RESOLUTION OF SINGULARITIES FOR 3-FOLDS IN POSITIVE CHARACTERISTIC

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Abstract. A complete proof of resolution of singularities of 3-folds in positive characteristic > 5 is given which is short. Abhyankar first proved this theorem in 1966.

1. Introduction

The purpose of this paper is to give a complete proof of resolution of singularities of 3-folds in positive characteristic which is short. The paper is essentially self contained, only using some material from my introductory book, “Resolution of singularities” [C], for the proofs of the 2-dimensional resolution theorems Theorems 1.2 and 1.3.

Resolution of singularities of algebraic varieties over fields of characteristic zero was solved in all dimensions in a very complete form by Hironaka [H1] in 1964.

Resolution of singularities of varieties over fields of positive characteristic is significantly harder. One explanation for this is the lack of “hypersurfaces of maximal contact” in positive characteristic, which is the main technique used in characteristic zero proofs to reduce to a lower dimension (Sections 6.2 and 7.4 [C]).

There are several proofs of resolution of singularities of surfaces in positive characteristic. The first one is by Abhyankar [Ab8] which appeared in 1955. There are later proofs by Hironaka, outlined in his 1964 Wood’s Hole notes [H2], and by Lipman [L2] for excellent surfaces.

There is only one complete published proof of resolution of singularities of 3-folds of positive characteristic. This is Abhyankar’s 1966 proof which appears in his book [Ab1] and the papers [Ab4] - [Ab7]. The entire proof encompasses 508 pages.

Recently, there has been a resurgence of interest in resolution in positive characteristic. Some papers making progress on resolution in positive characteristic are [Co1], [Co2], [Co3], [CoP], [Hau1],[H3], [Jo], [Ka], [Ku],[M], [P], [S], [T]. Several simplifications of Hironaka’s proof of characteristic zero resolution have appeared, including [AJo], [BrM], [BP], [BEV], [Hau2], [Kn], [V], [W].

We prove the following theorem.

Theorem 1.1. Suppose that $V$ is a projective variety of dimension 3 over an algebraically closed field $k$ of characteristic $\neq 2,3$ or 5. Then there exists a nonsingular projective variety $W$ and a birational morphism $\phi : W \to V$, which is an isomorphism above the nonsingular locus of $V$.

The restrictions on the field $k$ in the statement of Theorem 1.1 are those of Abhyankar’s original statement.

Essential ingredients in the proof are the following strong theorems on embedded resolution of surface singularities.

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Theorem 1.2. (Embedded resolution of surface singularities) Suppose that $X$ is a nonsingular 3-dimensional variety over an algebraically closed field $k$, and $S$ is a reduced surface (a pure 2-dimensional reduced closed subscheme) in $X$. Then there exists a sequence of monoidal transforms

$$\pi : X_n \to X_{n-1} \to \cdots \to X$$

such that the strict transform $S_n$ of $S$ on $X_n$ is nonsingular, and the divisor $\pi^*(S)$ is a SNC divisor on $X_n$. Further, each $X_i \to X_{i-1}$ is the blow up of a point or nonsingular curve in the locus in $X_{i-1}$ where the preimage of $S$ is not a SNC divisor.

Theorem 1.3. (Principalization of ideals) Suppose that $X$ is a nonsingular 3-dimensional variety over an algebraically closed field $k$ and $I \subset O_X$ is an ideal sheaf on $X$. Then there exists a sequence of monoidal transforms

$$X_n \to X_{n-1} \to \cdots \to X$$

such that $IO_{X_n}$ is invertible. Further, each $X_i \to X_{i-1}$ is the blow up of a point or nonsingular curve in the locus in $X_{i-1}$ where $IO_{X_{i-1}}$ is not invertible.

Theorems 1.2 and 1.3 were first proven by Abhyankar in the book [Ab1], relying on material proven in [Ab4] - [Ab7].

The proof of Theorem 1.1 can be broken down into 5 steps, which we enumerate:

1. Prove Theorems 1.2 and 1.3 (embedded resolution of surface singularities and principalization of ideals).
2. Prove that a projective variety of dimension $n$ is birationally equivalent to a normal projective variety all of whose points have multiplicity $\leq n!$.
3. Prove that points of low multiplicity can be resolved by performing a generic projection to a nonsingular variety, and resolving the branch locus of the mapping to reduce to the case of toric singularities. Then resolve these singularities locally.
4. Use a generalization of the patching method of Zariski [Z1] to produce a nonsingular projective variety, birationally equivalent to our original variety.
5. Modify a resolution of singularities $W \to V$ to produce a resolution $W_1 \to V$ which is an isomorphism over the nonsingular locus of $V$.

We give new proofs of each of these steps. The first step is the most difficult. We start with our proof in Chapter 7 of [C], which is a working out of the Wood’s Hole notes of Hironaka [H2] on his proof of resolution of singularities of surfaces. We extend this to a proof of embedded resolution. The proof of Theorem 1.3 requires an extension of the methods of Chapter 7 of [C] from the case of a single equation to an ideal. Our proof also incorporates elements of Abhyankar’s “good point” method ([Ab9], [L1] and chapters 5, 6 of [C]). We indicate the changes which must be made in the proof. The first step is proven in Sections 3 – 5 of this paper.

Our proof of the second step is more geometric in flavor. The origin of this projection method is by Albanese [Alb]. Albanese’s proof is discussed on pages 19 - 22 of [Z2], and in [L1]. We use some techniques of intersection theory in the proof of this step. One of them is the use of a Bezout theorem for an intersection of hyperplane sections which is not necessarily pure dimensional ((12.3.1) [Ab1] or 12.3.1 [F]). The second step is proven in Section 6.

The proof of the third step given in this paper is close to Abhyankar’s proof. We have however made some simplifications in the proof, which are especially made possible by assuming from the outset that everything is over an algebraically closed ground field. We use Theorem 1.2 on embedded resolution of surface singularities in this step. The main method used in this step is a generalization by Abhyankar to
positive characteristic of a technique of Jung. Jung’s method [J] is discussed on pages 16 - 17 of [Z2]. The third step is proven in Sections 7 and 8.

The patching argument in the fourth step is proven directly. This is where the theorem on principalization of ideals (Theorem 1.3) is required. Zariski’s original proof made use of general Bertini theorems which are not valid in positive characteristic. The fourth step is proven in Section 9.

The final fifth step is given in Section 10. The original proof of Abhyankar does not produce a resolution which is an isomorphism over the nonsingular locus. The fact that a resolution can be birationally modified (in dimension 3) to produce a resolution which is an isomorphism over the nonsingular locus is due to Cossart [Co3]. Our proof of this section is based on his argument.

I dedicate this paper to Zariski, Abhyankar, Hironaka and Lipman.

2. Preliminaries

In this section we establish notation, and review a few results that we will use in the course of the proof.

If $R$ is a local ring, we will denote the maximal ideal of $R$ by $M(R)$. Suppose that $S$ is another local ring. We will say that $S$ dominates $R$ if there is an inclusion $R \rightarrow S$ of rings such that $M(S) \cap R = M(R)$.

Suppose that $\Lambda \subset R$ is a subset. We define the closed subset $V(\Lambda) = \{ P \in \text{Spec}(R) \mid \Lambda \subset P \}$ of $\text{Spec}(R)$.

We will consider the order $\nu_R(I)$ of an ideal $I$ in $R$. $\nu_R(I)$ is defined to be the largest integer $n$ such that $I \subset M(R)^n$. If $q$ is a point on a variety $W$, and $J$ is an ideal sheaf on $W$, then we denote $\nu_q(J) = \nu_{O_{W,q}}(J_q)$.

Multiplicity. Suppose that $R$ is a $d$ dimensional Noetherian local ring, and that $q$ is an $M(R)$ primary ideal of $R$. There exists a degree $d$ polynomial $P_q(n)$ which has the property that the length

$$\ell(R/q^n) = P_q(n)$$

for $n \gg 0$. The multiplicity $e(q)$ is defined to be $d!$ times the leading coefficient of $P_q(n)$. The multiplicity of $R$ is defined to be $e(R) = e(M(R))$. From the definition we infer that if $q' \subset q$ are $M(R)$-primary, then $e(q) \leq e(q')$. Suppose that $R^*$ is the $M(R)$-adic completion of $R$. We see that $e(qR^*) = e(q)$.

