1. Introduction

Suppose that \( f : X \rightarrow Y \) is a dominant morphism of algebraic varieties, over a field \( k \) of characteristic zero. If \( X \) and \( Y \) are nonsingular, \( f : X \rightarrow Y \) is toroidal if there are simple normal crossing divisors \( D_X \) on \( X \) and \( D_Y \) on \( Y \) such that \( f^{-1}(D_Y) = D_X \), and \( f \) is locally given by monomials in appropriate etale local parameters on \( X \). The precise definition of this concept is in \([AK]\), \([KKMS]\) and Definition 3.2 of this paper. The problem of toroidalization is to determine, given a dominant morphism \( f : X \rightarrow Y \), if there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are products of blow ups of nonsingular subvarieties, \( X_1 \) and \( Y_1 \) are nonsingular, and there exist simple normal crossing divisors \( D_{Y_1} \) on \( Y_1 \) and \( D_{X_1} = f_1^{-1}(D_{Y_1}) \) on \( X_1 \) such that \( f_1 \) is toroidal (with respect to \( D_{X_1} \) and \( D_{Y_1} \)). This is stated in Problem 6.2.1. of \([AKMW]\).

A stronger form of toroidalization is also asked for in Problem 6.2.1 \([AKMW]\), which we will call strong toroidalization. Suppose that \( f : X \rightarrow Y \) is a dominant morphism of nonsingular projective varieties, \( D_Y \) is a SNC divisor on \( Y \) and \( D_X = f^{-1}(D_Y) \) is a SNC divisor on \( X \) such that the locus \( \text{sing}(f) \) where the morphism \( f \) is not smooth is contained in \( D_X \). The problem of strong toroidalization is to determine if there exists a commutative diagram \((1)\) such that \( \Phi \) and \( \Psi \) are products of blow ups of nonsingular centers which are supported in the preimages of \( D_X \) and \( D_Y \) respectively, and make SNCs with the respective preimages of \( D_X \) and \( D_Y \), and \( f_1 \) is toroidal with respect to \( D_{Y_1} = \Psi^{-1}(D_Y) \) and \( D_{X_1} = \Phi^{-1}(D_X) \).

Toroidalization, and related concepts, have been considered earlier in different contexts, mostly for morphisms of surfaces. Strong toroidalization is the strongest structure theorem which could be true for general morphisms. The concept of toroidalization fails completely in positive characteristic. A simple example is shown in \([C3]\).

In the case when \( Y \) is a curve, toroidalization follows from embedded resolution of singularities (\([H]\)). When \( X \) and \( Y \) are surfaces, there are several proofs in print (\([AkK]\), Corollary 6.2.3 \([AKMW]\), \([CP]\), \([Mat]\)). They all make use of special properties of the birational geometry of surfaces. An outline of proofs of the above cases can be found in the introduction to \([C3]\).

In \([C3]\), strong toroidalization is solved in the case when \( X \) is a 3-fold and \( Y \) is a surface. In Theorem 0.1 of \([C5]\) we prove toroidalization of birational morphisms
of 3-folds. In this paper, we prove strong toroidalization for birational morphisms of 3-folds.

**Theorem 1.1.** Suppose that \( f : X \to Y \) is a birational morphism of nonsingular projective 3-folds over an algebraically closed field \( k \) of characteristic 0. Further suppose that there is a SNC divisor \( D_Y \) on \( Y \) such that \( D_X = f^{-1}(D_Y) \) is a SNC divisor which contains the singular locus of the map \( f \). Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( \Phi, \Psi \) are products of possible blow ups for the preimages of \( D_X \) and \( D_Y \) respectively, and \( f_1 \) is toroidal with respect to \( D_{Y_1} = \Psi^{-1}(D_Y) \) and \( D_{X_1} = \Phi^{-1}(D_X) \).

A possible blow up on a nonsingular 3-fold with toroidal structure is the blow up of a point or a nonsingular curve contained in the toroidal structure which makes SNCs with the toroidal structure.

As a consequence of Theorem 1.1, we find the following strong toroidalization theorem for morphisms of (possibly singular) varieties.

**Theorem 1.2.** Suppose that \( f : X \to Y \) is a birational morphism of 3-folds which are proper over an algebraically closed field \( k \) of characteristic 0. Further suppose that there is an equidimensional codimension 1 reduced subscheme \( D_Y \) of \( Y \) such that \( D_Y \) contains the singular locus of \( Y \), and \( D_X = f^{-1}(D_Y) \) contains the singular locus of the map \( f \). Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( \Phi, \Psi \) are products of blow ups of nonsingular curves and points supported above \( D_X \) and \( D_Y \) respectively, \( D_{Y_1} = \Psi^{-1}(D_Y) \) is a simple normal crossings divisor on \( Y_1 \), \( D_{X_1} = f_1^{-1}(D_{Y_1}) \) is a simple normal crossings divisor on \( X_1 \) and \( f_1 \) is toroidal with respect to \( D_{Y_1} \) and \( D_{X_1} \).

The bulk of this paper is devoted to proving the following theorem.

**Theorem 1.3.** Suppose that \( f : X \to Y \) is a dominant morphism of nonsingular projective 3-folds over an algebraically closed field \( k \) of characteristic zero, with toroidal structures determined by SNC divisors \( D_Y \) on \( Y \) and \( D_X = f^{-1}(D_Y) \) on \( X \) such that \( D_X \) contains the singular locus of \( f \). Then there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Psi \) and \( \Phi \) are products of possible blow ups for the preimages of \( D_Y, D_X \) respectively, such that \( f_1 \) is prepared for \( D_{Y_1} = \Psi^{-1}(D_Y) \) and \( D_{X_1} = \Phi^{-1}(D_X) \), and \( D_{X_1} \) is cuspidal for \( f_1 \).

The notation used in the statement of Theorem 1.3 is defined in Sections 2 and 3 of this paper. From Theorem 1.3 we easily deduce Theorems 1.1 and 1.2 from results in [C5].
Theorem 1.3 is applicable to arbitrary dominant morphisms of 3-folds, and is a significant step towards a proof of strong toroidalization of arbitrary dominant morphisms of 3-folds.

If we relax some of the restrictions in the definition of toroidalization, there are other constructions producing a toroidal morphism $f_1$, which are valid for arbitrary dimensions of $X$ and $Y$. In [AK] it is shown that a diagram (1) can be constructed where $\Phi$ is weakened to being a modification (an arbitrary birational morphism). In [C1], [C2] and [C4], it is shown that a diagram (1) can be constructed where $\Phi$ and $\Psi$ are locally products of blow ups of nonsingular centers and $f_1$ is locally toroidal, but the morphisms $\Phi$, $\Psi$ and $f_1$ may not be separated. This construction is obtained by patching local solutions, at least one of which contains the center of any given valuation.

It has been shown in [AKMW] and [W2] that weak factorization of birational morphisms holds in characteristic zero, and arbitrary dimension. That is, birational morphisms of complete varieties can be factored by an alternating sequence of blow ups and blow downs of non singular subvarieties. Weak factorization of birational (toric) morphisms of toric varieties, (and of birational toroidal morphisms) has been proven by Danilov [D1] and Ewald [E] (for 3-folds), and by Wlodarczyk [W1], Morelli [Mo] and Abramovich, Matsuki and Rashid [AMR] in general dimensions.

Our Theorem 1.1 on strong toroidalization (or the weaker Theorem 0.1 of [C5] on toroidalization), when combined with weak factorization for toroidal morphisms ([AMR]), gives a new proof of weak factorization of birational morphisms of 3-folds. We point out that our proof uses an analysis of the structure as power series of local germs of a mapping, as opposed to the entirely different proof of weak factorization, using geometric invariant theory, of [AKMW] and [W1].

The version of weak factorization which we get from Theorem 1.1 is stronger than that obtained in [AKMW], [W1] or [C5].

**Corollary 1.4.** Suppose that $f : X \to Y$ is a dominant birational morphism of nonsingular projective 3-folds over an algebraically closed field $k$ of characteristic zero, with toroidal structures determined by SNC divisors $D_Y$ on $Y$ and $D_X = f^{-1}(D_Y)$ on $X$ such that $D_X$ contains the singular locus of $f$. Then there exists a commutative diagram of morphisms factoring $f$,

\[
\begin{array}{ccccccc}
& & Z_1 & & Z_3 & & Z_n-1 & \ 
& \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_n & \\
X_1 & \Phi\downarrow & Z_2 & & Z_4 & & \cdots & \downarrow \Psi & Y_1 & \downarrow \\
\Phi\downarrow & & & & & & & \downarrow & & \\
X & & & & & & & Y & & \\
\end{array}
\]

such that

1. All varieties $X_1$, $Y_1$ and the $Z_i$ are nonsingular, with toroidal structures $D_{X_1}$, $D_{Y_1}$ and $D_{Z_i}$ respectively.
2. There is a toroidal morphism $f_1 : X_1 \to Y_1$ making a strong toroidalization

\[
\begin{array}{ccc}
X_1 & f_1 & Y_1 \\
\downarrow & & \downarrow \\
X & f & Y. \\
\end{array}
\]
3. The morphisms in the diagram

\[
\begin{array}{ccccccc}
X_1 & \alpha_1 & Z_1 & \alpha_2 & Z_2 & \ldots & \alpha_{n-1} & Z_{n-1} & \alpha_n & Y_1
\end{array}
\]

are toroidal with respect to their toroidal structures.

The proof of Corollary 1.4 is immediate from Theorem 1.1, which constructs the commutative diagram 2, and [AMR], [Mo] or [W1], which produces the diagram 3.

The problem of strong factorization, as proposed by Abhyankar [Ab2] and Hironaka [H], is to factor a birational morphism \( f : X \to Y \) by constructing a diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & Y
\end{array}
\]

where \( Z \to X \) and \( Z \to Y \) factor as products of blow ups of nonsingular subvarieties. Oda [O] has proposed the analogous problem for (toric) morphisms of toric varieties.

A birational morphism \( f : S \to Y \) of (nonsingular) surfaces can be directly factored by blowing up points (Zariski [Z1] and Abhyankar [Ab1]), but there are examples showing that a direct factorization is not possible in general for 3-folds (Shannon [Sh] and Sally [S]).

We also obtain as an immediate corollary of Theorem 1.1 the following new result, which reduces the problem of strong factorization of 3-folds to the case of morphisms of toric varieties

\textbf{Corollary 1.5.} Suppose that the Oda conjecture on strong factorization of birational morphisms of 3-dimensional toric varieties is true. Then the Abhyankar, Hironaka strong factorization conjecture of birational morphisms of complete (characteristic zero) 3-folds is true.

Abhyankar’s local factorization conjecture [Ab2], which is “strong factorization” along a valuation, follows from local monomialization (Theorem A [C2]), to reduce to a locally toroidal morphism, and local factorization for toroidal morphisms along a toroidal valuation Christensen [Ch] (for 3-folds), and Karu [K] in general dimensions. A proof in the spirit of [Ch] of local factorization of a toroidal morphism in all dimensions, using only elementary properties of determinants, is given in [CS].

2. Notation

Throughout this paper, \( k \) will be an algebraically closed field of characteristic zero. A curve, surface or 3-fold is a quasi-projective variety over \( k \) of respective dimension 1, 2 or 3. If \( X \) is a variety, and \( p \in X \) is a nonsingular point, then regular parameters at \( p \) are regular parameters in \( \mathcal{O}_{X,p} \). Formal regular parameters at \( p \) are regular parameters in \( \mathcal{O}_{X,p} \). If \( X \) is a variety and \( V \subset X \) is a subvariety, then \( \mathcal{I}_V \subset \mathcal{O}_X \) will denote the ideal sheaf of \( V \). If \( V \) and \( W \) are subvarieties of a variety \( X \), we denote the scheme theoretic intersection \( Y = \text{spec}(\mathcal{O}_X/\mathcal{I}_V + \mathcal{I}_W) \) by \( Y = V \cdot W \).

Let \( f : X \to Y \) be a morphism of varieties. We will denote by \( \text{sing}(f) \) the closed set of points \( p \in X \) such that \( f \) is not smooth at \( p \). If \( D \) is a Cartier divisor on \( Y \), then \( f^{-1}(D) \) will denote the reduced divisor \( f^*(D)_{\text{red}} \).

Suppose that \( a, b, c, d \in \mathbb{Q} \). Then we will write \( (a, b) \leq (c, d) \) if \( a \leq b \) and \( c \leq d \).

A toroidal structure on a nonsingular variety \( X \) is a simple normal crossing divisor (SNC divisor) \( D_X \) on \( X \).
We will say that a nonsingular curve $C$ which is a subvariety of a nonsingular 3-fold $X$ with toroidal structure $D_X$ makes simple normal crossings (SNCs) with $D_X$ if for all $p \in C$, there exist regular parameters $x, y, z$ at $p$ such that $x = y = 0$ are local equations of $C$, and $xyz = 0$ contains the support of $D_X$ at $p$.

Suppose that $X$ is a nonsingular 3-fold with toroidal structure $D_X$. If $p \in D_X$ is on the intersection of three components of $D_X$ then $p$ is called a 3-point. If $p \in D_X$ is on the intersection of two components of $D_X$ (and is not a 3-point) then $p$ is called a 2-point. If $p \in D_X$ is not a 2-point or a 3-point, then $p$ is called a 1-point. If $C$ is an irreducible component of the intersection of two components of $D_X$, then $C$ is called a 2-curve.

A possible center on a nonsingular 3-fold $X$ with toroidal structure defined by a SNC divisor $D_X$, is a point on $D_X$ or a nonsingular curve in $D_X$ which makes SNCs with $D_X$. A possible center on a nonsingular surface $S$ with toroidal structure defined by a SNC divisor $D_S$ is a point on $D_S$. We will also call the blow up of a possible center a possible blow up.

Observe that if $\Phi : X_1 \to X$ is the blow up of a possible center, then $D_{X_1} = \Phi^{-1}(D_X)$ is a SNC divisor on $X_1$. Thus $D_{X_1}$ defines a toroidal structure on $X_1$. All blow ups $\Phi : X_1 \to X$ considered in this paper will be of possible centers, and we will impose the toroidal structure on $X_1$ defined by $D_{X_1} = \Phi^{-1}(D_X)$.

By a general point $q$ of a variety $V$, we will mean a point $q$ which satisfies conditions which hold on some nontrivial open subset of $V$. The exact open condition which we require will generally be clear from context. By a general section of a coherent sheaf $\mathcal{F}$ on a projective variety $X$, we mean the section corresponding to a general point of the $k$-linear space $\Gamma(X, \mathcal{F})$.

