A SIMpler PROOF OF TOROIDALIZATION OF MORPHISMS FROM
3-FOLDS TO SURFACES

STEVEN DALE CUTKOSKY

Abstract. We give a simpler and more conceptual proof of toroidalization of morphisms of 3-folds to surfaces, over an algebraically closed field of characteristic zero. A toroidalization is obtained by performing sequences of blow ups of nonsingular subvarieties above the domain and range, to make a morphism toroidal. The original proof of toroidalization of morphisms of 3-folds to surfaces, which appeared in Springer Lecture Notes in Math. in 2002 [12], is much more complicated.

1. INTRODUCTION

Let \( k \) be an algebraically closed field of characteristic zero. If \( X \) is a nonsingular variety, then the choice of a simple normal crossings divisor (SNC divisor) on \( X \) makes \( X \) into a toroidal variety.

Suppose that \( \Phi : X \rightarrow Y \) is a dominant morphism of nonsingular \( k \)-varieties, and there is a SNC divisor \( D_Y \) on \( Y \) such that \( D_X = \Phi^{-1}(D_Y) \) is a SNC divisor on \( X \). Then \( \Phi \) is toroidal (with respect to \( D_Y \) and \( D_X \)) if and only if \( \Phi^*(\Omega^1_Y (\log D_Y)) \) is a subbundle of \( \Omega^1_X (\log D_X) \) (Lemma 1.5 [12]). A toroidal morphism can be expressed locally by monomials. All of the cases are written down for toroidal morphisms from a 3-fold to a surface in Lemma 19.3 [12].

The toroidalization problem is to determine, given a dominant morphism \( f : X \rightarrow Y \) of \( k \)-varieties, if there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are products of blow ups of nonsingular subvarieties, \( X_1 \) and \( Y_1 \) are nonsingular, and there exist SNC divisors \( D_{Y_1} \) on \( Y_1 \) and \( D_{X_1} = f^*(D_{Y_1}) \) on \( X_1 \) such that \( f_1 \) is toroidal (with respect to \( D_{X_1} \) and \( D_{Y_1} \)).

The toroidalization problem does not have a positive answer in positive characteristic \( p \), even for maps of curves; \( t = x^p + x^{p+1} \) gives a simple example.

In characteristic zero, the toroidalization problem has an affirmative answer if \( Y \) is a curve and \( X \) has arbitrary dimension; this is really embedded resolution of hypersurface singularities, so follows from resolution of singularities [24] (some of the simplified proofs are [5], [4] [15], [19] and [20]). Toroidalization is proven for morphisms from a 3-fold to a surface in [12] and for the case of a 3-fold to a 3-fold in [13]. Detailed history and references on the toroidalization problem are given in the introductions to [12] and [13].

We consider the problem of toroidalization as a resolution of singularities type problem. When the dimension of the base is larger than one, the problem shares many of the complexities of resolution of vector fields ([30], [6], [28]) and of resolution of singularities.

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in positive characteristic (some references are [1], [2], [25], [7], [8], [9], [3], [14], [18], [21], [22], [23], [26], [27], [31]). In particular, natural invariants do not have a “hypersurface of maximal contact” and are sometimes not upper semicontinuous.

Toroidalization, locally along a fixed valuation, is proven in all dimensions and relative dimensions in [10] and [11].

The proof of toroidalization of a dominant morphism from a 3-fold to a surface given in [12] consists of 2 steps.

The first step is to prove “strong preparation”. Suppose that \( X \) is a nonsingular variety, \( S \) is a nonsingular surface with a SNC divisor \( D_S \), and \( f : X \to S \) is a dominant morphism such that \( D_X = f^{-1}(D_S) \) is a SNC divisor on \( X \) which contains the locus where \( f \) is not smooth. \( f \) is strongly prepared if \( f^*(\Omega^2_S(\log D_S)) = IM \) where \( I \subset O_X \) is an ideal sheaf, and \( M \) is a subbundle of \( \Omega^2_X(\log D_X) \) (Lemma 1.7 [12]). A strongly prepared morphism has nice local forms which are close to being toroidal (page 7 of [12]).

Strong preparation is the construction of a commutative diagram

\[
\begin{array}{ccc}
X_1 & \rightarrow & S \\
\downarrow & & \\
X & \rightarrow & S
\end{array}
\]

where \( S \) is a nonsingular surface with a SNC divisor \( D_S \) such that \( D_X = f^*(D_S) \) is a SNC divisor on the nonsingular variety \( X \) which contains the locus where \( f \) is not smooth, the vertical arrow is a product of blow ups of nonsingular subvarieties so that \( X_1 \to S \) is strongly prepared. Strong preparation of morphisms from 3-folds to surfaces is proven in Theorem 17.3 of [12].

The second step is to prove that a strongly prepared morphism from a 3-fold to a surface can be toroidalized. This is proven in Sections 18 and 19 of [12].

This second step is generalized in [16] to prove that a strongly prepared morphism from an \( n \)-fold to a surface can be toroidalized. Thus to prove toroidalization of a morphism from an \( n \)-fold to a surface, it suffices to proof strong preparation.

The proof of strong preparation in [12] is extremely complicated, and does not readily generalize to higher dimensions. The proof of this result occupies 170 pages of [12]. We mention that that the main invariant considered in this paper, \( \nu \), can be interpreted as the adopted order of Section 1.2 of [6] of the 2-form \( du \wedge dv \).

In this paper, we give a significantly simpler and more conceptual proof of strong preparation of morphisms of 3-folds to surfaces. It is our hope that this proof can be extended to prove strong preparation for morphisms of \( n \)-folds to surfaces, for \( n > 3 \). The proof is built around a new upper semicontinuous invariant \( \sigma_D \), whose value is a natural number or \( \infty \). if \( \sigma_D(p) = 0 \) for all \( p \in X \), then \( X \to S \) is prepared (which is slightly stronger than being strongly prepared). A first step towards obtaining a reduction in \( \sigma_D \) is to make \( X \) 3-prepared, which is achieved in Section 3. This is a nicer local form, which is proved by making a local reduction to lower dimension. The proof proceeds by performing a toroidal morphism above \( X \) to obtain that \( X \) is 3-prepared at all points except for a finite number of 1-points. Then general curves through these points lying on \( D_X \) are blown up to achieve 3-preparation everywhere on \( X \). if \( X \) is 3-prepared at a point \( p \), then there exists an étale cover \( U_p \) of an affine neighborhood of \( p \) and a local toroidal structure \( \overline{D}_p \) at \( p \) (which contains \( D_X \)) such that there exists a projective toroidal morphism \( \Psi : U' \to U_p \) such that \( \sigma_D \) has dropped everywhere above \( p \) (Section 4). The final step of the proof is to make these local constructions algebraic, and to patch them. This is accomplished in Section 5.
In Section 6 we state and prove strong preparation for morphisms of 3-folds to surfaces (Theorem 6.1) and toroidalization of morphisms from 3-folds to surfaces (Theorem 6.2).

2. The invariant $\sigma_D$, 1-preparation and 2-preparation.

For the duration of the paper, $\kappa$ will be an algebraically closed field of characteristic zero. We will write curve (over $\kappa$) to mean a 1-dimensional $\kappa$-variety, and similarly for surfaces and 3-folds. We will assume that varieties are quasi-projective. This is not really a restriction, by the fact that after a sequence of blow ups of nonsingular subvarieties, all varieties satisfy this condition. By a general point of a $\kappa$-variety $Z$, we will mean a member of a nontrivial open subset of $Z$ on which some specified good condition holds.

A reduced divisor $D$ on a nonsingular variety $Z$ of dimension $n$ is a simple normal crossings divisor (SNC divisor) if all irreducible components of $D$ are nonsingular, and if $p \in Z$, then there exists a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{Z,p}$ such that $x_1 x_2 \cdots x_r = 0$ is a local equation of $D$ at $p$, where $r \leq n$ is the number of irreducible components of $D$ containing $p$. Two nonsingular subvarieties $X$ and $Y$ intersect transversally at $p \in X \cap Y$ if there exists a regular system of parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{Z,p}$ and subsets $I, J \subset \{1, \ldots, n\}$ such that $\mathcal{I}_X, p = (x_i \mid i \in I)$ and $\mathcal{I}_Y, p = (x_j \mid j \in J)$.

**Definition 2.1.** Let $S$ be a nonsingular surface over $\kappa$ with a reduced SNC divisor $D_S$. Suppose that $X$ is a nonsingular 3-fold, and $f : X \to S$ is a dominant morphism. $X$ is 1-prepared (with respect to $f$) if $D_X = f^{-1}(D_S)_{\text{red}}$ is a SNC divisor on $X$ which contains the locus where $f$ is not smooth, and if $C_1, C_2$ are the two components of $D_S$ whose intersection is nonempty, $T_1$ is a component of $X$ dominating $C_1$ and $T_2$ is a component of $D_X$ which dominates $C_2$, then $T_1$ and $T_2$ are disjoint.

The following lemma is an easy consequence of the main theorem on resolution of singularities.

**Lemma 2.2.** Suppose that $g : Y \to T$ is a dominant morphism of a 3-fold over $\kappa$ to a surface over $\kappa$ and $D_T$ is a 1-cycle on $T$ such that $g^{-1}(D_T)$ contains the locus where $g$ is not smooth. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
Y & \xrightarrow{g} & T \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
X & \xrightarrow{f} & T
\end{array}
$$

such that the vertical arrows are products of blow ups of nonsingular subvarieties contained in the preimage of $D_T$, $Y_1$ and $T_1$ are nonsingular and $D_{T_1} = \pi_1^{-1}(D_T)$ is a SNC divisor on $T_1$ such that $Y_1$ is 1-prepared with respect to $g_1$.

For the duration of this paper, $S$ will be a fixed nonsingular surface over $\kappa$, with a (reduced) SNC divisor $D_S$. To simplify notation, we will often write $D$ to denote $D_X$, if $f : X \to S$ is 1-prepared.

Suppose that $X$ is 1-prepared with respect to $f : X \to S$. A permissible blow up of $X$ is the blow up $\pi_1 : X_1 \to X$ of a point of $D_X$ or a nonsingular curve contained in $D_X$ which makes SNCs with $D_X$. Then $D_{X_1} = \pi_1^{-1}(D_X)_{\text{red}} = (f \circ \pi_1)^{-1}(X_S)_{\text{red}}$ is a SNC divisor on $X_1$ and $X_1$ is 1-prepared with respect to $f \circ \pi_1$.

Assume that $X$ is 1-prepared with respect to $D$. We will say that $p \in X$ is an $n$-point (for $D$) if $p$ is on exactly $n$ components of $D$. Suppose $q \in D_S$ and $u, v$ are regular parameters in $\mathcal{O}_{S,q}$ such that either $u = 0$ is a local equation of $D_S$ at $q$ or $uv = 0$ is a local equation of $D_S$ at $q$. $u, v$ are called permissible parameters at $q$. 
For \( p \in f^{-1}(q) \), we have regular parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that

1) If \( p \) is a 1-point,

\[
(1) \quad u = x^a, v = P(x) + x^b F
\]

where \( x = 0 \) is a local equation of \( D \), \( x \not\mid F \) and \( x^b F \) has no terms which are a power of \( x \).

2) If \( p \) is a 2-point, after possibly interchanging \( u \) and \( v \),

\[
(2) \quad u = (x^a y^b)^l, v = P(x^a y^b) + x^c y^d F
\]

where \( xy = 0 \) is a local equation of \( D \), \( a, b > 0, \gcd(a, b) = 1 \), \( x, y \not\mid F \) and \( x^c y^d F \) has no terms which are a power of \( x^a y^b \).

3) If \( p \) is a 3-point, after possibly interchanging \( u \) and \( v \),

\[
(3) \quad u = (x^a y^b z^c)^l, v = P(x^a y^b z^c) + x^d y^e z^f F
\]

where \( xyz = 0 \) is a local equation of \( D \), \( a, b, c > 0, \gcd(a, b, c) = 1 \), \( x, y, z \not\mid F \) and \( x^d y^e z^f F \) has no terms which are a power of \( x^a y^b z^c \).

Regular parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) giving forms (1), (2) or (3) are called permissible parameters at \( p \) for \( u, v \).

Suppose that \( X \) is 1-prepared. We define an ideal sheaf

\[
\mathcal{I} = \text{fitting ideal sheaf of the image of } f^* : \Omega_S^2 \rightarrow \Omega_X^2(\log(D))
\]

in \( \mathcal{O}_X \). \( \mathcal{I} = \mathcal{O}_X(-G)\mathcal{I} \) where \( G \) is an effective divisor supported on \( D \) and \( \mathcal{I} \) has height \( \geq 2 \).

Suppose that \( E_1, \ldots, E_n \) are the irreducible components of \( D \). For \( p \in X \), define

\[
\sigma_D(p) = \text{order}_{\mathcal{O}_{X,p}/(\sum_{i \in I} \mathcal{I}_{E_i,p})} \mathcal{I}_{p} \left( \mathcal{O}_{X,p}/\sum_{p \in E_i} \mathcal{I}_{E_i,p} \right) \in \mathbb{N} \cup \{\infty\}.
\]

**Lemma 2.3.** \( \sigma_D \) is upper semicontinuous in the Zariski topology of the scheme \( X \).

**Proof.** For a fixed subset \( J \subset \{1, 2, \ldots, n\} \), we have that the function

\[
\text{order}_{\mathcal{O}_{X,p}/(\sum_{i \in J} \mathcal{I}_{E_i,p})} \mathcal{I}_{p} \left( \mathcal{O}_{X,p}/\sum_{i \in J} \mathcal{I}_{E_i,p} \right)
\]

is upper semicontinuous, and if \( J \subset J' \subset \{1, 2, \ldots, n\} \), we have that

\[
\text{order}_{\mathcal{O}_{X,p}/(\sum_{i \in J} \mathcal{I}_{E_i,p})} \mathcal{I}_{p} \left( \mathcal{O}_{X,p}/\sum_{i \in J} \mathcal{I}_{E_i,p} \right) \leq \text{order}_{\mathcal{O}_{X,p}/(\sum_{i \in J'} \mathcal{I}_{E_i,p})} \mathcal{I}_{p} \left( \mathcal{O}_{X,p}/\sum_{i \in J'} \mathcal{I}_{E_i,p} \right).
\]

Thus for \( r \in \mathbb{N} \cup \{\infty\} \),

\[
\text{Sing}_r(X) = \{ p \in X \mid \sigma_D(p) \geq r \}
\]

is a closed subset of \( X \), which is supported on \( D \) and has dimension \( \leq 1 \) if \( r > 0 \).

**Definition 2.4.** A point \( p \in X \) is prepared if \( \sigma_D(p) = 0 \).
We have that $\sigma_D(p) = 0$ if and only if $\mathcal{I}_p = \mathcal{O}_{X_p}$. Further,

$$\text{Sing}_1(X) = \{ p \in X \mid \mathcal{I}_p \neq \mathcal{O}_{X_p} \}.$$

If $p \in X$ is a 1-point with an expression (1) we have

$$\left( \mathcal{I}_p + (x) \right) \hat{\mathcal{O}}_{X_p} = (x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}).$$

If $p \in X$ is a 2-point with an expression (2) we have

$$\left( \mathcal{I}_p + (x, y) \right) \hat{\mathcal{O}}_{X_p} = (x, y, (ad - bc)F, \frac{\partial F}{\partial z}).$$

If $p \in X$ is a 3-point with an expression (3) we have

$$\left( \mathcal{I}_p + (x, y, z) \right) \hat{\mathcal{O}}_{X_p} = (x, y, z, (ae - bd)F, (af - cd)F, (bf - ce)F).$$

If $p \in X$ is a 1-point with an expression (1), then $\sigma_D(p) = \text{ord } F(0, y, z) - 1$. We have $0 \leq \sigma_D(p) < \infty$ if $p$ is a 1-point. If $p \in X$ is a 2-point, we have

$$\sigma_D(p) = \begin{cases} 
0 & \text{if ord } F(0, 0, z) = 0 \text{ (in this case, } ad - bc \neq 0) \\
\text{ord } F(0, 0, z) - 1 & \text{if } 1 \leq \text{ord } F(0, 0, z) < \infty \\
\infty & \text{if } \text{ord } F(0, 0, z) = \infty.
\end{cases}$$

If $p \in X$ is a 3-point, let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}.$$ we have

$$\sigma_D(p) = \begin{cases} 
0 & \text{if ord } F(0, 0, 0) = 0 \text{ (in this case, rank}(A) = 2) \\
\infty & \text{if ord } F(0, 0, 0) = \infty.
\end{cases}$$

**Lemma 2.5.** Suppose that $X$ is 1-prepared and $\pi_1 : X_1 \to X$ is a toroidal morphism with respect to $D$. Then $X_1$ is 1-prepared and $\sigma_D(p_1) \leq \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.

**Proof.** Suppose that $p \in X$ is a 2-point and $p_1 \in \pi_1^{-1}(p)$. Then there exist permissible parameters $x, y, z$ at $p$ giving an expression (2). In $\hat{\mathcal{O}}_{X_1, p_1}$, there are regular parameters $x_1, y_1, z$ where

$$x = x_1^{a_{11}}(y_1 + \alpha)^{a_{12}}, \quad y = x_1^{a_{21}}(y_1 + \alpha)^{a_{22}}$$

with $\alpha \in \mathfrak{k}$ and $a_{11}a_{22} - a_{12}a_{22} = \pm 1$. If $\alpha = 0$, so that $p_1$ is a 2-point, then $x_1, y_1, z$ are permissible parameters at $p_1$ and substitution of (7) into (2) gives an expression of the form (2) at $p_1$, showing that $\sigma_D(p_1) \leq \sigma_D(p)$. If $\alpha \neq 0 \in \mathfrak{k}$, so that $p_1$ is a 1-point, set $\lambda = \frac{a_{12} + b_{02}}{a_{01} + b_{01}}$ and $\pi_1 = x_1(y_1 + \alpha)\lambda$. Then $\pi_1, y_1, z$ are permissible parameters at $p_1$. Substitution into (2) leads to a form (1) with $\sigma_D(p_1) \leq \sigma_D(p)$.

If $p \in X$ is a 3-point and $\sigma_D(p) \neq \infty$, then $\sigma_D(p) = 0$ so that $p$ is prepared. Thus there exist permissible parameters $x, y, z$ at $p$ giving an expression (3) with $F = 1$. Suppose that $p_1 \in \pi_1^{-1}(p)$. In $\hat{\mathcal{O}}_{X_1, p_1}$ there are regular parameters $x_1, y_1, z_1$ such that

$$x = (x_1 + \alpha)^{a_{11}}(y_1 + \beta)^{a_{12}}(z_1 + \gamma)^{a_{13}}$$
$$y = (x_1 + \alpha)^{a_{21}}(y_1 + \beta)^{a_{22}}(z_1 + \gamma)^{a_{23}}$$
$$z = (x_1 + \alpha)^{a_{31}}(y_1 + \beta)^{a_{32}}(z_1 + \gamma)^{a_{33}}$$

where at least one of $\alpha, \beta, \gamma \in \mathfrak{k}$ is zero. Substituting into (3), we find permissible parameters at $p_1$ giving a prepared form. $\square$
Suppose that $X$ is 1-prepared with respect to $f : X \rightarrow S$. Define

$$\Gamma_D(X) = \max\{\sigma_D(p) \mid p \in X\}.$$ 

Lemma 2.6. Suppose that $X$ is 1-prepared and $C$ is a 2-curve of $D$ and there exists $p \in C$ such that $\sigma_D(p) < \infty$. Then $\sigma_D(q) = 0$ at the generic point $q$ of $C$.

Proof. If $p$ is a 3-point then $\sigma_D(p) = 0$ and the lemma follows from upper semicontinuity of $\sigma_D$.

Suppose that $p$ is a 2-point. If $\sigma_D(p) = 0$ then the lemma follows from upper semicontinuity of $\sigma_D$, so suppose that $0 < \sigma_D(p) < \infty$. There exist permissible parameters $x, y, z$ at $p$ giving a form (2), such that $x, y, z$ are uniformizing parameters on an étale cover $U$ of an affine neighborhood of $p$. Thus for $\alpha$ in a Zariski open subset of $k, x, y, z = z - \alpha$ are permissible parameters at a 2-point $\overline{p}$ of $C$. After possibly replacing $U$ with a smaller neighborhood of $p$, we have

$$\frac{\partial F}{\partial z} = \frac{1}{x^c y^d} \frac{\partial v}{\partial z} \in \Gamma(U, \mathcal{O}_X)$$

and $\frac{\partial F}{\partial z}(0, 0, z) \neq 0$. Thus there exists a 2-point $\overline{p} \in C$ with permissible parameters $x, y, z = z - \alpha$ such that $\frac{\partial F}{\partial z}(0, 0, \alpha) \neq 0$, and thus there is an expression (2) at $\overline{p}$

$$u = (x^a y^b)^l$$

$$v = P_l(x^a y^b) + x^c y^d F_1(x, y, \overline{z})$$

with $F_1(0, 0, \overline{z}) = 0$ or 1, so that $\sigma_D(\overline{p}) = 0$. By upper semicontinuity of $\sigma_D$, $\sigma_D(q) = 0$. □

Proposition 2.7. Suppose that $X$ is 1-prepared with respect to $f : X \rightarrow S$. Then there exists a toroidal morphism $\pi_1 : X_1 \rightarrow X$ with respect to $D$, such that $\pi_1$ is a sequence of blow ups of 2-curves and 3-points, and

1) $\sigma_D(p) < \infty$ for all $p \in D_{X_1}$.

2) $X_1$ is prepared (with respect to $f_1 = f \circ \pi_1 : X_1 \rightarrow S$) at all 3-points and the generic point of all 2-curves of $D_{X_1}$.

Proof. By upper semicontinuity of $\sigma_D$, Lemma 2.6 and Lemma 2.5, we must show that if $p \in X$ is a 3-point with $\sigma_D(p) = \infty$ then there exists a toroidal morphism $\pi_1 : X_1 \rightarrow X$ such that $\sigma_D(p_1) = 0$ for all 3-points $p_1 \in \pi_1^{-1}(p)$ and if $p \in X$ is a 2-point with $\sigma_D(p) = \infty$ then there exists a toroidal morphism $\pi_1 : X_1 \rightarrow X$ such that $\sigma_D(p_1) < \infty$ for all 2-points $p_1 \in \pi_1^{-1}(p)$.

First suppose that $p$ is a 3-point with $\sigma_D(p) = \infty$. Let $x, y, z$ be permissible parameters at $p$ giving a form (3). There exist regular parameters $\tilde{x}, \tilde{y}, \tilde{z}$ in $\mathcal{O}_{X,p}$ and unit series $\alpha, \beta, \gamma \in \mathcal{O}_{X,p}$ such that $x = \alpha \tilde{x}$, $y = \beta \tilde{y}$, $z = \gamma \tilde{z}$. Write $F = \sum b_{ijk} x^i y^j z^k$ with $b_{ijk} \in k$. Let $I = (\tilde{x}^i \tilde{y}^j \tilde{z}^k \mid b_{ijk} \neq 0)$, an ideal in $\mathcal{O}_{X,p}$. Since $\tilde{x} \tilde{y} \tilde{z} = 0$ is a local equation of $D$ at $p$, there exists a toroidal morphism $\pi_1 : X_1 \rightarrow X$ with respect to $D$ such that $I \mathcal{O}_{X_1,p_1}$ is principal for all $p_1 \in \pi_1^{-1}(p)$. At a 3-point $p_1 \in \pi_1^{-1}(p)$, there exist permissible parameters $x_1, y_1, z_1$ such that

$$x = x_1^{a_{11}} y_1^{a_{12}} z_1^{a_{13}}$$

$$y = x_1^{a_{21}} y_1^{a_{22}} z_1^{a_{23}}$$

$$z = x_1^{a_{31}} y_1^{a_{32}} z_1^{a_{33}}$$
with \( \text{Det}(a_{ij}) = \pm 1 \). Substituting into (3), we obtain an expression (3) at \( p_1 \), where

\[
\begin{align*}
    u &= (x_1^{a_1} y_1^{b_1} z_1^{c_1})^l \\
    v &= P_1(x_1^{a_1} y_1^{b_1} z_1^{c_1}) + x_1^{d_1} y_1^{e_1} z_1^{f_1} F_1
\end{align*}
\]

where \( P_1(x_1^{a_1} y_1^{b_1} z_1^{c_1}) = P(x^a y^b z^c) \) and

\[ F(x, y, z) = x_1^{a_1} y_1^{b_1} z_1^{c_1} F_1(x_1, y_1, z_1). \]

with \( x_1^{a_1} y_1^{b_1} z_1^{c_1} \) a generator of \( I \mathcal{O}_{X_1, p_1} \) and \( F_1(0, 0, 0) \neq 0 \). Thus \( \sigma_D(p_1) = 0 \).

Now suppose that \( p \) is a 2-point and \( \sigma_D(p) = \infty \). There exist permissible parameters \( x, y, z \) at \( p \) giving a form (2). Write \( F = \sum a_i(x, y) z^i \), with \( a_i(x, y) \in \mathfrak{t}[[x, y]] \) for all \( i \). We necessarily have that no \( a_i(x, y) \) is a unit series.

