0, y_1, z_1) = 1 \). Thus \( \sigma_D(q) = 0 < \sigma_D(p) \).

Now suppose that \( \tau_m = 0 \), so that

\[
G = x_1^{r \lambda_1 + s \lambda_2} \left[ \left( z_1 + \alpha \right)^{c + r \lambda_m} y_1^l + \left( z_1 + \alpha \right)^{c + r \lambda_m} y_1^l + \left( z_1 + \alpha \right)^{c + r \lambda_m} y_1^l + \ldots \\
+ x_1^{l_1} \Omega + y_1 \Omega_1 \right]
\]

with \( 0 \neq g_m \in \mathfrak{t} \) and \( \Omega_1 \in \hat{\mathcal{O}}_{U_1,q} \). Thus \( \sigma_D(q) = 0 < \sigma_D(p) \).

The final case is when \( q \) is a 3-point for \( \psi^{-1}(D) \), so that \( q \) is a 3-point or a 2-point of \( D_{U_1} \). Then we have permissible parameters \( x_1, y_1, z_1 \) at \( q \) such that

\[
x = x_1^{e_{11}} y_1^{e_{12}} z_1^{e_{13}}, \quad y = x_1^{e_{21}} y_1^{e_{22}} z_1^{e_{23}}, \quad z = x_1^{e_{31}} y_1^{e_{32}} z_1^{e_{33}}
\]

with \( \omega = \text{Det}(e_{ij}) = \pm 1 \). Thus there is a unique element of the set \( S \) consisting of \( z^m \) and

\[
\{x^{i_1} y^{i_2} z^{i_3} \mid 2 \leq i \leq m \text{ and } \tau_i \neq 0\}
\]

which is a generator \( x_1^{e_{11}} y_1^{e_{21}} z_1^{e_{31}} \) of \( IO_{U_1,q} \). Thus \( \sigma_D(q) = 0 \) if \( q \) is a 3-point of \( D_{U_1} \). If \( q \) is a 2-point of \( D_{U_1} \), we may assume that \( e_{13} = e_{23} = 0 \). Then \( e_{33} = 1 \). Since \( \tau_m \neq 0 \) or \( \tau_{m-1} \neq 0 \), we calculate that \( \sigma_D(q) = 0 \).

\[\square\]
Suppose that \( p \in X \) is a 2-point such that \( X \) is 3-prepared at \( p \) and \( \sigma_D(p) = r > 0 \). We can then define a local resolver \((U_p, \overline{D}_p, I_p, \nu^1_p, \nu^2_p)\) as in Theorem 4.3, where \( \nu^1_p \) are valuations on \( U_p \) which dominate the two curves \( C_1, C_2 \) which are the intersection of \( E \) with \( DU_p \) on \( U_p \) (where \( \overline{D}_p = DU_p + E \)), and which have the property that if \( \pi : V \to U_p \) is a birational morphism, then the center \( C(V, \nu^1_p) \) on \( V \) is the unique curve on the strict transform of \( E \) on \( V \) which dominates \( C \). We will think of \( U_p \) as a germ, so we will feel free to replace \( U_p \) with a smaller neighborhood of \( p \) whenever it is convenient.

If \( \pi : Y \to X \) is a birational morphism, then the center \( C(Y, \nu^1_p) \) on \( Y \) is the closed curve which is the center of \( \nu^1_p \) on \( Y \). We define \( \Lambda(Y, \nu^1_p) \) to be the image in \( Y \) of \( C(Y \times X U_p, \nu^1_p) \cap \pi^{-1}(p) \). This defines a valuation which is composite with \( C(Y, \nu^1_p) \).

We define \( W(Y,p) \) to be the clopen locus on \( Y \) of the image of points in \( \pi^{-1}(U_p) = Y \times_X U_p \) such that \( I_p \mathcal{O}_Y | \pi^{-1}(U_p) \) is not invertible. Define \( \text{Preimage}(Y,Z) = \pi^{-1}(Z) \) for \( Z \) a subset of \( X \).

5. Global reduction of \( \sigma_D \)

**Lemma 5.1.** Suppose that \( X \) is 2-prepared and \( p \in X \) is 3-prepared. Suppose that \( r = \sigma_D(p) > 0 \).

a) Suppose that \( p \) is a 1-point. Then there exists a unique curve \( C \) in \( \text{Sing}_1(X) \) containing \( p \). The curve \( C \) is contained in \( \text{Sing}_q(X) \). If \( x,y,z \) are permissible parameters at \( p \) giving an expression (14) or (15) at \( p \), then \( z = z = 0 \) are formal local equations of \( C \) at \( p \).

b) Suppose that \( p \) is a 2-point and \( C \) is a curve in \( \text{Sing}_r(X) \) containing \( p \). If \( x,y,z \) are permissible parameters at \( p \) giving an expression (13) at \( p \), then \( x = z = 0 \) or \( y = z = 0 \) are formal local equations of \( C \) at \( p \).

**Proof.** We first prove a). Let \( I \subseteq \mathcal{O}_X \) be the ideal sheaf defining the reduced scheme \( \text{Sing}_1(X) \). Then \( I_p \mathcal{O}_{X,p} = \sqrt{(x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})} = (x, z) \) is an ideal on \( U \) defining \( \text{Sing}_1(U) \). Thus the unique curve \( C \) in \( \text{Sing}_1(X) \) through \( p \) has (formal) local equations \( x = z = 0 \) at \( p \). At points near \( p \) on \( C \), a form (14) or (15) continues to hold with \( m = r + 1 \). Thus the curve is in \( \text{Sing}_r(X) \).

We now prove b). Suppose that \( C \subseteq \text{Sing}_r(X) \) is a curve containing \( p \). By Theorem 4.3, there exists a toroidal morphism \( \Psi : U_1 \to U \) where \( U \) is an étale cover of an affine neighborhood of \( p \), and \( \overline{D} \) is the local toroidal structure on \( U \) defined (formally at \( p \)) by \( xyz = 0 \), such that all points \( q \) of \( U_1 \) satisfy \( \sigma_D(q) < r \). Hence the strict transform on \( U_1 \) of the preimage of \( C \) on \( U \) must be empty. Since \( \Psi \) is toroidal for \( \overline{D} \) and \( X \) is 3-prepared at \( p \), \( C \) must have local equations \( x = z = 0 \) or \( y = z = 0 \) at \( p \). \( \square \)

**Definition 5.2.** Suppose that \( X \) is 3-prepared. We define a canonical sequence of blow ups over a curve in \( X \).

1) Suppose that \( C \) is a curve in \( X \) such that \( t = \sigma_D(q) > 0 \) at the generic point \( q \) of \( C \), and all points of \( C \) are 1-points of \( D \). Then we have that \( C \) is nonsingular and \( \sigma_D(p) = t \) for all \( p \in C \) by Lemma 5.1. By Lemma 5.1 and Theorem 4.1 or 4.2, there exists a unique minimal sequence of permissible blow ups of sections over \( C \), \( \pi_1 : X_1 \to X \), such that \( X_1 \) is 2-prepared and \( \sigma_D(p) < t \) for all \( p \in \pi_1^{-1}(C) \). We will call the morphism \( \pi_1 \) the canonical sequence of blow ups over \( C \).