If $X$ is a variety, $k(X)$ will denote the function field of $X$. A 0-dimensional valuation $\nu$ of $k(X)$ is a valuation of $k(X)$ such that $k$ is contained in the valuation ring $V_\nu$ of $\nu$ and the residue field of $V_\nu$ is $k$. If $X$ is a projective variety which is birationally equivalent to $X$, then there exists a unique (closed) point $p_1 \in X_1$ such that $V_\nu$ dominates $\mathcal{O}_{X_1, p_1}$. $p_1$ is called the center of $\nu$ on $X_1$. If $p \in X$ is a (closed) point, then there exists a 0-dimensional valuation $\nu$ of $k(X)$ such that $V_\nu$ dominates $\mathcal{O}_{X, p}$ (Theorem 37, Section 16, Chapter VI [ZS]). For $a_1, \ldots, a_n \in k(X)$, $\nu(a_1), \ldots, \nu(a_n)$ are rationally dependent if there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$ which are not all zero, such that $\alpha_1 \nu(a_1) + \cdots + \alpha_n \nu(a_n) = 0$ (in the value group of $\nu$). Otherwise, $\nu(a_1), \ldots, \nu(a_n)$ are rationally independent.

3. Toroidal Morphisms and Prepared Morphisms

Suppose that $X$ is a nonsingular variety with toroidal structure $D_X$. We will say that an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is toroidal if $\mathcal{I}$ is locally generated by monomials in local equations of components of $D_X$.

Suppose that $q \in X$. We say that $u, v, w$ are (formal) permissible parameters at $q$ (for $D_X$) if $u, v, w$ are regular parameters in $\mathcal{O}_{X, q}$ such that

1. If $q$ is a 1-point, then $u \in \mathcal{O}_{X, q}$ and $u = 0$ is a local equation of $D_X$ at $q$.
2. If $q$ is a 2-point then $u, v \in \mathcal{O}_{X, q}$ and $uv = 0$ is a local equation of $D_X$ at $q$.
3. If $q$ is a 3-point then $u, v, w \in \mathcal{O}_{X, q}$ and $uvw = 0$ is a local equation of $D_X$ at $q$.

$u, v, w$ are algebraic permissible parameters if we further have that $u, v, w \in \mathcal{O}_{X, q}$.

**Definition 3.1.** Let $f : X \to Y$ be a dominant morphism of nonsingular 3-folds with toroidal structures $D_Y$ on $Y$ and $D_X = f^{-1}(D_Y)$ on $X$ such that $\text{sing}(f) \subset D_X$. Suppose that $u, v, w$ are (possibly formal) permissible parameters at $q \in Y$. Then $u, v$...
are toroidal forms at \( p \in f^{-1}(q) \) if there exist permissible parameters \( x, y, z \) in \( \hat{O}_{X,p} \) such that

1. \( q \) is a 2-point or a 3-point, \( p \) is a 1-point and
   \[ u = x^a, \quad v = x^b(\alpha + y) \]
   where \( 0 \neq \alpha \in k \).

2. \( q \) is 2-point or a 3-point, \( p \) is a 2-point and
   \[ u = x^a y^b, \quad v = x^c y^d \]
   with \( ad - bc \neq 0 \).

3. \( q \) is 2-point or a 3-point, \( p \) is a 2-point and
   \[ u = (x^a y^b)^k, \quad v = (x^a y^b)^t (\alpha + z) \]
   where \( 0 \neq \alpha \in k \), \( a, b, k, t > 0 \) and \( \gcd(a, b) = 1 \).

4. \( q \) is 2-point or a 3-point, \( p \) is a 3-point and
   \[ u = x^a y^b z^c, \quad v = x^d y^e z^f \]
   where
   \[ \text{rank} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = 2. \]

5. \( q \) is a 1-point, \( p \) is a 1-point and
   \[ u = x^a, \quad v = y \]

6. \( q \) is a 1-point, \( p \) is a 2-point and
   \[ u = (x^a y^b)^k, \quad v = z \]
   with \( a, b, k > 0 \) and \( \gcd(a, b) = 1 \).

Regular parameters \( x, y, z \) as in Definition 3.1 will be called permissible parameters for \( u, v, w \) at \( p \).

**Definition 3.2.** ([KKMS], [AK]) A normal variety \( \overline{X} \) with a SNC divisor \( D_\overline{X} \) on \( \overline{X} \) is called toroidal if for every point \( p \in \overline{X} \) there exists an affine toric variety \( X_\sigma \), a point \( p' \in X_\sigma \) and an isomorphism of \( k \)-algebras
\[ \hat{O}_{X,p} \cong \hat{O}_{X_\sigma,p'} \]
such that the ideal of \( D_\overline{X} \) corresponds to the ideal of \( X_\sigma - T \) (where \( T \) is the torus in \( X_\sigma \)). Such a pair \( (X_\sigma, p') \) is called a local model at \( p \in \overline{X} \). \( D_\overline{X} \) is called a toroidal structure on \( \overline{X} \).

A dominant morphism \( \Phi : \overline{X} \to Y \) of toroidal varieties with SNC divisors \( D_\overline{X} \) on \( \overline{X} \) and \( D_Y = \Phi^{-1}(D_Y) \) on \( Y \), is called toroidal at \( p \in \overline{X} \), and we will say that \( p \) is a toroidal point of \( \Phi \) if with \( q = \Phi(p) \), there exist local models \( (X_\sigma, p') \) at \( p \), \( (Y_\tau, q') \) at \( q \) and a toric morphism \( \Psi : X_\sigma \to Y_\tau \) such that the following diagram commutes:
\[
\begin{array}{c}
\hat{O}_{X,p} \leftarrow \hat{O}_{X_\sigma,p'} \\
\hat{O}_{\overline{X},q} \rightarrow \hat{O}_{Y,q'} \\
\Phi^* \uparrow \quad \Psi^* \uparrow
\end{array}
\]
\( \Phi : \overline{X} \to Y \) is called toroidal (with respect to \( D_\overline{X} \) and \( D_Y \)) if \( \Phi \) is toroidal at all \( p \in \overline{X} \).
The following is the list of toroidal forms for a dominant morphism $f : X \rightarrow Y$ of nonsingular 3-folds with toroidal structure $D_Y$ and $D_X = f^{-1}(D_X)$. Suppose that $p \in D_X$, $q = f(p) \in D_Y$, and $f$ is toroidal at $p$. Then there exist permissible parameters $u, v, w$ at $q$ and permissible parameters $x, y, z$ for $u, v, w$ at $p$ such that one of the following forms hold:

1. $p$ is a 3-point and $q$ is a 3-point,
   \[
   u = x^a y^b z^c, \\
   v = x^d y^e z^f, \\
   w = x^g y^h z^i,
   \]
   where $a, b, d, e, f, g, h, i \in \mathbb{N}$ and
   \[
   \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.
   \]
2. $p$ is a 2-point and $q$ is a 3-point,
   \[
   u = x^a y^b, \\
   v = x^d y^e, \\
   w = x^g y^h (z + \alpha)
   \]
   with $0 \neq \alpha \in k$ and $a, b, d, e, g, h \in \mathbb{N}$ satisfy $ae - bd \neq 0$.
3. $p$ is a 1-point and $q$ is a 3-point,
   \[
   u = x^a, \\
   v = x^d (y + \alpha), \\
   w = x^g (z + \beta)
   \]
   with $0 \neq \alpha, \beta \in k$, $a, d, g > 0$.
4. $p$ is a 2-point and $q$ is a 2-point,
   \[
   u = x^a y^b, \\
   v = x^d y^e, \\
   w = z
   \]
   with $ae - bd \neq 0$.
5. $p$ is a 1-point and $q$ is a 2-point,
   \[
   u = x^a, \\
   v = x^d (y + \alpha), \\
   w = z
   \]
   with $0 \neq \alpha \in k$, $a, d > 0$.
6. $p$ is a 1-point and $q$ is a 1-point,
   \[
   u = x^a, \\
   v = y, \\
   w = z
   \]
   with $a > 0$.

A prepared morphism $\Phi_X : X \rightarrow S$ from a nonsingular 3-fold $X$ to a nonsingular surface $S$ (with respect to toroidal structures $D_S$ and $D_X = \Phi_X^{-1}(D_S)$) is defined in Definition 6.5 [C3]. A prepared morphism from a 3-fold to a 3-fold is defined in Definition 2.4 [C5]. This definition assumes that $f : X \rightarrow Y$ is birational, but this definition is perfectly valid for a generically finite morphism of 3-folds.
Remark 3.3. Suppose that $f : X \to Y$ is a dominant proper morphism of nonsingular 3-folds with toroidal structures determined by SNC divisors $D_Y$ on $Y$ and $D_X = f^{-1}(D_Y)$ on $X$, and $D_X$ contains the singular locus of the morphism $f$. With our assumptions on $f$, $f$ is generically finite. Recall that the fundamental locus of a generically finite morphism $f : X \to Y$ of nonsingular proper varieties is \{ $p \in Y$ | $\dim f^{-1}(p) > 0$ \}. The fundamental locus is a closed set of codimension $\geq 2$ in $Y$. Let $\overline{X}$ be the normalization of $Y$ in the function field of $X$, with induced finite morphism $\lambda : \overline{X} \to Y$. The branch locus of $\lambda$ is contained in the SNC divisor $D_Y$. Let $E$ be an irreducible component of $D_Y$. By Abhyankar’s lemma, the irreducible components of $\lambda^{-1}(E)$ are disjoint. Thus the irreducible components of $D_X$ which dominate $E$ are disjoint.

Definition 3.4. A dominant morphism $f : X \to Y$ of nonsingular 3-folds with toroidal structures determined by SNC divisors $D_Y$ on $Y$, and $D_X = f^{-1}(D_Y)$ on $X$ such that the singular locus of $f$ is contained in $D_X$ is prepared for $D_Y$ and $D_X$ if:

1. If $q \in Y$ is a 3-point, $u, v, w$ are permissible parameters at $q$ and $p \in f^{-1}(q)$, then $u, v$ and $w$ are each a unit (in $\mathcal{O}_{X,p}$) times a monomial in local equations of the toroidal structure $D_X$ at $p$. Furthermore, there exists a permutation of $u, v, w$ such that $u, v$ are toroidal forms at $p$.
2. If $q \in Y$ is a 2-point, $u, v, w$ are permissible parameters at $q$ and $p \in f^{-1}(q)$, then either
   (a) $u, v$ are toroidal forms at $p$ or
   (b) $p$ is a 1-point and there exist regular parameters $x, y, z \in \mathcal{O}_{X,p}$ such that there is an expression
       
       $u = x^a$
       $v = x^c(\gamma(x, y) + x^dy)$
       $w = y$

       where $\gamma$ is a unit series and $x = 0$ is a local equation of $D_X$, or
   (c) $p$ is a 2-point and there exist regular parameters $x, y, z$ in $\mathcal{O}_{X,p}$ such that there is an expression
       
       $u = (x^ay^b)^k$
       $v = (x^ay^b)^l(\gamma(x^ay^b, z) + x^cy^d)$
       $w = z$

       where $a, b > 0$, $gcd(a, b) = 1$, $ad - bc \neq 0$, $\gamma$ is a unit series and $xy = 0$ is a local equation of $D_X$.
3. If $q \in Y$ is a 1-point, and $p \in f^{-1}(q)$, then there exist permissible parameters $u, v, w$ at $q$ such that $u, v$ are toroidal forms at $p$.

Definition 3.5. Suppose that $f : X \to Y$ is a prepared morphism of nonsingular 3-folds with toroidal structures $D_Y$ and $D_X = f^{-1}(D_Y)$. Then $D_X$ is cuspidal for $f$ if:

1. If $E$ is a component of $D_X$ which does not contain a 3-point then $f$ is toroidal in a Zariski open neighborhood of $E$.
2. If $C$ is a 2-curve of $X$ which does not contain a 3-point then $f$ is toroidal in a Zariski open neighborhood of $C$.

Definition 3.6. Suppose that $X$ is a nonsingular 3-fold with toroidal structure determined by a SNC divisor $D_X$. We will say that $D_X$ is strongly cuspidal if every component of $D_X$ contains a 3-point and every 2-curve of $D_X$ contains a 3-point.
4. Preparation above 2 and 3-points

**Lemma 4.1.** Suppose that \( f : X \rightarrow Y \) is a dominant morphism of nonsingular projective 3-folds with toroidal structures determined by SNC divisors \( D_Y \) and \( D_X = f^{-1}(D_Y) \) such that \( D_X \) contains the singular locus of \( f \). Then there exist a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 & \downarrow & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are products of blow ups of 2-curves and \( f_1 \) is toroidal above all 3-points of \( Y_1 \).

**Proof.** Suppose that \( \nu \) is a 0-dimensional valuation of \( k(X) \). We will say that \( \nu \) is resolved for \( f \) if the center of \( \nu \) on \( Y \) is not a 3-point or if the center of \( \nu \) on \( Y \) is a 3-point, and \( f \) is toroidal at the center of \( \nu \) on \( X \).

Being resolved is an open condition on the Zariski-Riemann manifold of \( X \), and if \( \nu \) is resolved for \( f \) and \( X_1 \xrightarrow{f_1} \rightarrow Y_1 \Phi_1 \downarrow \downarrow \Psi_1 X \xrightarrow{f} Y \) is a commutative diagram such that \( \Phi_1 \) and \( \Psi_1 \) are products of blow ups of 2-curves, then \( \nu \) is resolved for \( f_1 \).

Suppose that \( \nu \) is a 0-dimensional valuation of \( k(X) \) such that the center \( q \) of \( \nu \) on \( Y \) is a 3-point. Let \( p \) be the center of \( \nu \) on \( X \). Let \( u, v, w \) be permissible parameters at \( q \).

**Case 1** Suppose that \( \nu(u), \nu(v), \nu(w) \) are rationally independent. Since \(uvw = 0 \) is a local equation of \( D_X \) at \( p \), there exist regular parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that \( xyz = 0 \) contains the germ of \( D_X \) at \( p \), and we have an expression

\[
\begin{align*}
u(u) &= x^a y^b z^c \gamma_1 \\
\nu(v) &= x^d y^e z^f \gamma_2 \\
\nu(w) &= x^g y^h z^i \gamma_3
\end{align*}
\]

where the \( \gamma_i \) are units in \( \mathcal{O}_{X,p} \). Since \( \nu(u), \nu(v), \nu(w) \) are rationally independent, \( \nu(x), \nu(y), \nu(z) \) are also rationally independent and

\[
\text{Det} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0
\]

which implies that \( p \) is a 3-point and \( f \) is toroidal at \( p \). Thus \( \nu \) is resolved for \( f \).

**Case 2** Suppose that \( \nu(u), \nu(v) \) are rationally dependent. After possibly interchanging \( u, v, w \) we reduce to this case. Let \( C \) be the 2-curve of \( Y \) with local equation \( u = v = 0 \) at \( q \). There exists a sequence of blow ups of 2-curves \( \Psi_\nu : Y_\nu \rightarrow Y \) such that the center of \( \nu \) on \( Y_\nu \) is not a 3-point.

