ASYMPTOTIC GROWTH OF SATURATED POWERS AND EPSILON MULTIPlicity

STEVEN DALE CUTKOSKY

1. Introduction

In this paper, we study the growth of saturated powers of modules. In the case of an ideal $I$ in a local ring $(R, \mathfrak{m})$, the saturation of $I^k$ in $R$ is

$$(I^k)_{\text{sat}} = I^k :_{R} \mathfrak{m}^\infty = \bigcup_{n=1}^\infty I^k :_{R} \mathfrak{m}^n.$$  

There are examples showing that the algebra of saturated powers of $I$, $\bigoplus_{k\geq 0} (I^k)_{\text{sat}}$ is not a finitely generated $R$-algebra; for instance, in many cases the saturated powers are the symbolic powers. As such, it cannot be expected that the “Hilbert function”, giving the length of the $R$-module $(I^k)_{\text{sat}}/I^k$, is very well behaved for large $k$. However, it can be shown that it is bounded above by a polynomial in $k$ of degree $d$, where $d$ is the dimension of $R$. We show that in many cases, there is a reasonable asymptotic behavior of this length.

Suppose that $(R, \mathfrak{m})$ is a Noetherian local domain of dimension $d \geq 1$. Let $L$ be the quotient field of $R$. Let $\lambda(M)$ denote the length of an $R$-module $M$. Let $F$ be a finitely generated free $R$-module, and let $E$ be a submodule of $F$ of rank $e$. Let $S = R[F] = \text{Sym}(F) = \bigoplus_{k \geq 0} F^k$ and let $R[E] = \bigoplus_{k \geq 0} E^k$ be the $R$-subalgebra of $S$ generated by $E$. Let

$$E^k :_{F^k} \mathfrak{m}^\infty = \bigcup_{n=1}^\infty E^k :_{F^k} \mathfrak{m}^n$$

denote the saturation of $E^k$ in $F^k$. We prove the following theorem:

**Theorem 1.1.** Suppose that $(R, \mathfrak{m})$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/\mathfrak{m}$ is algebraic over $K$). Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit

$$\lim_{k \to \infty} \frac{\lambda(E^k :_{F^k} \mathfrak{m}^\infty/E^k)}{L^{d+e-1}} \in \mathbb{R}$$

exists.

The conclusions of this theorem follow from Theorem 3.2 and Remark 3.3.

Theorem 1.1 is proven in the case when $E = I$ is a homogeneous ideal and $R$ is a standard graded normal $K$-algebra in our paper [3] with Hà, Srinivasan and Theodorescu. The theorem is proven with the additional assumptions that $R$ is regular, $E = I$ is an ideal in $F = R$, and the singular locus of $\text{Spec}(R/I)$ is $\mathfrak{m}$ in our paper [4] with Herzog and Srinivasan. Kleiman [13] has proven Theorem 1.1 in the case that $E$ is a direct summand of $F$ locally at every nonmaximal prime of $R$. The theorem is proven for $E$ of low analytic deviation in [4], for the case of ideals, and by Ulrich and Validashti [19] for the case of modules; in the case of low analytic deviation, the limit is always zero. A generalization of

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this problem to the case of saturations with respect to non \( m \)-primary ideals is investigated by Herzog, Puthenpurakal and Verma in [10]; they show that an appropriate limit exists for monomial ideals.

An example in [3] shows that even in the case when \( E \) is an ideal \( I \) in a regular local ring \( R \), the limit may be irrational.

An important technique in the proof of Theorem 1.1 is to use a theorem of Lazarsfeld [14] showing that the volume of a line bundle on a complex projective variety can be expressed as a limit of numbers of global sections of powers of the line bundle; Lazarsfeld’s theorem is deduced from an approximation theorem of Fujita [6] (generalizations of Fujita’s result to positive characteristic are given in [17] and [15]).

We can interpret our results in terms of local cohomology. Let \( F_k^k = F_k^k \otimes_R L \), where \( L \) is the quotient field of \( R \), so that we have natural embeddings \( E_k^k \subset F_k^k \subset F_k^k_L \) for all \( k \). We have identities

\[
H^0_m(F_k^k/E_k^k) \cong E_k^k :_{F_k^k} m^\infty/E_k^k \quad \text{and} \quad H^1_m(E_k^k) \cong E_k^k :_{F_k^k_L} m^\infty/E_k^k.
\]

Further, these two modules are equal if \( R \) has depth \( \geq 2 \).

We thus obtain the following corollary to Theorem 1.1, which shows that the epsilon multiplicity \( \varepsilon(E) \) of a module, defined as a limsup in [19], actually exists as a limit.

**Corollary 1.2.** Suppose that \((R, m)\) is a local domain of depth \( \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/m \) is algebraic over \( K \)). Let \( d \) be the dimension of \( R \). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Then the limit

\[
\lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H^0_m(F_k^k/E_k^k)) \in \mathbb{R}
\]

exists. Thus the epsilon multiplicity \( \varepsilon(E) \) of \( E \) exists as a limit.

By the above identities of local cohomology, we see that (1) is equivalent to the statement that

\[
\lim_{k \to \infty} \frac{H^0_m(F_k^k/E_k^k)}{k^{d+e-1}} = \lim_{k \to \infty} \frac{H^1_m(E_k^k)}{k^{d+e-1}} \in \mathbb{R}
\]

exists when depth\((R) \geq 2 \).

In Section 4, we extend our results to domains of dimension \( d \geq 2 \). We prove the following extension of Theorem 1.1, which shows that the second limit of (2),

\[
\lim_{k \to \infty} \frac{H^1_m(E_k^k)}{k^{d+e-1}} \in \mathbb{R}
\]

exists when \( R \) is a domain of dimension \( d \geq 2 \).