Let \( I \) be the ideal \( I = (a_i(x, y) \mid i \in \mathbb{N}) \) in \( \mathfrak{t}[[x, y]] \). There exists a sequence of blow ups of 2-curves \( \pi_1 : X_1 \to X \) such that \( \mathcal{O}_{X_1, p_1} \) is principal at all 2-points \( p \in \pi_1^{-1}(p) \). There exist \( x_1, y_1, z_1 \in \mathcal{O}_{X_1, p_1} \) so that \( x_1, y_1, z_1 \) are permissible parameters at \( p_1 \), and

\[
    x = x_1^{a_{11}} y_1^{b_{11}}, \quad y = x_1^{a_{12}} y_1^{b_{12}}
\]

with \( a_1 a_{22} - a_1 a_{21} = \pm 1 \). Let \( x_1^{a_1} y_1^{b_1} \) be a generator of \( I \mathcal{O}_{T_1, q_1} \). Then \( F = x_1^{a_1} y_1^{b_1} F_1(x_1, y_1, z) \) where \( F_1(0, 0, z) \neq 0 \), and we have an expression (2) at \( p_1 \), where

\[
\begin{align*}
    u &= (x_1^{a_1} y_1^{b_1})^l \\
    v &= P_1(x_1^{a_1} y_1^{b_1}) + x_1^{d_1} y_1^{e_1} F_1
\end{align*}
\]

where \( P_1(x_1^{a_1} y_1^{b_1}) = P(x^a y^b) \). Thus \( \sigma_D(p_1) < \infty \) and \( \sigma_D(q) < \infty \) if \( q \) is the generic point of the 2-curve of \( D_{X_1} \), containing \( p_1 \).

\[ \square \]

We will say that \( X \) is 2-prepared (with respect to \( f : X \to S \)) if it satisfies the conclusions of Proposition 2.7. We then have that \( \Gamma_D(X) < \infty \).

If \( X \) is 2-prepared, we have that \( \text{Sing}_2(X) \) is a union of (closed) curves whose generic point is a 1-point and isolated 1-points and 2-points. Further, \( \text{Sing}_1(X) \) contains no 3-points.

3. 3-preparation

**Lemma 3.1.** Suppose that \( X \) is 2-prepared. Suppose that \( p \in X \) is such that \( \sigma_D(p) > 0 \). Let \( m = \sigma_D(p) + 1 \). Then there exist permissible parameters \( x, y, z \) at \( p \) such that there exist \( \tilde{x}, y \in \mathcal{O}_{X, p} \), an étale cover \( U \) of an affine neighborhood of \( p \), such that \( x, z \in \Gamma(U, \mathcal{O}_X) \) and \( x, y, z \) are uniformizing parameters on \( U \), and \( x = \gamma \tilde{x} \) for some unit series \( \gamma \in \mathcal{O}_{X, p} \).

We have an expression (1) or (2), if \( p \) is respectively a 1-point or a 2-point, with

\[
F = \tau z^m + a_2(x, y) z^{m-2} + \cdots + a_{m-1}(x, y) z + a_m(x, y)
\]

where \( m \geq 2 \) and \( \tau \in \mathcal{O}_{X_1, p} = \mathfrak{t}[[x, y, z]] \) is a unit, and \( a_i(x, y) \neq 0 \) for \( i = m - 1 \) or \( i = m \). Further, if \( p \) is a 1-point, then we can choose \( x, y, z \) so that \( x = y = 0 \) is a local equation of a generic curve through \( p \) on \( D \).

For all but finitely many points \( p \) in the set of 1-points of \( X \), there is an expression (9) where

\[
a_i \text{ is either zero or has an expression } a_i = \overline{a}_i x^{r_i} \text{ where } \overline{a}_i \text{ is a unit}
\]

and \( r_i > 0 \) for \( 2 \leq i \leq m \), and \( a_m = 0 \) or \( a_m = x^{r_m} \overline{a}_m \) where \( r_m > 0 \) and \( \text{ord}(\overline{a}_m(0, y)) = 1 \).
Proof. There exist regular parameters \( \tilde{x}, y, \tilde{z} \) in \( \mathcal{O}_{X,p} \) and a unit \( \gamma \in \hat{\mathcal{O}}_{X,p} \) such that \( x = \gamma \tilde{x}, y, \tilde{z} \) are permissible parameters at \( p \), with \( \text{ord}(F(0,0,\tilde{z})) = m \). Thus there exists an affine neighborhood \( \text{Spec}(A) \) of \( p \) such that \( V = \text{Spec}(R) \), where \( R = A[\gamma^\frac{1}{m}] \) is an \( \text{étale} \) cover of \( \text{Spec}(A) \), \( x, y, \tilde{z} \) are uniformizing parameters on \( V \), and \( u, v \in \Gamma(V, \mathcal{O}_X) \). Differentiating with respect to the uniformizing parameters \( x, y, \tilde{z} \) in \( R \), set

\[
\tilde{z} = \frac{\partial^{m-1} F}{\partial \tilde{z}^{m-1}} = \omega(\tilde{z} - \varphi(x,y))
\]

where \( \omega \in \hat{\mathcal{O}}_{X,p} \) is a unit series, and \( \varphi(x,y) \in \mathfrak{k}[[x,y]] \) is a nonunit series, by the formal implicit function theorem. Set \( z = \tilde{z} - \varphi(x,y) \). Since \( R \) is normal, after possibly replacing \( \text{Spec}(A) \) with a smaller affine neighborhood of \( p \),

\[
\tilde{z} = \frac{1}{x^b} \frac{\partial^{m-1} v}{\partial \tilde{z}^{m-1}} \in R.
\]

By Weierstrass preparation for Henselian local rings (Proposition 6.1 [29]), \( \varphi(x,y) \) is integral over the local ring \( \mathfrak{k}[[x,y]]_{(x,y)} \). Thus after possibly replacing \( A \) with a smaller affine neighborhood of \( p \), there exists an \( \text{étale} \) cover \( U \) of \( V \) such that \( \varphi(x,y) \in \Gamma(U, \mathcal{O}_X) \), and thus \( z \in \Gamma(U, \mathcal{O}_X) \).

Let \( G(x, y, z) = F(x, y, \tilde{z}) \). We have that

\[
G = G(x, y, 0) + \frac{\partial G}{\partial z}(x, y, 0) z + \cdots + \frac{1}{(m-1)!} \frac{\partial^{m-1} G}{\partial z^{m-1}}(x, y, 0) z^{m-1} + \frac{1}{m!} \frac{\partial^m G}{\partial z^m}(x, y, 0) z^m + \cdots
\]

We have

\[
\frac{\partial^{m-1} G}{\partial z^{m-1}}(x, y, 0) = \frac{\partial^{m-1} F}{\partial \tilde{z}^{m-1}}(x, y, \varphi(x,y)) = 0
\]

and

\[
\frac{\partial^m G}{\partial z^m}(x, y, 0) = \frac{\partial^m F}{\partial \tilde{z}^m}(x, y, \varphi(x,y))
\]

is a unit in \( \hat{\mathcal{O}}_{X,p} \). Thus we have the desired form (9), but we must still show that \( a_m \neq 0 \) or \( a_{m-1} \neq 0 \). If \( a_i(x,y) = 0 \) for \( i = m \) and \( i = m - 1 \), we have that \( z^2 | F \) in \( \hat{\mathcal{O}}_{X,p} \), since \( m \geq 2 \). This implies that the ideal of \( 2 \times 2 \) minors

\[
I_2 \left( \begin{array}{ccc}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}
\end{array} \right) \subset (z),
\]

which implies that \( z = 0 \) is a component of \( D \) which is impossible. Thus either \( a_{m-1} \neq 0 \) or \( a_m \neq 0 \).

Suppose that \( C \) is a curve in \( \text{Sing}_1(X) \) (containing a 1-point) and \( p \in C \) is a general point. Let \( r = \sigma_D(p) \). Set \( m = r + 1 \). Let \( x, y, \tilde{z} \) be permissible parameters at \( p \) with \( y, \tilde{z} \in \mathcal{O}_{X,p} \), which are uniformizing parameters on an \( \text{étale} \) cover \( U \) of an affine neighborhood of \( p \) such that \( x = \tilde{z} = 0 \) are local equations of \( C \) and we have a form (1) at \( p \) with

\[
F = r \tilde{z}^m + a_1(x,y) \tilde{z}^{m-1} + \cdots + a_m(x,y).
\]

For \( \alpha \) in a Zariski open subset of \( t \), \( x, \overline{y} = y - \alpha, \tilde{z} \) are permissible parameters at a point \( q \in C \cap U \). For most points \( q \) on the curve \( C \cap U \), we have that \( a_i(x,y) = x^n \overline{t_i}(x,y) \) where \( \overline{a}_i(x,y) \) is a unit or zero for \( 1 \leq i \leq m - 1 \) in \( \hat{\mathcal{O}}_{X,q} \). Since \( \sigma_D(p) = r \) at this point,
we have that $1 \leq r_i$ for all $i$. We further have that if $a_m \neq 0$, then $a_m = x^r a'$ where $a' = f(y) + x \Omega$ where $f(y)$ is non constant. Thus
\[
0 \neq \frac{\partial a_m}{\partial y}(0, y) = \frac{\partial F}{\partial y}(0, y, 0).
\]
After possibly replacing $U$ with a smaller neighborhood of $p$, we have
\[
\frac{\partial F}{\partial y} = \frac{1}{x^b} \frac{\partial v}{\partial y} \in \Gamma(U, \mathcal{O}_X).
\]
Thus $\frac{\partial a_m}{\partial y}(0, \alpha) \neq 0$ for most $\alpha \in \mathfrak{m}$. Since $r > 0$, we have that $r_m > 0$, and thus $r_i > 0$ for all $i$ in (12). We have
\[
\frac{\partial^{m-1} F}{\partial \xi^{m-1}} = \xi^r + a_1(x, y),
\]
where $\xi$ is a unit series. Comparing the above equation with (11), we observe that $\varphi(x, y)$ is a unit series in $x$ and $y$ times $a_1(x, y)$. Thus $x$ divides $\varphi(x, y)$. Setting $z = \frac{\varphi(z, y)}{m}$, we obtain an expression (9) such that $x$ divides $a_i$ for all $i$. Now argue as in the analysis of (12), after substituting $z = \frac{\varphi(x, y)}{m}$, to conclude that there is an expression (9), where (10) holds at most points $q \in C \cap U$. Thus a form (9) and (10) holds at all but finitely many 1-points of $X$.

\[\square\]

**Lemma 3.2.** Suppose that $X$ is 2-prepared, $C$ is a curve in $\text{Sing}_1(X)$ containing a 1-point and $p$ is a general point of $C$. Let $m = \sigma_D(p) + 1$. Suppose that $\tilde{x}, y \in \mathcal{O}_{X,p}$ are such that $\tilde{x} = 0$ is a local equation of $D$ at $p$ and the germ $\tilde{x} = y = 0$ intersects $C$ transversally at $p$. Then there exists an étale cover $U$ of an affine neighborhood of $p$ and $z \in \Gamma(U, \mathcal{O}_X)$ such that $\tilde{x}, y, z$ give a form (9) at $p$.

**Proof.** There exists $\bar{z} \in \mathcal{O}_{X,p}$ such that $\bar{x}, y, \bar{z}$ are regular parameters in $\mathcal{O}_{X,p}$ and $x = \bar{z} = 0$ is a local equation of $C$ at $p$. There exists a unit $\gamma \in \mathcal{O}_{X,p}$ such that $x = \gamma \bar{x}, y, \bar{z}$ are permissible parameters at $p$. We have an expression of the form (1),
\[
u = x^a, v = P(x) + x^b F
\]
at $p$. Write $F = f(y, \bar{z}) + x \Omega$ in $\mathcal{O}_{X,p}$. Let $I$ be the ideal in $\mathcal{O}_{X,p}$ generated by $x$ and
\[
\{ \frac{\partial^{i+j} f}{\partial y^i \partial \bar{z}^j} \mid 1 \leq i + j \leq m - 1 \}.
\]
The radical of $I$ is the ideal $(x, \bar{z})$, as $x = \bar{z} = 0$ is a local equation of $\text{Sing}_{m-1}(X)$ at $p$. Thus $\bar{z}$ divides $\frac{\partial^{i+j} f}{\partial y^i \partial \bar{z}^j}$ for $1 \leq i + j \leq m - 1$ (with $m \geq 2$). Expanding
\[
f = \sum_{i=0}^{\infty} b_i(y) \bar{z}^i
\]
(where $b_0(0) = 0$) we see that $\frac{\partial b_0}{\partial y} = 0$ (so that $b_0(y) = 0$) and $b_1(y) = 0$ for $1 \leq i \leq m - 1$. Thus $\bar{z}^m$ divides $f(y, \bar{z})$. Since $\sigma_D(p) = m - 1$, we have that $f = \tau \bar{z}^m$ where $\tau$ is a unit series. Thus $x, y, \bar{z}$ gives a form (1) with ord($F(0, 0, \bar{z})$) = $m$. Now the proof of Lemma 3.1 gives the desired conclusion.

\[\square\]

Let $\omega(m, r_2, \ldots, r_{m-1})$ be a function which associates a positive integer to a positive integer $m$, natural numbers $r_2, \ldots, r_{m-2}$ and a positive integer $r_{m-1}$. We will give a precise form of $\omega$ after Theorem 4.1.
Definition 3.3. $X$ is 3-prepared (with respect to $f : X \to S$) at a point $p \in D$ if $\sigma_D(p) = 0$ or if $\sigma_D(p) > 0$, $f$ is 2-prepared with respect to $D$ at $p$ and there are permissible parameters $x, y, z$ at $p$ such that $x, y, z$ are uniformizing parameters on an étale cover of an affine neighborhood of $p$ and we have one of the following forms, with $m = \sigma_D(p) + 1$:

1) $p$ is a 2-point, and we have an expression (2) with

$$F = \tau_0 z^m + \tau_2 x^{r_2} y^{s_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} y^{s_{m-1}} z + \tau_m x^{r_m} y^{s_m}$$

where $\tau_0 \in \mathcal{O}_{X,p}$ is a unit, $\tau_i \in \mathcal{O}_{X,p}$ are units (or zero), $r_i + s_i > 0$ whenever $\tau_i \neq 0$ and $(r_m + c) b - (s_m + d) a \neq 0$. Further, $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

2) $p$ is a 1-point, and we have an expression (1) with

$$F = \tau_0 z^m + \tau_2 x^{r_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} z + \tau_m x^{r_m}$$

where $\tau_0 \in \mathcal{O}_{X,p}$ is a unit, $\tau_i \in \mathcal{O}_{X,p}$ are units (or zero) for $2 \leq i \leq m - 1$, $\tau_m \in \mathcal{O}_{X,p}$ and $\text{ord}(\tau_m(0,y,0)) = 1$ (or $\tau_m = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$, and $\tau_{m-1} \neq 0$ or $\tau_m \neq 0$.

3) $p$ is a 1-point, and we have an expression (1) with

$$F = \tau_0 z^m + \tau_2 x^{r_2} z^{m-2} + \cdots + \tau_{m-1} x^{r_{m-1}} z + x^l \Omega$$

where $\tau_0 \in \mathcal{O}_{X,p}$ is a unit, $\tau_i \in \mathcal{O}_{X,p}$ are units (or zero) for $2 \leq i \leq m - 1$, $\Omega \in \mathcal{O}_{X,p}$, $\tau_{m-1} \neq 0$ and $t > \omega(m,r_2,\ldots,r_{m-1})$ (where we set $r_i = 0$ if $\tau_i = 0$). Further, $r_i > 0$ if $\tau_i \neq 0$.

$X$ is 3-prepared if $X$ is 3-prepared for all $p \in X$.

Lemma 3.4. Suppose that $X$ is 2-prepared with respect to $f : X \to S$. Then there exists a sequence of blow ups of 2-curves $\pi_1 : X \to X_1$ such that $X_1$ is 3-prepared with respect to $f \circ \pi_1$, except possibly at a finite number of 1-points.

Proof. The conclusions follow from Lemmas 3.1, 2.6 and 2.5, and the method of analysis above 2-points of the proof of 2.7. □

Lemma 3.5. Suppose that $u, \nu \in \mathcal{O}[[x,y]]$. Let $T_0 = \text{Spec}(\mathcal{O}[[x,y]])$. Suppose that $u = x^a$ for some $a \in \mathbb{Z}_+$, or $u = (x^a y^b)^l$ where $\gcd(a,b) = 1$ for some $a, b, l \in \mathbb{Z}_+$. Let $p \in T_0$ be the maximal ideal $(x,y)$. Suppose that $\nu \in (x,y)\mathcal{O}[[x,y]]$. Then either $\nu \in \mathcal{O}[[x]]$ or there exists a sequence of blow ups of points $\lambda : T_1 \to T_0$ such that for all $p_1 \in \lambda^{-1}(p)$, we have regular parameters $x_1, y_1$ in $\mathcal{O}_{T_1,p_1}$, regular parameters $\tilde{x}_1, \tilde{y}_1$ in $\mathcal{O}_{T_1,p_1}$ and a unit $\gamma_1 \in \mathcal{O}_{T_1,p_1}$ such that $x_1 = \gamma_1 \tilde{x}_1$, and one of the following holds:

1) $u = x_1^{a_1}, \nu = P(x_1) + x_1^{b_1} \tilde{y}_1^c$ with $c > 0$ or

2) There exists a unit $\gamma_2 \in \mathcal{O}_{T_1,p_1}$ such that $y_1 = \gamma_2 \tilde{y}_1$ and $u = (x_1^{a_1} \gamma_2 y_1)^{b_1}, \nu = P(x_1^{a_1} y_1) + x_1^{c_1} \tilde{y}_1^{d_1}$ with $\gcd(a_1,b_1) = 1$ and $a_1 d_1 - b_1 c_1 \neq 0$.

Proof. Let

$$J = \text{Det} \left( \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right).$$

\begin{align*}
& J = \text{Det} \left( \begin{array}{cc} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{array} \right). \\
\end{align*}
First suppose that \( J = 0 \). Expand \( v = \sum \gamma_{ij} x^i y^j \) with \( \gamma_{ij} \in \mathfrak{k} \). If \( u = x^a \), then \( \sum j \gamma_{ij} x^i y^{j-1} = 0 \) implies \( \gamma_{ij} = 0 \) if \( j > 0 \). Thus \( v = P(x) \in \mathfrak{k}[[x]] \). If \( u = (x^a y^b)^l \), then

\[
0 = J = l a^{l-1} x^b y^{b-1} \left( \sum_{i,j} (ja - ib) \gamma_{ij} x^i y^j \right)
\]

implies \( \gamma_{ij} = 0 \) if \( ja - ib \neq 0 \), which implies that \( v \in \mathfrak{k}[[x^a y^b]] \).

Now suppose that \( J \neq 0 \). Let \( E \) be the divisor \( u J = 0 \) on \( T_0 \). There exists a sequence of blow ups of points \( \lambda: T_1 \to T_0 \) such that \( \lambda^{-1}(E) \) is a SNC divisor on \( T_1 \). Suppose that \( p_1 \in \lambda^{-1}(p) \). There exist regular parameters \( \tilde{x}_1, \tilde{y}_1 \) in \( \hat{O}_{T_1, p_1} \) such that if

\[
J_1 = \det \left( \frac{\partial u}{\partial \tilde{x}_1}, \frac{\partial u}{\partial \tilde{y}_1} \right),
\]

then

\[
u = \tilde{x}_1^{a_1}, ~ J_1 = \delta \tilde{x}_1^{b_1} \tilde{y}_1^{c_1}\]

where \( a_1 > 0 \) and \( \delta \) is a unit in \( \hat{O}_{T_1, p_1} \), or

\[
u = (\tilde{x}_1^{a_1} \tilde{y}_1^{b_1})^{l_1}, ~ J_1 = \delta \tilde{x}_1^{e_1} \tilde{y}_1^{d_1}\]

where \( a_1, b_1 > 0 \), \( \gcd(a_1, b_1) = 1 \) and \( \delta \) is a unit in \( \hat{O}_{T_1, p_1} \). Expand \( v = \sum \gamma_{ij} \tilde{x}_1^i \tilde{y}_1^j \) with \( \gamma_{ij} \in \mathfrak{k} \).

First suppose (16) holds. Then

\[
a_1 x_1^{a_1-1} \left( \sum_{i,j} j \gamma_{ij} \tilde{x}_1^i \tilde{y}_1^{j-1} \right) = \delta \tilde{x}_1^{b_1} \tilde{y}_1^{c_1}.
\]

Thus \( v = P(\tilde{x}_1) + \varepsilon \tilde{x}_1^{d_1} \tilde{y}_1^{f_1} \) where \( P(\tilde{x}_1) \in \mathfrak{k}[[\tilde{x}_1]], \varepsilon = b_1 - a_1 + a, f = c_1 + 1 \) and \( \varepsilon \) is a unit series. Since \( f > 0 \), we can make a formal change of variables, multiplying \( \tilde{x}_1 \) by an appropriate unit series to get the form 1) of the conclusions of the lemma.

Now suppose that (17) holds. Then

\[
\tilde{x}_1^{a_1 l_1 - 1} \tilde{y}_1^{b_1 l_1 - 1} \left( \sum_{i,j} (a_1 l_1 j - b_1 l_1 i) \gamma_{ij} \tilde{x}_1^i \tilde{y}_1^j \right) = \delta \tilde{x}_1^{e_1} \tilde{y}_1^{d_1}.
\]

Thus \( v = P(\tilde{x}_1^{a_1 d_1}) + \varepsilon \tilde{x}_1^{d_1} \tilde{y}_1^{f_1} \), where \( P \) is a series in \( \tilde{x}_1^{a_1} \tilde{y}_1^{b_1} \), \( \varepsilon \) is a unit series, \( e = c_1 + 1 - a_1 l_1, f = d_1 + 1 - b_1 l_1 \). Since \( a_1 l_1 f - b_1 l_1 e \neq 0 \), we can make a formal change of variables to reach 2) of the conclusions of the lemma. \( \Box \)

**Lemma 3.6.** Suppose that \( X \) is 2-prepared with respect to \( f : X \to S \). Suppose that \( p \in D \) is a 1-point with \( m = \sigma_D(p) + 1 > 1 \). Let \( u, v \) be permissible parameters for \( f(p) \) and \( x, y, z \) be permissible parameters for \( D \) at \( p \) such that a form (9) holds at \( p \). Let \( U \) be an étale cover of an affine neighborhood of \( p \) such that \( x, y, z \) are uniformizing parameters on \( U \). Let \( C \) be the curve in \( U \) which has local equations \( x = y = 0 \) at \( p \).

Let \( T_0 = \text{Spec}(\mathfrak{k}[x, y]), \Lambda_0 : U \to T_0 \). Then there exists a sequence of quadratic transforms \( T_1 \to T_0 \) such that if \( U_1 = U \times_{T_0} T_1 \) and \( \psi_1 : U_1 \to U \) is the induced sequence of blow ups of sections over \( C, \Lambda_1 : U_1 \to T_1 \) is the projection, then \( U_1 \) is 2-prepared with respect to \( f \circ \psi_1 \) at all \( p_1 \in \psi_1^{-1}(p) \). Further, for every point \( p_1 \in \psi_1^{-1}(p) \), there exist regular parameters \( x_1, y_1 \) in \( \hat{O}_{T_1, \Lambda_1(p_1)} \) such that \( x_1, y_1, z \) are permissible parameters at \( p_1 \), and there exist regular parameters \( \tilde{x}_1, \tilde{y}_1 \) in \( \hat{O}_{T_1, \Lambda_1(p_1)} \) such that if \( p_1 \) is a 1-point,
\[ x_1 = \alpha(\tilde{x}_1, \tilde{y}_1) \tilde{x}_1 \] where \( \alpha(\tilde{x}_1, \tilde{y}_1) \in \tilde{O}_{T_1, \Lambda_1(p)} \) is a unit series and \( y_1 = \beta(\tilde{x}_1, \tilde{y}_1) \) with \( \beta(\tilde{x}_1, \tilde{y}_1) \in \tilde{O}_{T_1, \Lambda_1(p)} \), and if \( p_1 \) is a 2-point, then \( x_1 = \alpha(\tilde{x}_1, \tilde{y}_1) \tilde{x}_1 \) and \( y_1 = \beta(\tilde{x}_1, \tilde{y}_1) \tilde{y}_1 \), where \( \alpha(\tilde{x}_1, \tilde{y}_1), \beta(\tilde{x}_1, \tilde{y}_1) \in \tilde{O}_{T_1, \Lambda_1(p)} \) are unit series. We have one of the following forms:

1) \( p_1 \) is a 2-point, and we have an expression (2) with

\[ F = \tau z^m + \tilde{a}_2(x_1, y_1)x_1^2 y_1^{s_2} z^{m-2} + \cdots + \tilde{a}_{m-1}(x_1, y_1)x_1^{r_{m-1}} y_1^{s_{m-1}} z + \tilde{a}_m x_1^{r_m} y_1^{s_m} \]

where \( \tau \in \tilde{O}_{U_1, p_1} \) is a unit, \( \tilde{a}_i(x_1, y_1) \in \tau[[x_1, y_1]] \) are units (or zero) for \( 2 \leq i \leq m-1 \), \( \tilde{a}_m = 0 \) or 1 and if \( \tilde{a}_m = 0 \), then \( \tilde{a}_{m-1} \neq 0 \). Further, \( r_1 + s_i > 0 \) whenever \( \tilde{a}_i \neq 0 \) and \( a(r_m + c)b - (s_m + d)a \neq 0 \).

2) \( p_1 \) is a 1-point, and we have an expression (1) with

\[ F = \tau z^m + \tilde{a}_2(x_1, y_1)x_1^2 y_1^{s_2} z^{m-2} + \cdots + \tilde{a}_{m-1}(x_1, y_1)x_1^{r_{m-1}} y_1^{s_{m-1}} z + x_1^{r_m} y_1^m \]

where \( \tau \in \tilde{O}_{U_1, p_1} \) is a unit, \( \tilde{a}_i(x_1, y_1) \in \tau[[x_1, y_1]] \) are units (or zero) for \( 2 \leq i \leq m-1 \). Further, \( r_1 > 0 \) (whenever \( \tilde{a}_i \neq 0 \)).