\( Y_\nu \) is the blow up of a toroidal ideal sheaf \( \mathcal{J}_\nu \) of \( \mathcal{O}_Y \). Since \( f^{-1}(D_Y) = D_X, \mathcal{J}_\nu \mathcal{O}_X \) is also a toroidal ideal sheaf. By Lemma 2.11 [C5], there exists a sequence of blow ups
of 2-curves $\Phi_{\nu} : X_{\nu} \to X$ such that there is a commutative diagram of morphisms

$$
\begin{array}{ccc}
X_{\nu} & \xrightarrow{f_{\nu}} & Y_{\nu} \\
\Phi_{\nu} \downarrow & & \downarrow \Psi_{\nu} \\
X & \xrightarrow{f} & Y.
\end{array}
$$

Thus $\nu$ is resolved for $f_{\nu}$.

It follows from compactness of the Zariski Riemann manifold of $X$ [Z], that there exists a positive integer $n$ and commutative diagrams

$$
\begin{array}{ccc}
X_{i} & \xrightarrow{f_{i}} & Y_{i} \\
\Phi_{i} \downarrow & & \downarrow \Psi_{i} \\
X & \xrightarrow{f} & Y
\end{array}
$$

for $1 \leq i \leq n$ such that $\Phi_{i}$ and $\Psi_{i}$ are products of blow ups of 2-curves, and every 0-dimensional valuation $\nu$ of $k(X)$ is resolved for some $f_{i}$.

$Y_{i}$ is the blow up of a toroidal ideal sheaf $\mathcal{J}_{i}$ of $\mathcal{O}_{Y}$ and $X_{i}$ is the blow up of a toroidal ideal sheaf $\mathcal{I}_{i}$ of $\mathcal{O}_{X}$. Thus there exists a sequence of blow ups of 2-curves $Y^{*} \to Y$ such that $\prod \mathcal{J}_{i}\mathcal{O}_{Y^{*}}$ is invertible, by Lemma 2.11 [C5]. $Y^{*}$ is thus the blow up of a toroidal ideal sheaf $\mathcal{J} \subset \mathcal{O}_{Y}$, so that $\mathcal{J}\mathcal{O}_{X}$ is also a toroidal ideal sheaf. By Lemma 2.11 [C5], there exists a sequence of blow ups of 2-curves $X^{*} \to X$ such that $\mathcal{J}\prod \mathcal{I}_{i}\mathcal{O}_{X^{*}}$ is invertible. Thus for $1 \leq i \leq n$ there exist commutative diagrams of morphisms

$$
\begin{array}{ccc}
X^{*} & \xrightarrow{f^{*}} & Y^{*} \\
\Phi_{i}^{*} \downarrow & & \downarrow \Psi_{i}^{*} \\
X_{i} & \xrightarrow{f_{i}} & Y_{i} \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y.
\end{array}
$$

Suppose that $\nu$ is a 0-dimensional valuation of $k(X)$. If the center of $\nu$ on $Y^{*}$ is a 3-point, then the center of $\nu$ on $Y_{i}$ is a 3-point for all $i$, since $\Psi_{i}^{*}$ is toroidal. There exists an $i$ such that $\nu$ is resolved for $f_{i}$. Thus $f_{i}$ is toroidal at the center of $\nu$ on $X_{i}$. Since $\Phi_{i}^{*}$ and $\Psi_{i}^{*}$ are toroidal, $f^{*}$ is toroidal at the center of $\nu$. Thus $\nu$ is resolved for $f^{*}$. Since all 0-dimensional valuations of $k(X)$ are resolved for $f^{*}$, it follows that $f^{*}$ is toroidal above all 3-points of $Y^{*}$, and we have achieved the conclusions of the lemma. \hfill $\square$

**Lemma 4.2.** Suppose that $f : X \to Y$ is a dominant morphism of nonsingular projective 3-folds, with toroidal structures determined by SNC divisors $D_{Y}$ and $D_{X} = f^{-1}(D_{Y})$ such that $D_{X}$ contains the singular locus of $f$. Further suppose that $f$ is toroidal above all 3-points of $Y$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
X_{1} & \xrightarrow{f_{1}} & Y_{1} \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
$$

such that $\Psi$ and $\Psi$ are products of blow ups of 2-curves, $f_{1}$ is toroidal above all 3-points of $Y_{1}$, and $f_{1}$ is prepared (and satisfies 2a) of Definition 3.4) above all 2-points of $Y_{1}$.

**Proof.** Suppose that $\nu$ is a 0-dimensional valuation of $k(X)$. We will say that $\nu$ is resolved for $f$ if the center of $\nu$ on $Y$ is a 1-point or if the center of $\nu$ on $Y$ is a 2-point and $f$ is prepared at the center of $\nu$ on $X$ (and satisfies 2a) of Definition 3.4), or if the center of $\nu$ on $Y$ is a 3-point, and $f$ is toroidal at the center of $\nu$ on $X$. 
Being resolved is an open condition on the Zariski-Riemann manifold of $X$. Suppose that

$$
\begin{align*}
X_1 & \xrightarrow{f_1} Y_1 \\
\Phi & \downarrow \quad \downarrow \Psi \\
X & \xrightarrow{f} Y
\end{align*}
$$

is a commutative diagram of morphisms such that $\Phi$ and $\Psi$ are products of blow ups of 2-curves. If $\nu$ is a 0-dimensional valuation of $k(X)$ such that $\nu$ is resolved for $f$, then $\nu$ is resolved for $f_1$.

Suppose that $q \in Y$ is a 2-point, and $\nu$ is a 0-dimensional valuation of $k(X)$ such that $q$ is the center of $\nu$ on $Y$. Let $p$ be the center of $\nu$ on $X$. Let $u, v, w$ be permissible parameters at $q$, so that $u = v = 0$ are local equations of the 2-curve $C$ through $q$.

**Case 1** Suppose that $\nu(u), \nu(v)$ are rationally independent. Since $uv = 0$ is a local equation of $D_X$ at $p$, there exist regular parameters $x, y, z$ in $\mathcal{O}_{X,p}$ such that $xyz = 0$ contains the germ of $D_X$ in $\mathcal{O}_{X,p}$, and we have an expression

$$
\begin{align*}
u(u) &= x^a y^b z^c \
u(v) &= x^d y^e z^{f}\gamma_2
\end{align*}
$$

where the $\gamma_i$ are units in $\mathcal{O}_{X,p}$. Since $\nu(u), \nu(v)$ are rationally independent,

$$
\text{rank} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = 2
$$

which implies that $p$ is a 2 or 3-point and $f$ is prepared at $p$ (and satisfies 2a) of Definition 3.4). Thus $\nu$ is resolved for $f$.

**Case 2** Suppose that $\nu(u), \nu(v)$ are rationally dependent. There exists a sequence of blow ups of 2-curves $\Psi : Y_\nu \rightarrow Y$ such that the center of $\nu$ on $Y_\nu$ is a 1-point.

$Y_\nu$ is the blow up of a toroidal ideal sheaf $J_\nu$ of $\mathcal{O}_Y$. Since $f^{-1}(D_Y) = D_X$, $J_\nu$ is also a toroidal ideal sheaf. By Lemma 2.11 [C5], and induction on the number of 2-curves blown up by $\Psi_\nu$, there exists a sequence of blow ups of 2-curves $\Phi_\nu : X_\nu \rightarrow X$ such that there is a commutative diagram of morphisms

$$
\begin{align*}
X_\nu & \xrightarrow{f_\nu} Y_\nu \\
\Phi_\nu & \downarrow \quad \downarrow \Psi_\nu \\
X & \xrightarrow{f} Y
\end{align*}
$$

Thus $\nu$ is resolved for $f_\nu$.

It follows from compactness of the Zariski Riemann manifold of $X$ [Z] that there exists a positive integer $n$ and commutative diagrams

$$
\begin{align*}
X_i & \xrightarrow{f_i} Y_i \\
\Phi_i & \downarrow \quad \downarrow \Psi_i \\
X & \xrightarrow{f} Y
\end{align*}
$$

for $1 \leq i \leq n$ such that $\Phi_i$ and $\Psi_i$ are products of blow ups of 2-curves, and every valuation $\nu$ of $k(X)$ is resolved for some $f_i$.

$Y_i$ is the blow up of a toroidal ideal sheaf $J_i$ of $\mathcal{O}_Y$ and $X_i$ is the blow up of a toroidal ideal sheaf $I_i$ of $\mathcal{O}_X$. Thus there exists a sequence of blow ups of 2-curves $Y^* \rightarrow Y$ such that $\prod J_i \mathcal{O}_{Y^*}$ is invertible, by Lemma 2.11 [C5]. $Y^*$ is thus the blow up of a toroidal ideal sheaf $J \subset \mathcal{O}_Y$. Thus $J \mathcal{O}_X$ is also a toroidal ideal sheaf. By Lemma 2.11 [C5], there exists a sequence of blow ups of 2-curves $X^* \rightarrow X$ such that
$J \prod J_i \mathcal{O}_{X_i}$ is invertible. Thus for $1 \leq i \leq n$, there exist commutative diagrams of morphisms

$$
\begin{array}{ccc}
X^* & \xrightarrow{f^*} & Y^* \\
\Phi^*_i & \downarrow & \Psi^*_i \\
X_i & \xrightarrow{f_i} & Y_i \\
\downarrow & & \downarrow \\
X & \rightarrow & Y.
\end{array}
$$

Since $\Phi^*_i$ and $\Psi^*_i$ are the blow ups of toroidal ideal sheaves, they are toroidal morphisms.

Suppose that $\nu$ is a 0-dimensional valuation of $k(X)$. If the center of $\nu$ on $Y$ is a 3-point then $f^*$ is resolved at the center of $\nu$ on $X^*$. In particular, if the center of $\nu$ on $Y^*$ is a 3-point, then the center of $\nu$ on $Y$ is a 3-point and $\nu$ is resolved for $f^*$. Suppose that the center of $\nu$ on $Y^*$ is 2-point, and the center of $\nu$ on $Y$ is not a 3-point. Then the center of $\nu$ on $Y_i$ is a 2-point for all $i$. There exists an $i$ such that $\nu$ is resolved for $f_i$. Thus $f_i$ is prepared (and satisfies 2a) of Definition 3.4) at the center of $\nu$ on $X_i$. Since $\Phi^*_i$ and $\Psi^*_i$ are toroidal, $f^*$ is prepared (and satisfies 2a) of Definition 3.4) at the center of $\nu$. Thus $\nu$ is resolved for $f^*$. Since all 0-dimensional valuations of $k(X)$ are resolved for $f^*$, it follows that $f^*$ is toroidal above all 3-points of $Y^*$, and prepared above all 2-point of $Y^*$, and we have achieved the conclusions of the lemma.

Lemma 4.3. Suppose that $f : X \rightarrow Y$ satisfies the conclusions of Lemma 4.2. Suppose that $H$ is a general hyperplane section of $Y$. Then $f$ is prepared above all points of $H$.

Proof. Bertini’s theorem implies that $H$ is nonsingular and makes SNCs with $D_Y$. Further, $H' = f^{-1}(H)$ is nonsingular and makes SNCs with $D_X$. Thus $H$ contains no 3-points of $Y$ and $H'$ contains no 3-points.

Suppose that $q \in H \cap D_Y$ is a 1-point, and $p \in f^{-1}(q)$. Let $u, v, w$ be regular parameters in $\mathcal{O}_{Y,q}$ such that $u = 0$ is a local equation of $D_Y$ at $q$, and $w = 0$ is a local equation of $H$. Then we have regular parameters $x, y, z$ in $\mathcal{O}_{X,p}$ such that either $p$ is a 1-point with $x = 0$ a local equation of $D_X$ or $p$ is a 2-point with $xy = 0$ a local equation of $D_X$ at $p$. We then have an expression $u = x^a \gamma, w = z$ or $u = x^a y^b \gamma, w = z$ where $\gamma$ is a unit in $\mathcal{O}_{X,p}$. Thus $f$ is prepared at $p$.

Corollary 4.4. Suppose that $f : X \rightarrow Y$ satisfies the conclusions of Lemma 4.2. Then there exists a finite set of 1-points $\Omega \subset Y$ such that $f$ is prepared above $Y - \Omega$.

Proof. The locus of points in $X$ where $f$ is prepared is an open set. Since $f$ is proper, the image $\Omega$ of the closed set of points where $f$ is not prepared is closed in $Y$. Since a general hyperplane section of $Y$ is disjoint from $\Omega$ by Lemma 4.3, $\Omega$ must be a finite set of points.

Lemma 4.5. Suppose that $f : X \rightarrow Y$ is a proper dominant morphism of nonsingular 3-folds and $\pi : Y \rightarrow S$ is a smooth dominant morphism onto a nonsingular surface $S$. Let $g = \pi \circ f$. Suppose that $C$ is a nonsingular curve of $S$, $D = \pi^{-1}(C)$ and $D' = f^{-1}(D)$. Suppose that $D'$ is a SNC divisor on $X$ which contains the singular locus of $g$ and the singular locus of $f$. Suppose that $\bar{q} \in C$ is a point, and that $g$ is toroidal and prepared (with respect to $C$ and $D'$) away from points above finitely many points $\Omega = \{q_1, \ldots, q_m\} \subset \gamma = \pi^{-1}(\bar{q})$. Further suppose that $f$ is finite above
a general point of $\gamma$. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\Phi_1} & X \\
g_1 \downarrow & \downarrow g & \downarrow \Phi \\
S & & \end{array}
$$

such that $\Phi_1$ is a product of possible blow ups for the preimage of $D'$ supported above $\Omega$ and $g_1$ is prepared (with respect to $C$ and $\Phi_1^{-1}(D')$) in a neighborhood of all components $F$ of $\Phi_1^{-1}(D')$ which dominate $D$ and in a neighborhood of all components $F$ of $\Phi_1^{-1}(D')$ which dominate a curve of $Y$.

Proof. Let $u, w$ be regular parameters in $O_{S, 2}$ such that $u = 0$ is a local equation of $C$ at $\gamma$. Let $C'$ be the curve on $S$ with local equation $w = 0$ at $\gamma$. Let $A = \pi^{-1}(C')$.

Since it suffices to prove the lemma above a neighborhood of $\gamma$ in $S$, we may assume that $E = C + C'$ is a SNC divisor on $S$ whose only singular point is $\gamma$. Since $g$ is toroidal away from points above $\Omega$, we have that $g^{-1}(E)$ defines a SNC divisor on $X$ away from points above $\Omega$. There exists a morphism $\Phi_1 : X_1 \to X$ which is a sequence of possible blow ups for the preimage of $D'$ supported above $\Omega$ such that with $q_1 = g \circ \Phi_1 : X_1 \to S$, $g_1^{-1}(E)$ is a SNC divisor, and $(f \circ \Phi_1)^{-1}(q_i)$ are divisors for all $q_i \in \Omega$. We may further assume that the union $\overline{A}$ of codimension 1 subvarieties of $X_1$ which dominate $A$ are disjoint, since they are disjoint away from the preimage of $\Omega$.