**Theorem 1.3.** Suppose that \((R, m)\) is a local domain of dimension \( d \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/m \) is algebraic over \( K \)). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Then the limit

\[
\lim_{k \to \infty} \frac{\lambda\left(E_k^k :_{F_k^k_L} m^\infty/E_k^k\right)}{k^{d+e-1}} \in \mathbb{R}
\]

exists.
Theorem 1.3 follows from Theorem 4.1 and equations (24) and (6). We prove that the first limit of (2),
\[ \lim_{k \to \infty} \frac{H^0_m(F^k/E^k)}{k^{d+e-1}} \in \mathbb{R} \]
exists when \( R \) is a domain of dimension \( d \geq 2 \) and \( E \) is embedded in \( F \) of rank \( < d + e \). I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out this interesting consequence of Theorem 1.3.

**Corollary 1.4.** Suppose that \((R, \mathfrak{m})\) is a local domain of dimension \( d \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/\mathfrak{m} \) is algebraic over \( K \)). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Suppose that \( \gamma = \text{rank}(F) < d + e \). Then the limits
\[ \lim_{k \to \infty} \frac{\lambda(E^k : F^k \mathfrak{m}^\infty / E^k)}{k^{d+e-1}} \in \mathbb{R} \]
and
\[ \lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H^0_m(F^k/E^k)) \in \mathbb{R} \]
exist. In particular, the epsilon multiplicity \( \varepsilon(E) \) of \( E \) exists as a limit.

In the case when \( e = 1 \) and \( F = R \), we get the following statement.

**Corollary 1.5.** Suppose that \((R, \mathfrak{m})\) is a local domain of dimension \( d \geq 1 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/\mathfrak{m} \) is algebraic over \( K \)). Suppose that \( I \) is an ideal in \( R \). Let \((I^k)^{\text{sat}} = I^k : R \mathfrak{m}^\infty \) be the saturation of \( I^k \). Then the limit
\[ \lim_{k \to \infty} \frac{\lambda((I^k)^{\text{sat}}/I^k)}{k^d} \in \mathbb{R} \]
extists.

Asymptotic polynomial like behavior of the length of extension functions is studied by Katz and Theodorescu [12], Theodorescu [18] and Crabbe, Katz, Striuli and Theodorescu [2]. By the local duality theorem, we obtain the following corollary to Theorem 1.1.

**Corollary 1.6.** Suppose that \((R, \mathfrak{m})\) is a Gorenstein local domain of dimension \( d \geq 2 \) which is essentially of finite type over a field \( K \) of characteristic zero (or over a perfect field \( K \) such that \( R/\mathfrak{m} \) is algebraic over \( K \)). Suppose that \( E \) is a rank \( e \) submodule of a finitely generated free \( R \)-module \( F \). Then the limit
\[ \lim_{k \to \infty} \frac{\lambda(\text{Ext}_R^d(F^k/E^k, R))}{k^{d+e-1}} \in \mathbb{R} \]
extists.

2. Preliminaries

Suppose that \((R, \mathfrak{m})\) is a Noetherian local domain of dimension \( d \geq 1 \) with quotient field \( L \). Let \( \lambda_R(M) \) denote the length of an \( R \)-module \( M \). When there is no danger of confusion, we will denote \( \lambda_R(M) \) by \( \lambda(M) \).
Let $F$ be a finitely generated free $R$-module of rank $\gamma$, and let $E$ be a submodule of $F$ of rank $e$. Let $S = R[F] = \text{Sym}(F) = \bigoplus_{k \geq 0} F^k$, and let $R[E] = \bigoplus_{k \geq 0} E^k$ be the $R$-subalgebra of $S$ generated by $E$. Let
\[ E^k :_{F^k} m^\infty = \cup_{n=1}^{\infty} E^k :_{F^k} m^n \]
denote the saturation of $E^k$ in $F^k$.

Let $F^k_L = F^k \otimes_R L$ (where $L$ is the quotient field of $R$), so that we have natural embeddings $E^k \subset F^k \subset F^k_L$ for all $k$. Let $X = \text{Spec}(R)$, $E^k$ be the sheafification of $E$ on $X$ and let $u_1, \ldots, u_s$ be generators of the ideal $m$.

There are identities
\[ H^0(X \setminus \{m\}, \widetilde{E^k}) = \bigcap_{t=1}^{u} (E^k)_{u_t} = E^k :_{F^k_L} m^\infty. \]

From the exact sequence of cohomology groups
\[ 0 \to H^0_m(E^k) \to E^k \to H^0_m(X \setminus \{m\}, \widetilde{E^k}) \to H^1_m(E^k) \to 0, \]
we deduce that we have isomorphisms of $R$-modules
\[ H^1_m(E^k) \cong E^k :_{F^k} m^\infty/E^k \]
for $k \geq 0$. The same calculation for $F^k$ shows that
\[ H^1_m(F^k) \cong F^k :_{F^k_L} m^\infty/F^k. \]

From the left exact local cohomology sequence
\[ 0 \to H^0_m(F^k/E^k) \to H^1_m(E^k) \to H^1_m(F^k), \]
we have that
\[ H^0_m(F^k/E^k) \cong \left( E^k :_{F^k_L} m^\infty \right) \cap F^k = E^k :_{F^k} m^\infty/E^k. \]

From (6), and the fact that $F^k$ is a free $R$-module, we have that $H^0(X \setminus \{m\}, \widetilde{F^k}) = F^k$ and
\[ E^k :_{F^k_L} m^\infty = E^k :_{F^k} m^\infty \text{ if } R \text{ has depth } \geq 2. \]

Let $ES$ be the ideal of $S$ generated by $E$. We compute the degree $n$ part of $(ES)^n$ from the formula
\[ [(ES)^n]_n = E^n. \]

Let $R[mE] = \bigoplus_{n \geq 0} (mE)^n$ be the $R$-subalgebra of $S$ generated by $mE$.