2) Suppose that \( C \) is a permissible curve in \( X \) which contains a 1-point such that \( \sigma_D(p) = 0 \) for all \( p \in C \), and a condition 1, 3 or 5 of Lemma 5.10 holds at all
\( p \in C \). Let \( \pi_1 : X_1 \to X \) be the blow up of \( C \). Then by Lemma 3.12, \( X_1 \) is 3-prepared and \( \sigma_D(p) = 0 \) for \( p \in \pi_1^{-1}(C) \). We will call the morphism \( \pi_1 \) the canonical blow up of \( C \).

**Theorem 5.3.** Suppose that \( X \) is 2-prepared. Then there exists a sequence of permissible blowups \( \psi : X_1 \to X \) such that \( X_1 \) is prepared.

**Proof.** By Proposition 3.13, there exists a sequence of permissible blow ups \( X^0 \to X \) such that \( X^0 \) is 3-prepared. Let \( r = \Gamma_D(X^0) \). Since \( X^0 \) is prepared if \( r = 0 \), we may assume that \( r > 0 \). Let \( T_0 = \{ p \in X^0 \mid X^0 \text{ is a 2-point for } D \text{ with } \sigma_D(p) = r \} \).

For \( p \in T_0 \), choose \( (U_p, \overline{D}_p, I_p, \nu^1_p, \nu^2_p) \). Let \( \Gamma_0 \) be the union of the set of curves

\[ \{ (X^0, \nu^2_p) \mid p \in T_0 \text{ and } \sigma_D(p) = r \text{ for } p \in C(X^0, \nu^2_p) \text{ the generic point} \} \]

and any remaining curves \( C \) in \( \text{Sing}_r(X^0) \) (which necessarily contain no 2-points).

By Lemma 5.1, all curves in \( \text{Sing}_r(X^0) \) are nonsingular, and if a curve \( C \in \text{Sing}_r(X^0) \) contains a 2-point \( p \in T_0 \), then \( C = C(X^0, \nu^2_p) \) for some \( j \).

Let \( Y_0 \to X^0 \) be the product of canonical sequences of blowups over the curves in \( \Gamma_0 \) (which are necessarily the curves in \( \text{Sing}_r(X^0) \)), so that \( Y_0 \setminus \cup_{p \in \Gamma_0} W(Y_0, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_0 \setminus \cup_{p \in \Gamma_0} W(Y_0, p) \).

Let \( Y_{0, 1} \to Y_0 \) be a toroidal morphism for \( D_{Y_0} \) so that the components of \( D_{Y_{0, 1}} \) containing some curve \( C(Y_{0, 1}, \nu^j_q) \) for \( p \in T_0 \) are pairwise disjoint, and if \( p \in T_0 \), then \( W(Y_{0, 1}, p) \) is contained in \( C(Y_{0, 1}, \nu^1_p) \cup C(Y_{0, 1}, \nu^2_p) \cup \text{Preimage}(Y_{0, 1}, p) \).

Let \( E \) be a component of \( D_{Y_{0, 1}} \) which contains \( C(Y_{0, 1}, \nu^j_q) \) for some \( p \in T_0 \) and some \( j \). Then there exists \( Y_{0, 2} \to Y_{0, 1} \) which is an isomorphism over \( Y_{0, 1} \setminus E \cap (\cup_{p \in T_0} W(Y_{0, 1}, p)) \), is toroidal for \( D_q \) over \( W(Y_{0, 1}, q) \cap E \) for \( q \in T_0 \), is an isomorphism over \( C(Y_{0, 1}, \nu^j_q) \setminus \text{Preimage}(q) \) for all \( q \in T_0 \), and so that if \( E \) is the strict transform of \( E \) on \( Y_{0, 1} \), then for \( p \in T_0 \), one of the following holds:

\[ W(Y_{0, 2}, p) \cap E = \emptyset \]

or

There exists a unique \( j \) such that

\[ W(Y_{0, 2}, p) \cap E \subset C(Y_{0, 2}, \nu^j_q) \subset E, \]

and if \( \overline{p}_j = \Lambda(Y_{0, 2}, \nu^j_q) \), then \( C(Y_{0, 2}, \nu^j_q) \) is smooth at \( \overline{p}_j \),

\[ (51) \]

and either \( \overline{p}_j \) is an isolated point in \( \text{Sing}_1(Y_{0, 2}) \) or \( C(Y_{0, 2}, \nu^j_q) \) is the only curve in \( \text{Sing}_1(Y_{0, 2}) \) which is contained in \( E \) and contains \( \overline{p}_j \),

and \( p_j \in C(Y_{0, 2}, \nu^k_{p'}) \) for some \( p' \in T_0 \) implies \( C(Y_{0, 2}, \nu^k_{p'}) = C(Y_{0, 2}, \nu^j_q) \).

We further have that \( Y_{0, 2} \setminus \cup_{p \in T_0} W(Y_{0, 2}, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_{0, 2} \setminus \cup_{p \in T_0} W(Y_{0, 2}, p) \).

Now repeat this procedure for other components of \( D_{Y_{0, 2}} \) which contain a curve \( C(Y_{0, 2}, \nu^j_q) \) for some \( j \) to construct \( Y_{0, 3} \to Y_{0, 2} \) so that condition (50) or (51) hold for all components \( E \) of \( D_{Y_{0, 3}} \) containing a curve \( C(Y_{0, 3}, \nu^j_q) \). We have that \( Y_{0, 3} \setminus \cup_{p \in T_0} W(Y_{0, 3}, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_{0, 3} \setminus \cup_{p \in T_0} W(Y_{0, 3}, p) \).
Now, by Lemma 3.4, let \( Y_{0,4} \to Y_{0,3} \) be a sequence of blow ups of 3-points for \( D \) and 2-curves of \( D \) on the strict transform of components \( E \) of \( D \) which contain \( C(Y_{0,3}, \nu_p^j) \) for some \( p \in T_0 \), so that if \( E \) is a component of \( D_{Y_{0,4}} \) which contains a curve \( C(Y_{0,4}, \nu_p^j) \), then \( Y_{0,4} \) is 3-prepared at all 2-points and 3-points of \( E \). We have that \( Y_{0,4} \setminus \cup_{p \in T_0} W(Y_{0,4}, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_{0,4} \setminus \cup_{p \in T_0} W(Y_{0,4}, p) \). We further have that for all \( p \in T_0 \), (50) or (51) holds on \( E \).

Now let \( E \) be a component of \( D_{Y_{0,4}} \) which contains a curve \( C(Y_{0,4}, \nu_p^j) \). Since one of the conditions (50) or (51) hold for all \( p \in T_0 \) on \( E \), we may apply Proposition 3.14 to \( E \) and the finitely many points

\[
A = \{ q \in E \mid Y_{0,4} \text{ is not 3-prepared at } q \},
\]

which are necessarily 1-points for \( D \), being sure that none of the finitely many 2-points for \( D \)

\[
B = \{ \Lambda(Y_{0,4}, \nu_p^j) \mid p \in T_0 \}
\]

are in the image of the general curves blown up, to construct a sequence of permissible blow ups \( Y_{0,5} \to Y_{0,4} \) so that \( Y_{0,5} \to Y_{0,4} \) is an isomorphism in a neighborhood of \( \cup_{p \in T_0} W(Y_{0,4}, p) \) and over \( Y_{0,4} \setminus E \), and \( Y_{0,5} \) is 3-prepared over \( E \setminus \cup_{p \in T_0} \Lambda(Y_{0,4}, \nu_p^j) \). We have that \( Y_{0,5} \setminus \cup_{p \in T_0} W(Y_{0,5}, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_{0,5} \setminus \cup_{p \in T_0} W(Y_{0,5}, p) \). We further have that for all \( p \in T_0 \), (50) or (51) hold on the strict transform \( E \) on \( Y_{0,5} \).