The multiplicity has the important property that $e(R) = 1$ if and only if $R$ is regular (Theorem 23, Section 10, Chapter VIII [ZS]).

**Theorem 2.1.** Suppose that $R/M(R)$ is infinite. Then there exists an ideal $q' \subset q$ generated by a system of parameters such that $e(q') = e(q)$.

This is proven in Theorem 22, Section 10, Chapter VIII [ZS].

**Theorem 2.2.** Suppose that $R$ is a local domain, and $T$ is an overring of $R$ which is a domain and a finite $R$ module. Then $T$ is a semilocal ring. Let $p_1, \ldots, p_s$ be the maximal ideals of $T$. Let $K$ be the quotient field of $R$ and $M$ be the quotient field of
Then
\[ [M : K]e(q) = \sum_{i=1}^{s} \frac{T/p_i : R/M(R)}{e(q_{T/p_i})}. \]

This is proven in Corollary 1 to Theorem 24, Section 10, Chapter VIII [ZS].

**Intersection multiplicity and degree.** We use the notation of Fulton [F] for intersection theory. The intersection class of a \( k \)-cycle \( \alpha \) with a Cartier divisor \( D \) on a scheme \( X \) is denoted by \( D \cdot \alpha \). If \( D_1, \ldots, D_k \) are Cartier divisors on \( X \), the intersection number of \( D_1, \ldots, D_k \) and \( \alpha \) is denoted by

\[ \int_X D_1 \cdots D_k \cdot \alpha. \]

The degree \( \deg Y \) of a subscheme \( Y \) of \( \mathbb{P}^n \) of dimension \( d \) is \( d! \) times the leading coefficient of the Hilbert Polynomial \( P_Y(n) \) of the homogeneous coordinate ring of \( Y \). It can also be computed as \( \deg Y = \int_{\mathbb{P}^n} H^d \cdot Y \) where \( H \) is a hyperplane of \( \mathbb{P}^n \).

Proofs are given in Section 7 of Chapter 1 [Ha] and the Corollary of Section 8.5 [I], or in Examples 2.5.1 and 2.5.2 [F].

**Lemma 2.3.** Suppose that \( R \) is a local ring of dimension \( d \), and let \( Y = \text{Proj}(\oplus_{n\geq 0} M(R)^n/M(R)^{n+1}) \).

Then \( \deg Y = e(R) \).

This follows from comparing the Hilbert polynomial of \( Y \), which is equal to

\[ \ell(M(R)^n/M(R)^{n+1}) \]

for all large \( n \), and the characteristic polynomial \( P_{M(R)}(n) \) of \( R \).

**Completion.** Suppose that \( k \) is an algebraically closed field, and \( R \) is a normal local ring, which is essentially of finite type over \( k \) (a localization of a ring which is of finite type over \( k \)). The \( M(R) \)-adic completion \( R^* \) of \( R \) is a normal local ring (Theorem 32, Section 13, Chapter VIII [ZS]). Suppose that \( L \) is a finite field extension of the quotient field of \( R^* \). Then the integral closure of \( R^* \) in \( L \) is a local domain (since \( R^* \) is Henselian, Corollary 2 to Theorem 18, Section 7, Chapter VIII [ZS]).

We also state here the Zariski subspace theorem. For a proof, see (10.10) [Ab1].

**Theorem 2.4.** Suppose that \( R \) and \( S \) are normal local rings which are locally of finite type over \( k \). Suppose that \( S \) dominates \( R \). Let \( R^* \) be the \( M(R) \)-adic completion of \( R \) and \( S^* \) be the \( M(S) \)-adic completion of \( S \). Then the natural morphism \( R^* \to S^* \) is an inclusion.

**Birational geometry.** In this paper, a variety is an open subset of an integral, closed subscheme of \( \mathbb{P}^n \). A curve, surface or 3-fold is a variety of the corresponding dimension.

The definition of variety in [C] is a little more general. In [C], a variety is an open subset of a reduced, equidimensional subscheme of \( \mathbb{P}^n \). The resolution theorems for varieties in [C] are thus valid for reduced equidimensional subschemes of \( \mathbb{P}^n \).

Suppose that \( X \) is a projective variety over an algebraically closed field \( k \). Let \( K \) be the function field of \( X \). Suppose that \( V \) is a valuation ring of \( K \) which contains \( k \). Then there exists a unique (not necessarily closed) point \( p \in X \) such that \( V \) dominates the local ring \( \mathcal{O}_{X,p} \). \( p \) is called the center of \( V \) on \( X \).

Suppose that \( \Phi : X \to Y \) is a birational morphism of normal projective varieties. The exceptional locus of \( \Phi \) is the closed subscheme of \( X \) on which \( \Psi \) is not an
isomorphism. Irreducible components $T$ of the exceptional locus may have any dimension $1 \leq \dim T \leq \dim X - 1$ (If $Y$ is nonsingular all components have dimension $\dim X - 1$.)

Now suppose that $\Phi : X \to Y$ is a birational map of normal projective varieties. There is a largest open set $U$ of $X$ such that $\Phi \mid U$ is a morphism. The complement, $F = X - U$ is called the fundamental locus of $\Phi$. By Zariski’s main theorem (Lemma V.5.1 and Theorem V.5.2 [Ha]), the fundamental locus $F$ has codimension $\geq 2$ in $X$.

Suppose that $X$ is a nonsingular variety and $Y$ is a nonsingular subvariety. The monoidal transform of $X$ with center $Y$ is the blow up $X_1 \to X$ of $Y$. $X_1$ is a nonsingular variety.

Suppose that $V$ is a valuation ring of the function field $K$ of $X$ and the center of $V$ on $X$ is $p$ and the center of $V$ on $X_1$ is $p_1$. Then $\mathcal{O}_{X,p} \to \mathcal{O}_{X_1,p_1}$ is called a monodial transform along $V$.

We will make use of the fact that any birational morphism $X \to Y$ of projective varieties is the blow up of an ideal sheaf (Theorem 7.17 [Ha]), and the “universal property of blow ups” (Proposition 7.14 [Ha]).

Suppose that $X \to Y$ is a dominant rational map of varieties. If the extension of function fields $k(Y) \to k(X)$ is finite, we denote $[X : Y] = [k(X) : k(Y)]$.

Suppose that $X$ is a nonsingular variety. An effective divisor $D$ on $X$ is a simple normal crossings divisor on $X$ if for each $p \in X$ there exist regular parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{X,p}$ and $a_1, \ldots, a_n \in \mathbb{N}$ such that $D$ is the divisor $x_1^{a_1} \cdots x_n^{a_n} = 0$ in a neighborhood of $p$. We will abbreviate a simple normal crossings divisor as a SNC divisor. Suppose that $X$ is a nonsingular variety, $D$ is a SNC divisor on $X$ and $Y$ is a subscheme of $X$. We will say that $Y$ makes SNCs with $D$ if $Y$ is nonsingular, and for $p \in Y \cap D$, there exist regular parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{X,p}$, $r \leq n$ and $a_1, \ldots, a_n \in \mathbb{N}$ such that $D$ is the divisor $x_1^{a_1} \cdots x_n^{a_n} = 0$ in a neighborhood of $p$, and $V(x_1, \ldots, x_r) \subset \text{Spec}(\mathcal{O}_{X,p})$ is the germ of $Y$.

If $D_1, D_2$ are two Cartier divisors on a variety $X$, we will write $D_1 \sim D_2$ if $D_1$ is linearly equivalent to $D_2$.

3. The $\nu$ and $\tau$ invariants

Suppose that $V$ is a nonsingular variety over an algebraically closed field $k$ of characteristic zero, and $I$ is an ideal sheaf of $V$. Suppose that $q \in V$ is a (not necessarily closed) point. If $Z$ is a subscheme of $V$ with ideal sheaf $\mathcal{I}_Z$, define $\nu_q(Z) = \nu_q(\mathcal{I}_Z)$ (with notation as in Section 2).