3) \( p_1 \) is a 1-point, and we have an expression (1) with

\[ F = \tau z^m + \tilde{a}_2(x_1, y_1)x_1^2 y_1^{s_2} z^{m-2} + \cdots + \tilde{a}_{m-1}(x_1, y_1)x_1^{r_{m-1}} y_1^{s_{m-1}} z + x_1^t y_1^\Omega \]

where \( \tau \in \tilde{O}_{U_1, p_1} \) is a unit, \( \tilde{a}_i(x_1, y_1) \in \tau[[x_1, y_1]] \) are units (or zero) for \( 2 \leq i \leq m-1 \) and \( r_1 > 0 \) whenever \( \tilde{a}_i \neq 0 \). We also have \( t > \omega(m, r_2, \ldots, r_{m-1}) \). Further, \( \tilde{a}_{m-1} \neq 0 \) and \( \Omega \in \tilde{O}_{U_1, p_1} \).

**Proof.** Let \( \tilde{p} = \Lambda_0(p) \). Let \( T = \{ i \mid a_i(x, y) \neq 0 \text{ and } 2 \leq i < m \} \). There exists a sequence of blow ups \( \varphi_1 : T_1 \to T_0 \) of points over \( \tilde{p} \) such that at all points \( q \in \psi_1^{-1}(p) \), we have permissible parameters \( x_1, y_1, z \) such that \( x_1, y_1 \) are regular parameters in \( \tilde{O}_{T_1, \Lambda_1(q)} \) and we have that \( u \) is a monomial in \( x_1 \) and \( y_1 \) times a unit in \( \tilde{O}_{T_1, \Lambda_1(q)} \), where \( q = \prod_{i \in T} a_i(x, y) \).

Suppose that \( a_m(x, y) \neq 0 \). Let \( \tilde{v} = x^a y^b \) if (1) holds and \( \tilde{v} = x^c y^d \) if (2) holds. We have \( \tilde{v} \notin \tilde{M}[[x]] \) (respectively \( \tilde{v} \notin \tilde{M}[[x^a y^b]] \)). Then by Theorem 3.5 applied to \( u, \tilde{v} \), we have that there exists a further sequence of blow ups \( \varphi_2 : T_2 \to T_1 \) of points over \( \tilde{p} \) such that at all points \( q \in (\psi_1 \circ \psi_2)^{-1}(p) \), we have permissible parameters \( x_2, y_2, z \) such that \( x_2, y_2 \) are regular parameters in \( \tilde{O}_{T_2, \Lambda_2(q)} \) such that \( u \neq 0 \) is a SNC divisor and either

\[ u = x_2^r \tilde{v} = \tilde{P}(x_2) + x_2^r y_2^r \]

with \( r > 0 \) or

\[ u = (x_2^r y_2^r)^t \tilde{v} = \tilde{P}(x_2^r y_2^r) + x_2^r y_2^r \]

where \( \tilde{P} - \tilde{P} \neq 0 \).

If \( q \) is a 2-point, we have thus achieved the conclusions of the lemma. Further, there are only finitely many 1-points \( q \) above \( p \) on \( U_2 \) where the conclusions of the lemma do not hold. At such a 1-point \( q \), \( F \) has an expression

\[ F = \tau z^m + \tilde{a}_2(x_2, y_2)x_2^2 y_2^{s_2} z^{m-2} + \cdots + \tilde{a}_{m-1}(x_2, y_2)x_2^{r_{m-1}} y_2^{s_{m-1}} z + \tilde{a}_m x_2^{r_m} y_2^{s_m} \]

where \( \tilde{a}_m = 0 \) or 1, \( \tilde{a}_i \) are units (or zero) for \( 2 \leq i \leq m \).

Let

\[ J = I_2 \left( \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial y_2}, \frac{\partial u}{\partial z} \right) = x^n \left( \frac{\partial F}{\partial y_2}, \frac{\partial F}{\partial z} \right) \]

for some positive integer \( n \). Since \( D \) contains the locus where \( f \) is not smooth, we have that the localization \( J_p = (\tilde{O}_{U_2, q})_p \), where \( p \) is the prime ideal \((y_2, z_2)\) in \( \tilde{O}_{U_2, q} \).
We compute
\[ \frac{\partial F}{\partial z} = \bar{a}_{m-1}x_2^{r_{m-1}}y_2^{s_{m-1}} + \Lambda_1 z \]
and
\[ \frac{\partial F}{\partial y_2} = s_m\bar{a}_m y_2^{s_{m-1}}x_2^{r_m} + \Lambda_2 z \]
for some \( \Lambda_1, \Lambda_2 \in \hat{O}_{U_{2,q}} \), to see that either \( \bar{a}_{m-1} \neq 0 \) and \( s_{m-1} = 0 \), or \( \bar{a}_m \neq 0 \) and \( s_m = 1 \).

Let \( q \) be one of these points, and let \( \varphi_3 : T_3 \to T_2 \) be the blow up of \( \Lambda_2(q) \). We then have that the conclusions of the lemma hold in the form (18) at the 2-point which has permissible parameters \( x_3, y_3 \), \( z \) defined by \( x_2 = x_3y_3 \) and \( y_2 = y_3 \). At a 1-point which has permissible parameters \( x_3, y_3, z \) defined by \( x_2 = x_3, y_2 = x_3(y_3 + \alpha) \) with \( \alpha \neq 0 \), we have that a form (19) holds. Thus the only case where we may possibly have not achieved the conclusions of the lemma is at the 1-point which has permissible parameters \( x_3, y_3, z \) defined by \( x_2 = x_3 \) and \( y_2 = x_3y_3 \). We continue to blow up, so that there is at most one point where the conclusions of the lemma do not hold. This point is a 1-point, which has permissible parameters \( x_3, y_3, z \) where \( x_2 = x_3 \) and \( y_2 = x_3^ny_3 \) where we can take \( n \) as large as we like. We thus have a form
\[ u = x_3^a, \quad v = P(x_3) + x_3^bF_3 \]
with \( F_3 = \tau z^m + \bar{b}_2x_3^{r_2}z^{m-2} + \cdots + \bar{b}_{m-1}x_3^{r_{m-2}}z + x_3^{r_m} \Omega \), where either \( \bar{b}_i(x_3, y_3) \) is a unit or is zero, \( \bar{b}_{m-1} \neq 0 \), and \( t > \omega(m, r_2, \ldots, r_{m-1}) \) if \( \bar{a}_{m-1} \neq 0 \) and \( s_{m-1} = 0 \) which is of the form of (20), or we have a form (19) (after replacing \( y_3 \) with \( y_3 \) times a unit series in \( x_3 \) and \( y_3 \)) if \( \bar{a}_m \neq 0 \) and \( s_m = 1 \).

Lemma 3.7. Suppose that \( X \) is 2-prepared with respect to \( f : X \to S \). Suppose that \( p \in D \) is a 1-point with \( \sigma_D(p) > 0 \). Let \( m = \sigma_D(p) + 1 \). Let \( x, y, z \) be permissible parameters for \( D \) at \( p \) such that a form (9) holds at \( p \).

Let notation be as in Lemma 3.6. For \( p_1 \in \psi_1^{-1}(p) \) let \( \tau(p_1) = m + 1 + r_m \), if a form (19) holds at \( p_1 \), and
\[ \tau(p_1) = \begin{cases} \max\{m + 1 + r_m, m + 1 + s_m\} & \text{if } \bar{a}_m = 1 \\ \max\{m + 1 + r_{m-1}, m + 1 + s_{m-1}\} & \text{if } \bar{a}_m = 0 \end{cases} \]
if a form (18) holds at \( p_1 \). Let \( \tau(p_1) = m + 1 + r_{m-1} \) if a form (20) holds at \( p_1 \).

Let \( r' = \max\{\tau(p_1) \mid p_1 \in \psi_1^{-1}(p)\} \). Let
\[ r = r(p) = m + 1 + r'. \]

Suppose that \( x^* \in \mathcal{O}_{X,p} \) is such that \( x = \tau x^* \) for some unit \( \tau \in \hat{O}_{X,p} \) with \( \tau \equiv 1 \mod m^\mu \hat{O}_{X,p} \).

Let \( V \) be an affine neighborhood of \( p \) such that \( x^*, y \in \Gamma(V, \mathcal{O}_X) \), and let \( C^* \) be the curve in \( V \) which has local equations \( x^* = y = 0 \) at \( p \).

Let \( T_0^* = \text{Spec}(\mathbb{C}[x^*, y]) \). Then there exists a sequence of blow ups of points \( T_1^* \to T_0^* \) above \( (x^*, y) \) such that if \( V_1 = V \times_{T_0^*} T_1^* \) and \( \psi_1^* : V_1 \to V \) is the induced sequence of blow ups of sections over \( C^* \), \( \Lambda_1^* : V_1 \to T_1^* \) is the projection, then \( V_1 \) is 2-prepared at all \( p_1^* \in (\psi_1^*)^{-1}(p) \). Further, for every point \( p_1^* \in (\psi_1^*)^{-1}(p) \), there exist \( \bar{x}_1, \bar{y}_1 \in \hat{O}_{V_1,p_1^*} \) such that \( \bar{x}_1, \bar{y}_1, z \) are permissible parameters at \( p_1^* \) and we have one of the following forms:

1) \( p_1^* \) is a 2-point, and we have an expression (2) with
\[ F = \tau_0 z^m + \tau_2 \bar{x}_1^{r_m} \bar{y}_1^{s_m} z^{m-2} + \cdots + \tau_{m-1} \bar{x}_1^{r_{m-1}} \bar{y}_1^{s_{m-1}} z + \bar{a}_m \bar{x}_1^{r_m} \bar{y}_1^{s_m} \]
where $\tau_0 \in \hat{O}_{V_1,p_1^*}$ is a unit, $\tau_i \in \hat{O}_{V_1,p_i^*}$ are units (or zero) for $0 \leq i \leq m - 1$, $\tau_m$ is zero or 1, $\tau_{m-1} \neq 0$ if $\tau_m = 0$, $r_i + s_i > 0$ if $\tau_i \neq 0$, and

$$(r_m + c)b - (s_m + d)a \neq 0.$$  

2) $p_1^*$ is a 1-point, and we have an expression (1) with

$$F = \tau_0 z^m + \tau_2 \hat{x}_1^2 z^{m-2} + \cdots + \tau_m \hat{x}_1^{r_m - 1} z + \tau_m \hat{x}_1^{r_m}$$

where $\tau_0 \in \hat{O}_{V_1,p_1^*}$ is a unit, $\tau_i \in \hat{O}_{V_1,p_i^*}$ are units (or zero), and $\text{ord}(\tau_m, \hat{y}_1, 0) = 1$. Further, $r_i > 0$ if $\tau_i \neq 0$.

3) $p_1^*$ is a 1-point, and we have an expression (1) with

$$F = \tau_0 z^m + \tau_2 \hat{x}_1^2 z^{m-2} + \cdots + \tau_m \hat{x}_1^{r_m - 1} z + \tau_1 \Omega$$

where $\tau_0 \in \hat{O}_{V_1,p_1^*}$ is a unit, $\tau_i \in \hat{O}_{V_1,p_i^*}$ are units (or zero), $\Omega \in \hat{O}_{V_1,p_1^*}$, $\tau_{m-1} \neq 0$ and $t > \omega(m, r_2, \ldots, r_{m-1})$. Further, $r_i > 0$ if $\tau_i \neq 0$.

Proof. The isomorphism $T_0^* \to T_0$ obtained by substitution of $x^*$ for $x$ and subsequent base change by the morphism $T_1 \to T_0$ of Lemma 3.6, induces a sequence of blow ups of points $T_1^* \to T_0^*$. The base change $\psi_1^*: V_1 = V \times T_0^* T_1^* \to V \cong V \times T_0^* T_0^*$ factors as a sequence of blow ups of sections over $C^*$. Let $\Lambda_1^*: V_1 \to T_1^*$ be the natural projection.

Let $p_1^* \in (\psi_1^*)^{-1}(p)$, and let $p_1 \in \psi_1^{-1}(p) \subseteq U_1$ be the corresponding point.

First suppose that $p_1$ has a form (19). With the notation of Lemma 3.6, we have polynomials $\varphi, \psi$ such that

$$x = \varphi(\hat{x}_1, \hat{y}_1), y = \psi(\hat{x}_1, \hat{y}_1)$$

determines the birational extension $O_{T_0,p_0} \to O_{T_1,\Lambda_1(p_1)}$, and we have a formal change of variables

$$x_1 = \alpha(\hat{x}_1, \hat{y}_1)\hat{x}_1, y_1 = \beta(\hat{x}_1, \hat{y}_1)$$

for some unit series $\alpha$ and series $\beta$. We further have expansions

$$a_i(x, y) = x_1^{r_i} a_i(x_1, y_1)$$

for $2 \leq i \leq m - 1$ where $a_i(x_1, y_1)$ are unit series or zero, and

$$a_m(x, y) = x_1^{r_m} y_1.$$  

We have $x = \tau x^*$ with $\tau \equiv 1$ mod $m_p^* O_{V,p}$. Set $y^* = y$. At $p_1^*$, we have regular parameters $x_1^*, y_1^*$ in $O_{T_1^*,\Lambda_1^*(p_1^*)}$ such that

$$x^* = \varphi(x_1^*, y_1^*), y^* = \psi(x_1^*, y_1^*),$$

and $x_1^*, y_1^*, \hat{z}$ are regular parameters in $O_{V_1,\hat{z}}$ (recall that $z = \sigma \hat{z}$ in Lemma 3.1). We have regular parameters $\tau_1, \bar{y}_1, \in \hat{O}_{T_1^*,\Lambda_1^*(p_1^*)}$ defined by

$$\tau_1 = \alpha(x_1^*, y_1^*) x_1^*, \bar{y}_1 = \beta(x_1^*, y_1^*).$$

We have $u = x^a = x_1^{a_1}$ where $a_1 = ad$ for some $d \in \mathbb{Z}_+$. Since $[\alpha(\hat{x}_1, \hat{y}_1)]d = x$, we have that $[\alpha(x_1^*, y_1^*)]d = x^*$. Set $\hat{x}_1 = \tau_1^{1/2} x_1 = \tau_1^a \alpha(x_1^*, y_1^*) x_1^*$. We have that $\tau_1^{1/2} \alpha(x_1^*, y_1^*)$ is a unit in $O_{V_1,p_1^*}$, and $x = \hat{x}_1^d$. Thus $x_1 = \hat{x}_1$ (with an appropriate choice of root $\tau_1^{1/2}$). We have $u = x_1^{ad}$, so that $\hat{x}_1, \bar{y}_1, z$ are permissible parameters at $p_1^*$.

For $2 \leq i \leq m - 1$, we have

$$a_i(x, y) = a_i(\tau x^*, y^*) \equiv a_i(x^*, y^*) \mod m_p^* O_{V,p}$$
and
\[
a_i(x^*, y^*) = a_i(\varphi(x_i^1, y_i^1), \psi(x_i^1, y_i^1)) = x_i^1 \tau_i^1 (x_1, y_1) \equiv x_i^1 \tau_i (x_1, y_1) \mod m_p^r \mathcal{O}_{V_i, p_i^*}.
\]

We further have
\[
a_m(x^*, y^*) \equiv x_1^r y_1 \mod m_p^r \mathcal{O}_{V_i, p_i^*}.
\]

Thus we have expressions
\[
\begin{align*}
  u &= x_1^{d_a} \\
  v &= P(x_1^d) + x_1^{b_d} P_1(x_1) + x_1^{b_d} (\tau z^m + x_1^r \tau_2 (x_1, y_1) z^{m-2} + \cdots + x_1^r y_1 + h)
\end{align*}
\]
where \( \tau \in \hat{\mathcal{O}}_{V_i, p_i^*} \) is a unit series and
\[
h \in m_p^r \hat{\mathcal{O}}_{V_i, p_i^*} \subset (x_1, z)^r.
\]

Set \( s = r - m \), and write
\[
h = z^m \Lambda_0 (x_1, y_1, z) + z^{m-1} x_1^{1+s} \Lambda_1 (x_1, y_1) + z^{m-2} x_1^{2+s} \Lambda_2 (x_1, y_1) + \cdots + z x_1^{(m-1)+s} \Lambda_{m-1} (x_1, y_1) + x_1^m \Lambda_m (x_1, y_1)
\]
with \( \Lambda_0 \in m_{p_i^*} \hat{\mathcal{O}}_{V_i, p_i^*} \) and \( \Lambda_i \in \mathfrak{r}[[x_1, y_1]] \) for \( 1 \leq i \leq m \).

Substituting into (27), we obtain an expression
\[
\begin{align*}
  u &= x_1^{d_a} \\
  v &= P(x_1^d) + x_1^{b_d} P_1(x_1) + x_1^{b_d} (\tau z^m + x_1^r \tau_2 (x_1, y_1) z^{m-2} + \cdots + x_1^r \tau_{m-1} (x_1, y_1) z + x_1^r \tau_m)
\end{align*}
\]
where \( \tau_0 \in \hat{\mathcal{O}}_{V_i, p_i^*} \) is a unit, \( \tau_i \in \hat{\mathcal{O}}_{V_i, p_i^*} \) are units (or zero), for \( 1 \leq i \leq m - 1 \) and \( \tau_m \in \mathfrak{r}[[x_1, y_1]] \) with \( \text{ord}(\tau_m (0, y_1)) = 1 \).

We have \( \tau_0 = \bar{\tau} + \Lambda_0, \tau_i = \bar{\tau}_i (x_1, y_1) \) for \( 2 \leq i \leq m - 1 \), and
\[
\tau_m = \bar{\tau}_1 + z^{m-1} x_1^{1+s-r_m} \Lambda_1 (x_1, y_1) + \cdots + x_1^{m+s-r_m} \Lambda_m (x_1, y_1).
\]

We thus have the desired form (25).

In the case when \( p_1 \) has a form (20), a similar argument to the analysis of (19) shows that \( p_i^* \) has a form (26).

Now suppose that \( p_1 \) has a form (18). We then have
\[
m_p \mathcal{O}_{U_i, p_i} \subset (x_1 y_1, z) \mathcal{O}_{U_i, p_i},
\]
unless there exist regular parameters \( x_i^1, y_i^1 \in \mathcal{O}_{T_i, \Lambda_i(p_i)} \) such that \( x_i^1, y_i^1, z \) are regular parameters in \( \mathcal{O}_{U_i, p_i} \) and
\[
x = x_i^1, y = (x_i^1)^n y_i^1
\]
or
\[
x = x_i^1 (y_i^1)^n, y = y_i^1
\]
for some \( n \in \mathbb{N} \). If (29) or (30) holds, then \( \hat{\mathcal{O}}_{V_i, p_i^*} = \hat{\mathcal{O}}_{U_i, p_i}, \) and (taking \( \bar{x}_1 = x_1, \bar{y}_1 = y_1 \)) we have that a form (24) holds at \( p_i^* \). We may thus assume that (28) holds.

With the notation of Lemma 3.6, we have polynomials \( \varphi, \psi \) such that
\[
x = \varphi(\bar{x}_1, \bar{y}_1), y = \psi(\bar{x}_1, \bar{y}_1)
\]
determines the birational extension \( \mathcal{O}_{T_0, p_0} \to \mathcal{O}_{T_i, \Lambda_i(p_i)}, \) and we have a formal change of variables
\[
x_1 = \alpha(\bar{x}_1, \bar{y}_1) \bar{x}_1, y_1 = \beta(\bar{x}_1, \bar{y}_1) \bar{y}_1
\]
for some unit series $\alpha$ and $\beta$. We further have expansions
\[ a_i(x, y) = \tilde{x}_i^a y_i^b \tilde{a}_i(x_1, y_1) \]
for $2 \leq i \leq m - 1$ where $\tilde{a}_i(x_1, y_1)$ are unit series or zero, and
\[ a_m(x, y) = \tilde{x}_m y_m^s \tilde{a}_m, \]
where $\tilde{a}_m = 0$ or 1. We have $x = \tilde{x} x^*$ with $\tilde{x} \equiv 1 \mod m_p \hat{O}_{X,p}$. Set $y^* = \psi$. At $p_1$, we have regular parameters $x_i^*, y_i^*$ in $O_{T_1^*, \Lambda_1^*}(p_1)$ such that
\[ x^* = \varphi(x_1^*, y_1^*), y^* = \psi(x_1^*, y_1^*), \]
and $x_1^*, y_1^*, \tilde{z}$ are regular parameters in $O_{V_1^*}$ (recall that $z = \sigma \tilde{z}$ in Lemma 3.1). We have regular parameters $\tilde{x}_1, \tilde{y}_1, \tilde{a}_i, \tilde{b}_i, \tilde{m}_i$ defined by
\[ \tilde{x}_1 = \alpha(x_1^*, y_1^*)x_1^*, \tilde{y}_1 = \beta(x_1^*, y_1^*)y_1^*. \]
We calculate
\[ u = x^a = (x_1^a y_1^b) \tilde{t}_1 = [\alpha(\tilde{x}_1, \tilde{y}_1) \tilde{x}_1]^{a_1 t_1} [\beta(\tilde{x}_1, \tilde{y}_1) \tilde{y}_1]^{b_1 t_1} \]
which implies
\[ (x^*)^a = [\alpha(x_1^*, y_1^*)]^{a_1 t_1} [\beta(x_1^*, y_1^*)]^{b_1 t_1} \]
Set $\tilde{x}_1 = \tilde{x}^{a_1 t_1} \tilde{x}_1$ to get $u = (\tilde{x}_1^a \tilde{y}_1^b) \tilde{t}_1$, so that $\tilde{x}_1, \tilde{y}_1, \tilde{z}$ are permissible parameters at $p_1^*$. For $2 \leq i \leq m$, we have
\[ a_i(x, y) = a_i(\tilde{x}^a y^*, y^*) \equiv a_i(x^*, y^*) \mod m_p \hat{O}_{V,p} \]
and
\[ a_i(x^*, y^*) = a_i(\varphi(x_1^*, y_1^*), \psi(x_1^*, y_1^*)) \]
\[ \equiv \tilde{x}_i^a \tilde{y}_i^b \tilde{a}_i(\tilde{x}_1, \tilde{y}_1) \]
\[ \equiv \tilde{x}_i^a \tilde{y}_i^b \tilde{a}_i(\tilde{x}_1, \tilde{y}_1) \mod m_p \hat{O}_{V_i,p_i^*}. \]
Thus we have expressions
\[ u = (\tilde{x}_1^a \tilde{y}_1^b)^{\tilde{t}_1}, \]
\[ v = P((\tilde{x}_1^a \tilde{y}_1^b) \tilde{x}_1) + (\tilde{x}_1^a \tilde{y}_1^b) \tilde{a}_i(\tilde{x}_1, \tilde{y}_1) \tilde{b}_i P_i(\tilde{x}_1^a \tilde{y}_1^b) + \tilde{x}_1^a \tilde{y}_1^b \tilde{m}_i \tilde{a}_i(\tilde{x}_1, \tilde{y}_1) \tilde{z}_i \tilde{m}_i \tilde{b}_i \tilde{t}_1 \]
where $\tilde{t}_1 \in \hat{O}_{V_1,p_1^*}$ is a unit series and
\[ h = m_p \hat{O}_{V_i,p_i^*} \subset (\tilde{x}_1 \tilde{y}_1) \tilde{r}. \]
Set $s = r - m$, and write
\[ h = z^m \Lambda_0(x_1, \tilde{y}_1, z) + z^{m-1}(\tilde{x}_1 \tilde{y}_1) + z^{m-2}(\tilde{x}_1 \tilde{y}_1)^2 \Lambda_1(x_1, \tilde{y}_1) + \cdots \]
\[ + z(\tilde{x}_1 \tilde{y}_1)^{(m-1)+s} \Lambda_{m-1}(x_1, \tilde{y}_1) + (\tilde{x}_1 \tilde{y}_1)^{m+s} \Lambda_m(\tilde{x}_1, \tilde{y}_1) \]
with $\Lambda_0 \in m_{p_1} \hat{O}_{V_1,p_1^*}$ and $\Lambda_i \in \mathfrak{t}(\tilde{x}_1, \tilde{y}_1)$ for $1 \leq i \leq m$. First suppose that $\tilde{a}_m = 1$. Substituting into (31), we obtain an expression
\[ u = (\tilde{x}_1^a \tilde{y}_1^b)^{\tilde{t}_1}, \]
\[ v = P((\tilde{x}_1^a \tilde{y}_1^b) \tilde{x}_1) + (\tilde{x}_1^a \tilde{y}_1^b) \tilde{a}_i(\tilde{x}_1, \tilde{y}_1) \tilde{b}_i P_i(\tilde{x}_1^a \tilde{y}_1^b) \]
\[ + (\tilde{x}_1^a \tilde{y}_1^b) \tilde{m}_i \tilde{a}_i(\tilde{x}_1, \tilde{y}_1) \tilde{z}_i \tilde{m}_i \tilde{b}_i \tilde{t}_1 \]
where $\tilde{t}_1, \tilde{t}_m \in \hat{O}_{V_1,p_1^*}$ are units, $\tilde{t}_i \in \hat{O}_{V_i,p_i^*}$ are units (or zero) for $2 \leq i \leq m - 1$. We have $\tau_0 = \tilde{t}_1 + \Lambda_0, \tau_i = \tilde{a}_i(\tilde{x}_1, \tilde{y}_1)$ for $2 \leq i \leq m - 1$, and
\[ \tau_m = \tilde{a}_m + z^{m-1} \tilde{x}_1 \tilde{y}_1^{1+s-r_m} \Lambda_1(x_1, \tilde{y}_1) + \cdots + \tilde{x}_1^{m+s-r_m} \tilde{y}_1^{m+s-r_m} \Lambda_m(\tilde{x}_1, \tilde{y}_1). \]
We thus have the desired form (24).