Now repeat this procedure for other components of \( D_{Y_{0,5}} \) which contain a curve \( C(Y_{0,5}, \nu_p^j) \) for some \( j \) to construct \( X_1 \to Y_{0,5} \) so that \( X_1 \) is 3-prepared over \( E \setminus \cup_{p \in T_0} \Lambda(Y_{0,4}, \nu_p^j) \) for all components \( E \) of \( D_{Y_{0,5}} \) which contain a curve \( C(Y_{0,5}, \nu_p^j) \) for some \( p \in T_0 \). We then have that the following holds.

1.1) \( X_1 \to X^0 \) is the canonical sequence of blow ups above a general point \( \eta \) of a curve in \( \Gamma_0 \) (so that \( \sigma_D(\eta) = r \)).

1.2) \( X_1 \to X^0 \) is toroidal for \( \overline{D}_p \) in a neighborhood of \( W(X_1, p) \), for \( p \in T_0 \).

1.3) \( X_1 \setminus \cup_{p \in T_0} W(X_1, p) \) is 2-prepared and \( \sigma_D(q) < r \) for \( q \in X_1 \setminus \cup_{p \in T_0} W(X_1, p) \).

1.4) If \( p \in T_0 \) then \( \sigma_D(q) \leq r - 1 \) and \( X_1 \) is 3-prepared at \( q \) for

\[
qu \in C(X_1, \nu_p^j) \setminus \cup_{p' \in T_0 \cap (C(X_1, \nu_p^j)) = C(X_1, \nu_p^k)} \text{ for some } k \text{ Preimage}(X_1, p').
\]

1.5) Let

\[
T_1 = \begin{cases} 
2\text{-points } q \text{ for } D \text{ of} \\
C(X_1, \nu_p^j) \setminus \cup_{p' \in T_0 \cap (C(X_1, \nu_p^j)) = C(X_1, \nu_p^k)} \text{ for some } k \text{ Preimage}(X_1, p') 
\end{cases}
\]

such that \( \sigma_D(q) > 0 \) and such that \( p \in T_0 \) with \( \sigma_D(\eta) = r - 1 \) for \( \eta \in C(X_1, \nu_p^j) \) the generic point.

\( X_1 \) is 3-prepared at \( p \in T_1 \). For \( q \in T_1 \), choose \( (U_q, \overline{D}_q, I_q, \nu_q^1, \nu_q^2) \). We have \( 0 < \sigma_D(q) \leq r - 1 \) for \( q \in T_1 \).

1.6) Suppose that \( p \in T_0 \) and \( C(X_1, \nu_p^j) \) is such that \( \sigma_D(\eta) = r - 1 \) for \( \eta \in C(X_1, \nu_p^j) \) the generic point. Then \( \sigma_D(q) = r - 1 \) for \( q \in C(X_1, \nu_p^j) \setminus \cup_{p' \in T_0 \cup T_1} W(X_1, p') \). If \( q \in T_0 \cup T_1 \) and \( W(X_1, q) \cap C(X_1, \nu_p^j) \neq \emptyset \), then \( C(X_1, \nu_p^j) = C(X_1, \nu_q^i) \) for some \( i \). (This follows from Lemma 5.1 since \( \sigma_D(q) \leq r - 1 \) for \( q \in T_1 \).)
Now for \( m \geq r \), we inductively construct

\[
X_{m,r-1} \rightarrow \cdots \rightarrow X_{m,0} \rightarrow \cdots \rightarrow X_{r+1,r-1} \rightarrow \cdots \rightarrow X_{r+1,0} \rightarrow \\
X_{r,r-1} \rightarrow X_{r,r-2} \rightarrow \cdots \rightarrow X_{r,0} \rightarrow X_{r-1,r-2} \rightarrow \cdots \rightarrow X_{3,0} \rightarrow X_{2,1} \rightarrow X_{2,0} \rightarrow X_{1,0} = X_1 \rightarrow X^0
\]
so that

2.1) \( X_{1,0} = X_1 \rightarrow X^0 \) is the canonical sequence of blow ups above a general point \( \eta \) of a curve in \( \Gamma_0 \) (so that \( \sigma_D(\eta) = r \)), and for \( i > 0 \),

\[
X_{i+1,0} \rightarrow X_{i,\min\{i-1,r-1\}}
\]

is the canonical sequence of blowups above a general point \( \eta \) of a curve \( C(X_{i,\min\{i-1,r-1\}}, \nu^j_p) \) with \( p \in T_0 \) and such that \( \sigma_D(\eta) = \max\{0, r-i\} \),

and the following properties hold on \( X_{i,l} \).

2.2) \( X_{i,l} \rightarrow X_{j,k} \) is toroidal for \( D_p \) in a neighborhood of \( W(X_{i,l}, p) \), for \( p \in T_{j,k} \) with \( T_{j,k} = T_0 \), or \( 1 \leq j \leq i-1 \) and \( 0 \leq k \leq \min\{j-1, r-1\} \), or \( j = i \) and \( 0 \leq k \leq l-1 \).

2.3) \( X_{i,l} \cap \bigcup_{p \in T_0} \bigcup_{\{j=1, \ldots, l\}} \bigcup_{\{k=0, \ldots, j-1\}} W(X_{i,l}, p) \) is 2-prepared and \( \sigma_D(q) < r \) for \( q \in X_{i,l} \cap \bigcup_{p \in T_0} \bigcup_{\{j=1, \ldots, l\}} \bigcup_{\{k=0, \ldots, j-1\}} W(X_{i,l}, p) \).

2.4) If \( p \in T_0 \) then \( \sigma_D(\eta) \leq \max\{0, r-i\} \) for \( \eta \in C(X_{i,l}, \nu^j_p) \) the generic point, and \( X_{i,l} \) is 3-prepared at \( q \) for

\[
q \in C(X_{i,l}, \nu^j_p) \cap \bigcup_{p \in \Omega} \text{Preimage}(X_{i,l}, p')
\]

where

\[
\Omega = \{ p' \in T_0 \cap \bigcup_{\{j=1, \ldots, l\}} \bigcup_{\{k=0, \ldots, j-1\}} W(X_{i,l}, p') \} \quad \text{for some } k.
\]

2.5) We have the set

\[
T_{i,l} = \left\{ p' \in T_0 \cup \bigcup_{\{j=1, \ldots, l\}} \bigcup_{\{k=0, \ldots, j-1\}} W(X_{i,l}, p') \mid C(X_{i,l}, \nu^j_p) = C(X_{i,l}, \nu^k_p) \text{ for some } k \right\}
\]

such that \( \sigma_D(\eta) > 0 \) and such that \( p \in T_0 \) with

\[
\sigma_D(\eta) = \max\{0, r-i\} \quad \text{for } \eta \in C(X_{i,l}, \nu^j_p) \quad \text{the generic point.}
\]

\( X_{i,l} \) is 3-prepared at \( p \in T_{i,l} \). We have local resolvers \( (U_p, D_p, I_p, \nu^1_p, \nu^1_p) \) at \( p \in T_{i,l} \).