Let $X = \text{Spec}(R)$, $Y = \text{Proj}(R[mE])$ and $Z = \text{Proj}(R[E])$.

Write $R[E] = R[\overline{x}_1, \ldots, \overline{x}_t]$ as a standard graded $R$-algebra, with $\deg \overline{x}_i = 1$ for all $i$. For $1 \leq i \leq t$, let
\[ R_i = R[\overline{x}_1, \ldots, \overline{x}_i], \]
and let $V_i = \text{Spec}(R_i)$ for $1 \leq i \leq t$. $\{V_i\}$ is an affine cover of $Z$. Let $u_1, \ldots, u_s$ be generators of the ideal $m$. For $1 \leq i \leq s$ and $1 \leq j \leq t$, let
\[ R_{i,j} = R[\frac{u_\alpha \overline{x}_\beta}{u_i \overline{x}_j} \mid 1 \leq \alpha \leq s, 1 \leq \beta \leq t], \]
and $U_{i,j} = \text{Spec}(R_{i,j})$. Then $\{U_{i,j}\}$ is an affine cover of $Y$. Since
\[ R_j[\frac{u_1}{u_i}, \ldots, \frac{u_s}{u_i}] = R_{i,j}, \]
we see that $Y$ is the blow up of the ideal sheaf $m\mathcal{O}_Z$.

The structure morphism $f : Y \to X$ factors as a sequence of projective morphisms

$$Y \xrightarrow{\alpha} Z \xrightarrow{h} X,$$

where $Y$ is the blow up the ideal sheaf $m\mathcal{O}_Z$. Define line bundles on $Y$ by $\mathcal{L} = g^*\mathcal{O}_Z(1)$ and $\mathcal{M} = m\mathcal{O}_Y$. Then $\mathcal{O}_Y(1) \cong \mathcal{M} \otimes \mathcal{L}$.

We have $\mathcal{O}_Z(1)|_{V_j} = \tau_j\mathcal{O}_{V_j}$, $\mathcal{L}|_{U_{i,j}} = \tau_j\mathcal{O}_{U_{i,j}}$ and $\mathcal{M}|_{U_{i,j}} = u_i\mathcal{O}_{U_{i,j}}$.

We give three consequences (Proposition 2.1, Proposition 2.2 and Corollary 2.3) of Serre’s fundamental theorem for projective morphisms which will be useful.

**Proposition 2.1.** $\bigoplus_{k \geq 0} H^i(Y, \mathcal{L}^k)$ are finitely generated $R[E]$-modules for all $i \in \mathbb{N}$.

**Proof.** Let $\mathcal{E}^k$ be the sheafication of $E^k$ on $X$. From the natural surjections for $k \geq 0$ of $\mathcal{O}_Z$-modules $g^*(\mathcal{E}^k) \to \mathcal{O}_Z(k)$, we obtain surjections $f^*(\mathcal{E}^k) \to \mathcal{L}^k$ of $\mathcal{O}_Y$-modules, and a surjection $f^*(\bigoplus \mathcal{E}^k) \to \bigoplus \mathcal{L}^k$. Hence $\bigoplus_{k \geq 0} \mathcal{L}^k$ is a finitely generated $f^*(\bigoplus_{k \geq 0} \mathcal{E}^k)$-module. By Theorem III.2.4.1 [8], $R^if_*\bigoplus_{k \geq 0} \mathcal{L}^k$ is a finitely generated $\bigoplus_{k \geq 0} \mathcal{E}_k$-module for $i \in \mathbb{N}$. Taking global sections on the affine $X$, we obtain the conclusions of the proposition. $\square$

**Proposition 2.2.** Suppose that $A$ is a Noetherian ring, and $B = \bigoplus_{k \geq 0} B_k$ is a finitely generated graded $A$-algebra, which is generated by $B_1$ as an $A$-algebra. Let $C = \text{Spec}(A)$ and $D = \text{Proj}(B)$. Let $\alpha : D \to C$ be the structure morphism. Then there exists a positive integer $\overline{k}$ such that $B_k = \Gamma(D, \mathcal{O}_D(k))$ for $k \geq \overline{k}$.

**Proof.** The ring $\bigoplus_{k \geq 0} \Gamma(D, \mathcal{O}_D(k))$ is a finitely generated graded $B$-module by Theorem III.2.4.1 [8]. Hence $(\bigoplus_{k \geq 0} \Gamma(D, \mathcal{O}_D(k)))/B$ is a finitely generated graded $B$-module. Since every element of this module is $B_+ = \bigoplus_{k \geq 0} B_k$ torsion, we have that $B_k/E_k = 0$ for $k \gg 0$. $\square$

Taking the maximum over the $\overline{k}$ obtained from the above proposition applied to a finite affine cover of $W$, we obtain the following generalization of Proposition 2.2.

**Corollary 2.3.** Suppose that $W$ is a Noetherian scheme and $B = \bigoplus_{k \geq 0} B_k$ is a finitely generated graded $\mathcal{O}_W$-algebra, which is locally generated by $B_1$ as an $\mathcal{O}_W$-algebra. Let $W' = \text{Proj}(B)$ and let $\alpha : W' \to W$ be the structure morphism. Then there exists a positive integer $\overline{k}$ such that $B_k = \alpha_* \mathcal{O}_{W'}(k)$ for $k \geq \overline{k}$.

3. **Asymptotic Growth**

**Proposition 3.1.** Let $(R, m)$ be a local domain of depth $\geq 2$. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ $R$-submodule of a finitely generated free $R$-module $F$. Let notation be as above. Then there exist positive integers $k_0$, $k_1$ and $\tau$ such that

1) for $k \geq k_0$, $n \in \mathbb{Z}$ and $p \in X \setminus \{m\}$, $\Gamma(Y, \mathcal{M}^n \otimes \mathcal{L}^k)_p = (E^k)_p$.