Let $\overline{D}$ be the union of codimension 1 subvarieties of $X_1$ which dominate $D$, so that $\overline{D}$ is a disjoint union of irreducible components of $D'' = g_1^{-1}(C)$ (by Remark 3.3).

Suppose that $p \in \overline{D}$ and $f \circ \Phi_1(p) = q_i \in \Omega$. Then $p$ must be a 2-point or a 3-point. We have regular parameters $x, y, z$ in $O_{X_1, p}$ such that one of the following cases hold:

1. $p$ is a 2-point and

   $$u = x^ay^b, w = y^c$$

   where $x = 0$ is a local equation of $\overline{D}$, $u = 0$ is a local equation of $D''$ and $a, b > 0$.

2. $p$ is a 2-point,

   $$u = x^ay^b, w = y^ez$$

   where $x = 0$ is a local equation of $\overline{D}$, $u = 0$ is a local equation of $D''$, $a, b, c > 0$ and $z = 0$ is a local equation of $\overline{A}$.

3. $p$ is a 3-point and

   $$u = x^ay^bz^e, w = y^dz^e$$

   where $x = 0$ is a local equation of $\overline{D}$, $u = 0$ is a local equation of $D''$ and $a, b, c, d, e > 0$.

Thus $g_1$ is prepared in a neighborhood of $\overline{D}$.

Now suppose that $F$ is a component of $D''$ which dominates a curve of $Y$ and $p \in F$ is such that $f \circ \Phi_1(p) = q_i \in \Omega$. Then $p$ must be a 2-point or a 3-point. By our assumption that $f$ is finite above a general point of $\gamma$, $F$ dominates the curve $C$ of $S$. Thus we have regular parameters $x, y, z$ in $O_{X_1, p}$ such that one of the following cases hold:

1. $p$ is a 2-point and

   $$u = x^ay^b, w = y^c$$

   where $x = 0$ is a local equation of $F$, $u = 0$ is a local equation of $D''$ and $a, b > 0$. 
2. $p$ is a 2-point,

$$u = x^ay^b, w = y^cz$$

where $x = 0$ is a local equation of $F$, $u = 0$ is a local equation of $D''$, $a, b > 0$ and $z = 0$ is a local equation of $A$.

3. $p$ is a 3-point and

$$u = x^ay^bz^c, w = y^dz^e$$

where $x = 0$ is a local equation of $F$, $u = 0$ is a local equation of $D''$ and $a, b, c > 0$.

Thus $g_1$ is prepared in a neighborhood of $F$. □

Lemma 4.6. Suppose that $f : X \to Y$ is a dominant morphism of nonsingular 3-folds with toroidal structures determined by SNC divisors $D_Y$ and $D_X = f^{-1}(D_Y)$ such that $D_X$ contains the singular locus of $f$. Further suppose that $f : X \to Y$ is toroidal and $q \in Y$ is a 2-point. Let $\Psi : Y_1 \to Y$ be the blow up of $q$. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
$$

such that $\Phi$ is a sequence of possible blow ups for the preimage of $D_X$ supported above $q$ and $f_1$ is toroidal with respect to $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$.

Proof. There exist permissible parameters $u, v, w$ at $q$ such that if $p \in f^{-1}(q)$ then there exist permissible parameters $x, y, z$ for $u, v, w$ such that if $p$ is a 1-point, then we have a form

$$u = x^a, v = x^b(\alpha + y), w = z$$

with $0 \neq \alpha \in k$, and if $p$ is a 2-point,

$$u = x^ay^b, v = x^cy^d, w = z,$$

with $ad - bc \neq 0$. We first show that there exists a sequence of possible blow ups

$$X_m \xrightarrow{\Phi_m} X_{m-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\Phi_1} X$$

obtained by blow ups of possible centers supported above $q$ such that the rational map $X_m \to Y_1$ is toroidal wherever it is defined, and if $\mathcal{I}_q \mathcal{O}_{X_{m-p}}$ is not invertible, then there exist regular parameters $x, y, z$ in $\mathcal{O}_{X_m, p}$ such that one of the following forms hold:

- $p$ is a 1-point

$$u = x^a, v = x^b(\alpha + y), w = x^cz$$

with $\alpha \neq 0$, and $c = 0$ or 1, or $p$ is a 2-point

$$u = x^ay^b, v = x^cy^d, w = xz$$

with $a, c \geq 1$, or $p$ is a 2-point

$$u = x^ay^b, v = x^cy^d, w = xyz$$

with $a, c \geq 1$ and $b, d \geq 1$.

The points $p \in f^{-1}(p)$ such that $u, v, w$ do not have a form (11), (12) or (13) at $p$ are 2-points of one of the following forms:

$$u = x^a, v = y^b, w = z.$$
in which case \( V(x, y, z) \) is the locus in \( \text{spec}(\hat{O}_{X,p}) \) where \( \mathcal{I}_q\hat{O}_{X,p} \) is not invertible,
\[
u = x^a, v = x^b y^c, w = z
\]
with \( b, c > 0 \), in which case \( V(x, z) \) is the locus in \( \text{spec}(\hat{O}_{X,p}) \) where \( \mathcal{I}_q\hat{O}_{X,p} \) is not invertible,
\[
u = x^a y^b, v = x^c y^d, w = z
\]
with \( a, b, c, d > 0 \) in which case \( V(x, z) \cup V(y, z) \) is the locus in \( \text{spec}(\hat{O}_{X,p}) \) where \( \mathcal{I}_q\hat{O}_{X,p} \) is not invertible.

Let \( Z \) be the closed locus of points \( r \) in \( X \) such that \( \mathcal{I}_q\hat{O}_{X,r} \) is not invertible. The isolated points \( p \) in \( Z \) have a form (14). If \( p \) is a non isolated point in \( Z \) which is a 2-point, then \( p \) has a form (15) or (16).

Suppose that \( E \) is a curve in \( Z \) such that \( E \) contains a 2-point \( p \) satisfying (15) or (16). Then a generic point of \( E \) satisfies (8) and all 2-points of \( E \) must have a form (15) or (16).

Let \( \Phi_1 : X_1 \to X \) be the blow up of the finitely many points \( p \in X \) of the form (14). Suppose that \( p \in X \) is such a point, and \( p_1 \in \Phi_1^{-1}(p) \). Without loss of generality, we may assume that \( a \leq b \) in (14). There are regular parameters \( x_1, y_1, z_1 \) in \( \hat{O}_{X_1,p_1} \) of one of the following forms:
\[
x = x_1, y = x_1(y_1 + \alpha), z = x_1(z_1 + \beta)
\]
with \( \alpha, \beta \in k \),
\[
x = x_1y_1, y = y_1, z = y_1(z_1 + \alpha)
\]
with \( \alpha \in k \) or
\[
x = x_1z_1, y = y_1z_1, z = z_1.
\]
Suppose that (17) holds. Then \( u, v, w \) have a form
\[
u = x_1^a, v = x_1^b(y_1 + \alpha)^b, w = x_1(z_1 + \beta)
\]
at \( p_1 \). If \( a = 1 \), then \( f \circ \Phi_1 \) factors through \( Y_1 \) at \( p_1 \) and we have one of the following toroidal forms:

1-point maps to 2-point:
\[
u_1 = u = x_1, v_1 = \frac{v}{u} = x_1^{b-1}(y_1 + \alpha)^b, w_1 = \frac{w}{u} - \beta = z_1
\]
if \( b > a = 1 \) and \( \alpha \neq 0 \),

1-point maps to 1-point:
\[
u_1 = u = x_1, v_1 = \frac{v}{u} - \alpha = y_1, w_1 = \frac{w}{u} - \beta = z_1
\]
if \( b = a = 1 \), \( \alpha \neq 0 \),

2-point maps to 2-point:
\[
u_1 = u = x_1, v_1 = \frac{v}{u} = x_1^{b-1}y_1^b, w_1 = \frac{w}{u} - \beta = z_1
\]
if \( a = 1 \) and \( \alpha = 0 \).

Suppose that (17) holds and \( a > 1 \). If \( \beta \neq 0 \), we have that \( \Phi_1 \circ f \) factors through \( Y_1 \) at \( p_1 \) and we have a toroidal form, obtained from a change of variable in
\[
u_1 = \frac{u}{w} = x_1^{-1}(z_1 + \beta)^{-1}, v_1 = \frac{v}{w} = x_1^{b-1}(z_1 + \beta)^{-1}(y_1 + \alpha)^b, w_1 = w = x_1(z_1 + \beta)
\]
where \( p_1 \) is 1-point mapping to a 3-point if \( \alpha \neq 0 \) and \( p_1 \) is 2-point mapping to a 3-point if \( \alpha = 0 \).
If $\beta = 0$ (and $a > 1$) then we have 
\[ u = x_1^a, \quad v = x_1(y_1 + \alpha), \quad w = x_1 z_1 \]
of the form (11) if $\alpha \neq 0$ and of the form (12) if $\alpha = 0$.
Suppose that (18) holds. Then at $p_1$, $u, v, w$ have a form:
\[ u = x_1^a y_1^a, \quad v = y_1^b, \quad w = y_1(z_1 + \alpha). \]

Assume $b = 1$ (which implies $a = 1$). then $f \circ \Phi_1$ factors through $Y_1$ at $p_1$, and there is a toroidal form:
\[ u_1 = u/v = x_1, \quad v_1 = v = y_1, \quad w_1 = w = z_1 \]
where $p_1$ is 2-point mapping to a 2-point.
Assume that $b > 1$ and $\alpha \neq 0$. Then $f \circ \Phi_1$ factors through $Y_1$ at $p_1$, and there is a toroidal form, obtained from a change of variable in
\[ u_1 = u/w = x_1^a y_1^{a-1}(z_1 + \alpha)^{-1}, \quad v_1 = v/w = y_1^{b-1}(z_1 + \alpha)^{-1}, \quad w_1 = w = y_1(z_1 + \alpha) \]
where $p_1$ is a 2-point mapping to a 3-point.
If $b > 1$ and $\alpha = 0$, then we have a form:
\[ u = x_1^a y_1^a, \quad v = y_1^b, \quad w = y_1 z_1 \]
of the form (12).
Suppose that (19) holds. Then $p_1$ is a 3-point and $u, v, w$ have a form
\[ u = x_1^a z_1^a, \quad v = y_1^b z_1^b, \quad w = z_1. \]
Thus $\Phi_1 \circ f$ factors through $Y_1$ at $p_1$ by
\[ u_1 = u/w = x_1^{a-1}, \quad v_1 = v/w = y_1^{b-1}, \quad w_1 = w = z_1, \]
where $p_1$ is a 3-point mapping to a 3-point.
We have thus completed the analysis of $\Phi_1$.
We now construct (10) by induction. Each $X_i$ will be such that the rational map $X_i \to Y_1$ is toroidal wherever it is defined, and if $p \in X_i$ is a 2-point such that $\mathcal{T}_q \mathcal{O}_{X_i,p}$ is not invertible, then there exist regular parameters $x, y, z$ at $p$ such that $u, v, w$ have one of the forms (11), (12), (13), (15) or (16) at $p$. If a form (15) or (16) holds at $p$, then $\Phi_1 \circ \cdots \circ \Phi_i$ is an isomorphism near $p$.
Each $\Phi_{i+1} : X_{i+1} \to X_i$ for $i \geq 1$ will be the blow up of a curve $E_i$ which is a possible center and is the strict transform of a component of $Z \subset X$.
Suppose that we have constructed (10) out to $X_i$, and $p \in X_i$ is a 2-point such that $\mathcal{T}_q \mathcal{O}_{X_i,p}$ is not invertible, and $u, v, w$ do not have a form (11), (12) or (13) at $p$. Then $u, v, w$ have a form (15) or (16) at $p$. Let $E = E_i$ be a curve in the locus where $\mathcal{T}_q \mathcal{O}_{X_i}$ is not invertible which contains $p$. Let $F$ be the component of $D_{X_i}$ containing $E_i$. We necessarily have $\text{ord}_F w = 0$ and $\text{ord}_F u > 0$, $\text{ord}_F v > 0$. Further, $\Phi_1 \circ \cdots \circ \Phi_i$ is an isomorphism near $p$. Thus $E$ is the strict transform of a component of $Z$.
Suppose that $p' \in E_i$ is another 2-point. Then at $p'$, since $\text{ord}_F w = 0$, $u, v, w$ must have a form (15), (16) or (12), where in this last case, $y = z = 0$ is a local equation of $E$ and $b, d \geq 1$ (since $\text{ord}_F w = 0$, $\text{ord}_F u > 0$ and $\text{ord}_F v > 0$). If $p' \in E_i$ is a 1-point, then $u, v, w$ have a form (8) at $p'$, since $\text{ord}_F w = 0$.
Let $\Phi_{i+1} : X_{i+1} \to X_i$ be the blow up of $E$ and $\Phi_{i+1} = \Phi_1 \circ \cdots \circ \Phi_{i+1}$.
Suppose that $p \in E$ is a 1-point and $p_1 \in \Phi_{i+1}^{-1}(p)$. Then $f \circ \Phi_{i+1}$ is toroidal whenever it is defined, and points above $p$ where $f \circ \Phi_{i+1}$ does not factor through $Y_i$ have a form (11). A detailed analysis of a case including this one is given later in the proof, after (26).
Suppose that \( p \in E \) is a 2-point of the form (16) and \( p_1 \in \Phi_{i+1}^{-1}(p) \).

There are regular parameters \( x_1, y_1, z_1 \) in \( \mathcal{O}_{X_r, p_1} \) of one of the following forms:

\[
x = x_1, \quad z = x_1(z_1 + \alpha)
\]

with \( \alpha \in k \) or

\[
x = x_1z_1, \quad z = z_1.
\]

Suppose that (20) holds. We have that \( p_1 \) is a 2-point, and

\[
u = x_1^a y^b, \quad v = x_1^c y^d, \quad w = x_1(z_1 + \alpha).
\]

If \( \alpha \neq 0 \), we have that \( f \circ \Phi_{i+1} \) factors through \( Y_1 \) at \( p_1 \). We have a form:

\[
u_1 = \frac{u}{w} = x_1^{a-1} y^b(z_1 + \alpha)^{-1}, \quad v_1 = \frac{v}{w} = x_1^{c-1} y^d(z_1 + \alpha)^{-1}, \quad w_1 = x_1(z_1 + \alpha)
\]

at the 2-point \( p_1 \), which maps to a 3-point, and thus is toroidal, after a change of variables.