We have \( \max\{1, r-i\} \leq \sigma_D(\eta) \leq r-l-1 \) for \( q \in T_{i,l} \).

2.6) Suppose that \( p \in T_0 \) and \( C(X_{i,l}, \nu^j_p) \) is such that \( \sigma_D(\eta) = \max\{0, r-i\} \) for \( \eta \in C(X_{i,l}, \nu^j_p) \) the generic point. Then \( \sigma_D(\eta) = \max\{0, r-i\} \) for

\[
q \in C(X_{i,l}, \nu^j_p) \cap \bigcup_{p' \in T_0} \bigcup_{\{j=1, \ldots, l\}} \bigcup_{\{k=0, \ldots, j-1\}} W(X_{i,l}, p')
\]

Further,

a) If \( q \in T_0 \cap \bigcup_{\{j=1, \ldots, l\}} \bigcup_{\{k=0, \ldots, j-1\}} W(X_{i,l}, q) \cap C(X_{i,l}, \nu^j_p) \neq \emptyset \), then \( C(X_{i,l}, \nu^j_p) = C(X_{i,l}, \nu^k_p) \) for some \( k \).

b) If \( q \in T_{i,l} \) and \( q \in C(X_{i,l}, \nu^j_p) \), then either \( C(X_{i,l}, \nu^j_p) = C(X_{i,l}, \nu^k_p) \) for some \( k \) or \( \max\{0, r-i\} < \sigma_D(q) \leq r-l-1 \).
Note that the condition “$\sigma_D(q) > 0$” in the definition of $T_{i,l}$ is automatically satisfied if $i < r$. If $l = \min\{i - 1, r - 1\}$, condition 2.6) becomes “Suppose that $p \in T_0$ and $C(X_{i,l}, \nu^p_i)$ is such that $\sigma_D(\eta) = \max\{0, r - i\}$ for $\eta \in C(X_{i,l}, \nu^p_i)$ the generic point. Then if $q \in T_0 \cup \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) \cup \left(\cup_{n=0}^{r-i} T_{i,n}\right)$ and $W(X_{i,l}, q) \cap C(X_{i,l}, \nu^p_i) \neq \emptyset$, then $C(X_{i,l}, \nu^p_i) = C(X_{i,l}, \nu^q_i)$ for some $k$”.

We now prove the above inductive construction of (52). Suppose that we have made the construction out to $X_{i,t}$.

**Case 1.** Suppose that $l = \min\{i - 1, r - 1\}$. We will construct $X_{i+1,0} \to X_{i,\min\{i-1,r-1\}}$.

First suppose that $r > i$. Let $Y_i \to X_{i,i-1}$ be the product of the canonical sequences of blow ups above all curves $C(X_{i,j}, \nu^p_j)$ for $p \in T_0$ such that $\sigma_D(\eta) = r - i$ at a generic point $\eta \in C(X_{i,j}, \nu^p_j)$. This is a permissible sequence of blow ups by the comment following 2.6) above. We have that $Y_i \setminus \cup_{p \in T_0} \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) W(Y_i, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_i \setminus \cup_{p \in T_0} \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) W(Y_i, p)$. Further, $Y_i \to X_{i,i-1}$ is toroidal for $D_p$ in a neighborhood of $W(Y_i, p)$ for $p \in T_0 \cup \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right)$.

Now suppose that $r \leq i$. On $X_{i,r-1}$, we have that $\sigma_D(q) = 0$ for $p \in T_0$ and $q \in C(X_{i,r-1}, \nu^p_r) \setminus \cup_{p' \in T_0} \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) W(X_{i,r-1}, p')$. By Lemmas 3.9, 3.10, 3.11 and 3.12, there exists a sequence $Y_i \to X_{i,r-1}$ of blow ups of prepared points on the strict transform of curves $C(X_{i,r-1}, \nu^p_r)$ with $p \in T_0$, followed by the blow ups of the strict transforms of these $C(X_{i,r-1}, \nu^p_r)$, so that for $q \in T_0 \cup \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) W(Y_i, p)$ is 2-prepared and $\sigma_D(q) < r$ for

$$q \in Y_i \setminus \cup_{p \in T_0} \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) W(Y_i, p).$$

Further, $Y_i \to X_{i,r-1}$ is toroidal for $D_p$ in a neighborhood of $W(Y_i, p)$ for $p \in T_0 \cup \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right)$.

From now on, we consider both cases $r > i$ and $r \leq i$ simultaneously. Let $Y_{i,1} \to Y_i$ be a torodial morphism for $D$ so that the components of $D$ containing some curve $C(Y_{i,1}, \nu^p_i)$ for $p \in T_0 \cup \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right)$ are pairwise disjoint, and if

$$p \in \cup_{p' \in T_0} \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) W(Y_{i,1}, p')$$

then $W(Y_{i,1}, p)$ is contained in $C(Y_{i,1}, \nu^1_i) \cup C(Y_{i,1}, \nu^2_i) \cup \text{Preimage}(Y_{i,1}, p)$.

Let $E$ be a component of $D$ on $Y_{i,1}$ which contains $C(Y_{i,1}, \nu^\alpha_i)$ for some $\alpha \in T_0$ and some $j$. Then there exists $Y_{i,2} \to Y_{i,1}$ which is an isomorphism over

$$Y_{i,1} \setminus E \cap \left(\cup_{p' \in T_0} \left(\cup_{j=1}^{r-i-1} \cup_{k=0}^{\min(j-1,r-1)} T_{j,k}\right) W(Y_{i,1}, p')\right),$$

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is toroidal for $D_q$ over $W(Y_{i,1}, q) \cap E$ for $q \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)$, is an isomorphism over $C(Y_{i,1}, \nu^q_j) \setminus \text{Preimage}(Y_{i,1}, q)$ for all $q \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)$, and so that if $E$ is the strict transform of $E$ on $Y_{i,2}$, then for $p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)$, one of the following holds:

\begin{equation}
W(Y_{i,2}, p) \cap \overline{E} = \emptyset
\end{equation}

or

\begin{equation}
\text{There exists a unique } j \text{ such that}
\end{equation}

\[ W(Y_{i,2}, p) \cap \overline{E} \subset C(Y_{i,2}, \nu^p_j) \subset \overline{E}, \]

and

if $\overline{p}_j = \Lambda(Y_{i,2}, \nu^j_2)$, then $C(Y_{i,2}, \nu^p_j)$ is smooth at $\overline{p}_j$, and either $\overline{p}_j$ is an isolated point in $\text{Sing}_1(Y_{i,2})$ or $C(Y_{i,2}, \nu^j_2)$ is the only curve in $\text{Sing}_1(Y_{i,2})$ which is contained in $\overline{E}$ and contains $\overline{p}_j$, and

\[ \overline{p}_j \in C(Y_{i,2}, \nu^k_p) \text{ for some } p' \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right) \] implies $C(Y_{i,2}, \nu^k_p) = C(Y_{i,2}, \nu^j_2)$. We have that $Y_{i,2} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,2}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{i,2} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,2}, p)$.