2) For $k \geq k_1$, $E^k :_F m^\infty = \Gamma(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k)$.

**Proof.** We first establish 1). $U_i = \text{Spec}(R_{u_i})$ for $1 \leq i \leq s$ is an affine cover of $X \setminus \{m\}$. $g|f^{-1}(U_i)$ is an isomorphism; in fact

$$f^{-1}(U_i) = \text{Proj}(R[me]_{u_i}) = \text{Proj}(R[E]_{u_i}) = h^{-1}(U_i).$$
By Proposition 2.2, there exist positive integers $a_i$ such that
\[ \Gamma(f^{-1}(U_i), \mathcal{M}^{-n} \otimes \mathcal{L}^k) = \Gamma(h^{-1}(U_i), \mathcal{O}_Z(k)) = (E^k)_{u_i} \]
for $k \geq a_i$. Let $k_0 = \max\{a_1, \ldots, a_s\}$. Then for $p \in U_i$
\[ \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k)_p = \Gamma(f^{-1}(U_i), \mathcal{M}^n \otimes \mathcal{L}^k)_p = (E^k)_p \]
for $k \geq k_0$, establishing 1).

We now establish 2). Suppose that $n \geq 0$, and $k \geq 0$. Suppose that $\sigma \in E^k : \colon F_k \in \mathfrak{m}^n$. Let $i, j$ be such that $1 \leq i \leq s$ and $1 \leq j \leq t$. $\sigma \mathfrak{m}^n \subseteq E^k$ implies $u_i^n \sigma \in E^k$ which implies there is an expansion
\[ u_i^n \sigma = \sum_{n_1 + \cdots + n_t = k} r_{n_1, \ldots, n_t} \overline{x}_1^{n_1} \cdots \overline{x}_t^{n_t} \]
with $r_{n_1, \ldots, n_t} \in R$. Thus
\[ u_i^n \sigma = \overline{x}_j^k \left( \sum_{n_1 + \cdots + n_t = k} r_{n_1, \ldots, n_t} (\overline{x}_1)^{n_1} \cdots (\overline{x}_t)^{n_t} \right) \]
so that $\sigma \in u_i^{-n} \overline{x}_j^k R_{i,j}$. Thus
\[ \sigma \in \cap_{i,j} u_i^{-n} \overline{x}_j^k R_{i,j} = \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k). \]

We have established that for $k \geq 0$ and $n \geq 0,$
\[ E^k : \colon F_k \in \mathfrak{m}^n \subseteq \Gamma(Y, \mathcal{M}^{-n} \otimes \mathcal{L}^k). \]

Recall that $S$ is a polynomial ring $S = R[y_1, \ldots, y_t]$ over $R$, where $\gamma$ is the rank of $F$. Let $W = \text{Proj}(S)$, with natural morphism $\alpha : W \rightarrow X$. Let $\mathcal{I}$ be the sheafication of the graded ideal $ES$ on $W$. We have expansions
\[ \overline{x}_i = \sum_{l=1}^{\gamma} f_{il} y_l \]
with $f_{il} \in R$.

The inclusion $R[E] \subseteq S$ induces a rational map from $W$ to $Z$.

Let $\beta : W' \rightarrow W$ be the blow up of the ideal sheaf $\mathcal{I}$. Let $N = \mathcal{I}_W'$ be the induced line bundle. $W'$ has an affine cover $A_{i,j} = \text{Spec}(T_{ij})$ for $1 \leq i \leq s$ and $1 \leq j \leq t$ with
\[ T_{ij} = R[y_1/y_j, \ldots, y_t/y_j, \overline{x}_i, \ldots, \overline{x}_t]. \]
From the inclusions
\[ R_i = R[\overline{x}_1/\overline{x}_i, \ldots, \overline{x}_t/\overline{x}_i] \subseteq T_{ij} \]
we have induced morphisms $A_{i,j} \rightarrow V_i = \text{Spec}(R_i)$ which patch to give a morphism $\varphi : W' \rightarrow Z$ which is a resolution of indeterminacy of the rational map from $W$ to $Z$.

We calculate for all $i,j$,
\[ \varphi^*(\mathcal{O}_Z(1)) | A_{i,j} = \varphi_* \mathcal{O}_{A_{i,j}} = y_j \left( \sum_{l} f_{il} y_l \right) \mathcal{O}_{A_{i,j}} = (\beta^* \mathcal{O}_W(1)) \mathcal{I} | A_{i,j}, \]
to see that
\[ (\beta^* \mathcal{O}_W(1)) \otimes N \cong \varphi^* \mathcal{O}_Z(1). \]
By Corollary 2.3, there exists a positive integer $k_1 \geq k_0$ such that $\beta_s N^k = T^k$ for $k \geq k_1$. From the natural inclusion $\mathcal{O}_Z(k) \subset \varphi^* \mathcal{O}_Z(k)$, we have by the projection formula that for $k \geq k_1$,

$$
h_* \mathcal{O}_Z(k) \subset h_* \varphi_*(\varphi^* \mathcal{O}_Z(k)) = \alpha_* \beta_*(\beta^* \mathcal{O}_W(k) \otimes N^k)
$$

(13)

$$
= \alpha_* [\mathcal{O}_W(k) \otimes \beta_* N^k] = \alpha_* [\mathcal{O}_W(k) \otimes T^k]
$$

$$
\subset \alpha_* \mathcal{O}_W(k) = \tilde{F}^k,
$$

where $\tilde{F}^k$ is the sheafification of the $R$-module $F$ on $X$. Now we have

$$
\Gamma(Y, M^{-n} \otimes \mathcal{L}^k) = \Gamma(X, f_* (M^{-n} \otimes \mathcal{L}^k))
$$

(14)

$$
\subset \Gamma(X \setminus \{m\}, f_* (M^{-n} \otimes \mathcal{L}^k)) = \Gamma(X \setminus \{m\}, h_* \mathcal{O}_Z(k))
$$

$$
\subset \Gamma(X \setminus \{m\}, \tilde{F}^k) = F^k
$$

since $R$, and hence the free $R$-module $F^k$, have depth $\geq 2$.