If \( \alpha = 0 \) in (20), we have

\[
u = x_1^a y^b, \quad v = x_1^c y^d, \quad w = x_1z_1
\]

of the form (12).

If (21) holds, then \( p_1 \) is a 3-point and

\[
u = x_1^a y^b z_1, \quad v = x_1^c y^d z_1, \quad w = z_1.
\]

Thus \( f \circ \Phi_{i+1} \) factors through \( Y_1 \) at \( p_1 \), and we have a toroidal form:

\[
u_1 = \frac{u}{w} = x_1^{a-1} y^b z_1, \quad v_1 = \frac{v}{w} = x_1^{c-1} y^d z_1, \quad w_1 = w = z_1
\]

at the 3-point \( p_1 \), which maps to a 3-point.

The analysis of \( \Phi_{i+1} \) above points (15) and above points satisfying (12) where \( y = z = 0 \) are local equations of \( E \) (and \( b, d \geq 1 \)) is similar. This last case will lead to a form (13). Since \( Z \) has only finitely many components, we inductively construct (10).

There now exists a sequence of blow ups of 2-curves \( X_r \to X_m \) which are supported above \( q \) such that the rational map \( X_r \to Y_1 \) is toroidal where ever it is defined, and if \( \mathcal{I}_q \mathcal{O}_{X_r, p} \) is not invertible, then there there exist permissible parameters \( x, y, z \) at \( p \) for \( u, v, w \) such that one of the following forms hold:

\[
u = x^a, \quad v = x^b(\alpha + y), \quad w = x^d z
\]

with \( 0 \neq \alpha \in k \) and \( d < \min\{a, b\} \) or

\[
u = x^a y^b, \quad v = x^c y^d, \quad w = x^e y^f z
\]

with \( (e, f) < (a, b) \) or \( (c, f) < (c, d) \) or \( (a, f) < (a, b) \).

We accomplish this as follows. We first consider \( u \) and \( v \). Suppose that \( p \in X_m \) is a 2-point such that \( \mathcal{I}_q \mathcal{O}_{X_m, p} \) is not invertible. We have forms

\[
u = x^a y^b, \quad v = x^c y^d, \quad w = x^e y^f z
\]

with \( e + f > 0 \) at 2-points \( p_i \) above \( p \) in the construction of the sequence \( X_r \to X_m \).

At \( p_i \) we have an invariant \( (a - c)(b - d) \). This is a nonnegative integer if and only if \( (a, b) \leq (c, d) \) or \( (c, f) \leq (a, b) \). Further, if \( (a - c)(b - d) < 0 \), then after blowing up the 2-curve \( E \) which has local equations \( x = y = 0 \) at \( p_i \), we obtain that all 2-points above \( p_i \) have a form (24), but \( (a - c)(b - d) \) has increased. Further \( E \) contracts to \( q \) on \( Y \) since \( e + f > 0 \).
After a finite number of blow ups of 2-curves above $X_m$ (which must contract to $q$) we achieve that all 2-points $p_i$ above a 2-point $p \in X_m$ such that $\mathcal{I}_q \mathcal{O}_{X_m, p}$ is not invertible have a form (24) with $(a, b) \leq (c, d)$ or $(c, d) \leq (a, b)$.

We now apply this algorithm to the pairs $u, x^e y^f$ and $v, x^m y^n$ in (24) to construct $X_r \to X_m$.

We will now inductively construct $X_n \to X_r$ so that $\mathcal{I}_q \mathcal{O}_{X_n}$ is invertible everywhere and the morphism $X_n \to Y$ is toroidal. We will construct a sequence of blow ups

$$X_n \to X_{n-1} \to \cdots \to X_r \to X_r,$$

so that each $\Phi_i : X_{i+1} \to X_i$ is the blow up of a nonsingular curve $\lambda_i$ which is a possible center and is contained in the locus where $\mathcal{I}_q \mathcal{O}_{X_i}$ is not invertible. We will have that the rational map $f_i : X_i \to Y$ is toroidal where ever it is defined, and all points $p \in X_i$ where $\mathcal{I}_q \mathcal{O}_{X_i, p}$ is not invertible have a form (22) or (23).

Suppose that we have inductively constructed (25) up to $X_i$ and $\mathcal{I}_q \mathcal{O}_{X_i}$ is not invertible. We will construct $\Phi_i+1 : X_{i+1} \to X_i$.

Inspection of the forms (22) and (23) shows that the locus in $X_i$ where $\mathcal{I}_q \mathcal{O}_{X_i}$ is not invertible is a union of nonsingular curves which are possible centers. For such a curve $\lambda_i$, let $\eta$ be a general point of $\lambda_i$ so that a form (22) holds at $\eta$. Let $A(\lambda_i) = \min\{a, b\} - d > 0$.

Choose a curve $\lambda_i$ which maximizes $A(\lambda)$ on $X_i$. Let $\Phi_i+1 : X_{i+1} \to X_i$ be the blow up of $\lambda_i$. Suppose that $p_i \in \lambda_i$ and $p_i+1 \in \Phi_i+1(\lambda_i)$.

First suppose that $p_i$ has the form (22). Without loss of generality, we may assume that $a \leq b$. There are regular parameters $x_1, y, z_1$ in $\hat{\mathcal{O}}_{X_{i+1}, p_i+1}$ satisfying

$$x = x_1, z = x_1(z_1 + \beta)$$

or

$$x = x_1z_1, z = z_1.$$

Suppose that (26) holds. $p_i+1$ is then a 1-point, and

$$u = x_1^a, v = x_1^b(\alpha + y), w = x_1^{d_1}(z_1 + \beta).$$

If $d + 1 = a = b$ in (28), then $X_{i+1} \to Y_1$ is a morphism near $p_i+1$, which maps $p_i+1$ to a 1-point, and has a toroidal form

$$u_1 = u = x_1^a, v_1 = \frac{v}{u} = x_1^{b-a}(\alpha + y), w_1 = \frac{w}{u} - \beta = z_1.$$

If $d + 1 = a < b$ in (28), then $X_{i+1} \to Y_1$ is a morphism near $p_i+1$, which maps $p_i+1$ to a 2-point, and has a toroidal form

$$u_1 = u = x_1^a, v_1 = \frac{v}{u} = x_1^{b-a}(\alpha + y), w_1 = \frac{w}{u} - \beta = z_1.$$

If $d + 1 < a \leq b$ and $\beta \neq 0$ in (28) then $X_{i+1} \to Y_1$ is a morphism near $p_i+1$, which maps $p_i+1$ to a 3-point, and has a toroidal form obtained from a change of variable in

$$u_1 = \frac{u}{w} = x_1^{a-d-1}(z_1 + \beta)^{-1}, v_1 = \frac{v}{w} = x_1^{b-d-1}(\alpha + y)(z_1 + \beta)^{-1}, w_1 = w = x_1^{d+1}(z_1 + \beta).$$

If $d + 1 < a \leq b$ and $\beta = 0$ then (28) has a form (22) with $d < d + 1 < \min\{a, b\}$.

Suppose that (27) holds. $p_i+1$ is then a 2-point, and

$$u = x_1^a, v = x_1^b(\alpha + y), w = x_1^{d_1}. d_1 + 1.$$

$X_{i+1} \to Y_1$ is thus a morphism near $p_i+1$, which maps $p_i+1$ to a 3-point, and has a toroidal form

$$u_1 = \frac{u}{w} = x_1^{a-d}z_1^{a-d-1}, v_1 = \frac{v}{w} = x_1^{b-d}z_1^{b-d-1}(\alpha + y), w_1 = w = x_1^{d_1}. d_1 + 1.$$
Now suppose that \( p_i \) has the form (23). After possibly interchanging \( u \) and \( v \), we may assume that \( (a, b) < (c, d) \). After possibly interchanging \( x \) and \( y \), we may assume that there are regular parameters \( x_1, y, z_1 \) in \( \mathcal{O}_{X_{i+1}, p_{i+1}} \) satisfying (26) or (27) (so that \( e < a \)).

Suppose that (26) holds. Then \( p_{i+1} \) is a 2-point. We have

\[
\begin{align*}
 u &= x_1^a y^b, \\
v &= x_1^c y^d, \\
w &= x_1^{e+1} y^f (z_1 + \beta).
\end{align*}
\]  

(29)

If \( (e+1, f) = (a, b) \) in (29), then \( X_{i+1} \rightarrow Y_1 \) is a morphism near \( p_{i+1} \), which maps \( p_{i+1} \) to a 2-point, and has a toroidal form

\[
u_1 = u = x_1^a y^b, v_1 = \frac{v}{u} = x_1^{c-a} y^{d-b}, w_1 = \frac{w}{u} - \beta = z_1.
\]

If \( (e+1, f) < (a, b) \) and \( \beta \neq 0 \) in (29), then \( X_{i+1} \rightarrow Y_1 \) is a morphism near \( p_{i+1} \), which maps \( p_{i+1} \) to a 3-point, and has a toroidal form obtained from a change of variable in

\[
u_1 = u = x_1^{a-e-1} y^{b-f} (z_1 + \beta)^{-1}, v_1 = \frac{v}{u} = x_1^{c-e-1} y^{d-f} (z_1 + \beta)^{-1}, w_1 = w = x_1^{e+1} y^f (z_1 + \beta).
\]

If \( (e+1, f) < (a, b) \) and \( \beta = 0 \) then (29) has a form (23) with \( (e, f) < (e+1, f) < (a, b) \).

Suppose that (27) holds. \( p_{i+1} \) is then a 3-point, and

\[
u = x_1^a y^b z_1^a, v = x_1^c y^d z_1^c, w = x_1^e y^f z_1^{e+1}.
\]

\( X_{i+1} \rightarrow Y_1 \) is thus a morphism near \( p_{i+1} \), which maps \( p_{i+1} \) to a 3-point, and has a toroidal form

\[
u_1 = u = x_1^{a-e} y^{b-f} z_1^{a-e-1}, v_1 = \frac{v}{u} = x_1^{c-e} y^{d-f} z_1^{c-e-1}, w_1 = w = x_1^e y^f z_1^{e+1}.
\]

In summary, we have that all points where \( \mathcal{I}_q \mathcal{O}_{X_{i+1}} \) is not invertible have a form (22) or (23) and if \( \lambda_{i+1} \subset \Phi_{i}^{-1}(\lambda_i) \) is a curve such that \( \mathcal{I}_q \mathcal{O}_{X_{i+1}} \) is not invertible along \( \lambda_i \), we have \( 0 < A(\lambda_{i+1}) < A(\lambda_i) \). Thus after a finite number of blow ups, we construct the desired sequence (25), completing the proof of the lemma.

**Lemma 4.7.** Suppose that \( f : X \rightarrow Y \) is a dominant morphism of nonsingular 3-folds with toroidal structures determined by SNC divisors \( D_Y \) and \( D_X = f^{-1}(D_Y) \) such that \( D_X \) contains the singular locus of \( f \). Further suppose that \( f : X \rightarrow Y \) is toroidal and \( C \subset Y \) is a possible center for \( D_Y \) which contains a 1-point. Let \( \Psi : Y_1 \rightarrow Y \) be the blow up of \( C \). Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
\overline{X}_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) is a sequence of possible blow ups for the preimage of \( D_X \) supported above \( C \) and \( f_1 \) is toroidal with respect to \( D_{Y_1} = \Psi^{-1}(D_Y) \) and \( D_{\overline{X}_1} = \Phi^{-1}(D_X) \).

Further, \( \Phi \) has a factorization

\[
\overline{X}_1 = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X
\]

where each \( \Phi_{i+1} : X_{i+1} \rightarrow X_i \) is either the blow up of a section \( E_i \) over \( C \) such that \( \mathcal{I}_C \mathcal{O}_{X_i} \) is not invertible, or \( \Phi_{i+1} : X_{i+1} \rightarrow X_i \) is the blow up of a curve \( E_i \) which maps to a 2-point of \( Y \) and such that \( E_i \) is contained in the locus where \( \mathcal{I}_C \mathcal{O}_{X_i} \) is invertible.
Proof. We follow the algorithm of Lemma 18.17 [C3] to construct Φ.

Suppose that \( q \in C \) and \( p \in f^{-1}(q) \). Then there are permissible parameters \( u, v, w \) for \( D_Y \) in \( \mathcal{O}_{Y,q} \) and regular parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that one of the following cases holds:

- \( q \) is a 2-point and \( p \) is a 2-point,
  \[
  u = x^a y^b, v = x^d y^e, w = z
  \]
  where \( uv = 0 \) is a local equation of \( D_Y \) and \( u = w = 0 \) is a local equation of \( C \).

- \( q \) is a 2-point and \( p \) is a 1-point,
  \[
  u = x^a, v = x^{b(y + \alpha)}, w = z
  \]
  where \( 0 \neq \alpha \in k, uv = 0 \) is a local equation of \( D_Y \) and \( u = w = 0 \) is a local equation of \( C \).

- \( q \) is a 1-point and \( p \) is a 1-point,
  \[
  u = x^a, v = y, w = z
  \]
  where \( u = 0 \) is a local equation of \( D_Y \) and \( u = w = 0 \) is a local equation of \( C \).

We will construct a sequence of morphisms

\[
\cdots \to X_n \xrightarrow{\Phi_n} X_{n-1} \xrightarrow{\Phi_{n-1}} \cdots \to X_1 \xrightarrow{\Phi_1} X
\]

where each \( \Phi_{i+1} \) is the blow up of a nonsingular curve \( E_i \) contained in the locus where \( \mathcal{I}_C \mathcal{O}_{X_i} \) is not invertible, and for each \( q \in C \) and \( p \in (f \circ \Phi_1 \circ \cdots \circ \Phi_i)^{-1}(q) \) such that \( \mathcal{I}_C \mathcal{O}_{X_i,p} \) is not invertible, there are permissible parameters \( u, v, w \) for \( D_Y \) in \( \mathcal{O}_{Y,q} \) and permissible parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that one of the forms (30) - (36) below hold.

- \( q \) a 2-point, \( p \) a 2-point
  \[
  u = x^a y^b, v = x^d y^e, w = x^b y^h z
  \]
  with \( ac - ba \neq 0 \), \( (g, h) < (a, b) \).

- \( q \) a 2-point, \( p \) a 1-point
  \[
  u = x^a, v = x^{b(y + \alpha)}, w = x^d z
  \]
  with \( 0 \neq \alpha \in k, d < a \),

- \( q \) a 1-point, \( p \) a 1-point
  \[
  u = x^a, v = y, w = x^d z
  \]
  with \( d < a \). Further in the locus where the rational map \( X_i \to Y_1 \) is a morphism, \( X_i \to Y_1 \) is toroidal.

Observe that the forms (30), (31) and (32) are special cases of (34), (35) and (36) respectively.