Now repeat this procedure for other components of $D$ for $Y_{i,2}$ which contain a curve $C(Y_{i,2}, \nu^\alpha_2)$ with $\alpha \in T_0$ for some $j$ to construct $Y_{i,3} \to Y_{i,2}$ so that condition (53) or (54) hold for all $p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)$ and components $E$ of $D$ for $Y_{i,3}$ containing a curve $C(Y_{i,3}, \nu^\alpha_2)$ with $\alpha \in T_0$. We have that $Y_{i,3} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,3}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{i,3} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,3}, p)$.

Now, by Lemma 3.4, let $Y_{i,4} \to Y_{i,3}$ be a sequence of blow ups of 2-curves of $D$ on the strict transform of components $E$ of $D$ which contain $C(Y_{i,3}, \nu^\alpha_2)$ for some $\alpha \in T_0$, so that if $E$ is a component of $D_{Y_{i,4}}$ which contains a curve $C(Y_{i,4}, \nu^\alpha_2)$ with $\alpha \in T_0$, and if $p \in E \setminus \bigcup_{q \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)} \Lambda(Y_{i,4}, \nu^j_2)$ is a 2-point, then $p$ is 3-prepared.

We have that $Y_{i,4} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,4}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{i,4} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,4}, p)$. We further have that for all $p \in T_0 \cup \left( \bigcup_{j=1}^{\min(j-1,r-1)} T_{j,k} \right)$, (53) or (54) holds on $E$.

Now let $E$ be a component of $D$ for $Y_{i,4}$ which contains a curve $C(Y_{i,4}, \nu^\alpha_2)$ with $\alpha \in T_0$. Let

\[ T = \{ q \in E \mid Y_{i,4} \text{ is not 3-prepared at } q \}. \]

If $r \leq i$, let

\[ T' = \left\{ 1\text{-points } q \text{ of } D \text{ contained in } E \text{ such that} \right. \]

\[ q \in C(Y_{i,4}, \nu^j_2) \text{ for some } p \in T_0 \text{ and } \sigma_D(q) > 0 \left. \right\}. \]
Since one of the conditions (53) or (54) hold for all \( p \in T_0 \cup \left( \bigcup_{j=1}^{\infty} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right) \) on \( E \), we may apply Proposition 3.14 to \( E \) and the finite set of points \( A = T \), if \( r > i \) or \( A = T \cup T' \) if \( r \leq i \), which are necessarily 1-points for \( D \) lying on \( E \), being sure that none of the finitely many points 2-points of \( D \)

\[
B = \{ \Lambda(Y_{i,4}, \nu_{p}^{j}) \mid p \in T_0 \cup \left( \bigcup_{j=1}^{\infty} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right) \}
\]

are in the image of the general curves blown up, to construct a sequence of permissible transforms \( Y_{i,5} \rightarrow Y_{i,4} \) so that \( Y_{i,5} \rightarrow Y_{i,4} \) is an isomorphism in a neighborhood of \( \cup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\infty} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,4}, p) \) and over \( Y_{i,4} \setminus E \), and \( Y_{i,5} \) is 3-prepared over \( E \).

Now we may construct, using the method of Case 1, a morphism \( X_{i+1,0} \rightarrow Y_{i,5} \) with \( \alpha \in T_0 \) for some \( j \) to construct \( X_{i+1,0} \rightarrow Y_{i,5} \) so that \( X_{i+1,0} \) is 3-prepared over \( E \setminus \cup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\infty} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} \Lambda(Y_{i,5}, \nu_{p}^{j}) \) for all components \( E \) of \( D \) for \( Y_{i,5} \) which contain a curve \( C(Y_{i,5}, \nu_{p}^{j}) \) with \( \alpha \in T_0 \).

Let

\[
T_{i+1,0} = \left\{ \begin{array}{l}
2\text{-points } q \text{ for } D \text{ of } C(X_{i+1,0}, \nu_{p}^{j}) \setminus \cup_{p' \in \Omega} \text{Preimage}(X_{i+1,0}, p')
\end{array} \right\}
\]

where \( \Omega = \{ p' \in T_0 \cup \left( \bigcup_{j=1}^{\infty} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right) \mid C(X_{i+1,0}, \nu_{p}^{j}) = C(X_{i+1,0}, \nu_{p'}^{j}) \text{ for some } l \} \)

such that \( \sigma_{D}(q) > 0 \) and such that \( p \in T_0 \) with

\[
\sigma_{D}(\eta) = \max \{ 0, r - i - 1 \} \text{ for } \eta \in C(X_{i+1,0}, \nu_{p}^{j}) \text{ a general point}.
\]

Now repeat this procedure for other components of \( D_{Y_{i,5}} \) with \( \alpha \in T_0 \) for some \( j \) to construct \( X_{i+1,0} \rightarrow Y_{i,5} \) so that \( X_{i+1,0} \) is 3-prepared over \( E \setminus \cup_{p \in T_0 \cup \left( \bigcup_{j=1}^{\infty} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} \Lambda(Y_{i,5}, \nu_{p}^{j}) \) for all components \( E \) of \( D \) for \( Y_{i,5} \) which contain a curve \( C(Y_{i,5}, \nu_{p}^{j}) \) with \( \alpha \in T_0 \).

**Case 2** Now suppose that \( l < \min\{i - 1, r - 1\} \). We will construct \( X_{i,l+1} \rightarrow X_{i,l} \). Let \( \Omega \) be the set of points \( q \in T_{i,l} \) such that \( q \) is contained in a curve \( C(X_{i,l}, \nu_{p}^{j}) \) where \( p \in T_0 \) and \( \sigma_{D}(\eta) = \max \{ 0, r - i \} \) for \( \eta \in C(X_{i,l}, \nu_{p}^{j}) \) a general point. By condition 2.5) satisfied by \( X_{i,l} \),

\[
\max \{ 1, r - i \} \leq \sigma_{D}(q) \leq r - l - 1
\]

for \( q \in \Omega \). Let \( Y \rightarrow X_{i,l} \) be a morphism which is an isomorphism over \( X_{i,l} \setminus \Omega \) and is toroidal for \( D_{Y} \) above \( q \in \Omega \) and such that \( C(Y, \nu_{p}^{j}) \cap W(Y, q) = \emptyset \) if \( C(Y, \nu_{p}^{j}) \) is such that \( p \in T_0 \), \( \sigma_{D}(\eta) = \max \{ 0, r - i \} \) if \( \eta \in C(Y, \nu_{p}^{j}) \) is a general point, and \( C(Y, \nu_{p}^{j}) \neq C(Y, \nu_{p}^{j}) \) for any \( k \). For such a case we have by (55), that \( \sigma_{D}(\eta) \leq \max \{ 0, r - l - 2 \} \) if \( \eta = \Lambda(Y, \nu_{p}^{j}) \).

Now we may construct, using the method of Case 1, a morphism \( X_{i,l+1} \rightarrow Y \) such that
$X_{i,l+1} \to X_{i,l}$ is toroidal for $D$ above $X_{i,l} \setminus \Omega$, and the conditions 2.2) - 2.6) following (52) hold. This completes the inductive construction of (52).