Now suppose that $\bar{a}_m = 0$. Then $\bar{a}_{m−1} \neq 0$, and $z$ divides $h$ in (31), so that $\Lambda_m = 0$ in (32). Substituting into (31), we obtain an expression

$$
u = P((\hat{x}^{a_1} y_1^{m_1})^{t_1}) + \frac{\bar{a}_1}{\bar{a}_0} P_1(\hat{x}^{a_1} y_1^{m_1})$$

where $\bar{a}_0, \bar{a}_1 \in \mathcal{O}_{V_1, p_1}$ are units, $\bar{a}_i \in \mathcal{O}_{V_1, p_1}$ are units (or zero) for $2 \leq i \leq m - 2$.

We have

$$\tau_0 = \tau + \Lambda_0, \tau_i = \tau_i(\hat{x}_1, y_1) \text{ for } 2 \leq i \leq m - 2,$$

and

$$\tau_{m-1} = a_{m-1} + z^{m-1} \hat{x}_1^{1+s-r_{m-1}-1} y_1^{1+s-m-1} \Lambda_1(\hat{x}_1, y_1) + \cdots + \hat{x}_1^{1+s-r_{m-1}-1} y_1^{1+s-m-1} \Lambda_{m-1}(\hat{x}_1, y_1).$$

We thus have the form (24).

\[ \square \]

**Lemma 3.8.** Suppose that $X$ is 2-prepared. Suppose that $p \in X$ is a 1-point with $\sigma_p(p) > 0$ and $E$ is the component of $D$ containing $p$. Suppose that $Y$ is a finite set of points in $X$ (not containing $p$). Then there exists an affine neighborhood $U$ of $p$ in $X$ such that

1. $Y \cap U = \emptyset$.
2. $[E - U \cap E] \cap \text{Sing}_1(X)$ is a finite set of points.
3. $U \cap D = U \cap E$ and there exists $\tau \in \Gamma(U, \mathcal{O}_X)$ such that $\tau = 0$ is a local equation of $E$ in $U$.
4. There exists an étale map $\pi : U \to \mathbb{A}^3 = \text{Spec}(\mathbb{C}[x, y, z])$.
5. The Zariski closure $C$ in $X$ of the curve in $U$ with local equations $\pi = \overline{y} = 0$ satisfies the following:
   i) $C$ is a nonsingular curve through $p$.
   ii) $C$ contains no 3-points of $D$.
   iii) $C$ intersects 2-curves of $D$ transversally at prepared points.
   iv) $C \cap \text{Sing}_1(X) \cap (X - U) = \emptyset$.
   v) $C \cap Y = \emptyset$.
   vi) $C$ intersects $\text{Sing}_1(X) - \{p\}$ transversally at general points of curves in $\text{Sing}_1(X)$.
   vii) There exist permissible parameters $x, y, z$ at $p$, with $\hat{x} = \tau, y = \overline{y}$, which satisfy the hypotheses of lemma 3.1.

**Proof.** Let $H$ be an effective, very ample divisor on $X$ such that $H$ contains $Y$ and $D - E$, but $H$ does not contain $p$ and does not contain any one dimensional components of $\text{Sing}_1(X, D) \cap E$. There exists $n > 0$ such that $E + nH$ is ample, $\mathcal{O}_X(E + nH)$ is generated by global sections and a general member $H'$ of the linear system $|E + nH|$ does not contain any one dimensional components of $\text{Sing}_1(X, D) \cap E$, and does not contain $p$. $H + H'$ is ample, so $V = X - (H + H')$ is affine. Further, there exists $f \in \mathcal{O}(X)$, the function field of $X$, such that $(f) = H' - (E + nH)$. Thus $\tau = \frac{1}{f} \in \Gamma(V, \mathcal{O}_X)$ as $X$ is normal and $\tau$ has no poles on $V$. $\tau = 0$ is a local equation of $E$ on $V$. We have that $V$ satisfies the conclusions 1), 2) and 3) of the lemma.

Let $R = \Gamma(V, \mathcal{O}_X)$. $R = \bigcup_{s \in \mathbb{Z} \geq 1} \Gamma(X, \mathcal{O}_X(s(H + H')))$. $R$ is a finitely generated $\mathbb{C}$-algebra. Thus for $s \gg 0$, $R$ is generated by $\Gamma(X, \mathcal{O}_X(s(H + H')))$. $R$ is a $\mathbb{C}$-algebra for all $s \gg 0$.

From the exact sequences

$$0 \to \Gamma(X, \mathcal{O}_X(s(H + H'))) \otimes \mathcal{I}_p \to \Gamma(X, \mathcal{O}_X(s(H + H'))) \to \mathcal{O}_{X,p}/m_p \cong k$$

and the fact that $1 \in \Gamma(X, \mathcal{O}_X(s(H + H')))$, we have that $R$ is generated by $\Gamma(X, \mathcal{O}_X(s(H + H'))) \otimes \mathcal{I}_p$ as a $\mathbb{C}$-algebra for all $s \gg 0$. 17
For $s \gg 0$, and a general member $\sigma$ of $\Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p)$ we have that the curve $\mathcal{C} = B \cdot E$, where $B$ is the divisor $B = (\sigma) + s(H + H')$, satisfies the conclusions of 5) of the lemma; since each of the conditions 5i) through 5vii) is an open condition on $\Gamma(X, \mathcal{O}_X(s(H + H') \otimes \mathcal{I}_p))$, we need only establish that each condition holds on a nonempty subset. This follows from the fact that $H + H'$ is ample, Bertini’s theorem applied to the base point free linear system $|\varphi^*(s(H + H')) - A|$, where $\varphi : W \to X$ is the blow up of $p$ with exceptional divisor $A$, and the fact that

$$\varphi^*(\mathcal{O}_{W}(\varphi^*(s(H + H') - A)) = \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p.$$  

For fixed $s \gg 0$, let $\mathfrak{t}, \mathfrak{g}_1, \ldots, \mathfrak{g}_n$ be a $\mathfrak{t}$-basis of $\Gamma(X, \mathcal{O}_X(s(H + H')) \otimes \mathcal{I}_p)$, so that $R = \mathfrak{t}[\mathfrak{t}, \mathfrak{g}_1, \ldots, \mathfrak{g}_n]$. We have shown that there exists a Zariski open set $\mathcal{Z}$ of $k^n$ such that for $(b_1, \ldots, b_n) \in \mathcal{Z}$, the curve $C$ in $X$ which is the Zariski closure of the curve with local equation $\mathfrak{t} = b_1 \mathfrak{g}_1 + \cdots + b_n \mathfrak{g}_n = 0$ in $V$ satisfies 5) of the conclusions of the lemma.

Let $C_1, \ldots, C_t$ be the curves in $\text{Sing}_1(X) \cap V$, and let $p_i \in C_i$ be closed points such that $p_i, p_{i+1}, \ldots, p_{i+t}$ are distinct. Let $Q_0$ be the maximal ideal of $p$ in $R$, and $Q_i$ be the maximal ideal in $R$ of $p_t$ for $1 \leq i \leq t$. We have that $\mathfrak{t}$ is nonzero in $Q_i/Q_i^2$ for all $i$. For a matrix $A = (a_{ij}) \in \mathfrak{t}^{2n}$, and $1 \leq i \leq 2$, let

$$L_i^A(\mathfrak{g}_1, \ldots, \mathfrak{g}_n) = \sum_{j=1}^{n} a_{ij} \mathfrak{g}_j.$$  

There exist $\alpha_{jk} \in \mathfrak{t}$ such that $Q_k = (\mathfrak{g}_1 - \alpha_{1,k}, \ldots, \mathfrak{g}_n - \alpha_{n,k})$ for $0 \leq k \leq t$. By our construction, we have $\alpha_{1,0} = \cdots = \alpha_{n,0} = 0$. For each $0 \leq k \leq t$, there exists a non empty Zariski open subset $Z_k$ of $k^{2n}$ such that

$$\mathfrak{t}, L_1^A(\mathfrak{g}_1, \ldots, \mathfrak{g}_n) - L_1^A(\alpha_{1,k}, \ldots, \alpha_{n,k}), L_2^A(\mathfrak{g}_1, \ldots, \mathfrak{g}_n) - L_2^A(\alpha_{1,k}, \ldots, \alpha_{n,k})$$

is a $\mathfrak{t}$-basis of $Q_k/Q_k^{2+1}$. Suppose $(a_{1,1}, \ldots, a_{1,n}) \in \mathcal{Z}$ and $A \in Z_0 \cap \cdots \cap Z_t$.

We will show that $\mathfrak{t}, L_1^A, L_2^A$ are algebraically independent over $\mathfrak{t}$. Suppose not. Then there exists a nonzero polynomial $h \in \mathfrak{t}[t_1, t_2, t_3]$ such that $h(\mathfrak{t}, L_1^A, L_2^A) = 0$. Write $h = H + h'$ where $H$ is the leading form of $h$, and $h' = h - H$ is a polynomial of larger order than the degree of $H$. Now $H(\mathfrak{t}, L_1^A, L_2^A) = -h(\mathfrak{t}, L_1^A, L_2^A)$, so that $H(\mathfrak{t}, L_1^A, L_2^A) = 0$ in $Q_0^{r+1}$. Thus $H = 0$, since $R_{Q_0}$ is a regular local ring, which is a contradiction. Thus $\mathfrak{t}, L_1^A, L_2^A$ are algebraically independent. Without loss of generality, we may assume that $L_i^A = \mathfrak{g}_i$ for $1 \leq i \leq 2$.

Let $S = \mathfrak{t}[\mathfrak{t}, \mathfrak{g}_1, \mathfrak{g}_2]$, a polynomial ring in 3 variables over $\mathfrak{t}$. $S \to R$ is unramified at $Q_i$ for $0 \leq i \leq t$ since

$$(\mathfrak{t}, \mathfrak{g}_1 - \alpha_{1,i}, \mathfrak{g}_2 - \alpha_{2,i})R_{Q_i} = Q_iR_{Q_i},$$

for $0 \leq i \leq t$.

Let $W$ be the closed locus in $V$ where $V \to \text{Spec}(S)$ is not étale. We have that $p, p_1, \ldots, p_t \not\in W$, so there exists an ample effective divisor $\overline{H}$ on $X$ such that $W \subset \overline{H}$ and $p, p_1, \ldots, p_t \not\in \overline{H}$. Let $U = V - \overline{H}$. $U$ is affine, and $U \to \text{Spec}(S) \cong \mathbb{A}^3$ is étale, so satisfies 4) of the conclusions of the lemma.

\[\square\]

**Lemma 3.9.** Suppose $X$ is 2-prepared with respect to $f : X \to S$, $p \in D$ is a prepared point, and $\pi_1 : X_1 \to X$ is the blow up of $p$. Then all points of $\pi_1^{-1}(p)$ are prepared.

**Proof.** The conclusions follow from substitution of local equations of the blow up of a point into a prepared form (1), (2) or (3). \[\square\]
Lemma 3.10. Suppose that $X$ is 2-prepared with respect to $f : X \to S$, and that $C$ is a permissible curve for $D$, which is not a 2-curve. Suppose that $p \in C$ satisfies $\sigma_D(p) = 0$. Then there exist permissible parameters $x, y, z$ at $p$ such that one of the following forms hold:

1) $p$ is a 1-point of $D$ of the form of (1), $F = z$ and $x = y = 0$ are formal local equations of $C$ at $p$.
2) $p$ is a 1-point of $D$ of the form of (1), $F = z$ and $x = z = 0$ are formal local equations of $C$ at $p$.
3) $p$ is a 1-point of $D$ of the form of (1), $F = z, x = z + y^r \sigma(y) = 0$ are formal local equations of $C$ at $p$, where $r > 1$ and $\sigma$ is a unit series.
4) $p$ is a 2-point of $D$ of the form of (2), $F = z, x = z = 0$ are formal local equations of $C$ at $p$.
5) $p$ is a 2-point of $D$ of the form of (2), $F = z, x = f(y, z) = 0$ are formal local equations of $C$ at $p$, where $f(y, z)$ is not divisible by $z$.
6) $p$ is a 2-point of $D$ of the form of (2), $F = 1$ (so that $ad - bc \neq 0$) and $x = z = 0$ are formal local equations of $C$ at $p$.

Further, there are at most a finite number of 1-points on $C$ satisfying condition 3) (and not satisfying condition 1) or 2)).

Proof. Suppose that $p$ is a 1-point. We have permissible parameters $x, y, z$ at $p$ such that a form (1) holds at $p$ with $F = z$. There exists a series $f(y, z)$ such that $x = f = 0$ are formal local equations of $C$ at $p$. By the formal implicit function theorem, we get one of the forms 1), 2) or 3). A similar argument shows that one of the forms 4), 5) or 6) must hold if $p$ is a 2-point.

Now suppose that $p \in C$ is a 1-point, $\sigma_D(p) = 0$ and a form 3) holds at $p$. There exist permissible parameters $x, y, z$ at $p$, with an expression (1), such that $x = z = 0$ are formal local equations of $C$ at $p$ and $x, y, z$ are uniformizing parameters on an étale cover $U$ of an neighborhood of $p$, where we can choose $U$ so that

$$
\frac{\partial F}{\partial y} = \frac{1}{x^b} \frac{\partial v}{\partial y} \in \Gamma(U, \mathcal{O}_X).
$$

Since there is not a form 2) at $p$, we have that $z$ does not divide $F(0, y, z)$, so that $F(0, y, 0) \neq 0$. Since $F$ has no constant term, we have that $\frac{\partial F}{\partial y}(0, y, 0) \neq 0$. There exists a Zariski open subset of $\mathfrak{f}$ such that $\alpha \in \mathfrak{f}$ implies $x, y - \alpha, z$ are regular parameters at a point $q \in U$. There exists a Zariski open subset of $\mathfrak{f}$ of such $\alpha$ so that $\frac{\partial F}{\partial y}(0, \alpha, 0) \neq 0$. Thus $x, y - \alpha, z$ are permissible parameters at $q$ giving a form 1) at $q \in C$.

Lemma 3.11. Suppose that $X$ is 2-prepared. Suppose that $C$ is a permissible curve on $X$ which is not a 2-curve and $p \in C$ satisfies $\sigma_D(p) = 0$. Further suppose that either a form 3) or 5) of the conclusions of Lemma 3.10 hold at $p$. Then there exists a sequence of blow ups of points $\pi_1 : X_1 \to X$ above $p$ such that $X_1$ is 2-prepared and $\sigma_D(p_1) = 0$ for all $p_1 \in \pi_1^{-1}(p)$, and the strict transform of $C$ on $X_1$ is permissible, and has the form 4) or 6) of Lemma 3.10 at the point above $p$.

Proof. If $p$ is a 1-point, let $\pi' : X' \to X$ be the blow ups of $p$, and let $C'$ be the strict transform of $C$ on $X'$. Let $p'$ be the point on $C'$ above $p$. Then $p'$ is a 2-point and $\sigma_D(p') = 0$. We may thus assume that $p$ is a 2-point and a form 5) holds at $p$. For $r \in \mathbb{Z}_+$, let

$$
X_r \to X_{r-1} \to \cdots \to X_1 \to X
$$
be the sequence of blow ups of the point $p_i$ which is the intersection of the strict transform $C_i$ of $C$ on $X_i$ with the preimage of $p$.

There exist permissible parameters $x, y, z$ at $p$ such that $x = z = 0$ are formal local equations of $C$ at $p$, and a form (2) holds at $p$ with $F = x\Omega + f(y, z)$. We have that \[ \text{ord } f(y, z) = 1, \text{ord } \Omega(0, y, z) \geq 1, y \text{ does not divide } f(y, z) \text{ and } z \text{ does not divide } f(y, z). \]

At $p_r$, we have permissible parameters $x_r, y_r, z_r$ such that

\[ x = x_r y_r^r, \quad y = y_r, \quad z = z_r y_r. \]

$x_r = z_r = 0$ are local equations of $C_r$ at $p_r$. We have a form (2) at $p_r$ with

\[
\begin{align*}
 u & = (x_r y_r)^{a r + b} \\
 v & = P(x_r^r y_r^{a r + b}) + x_r^r y_r^{r + d + r} F'
\end{align*}
\]

where

\[ F' = x_r \Omega + \frac{f(y_r, z_r y_r^r)}{y_r^r}, \]

if \( f(y_r^{-1}, x_r^{-1} y_r^{-1}) \) is not a unit series. Thus for $r$ sufficiently large, we have that $F'$ is a unit, so that a form 6) holds at $p_r$.

Lemma 3.12. Suppose that $X$ is 2-prepared and that $C_1$ is a permissible curve on $X$. Suppose that $q \in C$ is a point with $\sigma_D(q) = 0$ which has a form 1), 4) or 6) of Lemma 3.10. Let $\pi_1 : X_1 \to X$ be the blow up of $C$. Then $X_1$ is 3-prepared in a neighborhood of $\pi_1^{-1}(q)$. Further, $\sigma_D(q_1) = 0$ for all $q_1 \in \pi_1^{-1}(q)$.

Proof. The conclusions follow from substitution of local equations of the blow up of $C$ into the forms 1), 4) and 6) of Lemma 3.10.

Proposition 3.13. Suppose that $X$ is 2-prepared. Then there exists a sequence of permissible blow ups $\pi_1 : X_1 \to X$, such that $X_1$ is 3-prepared. We further have that $\sigma_D(p_1) \leq \sigma_D(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.

Proof. Let $T$ be the points $p \in X$ such that $X$ is not 3-prepared at $p$. By Lemmas 3.4 and 2.5, after we perform a sequence of blow ups of 2-curves, we may assume that $T$ is a finite set consisting of 1-points of $D$.

Suppose that $p \in T$. Let $T' = T \setminus \{p\}$. Let $U = \text{Spec}(R)$ be the affine neighborhood of $p$ in $X$ and let $C$ be the curve in the conclusions of Lemma 3.8 (with $Y = T'$), so that $C$ has local equations $\tau = \sigma = 0$ in $U$.

Let $\Sigma_1 = C \cap \text{Sing}_1(X)$. $\Sigma_1 = \{p = p_0, \ldots, p_r\}$ is the union of $\{p\}$ and a finite set of general points of curves in $\text{Sing}_1(X)$, which must be 1-points. We have that $\Sigma_1 \subset U$. Let

\[ \Sigma_2 = \{q \in C \cap U \mid \sigma_D(q) = 0 \text{ and a form 2) of Lemma 3.10 holds at } q\}. \]

$\Sigma_2$ is a finite set by Lemma 3.10. Let $\Sigma_3 = C \setminus U$, a finite set of 1-points and 2-points which are prepared.

Set $U' = U \setminus \Sigma_2$. There exists a unit $\tau \in R$ and $a \in \mathbb{Z}_+$ such that $u = \tau \tilde{x}^a$.

By 5 vi), 5 vii) of Lemma 3.8 and Lemma 3.2, there exist $z_i \in \hat{O}_{X, p_i}$ such that for all $p_i \in \Sigma_1$, $x = \tau^\frac{1}{a} \tilde{x}, y, z_i$ are permissible parameters at $p_i$ giving a form (9).

Let $t = \max\{r(p_i) \mid 0 \leq i \leq r\}$, where $r(p_i)$ are calculated from (23)) of Lemma 3.7. There exists $\lambda \in R$ such that $\lambda \equiv \tau^{-\frac{1}{a}} \mod m_{p_i}^t \hat{O}_{X, p_i}$ for $0 \leq i \leq r$. Let $x^* = \lambda^{-1} \tilde{x}$, $\gamma = \tau^{\frac{1}{a}} \lambda$. Then $x = \tau^{\frac{1}{a}} \tilde{x} = \gamma x^*$ with $\gamma \equiv 1 \mod m_{p_i}^t \hat{O}_{X, p_i}$ for $0 \leq i \leq r$. Let $U' = U \setminus \Sigma_2$. 
Let $T_0^* = \text{Spec}(\mathbb{k}[x^*, y^*])$, and let $T_1^* \to T_0^*$ be a sequence of blow ups of points above $(x^*, y^*)$ such that the conclusions of Lemma 3.7 hold on $U_i' = U' \times T_0^* T_1^*$ above all $p_i$ with $0 \leq i \leq r$. The projection $\lambda_1 : U_1' \to U'$ is a sequence of blow ups of sections over $C$. $\lambda_1$ is permissible and $\lambda_1^{-1}(C \cap (U' \setminus \Sigma_i))$ is prepared by Lemma 3.12.

All points of $\Sigma_2 \cup \Sigma_3$ are prepared. Thus by Lemma 3.9, Lemmas 3.11 and Lemma 3.12, by interchanging some blowups of points above $\Sigma_2 \cup \Sigma_4$ between blow ups of sections over $C$, we may extend $\lambda_1$ to a sequence of permissible blow ups over $X$ to obtain the desired sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that $X_1$ is 2-prepared. $\pi_1$ is an isomorphism over $T'$, $X_1$ is 3-prepared over $\pi_1^{-1}(X_1 \setminus T')$, and $\sigma_\mathcal{D}(p_1) \leq \sigma_\mathcal{D}(p)$ for all $p \in X_1 \setminus T'$.

By induction on $|T|$, we may iterate this procedure a finite number of times to obtain the conclusions of Proposition 3.13.

\[ \square \]

The following proposition is proven in a similar way.

**Proposition 3.14.** Suppose that $X$ is 1-prepared and $\mathcal{D}'$ is a union of irreducible components of $\mathcal{D}$. Suppose that there exists a neighborhood $V$ of $\mathcal{D}'$ such that $V$ is 2-prepared and $V$ is 3-prepared at all 2-points and 3-points of $V$.

Let $A$ be a finite set of 1-points of $\mathcal{D}'$, such that $A$ is contained in $\text{Sing}_1(X)$ and $A$ contains the points where $V$ is not 3-prepared, and let $B$ be a finite set of 2-points of $\mathcal{D}'$. Then there exists a sequence of permissible blow ups $\pi_1 : X_1 \to X$ such that

1) $X_1$ is 3-prepared in a neighborhood of $\pi_1^{-1}(\mathcal{D}')$.
2) $\pi_1$ is an isomorphism over $X_1 \setminus \mathcal{D}'$.
3) $\pi_1$ is an isomorphism in a neighborhood of $B$.
4) $\pi_1$ is an isomorphism over generic points of 2-curves on $\mathcal{D}'$ and over 3-points of $\mathcal{D}'$.
5) Points on the intersection of the strict transform of $\mathcal{D}'$ on $X_1$ with $\pi_1^{-1}(A)$ are 2-points of $\mathcal{D}X_1$.
6) $\sigma_\mathcal{D}(p_1) \leq \sigma_\mathcal{D}(p)$ for all $p \in X$ and $p_1 \in \pi_1^{-1}(p)$.

\[ \text{4. Reduction of } \sigma_\mathcal{D} \text{ above a 3-prepared point.} \]

**Theorem 4.1.** Suppose that $p \in X$ is a 1-point such that $X$ is 3-prepared at $p$, and $\sigma_\mathcal{D}(p) > 0$. Let $(x,y,z)$ be permissible parameters at $p$ giving a form (14) at $p$. Let $\mathcal{U}$ be an étale cover of an affine neighborhood of $p$ in which $x,y,z$ are uniformizing parameters. Then $xz = 0$ gives a toroidal structure $\mathcal{D}$ on $\mathcal{U}$. Let $\mathcal{T}$ be the ideal in $\Gamma(U, \mathcal{O}_X)$ generated by $z^m$, $x^r m$ if $\tau_m \neq 0$, and by

\[ \{x^i z^m - i \mid 2 \leq i \leq m - 1 \text{ and } \tau_i \neq 0\} \]

Suppose that $\psi : U' \to U$ is a toroidal morphism with respect to $\mathcal{D}$ such that $U'$ is nonsingular and $I\mathcal{O}_{U'}$ is locally principal. Then (after possibly replacing $U$ with a smaller neighborhood of $p$) $U'$ is 2-prepared and $\sigma_\mathcal{D}(q) < \sigma_\mathcal{D}(p)$ for all $q \in U'$.

There is (after possibly replacing $U$ with a smaller neighborhood of $p$) a unique, minimal toroidal morphism $\psi : U' \to U$ with respect to $\mathcal{D}$ with the property that $U'$ is nonsingular, 2-prepared and $\Gamma(U') < \sigma_\mathcal{D}(p)$. This map $\psi$ factors as a sequence of permissible blowups $\pi_i : U_i \to U_{i-1}$ of sections $C_i$ over the two curve $C$ of $\mathcal{D}$. $U_i$ is 1-prepared for $U_i \to S$. We have that the curve $C_i$ blown up in $U_{i+1} \to U_i$ is in $\text{Sing}_{\pi_i}(U_{i+1})$ if $C_i$ is not a 2-curve of $D_{U_i}$, and that $C_i$ is in $\text{Sing}_{\pi_i}(U_i)$ if $C_i$ is a 2-curve of $D_{U_i}$.

21
Proof. Suppose that \( \psi : U' \to U \) is toroidal for \( \overline{D} \) and \( U' \) is nonsingular. Let \( \overline{D}' = \psi^{-1}(\overline{D}) \).

The set of 2-curves of \( \overline{D}' \) is the disjoint union of the 2-curves of \( D_{U'} \) and the 2-curve which is the intersection of the strict transform of the surface \( z = 0 \) on \( U' \) with \( D_{U'} \). \( \psi \) factors as a sequence of blow ups of 2-curves of (the preimage of) \( \overline{D} \). We will verify the following three statements, from which the conclusions of the theorem follow.

\[
\text{If } q \in \psi^{-1}(p) \text{ and } I\mathcal{O}_{U',q} \text{ is principal, then } \sigma_D(q) < \sigma_D(p). \tag{33}
\]

In particular, \( \sigma_D(q) < \sigma_D(p) \) if \( q \) is a 1-point of \( \overline{D}' \).

\[
\text{If } C' \text{ is a 2-curve of } D_{U'}, \text{ then } U' \text{ is prepared at } q = C' \cap \psi^{-1}(p)
\text{ if and only if } \sigma_D(q) < \infty
\text{ if and only if } I\mathcal{O}_{U',q} \text{ is principal}
\text{ if and only if } U' \text{ is prepared at all } q' \in C' \text{ in a neighborhood of } q. \tag{34}
\]

If \( C' \) is the 2-curve of \( \overline{D}' \) which is the intersection of \( D_{U'} \) with the strict transform of \( \tilde{z} = 0 \) in \( U' \), then \( \sigma_D(q) \leq \sigma_D(p) \) if \( q = C' \cap \psi^{-1}(p) \), and \( \sigma_D(q') = \sigma_D(q) \) for \( q' \in C' \) in a neighborhood of \( q \).