For $t \in \mathbb{N}$, let

$$\text{Sing}_q(I) = \{p \in V \mid \nu_p(I) \geq t\},$$

which is a Zariski closed subset of $V$ (Theorem A.19 [C]). Let

$$r = \max\{t \mid \text{Sing}_q(I) \neq \emptyset\} = \max\{\nu_p(I) \mid p \in V\}.$$

Suppose that $Y$ is a nonsingular, integral subvariety of $V$. Let $\pi_1 : V_1 \to V$ be the blow up of $Y$ with exceptional divisor $F$. Let $t = \nu_q(\mathcal{I}_q)$, where $q$ is the generic point of $Y$. Then the weak transform $I_1$ of $I$ on $V_1$ is (page 65 [C])

$$I_1 = \mathcal{O}_{V_1}(-tF) \mathcal{I} \mathcal{O}_{V_1}.$$

If $Z$ is the subscheme of $V$ defined by $I$, then the weak transform $Z_1$ of $Z$ is the subscheme of $V_1$ defined by $I_1$. If $Z$ is a divisor on $V$, then the weak transform of $Z$ is the strict transform of $Z$ (page 38, page 65 [C]).
Suppose that \( p \in \text{Sing}_r(\mathcal{I}) \) is a closed point, \( x_1, \ldots, x_n \) are regular parameters in \( \mathcal{O}_{V,p} \) and \( f \in \mathcal{I}_p \) is such that \( \nu_p(f) = r \). There is an expansion

\[
f = \sum_{i_1 + \cdots + i_n \geq r} a_{i_1, \ldots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}
\]

with \( a_{i_1, i_2, \ldots, i_n} \in k \) in \( \mathcal{O}_{V,p} = k[[x_1, \ldots, x_n]] \). The leading form \( L \) of \( f \) is defined to be

\[
L(x_1, x_2, \ldots, x_n) = \sum_{i_1 + \cdots + i_n = r} a_{i_1, i_2, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n}.
\]

Define \( \tau(p) = \tau_r(p) \) to be the dimension of the smallest linear subspace \( M \) of the \( k \)-subspace spanned by \( x_1, x_2, \ldots, x_n \) in \( k[[x_1, \ldots, x_n]] \) such that \( L \in k[M] \) for all \( f \in \mathcal{I}_p \) with \( \nu_p(f) = r \). We will call the subvariety \( D_p = V(M) \) of \( \text{Spec}(\mathcal{O}_{V,p}) \) an approximate manifold to \( \mathcal{I} \) at \( p \). \( M \) depends on our choice of regular parameters at \( p \), although its dimension, \( n - \tau(p) \), does not. After making a choice of \( x, y, z \), \( M \) is uniquely determined. It follows that if \( C \subset \text{Sing}_r(\mathcal{I}) \) is a nonsingular subvariety with \( p \in C \), then there exists an approximate manifold \( D_p \) to \( \mathcal{I} \) at \( p \) such that the germ of \( C \) at \( p \) is contained in \( D_p \).

**Lemma 3.1.** Suppose that \( p \in \text{Sing}_r(\mathcal{I}) \), \( \pi_1 : V_1 \to V \) is the blow up of \( p \), \( \mathcal{I}_1 \) is the weak transform of \( \mathcal{I} \) on \( V_1 \), \( D_p \) is an approximate manifold to \( \mathcal{I} \) at \( p \), and \( q \in \pi_1^{-1}(p) \). Then

1. \( \nu_q(\mathcal{I}_1) \leq r \)
2. \( \nu_q(\mathcal{I}_1) = r \) implies \( q \) is on the strict transform of \( D_p \) and \( \tau(p) \leq \tau(q) \)

**Proof.** This follows from Lemma 6.4 and Lemma 7.4 [C]. \( \square \)

With the hypotheses of Lemma 3.1, we immediately conclude that \( \tau(p) = n \) implies \( \nu_q(\mathcal{I}_1) < r \).

**Lemma 3.2.** Suppose that \( \text{dim}(V) = 3 \), \( C \subset \text{Sing}_r(\mathcal{I}) \) is a nonsingular curve, \( \pi_1 : V_1 \to V \) is the blow up of \( C \), \( \mathcal{I}_1 \) is the weak transform of \( \mathcal{I} \) on \( V_1 \), \( p \in C \), \( D_p \) is an approximate manifold to \( \mathcal{I} \) at \( p \) containing the germ of \( C \) at \( p \), and \( q \in \pi_1^{-1}(p) \). Then

1. \( \nu_q(\mathcal{I}_1) \leq r \)
2. \( \nu_q(\mathcal{I}_1) = r \) implies \( q \) is on the strict transform of \( D_p \) and \( \tau(p) \leq \tau(q) \)

**Proof.** This follows from Lemma 6.4 and Lemma 7.5 [C]. \( \square \)

With the hypotheses of Lemma 3.2, we immediately conclude that \( \tau(p) \leq 2 \) and \( \tau(p) = 2 \) implies \( \nu_q(\mathcal{I}_1) < r \).

## 4. Embedded resolution of surfaces

In this section, we suppose that \( V \) is a nonsingular 3-fold (a 3-dimensional variety over an algebraically closed field \( k \)), \( S \) is a reduced surface (a pure 2-dimensional reduced closed subscheme) in \( V \). We will prove embedded resolution of surface singularities, which is stated in Theorem 1.2 of the introduction. We will make use of material and notation from the previous section.

**Lemma 4.1.** Suppose that \( T = \text{Spec}(R) \) where \( R \) is a 2-dimensional power series ring, over an algebraically closed field \( k \), and \( C \) is a reduced divisor on \( T \). There exists a sequence of blow ups of closed points \( \pi : T_1 \to T \) such that the strict transform \( C_1 \) of \( C \) on \( T_1 \) is nonsingular, and the divisor \( \pi^*(C_1) \) is a SNC divisor on \( T_1 \).
Proof. By Theorem 3.15 [C], there exists a sequence of blow ups of points \( \pi_1 : T_1 \to T \) such that the strict transform \( C_1 \) of \( C \) is nonsingular. Then \( \pi_1^* (C) = E + C_1 \) where \( E \) is a divisor supported on the exceptional locus of \( \pi_1 \), and is consequently a SNC divisor on \( T_1 \). Suppose that \( p \in T_1 \) is a point where \( \pi_1^* (C) \) is not a SNC divisor. By the Weierstrass preparation theorem (Theorem 5, Section 1, Chapter VII [ZS]), there exist regular parameters \( x, y \) in \( \mathcal{O}_{T_1, p} \) such that in \( \mathcal{O}_{T_1, p} \), a local equation \( f = 0 \) of \( \pi_1^* (C) \) has the form

\[
(1) \quad f = x^a y^b (y + h(x)) = 0.
\]

Let \( r \geq 1 \) be the order of \( h(x) \).

Let \( \pi_2 : T_2 \to T_1 \) be the blow up of \( p \). Let \( D = (\pi_1 \circ \pi_2)^* (C) \). Suppose that \( p_1 \in \pi_2^{-1} (p) \). Then there are regular parameters \( x_1, y_1 \) in \( \mathcal{O}_{T_2, p_1} \) such that

\[
(2) \quad x = x_1, y = x_1 y_1
\]
or

\[
(3) \quad y = y_1, x = y_1 (x_1 + \alpha)
\]

for some \( \alpha \in k \).

Substituting these forms into (1), we see that if (2) holds, then \( D \) is a SNC divisor at \( p_1 \) if \( r = 1 \), and there is a form (1) at \( p_1 \) with a reduction of \( r \) by one other otherwise. Further, \( D \) is a SNC divisor at \( p_1 \) if (3) holds. Thus after a finite number of blowups of points, we reach the conclusion of the lemma.