For $m$ sufficiently large in (52), we have that for $p \in T_0$, $I_p \mathcal{O}_{X,m,r-1,p}$ is locally principal at a general point $\eta$ of a curve $C(X,m,r-1,\nu_p^1)$.

After possibly performing a toroidal morphism for $D$, we have that the locus where $I_p(\mathcal{O}_{X,m,r-1}| \text{Preimage}(X,m,r-1, U_p))$ is not locally principal is supported above $p$ for $p \in T_0$. Thus toroidal morphisms for $D_p$ above Preimage$(X,m,r-1, U_p)$ which principalize $I_p$ above $U_p$ for $p \in T_0$ extend to a morphism $Z^1 \to X_{m,r-1}$ which is an isomorphism over $X_{m,r-1} \setminus \cup_{p \in T_0} \text{Preimage}(X_{m,r-1}, p)$. We have that $W(Z^1, p) = \emptyset$ for $p \in T_0$. We have that $Z^1$ is 2-prepared at $q \in Z^1 \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}} W(Z^1, p)$ and $\sigma_D(q) \leq r - 1$ for $q \in Z^1 \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}} W(Z^1, p)$.

If $r = 1$, then $Z^1$ is prepared. In this case let $X_1 = Z^1$. Suppose that $r > 1$. Let $Z^1 \to Z^1$ be a toroidal morphism for $D$ so that components of $D$ containing curves $C(Z^1, \nu_p^1)$ for $p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}$ are pairwise disjoint, and that if $p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}$, then $W(Z^1, p)$ is contained in $C(Z^1, \nu_p^1) \cup C(Z^1, \nu_p^2) \cup \text{Preimage}(Z^1, p)$.

Let $E$ be a component of $D$ on $Z^1$ which contains $C(Z^1, \nu_p^1)$ for some $p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}$ or contains a point $q \in E \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}} W(Z^1, p)$ such that $\sigma_D(q) = r - 1$.

Then there exists $Z_2^1 \to Z^1$ which is an isomorphism over

$$Z^1 \setminus E \cap (\cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}} W(Z^1, p)),$$

is toroidal for $D_q$ over $W(Z^1, q) \cap E$ for $q \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}$, is an isomorphism over $C(Z^1, \nu_q^1) \setminus \text{Preimage}(Z^1, q)$ for all $q \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}$ and factors as a sequence of permissible blow ups of points and curves

$$Z_2^1 = Z_2^{1,n} \to Z_2^{1,n-1} \to \cdots \to Z_2^{1,1} \to Z^1,$$

such that the center blown up in $Z_2^{1,t} \to Z_2^{1,t-1}$ is a curve or point contained in $W(Z_2^{1,t-1}, p)$ for some $p \in \cup_{j=1}^{m} \cup_{l=1}^{\min(j-1, r-1)} T_{j,l}$, and so that if $E$ is the strict transform of $E$ on $Z_2^1$, then for $p \in \cup_{j=1}^{m} \cup_{k=1}^{\min(j-1, r-1)} T_{j,k}$, one of the following holds:

\begin{equation}
W(Z_2^1, p) \cap E = \emptyset
\end{equation}
or

(57)

There exists a unique \( j \) such that

\[ W(Z_2^1, p) \cap \mathcal{E} \subset C(Z_2^1, \nu^j_0) \subset \mathcal{E}, \]

and

if \( \overline{p}_j = \Lambda(Z_2^1, \nu^j_0) \), then \( C(Z_2^1, \nu^j_0) \) is smooth at \( \overline{p}_j \),

and either \( \overline{p}_j \) is an isolated point in \( \text{Sing}_1(Z_2^1) \) or \( C(Z_2^1, \nu^j_0) \)

is the only curve in \( \text{Sing}_1(Z_2^1) \) which is contained in \( \mathcal{E} \) and contains \( \overline{p}_j \),

and

\( \overline{p}_j \in C(Z_2^1, \nu^j_0) \) for some \( p' \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \) implies \( C(Z_2^1, \nu^j_0) = C(Z_2^1, \nu^j_0) \)

and

If \( \gamma \) is a 2-curve of \( D \) on \( E \) which contains \( \overline{p}_j \),

then \( \sigma_D(q) \leq r - 2 \) for \( q \in \gamma \setminus \{ \overline{p}_j \} \).

Note that no new components of \( D \) containing points

\[ p \in D \setminus \left( \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_2^1, p) \right) \]

with \( \sigma_D(p) = r - 1 \) can be created as

\[ q \in \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} \text{(Preimage}(Z_2^1, W(Z_1^1, p)) \setminus W(Z_2^1, p)) \]

implies \( \sigma_D(q) \leq r - 2 \).

We further have that \( Z_2^1 \) is 2-prepared at \( q \in Z_2^1 \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_2^1, p) \) and

\( \sigma_D(q) \leq r - 1 \) for \( q \in Z_2^1 \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_2^1, p) \).

Now repeat this procedure for other such components \( E \) of \( D \) for \( Z_2^1 \) which contain

\( C(Z_2^1, \nu^j_0) \) for some \( p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \) or contain a point

\[ q \in E \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_2^1, p) \]

with \( \sigma_D(q) = r - 1 \) (which are necessarily the strict transform of a component of \( D \)

on \( Z_1^1 \)) to construct \( Z_3^1 \to Z_2^1 \) so that for all \( p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \), condition

(56) or (57) hold for all components \( E \) of \( D \) for \( Z_3^1 \) which contain \( C(Z_3^1, \nu^j_0) \) for some

\( p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \) or contain a point \( q \in E \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_3^1, p) \) with

\( \sigma_D(q) = r - 1 \). We have that \( Z_3^1 \) is 2-prepared at \( q \in Z_3^1 \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_3^1, p) \)

and \( \sigma_D(q) \leq r - 1 \) for \( q \in Z_3^1 \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_3^1, p) \).

Now by Lemma 3.4, we can perform a toroidal morphism for \( D \) (which is a sequence of blowups of 2-curves for \( D \)) \( Z_3^1 \to Z_3^1 \), so that we further have that if \( G \) is a component of \( D_{Z_4^1} \) containing a curve \( C(Z_4^1, p) \) for some \( p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \) or

\( G \setminus \cup_{p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}} W(Z_4^1, p) \) contains a point \( q \) with \( \sigma_D(q) = r - 1 \), then \( Z_4^1 \) is 3-

prepared at all 2-points and 3-points of \( G \). We further have that for all \( p \in \cup_{j=1}^{m} \cup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \), (56) or (57) holds on \( G \).
We now may apply Proposition 3.14 to the union $H$ of components $E$ of $D$ for $Z^1_4$ containing a curve $C(Z^1_4, \nu^q_p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, or containing a point $q$ with $\sigma_D(q) = r - 1$

$$A = \{ q \in H \ | \ Z^1_4 \text{ is not 3-prepared at } q \text{ (which are necessarily one points of } D) \}$$

being sure that none of the finitely many 2-points for $D

$$B = \{ \Lambda(Z^1_4, \nu^q_j) \ | \ p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k} \}$$

are in the image of the general curves blown up, to construct $X^1 \to Z^1_4$ so that $X^1$

is 3-prepared over $E \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}} \Lambda(X^1, \nu^q_p)$ for all components $E$ of $D$ for $X^1$ which contain a curve $C(X^1, \nu^q_p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, or contain a point $q \in X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$ with $\sigma_D(q) = r - 1$. Further, for all

$$p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k},$$

condition (56) or (57) hold on components $F$ of $D$ for $X^1$

containing a curve $C(X^1, \nu^q_p)$ or a point $q \in X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$ such that $\sigma_D(q) = r - 1$.