Suppose that \( q \in \psi^{-1}(p) \) is a 1-point for \( \overline{D}' \). Then \( I\mathcal{O}_{U',q} \) is principal. At \( q \), we have permissible parameters \( x_1, y, z_1 \) defined by

\[
x = x_1^{a_1}, z = x_1^{b_1}(z_1 + \alpha)
\]

for some \( a_1, b_1 \in \mathbb{Z}_+ \) and \( 0 \neq \alpha \in \mathfrak{t} \). Substituting into (14), we have

\[
u = x_1^{a_1}, v = P(x_1^{a_1}) + x_1^{b_1}G
\]

where

\[
G = \tau_0 x_1^{b_1 - 1}(z_1 + \alpha)^m + \tau_2 x_1^{a_1 r_2 + b_1 (m - 2)}(z_1 + \alpha)^{m - 2} + \cdots + \tau_{m - 1} x_1^{a_1 r_{m - 1} + b_1}(z_1 + \alpha) + \tau_m x_1^{a_1 r_m}.
\]

Let \( x_1^e \) be a local generator of \( I\mathcal{O}_{U',q} \). Let \( G' = \frac{G}{x_1^e} \).

If \( z^m \) is a local generator of \( I\mathcal{O}_{U',q} \), then \( G' \) has an expansion

\[
G' = \tau'(z_1 + \alpha)^m + g_2(z_1 + \alpha)^{m - 2} + \cdots + g_{m - 1}(z_1 + \alpha) + g_m + x_1 \Omega_1 + y \Omega_2
\]

where \( 0 \neq \tau' = \tau(0, 0, 0) \in \mathfrak{t}, g_2, \ldots, g_m \in \mathfrak{f} \) and \( \Omega_1, \Omega_2 \in \mathfrak{O}_{U',q} \). We have \( \text{ord}(G'(0, 0, z_1)) \leq m - 1 \). Setting \( F' = G' - G'(x_1, 0, 0) \) and \( P'(x_1) = P(x_1^{a_1}) + x_1^{b_1}G'(x_1, 0, 0) \), we have an expression

\[
u = x_1^{a_1}, v = P'(x_1) + x_1^{b_1}F'
\]

of the form of (1). Thus \( U' \) is 2-prepared at \( q \) with \( \sigma_{D'}(q) < m - 1 = \sigma_D(p) \).

Suppose that \( z^m \) is not a local generator of \( I\mathcal{O}_{U',q} \), but there exists some \( i \) with \( 2 \leq i \leq m - 1 \) such that \( x_1^{r_i} z^{m - i} \) is a local generator of \( I\mathcal{O}_{U',q} \). Let \( h \) be the smallest \( i \) with this property. Then \( G' \) has an expression

\[
G' = g_1(z_1 + \alpha)^{m - h} + \cdots + g_m + x_1 \Omega_1 + y_1 \Omega_2
\]

for some \( g_i \in \mathfrak{f} \) with \( g_h \neq 0 \) and \( \Omega_1, \Omega_2 \in \mathfrak{O}_{U',q} \). As in the previous case, we have that \( U' \) is 2-prepared at \( q \) with \( \sigma_D(q) < m - h - 1 < m - 1 = \sigma_D(p) \).
Suppose that $z^m$ is not a local generator of $I\hat{O}_{U',q}$ and $x^{r_1}z^{m-i}$ is not a local generator of $I\hat{O}_{U',q}$ for $2 \leq i \leq m - 1$. Then $x_1^{r_1}$ is a local generator of $I\hat{O}_{U',q}$, and we have an expression

$$G' = \Lambda + x_1 \Omega_1,$$

where $\Lambda(x_1, y, z_1) = \tau_m(x_1^{a_1}, y, x_1^{b_1}(z_1 + \alpha))$ and $\Omega_1 \in \hat{O}_{U',q}$. Then

$$\text{ord } \Lambda(0, y, 0) = \text{ord } \tau_m(0, y, 0) = 1,$$

and we have that $U'$ is prepared at $q$.

Now suppose that $q \in \psi^{-1}(p)$ is a 2-point for $D_{U'}$. We have permissible parameters $x_1, y, z_1$ in $\hat{O}_{U',q}$ such that

$$x = x_1^{a_1}z_1^{b_1}, z = x_1^{c_1}z_1^{d_1}$$

with $a_1, b_1 > 0$ and $a_1d_1 - b_1c_1 = \pm 1$. Substituting into (14), we have

$$u = x_1^{a_1}z_1^{b_1}, v = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b_1z_1^{b_1}}G$$

where

$$G = \tau_0x_1^{c_1m}z_1^{d_1m} + \tau_2x_1^{r_2a_1+c_1(m-2)}z_1^{r_2b_1+d_2(m-2)} + \cdots + \tau_{m-1}x_1^{a_1r_{m-1}+c_1b_{1r_{m-1}}+d_1} + \tau_mx_1^{a_1r_m}z_1^{b_1r_m}.$$

Let $C'$ be the 2-curve of $D_{U'}$ containing $q$. Since ord $\tau_m(0, y, 0) = 1$ (if $\tau_m \neq 0$) we see that the three statements $\sigma_D(q) < \infty$, $\sigma_D(q) = 0$ and $I\hat{O}_{U',q}$ is principal are equivalent. Further, we have that $\sigma_D(q') = \sigma_D(q)$ for $q' \in C'$ in a neighborhood of $q$.

Suppose that $I\hat{O}_{U',q}$ is principal and let $x_1^az_1^t$ be a local generator of $I\hat{O}_{U',q}$. Let $G' = G/x_1^az_1^t$. We have that

$$u = (x_1^{a_1}z_1^{b_1})^{a_1}z_1^{b_1} = P(x_1^{a_1}z_1^{b_1}) + x_1^{a_1b_1z_1^{b_1}}G$$

has the form (2), since we have made a monomial substitution in $x$ and $z$. If $z^m$ or $x^{r_1}z^{m-i}$ for some $i < m$ is a local generator of $I\hat{O}_{U',q}$, then $G'$ is a unit in $\hat{O}_{U',q}$. If none of $z^m$, $x^{r_1}z^{m-i}$ for $i < m$ are local generators of $I\hat{O}_{U',q}$, then

$$G' = \Lambda + x_1 \Omega_1 + z_1 \Omega_2,$$

where

$$\Lambda(x_1, y_1, z_1) = \tau_m(x_1^{a_1}z_1^{b_1}, y, x_1^{c_1}z_1^{d_1})$$

and $\Omega_1, \Omega_2 \in \hat{O}_{U',q}$. Thus

$$\text{ord } \Lambda(0, y, 0) = \text{ord } \tau_m(0, y, 0) = 1.$$

We thus have that $U'$ is prepared at $q$.

The final case is when $q \in \psi^{-1}(p)$ is on the 2-curve $C'$ of $\overline{D'}$ which is the intersection of $D_{U'}$ with the strict transform of $z = 0$ in $U'$. Then there exist permissible parameters $x_1, y, z_1$ at $q$ such that

$$x = x_1, z = \hat{x}_1^{b_1}z_1$$

for some $b_1 \in \mathbb{Z}_+$. The equations $x_1 = z_1 = 0$ are local equations of $C'$ at $q$. Let

$$s = \min\{b_1s_{r_1} + b_1(m - i) \text{ with } \tau_i \neq 0 \text{ for } 2 \leq i \leq m - 1, r_m \text{ if } \tau_i \neq 0\}.$$

We have an expression of the form (1) at $q$,

$$u = x_1^a, v = P(x_1^q) + x_1^{a_1b_1}G'$$
with
\[ G' = \tau_0 x_1^{b_1 m - s} z_1^m + \tau_2 x_1^{r_2 + b_1 (m-2) - s} z_1^{m-2} + \ldots + \tau_{m-1} x_1^{r_{m-1} - b_1 - s} z_1 + \tau_m x_1^{r_m - s}. \]

We see that \( \sigma_D(q) \leq \sigma_D(p) \) (with \( \sigma_D(q) < \sigma_D(p) \) if \( s = r_i + b_1 (m - i) \) for some \( i \) with \( 2 \leq i \leq m - 1 \) or \( s = r_m \)) and \( \sigma_D(q') = \sigma_D(q) \) for \( q' \) in a neighborhood of \( q \) on \( C' \).

Suppose that \( IO_U,\dot{q} \) is principal. Then \( x^m \) generates \( IO'_{U',q} \). We have that \( G' = x^m \) where \( \Omega \in \hat{\Omega}_{U',q} \) satisfies \( ord \Omega(0,0,0) = 1 \). Thus \( U' \) is prepared at \( q \).

We will now construct the function \( \omega(m, r_2, \ldots, r_{m-1}) \) where \( m > 1, r_i \in \mathbb{N} \) for \( 2 \leq i \leq m - 1 \) and \( r_{m-1} > 0 \).

Let \( I \) be the ideal in the polynomial ring \( \mathfrak{t}[x, z] \) generated by \( z^m \) and \( x^r z^{m-i} \) for all \( i \) such that \( 2 \leq i \leq m - 1 \) and \( r_i > 0 \). Let \( m = (x, z) \) be the maximal ideal of \( k[x, z] \). Let \( \Phi : V_1 \to V = \text{Spec}(\mathfrak{t}[x, z]) \) be the toroidal morphism with respect to the divisor \( xz = 0 \) on \( V \) such that \( V_1 \) is the minimal nonsingular surface such that

1. \( IO_{V_1,q} \) is principal if \( q \in \Phi^{-1}(m) \) is not on the strict transform of \( z = 0 \).
2. If \( q \) is the intersection point of the strict transform of \( z = 0 \) and \( \Phi^{-1}(m) \), so that \( q \) has regular parameters \( x_1, z_1 \), with \( x = x_1, z = x_1^b z_1 \) for some \( b \in \mathbb{Z}_+ \), then \( r_i + b_1 (m - i) < b_1 m \) for some \( 2 \leq i \leq m - 1 \) with \( r_i > 0 \).

Every \( q \in \Phi^{-1}(m) \) which is not on the strict transform of \( z = 0 \) has regular parameters \( x_1, z_1 \) at \( q \) which are related to \( x, z \) by one of the following expressions:

\[ x = x_1^{a_1}, \quad z = x_1^{b_1} (z_1 + \alpha) \]

for some \( 0 \neq \alpha \in \mathfrak{t} \) and \( a_1, b_1 > 0 \), or

\[ x = x_1^{a_1}, \quad z = x_1^{b_1} z_1 \]

with \( a_1, b_1 > 0 \) and \( a_1 d_1 - b_1 c_1 = \pm 1 \). There are only finitely many values of \( a_1, b_1 \) occurring in expressions (39), and \( a_1, b_1, c_1, d_1 \) occurring in expressions (40).

The point \( q \) on the intersection of the strict transform of \( z = 0 \) and \( \Phi^{-1}(m) \) has regular parameters \( x_1, z_1 \) defined by

\[ x = x_1, \quad z = x_1^{b_1} z_1 \]

for some \( b_1 > 0 \).

Now we define \( \omega = \omega(m, r_2, \ldots, r_{m-1}) \) to be a number such that

\[ \omega > \max\left\{ \frac{b_1}{a_1} m, r_i + \frac{b_1}{a_1} (m - i) \right\} \text{ for } 2 \leq i \leq m - 1 \text{ such that } r_i > 0 \].

For all expressions (39),

\[ \omega > \max\left\{ \frac{c_1}{a_1} m, \frac{d_1}{b_1} m, r_i + \frac{c_1}{a_1} (m - i), r_i + \frac{d_1}{b_1} (m - i) \right\} \text{ for } 2 \leq i \leq m - 1 \text{ such that } r_i > 0 \]

for all expressions (40), and

\[ \omega > \max\left\{ b_1 m, r_i + b_1 (m - i) \right\} \text{ for } 2 \leq i \leq m - 1 \text{ such that } r_i > 0 \]

in (41).

**Theorem 4.2.** Suppose that \( p \in \text{Sing}_1(X) \) is a 1-point and \( X \) is 3-prepared at \( p \). Let \( x, y, z \) be permissible parameters at \( p \) giving a form (15) at \( p \). Let \( U \) be an étale cover of an affine neighborhood of \( p \) in which \( x, y, z \) are uniformizing parameters. Then \( xz = 0 \) gives a toroidal structure \( D \) on \( U \).

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24
There is (after possibly replacing \(U\) with a smaller neighborhood of \(p\)) a unique, minimal toroidal morphism \(\psi : U' \to U\) with respect to \(\overline{D}\) with has the property that \(U'\) is nonsingular, 2-prepared and \(\Gamma_D(U') < \sigma_D(p)\). This map \(\psi\) factors as a sequence of permissible blowups \(\pi_i : U_i \to U_{i-1}\) of sections \(C_i\) over the two curve \(C\) of \(\overline{D}\), \(U_i\) is 1-prepared for \(U_i \to S\). We have that the curve \(C_i\) blown up in \(U_{i+1} \to U_i\) is in \(\text{Sing}_{\sigma_D(p)}(U_i)\) if \(C_i\) is not a 2-curve of \(D_{U_i}\), and that \(C_i\) is in \(\text{Sing}(U_i)\) if \(C_i\) is a 2-curve of \(D_{U_i}\).

Proof. The proof is similar to that of Theorem 4.1, using the fact that \(t > \omega(m, r_2, \ldots, r_{m-1})\) as defined above.

**Theorem 4.3.** Suppose that \(p \in X\) is a 2-point and \(X\) is 3-prepared at \(p\) with \(\sigma_D(p) > 0\). Let \(x, y, z\) be permissible parameters at \(p\) giving a form (13) at \(p\). Let \(U\) be an étale cover of an affine neighborhood of \(p\) in which \(x, y, z\) are uniformizing parameters on \(U\). Then \(xyz = 0\) gives a toroidal structure \(\overline{D}\) on \(U\). Let \(I\) be the ideal in \(\Gamma(U, \mathcal{O}_X)\) generated by \(z^m, x^m y^m\) if \(\tau_m \neq 0\)

\[
\{x^i y^i z^{m-i} \mid 2 \leq i \leq m-1 \text{ and } \tau_i \neq 0\}. 
\]

Suppose that \(\psi : U_1 \to U\) is a toroidal morphism with respect to \(\overline{D}\) such that \(U_1\) is nonsingular and \(\text{IO}_{U_1}\) is locally principal. Then (after possibly replacing \(U\) with a smaller neighborhood of \(p\)) \(U_1\) is 2-prepared for \(U_1 \to S\), with \(\sigma_D(q) < \sigma_D(p)\) for all \(q \in U_1\).

Proof. Suppose that \(q \in \psi^{-1}(p)\) is a 1-point for \(\psi^{-1}(\overline{D})\). Then \(q\) is also a 1-point for \(D_{U_1}\). Since \(\psi\) is toroidal with respect to \(\overline{D}\), there exist regular parameters \(\hat{x}_1, \hat{y}_1, \hat{z}_1\) in \(\mathcal{O}_{X_1, \hat{q}}\) and a matrix \(A = (a_{ij})\) with nonnegative integers as coefficients such that \(\text{Det} A = \pm 1\), and we have an expression

\[
\begin{align*}
    x &= \hat{x}_1^{a_{11}} (\hat{y}_1 + \alpha)^{a_{12}} (\hat{z}_1 + \beta)^{a_{13}}, \\
    y &= \hat{x}_1^{a_{21}} (\hat{y}_1 + \alpha)^{a_{22}} (\hat{z}_1 + \beta)^{a_{23}}, \\
    z &= \hat{x}_1^{a_{31}} (\hat{y}_1 + \alpha)^{a_{32}} (\hat{z}_1 + \beta)^{a_{33}},
\end{align*}
\]

with \(a_{11}, a_{21}, a_{31} \neq 0\) and \(0 \neq \alpha, \beta \in \mathfrak{k}\). Set

\[
\overline{x}_1 = \hat{x}_1^{a_{11}} (\hat{y}_1 + \alpha)^{a_{12}} (\hat{z}_1 + \beta)^{a_{13}} \in \mathcal{O}_{X_1, \hat{q}}.
\]

Substituting into (42), we have

\[
\begin{align*}
    x &= \overline{x}_1^{a_{11}}, \\
    y &= \overline{x}_1^{a_{21}} (\hat{y}_1 + \alpha)^{a_{22}} - \frac{a_{21} a_{12}}{a_{11}} (\hat{z}_1 + \beta)^{a_{23}} - \frac{a_{21} a_{13}}{a_{11}} \overline{x}_1^{a_{11}}, \\
    z &= \overline{x}_1^{a_{31}} (\hat{y}_1 + \alpha)^{a_{32}} - \frac{a_{31} a_{12}}{a_{11}} (\hat{z}_1 + \beta)^{a_{33}} - \frac{a_{31} a_{13}}{a_{11}} \overline{x}_1^{a_{11}}.
\end{align*}
\]

Let \(B = (b_{ij})\) be the adjoint matrix of \(A\). Let \(\overline{\alpha} = \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta^{\frac{a_{33}}{a_{11}}}\), \(\overline{\beta} = \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta^{\frac{a_{33}}{a_{11}}}\). Set

\[
\overline{y}_1 = \frac{y}{\overline{x}_1^{a_{21}}} - \overline{\alpha}, \overline{z}_1 = \frac{z}{\overline{x}_1^{a_{31}}} - \overline{\beta}.
\]

We will show that \(\overline{x}_1, \overline{y}_1, \overline{z}_1\) are regular parameters in \(\mathcal{O}_{X_1, \hat{q}}\). We have that

\[
\begin{align*}
(\hat{y}_1 + \alpha)^{a_{22}} - \frac{a_{21} a_{12}}{a_{11}} (\hat{z}_1 + \beta)^{a_{23}} - \frac{a_{21} a_{13}}{a_{11}} \overline{x}_1^{a_{11}} &= \overline{\alpha} + \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta^{\frac{a_{33}}{a_{11}}} \overline{y}_1 - \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta \overline{z}_1 + \cdots, \\
(\hat{y}_1 + \alpha)^{a_{32}} - \frac{a_{31} a_{12}}{a_{11}} (\hat{z}_1 + \beta)^{a_{33}} - \frac{a_{31} a_{13}}{a_{11}} \overline{x}_1^{a_{11}} &= \overline{\beta} - \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta \overline{y}_1 + \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta \overline{z}_1 + \cdots.
\end{align*}
\]

Let

\[
C = \begin{pmatrix}
    \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta^{\frac{a_{33}}{a_{11}}} & -\frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta \overline{z}_1 \\
    -\frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta^{\frac{a_{33}}{a_{11}}} & \frac{b_{33}}{a_{11}} \beta - \frac{b_{32}}{a_{11}} \beta \overline{z}_1
\end{pmatrix}.
\]
We must show that $C$ has rank 2. $C$ has the same rank as
\[
\begin{pmatrix}
  b_{33}\beta & -b_{23}\alpha \\
b_{32}\beta & -b_{22}\alpha
\end{pmatrix}
= \begin{pmatrix}
b_{33} & b_{23} \\
b_{32} & b_{22}
\end{pmatrix}
\begin{pmatrix}
\beta & 0 \\
0 & -\alpha
\end{pmatrix}.
\]
Since $\alpha, \beta \neq 0$, $C$ has the same rank as
\[
B' = \begin{pmatrix}
b_{33} & b_{23} \\
b_{32} & b_{22}
\end{pmatrix}.
\]
Since $B$ has rank 3,
\[
\begin{pmatrix}
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{pmatrix}
\]
has rank 2. Since
\[
\begin{pmatrix}
b_{21} \\
b_{31}
\end{pmatrix} = \frac{a_{21}}{a_{11}} \begin{pmatrix}
b_{22} \\
b_{32}
\end{pmatrix} + \frac{a_{31}}{a_{11}} \begin{pmatrix}
b_{23} \\
b_{33}
\end{pmatrix},
\]
we have that $B'$ has rank 2, and hence $C$ has rank 2. Thus $\overline{x}_1, \overline{y}_1, \overline{z}_1$ are regular parameters in $\mathcal{O}_{X_1, q}$. We have
\[
x = x_1^{a_1}, \quad y = x_1^{a_2}(\overline{y}_1 + \overline{\alpha}), \quad z = x_1^{a_3}(\overline{z}_1 + \overline{\beta}).
\]
We have that $u = (x^a y^b)^t$. Let
\[
t = -\frac{b}{a_{11}a + a_{21}b},
\]
and set $x_1 = x_1(y_1 + \alpha)^t$. Define $\overline{y}_1 = y_1, \overline{\alpha} = \overline{\alpha}, \overline{\beta} = \overline{\alpha}^{a_{21}}\overline{\beta}$ and $z_1 = (\overline{y}_1 + \overline{\alpha})^{a_{31}}(z_1 + \overline{\beta}) - \overline{\beta}$. Then $x_1, y_1, z_1$ are permissible parameters at $q$, with $u = x_1^{(a_{11}a + a_{21}b)t}$,
\[
x = x_1^{a_1}(y_1 + \alpha)^{a_2}, \quad y = x_1^{a_2}(y_1 + \alpha)^{a_3}, \quad z = x_1^{a_3}(z_1 + \overline{\beta}).
\]
Thus we have shown that there exist (formal) permissible parameters $x_1, y_1, z_1$ at $q$ such that
\[
x = x_1^{e_1}(y_1 + \alpha)^{\lambda_1}, \quad y = x_1^{e_2}(y_1 + \alpha)^{\lambda_2}, \quad z = x_1^{e_3}(z_1 + \overline{\beta}),
\]
where $e_1, e_2, e_3 \in \mathbb{Z}_+$, $\alpha, \beta \in \mathfrak{t}$ are nonzero, $\lambda_1, \lambda_2 \in \mathbb{Q}$ are both nonzero, and $u = x_1^{b_1}$, where $b_1 = a_1e_1 + b_1e_2, a\lambda_1 + b\lambda_2 = 0$. We then have an expression
\[
v = P(x_1^{e_1 + de_2}) + x_1^{e_1 + de_2}G,
\]
where
\[
G = (y_1 + \alpha)^{d\lambda_1 + d\lambda_2}[\tau_0 x_1^{e_3m}(z_1 + \overline{\beta})^m + \tau_2 x_1^{r_2e_1 + r_2e_2 + (m-2)e_3}(y_1 + \alpha)^{r_2\lambda_1 + \lambda_2}(z_1 + \overline{\beta})^{m-2} + \cdots + \tau_{m-1} x_1^{r_{m-1}e_1 + s_{m-1}e_2 + e_3}(y_1 + \alpha)^{r_{m-1}\lambda_1 + s_{m-1}\lambda_2}(z_1 + \overline{\beta}) + \tau_m x_1^{r_m e_1 + s_m e_2}y_1^{r_m\lambda_1 + s_m\lambda_2}].
\]
Let $\tau' = \tau_0(0, 0, 0)$. Let $x_1^{b_1}$ be a generator of $I\mathcal{O}_{U_1, q}$. Let $G' = F(x_1^t)$. If $z^m$ is a local generator of $I\mathcal{O}_{U_1, q}$, then $G'$ has an expression
\[
G' = \tau' z^m = g_1(z_1 + \overline{\beta})^m + g_2(z_1 + \overline{\beta})^{m-2} + \cdots + g_{m-1}(z_1 + \overline{\beta}) + g_m + x_1\Omega_1 + y_1\Omega_2
\]
for some $g_i \in \mathfrak{t}$ and $\Omega_1, \Omega_2 \in \mathcal{O}_{U_1, q}$, where $\varphi = c\lambda_1 + d\lambda_2$. Setting $F' = G' - G'(x_1, 0, 0)$, and $P'(x_1) = P(x_1^{e_1 + de_2}) + x_1^{e_1 + de_2 + s}G'(x_1, 0, 0)$, we have that
\[
u = x_1^{b_1}, \quad v = P'(x_1), \quad x_1^{e_1 + de_2 + s} F'
\]
has the form (1) and $\sigma_D(q) \leq \text{ord } F'(0, 0, z_1) - 1 \leq m - 2 < m - 1 = \sigma_D(p)$ since $0 \neq \overline{\beta}$. 

Suppose that $z^m$ is not a local generator of $I\hat{O}_{U_1,q}$, but there exists some $i$ with $2 \leq i \leq m - 1$ such that $\tau_i x^i y^s z^{m-i}$ is a local generator of $I\hat{O}_{U_1,q}$. Let $h$ be the smallest $i$ with this property. Then $G'$ has an expression

$$G' = g_h(z_1 + \hat{\beta})^{m-h} + \cdots + g_{m-1}(z_1 + \hat{\beta}) + g_m + x_1 \Omega + y_2 \Omega$$

for some $g_i \in \mathfrak{k}$ with $g_h \neq 0$. As in the previous case, we have

$$\sigma_D(q) \leq m - h - 1 < m - 1 = \sigma_D(p).$$

Suppose that $z^m$ is not a local generator of $I\hat{O}_{U_1,q}$, and $\tau_i x^i y^s z^{m-i}$ is not a local generator of $I\hat{O}_{U_1,q}$ for $2 \leq i \leq m$. Then $x^i y^s$ is a local generator of $I\hat{O}_{U_1,q}$, and $G'$ has an expression

$$G' = \tau'_m(y_1 + \hat{\alpha})^{e + r_m \lambda_1 + s_m \lambda_2} + x_1 \Omega$$

where $\tau'_m = \tau_m(0,0,0)$ for some $\Omega \in \hat{O}_{U_1,q}$. Suppose, if possible, that $\varphi + r_m \lambda_1 + s_m \lambda_2 = 0$. Since $\varphi + r_m \lambda_1 + s_m \lambda_2 = (c + r_m) \lambda_1 + (d + s_m) \lambda_2$, we then have that the nonzero vector $(\lambda_1, \lambda_2)$ satisfies $\alpha \lambda_1 + \beta \lambda_2 = (c + r_m) \lambda_1 + (d + s_m) \lambda_2 = 0$. Thus the determinant $\alpha(d + s_m) - \beta(c + r_m) = 0$, a contradiction to our assumption that $F$ satisfies (2).