**Definition 4.2.** A resolution datum \( \mathcal{R} = (E^+, E^-, S, V) \) is a 4-tuple where \( E^+ \) and \( E^- \) are effective divisors on a nonsingular 3 dimensional variety \( V \) over an algebraically closed field \( k \), such that \( E^+ + E^- \) is a SNC divisor, and \( S \) is a reduced surface (a pure 2-dimensional reduced closed subscheme) in \( V \).

\( \mathcal{R} \) is resolved at \( p \in S \) if \( S \) is nonsingular at \( p \) and \( E^+ + E^- + S \) is a SNC divisor at \( p \). For \( r \geq 1 \), let

\[
\text{Sing}_{r} (\mathcal{R}) = \{ p \in S \mid \nu_p(S) \geq r \text{ and } \mathcal{R} \text{ is not resolved at } p \}.
\]

\( \text{Sing}_{r} (\mathcal{R}) \) is a proper Zariski closed subset of \( S \) for \( r \geq 1 \). For \( p \in S \), let

\[
\eta(p) = \text{ the number of components of } E^- \text{ containing } p.
\]

We have \( 0 \leq \eta(p) \leq 3 \).

Let

\[
r = \nu(\mathcal{R}) = \nu(S, E^+ + E^-) = \max \{ \nu_p(S) \mid p \in S \text{ and } \mathcal{R} \text{ is not resolved at } p \}.
\]

Let \( \tau(p) = \tau_{E^-} (p) \) for \( p \in S \).

If \( r = 1 \) and \( p \in \text{Sing}_r (\mathcal{R}) \), then \( p \in S \) is a nonsingular point, so that the tangent plane to \( S \) at \( p \) is an approximate manifold.

**Definition 4.3.** A permissible transform of \( \mathcal{R} \) is a monoidal transform \( \pi_1 : V_1 \to V \) whose center is either a point of \( \text{Sing}_r (\mathcal{R}) \) or a nonsingular curve \( C \subset \text{Sing}_r (\mathcal{R}) \) such that \( C \) makes SNCs with \( E^+ + E^- \).

Let \( F \) be the exceptional divisor of a permissible transform \( \pi : V_1 \to V \) of \( \mathcal{R} \). Then \( \pi_1^* (E^+ + E^-, \text{red}) + F \) is a SNC divisor. We define the transform \( \mathcal{R}_1 \) of \( \mathcal{R} \) to be

\[
\mathcal{R}_1 = (E^+_1, E^-_1, S_1, V_1)
\]

where \( S_1 \) is the strict transform of \( S \), and

\[
E^+_1 = \pi_1^* (E^+ \text{red}) + F, E^-_1 = \text{the strict transform of } E^-\]

if \( \nu(S_1, \pi_1^* ((E^+_1 + E^-_1) \text{red}) + F) = \nu(\mathcal{R}), \)

\[
E^+_1 = \emptyset, E^-_1 = \pi_1^* (E^+ + E^-) \text{red} + F
\]
if $\nu(S_1, \pi_1^*((E_1^+ + E_1^-)_\text{red} + F) < \nu(R)$.

**Lemma 4.4.** With the notation of Definition 4.3,

$$\nu(R_1) \leq \nu(R)$$

and

$$\nu(R_1) = \nu(R) \text{ implies } \eta(R_1) \leq \eta(R).$$

The proof is as in Lemma 5.22 [C].

**Theorem 4.5.** Suppose that $p \in \text{Sing}_r(R)$, $\tau(p) = 1$, and an approximate manifold $D_p$ of $S$ at $p$ makes SNCs with $E^+$ and is not a component of $E^-$. Suppose that $\pi_1 : V_1 \to V$ is the blow up of $p$ on a nonsingular curve in $\text{Sing}_r(R)$ which makes SNCs with $E^+ + E^-$ and contains $p$. Let $R_1$ be the transform of $R$ on $V_1$. Suppose that $p_1 \in \pi_1^{-1}(p) \cap \text{Sing}_r(R_1)$ and $\tau(p_1) = 1$. Let $D_{p_1}$ be an approximate manifold of $S_1$ at $p_1$. Then $D_{p_1}$ makes SNCs with $E_1^+$, and is not a component of $E_1^+$. 

**Proof.** First suppose that $\pi_1$ is the blow up of $p$. There exist regular parameters $x, y, z$ in $\mathcal{O}_{V,p}$ such that $z = 0$ is an approximate manifold at $p$, and if $p \in E^+$, then $x = 0$, $y = 0$ or $xy = 0$ is a local equation of $E^+$ at $p$. Let $f(x, y, z) = 0$ be a local equation of $S$ at $p$ in $\mathcal{O}_{V,p}$. The leading form $L$ of $f$ is $L = a x^r$ for some $0 \neq a \in k$.

At $p_1$, by Lemma 3.1, we have regular parameters $x_1, y_1, z_1$ in $\mathcal{O}_{V_1,p_1}$, such that either

(4) 
$$x = x_1, y = x_1(y_1 + \alpha), z = x_1z_1$$

for some $\alpha \in k$, or

(5) 
$$x = x_1y_1, y = y_1, z = x_1z_1.$$ 

After possibly interchanging $x$ and $y$, we may assume that (4) holds. We see that a local equation for $S_1$ at $p_1$ in $\mathcal{O}_{V_1,p_1}$ is then $f_1(x_1, y_1, z_1) = \frac{f(x, y, z)}{x_1^r} = 0$. Since $f_1$ is assumed to have multiplicity $r$ at $p_1$ and $\tau(p_1) = 1$, we see that there exists $\beta \in k$ such that $z_1 - \beta x_1 = 0$ is an approximate manifold for $S_1$ at $p_1$. Since $E_1^+$ is supported on $x_1y_1 = 0$, we see that $E_1^+$ makes SNCs with $D_{p_1}$.

Now suppose that $\pi_1$ is the blow up of a curve $C$. There exist regular parameters $x, y, z$ in $\mathcal{O}_{V,p}$ such that $z = 0$ is an approximate manifold at $p$, $x = z = 0$ are local equations of $C$ at $p$, and if $p \in E^+$, then $x = 0$, $y = 0$ or $xy = 0$ is a local equation of $E^+$ at $p$. Let $f(x, y, z) = 0$ be a local equation of $S$ at $p$ in $\mathcal{O}_{V,p}$. The leading form $L$ of $f$ is $L = a x^r$ for some $0 \neq a \in k$.

At $p_1$, by Lemma 3.2, we have regular parameters $x_1, y_1, z_1$ in $\mathcal{O}_{V_1,p_1}$ such that

(6) 
$$x = x_1, y = y_1, z = x_1z_1.$$ 

A local equation for $S_1$ at $p_1$ in $\mathcal{O}_{V_1,p_1}$ is $f_1(x_1, y_1, z_1) = \frac{f(x, y, z)}{x_1^r} = 0$. Since $f_1$ is assumed to have multiplicity $r$ at $p_1$ and $\tau(p_1) = 1$, we see that there exists $\beta \in k$ such that $z_1 - \beta x_1 = 0$ is an approximate manifold for $S_1$ at $p_1$. Since $E_1^+$ is supported on $x_1y_1 = 0$, we see that $E_1^+$ makes SNCs with $D_{p_1}$. 

**Theorem 4.6.** Let $R = (\emptyset, E, S, V)$. Then there exists a sequence of permissible transforms $\pi : V_1 \to V$ such that if $R_1$ is the transform of $R$ on $V_1$, then

1. $\text{Sing}_r(R_1) \subset E_1^+$
2. For $p \in \text{Sing}_r(R_1)$ such that $\tau(p) = 1$, an approximate manifold $D_p$ of $S_1$ at $p$ makes SNCs with $E_1^+$ and is not a component of $E_1^+$. 


Proof. There exists by Corollary 4.4 [C], a sequence of blow ups of points \( \psi_1 : W_1 \to V \), with transform \( \mathcal{R}_1' = ((E_1^+)'(E_1^-)'(S_1', W_1)) \) of \( \mathcal{R} \) on \( W_1 \), such that the strict transform of \( \text{Sing}_r(\mathcal{R}) \) on \( W_1 \) is a disjoint union of nonsingular curves and points.