We now have (using Lemma 5.1) the following:

3.1) $X^1 \to X_{j,k}$ is toroidal for $D_p$ for $p \in T_{j,k}$ with $1 \leq j \leq m$, $0 \leq k \leq \min\{j-1,r-1\}$ in a neighborhood of $W(X^1, p)$.

3.2) $X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$ is 2-prepared and $\sigma_D(q) \leq r - 1$ for $q \in X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$.

3.3) Suppose that $1 < r$. Then

a) $X^1$ is 3-prepared at all points

$$q \in C(X^1, \nu^k_p) \setminus \bigcup_{p' \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p'),$$

for $p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}$.

b) $X^1$ is 3-prepared at all points of

$$\left( X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p) \right) \cap \text{Sing}_{r-1}(X^1),$$

and if $C \subset \text{Sing}_{r-1}(X^1)$ is not equal to a curve $C(X^1, \nu^k_p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}$, then

$$C \cap \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p) = \emptyset.$$

3.4) Suppose that $1 < r$. Let

$$T^1_0 = \left\{ \begin{array}{ll} 2\text{-points } q \text{ of } X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p) & \text{such that } \sigma_D(q) = r - 1, \end{array} \right\}$$
For \( p \in T_0^1 \), let \((U_p, \overline{\mathcal{D}}_p, \nu_p^1, \nu_p^2)\) be associated local resolvers. Let \( \Gamma_1 \) be the union of the curves
\[
\left\{ C(X^1, \nu_p^1) \text{ such that } p \in \left( \bigcup_{j=1}^{m} \bigcup_{j=0}^{\min\{j-1, r-j\}} T_{j,k} \right) \cup T_0^1 \right\}
\]
and \( \sigma_D(\eta) = r - 1 \) for \( \eta \in C(X^1, \nu_p^1) \) a general point
and any remaining curves \( C \) in
\[
\text{Sing}_{r-1}(X^1 \setminus \left( \bigcup_{j=1}^{m} \bigcup_{j=0}^{\min\{j-1, r-j\}} T_{j,k} \right) \cup T_0^1)
\]
(which are necessarily closed in \( X^1 \) and do not contain 2-points).

3.5) Suppose that \( 1 < r \). Suppose that
\[
p \in \left( \bigcup_{j=1}^{m} \bigcup_{j=0}^{\min\{j-1, r-j\}} T_{j,k} \right) \cup T_0^1
\]
and \( C(X^1, \nu_p^1) \) is such that \( \sigma_D(\eta) = r - 1 \) for \( \eta \in C(X^1, \nu_p^1) \) the generic point. Then \( \sigma_D(q) = r - 1 \) for
\[
q \in C(X^1, \nu_p^1) \setminus \left( \bigcup_{p' \in \left( \bigcup_{j=1}^{m} \bigcup_{j=0}^{\min\{j-1, r-j\}} T_{j,k} \right) \cup T_0^1} W(X^1, p') \right).
\]
Further, if \( q \in \left( \bigcup_{j=1}^{m} \bigcup_{j=0}^{\min\{j-1, r-j\}} T_{j,k} \right) \cup T_0^1 \) and \( W(X^1, q) \cap C(X^1, \nu_p^1) \neq \emptyset \), then
\( C(X^1, \nu_p^1) = C(X^1, \nu_q^n) \) for some \( n \).

Now we proceed in this way to inductively construct sequences of blow ups for \( 0 \leq j \leq r - 1 \) (as in the algorithm of (52)), where we identify \( X_0^0 \) with \( X_{i,l} \),
\[
X^j \to X^j_{m_j-1, r-j-1}
\]
for \( 1 \leq j \leq r \) (as in the construction of \( X^1 \)) such that for \( 1 \leq j \leq r \),
\begin{enumerate}
\item \( X^j \to X^j_{i,k} \) is toroidal for \( \overline{\mathcal{D}}_p \) for \( p \in T^j_{i,k} \) with \( 1 \leq i \leq m_{j-1}, 0 \leq k \leq \min\{i-1, r-j\} \) in a neighborhood of \( W(X^j, p) \).
\item \( X^j \setminus \bigcup_{p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\} T_{i,k}}} W(X^j, p) \) is 2-prepared and \( \sigma_D(q) \leq r - j \) for \( q \in \bigcup_{p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\} T_{i,k}}} W(X^j, p) \).
\item \( X^j \) is 3-prepared at all points
\end{enumerate}

\begin{enumerate}
\item \( q \in C(X^j, \nu_p^k) \setminus \bigcup_{p' \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\} T_{i,k}}} W(X^j, p') \) for some \( l \) \( \text{Preimage}(X^j, p') \)
\item \( X^j \) is 3-prepared at all points of
\[
\left( X^j \setminus \bigcup_{p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\} T_{i,k}}} W(X^j, p) \right) \cap \text{Sing}_{r-j}(X^j).
\]
\end{enumerate}
and if $C \subset \text{Sing}_{r-j}(X^j)$ is not equal to a curve $C(X^j, \nu^j_p)$ for some $p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1}$, then

$$C \cap \bigcup_{p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1}} W(X^j, p) = \emptyset.$$ 

4.4) Suppose that $j < r$. Let

$$T_0^j = \left\{ \begin{array}{ll}
2\text{-points } q \text{ of } X^j - \bigcup_{p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1}} W(X^j, p) & \\
\text{such that } \sigma_{D}(q) = r - j & 
\end{array} \right\}$$

For $p \in T_0^j$, let $(U_p, D_p, \nu^j_p, \nu^n_p)$ be associated local resolvers.

Let $\Gamma_j$ be the union of the curves

$$\left\{ (X^j, \nu^j_i) \text{ such that } p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1} \cup T_0^j \text{ and } \sigma_{D}(\eta) = r - j \text{ for } \eta \in (X^j, \nu^j_p) \text{ a general point} \right\}$$

and any remaining curves $C$ in

$$\text{Sing}_{r-j}(X^j \setminus \left( \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1} \cup T_0^j \right))$$

(which are necessarily closed in $X^j$ and do not contain 2-points).

4.5) Suppose that $j < r$. Suppose that

$$p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1} \cup T_0^j$$

and $C(X^j, \nu^j_p)$ is such that $\sigma_{D}(\eta) = r - j$ for $\eta \in C(X^j, \nu^j_p)$ the generic point. Then $\sigma_{D}(q) = r - j$ for

$$q \in C(X^j, \nu^j_p) \setminus \bigcup_{p' \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1} \cup T_0^j} W(X^j, p').$$

Further, if $q \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min(i-1,r-j)} T_{i,k}^{j-1} \cup T_0^j$ and $W(X^j, q) \cap C(X^j, \nu^j_p) \neq \emptyset$, then $C(X^j, \nu^j_p) = C(X^j, \nu^n_q)$ for some $n$.