The locus of points where \( \mathcal{I}_C \mathcal{O}_{X_i} \) is not invertible is a union of nonsingular curves which intersect transversally. If \( E \) is a curve in this locus, and \( p' \in E \) is a general point, then \( u, v, w \) have a form (35) or (36) at \( p' \). In either case, we define an invariant

\[
\Omega(E) = a - d > 0.
\]

Let \( \Phi_{i+1} : X_{i+1} \to X_i \) be the blow up of a curve \( E_i \) such that \( \Omega(E_i) \) is maximal.

Suppose that \( p_1 \in E_i, p_2 \in \Phi_{i+1}^{-1}(p_1) \) and \( q = (f \circ \Phi_1 \circ \cdots \circ \Phi_i)(p_1) \).

Suppose that \( p_1 \) has a form (35). Then \( \mathcal{O}_{X_{i+1},p_2} \) has regular parameters \( x_1, y, z_1 \) such that

\[
x = x_1, z = x_1(z_1 + \beta)
\]
with $\beta \in k$ or

$$x = x_1 z_1, \ z = z_1.$$  \hspace{1cm} (38)

Suppose that (37) holds. Then $p_2$ is a 1-point,

$$u = x_1^a, \ v = x_1^b(y + \alpha), \ w = x_1^{d+1}(z_1 + \beta).$$  \hspace{1cm} (39)

If $d + 1 = a$ in (39), then $X_{i+1} \to Y_1$ is a morphism near $p_2$, mapping $p_2$ to a 2-point, and at $p_2$, we have a toroidal form

$$u_1 = u = x_1^a, \ v = x_1^b(y + \alpha), \ w_1 = w/u - \beta = z_1.$$  

If $d + 1 < a$ and $\beta \neq 0$ in (39) then $X_{i+1} \to Y_1$ is a morphism near $p_2$, mapping $p_2$ to a 3-point, and at $p_2$, we have a toroidal form obtained from a change of variable in

$$u_1 = \frac{u}{w} = x_1^{a-d}(z_1 + \beta)^{-1}, \ v = x_1^b(y_1 + \alpha), \ w_1 = w = x_1^{d+1}(z_1 + \beta).$$

If $d + 1 < a$ and $\beta = 0$ in (39), then we have a form (35) with $d$ increased to $d + 1$. The curve $E'$ containing $p_2$ in the locus where $\mathcal{I}_E \mathcal{O}_{X_{i+1}}$ is not invertible satisfies

$$0 < \Omega(E') = a - (d + 1) < \Omega(E).$$

Suppose that (38) holds. Then $p_2$ is a 2-point.

$$u = x_1^a z_1^a, \ v = x_1^b z_1^b(y + \alpha), \ w = x_1^{d+1} z_1^{d+1}.$$  \hspace{1cm} (40)

Further, $X_{i+1} \to Y_1$ is a morphism near $p_2$, mapping $p_2$ to a 3-point, and at $p_2$, we have a toroidal form

$$u_1 = \frac{u}{w} = x_1^{a-d} z_1^{a-d}, \ v = x_1^b z_1^b(y + \alpha), \ w_1 = w = x_1^{d+1} z_1^{d+1}.$$  

There is a similar argument if $p_1$ satisfies (36).

Suppose that $p_1$ has a form (34) and $x = z = 0$ are local equations of $E_i$ (so that $g < \alpha$). $\mathcal{O}_{X_{i+1-p_2}}$ has regular parameters $x_1, y_1, z_1$ satisfying (37) or (38).

Suppose that (37) holds. Then $p_2$ is a 2-point,

$$u = x_1^a y_1^b, \ v = x_1^d y_1^e, \ w = x_1^{a+1} y_1^h(z_1 + \beta).$$  \hspace{1cm} (41)

If $(g + 1, h) = (a, b)$ in (41), then $X_{i+1} \to Y_1$ is a morphism near $p_2$, mapping $p_2$ to a 2-point, and at $p_2$, we have a toroidal form

$$u_1 = u = x_1^a y_1^b, \ v_1 = v = x_1^d y_1^e, \ w_1 = w/u - \beta = z_1.$$  

If $(g + 1, h) < (a, b)$ and $\beta \neq 0$ in (41), then $X_{i+1} \to Y_1$ is a morphism near $p_2$, mapping $p_2$ to a 3-point, and at $p_2$, we have a toroidal form obtained from a change of variable in

$$u_1 = \frac{u}{w} = x_1^{a-g-1} y_1^{b-h} (z_1 + \beta)^{-1}, \ v = x_1^d y_1^e, \ w = w = x_1^{a+1} y_1^h(z_1 + \beta).$$

If $(g + 1, h) < (a, b)$ and $\beta = 0$ in (41), then (41) has the form (34) with $g$ increased to $g + 1$.

Suppose that (38) holds. Then $p_2$ is a 3-point,

$$u = x_1^a y_1^b z_1^a, \ v = x_1^d y_1^e z_1^d, \ w = x_1^{a+1} y_1^h z_1^{d+1}.$$  

$X_{i+1} \to Y_1$ is a morphism near $p_2$, mapping $p_2$ to a 3-point, and at $p_2$, we have a toroidal form

$$u_1 = \frac{u}{w} = x_1^{a-g} y_1^{b-h} z_1^{a-g-1}, \ v = x_1^d y_1^e z_1^d, \ w_1 = w = x_1^{a+1} y_1^h z_1^{d+1}.$$  

By descending induction on $\max(\Omega(E))$, we see that the sequence (33) must terminate after a finite number of blow ups, and we complete the proof of the lemma. }
5. Preparation

In this section we prove Theorem 1.3.

We may assume (after possibly blowing up points on $X$) that $D_X$ is strongly cuspidal.

To prove this theorem, we may assume by Lemmas 4.1 and 4.2 that $f$ is prepared (of type 2 a) of Definition 3.4) above 2 points and toroidal above 3-points of $Y$. By Corollary 4.4, $f$ only fails to be prepared above a finite set of 1-points $\Sigma \subset Y$. Since this reduction involves only blow ups of 2-curves we continue to have the condition that $D_X$ is strongly cuspidal.

Suppose that $q \in \Sigma$. Let $D$ be the component of $D_Y$ containing $q$. There exists a very ample effective divisor $L$ on $Y$ such that $q \not\in L$ and $D + L \sim H$ where $H$ is a very ample effective divisor such that $q \not\in H$. Let $\alpha : Z \to Y$ be the blow up of $q$, with exceptional divisor $E$. We may replace $L$ with a high multiple of $L$ so that $\alpha^*H - E$ is very ample on $Z$. Let $N$ be a general member of $\alpha^*H - E$. By Bertini’s theorem, $N$ is nonsingular, makes SNCs with $D_Z = \alpha^{-1}(D_Y)$, intersects 2-curves of $D_Z$ transversally at general points, does not contain a component of the strict transform on $Z$ of the fundamental locus of $f$, and is disjoint from $\alpha^{-1}(\Sigma - \{q\})$.

Let $M = \alpha(N)$. Then $M \sim H$, $M$ is nonsingular and intersects $D$ transversally in a nonsingular curve $\overline{\gamma}$ which contains $q$, $M$ contains no other points of $\Sigma$, contains no 3-points of $D_Y$, intersects 2-curves of $D_Y$ transversally at general points, does not contain a component of the fundamental locus of $f$ and by Bertini’s theorem, at points which are not above $q$. $f^*(M)$ is nonsingular and $f^*(M) + D_X$ is a SNC divisor.

After possibly replacing $L$ and $H$ with effective divisors linearly equivalent to $L$ and $H$ respectively, we may assume that $\overline{\gamma} \cap (L + H)$ consists of 1-points of $D_Y$ and is disjoint from the fundamental locus of $f$.

$U = Y - (L + H)$ is an affine neighborhood of $q$. Let $\gamma = \overline{\gamma} \cap U$. There exist $\overline{f}, \overline{g} \in \Gamma(Y, \mathcal{O}_Y(H))$ such that $\overline{(f)} = D + L - H$ and $\overline{g} = M - H$. We can thus define a morphism $\pi : U \to S = \mathbb{A}^2$ by $\pi(a) = (\overline{f}(a), \overline{g}(a))$ for $a \in U$. Let $\overline{\pi} = \pi(q)$. $\pi^{-1}(\overline{\gamma}) = \gamma$ (scheme theoretically) so $\pi$ is smooth in a neighborhood of $\gamma$. We may thus replace $U$ with an open neighborhood of $\gamma$ so that $\pi$ is smooth.

Let $\overline{X} = f^{-1}(U)$, and $\overline{f} = f \mid \overline{X}$. Let $D_U = D_Y \cap U$, $D_U^* = D \cap U$, $D_{\overline{X}}^* = \pi(D_U^*)$, $g = \pi \circ \overline{f} : \overline{X} \to S$, $D_{\overline{X}} = g^{-1}(D_{\overline{S}})$, $D_{\overline{X}} = D_X \cap \overline{X}$. The map $\pi$ is toroidal with respect to $D_{\overline{S}}^*$ and $D_{\overline{U}}^*$.

Let $D_{1}, \ldots, D_m$ be the components of $D_Y$ other than $D$ which intersect $\gamma$. Since $\gamma$ intersects these components transversally, we may assume then that $\pi \mid D_i \cap U$ is étale onto its image for $1 \leq i \leq m$. We further may assume that $\Sigma \cap U = \{q\}$, and (by Bertini’s theorem) for $\overline{q} \in D_{\overline{S}}^* - \{\overline{\gamma}\}$, there exist regular parameters $u, w$ at $\overline{q}$ such that $u = 0$ is a local equation of $D_{\overline{S}}^*$, and if $E$ is the curve $w = 0$ on $S$, then $E$ is nonsingular, $D_{\overline{S}}^* + E$ is a SNC divisor, $g^{-1}(E)$ is nonsingular, and $g^{-1}(E) + D_{\overline{X}}$ is a SNC divisor on $\overline{X}$. Thus if $q' \in \pi^{-1}(\overline{q})$, there exist permissible parameters $u, v, w$ in $\mathcal{O}_{U, q'}$ (for $D_U$) such that if $p \in \overline{f}^{-1}(\overline{q'})$ then there exist regular parameters $x, y, z$ in $\mathcal{O}_{\overline{X}, p}$ such that

$$u = x^ay^b, w = z$$ \hspace{1cm} (42)

where $u = x^ay^b = 0$ is a local equation of $D_{\overline{X}}$ at $p$ (with $a > 0, b \geq 0$) if $q' \in D - \cup D_i$ and

$$u = x^ay^b, v = x^cy^d, w = z$$ \hspace{1cm} (43)
where \( \gamma \in \mathcal{O}_{X,p} \) is a unit and \( uv = x^{a+c}y^{b+d} = 0 \) is a local equation of \( D_X \) at \( p \) if \( q' \in D \cap D_i \) for some \( i \).

Since \( \gamma \) intersects the 2-curves \( D_i \cap D \cap U \) of \( U \) at general points of the 2-curves, after possibly replacing \( U \) with a smaller open neighborhood of \( \gamma \), we have that the intersection of the fundamental locus of \( f \) with \( U \) is contained in \( D \cap U \).

We will now establish that \( g \) is toroidal and prepared with respect to \( D^*_S \) and \( D^*_X \) away from the preimages of finitely many 1-points \( \Omega \subset \gamma \) of \( D_U \).

Suppose that \( q' \in (D_i - D) \cap U \), \( p \in \overline{f}^{-1}(q') \), and \( \overline{q}' = \pi(q') \), which implies that there exist regular parameters \( u, w \) at \( \overline{q}' \), \( u, v, w \) at \( q' \) such that \( v = 0 \) is a local equation of \( D_i \). \( q' \) is not in the fundamental locus of \( \overline{f} \), and \( q' \) is a 1-point of \( D_U \), so by Abhyankar’s lemma there exist regular parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that

\[
q = x, v = y^b, w = z.
\]

Suppose that \( q' \in (D - \gamma) \cap U \), \( p \in \overline{f}^{-1}(q') \), \( \overline{q}' = \pi(q') \). Then we have a form \((42)\) or \((43)\) at \( p \), so that \( q \) is prepared and toroidal for \( D^*_S \) and \( D^*_X \) at \( p \).

Let \( \delta = D \cap D_i \cap U \) for some \( 1 \leq i \leq m \). Suppose that \( q' \in \delta \cap \gamma \) and \( p \in \overline{f}^{-1}(q') \). Then \( \pi(q') = \overline{q} \). Recall that \( q' \) is a general point of the 2-curve \( \delta \). There exist regular parameters \( u, w \) in \( \mathcal{O}_{S}, \pi \) such that \( u = 0 \) is a local equation of \( D \) on \( U \), \( w = 0 \) is a local equation of \( M \) on \( U \), and there exists \( v \in \mathcal{O}_{U,q'} \) such that \( v = 0 \) is a local equation of \( D_i \) and \( u, v, w \) are regular parameters in \( \mathcal{O}_{U,q'} \). By our choice of \( M, \overline{f}(M) \) is nonsingular and makes SNCs with \( D_X \) at \( p \). Since \( uv = 0 \) is a local equation of \( D_X \) at \( p \), and \( w = 0 \) is a local equation of \( \overline{f}^*(M) \) at \( p \), there exist permissible parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that

\[
u = x^a y^b \gamma_1, v = x^c y^d \gamma_2, w = z
\]

with \( \gamma_1, \gamma_2 \) units in \( \mathcal{O}_{X,p} \). Thus \( q \) is prepared and toroidal for \( D^*_S \) and \( D^*_X \) at \( p \).

Suppose that \( q' \in \gamma \) is a general point. Then \( q' \) is a 1-point of \( D_U \) and \( \overline{f} \) is finite above \( q' \). There exist regular parameters \( u, v, w \) in \( \mathcal{O}_{U,q'} \) such that \( u, w \) are permissible parameters for \( D^*_S \) at \( \overline{q} = \pi(q') \), and if \( p \in \overline{f}^{-1}(q') \), then there exist permissible parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that

\[
u = x^a, v = y, w = z
\]

by Abhyankar’s lemma, which implies that \( q \) is prepared and toroidal at \( p \) for \( D^*_S \) and \( D^*_X \).

We conclude that \( g \) is toroidal and prepared with respect to \( D^*_S \) and \( D^*_X \) away from points above finitely many 1-points \( \Omega \subset \gamma \) of \( D_U \).

Recall that there are no 3-points of \( X \) supported above \( D_i \cap U \) for \( 1 \leq i \leq m \). After blowing up points supported above \( \Omega \), we obtain that the irreducible components \( F \) of \( D_X \) which do not contain a 3-point are precisely the components which dominate \( D_i \) for some \( i \) or dominate \( D_i \cap D \) for some \( i \), and the 2-curves \( C \) of \( D_X \) which do not contain a 3-point are precisely the 2-curves which dominate a 2-curve \( D_i \cap D \).