Suppose that \( C \) is an integral, one dimensional component of the strict transform of \( \text{Sing}_r(\mathcal{R}) \).

First suppose that \( \psi_1(C) \not\subset E \). Suppose that \( p \in W_1 \) is a point where \( C \) does not make SNCs with \((E_1^+)' + (E_1^-)'\). There exist regular parameters \( x, y, z \) in \( \mathcal{O}_{W_1, p} \) such that \( x = y = 0 \) are local equations of the strict transform of \( \text{Sing}_r(\mathcal{R}) \) at \( p \). Let \( f(x, y, z) = 0 \) be a local equation of \((E_1^+)' + (E_1^-)'\) at \( p \) in \( \mathcal{O}_{W_1, p} \). We necessarily have that \( f \not\in (x, y)\mathcal{O}_{W_1, p} \). Let \( \psi_2 : W_2 \to W_1 \) be the blow up of \( p \), and suppose that \( p_1 \in \pi_2^{-1}(p) \) is on the strict transform of \( \text{Sing}_r(\mathcal{R}) \). Then there exist regular parameters \( x_1, y_1, z_1 \) in \( \mathcal{O}_{W_2, p_1} \) such that \( x = x_1 z_1, y = y_1 z_1, z = z_1 \).

Since \( f \not\in (x, y) \), we see by substituting \( x = x_n z_n, y = y_n z_n, z = z_n \) into \( f(x, y, z) \), that after finitely many blow ups of points \( \psi_2 : W_2 \to W_1 \), the strict transform of \( C \) is disjoint from the strict transform of \((E_1^+)' + (E_1^-)'\).

Now suppose that \( \psi_1(C) \subset E \). If \( \psi_1(C) \) is contained in two components of \( E \), then \( C \) necessarily makes SNCs with \((E_1^+)' + (E_1^-)'\). Suppose that \( \psi_1(C) \) is contained in only one component \( D \) of \( E \). Let \( D_1 \) be the strict transform of \( D \) on \( W_1 \). By Lemma 4.1, there exists a finite sequence of blow ups of points \( \psi_2 : W_2 \to W_1 \) such that \( \mathcal{R}_2' = ((E_2^+_1)'(E_2^-_1)'(S_2', W_2)) \) is the transform of \( \mathcal{R}_1' \) on \( W_2 \), and \( D_2 \) is the strict transform of \( D_1 \) on \( W_2 \), then \((E_2^+_1)' + (E_2^-_1)'' - D_2 \subset D_2 \) makes SNCs with \( C \) on \( D_2 \). Thus \( C \) makes SNCs with \((E_2^+_1)' + (E_2^-_1)'' \) on \( W_2 \).

It follows that there exists a finite sequence of blow ups of points \( \pi_1 : V_1 \to W_1 \), such that if \( \mathcal{R}_3 = (E_1^+, E_1^-, S_1, V_1) \) is the transform of \( \mathcal{R}' \) on \( V_1 \), then the strict transform of \( \text{Sing}_r(\mathcal{R}) \) on \( V_1 \) is a disjoint union of isolated points and nonsingular curves which make SNCs with \( E_1^+ + E_1^- \). Then the blow up \( \pi_2 : V_2 \to V_1 \) of these curves and points is a permissible transform.

Let \( \mathcal{R}_2 = (E_2^+, E_2^-, S_2, V_2) \) be the transform of \( \mathcal{R}_1 \) on \( V_2 \). By our construction, \( \text{Sing}_r(\mathcal{R}_2) \subset E_2^+ \), so that 1 of the conclusions of Theorem 4.6 holds. By Theorem 4.5, 2 of the conclusions of Theorem 4.6 holds.

\[ \square \]

Suppose that \( \mathcal{R} \) satisfies the conclusions of Theorem 4.6. Then 1 of Theorem 4.6 and Lemmas 3.1 and 3.2 imply that any curve \( C \subset \text{Sing}_r(\mathcal{R}) \) is nonsingular.

**Theorem 4.7.** Suppose that the conclusions of Theorem 4.6 hold for \( \mathcal{R} = (E^+, E^-, S, V) \). Then there exists a sequence of permissible transforms \( \pi_1 : V_1 \to V \) such that if \( \mathcal{R}_4 \) is the transform of \( \mathcal{R} \) on \( V_1 \), then

1. \( \text{Sing}_r(\mathcal{R}_4) \subset E_1^+ \).
2. If \( p \in \text{Sing}_r(\mathcal{R}_4) \), where \( \mathcal{R}_4 \) is the transform of \( \mathcal{R} \) on \( V_1 \), then \( \eta(p) = 0 \) (so that \( p \not\in E_1^- \)).
3. If \( p \in \text{Sing}_r(\mathcal{R}_4) \) and \( \tau(p) = 1 \), then an approximate manifold \( D_p \) of \( S_1 \) at \( p \) makes SNCs with \( E_1^+ \), and is not a component of \( E_1^+ \).

**Proof.** There are at most a finite number of points \( p \in \text{Sing}_r(\mathcal{R}) \) with \( \eta(p) = 3 \). Let \( \pi_1 : V_1 \to V \) be the blow up of these points. Then \( \eta(p_1) \leq 2 \) for all \( p_1 \in \text{Sing}_r(\mathcal{R}_4) \).
Now suppose that $\eta(p) \leq 2$ for all $p \in \text{Sing}_{r}(\mathcal{R})$. Since $\text{Sing}_{r}(\mathcal{R}) \subset E^{+}$ and $E^{+} + E^{-}$ is a SNC divisor, there are only finitely many points $p \in \text{Sing}_{r}(\mathcal{R})$ with $\eta(p) = 2$.

Suppose that $\eta(p) = 2$. There exist regular parameters $x, y, z$ at $p$ in $\mathcal{O}_{V,p}$ such that $xy = 0$ is a local equation of $E^{-}$. Let $\pi_{1} : V_{1} \to V$ be the blow up of $p$. There is at most one point $p_{1} \in \pi_{1}^{-1}(p)$ which is in $\text{Sing}_{r}(\mathcal{R}_{1})$ and satisfies $\eta(p_{1}) = 2$. $p_{1}$ has regular parameters $x_{1}, y_{1}, z_{1}$ in $\mathcal{O}_{V_{1},p_{1}}$ such that $x = x_{1}z_{1}, y = y_{1}z_{1}, z = z_{1}$. $x_{1}y_{1} = 0$ is a local equation of $E_{1}^{-}$.

We can iterate this construction, giving a sequence of blow ups of points

$$(7) \quad V_{1} \overset{\pi_{1}}{\to} \cdots \to V_{k} \overset{\pi_{k}}{\to} V$$

where the center of $\pi_{i}$ is the point $p_{i−1}$ on the strict transform of the curve germ $x = y = 0$, as long as $p_{1−1} \in \text{Sing}_{r}(\mathcal{R}_{i−1})$ and $\eta(p_{i−1}) = 2$.

Let $f = \sum_{i+j+k \geq r} a_{ijk} x^{i}y^{j}z^{k} = 0$ be a local equation of $S$ at $p$ in $\hat{\mathcal{O}}_{V,p}$. We have regular parameters $x_{1}, y_{1}, z_{1}$ at $p_{1}$ such that

$$x = x_{1}z_{1}^{r}, y = y_{1}z_{1}^{r}, z = z_{1},$$

$x_{1}y_{1} = 0$ is a local equation of $E_{1}^{-}$ and

$$f_{t} = \sum_{i+j+k \geq r} a_{ijk} x_{1}^{i}y_{1}^{j}z_{1}^{(i+j-r)t+k} = 0$$

is a local equation of the strict transform $S_{t}$ of $S$.

If the multiplicity of $S_{t}$ at $p_{i}$ is $r$, we have

$$(i + j - r)t + k \geq r$$

for all $i, j, k$ such that $a_{ijk} \neq 0$. If the sequence (7) does not terminate after a finite number of blow ups, we have that $i + j - r \geq r$ for all $i, j$ such that $a_{ijk} \neq 0$, which implies that $f \in (x, y)^{r}$, a contradiction. We must then eventually achieve a reduction in $r$ in the sequence (7).