For $0 \leq j \leq r - 1$, $0 \leq i \leq m_j$ and $0 \leq k \leq \min\{i-1, r-j-1\}$,

5.1) $X^j_{i,0} \to X^j$ is the canonical sequence of blow ups above a general point $\eta$ of a curve in $\Gamma_j$ (so that $\sigma_{D}(\eta) = r - j$), and for $i > 0$,

$$X^j_{i+1,0} \to X^j_{i,\min\{i-1, r-j-1\}}$$

is the canonical sequence of blow ups above a general point $\eta$ of a curve

$$C(X^j_{i,\min\{i-1, r-j-1\}}, \nu^j_p)$$

with $p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1} \cup T_0^j$ and $\sigma_{D}(\eta) = \max\{0, r - i - j\}$,

and the following properties hold. Let

$$S_{i,k}^j = \left( \bigcup_{l=1}^{m_j} \bigcup_{n=0}^{\min\{l-1, r-j\}} T_{l,n}^{j-1} \cup T_0^j \cup \left( \bigcup_{l=1}^{i-1} \bigcup_{n=0}^{\min\{l-1, r-j-1\}} T_{l,n}^j \right) \cup \left( \bigcup_{n=0}^{k-1} T_{i,n}^j \right) \right).$$
5.2) $X_{i,k}^j \to X_{l,n}^s$ is toroidal for $D_p$ in a neighborhood of $W(X_{i,k}^j, p)$ for $p \in S_{i,k}^j$ (with $p \in X_{l,n}^s$).

5.3) $X_{i,k}^j \setminus (\bigcup_{p \in S_{i,k}^j} W(X_{i,k}^j, p))$ is 2-prepared and $\sigma_D(p) < r - j$ for $q \in X_{i,k}^j \setminus (\bigcup_{p \in S_{i,k}^j} W(X_{i,k}^j, p))$.

5.4) If $p \in \left( \bigcup_{l=1}^{m_j-1} \bigcup_{n=0}^{\min(l-1,r-j)} T_{i,n}^{j-1} \right) \cup T_0^j$, then $\sigma_D(\eta) \leq \max\{0, r - i - j\}$ for $\eta \in C(X_{i,k}^j, \nu_p^j)$ the generic point and $X_{i,k}^j$ is 3-prepared at $q$ for

$$q \in C(X_{i,k}^j, \nu_p^j) \setminus \bigcup_{p' \in S_{i,k}^j} \setminus C(X_{i,k}^j, \nu_{p'}^j) = C(X_{i,k}^j, \nu_p^j)$$

for some $i$.

5.5) We have the set

$$T_{i,n}^j = \left\{ \begin{array}{l}
2\text{-points } q \text{ for } D \text{ of } C(X_{i,k}^j, \nu_p^j) \setminus \bigcup_{p' \in \Omega} \text{Preimage}(X_{i,k}^j, p'), \\
\text{such that } \sigma_D(q) > 0 \text{ and such that } \\
p \in \left( \bigcup_{l=1}^{m_j-1} \bigcup_{n=0}^{\min(l-1,r-j)} T_{i,n}^{j-1} \right) \cup T_0^j \\
\text{with } \sigma_D(\eta) = \max\{0, r - i - j\} \text{ for } \eta \in C(X_{i,k}^j, \nu_p^j) \text{ the generic point.}
\end{array} \right\}$$

$X_{i,k}^j$ is 3-prepared at $p \in T_{i,k}^j$. We have local resolvers $(U_p, D_p, I_p, \nu_p^j, \nu_p^2)$ at $p \in T_{i,k}^j$. We have $\max\{1, r - i - j\} \leq \sigma_D(q) \leq r - j - k - 1$ for $q \in T_{i,k}^j$.

5.6) Suppose that

$$p \in \left( \bigcup_{l=1}^{m_j-1} \bigcup_{n=0}^{\min(l-1,r-j)} T_{i,n}^{j-1} \right) \cup T_0^j$$

and $C(X_{i,k}^j, \nu_p^j)$ is such that $\sigma_D(\eta) = \max\{0, r - i - j\}$ for $\eta \in C(X_{i,k}^j, \nu_p^j)$ a general point. Then $\sigma_D(q) = \max\{0, r - i - j\}$ for $q \in C(X_{i,k}^j, \nu_p^j) \setminus \bigcup_{p' \in S_{i,k}^j \setminus T_{i,k}^j} W(X_{i,k}^j, p')$.

Further,

a) If $q \in S_{i,k}^j$ and $W(X_{i,k}^j, q) \cap C(X_{i,k}^j, \nu_p^j) \neq \emptyset$, then $C(X_{i,k}^j, \nu_p^j) = C(X_{i,k}^j, \nu_q^n)$ for some $n$.

b) If $q \in T_{i,k}^j$ and $q \in C(X_{i,k}^j, \nu_p^j)$, then either $C(X_{i,k}^j, \nu_p^j) = C(X_{i,k}^j, \nu_q^n)$ for some $n$ or

$$\max\{0, r - i - j\} < \sigma_D(q) \leq r - k - j - 1.$$

By the definition of $T_{i,k}^j$ in 5.5) above, we have that $\bigcup_{l=1}^{m_j-1} \bigcup_{n=0}^{\min(l-1,r-j)} T_{i,k}^{j-1} = \emptyset$. Thus 4.2), following (59), implies that $X^r$ is prepared.

\[\square\]

6. Proof of Toroidalization

**Theorem 6.1.** Suppose that $k$ is an algebraically closed field of characteristic zero, and $f : X \to S$ is a dominant morphism from a nonsingular 3-fold over $k$ to a nonsingular surface $S$ over $k$ and $D_S$ is a reduced SNC divisor on $S$ such that $D_X = f^{-1}(D_S)_{\text{red}}$ is a SNC divisor on $X$ which contains the locus where $f$ is not smooth. Further suppose that $f$ is 1-prepared. Then there exists a sequence of blow ups of points and nonsingular curves $\pi_2 : X_1 \to X$, which are contained in the preimage of $D_X$, such that the induced morphism $f_1 : X_1 \to S$ is prepared with respect to $D_S$.

**Proof.** The proof is immediate from Lemma 2.2, Proposition 2.7 and Theorem 5.3. \[\square\]
Theorem 6.1 is a slight restatement of Theorem 17.3 of [15]. Theorem 17.3 [15] easily follows from Lemma 2.2 and Theorem 6.1.

**Theorem 6.2.** Suppose that $k$ is an algebraically closed field of characteristic zero, and $f : X \to S$ is a dominant morphism from a nonsingular 3-fold over $k$ to a nonsingular surface $S$ over $k$ and $D_S$ is a reduced SNC divisor on $S$ such that $D_X = f^{-1}(D_S)_{\text{red}}$ is a SNC divisor on $X$ which contains the locus where $f$ is not smooth. Then there exists a sequence of blow ups of points and nonsingular curves $\pi_2 : X_1 \to X$, which are contained in the preimage of $D_X$, and a sequence of blow ups of points $\pi_1 : S_1 \to S$ which are in the preimage of $D_S$, such that the induced rational map $f_1 : X_1 \to S_1$ is a morphism which is toroidal with respect to $D_{S_1} = \pi_1^{-1}(D_S)$.

**Proof.** The proof follows immediately from Theorem 6.1, and Theorems 18.19, 19.9 and 19.10 of [15].


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