Now since $\varphi + r_m \lambda_1 + s_m \lambda_2 \neq 0$ and $\hat{\alpha} \neq 0$, we have $1 = \text{ord} G'(0,y_1,0) < m$, so that $\sigma_D(q) = 0 < m - 1 = \sigma_D(p)$.

Suppose that $q \in \psi^{-1}(p)$ is a 2-point of $\psi^{-1}(D)$. Then there exist (formal) permissible parameters $\hat{x}_1, \hat{y}_1, \hat{z}_1$ at $q$ such that

$$(44) \quad x = \hat{x}_1^{e_{11}} y_1^{e_{12}} (\hat{z}_1 + \hat{\alpha})^{e_{13}}, \quad y = \hat{x}_1^{e_{21}} y_1^{e_{22}} (\hat{z}_1 + \hat{\alpha})^{e_{23}}, \quad z = \hat{x}_1^{e_{31}} y_1^{e_{32}} (\hat{z}_1 + \hat{\alpha})^{e_{33}}$$

where $e_{ij} \in \mathbb{N}$, with $\text{Det}(e_{ij}) = \pm 1$, and $\hat{\alpha} \in \mathfrak{k}$ is nonzero. We further have

$$e_{11} + e_{12} > 0, e_{21} + e_{22} > 0 \text{ and } e_{31} + e_{32} > 0.$$  

First suppose that $e_{11} e_{22} - e_{12} e_{21} \neq 0$. Then $q$ is a 2-point of $D_{U_1}$.

There exist $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that upon setting

$$\hat{x}_1 = x_1(z_1 + \hat{\alpha})^{\lambda_1} \quad \text{and} \quad \hat{y}_1 = y_1(z_1 + \hat{\alpha})^{\lambda_2},$$

we have

$$x = x_1^{e_{11}} y_1^{e_{12}}, \quad y = x_1^{e_{21}} y_1^{e_{22}}, \quad z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \hat{\alpha})^r,$$

where

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}.$$  

By Cramer’s rule,

$$r = \pm \frac{1}{e_{11} e_{22} - e_{12} e_{21}} \neq 0.$$  

Now set $z_1 = (z_1 + \hat{\alpha})^r - \alpha^r$ and $\alpha = \alpha^r$ to obtain permissible parameters $x_1, y_1, z_1$ at $q$ with

$$x = x_1^{e_{11}} y_1^{e_{12}}, \quad y = x_1^{e_{21}} y_1^{e_{22}}, \quad z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \alpha).$$

We have an expression

$$u = ((x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b)^{\ell_1} = (x_1^{t_1} y_1^{t_2})^{\ell_1}$$

where $t_1, t_2, \ell_1 \in \mathbb{Z}_+$ and $\gcd(t_1, t_2) = 1$.

We then have an expression

$$v = P((x_1^{t_1} y_1^{t_2})^{\frac{\ell_1}{2}}) + x_1^{c e_{11} + d e_{21}} y_1^{c e_{12} + d e_{22}} G,$$
Let \( G \) then \( I \) has the form (2), and \( g \) satisfies (45), we have that
\[
\hat{F} = \left[ z \right] = \det(z + x_1 \Omega_1 + y_1 \Omega_2)
\]
for some \( z \in \mathfrak{L} \) and \( \Omega_1, \Omega_2 \in \hat{O}_{U_1} \). Let
\[
(45) \quad \hat{P}(x_1^{i_1} y_1^{j_1}) = \sum_{i_2-i_1j=0} 1 \frac{\partial (x_1^{i_1} y_1^{j_1} G)}{\partial x_1^{i_1} \partial y_1^{j_1}}(0, 0, 0)x_1^{i_1} y_1^{j_1}
\]
and \( F' = G' - \frac{\hat{P}(x_1^{i_1} y_1^{j_1})}{x_1^{i_1} y_1^{j_1}} \). Set \( P'(x_1^{i_1} y_1^{j_1}) = P((x_1^{i_1} y_1^{j_1}) + \hat{P}(x_1^{i_1} y_1^{j_1})). \) We have that
\[
u = (x_1^{i_1} y_1^{j_1}) + \frac{\partial (x_1^{i_1} y_1^{j_1} G)}{\partial x_1^{i_1} \partial y_1^{j_1}}(0, 0, 0)x_1^{i_1} y_1^{j_1}
\]
has the form (2), and \( \sigma_D(q) = \text{ord}(F'(0, 0, z_1) - 1) \leq m - 2 < m - 1 = \sigma_D(p) \) since \( 0 \neq \alpha \).

Suppose that \( z \) is not a local generator of \( O_{U_1} \), but there exists some \( i \) with \( 2 \leq i \leq m - 1 \) such that \( \tau_i x^m y^s z^{-i} \) is a local generator of \( O_{U_1} \). Let \( h \) be the smallest \( i \) with this property. Then \( G' \) has an expression
\[
G' = g_h(z_1 + \beta)^{m-h} + \cdots + g_m x_1 \Omega_1 + y_1 \Omega_2
\]
for some \( g_i \in \mathfrak{L} \) with \( g_h \neq 0 \). As in the previous case, we have \( \sigma_D(q) \leq m - h - 1 < m - 1 = \sigma_D(p) \).

Suppose that \( z \) is not a local generator of \( O_{U_1} \), and \( \tau_i x^m y^s z^{-i} \) is not a local generator of \( O_{U_1} \) for \( 2 \leq i \leq m - 1 \). Then \( x^m y^s \) is a local generator of \( O_{U_1} \), and then \( G' \) has an expression
\[
G' = 1 + x_1 \Omega_1 + y_1 \Omega_2
\]
for some \( \Omega_1, \Omega_2 \in \hat{O}_{U_1} \).

We now claim that after replacing \( G' \) with \( F' = G' - \frac{\hat{P}(x_1^{i_1} y_1^{j_1})}{x_1^{i_1} y_1^{j_1}} \), where \( \hat{P} \) is defined by (45), we have that \( F'(0, 0, 0) \neq 0 \). If this were not the case, we would have
\[
0 = \det \begin{pmatrix} a & d + s_m \\ e_1 & e_2 \end{pmatrix} \det \begin{pmatrix} a & d + s_m \\ e_1 & e_2 \end{pmatrix}.
\]
Since \( e_1 e_2 - e_2 e_1 = 0 \) (by our assumption), we get
\[
0 = \det \begin{pmatrix} a & d + s_m \\ e_1 & e_2 \end{pmatrix}
\]
which is a contradiction to our assumption that \( F \) satisfies (2). Since \( F'(0, 0, 0) \neq 0 \), we have that \( \sigma_D(q) = 0 < m - 1 = \sigma_D(p) \).

Now suppose that \( q \) is a 2-point of \( \psi^{-1}(D) \) with \( e_1 e_2 - e_2 e_1 = 0 \) in (44).
We make a substitution
\[\hat{x}_1 = x_1(z_1 + \alpha)^{\hat{x}_1}, \hat{y}_1 = y_1(z_1 + \alpha)^{\hat{y}_1}, \hat{z}_1 = z_1\]
where \( \alpha = \hat{\alpha} \) and \( \varphi_1, \varphi_2 \in \mathbb{Q} \) satisfy
\[
0 = a(\varphi_1 e_1 + \varphi_2 e_{12} + e_{13}) + b(\varphi_1 e_1 + \varphi_2 e_{22} + e_{23})
= \varphi_1(e_1 + b e_{21}) + \varphi_2(a e_{12} + b e_{22}) + a e_{13} + b e_{23}.
\]

We have \( a e_{11} + b e_{21} > 0 \) and \( a e_{12} + b e_{22} > 0 \) since \( a, b > 0 \) and by the condition satisfied by the \( e_{ij} \) stated after (44).

Let
\[
\lambda_1 = \varphi_1 e_{11} + \varphi_2 e_{12} + e_{13}, \quad \lambda_2 = \varphi_1 e_{21} + \varphi_2 e_{22} + e_{23}, \quad \lambda_3 = \varphi_1 e_{31} + \varphi_2 e_{32} + e_{33}.
\]

Then \( x_1, y_1, z_1 \) are permissible parameters at \( q \) such that
\[
x = x_1^{e_{11}} y_1^{e_{12}} (z_1 + \alpha)^{\lambda_1}, \quad y = x_1^{e_{21}} y_1^{e_{22}} (z_1 + \alpha)^{\lambda_2}, \quad z = x_1^{e_{31}} y_1^{e_{32}} (z_1 + \alpha)^{\lambda_3}
\]
with \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q} \), and \( a \lambda_1 + b \lambda_2 = 0 \).

Now suppose that \( e_{11} > 0 \) and \( e_{12} > 0 \), which is the case where \( q \) is a 2-point of \( D_{U_1} \).

Write
\[
u = ((x_1^{e_{11}} y_1^{e_{12}})^a (x_1^{e_{21}} y_1^{e_{22}})^b)^\ell = (x_1 t_1 y_1 t_2)^{\ell_1}
\]
where \( t_1, t_2, \ell_1 \in \mathbb{Z}_+ \) and \( \text{gcd}(t_1, t_2) = 1 \).

We then have an expression
\[
v = P((x_1^{t_1} y_1^{t_2})^{\ell_1}) + x_1^{c e_{11} + d e_{21}} y_1^{c e_{12} + d e_{22}} G,
\]
where
\[
G = (z_1 + \alpha)^{\lambda_1 + d \lambda_2} [r_0 x_1^{m e_{31}} y_1^{m e_{32}} (z_1 + \alpha)^{m \lambda_3}
+ r_2 x_1^{r_2 e_{11} + s_2 e_{21} + (m-2) e_{31}} y_1^{r_2 e_{12} + s_2 e_{22} + (m-2) e_{32}} (z_1 + \alpha)^{r_2 \lambda_1 + s_2 \lambda_2 + (m-2) \lambda_3} + \ldots
+ r_m x_1^{r_m e_{11} + s_m e_{21} + (m-1) e_{31}} y_1^{r_m e_{12} + s_m e_{22} + e_{32}} (z_1 + \alpha)^{r_m \lambda_1 + s_m \lambda_2 + \lambda_3}
+ r_m x_1^{r_m e_{11} + s_m e_{21} + (m-1) e_{31}} y_1^{r_m e_{12} + s_m e_{22} + e_{32}} (z_1 + \alpha)^{r_m \lambda_1 + s_m \lambda_2 + \lambda_3}].
\]

Let \( x_1^s y_1^t \) be a generator of \( I\hat{O}_{U_1, q} \). Let \( G' = \frac{F}{x_1^s y_1^t} \).

We will now establish that, with our assumptions, there is a unique element of the set \( S \) consisting of \( z^m \), and
\[
\{x^n y^i z^{m-i} \mid 2 \leq i \leq m \text{ and } \tau_i \neq 0\}
\]
which is a generator of \( I\hat{O}_{U_1, q} \); that is, equal to \( x_1^s y_1^t \) times a unit in \( \hat{O}_{U_1, q} \). Let \( r_0 = 0 \) and \( s_0 = 0 \). Suppose that \( x_1^s y_1^t z^{m-i} \) (with \( 0 \leq i \leq m \)) is a generator of \( I\hat{O}_{U_1, q} \). We have \( x_1^s y_1^t z^{m-i} = x_1 y_1^t (z_1 + \alpha)^{\gamma_i} \) where
\[
r_1 e_{11} + s_1 e_{21} + (m-i) e_{31} = s
r_1 e_{12} + s_1 e_{22} + (m-i) e_{32} = t
r_1 \lambda_1 + s_1 \lambda_2 + (m-i) \lambda_3 = \gamma_i.
\]

Let
\[
A = \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}.
\]

We have
\[
A \begin{pmatrix} r_1 \\ s_1 \\ m-i \end{pmatrix} = \begin{pmatrix} s \\ t \\ \gamma_i \end{pmatrix}.
\]
Let $\omega = \det(A)$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \varphi_1 & \varphi_2 & 1 \end{pmatrix} \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{pmatrix}$$

implies $\omega = \det(A) = \pm 1$.

By Cramer’s rule, we have

$$\omega(m-i) = \det\begin{pmatrix} e_{11} & e_{21} & s \\ e_{12} & e_{22} & t \\ \lambda_1 & \lambda_2 & \gamma_i \end{pmatrix} = s\det\begin{pmatrix} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{pmatrix} - t\det\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix} + \gamma_i\det\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$  

Since $e_{11}e_{21} - e_{12}e_{22} = 0$ by assumption, we have that

$$i = m - \frac{1}{\omega} \left( s\det\begin{pmatrix} e_{12} & e_{22} \\ \lambda_1 & \lambda_2 \end{pmatrix} - t\det\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix} + \gamma_i\det\begin{pmatrix} e_{11} & e_{21} \\ \lambda_1 & \lambda_2 \end{pmatrix} \right).$$

In particular, there is a unique element $x^i y^j z^m$ in $S$ which is a generator of $I\hat{O}_{U,q}$. We have $x^i y^j z^m = x_1^{i_1}(z_1 + \alpha)^{i_1}.\gamma_i$.

We thus have that $G = x^i y^j [g(z_1 + \alpha)^{i_1} + \delta_1 + x_1 + y_1 \Omega_2]$ for some $\Omega_1, \Omega_2 \in \hat{O}_{U,q}$ and $0 \neq g \in \mathfrak{t}$.

We now establish that we cannot have that $\gamma_i + \delta_1 + x_1 = 0$ and $x_1^{c_{11} + d_{21} + e_{12} + d_{22} + i}$ is a power of $x_1^{c_1} y_1^{d_2}$. We will suppose that both of these conditions do hold, and derive a contradiction. Now we know that $x_1^{c_1} y_1^{d_2} = x_1^{c_{11} + d_{21} + e_{12} + d_{22}}$ is a power of $x_1^{c_1} y_1^{d_2}$. By (47), (48) and our assumptions, we have that

$$A\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

and

$$A\begin{pmatrix} c + r_i \\ d + s_i \\ m - i \end{pmatrix}$$

are rational multiples of

$$\begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix}.$$  

Since $\omega = \det(A) \neq 0$, we have that $(c + r_i, d + s_i, m - i)$ is a rational multiple of $(a, b, 0)$. Thus $x^{i_1} y^{j_1} z^{m} = x_1^{c_{11} + d_{21} + e_{12} + d_{22}}$ is a rational multiple of $x^{a} y^{b}$, a contradiction to our assumption that $F$ satisfies (2).

Let

$$P(x_1^{i_1} y_1^{j_1}) = \sum_{i_2, j_1, j_0} \frac{1}{i! j_1!} \frac{\partial(x_1^{ce_{11} + de_{21}} y_1^{ce_{12} + de_{22}} G)}{\partial x_1^{i_1} y_1^{j_1}}(0, 0, 0) x_1^{i_1} y_1^{j_1},$$

and $F' = G' - \frac{P(x_1^{i_1} y_1^{j_1})}{x_1^{c_{11} + d_{21} + e_{12} + d_{22} + i}}$. Set

$$P'(x_1^{i_1} y_1^{j_1}) = P((x_1^{i_1} y_1^{j_1})^t) + \overline{P}(x_1^{i_1} y_1^{j_1}).$$  

We have that

$$u = (x_1^{i_1} y_1^{j_1})^{l_1}, v = P'(x_1^{i_1} y_1^{j_1}) + x_1^{ce_{11} + fe_{21}} y_1^{ce_{21} + de_{22}} F'. $$
has the form (2) and \( \sigma_D(q) = 0 \leq m - 2 = \sigma_D(p) \).

Now suppose that \( q \in \psi^{-1}(p) \) is a 2-point of \( \psi^{-1}(D) \), \( e_{11}e_{22} - e_{12}e_{21} = 0 \) in (44), and \( e_{11} = 0 \) or \( e_{12} = 0 \). Without loss of generality, we may assume that \( e_{12} = 0 \). \( q \) is a 1-point of \( D_{U_1} \), and we have permissible parameters (46) at \( q \). Since \( \text{Det}(e_{ij}) = \pm 1 \), we have that \( e_{32} = 1 \), and \( e_{11}e_{23} - e_{21}e_{13} = \pm 1 \). Replacing \( y_1 \) with \( y_1(z_1 + \alpha)^{\lambda_3} \) in (46), we find permissible parameters \( x_1, y_1, z_1 \) at \( q \) such that

\[
x = x_1^{e_{11}}(z_1 + \alpha)^{\lambda_1}, \quad y = x_1^{e_{21}}(z_1 + \alpha)^{\lambda_2}, \quad z = x_1^{e_{31}}y_1,
\]

with \( e_{11}, e_{21} > 0 \) and \( a\lambda_1 + b\lambda_2 = 0 \). We have

\[
u = P(x_1^{ae_{e11} + be_{e21}}) + x_1^{e_{e11} + de_{e21}} G
\]

where

\[
G = (z_1 + \alpha)^{c\lambda_1 + d\lambda_2} \tau_0 x_1^{me_{e11}} y_1^{m_1} + \tau_2 x_1^{r_2 e_{e11} + s_2 e_{21} + (m - 2)e_{31}} y_1^{m - 2}(z_1 + \alpha)^{r_2 \lambda_1 + s_2 \lambda_2} + \ldots
\]

\[
+ \tau_m x_1^{r_m e_{e11} + s_m e_{21}} y_1(z_1 + \alpha)^{r_m \lambda_1 + s_m \lambda_2}
\]

Since \( I \hat{\mathcal{O}}_{U_1, q} \) is principal and \( \tau_m \) or \( \tau_{m-1} \neq 0 \), we have that \( x_1^{r_{m} e_{e11} + s_{m} e_{21}} \) is a generator of \( I \hat{\mathcal{O}}_{U_1, q} \) if \( \tau_m \neq 0 \), and \( x_1^{r_{m-1} e_{e11} + s_{m-1} e_{21} + e_{31}} y_1 \) is a generator of \( I \hat{\mathcal{O}}_{U_1, q} \) if \( \tau_m = 0 \) and \( \tau_{m-1} \neq 0 \).

First suppose that \( \tau_m \neq 0 \) so that

\[
G = x_1^{r_m e_{e11} + s_m e_{21}} [g_m(z_1 + \alpha)^{(c+r_{m})\lambda_1 + (d+s_{m})\lambda_2} + x_1 \Omega + y_1 \Omega_2]
\]

with \( 0 \neq g_m \in \mathfrak{t} \), \( \Omega_1, \Omega_2 \in \hat{\mathcal{O}}_{U_1, q} \). Since \( \lambda_1, \lambda_2 \) are not both zero, \( a\lambda_1 + b\lambda_2 = 0 \) and \( a(d+s_m) - b(c+r_m) \neq 0 \), we have that \( (c + r_m)\lambda_1 + (d+s_{m})\lambda_2 \neq 0 \). Let \( \overline{P}(x_1) = G(x_1, 0, 0) \), and \( P'(x_1) = P(x_1^{e_{e11} + de_{e21}}) + \overline{P}(x_1) \) Let

\[
F' = \frac{1}{x_1^{e_{e11} + de_{e21}}}(G - \overline{P}(x_1)).
\]

Then

\[
\begin{align*}
u &= P'(x_1) + x_1^{e_{e11} + de_{e21}} F'
\end{align*}
\]

is of the form (1) with \( \text{ord} F'(0, y_1, z_1) = 1 \). Thus \( \sigma_D(q) = 0 < \sigma_D(p) \).

Now suppose that \( \tau_m = 0 \), so that

\[
G = x_1^{r_m e_{e11} + s_m e_{21} + e_{31}} [g_{m-1}(z_1 + \alpha)^{(c+r_{m-1})\lambda_1 + (d+s_{m-1})\lambda_2} + x_1 \Omega_1]
\]

with \( 0 \neq g_{m-1} \in \mathfrak{t} \) and \( \Omega_1 \in \hat{\mathcal{O}}_{U_1, q} \). Thus \( \sigma_D(q) = 0 < \sigma_D(p) \).

The final case is when \( q \) is a 3-point for \( \psi^{-1}(D) \), so that \( q \) is a 3-point or a 2-point of \( D_{U_1} \). Then we have permissible parameters \( x_1, y_1, z_1 \) at \( q \) such that

\[
x = x_1^{e_{11}}y_1^{e_{12}}z_1^{e_{13}}, \quad y = x_1^{e_{21}}y_1^{e_{22}}z_1^{e_{23}}, \quad z = x_1^{e_{31}}y_1^{e_{32}}z_1^{e_{33}}
\]

with \( \omega = \text{Det}(e_{ij}) = \pm 1 \). Thus there is a unique element of the set \( S \) consisting of \( z^m \) and

\[
\{x^{i_1}y^{i_2}z^{i_3} | 2 \leq i_1 \leq m \text{ and } \tau_i \neq 0\}
\]

which is a generator \( x_1^{e_{11}}y_1^{e_{21}}z_1^{e_{31}} \) of \( I \hat{\mathcal{O}}_{U_1, q} \). Thus \( \sigma_D(q) = 0 \) if \( q \) is a 3-point of \( D_{U_1} \). If \( q \) is a 2-point of \( D_{U_1} \), we may assume that \( e_{13} = e_{23} = 0 \). Then \( e_{33} = 1 \). Since \( \tau_m \neq 0 \) or \( \tau_{m-1} \neq 0 \), we calculate that \( \sigma_D(q) = 0 \).
Suppose that $p \in X$ is a 2-point such that $X$ is 3-prepared at $p$ and $\sigma_D(p) = r > 0$. We can then define a local resolver $(U_p, D_p, I_p, \nu^1_p, \nu^2_p)$ as in Theorem 4.3, where $\nu^i_p$ are valuations on $U_p$ which dominate the two curves $C_1, C_2$ which are the intersection of $E$ with $D_U$ on $U_p$ (where $D_p = D_U + E$), and which have the property that if $\pi : V \to U_p$ is a birational morphism, then the center $C(V, \nu^i_p)$ on $V$ is the unique curve on the strict transform of $E$ on $V$ which dominates $C_i$. We will think of $U_p$ as a germ, so we will feel free to replace $U_p$ with a smaller neighborhood of $p$ whenever it is convenient.

If $\pi : Y \to X$ is a birational morphism, then the center $C(Y, \nu_p^i)$ on $Y$ is the closed curve which is the center of $\nu^i_p$ on $Y$. We define $\Lambda(Y, \nu_p^i)$ to be the image in $Y$ of $C(Y \times X U_p, \nu^i_p) \cap \pi^{-1}(p)$. This defines a valuation which is composite with $C(Y, \nu_p^i)$.

We define $W(Y, p)$ to be the clopen locus on $Y$ of the image of points in $\pi^{-1}(U_p) = Y \times_X U_p$ such that $I_p \mathcal{O}_Y | \pi^{-1}(U_p)$ is not invertible. Define $\text{Preimage}(Y, Z) = \pi^{-1}(Z)$ for $Z$ a subset of $X$.

5. Global reduction of $\sigma_D$

**Lemma 5.1.** Suppose that $X$ is 2-prepared and $p \in X$ is 3-prepared. Suppose that $r = \sigma_D(p) > 0$.

a) Suppose that $p$ is a 1-point. Then there exists a unique curve $C$ in $\text{Sing}_1(X)$ containing $p$. The curve $C$ is contained in $\text{Sing}_r(X)$. If $x, y, z$ are permissible parameters at $p$ giving an expression (14) or (15) at $p$, then $x = y = z = 0$ are formal local equations of $C$ at $p$.

b) Suppose that $p$ is a 2-point and $C$ is a curve in $\text{Sing}_r(X)$ containing $p$. If $x, y, z$ are permissible parameters at $p$ giving an expression (13) at $p$, then $x = y = z = 0$ are formal local equations of $C$ at $p$.

**Proof.** We first prove a). Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf defining the reduced scheme $\text{Sing}_1(X)$. Then $I_p \mathcal{O}_{X,p} = \left(\sqrt{(x, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})} = (x, z)\right)$ is an ideal on $U$ defining $\text{Sing}_1(U)$. Thus the unique curve $C$ in $\text{Sing}_1(X)$ through $p$ has (formal) local equations $x = z = 0$ at $p$. At points near $p$ on $C$, a form (14) or (15) continues to hold with $m = r + 1$. Thus the curve is in $\text{Sing}_r(X)$.

We now prove b). Suppose that $C \subset \text{Sing}_r(X)$ is a curve containing $p$. By Theorem 4.3, there exists a toroidal morphism $\Psi : U_1 \to U$ where $U$ is an étale cover of an affine neighborhood of $p$, and $\overline{D}$ is the local toroidal structure on $U$ defined (formally at $p$) by $xyz = 0$, such that all points $q$ of $U_1$ satisfy $\sigma_D(q) < r$. Hence the strict transform on $U_1$ of the preimage of $C$ on $U$ must be empty. Since $\Psi$ is toroidal for $\overline{D}$ and $X$ is 3-prepared at $p$, $C$ must have local equations $x = z = 0$ or $y = z = 0$ at $p$. \hfill $\square$

**Definition 5.2.** Suppose that $X$ is 3-prepared. We define a canonical sequence of blow ups over a curve in $X$.

1) Suppose that $C$ is a curve in $X$ such that $t = \sigma_D(q) > 0$ at the generic point $q$ of $C$, and all points of $C$ are 1-points of $D$. Then we have that $C$ is nonsingular and $\sigma_D(p) = t$ for all $p \in C$ by Lemma 5.1. By Lemma 5.1 and Theorem 4.1 or 4.2, there exists a unique minimal sequence of permissible blow ups of sections over $C$, $\pi_1 : X_1 \to X$, such that $X_1$ is 2-prepared and $\sigma_D(p) < t$ for all $p \in \pi_1^{-1}(C)$. We will call the morphism $\pi_1$ the canonical sequence of blow ups over $C$.