We have reduced to the case where $\eta(p) \leq 1$ for all $p \in \text{Sing}_{r}(\mathcal{R})$.

Suppose that $p \in \text{Sing}_{r}(\mathcal{R})$ and $\eta(p) = 1$. There exist regular parameters $x, y, z$ at $p$ in $\mathcal{O}_{V,p}$ such that $z = 0$ is a local equation of $E^{-}$, and the germ of $E^{+} + E^{-}$ at $p$ is contained in $V(xy)$. Let

$$(8) \quad f = \sum_{j=1}^{r} a_{j}(x, y)z^{r-j} + z^{r}\Omega = 0$$

be a local equation of $S$ at $p$ in $\hat{\mathcal{O}}_{V,p}$. We have $a_{r}(x, y) \neq 0$ since $z \not\mid f$.

Suppose that $\prod_{j=1}^{r} a_{j}(x, y) = 0$ is not a SNC divisor in Spec$(k[[x, y]])$.

Let $\pi_{1} : V_{1} \to V$ be the blow up of $p$. Suppose that $p_{1} \in \pi_{1}^{-1}(p)$ is in $\text{Sing}_{r}(\mathcal{R}_{1})$ and satisfies $\eta(p_{1}) = 1$. Then $p_{1}$ is on the strict transform of $z = 0$.

$p_{1}$ thus has regular parameters $x_{1}, y_{1}, z_{1}$ in $\mathcal{O}_{V_{1},p_{1}}$ such that one of the following forms hold:

$$(9) \quad x = x_{1}, y = x_{1}(y_{1} + \alpha_{1}), z = x_{1}z_{1}$$

for some $\alpha_{1} \in k$, or

$$(10) \quad x = x_{1}y_{1}, y = y_{1}, z = y_{1}z_{1}.$$
of the strict transform $S_1$ of $S$ at $p_1$ in $\hat{O}_{V_1, p_1}$, where

$$a_{j,1}(x_1, y_1) = \frac{a_j(x_1, x_1(y_1 + \alpha_1))}{x_1^j}$$

if (9) holds, and

$$a_{j,1}(x_1, y_1) = \frac{a_j(x_1y_1, y_1)}{y_1^j}$$

if (10) holds.

Thus there are only finitely many points $p_1 \in \pi_1^{-1}(p)$ such that $\prod_{j=1}^n a_{j,1}(x_1, y_1) = 0$ is not a SNC divisor in $\text{Spec}(k[[x_1, y_1]])$.

By Lemma 4.1, we can construct a finite sequence of blow ups of points

$$V_t \overset{\pi_1}{\rightarrow} \cdots \overset{\pi_1}{\rightarrow} V_1 \overset{\pi_1}{\rightarrow} V$$

where the center of $\pi_1$ is a point $p_{i-1}$ on the strict transform of $z = 0$, $p_{i-1} \in \text{Sing}_r(\mathcal{R}_{i-1})$ and $\eta(p_{i-1}) = 1$, so that for all $p_i \in \text{Sing}_r(\mathcal{R}_t)$ above $p$, there are regular parameters $x_t, y_t, z_t$ at $p_t$ such that the strict transform $S_t$ of $S$ on $V_t$ at $p_t$ has a local equation

$$f_t = \sum_{j=1}^r x_t^{a_{jt}} y_t^{b_{jt}} z_t^{r-j} + z_t^r \Omega_t = 0$$

where the $\pi_j$ are units in $k[[x_t, y_t]]$ (or 0), $z_t = 0$ is a local equation of $E_t^-$, and the germ of $E_t^+$ at $p_t$ is contained in $V(x_t, y_t)$.

By further blowing up of points $p_i$ in $\text{Sing}_r(\mathcal{R}_t)$ with $\eta(p_i) = 1$, by Corollary 5.15 and Lemma 5.16 [C] (the combinatorial proof works in all characteristics), we can construct (11) so that for all $p_i$ above $p$ in $\text{Sing}_r(\mathcal{R}_t)$ with $\eta(p_i) = 1$, there exists $j$ with $1 \leq j \leq r$ such that $\pi_{jt} \neq 0$ in (12), and

1. \[ \frac{a_{jt}}{j} \leq \frac{a_{kt}}{k}, \quad \frac{b_{jt}}{j} \leq \frac{b_{kt}}{k} \]

for all $k$ with $1 \leq k \leq r$ and $\bar{a}_{kt} \neq 0$, while

2. \[ \left\{ \frac{a_{jt}}{j} \right\} + \left\{ \frac{b_{jt}}{j} \right\} < 1. \]

Here $\{ \lambda \}$ denotes the fractional part of $\lambda \in \mathbb{Q}$.

Since $a_{jt} + b_{jt} \geq j$, (14) implies that $a_{jt} \geq j$ or $b_{jt} \geq j$, and the curve with local equations $x_t = z_t = 0$ at $p_t$ is contained in $\text{Sing}_r(\mathcal{R}_t)$ or the curve with local equations $y_t = z_t = 0$ at $p_t$ is contained in $\text{Sing}_r(\mathcal{R}_t)$.

Suppose that $C \subset E^- \cap \text{Sing}_r(\mathcal{R})$ is a curve. Then $C \subset E^+$, so that $C$ makes SNCs with $E^+ + E^-$, and is (a connected component of) the intersection of a component of $E^+$ and a component of $E^+$. Thus the blow up $\pi_1 : V_1 \rightarrow V$ of $C$ is a permissible transform of $\mathcal{R}$. Let $\mathcal{R}_1$ be the transform of $\mathcal{R}$ on $V_1$. Let $C_1$ be the section over $C$ which is on the strict transform of $E^-$. If $C_1 \subset \text{Sing}_r(\mathcal{R}_1)$, then the blow up of $C_1$ is a permissible transform for $\mathcal{R}_1$. In this way, we construct a sequence of permissible transforms

$$V_n \overset{\pi_n}{\rightarrow} V_{n-1} \rightarrow \cdots \rightarrow V_1 \overset{\pi_1}{\rightarrow} V$$

of blow ups of sections over $C$ which are on the strict transform of $E^-$. The sequence must terminate after a finite number of steps with $C_n \not\subset \text{Sing}_r(\mathcal{R}_n)$, by the proof of Lemma 5.10 [C], which is valid in all characteristics.
There are thus only a finite number of points \(p^1, \ldots, p^n \in \text{Sing}_r(\mathcal{R})\) such that a sequence of permissible transforms (15) does not lead to a drop in \((\nu(\mathcal{R}), \eta(\mathcal{R}))\) (in the lex order). Above each \(p^k\) with \(1 \leq i \leq k\), we construct a sequence of blow ups of points (11), so that (13) and (14) hold at all points \(p^k_i\) above \(p_k\) such that \(p^k_i \in \text{Sing}_r(\mathcal{R}_i)\) and \(\eta(p^k_i) = 1\).

Let \(\pi : W \rightarrow V\) be the product of all of these sequences of blow ups of points (11) above \(p^1, \ldots, p^n\). Let \(\mathcal{R}^* = (E^*_1, E^*_2, S, W)\) be the transform of \(\mathcal{R}\) on \(W\).

By our construction, there can be no isolated points \(p\) in \(\text{Sing}_r(\mathcal{R}^*)\) which satisfy \(\eta(p) = 1\).

Suppose that \(C \subset E^*_1 \cap \text{Sing}_r(\mathcal{R}^*)\) is a curve such that \(\eta(p) = 1\) for \(p \in C\). Let \(\pi_1 : W_1 \rightarrow W\) be the blow up of \(C\). \(\pi_1\) is a permissible transform of \(\mathcal{R}^*\). Let \(\mathcal{R}_1^*\) be the transform of \(\mathcal{R}^*\) on \(W_1\).