From (11), we deduce that for $k, n \geq 0$,

$$
(\{ES\}^k : \mathcal{O} \{m^n S\}) \cap F^k = E^k :_{\mathcal{O}} \{m^n\}.
$$

(15)

By 1.5 [11] or Theorem 1.3 [16], there exists a positive integer $\tau$ such that

$$
(\{ES\}^k : \mathcal{O} \{m^\tau S\}) = (\{ES\}^k : \mathcal{O} \{mS\})
$$

for all $k \geq 0$. Thus from (15) we have that

$$
E^k :_{\mathcal{O}} \{m^\tau\} = E^k :_{\mathcal{O}} \{m^\infty\}
$$

(16)

for $k \geq 0$. From (16), (12) and (14), we have inclusions

$$
E^k :_{\mathcal{O}} \{m^\infty\} \subset \Gamma(Y, M^{-\infty} \otimes \mathcal{L}^k) \subset F^k
$$

for $k \geq k_1$. The conclusions of 2) of the proposition now follow from 1) of the proposition since $E^k :_{\mathcal{O}} \{m^\infty\}$ is the largest $R$-submodule $N$ of $F^k$ which has the property that $N_p = (E^k)_p$ for $p \in X \setminus \{m\}$.

$\square$

**Theorem 3.2.** Suppose that $(R, m)$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit

$$
\lim_{k \to \infty} \frac{\lambda(E^k :_{\mathcal{O}} \{m^\infty\} / E^k)}{k^{d+e-1}} \in \mathbb{R}
$$

exists.

**Proof.** Let notation be as above.

First consider the short exact sequences

$$
0 \to \Gamma(Y, \mathcal{L}^k) / E^k \to E^k :_{\mathcal{O}} \{m^\infty\} / E^k \to E^k :_{\mathcal{O}} \{m^\infty\} / \Gamma(Y, \mathcal{L}^k) \to 0.
$$

(17)

$\bigoplus_{k \geq 0} \Gamma(Y, \mathcal{L}^k)$ is a finitely generated $R[E]$-module by Lemma 2.1. By 1) of Proposition 3.1, the support of the $R$-module $\Gamma(Y, \mathcal{L}^k) / E^k$ is contained in $\{m\}$ for all $k$. Since $\bigoplus_{k \geq 0} \Gamma(Y, \mathcal{L}^k) / E^k$ is a finitely generated $R[E]$-module, there is a positive integer $r$ such that $m^r (\Gamma(Y, \mathcal{L}^k) / E^k) = 0$ for all $k$. Since $\dim R[E] / mR[E] \leq \dim R + \text{rank } E - 1 = d + e - 1$, and $R/m^r$ is an Artin local ring, we have by the Hilbert-Serre theorem that $\lambda(\Gamma(Y, \mathcal{L}^k) / E^k)$ is a polynomial of degree less than or equal to $d + e - 2$ for $k \gg 0$. Thus
there exists a constant \( \alpha \) such that \( \lambda(\Gamma(Y, \mathcal{L}^k)/E^k) \leq \alpha k^{d+e-2} \) for all \( k \). From (17), we are now reduced to showing that the limit

\[
\lim_{k \to \infty} \frac{\lambda(E^k : F_k \mathfrak{m}^\infty / \Gamma(Y, \mathcal{L}^k))}{k^{d+e-1}}
\]

exists, from which we will have

\[
\lim_{k \to \infty} \frac{\lambda(E^k : F_k \mathfrak{m}^\infty / \Gamma(Y, \mathcal{L}^k))}{k^{d+e-1}} = \lim_{k \to \infty} \frac{\lambda(E^k : F_k \mathfrak{m}^\infty / E^k)}{k^{d+e-1}}.
\]

Taking global sections of the short exact sequences

\[
0 \to \mathcal{L}^k \to \mathcal{M}^{-k^r} \otimes \mathcal{L}^k \to \mathcal{M}^{-k^r} \otimes \mathcal{L}^k \otimes (\mathcal{O}_Y / \mathfrak{m}^{k^r} \mathcal{O}_Y) \to 0,
\]

we obtain by Proposition 3.1 left exact sequences

\[
0 \to E^k : F_k \mathfrak{m}^\infty / \Gamma(Y, \mathcal{L}^k) \to \Gamma(Y, \mathcal{M}^{-k^r} \otimes \mathcal{L}^k \otimes (\mathcal{O}_Y / \mathfrak{m}^{k^r} \mathcal{O}_Y)) \to H^1(Y, \mathcal{L}^k)
\]

for \( k \geq k_1 \).

Let \( u_1, \ldots, u_s \) be generators of the ideal \( \mathfrak{m} \), and set \( U_i = \text{Spec}(R_{u_i}) \), so that \( \{U_1, \ldots, U_s\} \) is an affine cover of \( X \setminus \{m\} \). Then \( \mathcal{L}|f^{-1}(U_i) \) is ample, so there exist positive integers \( b_i \) such that \( R^{1} f_*(\mathcal{L}^k) | U_i = 0 \) for \( k \geq b_i \). Let \( k_2 = \max\{b_1, \ldots, b_s\} \). We have that the support of \( H^1(Y, \mathcal{L}^k) \) is contained in \( \{m\} \) for \( k \geq k_2 \).