Suppose that \( \Lambda : Z \to U \) is a dominant morphism of 3-folds, and \( D_Z \) is a SNC divisor on \( U \). We will say that \( D_Z \) is \( U \) cuspidal if all irreducible components \( F \) of \( D_Z \) which do not contain a 3-point dominate \( D_i \) for some \( i \), or dominate \( D_i \cap D \) for some \( i \), and the 2-curves \( C \) of \( D_Z \) which do not contain a 3-point dominate a 2-curve \( D_i \cap D \).
By Lemma 4.5, there exists a morphism $\Phi_1 : \overline{X}_1 \to \overline{X}$ such that $\Phi_1$ is a sequence of possible blow ups for the preimage of $D_{X_1}^*$ of points and nonsingular curves supported above $\Omega$ such that if $g_1 = g \circ \Phi_1 : \overline{X}_1 \to S$ and $f_1 = \overline{f} \circ \Phi_1 : \overline{X}_1 \to U$, then $g_1$ is prepared for $D_S^*$ and $D_{X_1}^* = \Phi_1^{-1}(D_{X_1}^*)$ in a neighborhood of all components of $D_{X_1}^*$ which do not map to a point of $\Omega$.

By blowing up points on components of $D_{X_1}^*$ which dominate a point of $\Omega$, we may suppose that $D_{X_1}^* = \Phi_1^{-1}(D_{\overline{X}})$ is $U$ cuspidal.

By 1 of Theorem 3.1 [C5], there exists a morphism $\Phi_2 : \overline{X}_2 \to \overline{X}_1$ which is a sequence of possible blow ups for the preimage of $D_{X_1}^*$ of points and nonsingular curves supported above $\Omega$, such that $g_2 = \pi \circ f_1 \circ \Phi_2 : \overline{X}_2 \to S$ is prepared for $D_S^*$ and $D_{X_2}^* = \Phi_2^{-1}(D_{X_2}^*)$. Let $f_2 = f_1 \circ \Phi_2 : \overline{X}_2 \to U$. We further have that $D_{X_2}^* = \Phi_2^{-1}(D_{\overline{X}_2})$ is $U$ cuspidal.

Now by 2 of Theorem 3.1 [C5], there exists a commutative diagram

$$\begin{array}{ccc}
\overline{X}_3 & \xrightarrow{g_3} & S_1 \\
\overline{\Phi}_3 \downarrow & & \downarrow \lambda_1 \\
\overline{X}_2 & \xrightarrow{g_2} & S
\end{array}$$

such that $\lambda_1$ is a sequence of possible blow ups for the preimage of $D_S^*$ of points supported above $\overline{\gamma}_3$, $\overline{\Phi}_3$ is a sequence of possible blow ups for the preimage of $D_{X_1}^*$ of points and nonsingular curves supported above $\gamma$, and $g_3$ is toroidal with respect to $D_{X_1}^* = \lambda_1^{-1}(D_S^*)$ and $D_{X_2}^* = \Phi_3^{-1}(D_{X_2}^*)$. We further have that $D_{X_2}^*$ is $U$ cuspidal.

Let $f_3 = f_2 \circ \overline{\Phi}_3 : \overline{X}_3 \to U$.

Consider the commutative diagram

$$\begin{array}{ccc}
\overline{X}_3 & \xrightarrow{\overline{f}_3} & \overline{X}_1 \\
\overline{\Phi} \downarrow & & \downarrow \pi_1 \\
\overline{X} & \xrightarrow{\overline{f}} & U \\
\overline{\Phi}_3 \downarrow & & \downarrow \lambda_1 \\
\overline{\Phi}_3 \circ \overline{\Phi}_3 \downarrow & & \downarrow \lambda_1 \\
\overline{X}_2 & \xrightarrow{g_2} & S
\end{array}$$

where $\overline{\Phi} = \Phi_1 \circ \Phi_2 \circ \overline{\Phi}_3$, $\overline{X}_1 = U \times S_1$ and $\overline{\Phi}_1 : \overline{X}_1 \to U$, $\pi_1 : \overline{X}_1 \to S_1$ are the natural projections, and $\overline{f}_3 = f_3 \times g_3$. $D_{\overline{X}_1}^* = \overline{\Phi}_1^{-1}(D_U)$ and $D_{\overline{X}_1}^* = \overline{\Phi}_1^{-1}(D_U)$ are SNC divisors. $\overline{X}_1$ is nonsingular, and is obtained from $U$ by possible blow ups for the preimage of $D_U$ of sections over $\gamma$. Since $g_3$ is toroidal with respect to $D_{X_1}^*$ and $D_{X_2}^*$, $\overline{f}_3$ is prepared with respect to $D_{\overline{X}_1}^*$ and $D_{\overline{X}_2}^*$. Over a general point of $\gamma$, $\overline{f}_3$ is toroidal with respect to $D_{\overline{X}_1}^*$ and $D_{\overline{X}_2}^*$. Also, over a general point of $\gamma$, $\overline{\Phi}$ is a sequence of possible blow ups for the preimages of $D_{X_1}^*$ of sections over $\gamma$.

Recall that $D_{U_1} = D + D_1 + \cdots + D_m + G$, where $G$ consists of the components of $D_{U_1}$ disjoint from $U$, and that $D_i \cap U$ are étale over their images in $S$. Let $D_i$ be the strict transform of $D_i$ on $\overline{X}_1$ for $1 \leq i \leq m$.

$$D_{\overline{X}_1}^* = \overline{\Phi}_1^{-1}(D_U) = D_{\overline{X}_1}^* + D_{\overline{X}_1} + \cdots + D_m.
$$

Let $D_{\overline{X}_3} = \overline{\Phi}_3^{-1}(D_{\overline{X}})$.

We will now verify that $D_{\overline{X}_3}$ is a $U$ cuspidal SNC divisor on $\overline{X}_3$ and that $\overline{f}_3$ is prepared for $D_{\overline{X}_1}$ and $D_{\overline{X}_3}$. Since $\overline{f}_3$ is prepared for $D_{\overline{X}_1}^*$ and $D_{\overline{X}_3}^*$, we need only verify that $\overline{f}_3$ is prepared for $D_{\overline{X}_1}$ and $D_{\overline{X}_3}$ at points $p' \in \overline{X}_3$ such that $q' = \overline{f}_3 \circ \overline{\Phi}(p') \in D_i$ for some $i$.

First suppose that $q' \in D_i - \gamma$ for some $i$. Then $\overline{\Phi}$ and $\overline{\Phi}_3$ are isomorphisms near $p'$ and $q'$ respectively. Suppose that $q' \not\in D$. Then we have permissible parameters
$v, u, w$ for $D_U$ at $q'$ which have an expression (44) at $p'$. Thus $\overline{\mathcal{F}}_3$ has an expression 3 of Definition 3.4 at $p'$. Suppose that $q' \in D \cap D_1 - \gamma$. Then $q'$ is a 2-point of $D_U$, so that $\overline{\mathcal{F}}$ is prepared above $q'$ for $D_U$ and $D_{X}$.

Suppose that $q' = \overline{\mathcal{F}} \circ \overline{\Phi}(p') \in \gamma \cap D_1$ for some $i$. Without loss of generality, we may assume that $D_i = D_1$. Recall that $q' \in \gamma \cap D_1$ is a general point of the 2-curve $D \cap D_1$, $\overline{\mathcal{F}}$ is prepared above $q'$ and $\overline{\mathcal{T}}(M)$ is nonsingular and makes SNCs with $D_{X}$ above $q'$. Since $q' \in D \cap D_1$ is a general point, there are no 3-points in $\overline{\mathcal{F}}^{-1}(q')$. Let $D'_1$ be the reduced divisor on $X$ whose components dominate $D_1$. The irreducible components of $D'_1$ are disjoint by Remark 3.3.

There exist permissible parameters $u, v, w$ in $\mathcal{O}_{U, q'}$ for the two point $q'$ of $D_U$ such that $u = 0$ is a local equation of $D$, $v = 0$ is a local equation of $D_1$, $w = 0$ is a local equation of $M$ on $U$, and $u, w$ are regular parameters in $\mathcal{O}_{X, q}$ such that if $p = \overline{\Phi}(p') \in \overline{\mathcal{F}}^{-1}(q')$, then there exist regular parameters $x, y, z$ in $\hat{\mathcal{O}}_{X, p}$ such that one of the following prepared forms for $\overline{\mathcal{F}}$ hold at $p$. $u, w$ are toroidal forms for $D_U$ and $D_{X}$ in all cases.

1. $p$ is a 1-point of $D_{X}$

\[ u = x^a \]
\[ v = x^b \gamma \]
\[ w = z \]

where $\gamma \in \hat{\mathcal{O}}_{X, p}$ is a unit and $x = 0$ is a local equation of $D_{X}$.

2. $p$ is a 2-point of $D_{X}$ which is not on $D'_1$

\[ u = x^a y^b \]
\[ v = x^c y^d \gamma \]
\[ w = z \]

with $a, b > 0$, $\gamma \in \hat{\mathcal{O}}_{X, p}$ is a unit and $xy = 0$ is a local equation of $D_{X}$.

3. $p$ is a 2-point which is on $D'_1$

\[ u = x^a \]
\[ v = x^b y^c \]
\[ w = z \]

where $xy = 0$ is a local equation of $D_{X}$ and $y = 0$ is a local equation of $D'_1$.

$\overline{\Phi}$ is the sequence of monodial transforms induced by a sequence of quadratic transforms,

\[ S_1 = \overline{S}_n \to \cdots \to \overline{S}_0 = S. \]

Each map $\overline{S}_{j+1} \to \overline{S}_j$ is the blow up of the ideal sheaf $m_j$ of a point $\overline{q}_j$ above $\overline{q}$. Let

\[ \overline{\mathcal{T}}_1 = \overline{Y}_n \to \cdots \to \overline{Y}_0 = U \]

be the induced factorization of $\overline{\Phi}$, where $\overline{\Phi}_{j+1} : \overline{Y}_{j+1} = U \times_S \overline{S}_{j+1} \to \overline{Y}_j = U \times_S \overline{S}_j$, is the blow up of a curve $C_j$. Let $\overline{\pi}_j : \overline{Y}_j \to \overline{S}_j$ be the natural projection.

$\overline{\Phi}$ is a sequence of morphisms

\[ \overline{X}_3 = \overline{X}_n \to \cdots \to \overline{X}_0 = \overline{X}_2 \to \overline{X}. \]

where $\overline{\Phi}_{j+1} : \overline{X}_{j+1} \to \overline{X}_j$ is a principalization of $m_j \mathcal{O}_{\overline{X}_j}$, with natural morphism $\overline{f}_j : \overline{X}_j \to \overline{Y}_j$. Let $D_{\overline{X}_j}, D_{\overline{Y}_j}$ be the respective preimages of $D_U$, and let $D^*_j, D^*_{\overline{S}_j}$ be the respective preimages of $D^*_U$. Let $D^*_{\overline{X}_j}$ be the preimage of $D^*_S$ in $\overline{S}_j$. The
principalizations $\hat{\Phi}_j$ are explicitly described in the proof of Theorem 3.1 [C5]. We have a factorization

$$\hat{\Phi}_{j+1} = \hat{\Phi}_{n,j} \to \cdots \to \hat{\Phi}_{0,j} = \hat{\Phi}_j.$$  \tag{49}$$

where each $\hat{\Phi}_{i+1,j} : \hat{X}_{i+1,j} \to \hat{X}_{i,j}$ is the blow up of a single curve or point $E_{ij}$ which is a possible center for the preimage $D^*_{ij}$ of $D^*_{ij}$ on $\hat{X}_{i,j}$. If $E_{ij}$ is a curve, then $E_{ij}$ is in the locus where $m_{ij}O_{\hat{X}_{i,j}}$ is not locally principal. If $E_{ij}$ is a point, then $E_{ij}$ is in the support of $m_jO_{\hat{X}_{i,j}}$ and $m_jO_{\hat{X}_{i,j},E_{ij}}$ is locally principal. Further, as is shown in the proof of Theorem 3.1 [C5], $D^*_{ij}$ is $U$ cuspidal (this is the reason for the point blow ups). Let $D_{\hat{X}_{ij}}$ be the preimage of $D_U$ on $\hat{X}_{ij}$.

Recall that we have a fixed choice of regular parameters $u = u_0, v, w = v_0$ in $O_{U,q^j}$, which are permissible parameters for $D_U$ at the 2-point $q^j$, and one of the forms (45) - (47) holds at all points of $\hat{X}$ above $q^j$.

Suppose by induction that $D_{\hat{X}_j}$ is a $U$ cuspidal SNC divisor, $\hat{f}_j : \hat{X}_j \to \hat{Y}_j$ is prepared for $D_{\hat{f}_j}$ and $D_{\hat{X}_j}$, and if $q_j \in \hat{Y}_j$ and $\hat{\Psi}_1 \circ \cdots \circ \hat{\Psi}_j(q_j) = q^j$, then

1. $q_j$ is a 2-point or a 3-point of $D_{\hat{y}_j}$ and there exist regular parameters $u_j, w_j$ in $O_{\hat{X}_j,q_j}$, such that $u_j, v, w_j$ are permissible parameters for $D_{\hat{X}_j}$ in $O_{\hat{X}_j,q_j}$.

2. If $q_j \in C_j$, then $u_j = w_j = 0$ are local equations of $C_j$ at $q_j$.

3. If $p_j \in f_j^{-1}(q_j)$, then there exist regular parameters $x_j, y_j, z_j$ in $\hat{O}_{\hat{X}_j,p_j}$ such that one of the following forms hold:

**Case 1.** $q_j$ is a 2-point of $D_{\hat{y}_j}$, and $u_j v = 0$ is a local equation of $D_{\hat{y}_j}$ (so that $\pi_j(q_j) = \pi_j$ is a 1-point), and $p_j$ is a 1-point of $D_{\hat{X}_j}$ with

$$u_j = x_j^a, v = x_j^b, w_j = z_j \tag{50}$$

where $x_j = 0$ is a local equation of $D_{\hat{X}_j}$, $\gamma_j$ is a unit in $\hat{O}_{\hat{X}_j,p_j}$ or $p_j$ is a 2-point of $D_{\hat{X}_j}$ with

$$u_j = x_j^a y_j^b, v = x_j^c y_j^d, w_j = z_j \tag{51}$$

where $x_j y_j = 0$ is a local equation of $D_{\hat{X}_j}$, $\gamma_j$ is a unit in $\hat{O}_{\hat{X}_j,p_j}$.