Suppose that \(q \in (\pi \circ \pi_1)^{-1}(p^k)\) for some \(p^k, q \in \text{Sing}_r(\mathcal{R}_1^*)\) and \(\eta(q) = 1\). Then (12) - (14) hold at \(p^k_{t}\). \(C\) has local equations \(x_i = z_i = 0\) or \(y_i = z_i = 0\) at \(p^k_{t}\). We may assume that \(x_i = z_i = 0\) are local equations of \(C\) at \(p^k_{t}\). Thus there are regular parameters \(x_s, y_s, z_s\) in \(\mathcal{O}_{W_1,q}\) such that \(x_s = x_s, y_s = y_s, z_s = z_s\). We see that (12) - (14) hold at \(q\) for a local equation of the strict transform of \(S\), but \(\frac{d^n}{dt^n}\) decreases by 1.

We may thus construct a sequence \(\pi_2 : V_2 \rightarrow V\) of blow ups of curves on the strict transform of \(E^-_1\) which are permissible transforms for the transform of \(\mathcal{R}^*\), so that the transform \(\mathcal{R}_2^*\) of \(\mathcal{R}^*\) satisfies \(\eta(p) = 0\) at all points in \(\text{Sing}_r(\mathcal{R}_2^*)\) above \(p\).

By Theorem 4.5, the conclusions of Theorem 4.7 now hold.

**Lemma 4.8.** Suppose that \(\mathcal{R} = (E^+, E^-, S, V)\) satisfies the conclusions of Theorem 4.7. Suppose that \(p \in \text{Sing}_r(\mathcal{R})\) is such that \(\tau(p) = 2\), and there exists a curve \(C\) in \(\text{Sing}_r(\mathcal{R})\) which contains \(p\) (so that \(C\) is nonsingular). Let \(\pi_1 : V_1 \rightarrow V\) be the blow up of \(p\), and \(\mathcal{R}_1 = (E^+_1, E^-_1, S_1, V_1)\) be the transform of \(\mathcal{R}\). Then \(\pi_1^{-1}(p) \cap \text{Sing}_r(\mathcal{R}_1)\) is a single point \(p_1\), which is on the strict transform of \(C\). We have that \(\tau(p_1) = 2\).

**Proof.** This is immediate from Lemma 3.1.

**Theorem 4.9.** Suppose that \(\mathcal{R} = (\emptyset, E, S, V)\). Let \(r = \nu(\mathcal{R})\). Then there exists a sequence of permissible transforms \(\pi : V_1 \rightarrow V\) of \(\mathcal{R}\), centered at points and curves contained in the locus where the strict transform of \(S\) has multiplicity \(r\), such that \(\text{Sing}_r(\mathcal{R}_1) = \emptyset\), where \(\mathcal{R}_1\) is the transform of \(\mathcal{R}\) by \(\pi\).

**Proof.** By Theorems 4.6 and 4.7, we may assume that the conclusions of Theorem 4.7 hold (but we now have \(\mathcal{R} = (E^+, E^-, S, V)\)).

First suppose that \(r = 1\). Then the tangent space to the nonsingular surface \(S\) is an approximate manifold at \(p\) for all \(p \in S\). By Theorem 4.7, \(S\) makes SNCs with \(E^+ + E^-\) at all \(p \in S\). Thus \(\text{Sing}_r(\mathcal{R}) = \emptyset\).

Now assume that \(r \geq 2\). Suppose that \(C \subset \text{Sing}_r(\mathcal{R})\) is a curve. Then \(C \cap E^- = \emptyset\) and \(C \subset E^+\). Further, \(C\) is nonsingular as remarked after Theorem 4.6.

\(C\) makes SNCs with \(E^+\) at all points with \(\tau(p) = 1\). Let \(q_1, \ldots, q_t \in C\) be the finitely many points where \(C\) does not make SNCs with \(E^+\). We have that \(\tau(q_i) = 2\) at these points. Let \(\pi : V_1 \rightarrow V\) be a sequence of blow ups of points above the set \(\{q_1, \ldots, q_t\}\), with transform \(\mathcal{R}_1 = (E^+_1, E^-_1, S_1, V_1)\) of \(\mathcal{R}\), so that that the strict transform \(\overline{C}\) of \(C\) makes SNCs with \(E^+_1\). The existence of such a map is shown in the proof of Theorem 4.6.

By Lemma 4.8, the curves in \(\text{Sing}_r(\mathcal{R}_1)\) are the strict transforms of curves in \(\text{Sing}_r(\mathcal{R})\).

Let \(\pi_2 : V_2 \rightarrow V_1\) be the blow up of \(\overline{C}\). then by Lemma 3.2, \(\text{Sing}_r(\mathcal{R}_2)\) contains no points above the \(q_i\). By Theorem 4.5, the conclusions of Theorem 4.7 hold for \(\mathcal{R}_2\).
We continue this way to produce a map \( \pi : V_n \to V \) where the conclusions of Theorem 4.7 hold for the transform \( \mathcal{R}_n \) of \( \mathcal{R} \) on \( V_n \), and \( \text{Sing}_r(\mathcal{R}_n) \) is a finite number of points (as in the proof of Theorem 7.7 [C]).

Now we apply the following algorithm.

1. If there exists a curve \( C \subset \text{Sing}_r(\mathcal{R}_n) \) then blow up all points in \( C \) where \( C \) does not make SNCs with \( E_n^+ \) (these are points \( p \) with \( r(p) = 2 \) disjoint from \( E_n^- \)), until the strict transform \( \overline{C} \) of \( C \) makes SNCs with \( E_n^+ + E_n^- \). Then blow up \( \overline{C} \).

2. Otherwise blow up a point which is isolated in \( \text{Sing}_r(\mathcal{R}_n) \).

The conclusions of Theorem 4.7 are preserved for the transform \( \mathcal{R}_{n+1} \) of \( \mathcal{R} \), after performing a sequence of permissible transforms of type 1 or 2.

By Theorem 7.8 [C] and Lemma 4.8, we achieve a reduction \( \nu(\mathcal{R}_n) < r \) after a finite number of iterations of the algorithm.

□

We now prove Theorem 1.2. We start with \( \mathcal{R} = (\emptyset, \emptyset, S, X) \). Let \( r = \nu(\mathcal{R}) \) and apply Theorem 4.9 to produce \( \mathcal{R}_1 \) with \( \nu(\mathcal{R}_1) < r \). We iterate application of Theorem 4.9 to achieve the conclusions of Theorem 1.2.

5. Principalization of ideals on nonsingular 3-folds

In this section we extend the proof of resolution of surface singularities in Chapter 7 of [C] to prove principalization of ideals (Theorem 1.3 of the introduction). We will use the notation and results of Section 3.

**Theorem 5.1.** Suppose that \( V \) is a nonsingular 3-dimensional variety over an algebraically closed field \( k \) and \( \mathcal{I} \subset \mathcal{O}_V \) is an ideal sheaf on \( V \). Let

\[
\nu = \max\{\nu(p) \mid p \in V\}.
\]

Suppose that \( \nu \geq 1 \) Then there exists a sequence of monoidal transforms

\[
V_n \to V_{n-1} \to \cdots \to V
\]

such that for all \( i \), \( \mathcal{I}_i \) is the weak transform of \( \mathcal{I}_{i-1} \) on \( V_i \), each \( V_i \to V_{i-1} \) is the blow up of a point, nonsingular curve or nonsingular surface in \( \text{Sing}_r(\mathcal{I}_{i-1}) \), and \( \text{Sing}_r(\mathcal{I}_n) = \emptyset \).

We first show that Theorem 1.3 follows from Theorem 5.1. We can factor \( \mathcal{I} = JK \) in \( \mathcal{O}_X \), where \( J \) is an invertible ideal sheaf, and the support of \( \mathcal{O}_X/K \) has codimension \( \geq 2 \) in \( X \). Now apply Theorem 5.1 to \( K \).

We will need the following lemma in the proof of Theorem 5.1. The proof is an extension of Theorem 3.15 [C].