\[ \bigoplus_{k \geq 0} H^1(Y, \mathcal{L}^k) \]

is a finitely generated \( R[E] \)-module by Lemma 2.1. Hence the submodule \( M = \bigoplus_{k \geq k_2} H^1(Y, \mathcal{L}^k) \) is a finitely generated graded \( R[E] \)-module. We have that \( \mathfrak{m}^r M = 0 \) for some positive integer \( r \). Since

\[
\dim R[E]/\mathfrak{m} R[E] \leq \dim R + \text{rank } E - 1 = d + e - 1,
\]

and \( R/\mathfrak{m}^r \) is an Artin local ring, we have by the Hilbert-Serre theorem that \( \lambda(H^1(Y, \mathcal{L}^k)) \) is a polynomial of degree less than or equal to \( d + e - 2 \) for \( k \gg 0 \). Thus there exists a constant \( c \) such that

\[
\lambda(H^1(Y, \mathcal{L}^k)) \leq ck^{d+e-2}
\]

for all \( k \geq 0 \). By consideration of (18) and (19), we are reduced to proving that the limit

\[
\lim_{k \to \infty} \frac{\lambda(H^0(Y, \mathcal{M}^{-k^r} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y / \mathfrak{m}^{k^r} \mathcal{O}_Y))}{k^{d+e-1}}
\]

exists.

If \( R/\mathfrak{m} \) is algebraic over \( K \), let \( K' = K \). If \( R/\mathfrak{m} \) is transcendental over \( K \), let \( t_1, \ldots, t_r \) be a lift of a transcendence basis of \( R/\mathfrak{m} \) over \( K \) to \( R \). The rational function field \( K(t_1, \ldots, t_r) \) is contained in \( R \). Let \( K' = K(t_1, \ldots, t_r) \). We have that \( R/\mathfrak{m} \) is finite algebraic over \( K' \).

There exists an affine \( K' \)-variety \( X' = \text{Spec}(A) \) such that \( R \) is the local ring of a closed point \( \alpha \) of \( X' \), and \( E \) extends to a submodule \( E' \) of \( A^\gamma \), where \( \gamma \) is the rank of the free \( R \)-module \( F \). We then have an inclusion of graded \( A \)-algebras \( A[E'] \subset \text{Sym}(A^\gamma) \) which extends \( R[E] \). Identify \( \mathfrak{m} \) with its extension to a maximal ideal of \( A \). The structure morphism \( Y' = \text{Proj}(A[\mathfrak{m} E']) \to X' \) is projective and its localization at \( \mathfrak{m} \) is \( f : Y \to X \).

Let \( \bar{X} \) be a projective closure of \( X' \) and let \( \bar{Y} \) be a projective closure of \( Y' \). \( X' \) is an open subset of \( \bar{X} \) and \( Y' \) is an open subset of \( \bar{Y} \). Let \( \bar{Y} \to \bar{X} \) be the blow up of an ideal sheaf which gives a resolution of indeterminacy of the rational map from \( \bar{Y} \) to \( \bar{X} \). We may assume that the morphism \( \bar{Y} \to \bar{X} \) is an isomorphism over the locus where the rational map is a morphism, and thus an isomorphism over the subset \( Y' \) of \( \bar{Y} \). Let \( \mathcal{F} : Y \to \bar{X} \) be the resulting morphism. We now establish that \( \mathcal{F}^{-1}(X') = Y' \). Suppose that \( p \in X' \) and \( q \in \mathcal{F}^{-1}(p) \). Let \( V \) be a valuation ring of the function field \( L \) of \( \bar{Y} \) (which is also the
function field of $Y'$) which dominates the local ring $\mathcal{O}_{\overline{Y}, q}$. By assumption, $V$ dominates the local ring $\mathcal{O}_{X', p}$. $V$ dominates the local ring of a point on $Y'$, by the valuative criterion for properness (Theorem II.4.7 [9]) applied to the proper morphism $Y' \to X'$. Since $V$ dominates the local ring of a unique point on $\overline{Y}$, we have that $q \in Y'$.

After possibly replacing $\overline{Y}$ with the blow up of an ideal sheaf on $\overline{Y}$ whose support is disjoint from $Y'$, we may assume that $\mathcal{L}$ extends to a line bundle on $\overline{Y}$ which we will also denote by $\mathcal{L}$. We will identify $m$ with its extension to the ideal sheaf of the point $\alpha$ on $\overline{X}$, and identify $\mathcal{M}$ with its extension $m\mathcal{O}_{\overline{Y}}$ to a line bundle on $\overline{Y}$. Let $A$ be an ample divisor on $\overline{X}$. Then there exists $l > 0$ such that $C = \mathcal{F}(A^l) \otimes \mathcal{L}$ is generated by global sections and is big.

Set $B = C \otimes M^{-\tau}$. Tensor the short exact sequences
\[ 0 \to \mathcal{M}^{k\tau} \to \mathcal{O}_{\overline{Y}} \to \mathcal{O}_{\overline{Y}}/m^{k\tau}\mathcal{O}_{\overline{Y}} \cong \mathcal{O}_Y/m^{k\tau}\mathcal{O}_Y \to 0 \]
with $B^k$ to obtain the short exact sequences
\[ 0 \to \mathcal{O}^k \to B^k \to \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/m^{k\tau}\mathcal{O}_Y \to 0 \]
for $k \geq 0$. Taking global sections, we have exact sequences
\[ 0 = H^0(\overline{Y}, \mathcal{O}^k) \to H^0(\overline{Y}, B^k) \to H^0(\overline{Y}, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/m^{k\tau}\mathcal{O}_Y) \to H^1(\overline{Y}, \mathcal{O}^k). \]

For a coherent sheaf $\mathcal{F}$ on $\overline{Y}$, let
\[ h^1(\overline{Y}, \mathcal{F}) = \dim_K H^1(\overline{Y}, \mathcal{F}). \]