2) Suppose that $C$ is a permissible curve in $X$ which contains a 1-point such that $\sigma_D(p) = 0$ for all $p \in C$, and a condition 1, 3 or 5 of Lemma 5.10 holds at all
Let \( \pi_1 : X_1 \to X \) be the blow up of \( C \). Then by Lemma 3.12, \( X_1 \) is 3-prepared and \( \sigma_D(p) = 0 \) for \( p \in \pi_1^{-1}(C) \). We will call the morphism \( \pi_1 \) the canonical blow up of \( C \).

**Theorem 5.3.** Suppose that \( X \) is 2-prepared. Then there exists a sequence of permissible blowups \( \psi : X_1 \to X \) such that \( X_1 \) is prepared.

**Proof.** By Proposition 3.13, there exists a sequence of permissible blow ups \( X^0 \to X \) such that \( X^0 \) is 3-prepared. Let \( r = \Gamma_D(X^0) \). Since \( X^0 \) is prepared if \( r = 0 \), we may assume that \( r > 0 \). Let

\[
T_0 = \{ p \in X^0 \mid X^0 \text{ is a 2-point for } D \text{ with } \sigma_D(p) = r \}.
\]

For \( p \in T_0 \), choose \((U_p, D_p, I_p, \nu^1_p, \nu^2_p)\). Let \( \Gamma_0 \) be the union of the set of curves

\[
\{ (X^0, \nu^1_p) \mid p \in T_0 \text{ and } \sigma_D(\eta) = r \text{ for } \eta \in (X^0, \nu^1_p) \}
\]

and any remaining curves \( C \in \text{Sing}_r(X^0) \) (which necessarily contain no 2-points).

By Lemma 5.1, all curves in \( \text{Sing}_r(X^0) \) are nonsingular, and if a curve \( C \) in \( \text{Sing}_r(X^0) \) contains a 2-point \( p \in T_0 \), then \( C = (X^0, \nu^1_p) \) for some \( j \).

Let \( Y_0 \to X^0 \) be the product of canonical sequences of blowups over the curves in \( \Gamma_0 \) (which are necessarily the curves in \( \text{Sing}_r(X^0) \)), so that \( Y_0 \setminus \cup_{p \in T_0} W(Y_0, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_0 \setminus \cup_{p \in T_0} W(Y_0, p) \).

Let \( Y_{0,1} \to Y_0 \) be a toroidal morphism for \( D_{Y_0} \) so that the components of \( D_{Y_{0,1}} \) containing some curve \( C(Y_{0,1}, \nu^j_q) \) for \( p \in T_0 \) are pairwise disjoint, and if \( p \in T_0 \), then \( W(Y_{0,1}, p) \) is contained in \((Y_0, \nu^1_p) \cup (Y_0, \nu^2_p) \cup \text{Preimage}(Y_{0,1}, p)\).

Let \( E \) be a component of \( D_{Y_{0,1}} \) which contains \( C(Y_{0,1}, \nu^j_q) \) for some \( p \in T_0 \) and some \( j \). Then there exists \( Y_{0,2} \to Y_{0,1} \) which is an isomorphism over \( Y_{0,1} \setminus E \cap (\cup_{p \in T_0} W(Y_{0,1}, p)) \), is toroidal for \( D_q \) over \( W(Y_{0,1}, q) \cap E \) for \( q \in T_0 \), is an isomorphism over \( C(Y_{0,1}, \nu^j_q) \setminus \text{Preimage}(q) \) for all \( q \in T_0 \), and so that if \( E \) is the strict transform of \( E \) on \( Y_{0,2} \), then for \( p \in T_0 \), one of the following holds:

\[
W(Y_{0,2}, p) \cap \overline{E} = \emptyset
\]

or

There exists a unique \( j \) such that

\[
W(Y_{0,2}, p) \cap \overline{E} \subset C(Y_{0,2}, \nu^j_p) \subset \overline{E},
\]

and

\[
\text{if } \overline{p}_j = \Lambda(Y_{0,2}, \nu^j_p), \text{ then } C(Y_{0,2}, \nu^j_p) \text{ is smooth at } \overline{p}_j,
\]

\[
\text{and either } \overline{p}_j \text{ is an isolated point in } \text{Sing}_1(Y_{0,2}) \text{ or } C(Y_{0,2}, \nu^j_p)
\]

is the only curve in \( \text{Sing}_1(Y_{0,2}) \) which is contained in \( \overline{E} \) and contains \( \overline{p}_j \), and

\[
\text{if } \overline{p}_j \in C(Y_{0,2}, \nu^k_{p'}) \text{ for some } p' \in T_0 \text{ implies } C(Y_{0,2}, \nu^k_{p'}) = C(Y_{0,2}, \nu^j_p).
\]

We further have that \( Y_{0,2} \setminus \cup_{p \in T_0} W(Y_{0,2}, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_{0,2} \setminus \cup_{p \in T_0} W(Y_{0,2}, p) \).

Now repeat this procedure for other components of \( D_{Y_{0,2}} \) which contain a curve \( C(Y_{0,2}, \nu^j_q) \) for some \( j \) to construct \( Y_{0,3} \to Y_{0,2} \) so that condition (50) or (51) hold for all components \( E \) of \( D_{Y_{0,3}} \) containing a curve \( C(Y_{0,3}, \nu^j_q) \). We have that \( Y_{0,3} \setminus \cup_{p \in T_0} W(Y_{0,3}, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_{0,3} \setminus \cup_{p \in T_0} W(Y_{0,3}, p) \).
Now, by Lemma 3.4, let $Y_{0.4} \to Y_{0.3}$ be a sequence of blow ups of 3-points for $D$ and 2-curves of $D$ on the strict transform of components $E$ of $D$ which contain $C(Y_{0.3}, \nu_0^j)$ for some $p \in T_0$, so that if $E$ is a component of $D_{Y_{0.4}}$, which contains a curve $C(Y_{0.4}, \nu_0^j)$, then $Y_{0.4}$ is 3-prepared at all 2-points and 3-points of $E$. We have that $Y_{0.4} \setminus \cup_{p \in T_0}W(Y_{0.4}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{0.4} \setminus \cup_{p \in T_0}W(Y_{0.4}, p)$. We further have that for all $p \in T_0$, (50) or (51) holds on $E$.

Now let $E$ be a component of $D_{Y_{0.4}}$ which contains a curve $C(Y_{0.4}, \nu_0^j)$. Since one of the conditions (50) or (51) holds for all $p \in T_0$ on $E$, we may apply Proposition 3.14 to $E$ and the finitely many points

$$A = \{ q \in E \mid Y_{0.4} \text{ is not 3-prepared at } q \},$$

which are necessarily 1-points for $D$, being sure that none of the finitely many 2-points for $D$

$$B = \{ \Lambda(Y_{0.4}, \nu_0^j) \mid p \in T_0 \}$$

are in the image of the general curves blown up, to construct a sequence of permissible blow ups $Y_{0.5} \to Y_{0.4}$ so that $Y_{0.5} \to Y_{0.4}$ is an isomorphism in a neighborhood of $\cup_{p \in T_0}W(Y_{0.4}, p)$ and over $Y_{0.4} \setminus E$, and $Y_{0.5}$ is 3-prepared over $E \setminus \cup_{p \in T_0}\Lambda(Y_{0.4}, \nu_0^j)$. We have that $Y_{0.5} \setminus \cup_{p \in T_0}W(Y_{0.5}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{0.5} \setminus \cup_{p \in T_0}W(Y_{0.5}, p)$. We further have that for all $p \in T_0$, (50) or (51) hold on the strict transform $E$ of $E$ on $Y_{0.5}$.

Now repeat this procedure for other components of $D_{Y_{0.5}}$ which contain a curve $C(Y_{0.5}, \nu_0^j)$ for some $j$ to construct $X_1 \to Y_{0.5}$ so that $X_1$ is 3-prepared over $E \setminus \cup_{p \in T_0}\Lambda(Y_{0.5}, \nu_0^j)$ for all components $E$ of $D_{Y_{0.5}}$ which contain a curve $C(Y_{0.5}, \nu_0^j)$ for some $p \in T_0$. We then have that the following holds.

1.1) $X_1 \to X^0$ is the canonical sequence of blow ups above a general point $\eta$ of a curve in $\Gamma_0$ (so that $\sigma_D(\eta) = r$).
1.2) $X_1 \to X^0$ is toroidal for $D_p$ in a neighborhood of $W(X_1, p)$, for $p \in T_0$.
1.3) $X_1 \setminus \cup_{p \in T_0}W(X_1, p)$ is 2-prepared and $\sigma_D(q) < r$ for $q \in X_1 \setminus \cup_{p \in T_0}W(X_1, p)$.
1.4) If $p \in T_0$ then $\sigma_D(q) \leq r - 1$ and $X_1$ is 3-prepared at $q$ for

$$q \in C(X_1, \nu_0^j) \setminus \cup_{p' \in T_0|C(X_1, \nu_0^j)\setminus C(X_1, \nu_0^k)} \text{ for some } k \text{ Preimage}(X_1, p').$$

1.5) Let

$$T_1 = \begin{cases} 2\text{-points } q \text{ for } D \colon C(X_1, \nu_0^j) \setminus \cup_{p' \in T_0|C(X_1, \nu_0^j)\setminus C(X_1, \nu_0^k)} \text{ for some } k \text{ Preimage}(X_1, p') \\ \text{such that } \sigma_D(q) > 0 \text{ and such that } p \in T_0 \text{ with} \\ \sigma_D(\eta) = r - 1 \text{ for } \eta \in C(X_1, \nu_0^j) \text{ the generic point.} \end{cases}$$

$X_1$ is 3-prepared at $p \in T_1$. For $q \in T_1$, choose $(U_q, D_q, I_q, \nu_0^1, \nu_0^2)$. We have $0 < \sigma_D(q) \leq r - 1$ for $q \in T_1$.

1.6) Suppose that $p \in T_0$ and $C(X_1, \nu_0^j)$ is such that $\sigma_D(\eta) = r - 1$ for $\eta \in C(X_1, \nu_0^j)$ the generic point. Then $\sigma_D(q) = r - 1$ for $q \in C(X_1, \nu_0^j) \setminus \cup_{p' \in T_0|T_0}W(X_1, p')$. If $q \in T_0 \cup T_1$ and $W(X_1, q) \cap C(X_1, \nu_0^j) \neq \emptyset$, then $C(X_1, \nu_0^j) = C(X_1, \nu_0^j)$ for some $i$. (This follows from Lemma 5.1 since $\sigma_D(q) \leq r - 1$ for $q \in T_1.$)
Now for \( m \geq r \), we inductively construct
\[
\begin{align*}
X_{m,r-1} \rightarrow \cdots \rightarrow X_{m,0} \rightarrow \cdots \rightarrow X_{r+1,r-1} \rightarrow \cdots \rightarrow X_{r+1,0} & \\
X_{r,r-1} \rightarrow X_{r,r-2} \rightarrow \cdots \rightarrow X_{r,0} \rightarrow X_{r-1,r-2} \rightarrow \cdots \rightarrow X_{3,0} \rightarrow X_{2,1} \rightarrow X_{2,0} \rightarrow X_{1,0} & = X_1 \rightarrow X^0
\end{align*}
\]
so that

2.1) \( X_{i,0} = X_1 \rightarrow X^0 \) is the canonical sequence of blow ups above a general point \( \eta \) of a curve in \( \Gamma_0 \) (so that \( \sigma_D(\eta) = r \)), and for \( i > 0 \),

\[
X_{i+1,0} \rightarrow X_{i,\min\{i-1,r-1\}}
\]

is the canonical sequence of blowups above a general point \( \eta \) of a curve \( C(X_{i,\min\{i-1,r-1\}}, \nu^j_p) \) with \( p \in T_0 \) and such that \( \sigma_D(\eta) = \max\{0, r-i\} \),

and the following properties hold on \( X_{i,\ell} \).

2.2) \( X_{i,\ell} \rightarrow X_{j,k} \) is toroidal for \( \overline{D}_p \) in a neighborhood of \( W(X_{i,\ell}, p) \), for \( p \in T_{j,k} \) with \( T_{j,k} = T_0 \), or \( 1 \leq j \leq i-1 \) and \( 0 \leq k \leq \min\{j-1,r-1\} \), or \( j = i \) and \( 0 \leq k \leq l-1 \).  

2.3) \( X_{i,\ell} \setminus \bigcup_{p \in T_0} \left( \bigcup_{j=1}^{i-1} \min\{j-1,r-1\} T_{j,k} \right) \cup \left( \bigcup_{n=0}^{j-1} T_{i,n} \right) \) is 2-prepared and \( \sigma_D(q) < r \) for \( q \in X_{i,\ell} \setminus \bigcup_{p \in \Omega} \text{Preimage}(X_{i,\ell}, p') \) for some \( k \).

2.4) If \( p \in T_0 \) then \( \sigma_D(\eta) \leq \max\{0, r-i\} \) for \( \eta \in C(X_{i,\ell}, \nu^j_p) \) the generic point, and \( X_{i,\ell} \) is 3-prepared at \( q \) for

\[
q \in C(X_{i,\ell}, \nu^j_p) \setminus \bigcup_{p' \in \Omega} \text{Preimage}(X_{i,\ell}, p'),
\]

Where

\[
\Omega = \{ p' \in T_0 \cup \left( \bigcup_{j=1}^{i-1} \min\{j-1,r-1\} T_{j,k} \right) \cup \left( \bigcup_{n=0}^{j-1} T_{i,n} \right) \mid C(X_{i,\ell}, \nu^j_p) = C(X_{i,\ell}, \nu^k_p) \text{ for some } k \}.
\]

2.5) We have the set

\[
T_{i,\ell} = \left\{ \begin{array}{ll}
2\text{-points } q \text{ for } D \text{ of } C(X_{i,\ell}, \nu^j_p) \setminus \bigcup_{p' \in \Omega} \text{Preimage}(X_{i,\ell}, p') & \\
\text{where } \Omega = & \\
\{ p' \in T_0 \cup \left( \bigcup_{j=1}^{i-1} \min\{j-1,r-1\} T_{j,k} \right) \cup \left( \bigcup_{n=0}^{j-1} T_{i,n} \right) \mid C(X_{i,\ell}, \nu^j_p) = C(X_{i,\ell}, \nu^k_p) \text{ for some } k \} & \\
\text{such that } \sigma_D(q) > 0 \text{ and such that } p \in T_0 \text{ with } & \\
\sigma_D(\eta) = \max\{0, r-i\} \text{ for } \eta \in C(X_{i,\ell}, \nu^j_p) \text{ the generic point.} & \\
\end{array} \right.
\]

\( X_{i,\ell} \) is 3-prepared at \( p \in T_{i,\ell} \). We have local resolvers \( (U_p, D_p, I_p, \nu^1_p, \nu^2_p) \) at \( p \in T_{i,\ell} \). We have \( \max\{1, r-i\} \leq \sigma_D(q) \leq r-l-1 \) for \( q \in T_{i,\ell} \).

2.6) Suppose that \( p \in T_0 \) and \( C(X_{i,\ell}, \nu^j_p) \) is such that \( \sigma_D(\eta) = \max\{0, r-i\} \) for \( \eta \in C(X_{i,\ell}, \nu^j_p) \) the generic point. Then \( \sigma_D(q) = \max\{0, r-i\} \) for

\[
q \in C(X_{i,\ell}, \nu^j_p) \setminus \bigcup_{p' \in T_0} \left( \bigcup_{j=1}^{i-1} \min\{j-1,r-1\} T_{j,k} \right) \cup \left( \bigcup_{n=0}^{j-1} T_{i,n} \right) W(X_{i,\ell}, p').
\]

Further,

a) If \( q \in T_0 \cup \left( \bigcup_{j=1}^{i-1} \min\{j-1,r-1\} T_{j,k} \right) \cup \left( \bigcup_{n=0}^{j-1} T_{i,n} \right) \) and \( W(X_{i,\ell}, q) \cap C(X_{i,\ell}, \nu^j_p) \neq \emptyset \), then \( C(X_{i,\ell}, \nu^j_p) = C(X_{i,\ell}, \nu^k_p) \) for some \( k \).

b) If \( q \in T_{i,\ell} \) and \( q \in C(X_{i,\ell}, \nu^j_p) \), then either \( C(X_{i,\ell}, \nu^j_p) = C(X_{i,\ell}, \nu^k_p) \) for some \( k \) or \( \max\{0, r-i\} < \sigma_D(q) \leq r-l-1 \).
Note that the condition \( \sigma_D(q) > 0 \) in the definition of \( T_{i,l} \) is automatically satisfied if \( i < r \). If \( l = \min\{i - 1, r - 1\} \), condition 2.6 becomes “Suppose that \( p \in T_0 \) and \( C(X_{i,l}, \nu_p^l) \) is such that \( \sigma_D(\eta) = \max\{0, r - i\} \) for \( \eta \in C(X_{i,l}, \nu_p^l) \) the generic point. Then if \( q \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k} \cup \bigcup_{n=0}^l T_{i,n} \) and \( W(X_{i,l}, q) \cap C(X_{i,l}, \nu_p^l) \neq \emptyset \), then \( C(X_{i,l}, \nu_p^l) = C(X_{i,l}, \nu_k^l) \) for some \( k \).”

We now prove the above inductive construction of (52). Suppose that we have made the construction out to \( X_{i,l} \).

**Case 1.** Suppose that \( l = \min\{i - 1, r - 1\} \). We will construct \( X_{i+1,0} \to X_{i,\min\{i - 1, r - 1\}} \).

First suppose that \( r > i \). Let \( Y_i \to X_{i,i-1} \) be the product of the canonical sequences of blow ups above all curves \( C(X_{i,i-1}, \nu_p^l) \) for \( p \in T_0 \) such that \( \sigma_D(\eta) = r - i \) at a generic point \( \eta \in C(X_{i,i-1}, \nu_p^l) \). This is a permissible sequence of blow ups by the comment following 2.6 above. We have that \( Y_i \setminus \bigcup_{p \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k}} W(Y_i, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_i \setminus \bigcup_{p \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k}} W(Y_i, p) \). Further, \( Y_i \to X_{i,i-1} \) is toroidal for \( D_p \) in a neighborhood of \( W(Y_i, p) \) for \( p \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k} \).

Now suppose that \( r \leq i \). On \( X_{i,i-1} \), we have that \( \sigma_D(q) = 0 \) for \( p \in T_0 \) and \( q \in C(X_{i,i-1}, \nu_p^l) \setminus \bigcup_{j \in T_0} \bigcup_{k=0}^l \min\{j-1, r-1\} T_{j,k} \) \( W(X_{i,i-1}, p') \). By Lemmas 3.9, 3.10, 3.11 and 3.12, there exists a sequence \( Y_i \to X_{i,i-1} \) of blow ups of prepared points on the strict transform of curves \( C(X_{i,r-1}, \nu_p^l) \) with \( p \in T_0 \), followed by the blow ups of the strict transforms of these \( C(X_{i,r-1}, \nu_p^l) \), so that for \( p \in T_0 \), \( \sigma_D(q) = 0 \) at a point \( q \) of \( C(Y_i, \nu_p^l) \), \( Y_i \setminus \bigcup_{p \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k}} W(Y_i, p) \) is 2-prepared and \( \sigma_D(q) < r \) for

\[
q \in Y_i \setminus \bigcup_{p \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k}} W(Y_i, p).
\]

Further, \( Y_i \to X_{i,i-1} \) is toroidal for \( D_p \) in a neighborhood of \( W(Y_i, p) \) for \( p \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k} \).

From now on, we consider both cases \( r > i \) and \( r \leq i \) simultaneously. Let \( Y_{i,1} \to Y_i \) be a toroidal morphism for \( D \) so that the components of \( D \) containing some curve \( C(Y_{i,1}, \nu_p^l) \) for \( p \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k} \) are pairwise disjoint, and if

\[
p \in \bigcup_{p' \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k}} W(Y_{i,1}, p')
\]

then \( W(Y_{i,1}, p) \) is contained in \( C(Y_{i,1}, \nu_p^l) \cup C(Y_{i,1}, \nu_p^r) \cup \text{Preimage}(Y_{i,1}, p) \).

Let \( E \) be a component of \( D \) on \( Y_{i,1} \) which contains \( C(Y_{i,1}, \nu_p^l) \) for some \( \alpha \in T_0 \) and some \( \beta \). Then there exists \( Y_{i,1} \to Y_{i,1} \) which contains \( C(Y_{i,1}, \nu_p^l) \) for some \( \alpha \in T_0 \) and some \( j \). Then there exists \( Y_{i,2} \to Y_{i,1} \) which is an isomorphism over

\[
Y_{i,1} \setminus E \cap \bigcup_{p' \in T_0 \cup \bigcup_{j=1}^l \min\{j-1, r-1\} T_{j,k}} W(Y_{i,1}, p')
\]
is toroidal for $D_q$ over $W(Y_{i,1}, q) \cap E$ for $q \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)$, is an isomorphism over $C(Y_{i,1}, \nu_q) \setminus \text{Preimage}(Y_{i,1}, q)$ for all $q \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)$, and so that if $\overline{E}$ is the strict transform of $E$ on $Y_{i,2}$, then for $p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)$, one of the following holds:

(53) $W(Y_{i,2}, p) \cap \overline{E} = \emptyset$

or

(54) There exists a unique $j$ such that

$W(Y_{i,2}, p) \cap \overline{E} \subset C(Y_{i,2}, \nu_p) \subset \overline{E}$, and

if $p_j = \Lambda(Y_{i,2}, \nu_{p_j})$, then $C(Y_{i,2}, \nu_{p_j})$ is smooth at $p_j$, and either $p_j$ is an isolated point in $\text{Sing}_1(Y_{i,2})$ or $C(Y_{i,2}, \nu_{p_j})$ is the only curve in $\text{Sing}_1(Y_{i,2})$ which is contained in $\overline{E}$ and contains $p_j$, and

$p_j \in C(Y_{i,2}, \nu_p)$ for some $p' \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)$ implies $C(Y_{i,2}, \nu_{p'}) = C(Y_{i,2}, \nu_{p})$.

We have that $Y_{i,2} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,2}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{i,2} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,2}, p)$.

Now repeat this procedure for other components of $D$ for $Y_{i,2}$ which contain a curve $C(Y_{i,2}, \nu_\alpha)$ with $\alpha \in T_0$ for some $j$ to construct $Y_{i,3} \to Y_{i,2}$ so that condition (53) or (54) hold for all $p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)$ and components $E$ of $D$ for $Y_{i,3}$ containing a curve $C(Y_{i,3}, \nu_\alpha)$ with $\alpha \in T_0$. We have that $Y_{i,3} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,3}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{i,3} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,3}, p)$.

Now, by Lemma 3.4, let $Y_{i,4} \to Y_{i,3}$ be a sequence of blow ups of 2-curves of $D$ on the strict transform of components $E$ of $D$ which contain $C(Y_{i,3}, \nu_\alpha)$ for some $\alpha \in T_0$, so that if $E$ is a component of $D_{Y_{i,4}}$ which contains a curve $C(Y_{i,4}, \nu_\alpha)$ with $\alpha \in T_0$, and if $p \in E \setminus \bigcup_{q \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} \Lambda(Y_{i,4}, \nu_q)$ is a 2-point, then $p$ is 3-prepared.

We have that $Y_{i,4} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,4}, p)$ is 2-prepared, and $\sigma_D(q) < r$ for $q \in Y_{i,4} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,4}, p)$. We further have that for all $p \in T_0 \cup \left( \bigcup_{j=1}^i \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)$, (53) or (54) holds on $E$.

Now let $E$ be a component of $D$ for $Y_{i,4}$ which contains a curve $C(Y_{i,4}, \nu_\alpha)$ with $\alpha \in T_0$. Let

$$T = \{ q \in E \mid Y_{i,4} \text{ is not 3-prepared at } q \}.$$  

If $r \leq i$, let

$$T' = \left\{ 1\text{-points } q \text{ of } D \text{ contained in } E \text{ such that } q \in C(Y_{i,4}, \nu_q) \text{ for some } p \in T_0 \text{ and } \sigma_D(q) > 0 \right\}.$$  

37
Since one of the conditions (53) or (54) hold for all \( p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right) \) on \( E \), we may apply Proposition 3.14 to \( E \) and the finite set of points \( A = T \cup T' \) if \( r > i \) or \( A = T \cup T' \) if \( r \leq i \), which are necessarily 1-points for \( D \) lying on \( E \), being sure that none of the finitely many points 2-points of \( D \)

\[
B = \{ \Lambda(Y_{i,4}, \nu_p^i) \mid p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right) \}
\]

are in the image of the general curves blown up, to construct a sequence of permissible transforms \( Y_{i,5} \to Y_{i,4} \) so that \( Y_{i,5} \to Y_{i,4} \) is an isomorphism in a neighborhood of \( \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,4}, p) \) and over \( Y_{i,4} \setminus E \), and \( Y_{i,5} \) is 3-prepared over \( E \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} \Lambda(Y_{i,4}, \nu_p^i) \).