**Case 2.** $q_j$ is a 3-point of $D_{\hat{y}_j}$, and $u_j v w_j = 0$ is a local equation of $D_{\hat{y}_j}$ (so that $\pi_j(q_j) = \pi_j$ is a 2-point), and $p_j$ is a 1-point of $D_{\hat{X}_j}$ with

$$u_j = x_j^a, v = x_j^b \gamma_j, w_j = x_j^c (z_j + \beta) \tag{52}$$

where $x_j = 0$ is a local equation of $D_{\hat{X}_j}$, $\gamma_j$ is a unit in $\hat{O}_{\hat{X}_j,p_j}$ and $0 \neq \beta \in k$ or $p_j$ is a 2-point of $D_{\hat{X}_j}$ with

$$u_j = x_j^a z_j^b, v = x_j^c z_j^d, w_j = x_j^e z_j^f \tag{53}$$
where $af - be \neq 0$, $x_jz_j = 0$ is a local equation of $D_{\tilde{X}_j}$, $\gamma_j$ is a unit in $\tilde{O}_{\tilde{X}_j, p_j}$ or $p_j$ is a 2-point of $D_{\tilde{X}_j}$ with

$$u_j = (x_j^a y_j^b)^k, v = x_j^d y_j^e \gamma_j, w_j = (x_j^a y_j^b)^l (z_j + \beta)$$  \hspace{1cm} (54)$$

where $x_jy_j = 0$ is a local equation of $D_{\tilde{X}_j}$, $\gamma_j$ is a unit in $\tilde{O}_{\tilde{X}_j, p_j}$, $\gcd(a, b) = 1$ and $0 \neq \beta \in k$ or $p_j$ is a 3-point of $D_{\tilde{X}_j}$ with

$$u_j = x_j^a y_j^b \gamma_j, v = x_j^d y_j^e \gamma_j, w_j = x_j^g y_j^h \gamma_j$$  \hspace{1cm} (55)$$

where $x_jy_jz_j = 0$ is a local equation of $D_{\tilde{X}_j}$, $\gamma_j$ is a unit in $\tilde{O}_{\tilde{X}_j, p_j}$ and

$$\text{rank} \begin{pmatrix} a & b & c \\ g & h & i \end{pmatrix} = 2.$$ 

We will prove that the above statements hold for $\tilde{f}_{j+1} : \tilde{X}_{j+1} \rightarrow \tilde{Y}_{j+1}$.

Suppose that $q_j \in C_j$ is a 2-point (and $\tilde{\Psi}_1 \circ \cdots \circ \tilde{\Psi}_j(q_j) = q'$), so that Case 1 holds, and $p_j \in \tilde{f}_{j-1}^{-1}(q_j)$.

If $C_j, \tilde{O}_{\tilde{X}_j, p_j}$ is not invertible, and $u_j, w_j$ satisfy (185) [C3] at $p_j$ if (50) holds, $u_j, w_j$ satisfy (190) [C3] at $p_j$ if (51) holds and $a, b > 0$ (so that $p_j$ is a 2-point of $D_{\tilde{X}_{j+1}}$), $u_j, w_j$ satisfy (185) [C3] at $p_j$ if (51) holds and $b = 0$ (so that $p_j$ is a 1-point of $D_{\tilde{X}_j}$).

The algorithm of Lemma 18.17 [C3] (as modified after (23) in the proof of Theorem 3.1 of [C5] by adding appropriate point blow ups to ensure that $D_{\tilde{X}_{j+1}}$ is $U$ cuspidal) is applied to construct $\tilde{\Phi}_{j+1} : \tilde{X}_{j+1} \rightarrow \tilde{X}_j$ and $\tilde{f}_{j+1} : \tilde{X}_{j+1} \rightarrow \tilde{Y}_{j+1}$ above $q_j$. Suppose that $q_{j+1} \in \tilde{\Psi}^{-1}_{j+1}(q_j)$, and $\pi_{j+1}(q_{j+1}) = \pi_{j+1}^\prime(q_{j+1}) = \tilde{\Psi}^{-1}_{j+1}(q_j)$.

Then there exist regular parameters $u_{j+1}, w_{j+1}$ in $\tilde{O}_{\tilde{X}_{j+1}, \pi_{j+1}}$ such that $u_{j+1}, v, w_{j+1}$ are regular parameters in $\tilde{O}_{\tilde{X}_{j+1}, q_{j+1}}$ and one of the following forms hold:

$$\tilde{q}_{j+1}^\prime = \text{a 1-point of } D_{\tilde{X}_{j+1}}$$

$$u_j = u_{j+1}, w_j = u_{j+1}(w_{j+1} + \alpha)$$  \hspace{1cm} (56)$$

with $\alpha \in k$, or $\tilde{q}_{j+1}^\prime$ is a 2-point for $D_{\tilde{X}_{j+1}}$

$$u_j = u_{j+1} w_{j+1}, w_j = w_{j+1}.$$  \hspace{1cm} (57)$$

If (56) holds at $\tilde{q}_{j+1}^\prime$ and $p_{j+1} \in \tilde{f}_{j+1}^{-1}(q_{j+1})$, then an analysis of the algorithm of Lemma 18.17 [C3] and Theorem 3.1 [C5] shows that $u_{j+1}, v, w_{j+1}$ satisfy one of the forms (50) or (51) at $p_{j+1}$.

If (57) holds at $\tilde{q}_{j+1}^\prime$, and $p_{j+1} \in \tilde{f}_{j+1}^{-1}(q_{j+1})$, then $u_{j+1}, v, w_{j+1}$ satisfy one of the forms (52) - (55) at $p_{j+1}$.

If $q_{j+1} \in C_{j+1}$, then $u_{j+1} = w_{j+1} = 0$ are local equations of $C_{j+1}$.

Now suppose that $q_j \in C_j$ is a 3-point (and $\tilde{\Psi}_1 \circ \cdots \circ \tilde{\Psi}_j(q_j) = q'$), so that Case 2 holds, and $p_j \in \tilde{f}_{j-1}^{-1}(q_j)$.

If $C_j, \tilde{O}_{\tilde{X}_j, p_j}$ is not invertible, then after possibly interchanging $u_j$ and $w_j$, then we have one of the following forms. $u_j, w_j$ satisfy (187) [C3] at $p_j$ if (53) holds and $a, b > 0$, $u_j, w_j$ satisfy (191) [C3] at $p_j$ if (53) holds and $b = 0$ (in both cases, $p_j$ is a 2-point of $D_{\tilde{X}_j}$), $u_j, w_j$ satisfy (187) or (191) [C3] if (55) holds (so that $p_j$ is a 2 point of $D_{\tilde{X}_j}$), $u_j, w_j$ satisfy (193), (194) or (195) [C3] at $p_j$ if (55) holds.

The algorithm of Lemma 18.18 [C3] is then applied to construct $\tilde{\Phi}_{j+1} : \tilde{X}_{j+1} \rightarrow \tilde{X}_j$ and $\tilde{f}_{j+1} : \tilde{X}_{j+1} \rightarrow \tilde{Y}_{j+1}$ above $q_j$. Suppose that $q_{j+1} \in \tilde{\Psi}^{-1}_{j+1}(q_j)$, and $\pi(q_{j+1}) = \pi_{j+1}^\prime(q_{j+1}) = \pi_{j+1}^\prime(p_{j+1}) = \pi_{j+1}^\prime(q_{j+1})$.

...
In either case, all 2-curves of $D$, property. All points blown up in the construction of $\tilde{\Phi}$ are 2-points which are disjoint from $D$ or are 2-points which are disjoint from $D$. Suppose that $E$ contains a 1-point of $D$, cuspidal, and we know that $D$ is prepared for $X$, and that $\tilde{\Phi} = \tilde{\Phi}_j + 1$ is a SNC divisor above $q_j$. We will now verify that $D_{\tilde{X}_j}$ is a curve. We may thus assume that the center $E_{i,j}$ blown up by $\tilde{\Phi}_{i+1,j}$ is a curve.

If $E_{i,j}$ contains a 1-point of $X$, then $E_{i,j}$ intersects $D_{\tilde{X}_j}$ transversally at 2-points of $D_{\tilde{X}_j}$, and thus all 2-curves of $D_{\tilde{X}_j}$ contained in $D_{\tilde{X}_j}$ contain a 3-point. Suppose that $E_{i,j}$ is a 2-curve of $D_{\tilde{X}_j}$, and $\Lambda$ is an irreducible component of $D_{\tilde{X}_j}$. Then either $E_{i,j}$ is contained in $\Lambda$, so that $\Lambda$ contains a 3-point of $E_{i+1,j}$, as $D_{\tilde{X}_j}$ is by assumption $U$ cuspidal, or else $E_{i,j}$ intersects $\Lambda$ transversally at 3-points of $D_{\tilde{X}_j}$. In either case, all 2-curves of $D_{\tilde{X}_j}$ contained in $D_{\tilde{X}_j}$ contain a 3-point.

We conclude that $D_{\tilde{X}_j}$ is $U$ cuspidal.

We have thus established that $\tilde{f}_j$ is prepared for $D_{\tilde{X}_j}$ and $D_{\tilde{X}_j}$, and $D_{\tilde{X}_j}$ is $U$ cuspidal.

Recall that $\tilde{\Phi}_{i+1} : \tilde{X}_{i+1} \to \tilde{X}_i$ is a principalization of $m_i \mathcal{O}_{\tilde{X}_i}$ which in a neighborhood of a general point of $\gamma$ is a sequence of blow ups of sections over $\gamma$ where $m_i \mathcal{O}_{\tilde{X}_i}$ is not invertible.

Each $\tilde{\Psi}_{i+1} : \tilde{Y}_{i+1} \to \tilde{Y}_i$ is the blow up of a curve $C_i$ which is a section over $\gamma$ and is a possible center for $D_{\tilde{Y}_i}$.
We will construct $\Psi_1 : Y_1 \to Y$ such that $\Psi_1^{-1}(U) \cong \Psi_1^{-1}(U)$, $\Psi_1 | \Psi_1^{-1}(U) = \Psi$ and $\Psi_1^{-1}(D_Y)$ is a SNC divisor by constructing a sequence of morphisms
\[
Y_1 = \hat{Y}_n \circ \cdots \circ \hat{Y}_1 \to \cdots \to \Psi_1 \to Y
\] (61)
where each $\hat{Y}_{i+1}$ is a product of blow ups of possible centers for the preimage of $D_Y$, and $\Psi_1^{-1}(Y_i) \cong \hat{Y}_{i+1}$, $\hat{Y}_{i+1} | \hat{Y}_{i+1} = \Psi_{i+1}$ for all $i$.

We will inductively construct (61). Suppose that we have constructed $\hat{\Psi}_i : \hat{Y}_i \to \hat{Y}_{i-1}$.

Let $\gamma_i$ be the Zariski closure of $C_i$ in $\hat{Y}_i$. Then $\gamma_{i+1}$ is a section over $\gamma$, and is thus a nonsingular curve. We construct $\hat{\Psi}_{i+1}$ by first blowing up points on (the strict transform of) $\gamma_i$ above $\gamma - \gamma$ where (the strict transform of) $\gamma_i$ does not make SNCs with (the preimage of) $D_{Y_i}$, and then blowing up the strict transform of $\gamma_i$.

$X_2 \to X$ is an isomorphism away from the preimage of $\Omega$. Thus the sequence of blow ups $X_2 \to X$ extends trivially to a morphism $X_2 \to X$, so that $X_2 \to X$ is an isomorphism away from the preimage of $\Omega$.

Now we construct $\Phi_2 : X_3 \to X_2$ such that $\Phi_3^{-1}(X_2) = X_3$, $\Phi_3 | X_3 = \Phi_3$ and $\Phi_3^{-1}(D_{X_2})$ is a SNC divisor by constructing a sequence of morphisms
\[
X_3 = \hat{X}_n \circ \cdots \circ \hat{X}_1 \to \cdots \to \hat{X}_0 = X_2
\]
where $\Phi_3^{-1}(\hat{X}_i) \cong \hat{X}_i$ and $\hat{X}_i = \Phi_i$ for all $i$, and so that there are morphisms $\hat{X}_i \to \hat{Y}_i$ which are toroidal (with respect to the preimages of $D_Y$ and $D_X$) over points of $X - \gamma$. This follows from application of Lemmas 4.6 and 4.7, and the fact that the case when $\gamma_i$ is a 2-curve (or a 3-point is blown up) extends directly to a toroidal morphism.

The resulting morphism $\Phi_3$ is an isomorphism away from the preimage of $\gamma$.

We have constructed a diagram
\[
\begin{array}{ccc}
X_3 & \xrightarrow{f_3} & Y_1 \\
\Phi & \downarrow & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]
such that $\Phi$ and $\Psi_1$ are isomorphisms away from the preimage of $\gamma$ and $f_3$ is prepared with respect to $D_{Y_1} = \Psi_1^{-1}(D_Y)$ and $D_{X_3} = \Phi^{-1}(D_X)$ away from the points in $\Sigma - \{q\}$. Further, all components of $D_{X_3}$ which do not contain a 3-point and all 2-curves of $D_{X_3}$ which do not contain a 2-point must contract to points of $\gamma$. In particular, $D_{X_3}$ is cuspidal for $f_3$. By induction on $|\Sigma|$, we repeat this construction to prove Theorem 1.3.

6. toroidalization

In this section we prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1**

By Theorem 1.3, we can construct a commutative diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 & \downarrow & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]
such that $\Phi_1$ and $\Psi_1$ are products of possible blow ups such that $f_1$ is prepared for $D_{Y_1} = \Psi_1^{-1}(D_Y)$ and $D_{X_1} = \Phi_1^{-1}(D_X)$ and $D_{X_1}$ is cuspidal for $f_1$. 


Now, we conclude the proof as in the proof of Theorem 0.1 of [C5].
By descending induction on $\tau(X_2)$ (Definition 2.9 [C5]) and by Theorems 7.11 and 8.1 [C5], there exists a commutative diagram

$$
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & Y_2 \\
\Phi_2 \downarrow & & \downarrow \Psi_2 \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
$$

such that $\Phi_2$ and $\Psi_2$ are products of possible blow ups, $f_2$ is prepared for $D_{Y_2} = \Psi_2^{-1}(D_{Y_1})$ and $D_{X_2} = \Phi_2^{-1}(D_{X_1})$, $D_{X_2}$ is cuspidal for $f_2$ and $\tau f_2(X_2) = -\infty$.

By Theorem 8.2 [C5], $f_2$ is toroidal, and the conclusions of the theorem follow.

**Proof of Theorem 1.2**

By resolution of singularities and resolution of indeterminacy [H] (cf. Section 6.8 [C6]), and by [M], there exists a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
$$

where $\Phi_1$, $\Psi_1$ are products of blow ups of points and nonsingular curves supported above $D_Y$, such that $X_1$ and $Y_1$ are nonsingular and projective, and $D_{X_1} = \Phi_1^{-1}(D_X)$ and $D_{Y_1} = \Psi_1^{-1}(D_Y)$ are SNC divisors. The proof of Theorem 1.2 now follows from Theorem 1.1.

**References**


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