Since $C$ is semiample (generated by global sections and big) and $Y$ has dimension $d + e - 1$, we have that
\[ \lim_{k \to \infty} \frac{h^1(\overline{Y}, \mathcal{O}^k)}{k^{d+e-1}} = 0. \]

This follows for instance from [5]. Since $\bigoplus_{k \geq 0} H^0(\overline{Y}, \mathcal{O}^k)$ is a finitely generated $K'$ algebra of dimension $d + e$, as $C$ is generated by global sections and is big (or by the Riemann Roch theorem and the vanishing theorem of [5]) we have that the limit
\[ \lim_{k \to \infty} \frac{h^0(\overline{Y}, \mathcal{O}^k)}{k^{d+e-1}} \in \mathbb{Q} \]
exists. Since $B$ is big, by the corollary to [6] given in Example 11.4.7 [14] or [3], we have that the limit
\[ \lim_{k \to \infty} \frac{h^0(\overline{Y}, B^k)}{k^{d+e-1}} \in \mathbb{R} \]
exists. From the sequence (21), we see that
\[ \lim_{k \to \infty} \frac{h^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/m^{k\tau}\mathcal{O}_Y)}{k^{d+e-1}} \in \mathbb{R} \]
exists. The conclusions of the theorem now follow from (20) and the formula
\[ h^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/m^{k\tau}\mathcal{O}_Y) = \dim_K H^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/m^{k\tau}\mathcal{O}_Y) = [R/m : K'] \lambda(H^0(Y, \mathcal{M}^{-k\tau} \otimes \mathcal{L}^k \otimes \mathcal{O}_Y/m^{k\tau}\mathcal{O}_Y)). \]

\[ \square \]

**Remark 3.3.** The conclusions of Theorem 3.2 are also true if $K$ is a perfect field of positive characteristic and $R/m$ is algebraic over $K$. In this case we have that $K' = K$ in the proof of Theorem 3.2. Let $\overline{K}$ be an algebraic closure of $K$. Since $K$ is perfect, $\overline{Y} \times_K \overline{K}$ is reduced, and to compute the limit, we reduce to computing the sections of the
pullback of $B^k$ on the disjoint union of the irreducible (integral) components of $\bar{Y} \times_K \bar{R}$. Fujita’s approximation theorem is valid on varieties over an algebraically closed field of positive characteristic, as was shown by Takagi [17], from which the existence of the limit now follows.

**Remark 3.4.** Theorem 3.2 is proven for graded ideals in [3]. An example where the limit is an irrational number is given in [3]. The theorem is proven with the additional assumptions that $R$ is regular, $E = I$ is an ideal in $F = R$, and the singular locus of $\text{Spec}(R/I)$ is $m$ in [4]. Kleiman [13] has proven Theorem 3.2 in the case that $E$ is a direct summand of $F$ locally at every nonmaximal prime of $R$.

**Corollary 3.5.** Suppose that $(R, m)$ is a local domain of depth $\geq 2$ which is essentially of finite type over a field $K$ of characteristic zero. Let $d$ be the dimension of $R$. Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit

$$\lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H^0_m(F^k/E^k)) \in \mathbb{R}$$

exists. Thus the epsilon multiplicity $\varepsilon(E)$ of the module $E$, defined in [19] as a limsup, actually exists as a limit.

The example of [3] shows that $\varepsilon(E)$ may be an irrational number.

*Proof.* The corollary is immediate from Theorem 3.2 and (9). □

**Remark 3.6.** The conclusions of Corollary 3.5 are valid if $K$ is a perfect field of positive characteristic and $R/m$ is algebraic over $K$, by Remark 3.3.

### 4. Extension to domains of dimension $\geq 2$.

In this section, we prove extensions of Theorem 1.1 and Corollary 1.2 to domains of dimension $\geq 2$. Let notation be as in Section 2.

Suppose that $R$ is a domain of dimension $d \geq 2$ with a dualizing module. By the Theorem of Finiteness, Theorem VIII.2.1 (and footnote) [7],

$$\tilde{R} = \Gamma(X \setminus \{m\}, \mathcal{O}_X) = \cap_{p \in X \setminus \{m\}} R_p$$

is a finitely generated $R$-module, which lies between $R$ and its quotient field. Since $\tilde{R}/R$ is $m$-torsion,

$$\lambda_R(\tilde{R}/R) < \infty.$$  

Let $m_1, \ldots, m_\alpha$ be the maximal ideals of $\tilde{R}$ which lie over $m$. By our construction,

$$0 = H^1_m(\tilde{R}) = H^1_{mR}(\tilde{R}) = \bigoplus_{i=1}^\alpha H^1_{m_i\tilde{R}}(\tilde{R}),$$

so

$$H^1_{m_i\tilde{R}_{m_i}}(\tilde{R}_{m_i}) = H^1_{m_i\tilde{R}}(\tilde{R}) \otimes_{\tilde{R}} \tilde{R}_{m_i} = 0$$

for $1 \leq i \leq \alpha$, and thus $\text{depth}(\tilde{R}_{m_i}) \geq 2$ for $1 \leq i \leq \alpha$.

Let $F = F \otimes_R \tilde{R}$ and $\tilde{R}[F] = \bigoplus_{k \geq 0} \tilde{F}^k$, so that $\tilde{F}^k \cong F^k \otimes_R \tilde{R}$ for all $k$. Let $\tilde{E} = \tilde{R}E$ be the $\tilde{R}$-submodule of $\tilde{F}$ generated by $E$. Let $\tilde{R}[\tilde{E}] = \bigoplus_{k \geq 0} \tilde{E}^k$ be the $\tilde{R}$-subalgebra of $\tilde{R}[F]$ generated by $\tilde{E}$.

Let $u_1, \ldots, u_s$ be generators of the ideal $m$. For $k \in \mathbb{N}$, let $\tilde{E}^k$ be the sheafification of $E^k$ on $X = \text{Spec}(R)$. 