We have that \( Y_{i,5} \setminus E \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,5}, p) \) is 2-prepared, and \( \sigma_D(q) < r \) for \( q \in Y_{i,5} \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} W(Y_{i,5}, p) \). If \( r \leq i \) and \( p \in T_0 \), then

\[
\sigma_D(q) = 0 \text{ if } q \in C(Y_{i,5}, \nu_q^j) \text{ is a 1-point for } D. \text{ We further have that for all } p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right), (53) \text{ or (54) hold on the strict transform } \overline{E} \text{ of } E \text{ on } Y_{i,5}.
\]

Now repeat this procedure for other components of \( D_{Y_{i,5}} \) which contain a curve \( C(Y_{i,5}, \nu_{\alpha}^j) \) with \( \alpha \in T_0 \) for some \( j \) to construct \( X_{i+1,0} \to Y_{i,5} \) so that \( X_{i+1,0} \) is 3-prepared over \( E \setminus \bigcup_{p \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right)} \Lambda(Y_{i,5}, \nu_p^i) \) for all components \( E \) of \( D \) for \( Y_{i,5} \) which contain a curve \( C(Y_{i,5}, \nu_{\alpha}^j) \) with \( \alpha \in T_0 \).

Let

\[
T_{i+1,0} = \left\{ p' \in T_0 \cup \left( \bigcup_{j=1}^{i} \bigcup_{k=0}^{\min(j-1,r-1)} T_{j,k} \right) \mid C(X_{i+1,0}, \nu_p^i) = C(X_{i+1,0}, \nu_{p'}^i) \text{ for some } l \right\}
\]

where \( \Omega = \{ 2\text{-points } q \text{ for } D \text{ of } C(X_{i+1,0}, \nu_p^i) \setminus \bigcup_{p' \in \Omega} \text{Preimage}(X_{i+1,0}, p') \} \). Then \( X_{i+1,0} \) satisfies the conclusions 2.1 - 2.6.

**Case 2** Now suppose that \( l < \min\{i-1, r-1\} \). We will construct \( X_{i,l+1} \to X_{i,l} \). Let \( \Omega \) be the set of points \( q \in T_{i,l} \) such that \( q \) is contained in a curve \( C(X_{i,l}, \nu_p^i) \) where \( p \in T_0 \) and \( \sigma_D(\eta) = \max\{0, r - i\} \) for \( \eta \in C(X_{i,l}, \nu_p^i) \) a general point. By condition 2.5) satisfied by \( X_{i,l} \),

\[
\max\{1, r - i\} \leq \sigma_D(q) \leq r - l - 1
\]

for \( q \in \Omega \). Let \( Y \to X_{i,l} \) be a morphism which is an isomorphism over \( X_{i,l} \setminus \Omega \) and is toroidal for \( \overline{D}_{Y} \) above \( q \in \Omega \) and such that \( C(Y, \nu_p^i) \cap W(Y, q) = \emptyset \) if \( C(Y, \nu_p^i) \) is such that \( p \in T_0 \), \( \sigma_D(\eta) = \max\{0, r - i\} \) if \( \eta \in C(Y, \nu_p^i) \) is a general point, and \( C(Y, \nu_p^i) \not= C(Y, \nu_{k}^i) \) for any \( k \). For such a case we have by (55), that \( \sigma_D(\overline{q}) \leq \max\{0, r - l - 2\} \) if \( \overline{q} = \Lambda(Y, \nu_p^i) \).

Now we may construct, using the method of Case 1, a morphism \( X_{i,l+1} \to Y \) such that
$X_{i,j+1} \to X_{i,l}$ is toroidal for $D$ above $X_{i,l} \setminus \Omega$, and the conditions 2.2) - 2.6) following (52) hold. This completes the inductive construction of (52).

For $m$ sufficiently large in (52), we have that for $p \in T_0$, $I_p\mathcal{O}_{X_{m,r-1},\eta}$ is locally principal at a general point $\eta$ of a curve $C(X_{m,r-1},\nu_p)$.

After possibly performing a toroidal morphism for $D$, we have that the locus where $I_p(\mathcal{O}_{X_{m,r-1}}|\text{Preimage}(X_{m,r-1},U_p))$ is not locally principal is supported above $p$ for $p \in T_0$. Thus toroidal morphisms for $D_p$ above $\text{Preimage}(X_{m,r-1},U_p)$ which principalize $I_p$ above $U_p$ for $p \in T_0$ extend to a morphism $Z^1 \to X_{m,r-1}$ which is an isomorphism over $X_{m,r-1} \setminus \cup_{p \in T_0}\text{Preimage}(X_{m,r-1},p)$. We have that $W(Z^1,p) = \emptyset$ for $p \in T_0$. We have that $Z^1$ is 2-prepared at $q \in Z^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}} W(Z^1,p)$ and $\sigma_D(q) \leq r - 1$ for $q \in Z^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}} W(Z^1,p)$.

If $r = 1$, then $Z^1$ is prepared. In this case let $X_1 = Z^1$. Suppose that $r > 1$. Let $Z^1 \to Z^1$ be a toroidal morphism for $D$ so that components of $D$ containing curves $C(Z^1,\nu_p)$ for $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}$ are pairwise disjoint, and that if $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}$, then $W(Z^1,p)$ is contained in $C(Z^1,\nu_p) \cup C(Z^1,\nu_q^2) \cup \text{Preimage}(Z^1,p)$.

Let $E$ be a component of $D$ on $Z^1$ which contains $C(Z^1,\nu_p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}$ or contains a point $q \in E \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}} W(Z^1,p)$ such that $\sigma_D(q) = r - 1$.

Then there exists $Z^2 \to Z^1$ which is an isomorphism over

$$Z^1 \setminus E \cap \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}} W(Z^1,p),$$

is toroidal for $D_q$ over $W(Z^1,q) \cap E$ for $q \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}$, is an isomorphism over $C(Z^1,\nu_q^2) \setminus \text{Preimage}(Z^1,q)$ for all $q \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}$ and factors as a sequence of permissible blow ups of points and curves

$$Z^2 = Z^2_n \to Z^2_{n-1} \to \cdots \to Z^2_1 \to Z^1$$

such that the center blown up in $Z^2_t \to Z^2_{t-1}$ is a curve or point contained in $W(Z^1_{t-1},p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{l=1}^{\min(j-1,r-1)} T_{j,l}$, and so that if $E$ is the strict transform of $E$ on $Z^2$, then for $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min(j-1,r-1)} T_{j,k}$, one of the following holds:

$$(56) \quad W(Z^2,p) \cap E = \emptyset$$
There exists a unique \( j \) such that
\[
W(Z_1^2, p) \cap E \subset C(Z_1^2, \nu_1^j) \subset E,
\]
and
if \( \nu_j = \Lambda(Z_1^2, \nu_1^j) \), then \( C(Z_1^2, \nu_1^j) \) is smooth at \( \nu_j \),
and either \( \nu_j \) is an isolated point in \( \text{Sing}_1(Z_1^2) \) or \( C(Z_1^2, \nu_1^j) \)
is the only curve in \( \text{Sing}_1(Z_1^2) \) which is contained in \( E \) and contains \( \nu_j \),
and
\[
\nu_j \in C(Z_1^1, \nu_1^k) \text{ for some } \nu_1^k \in \bigcup_{j=1}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} T_{j,k}
\]
implies \( C(Z_1^2, \nu_1^k) = C(Z_1^2, \nu_1^j) \)
and
If \( \gamma \) is a 2-curve of \( E \) which contains \( \nu_j \),
then \( \sigma_D(q) \leq r - 2 \) for \( q \in \gamma \setminus \{\nu_j\} \).

Note that no new components of \( D \) containing points
\[
p \in D \setminus \left( \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_2^1, p) \right)
\]
with \( \sigma_D(p) = r - 1 \) can be created as
\[
q \in \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} (\text{Preimage}(Z_2^1, W(Z_1^2, p)) \setminus W(Z_2^1, p))
\]
implies \( \sigma_D(q) \leq r - 2 \).

We further have that \( Z_2^1 \) is 2-prepared at \( q \in Z_2^1 \setminus \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_1^2, p) \)
and
\[
\sigma_D(q) \leq r - 1 \text{ for } q \in Z_2^1 \setminus \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_1^2, p).
\]

Now repeat this procedure for other such components \( E \) of \( D \) for \( Z_2^1 \) which contain
\( C(Z_2^1, \nu_2^k) \) for some \( p \in \bigcup_{j=1}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \) or contain a point
\[
q \in E \setminus \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_2^1, p)
\]
with \( \sigma_D(q) = r - 1 \) (which are necessarily the strict transform of a component of \( D \)
on \( Z_1^1 \)) to construct \( Z_3^1 \) so that for all \( p \in \bigcup_{j=1}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \), condition
(56) or (57) hold for all components \( E \) of \( D \) for \( Z_2^1 \) which contain \( C(Z_2^1, \nu_2^k) \) for some
\( p \in \bigcup_{j=1}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \) or contain a point \( q \in E \setminus \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_2^1, p) \)
with \( \sigma_D(q) = r - 1 \). We have that \( Z_3^1 \) is 2-prepared at \( q \in Z_3^1 \setminus \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_2^1, p) \)
and \( \sigma_D(q) \leq r - 1 \) for \( q \in Z_3^1 \setminus \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_2^1, p) \).

Now by Lemma 3.4, we can perform a toroidal morphism for \( D \) (which is a sequence of blowups of 2-curves for \( D \) \( Z_2^1 \) to \( Z_3^1 \), so that we further have that if \( G \) is a component of \( DZ_3^1 \) containing a curve \( C(Z_1^1, p) \) for some \( p \in \bigcup_{j=1}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \) or \( G \setminus \bigcup_{p \in \bigcup_{j=1}}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} W(Z_2^1, p) \) contains a point \( q \) with \( \sigma_D(q) = r - 1 \), then \( Z_3^1 \) is 3-prepared at all 2-points and 3-points of \( G \). We further have that for all \( p \in \bigcup_{j=1}^{m} \bigcup_{k=1}^{\min\{j-1, r-1\}} T_{j,k} \), (56) or (57) holds on \( G \).
We now may apply Proposition 3.14 to the union $H$ of components $E$ of $D$ for $Z^1_4$ containing a curve $C(Z^1_4, \nu^j_p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, or containing a point $q$ with $\sigma_D(q) = r - 1$ with

$$A = \{ q \in H \mid Z^1_4 \text{ is not 3-prepared at } q \text{ (which are necessarily one points of } D) \}$$

being sure that none of the finitely many 2-points for $D$

$$B = \{ \Lambda(Z^1_4, \nu^j_p) \mid p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k} \}$$

are in the image of the general curves blown up, to construct $X^1 \to Z^1_4$ so that $X^1$ is 3-prepared over $E \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}} \Lambda(X^1, \nu^j_p)$ for all components $E$ of $D$ for $X^1$ which contain a curve $C(X^1, \nu^j_p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, or contain a point $q \in X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$ with $\sigma_D(q) = r - 1$. Further, for all $p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}$, condition (56) or (57) hold on components $F$ of $D$ for $X^1$ containing a curve $C(X^1, \nu^j_p)$ or a point $q \in X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=1}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$ such that $\sigma_D(q) = r - 1$.

We now have (using Lemma 5.1) the following:

3.1) $X^1 \to X_{j,k}$ is toroidal for $D_p$ for $p \in T_{j,k}$ with $1 \leq j \leq m$, $0 \leq k \leq \min\{j-1,r-1\}$ in a neighborhood of $W(X^1, p)$.

3.2) $X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$ is 2-prepared and $\sigma_D(q) \leq r - 1$ for $q \in X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)$.

3.3) Suppose that $1 < r$. Then

a) $X^1$ is 3-prepared at all points $q \in C(X^1, \nu^k_p) \cap \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} \text{Preimage}(X^1, p)$ for $p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}$.

b) $X^1$ is 3-prepared at all points of

$$\left(X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p)\right) \cap \text{Sing}_{r-1}(X^1),$$

and if $C \subset \text{Sing}_{r-1}(X^1)$ is not equal to a curve $C(X^1, \nu^k_p)$ for some $p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}$, then

$$C \cap \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p) = \emptyset.$$ 

3.4) Suppose that $1 < r$. Let

$$T^1_0 = \left\{ \text{2-points } q \text{ of } X^1 \setminus \bigcup_{p \in \bigcup_{j=1}^m \bigcup_{k=0}^{\min\{j-1,r-1\}} T_{j,k}} W(X^1, p) \mid \text{such that } \sigma_D(q) = r - 1. \right\}$$
For \( p \in T^1_0 \), let \((U_p, \overline{D}_p, \nu^1_p, \nu^2_p)\) be associated local resolvers. Let \( \Gamma_1 \) be the union of the curves
\[
\begin{align*}
C(X^1, \nu^1_p) & \text{ such that } p \in \left( \bigcup_{j=1}^{m} \bigcup_{k=0}^{\min\{j-1, r-1\}} T_{j,k} \right) \cup T^1_0 \\
\text{and } \sigma_D(\eta) = r - 1 \text{ for } \eta \in C(X^1, \nu^1_p) \text{ a general point}
\end{align*}
\]
and any remaining curves \( C \) in
\[
\text{Sing}_{r-1}(X^1 \setminus \left( \bigcup_{j=1}^{m} \bigcup_{k=0}^{\min\{j-1, r-1\}} T_{j,k} \right) \cup T^1_0)
\]
(which are necessarily closed in \( X^1 \) and do not contain 2-points).

3.5) Suppose that \( 1 < r \). Suppose that
\[
p \in \left( \bigcup_{j=1}^{m} \bigcup_{k=0}^{\min\{j-1, r-1\}} T_{j,k} \right) \cup T^1_0
\]
and \( C(X^1, \nu^1_p) \) is such that \( \sigma_D(\eta) = r - 1 \) for \( \eta \in C(X^1, \nu^1_p) \) the generic point. Then \( \sigma_D(q) = r - 1 \) for
\[
q \in C(X^1, \nu^1_p) \setminus \left( \bigcup_{p'} \left( \bigcup_{j=1}^{m} \bigcup_{k=0}^{\min\{j-1, r-1\}} T_{j,k} \right) \cup T^1_0 \right) W(X^1, p').
\]
Further, if \( q \in \left( \bigcup_{j=1}^{m} \bigcup_{k=0}^{\min\{j-1, r-1\}} T_{j,k} \right) \cup T^1_0 \) and \( W(X^1, q) \cap C(X^1, \nu^1_p) \neq \emptyset \), then \( C(X^1, \nu^1_p) = C(X^1, \nu^n_q) \) for some \( n \).

Now we proceed in this way to inductively construct sequences of blow ups for \( 0 \leq j \leq r - 1 \) (as in the algorithm of (52)), where we identify \( X^0_{i,l} \) with \( X_{i,l} \),

\[
X^j_{m_j, r-j-1} \to \cdots \to X^j_{m_j, 0} \to \cdots \to X^j_{r-j, r-j-1} \to \cdots \to X^j_{r-j, 0} \to X^j_{r-j-1, r-j-2} \\
\to \cdots \to X^j_{3, 0} \to X^j_{2, 1} \to X^j_{2, 0} \to X^j_{1, 0} \to X^j
\]

and

\[
X^j \to X^j_{m_j-1, r-j-2}
\]
for \( 1 \leq j \leq r \) (as in the construction of \( X^1 \)) such that for \( 1 \leq j \leq r \),

4.1) \( X^j \to X^{j-1}_{i,k} \) is toroidal for \( \overline{D}_p \) for \( p \in T^{j-1}_{i,k} \) with \( 1 \leq i \leq m_{j-1}, 0 \leq k \leq \min\{i-1, r-j\} \) in a neighborhood of \( W(X^j, p) \).

4.2) \( X^j \setminus \bigcup_p \bigcup_{i=1}^{m_{j-1}} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k} W(X^j, p) \) is 2-prepared and \( \sigma_D(q) \leq r - j \) for \( q \in X^j \setminus \bigcup_p \bigcup_{i=1}^{m_{j-1}} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k} W(X^j, p) \).

4.3) Suppose that \( j < r \). Then
a) \( X^j \) is 3-prepared at all points \( q \in C(X^j, \nu^1_p) \setminus \bigcup_{p'} \bigcup_{i=1}^{m_{j-1}} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k} W(X^j, p') \) for some \( l \) and \( \text{Preimage}(X^j, p') \) for \( p \in \bigcup_{i=1}^{m_{j-1}} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k} \).

b) \( X^j \) is 3-prepared at all points of
\[
\left( X^j \setminus \bigcup_p \bigcup_{i=1}^{m_{j-1}} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k} W(X^j, p) \right) \cap \text{Sing}_{r-j}(X^j),
\]
and if $C \subset \operatorname{Sing}_{r-j}(X^j)$ is not equal to a curve $C(X^j, \nu_p^k)$ for some $p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1}$, then
\[ C \cap \bigcup_{p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1}} W(X^j, p) = \emptyset. \]

4.4) Suppose that $j < r$. Let
\[ T_0^j = \left\{ \begin{array}{l}
2\text{-points } q \text{ of } X^j - \bigcup_{p \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1}} W(X^j, p) \\
\text{such that } \sigma_D(q) = r - j
\end{array} \right\} \]

For $p \in T_0^j$, let $(U_p, D_p, \nu_p^1, \nu_p^2)$ be associated local resolvers. Let $\Gamma_j$ be the union of the curves
\[ C(X^j, \nu_p^j) \text{ such that } p \in \left( \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j \]
and any remaining curves $C$ in
\[ \operatorname{Sing}_{r-j}(X^j \setminus \left( \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j) \]
(which are necessarily closed in $X^j$ and do not contain 2-points).

4.5) Suppose that $j < r$. Suppose that
\[ p \in \left( \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j \]
and $C(X^j, \nu_p^j)$ is such that $\sigma_D(p) = r - j$ for $p \in C(X^j, \nu_p^j)$ the generic point. Then $\sigma_D(q) = r - j$ for
\[ q \in C(X^j, \nu_p^j) \setminus \left( \bigcup_{p' \in \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1}} W(X^j, p') \right). \]

Further, if $q \in \left( \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j$ and $W(X^j, q) \cap C(X^j, \nu_p^j) \neq \emptyset$, then $C(X^j, \nu_p^j) = C(X^j, \nu_q^n)$ for some $n$.

For $0 \leq j \leq r - 1$, $0 \leq i \leq m_j$ and $0 \leq k \leq \min\{i - 1, r - j - 1\}$,

5.1) $X^j_{i,0} \to X^j$ is the canonical sequence of blow ups above a general point $\eta$ of a curve in $\Gamma_j$ (so that $\sigma_D(\eta) = r - j$), and for $i > 0$,
\[ X^j_{i+1,0} \to X^j_{i,\min\{i-1, r-j-1\}} \]
is the canonical sequence of blow ups above a general point $\eta$ of a curve
\[ C(X^j_{i,\min\{i-1, r-j-1\}}, \nu_p^j) \]
with $p \in \left( \bigcup_{i=1}^{m_j} \bigcup_{k=0}^{\min\{i-1, r-j\}} T_{i,k}^{j-1} \right) \cup T_0^j$ and $\sigma_D(\eta) = \max\{0, r - i - j\}$, and

and the following properties hold. Let
\[ S_{i,k}^j = \left( \bigcup_{i=1}^{m_j} \bigcup_{n=0}^{\min\{i-1, r-j\}} T_{i,n}^{j-1} \right) \cup T_0^j \cup \left( \bigcup_{i=1}^{m_j} \bigcup_{n=0}^{\min\{i-1, r-j-1\}} T_{i,n}^j \right) \cup \left( \bigcup_{n=0}^{k-1} T_{i,n}^j \right). \]
5.2) \( X^{j}_{i,k} \to X^{j}_i \) is toroidal for \( D_p \) in a neighborhood of \( W(X^j_{i,k},p) \) for \( p \in S^j_{i,k} \) (with \( p \in X^j_{i,k} \)).

5.3) \( X^j_{i,k} \setminus (\cup_{p \in S^j_{i,k}} W(X^j_{i,k},p)) \) is 2-prepared and \( \sigma_D(p) < r-j \) for \( q \in X^j_{i,k} \setminus (\cup_{p \in S^j_{i,k}} W(X^j_{i,k},p)) \).

5.4) If \( p \in \left( \cup_{l=1}^{m_j-1} \cup_{n=0}^{\min(l-1,r-j)} T^{j-1}_{i,n} \right) \cup T^j_0 \), then \( \sigma_D(\eta) \leq \max\{0, r-i-j \} \) for \( \eta \in C(X^j_{i,k}, \nu^j_p) \) the generic point and \( X^j_{i,k} \) is 3-prepared at \( q \) for \( q \in C(X^j_{i,k}, \nu^j_p) \setminus \cup_{p' \in S^j_{i,k}} C(X^j_{i,k}, \nu^j_{p'}) = C(X^j_{i,k}, \nu^j_{p'}) \) for some \( i \)Preimage\( (X^j_{i,k}, p') \).

5.5) We have the set

\[
T^{j}_{i,k} = \begin{cases} 
2\text{-points } q \text{ for } D \text{ of} \\
C(X^j_{i,k}, \nu^j_p) \setminus \cup_{p' \in \Omega} \text{Preimage}(X^j_{i,k}, p'), \\
\text{where } \Omega = \{ p' \in S^j_{i,k} \mid C(X^j_{i,k}, \nu^j_{p'}) = C(X^j_{i,k}, \nu^j_{p'}) \text{ for some } l \} \\
\text{such that } \sigma_D(q) > 0 \text{ and such that} \\
p \in \left( \cup_{l=1}^{m_j-1} \cup_{n=0}^{\min(l-1,r-j)} T^{j-1}_{i,n} \right) \cup T^j_0 \\
\text{with } \sigma_D(q) = \max\{0, r-i-j \} \text{ for } \eta \in C(X^j_{i,k}, \nu^j_p) \text{ the generic point.} 
\end{cases}
\]

\( X^j_{i,k} \) is 3-prepared at \( p \in T^j_{i,k} \). We have local resolvers \( (U_p, D_p, I_p, \nu^j_p, \nu^2_p) \) at \( p \in T^j_{i,k} \).

We have \( \max\{1, r-i-j \} \leq \sigma_D(q) \leq r-j-k-1 \) for \( q \in T^j_{i,k} \).

5.6) Suppose that

\[
p \in \left( \cup_{l=1}^{m_j-1} \cup_{n=0}^{\min(l-1,r-j)} T^{j-1}_{i,n} \right) \cup T^j_0
\]

and \( C(X^j_{i,k}, \nu^j_p) \) is such that \( \sigma_D(q) = \max\{0, r-i-j \} \) for \( q \in C(X^j_{i,k}, \nu^j_p) \) a general point. Then \( \sigma_D(q) = \max\{0, r-i-j \} \) for \( q \in C(X^j_{i,k}, \nu^j_p) \setminus \left( \cup_{p' \in S^j_{i,k} \cup T^j_{i,k}} W(X^j_{i,k}, p') \right) \).

Further,

a) If \( q \in S^j_{i,k} \) and \( W(X^j_{i,k}, q) \cap C(X^j_{i,k}, \nu^j_p) \neq \emptyset \), then \( C(X^j_{i,k}, \nu^j_p) = C(X^j_{i,k}, \nu^j_q) \) for some \( n \).

b) If \( q \in T^j_{i,k} \) and \( q \in C(X^j_{i,k}, \nu^j_p) \), then either \( C(X^j_{i,k}, \nu^j_p) = C(X^j_{i,k}, \nu^j_q) \) for some \( n \) or

\[
\max\{0, r-i-j \} < \sigma_D(q) \leq r-j-k-1.
\]

By the definition of \( T^j_{i,k} \) in 5.5) above, we have that \( \cup_{l=1}^{m_j-1} \cup_{k=0}^{\min(l-1,r-j)} T^{j-1}_{i,k} = \emptyset \). Thus

4.2), following (59), implies that \( X^r \) is prepared.

\[\square\]

6. PROOF OF TOROIDALIZATION

**Theorem 6.1.** Suppose that \( \mathbb{K} \) is an algebraically closed field of characteristic zero, and \( f : X \to S \) is a dominant morphism from a nonsingular 3-fold over \( \mathbb{K} \) to a nonsingular surface \( S \) over \( \mathbb{K} \) and \( D_S \) is a reduced SNC divisor on \( S \) such that \( D_X = f^{-1}(D_S)_{\text{red}} \) is a SNC divisor on \( X \) which contains the locus where \( f \) is not smooth. Further suppose that \( f \) is 1-prepared. Then there exists a sequence of blow ups of points and nonsingular curves \( \pi_2 : X_1 \to X \), which are contained in the preimage of \( D_X \), such that the induced morphism \( f_1 : X_1 \to S \) is prepared with respect to \( D_S \).

**Proof.** The proof is immediate from Lemma 2.2, Proposition 2.7 and Theorem 5.3. \[\square\]
Theorem 6.1 is a slight restatement of Theorem 17.3 of [12]. Theorem 17.3 [12] easily follows from Lemma 2.2 and Theorem 6.1.

**Theorem 6.2.** Suppose that \( \mathfrak{k} \) is an algebraically closed field of characteristic zero, and \( f : X \to S \) is a dominant morphism from a nonsingular 3-fold over \( \mathfrak{k} \) to a nonsingular surface \( S \) over \( \mathfrak{k} \) and \( D_S \) is a reduced SNC divisor on \( S \) such that \( D_X = f^{-1}(D_S)_{\text{red}} \) is a SNC divisor on \( X \) which contains the locus where \( f \) is not smooth. Then there exists a sequence of blow ups of points and nonsingular curves \( \pi_2 : X_1 \to X \), which are contained in the preimage of \( D_X \), and a sequence of blow ups of points \( \pi_1 : S_1 \to S \) which are in the preimage of \( D_S \), such that the induced rational map \( f_1 : X_1 \to S_1 \) is a morphism which is toroidal with respect to \( D_{S_1} = \pi_1^{-1}(D_S) \).

**Proof.** The proof follows immediately from Theorem 6.1, and Theorems 18.19, 19.9 and 19.10 of [12]. □

Theorem 6.2 is a slight restatement of Theorem 19.11 of [10]. Theorem 19.11 [12] easily follows from Theorem 6.2.

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Steven Dale Cutkosky, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: cutkoskys@missouri.edu