There are identities
\[
H^0(X \setminus \{m\}, \tilde{E}^k) = \cap_{i=1}^e (E^k)_{u_i} = E^k : \mathfrak{p}^k m^\infty.
\]
From the exact sequence of cohomology groups
\[
0 \to H^0_m(E^k) \to E^k \to H^0_m(X \setminus \{m\}, \tilde{E}^k) \to H^1_m(E^k) \to 0,
\]
we deduce that we have isomorphisms of $R$-modules
\[
H^1_m(E^k) \cong E^k : \mathfrak{p}^k m^\infty / E^k
\]
for $k \geq 0$. The same calculation for $F^k$ shows that
\[
H^1_m(F^k) \cong F^k : \mathfrak{p}^k m^\infty / F^k.
\]
From the left exact local cohomology sequence
\[
0 \to H^0_m(F^k / E^k) \to H^1_m(E^k) \to H^1_m(F^k),
\]
we have that
\[
H^0_m(F^k / E^k) \cong \left( E^k : \mathfrak{p}^k m^\infty \right) \cap F^k / E^k = E^k : \mathfrak{p}^k m^\infty / F^k.
\]

**Theorem 4.1.** Suppose that $(R, m)$ is a local domain of dimension $d \geq 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/m$ is algebraic over $K$). Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Then the limit
\[
\lim_{k \to \infty} \lambda \left( E^k : \mathfrak{p}^k m^\infty / E^k \right) / k^{d+e-1} \in \mathbb{R}
\]
exists.

**Proof.** Since $E^k : \mathfrak{p}^k m^\infty / E^k$ are finitely generated $m\mathfrak{R}$-torsion $\mathfrak{R}$-modules, we have that
\[
E^k : \mathfrak{p}^k m^\infty / E^k \cong \bigoplus_{i=1}^\alpha \left( E^k_{m_i} : \mathfrak{p}^k m_i^\infty / E^k_{m_i} \right).
\]
By Theorem 1.1, we have that
\[
\lim_{k \to \infty} \lambda_{\mathfrak{R}_{m_i}} \left( E^k_{m_i} : \mathfrak{p}^k m_i^\infty / E^k_{m_i} \right) / k^{d+e-1}
\]
exists for $1 \leq i \leq \alpha$. Since for any $\mathfrak{R}_{m_i}$ module $M$ we have that
\[
\lambda_R(M) = [\mathfrak{R}/m_i : R/m] \lambda_{\mathfrak{R}_{m_i}}(M),
\]
we conclude that
\[
\lim_{k \to \infty} \lambda_R(E^k : \mathfrak{p}^k m^\infty / E^k) / k^{d+e-1}
\]
exists. We have
\[
E^k : \mathfrak{p}^k m^\infty = \cap_{i=1}^e (E^k)_{u_i} = \cap_{i=1}^e (E^k)_{u_i} = E^k : \mathfrak{p}^k m^\infty.
\]
Consider the short exact sequences
\[
0 \to E^k / E^k \to E^k : \mathfrak{p}^k m^\infty / E^k \to E^k : \mathfrak{p}^k m^\infty / E^k \to 0.
\]
Now $\overline{R[E]/R[E]}$ is a finitely generated $R[E]$-module, and the support of the $R$-module $E^k/E_k$ is contained in $\{m\}$ for all $k$, so there exists a positive integer $r$ such that $m^r$ annihilates $\overline{R[E]/R[E]}$. Thus (as in the argument following equation (17) in the proof of Theorem 3.2), we have that there exists a constant $\beta$ such that

$$\lambda_R(E^k/E_k) \leq \beta k^{d+e-2}$$

for all $k$. The conclusions of the proposition now follow from (29), (31) and (30).

I thank Craig Huneke, Bernd Ulrich and Javid Validashti for pointing out the following consequence of Theorem 4.1.

**Corollary 4.2.** Suppose that $(R, m)$ is a local domain of dimension $d \geq 2$ which is essentially of finite type over a field $K$ of characteristic zero (or over a perfect field $K$ such that $R/m$ is algebraic over $K$). Suppose that $E$ is a rank $e$ submodule of a finitely generated free $R$-module $F$. Suppose that $\gamma = \text{rank}(F) < d + e$. Then the limits

$$\lim_{k \to \infty} \frac{\lambda \left( E^k : F^k m^\infty / E^k \right)}{k^{d+e-1}} \in \mathbb{R}$$

and

$$\lim_{k \to \infty} \frac{(d + e - 1)!}{k^{d+e-1}} \lambda(H^0_m(F^k/E_k)) \in \mathbb{R}$$

exist. In particular, the epsilon multiplicity $\varepsilon(E)$ of $E$ exists as a limit.

**Proof.** We will establish that the limit (32) exists. We have exact sequences

$$0 \to E^k : F^k m^\infty / E^k \to E^k : F^k m^\infty / E^k : F^k m^\infty \to E^k : F^k m^\infty \to 0$$

and inclusions

$$E^k : F^k m^\infty / E^k : F^k m^\infty = E^k : F^k m^\infty / \left( (E^k : F^k m^\infty) \cap F^k \right) \to F^k : F^k m^\infty / F^k$$

for $k \geq 0$.

We have

$$F^k : F^k m^\infty / F^k = \overline{F^k / F^k} \cong (\overline{R/R})^{(k+\gamma-1)}.$$ 

Since $\gamma = \text{rank}(F) < d + e$, we have

$$\lim_{k \to \infty} \frac{\lambda_R \left( E^k : F^k m^\infty / E^k : F^k m^\infty \right)}{k^{d+e-1}} = 0.$$ 

The existence of the limit (32) now follows from (34) and Theorem 4.1. The existence of the limit (33) is immediate from (32) and (27).

**References**


Steven Dale Cutkosky, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail address: cutkoskys@missouri.edu