**Lemma 5.2.** Suppose that \( T \) is a nonsingular surface over a field \( L \), and \( \mathcal{I} \subset \mathcal{O}_T \) is an ideal sheaf on \( T \). Let \( \nu = \max\{\nu(p) \mid p \in T\} \). Suppose that \( \nu \geq 1 \). Then there exists a sequence of monoidal transforms

\[
T_n \to T_{n-1} \to \cdots \to T
\]

such that for all \( i \), \( \mathcal{I}_i \) is the weak transform of \( \mathcal{I}_{i-1} \) on \( T_i \), each \( T_i \to T_{i-1} \) is the blow up of a point or nonsingular curve in \( \text{Sing}_r(\mathcal{I}_{i-1}) \), and \( \text{Sing}_r(\mathcal{I}_n) = \emptyset \).

**Proof.** Suppose that \( C \subset \text{Sing}_r(\mathcal{I}) \) is a curve. Then \( C \) is nonsingular, and there exists a neighborhood \( U \) of \( C \) in \( T \) such that \( \mathcal{I} |_U = \mathcal{I}_C |_U \). Let \( \pi : T_1 \to T \) be the blow up of \( C, \mathcal{I}_1 \) be the weak transform of \( \mathcal{I} \) on \( T_1 \). Then \( \mathcal{I}_1 |_{\pi^{-1}(U)} = \mathcal{O}_{\pi_1^{-1}(U)} \). Thus \( \text{Sing}_r(\mathcal{I}_1) \cap \pi^{-1}(U) = \emptyset \).
We thus reduce to the case where $\text{Sing}_r(\mathcal{I})$ is a finite set of points.

Now the proof is an extension of the proof of Theorem 3.15 [C]. We construct a sequence of projective morphisms

$$(16) \quad \cdots \rightarrow T_n \xrightarrow{\pi_n} \cdots \rightarrow T_1 \xrightarrow{\pi_1} T_0 = T$$

where each $\pi_{n+1}: T_{n+1} \rightarrow T_n$ is the blow up of all points in $\text{Sing}_r(\mathcal{I}_n)$, where $\mathcal{I}_n$ is the weak transform of $\mathcal{I}_{n-1}$ on $T_n$. We must show that this sequence is finite.

Suppose that $(16)$ has infinite length. We can then find closed points $p_n \in T_n$ such that $\pi_n(p_n) = p_{n-1}$, and $p_n \in \text{Sing}_r(\mathcal{I}_n)$ for all $n$. We then have an infinite sequence of local rings

$$R = R_0 \rightarrow R_1 \rightarrow \cdots \rightarrow R_n \rightarrow \cdots$$

where $R_n = \hat{\mathcal{O}}_{T_n, p_n}$. Let $\mathcal{T}$ be the residue field of $R$. By Lemma 3.14 [C], the residue field of each of these local rings is equal to $\mathcal{T}$.

We will define $\delta_{p_i} \in \frac{1}{n!} \mathbb{N}$ so that $\delta_{p_i} = \delta_{p_{i-1}} - 1$ for all $i$, leading to a contradiction to the assumed infinite length of $(16)$.

Suppose that $x, y$ are regular parameters at $p_0$. For $f \in I = \hat{\mathcal{I}}_{p_0}$, we have an expansion

$$f = \sum_{i+j \geq r} a_{ij} x^i y^j$$

with $a_{ij} \in \mathcal{T}$. We will call $x, y$ good parameters for $I$ if there exists $f \in I$ such that $a_{0r} \neq 0$ (ord($f(0, y)$) = $r$). Suppose that $x, y$ are good parameters for $I$. For $f$ with ord($f(0, y)$) = $r$, we define

$$\delta(f; x, y) = \min \left\{ \frac{i}{r-j} | j < r \text{ and } a_{ij} \neq 0 \right\},$$

and

$$\delta(I; x, y) = \min \{ \delta(f; x, y) | f \in I \text{ and ord}(f(0, y)) = r \}.$$

Now define

$$\delta_{p_0} = \sup \{ \delta(I; x, y_1) | y = y_1 + \sum_{i=1}^{n} b_i x^i \text{ with } n \in \mathbb{N} \text{ and } b_i \in \mathcal{T} \}. $$

By a variation on the proof of Theorem 3.15 [C], we can define $\delta_{p_i}$ for all $i$ with the desired property. □

We now prove Theorem 5.1.

Suppose that $\text{Sing}_r(\mathcal{I})$ has dimension 2. Let $D$ be the reduced effective divisor on $V$ whose support is the union of dimension 2 components of $\text{Sing}_r(\mathcal{I})$. Then we have a factorization $\mathcal{I} = \mathcal{I}_D \mathcal{J}$ where $\mathcal{I}_D$ is the ideal sheaf of $D$, and the support of $\mathcal{O}_V/\mathcal{J}$ has codimension $\geq 2$ in $V$. Thus $D$ is a nonsingular (possibly not irreducible) surface in $V$, and the support of $\mathcal{O}_V/\mathcal{J}$ is disjoint from $D$. Let $\pi_1: V_1 \rightarrow V$ be the blow up of $D$, and let $\mathcal{I}_1$ be the weak transform of $\mathcal{I}$ on $V_1$. Then $\mathcal{I}_1$ is also the weak transform of $\mathcal{J}$, and $\text{Sing}_r(\mathcal{I}_1)$ has dimension less than 2. We may thus assume that $\text{Sing}_r(\mathcal{I})$ has dimension less than 2.

The proofs of Theorem 7.6 and Theorem 7.7 of [C] (with $S_1$ replaced by the weak transform $\mathcal{I}_1$ of $\mathcal{I}_{r-1}$) generalize to our situation, allowing us to assume that $\text{Sing}_r(\mathcal{I})$ is a finite set. In the proof of Theorem 7.7 [C], we must replace the reference to Theorem 3.15 with a reference to Lemma 5.2.

Theorem 5.1 now follows from Theorem 5.3 below, which is a generalization of Theorem 7.8 [C].
Theorem 5.3. Suppose that \( \text{Sing}_r(\mathcal{I}) \) is a finite set. Consider a sequence of monoidal transforms (eq34)
\[
V_n \to V_{n-1} \to \cdots \to V
\]
such that for all \( i \), \( \mathcal{I}_i \) is the weak transform of \( \mathcal{I}_{i-1} \) on \( V_i \), and \( V_i \to V_{i-1} \) is the blow up of a curve in \( \text{Sing}_r(\mathcal{I}_{i-1}) \) if such a curve exists, and is the blow up of a point in \( \text{Sing}_r(\mathcal{I}_{i-1}) \) otherwise. Then \( \text{Sing}_r(\mathcal{I}_i) \) is a union of nonsingular curves and isolated points for all \( i \). Further, this sequence is finite; that is, there exists an \( n \) such that \( \text{Sing}_r(\mathcal{I}_n) = \emptyset \).

The proof of Theorem 5.3 follows from Lemmas 3.1 and 3.2, and Theorem 5.4 below.

Theorem 5.4. Let the assumptions be as in the statement of Theorem 5.3, and suppose that \( q \in \text{Sing}_r(\mathcal{I}) \) is a closed point. Then there exists an \( n \) such that all closed points \( q_n \in V_n \) such that \( q_n \) maps to \( q \) and \( \nu_{q_n}(\mathcal{I}_n) = r \) satisfy \( \tau(q_n) > \tau(q) \).

The remainder of this section is devoted to the proof of Theorem 5.4. The proof is a modification of the proof of Theorem 7.9 [C].

The case where \( \tau(q) = 3 \) is immediate from Lemma 3.1.

Suppose that \( \tau(q) = 2 \). We indicate the necessary changes in the proof of Section 7.2 [C]. We may assume that we have a sequence
\[
R_0 \to R_1 \to \cdots
\]
of infinite length, where \( R_i = \hat{O}_{V_i,q_i} \) is the completion of the local ring of \( V_i \) at \( q_i \), \( \nu_{q_i}(\mathcal{I}_i) = r \) and \( \tau(q_i) = 2 \) for all \( i \). We may assume without loss of generality that \( R_i \to R_{i+1} \) is not an isomorphism for all \( i \).

If \( q_i \) is not isolated in \( \text{Sing}_r(\mathcal{I}_i) \) for some \( i \), we have that \( V_{i+1} \to V_i \) is the blow up of a curve in \( \text{Sing}_r(\mathcal{I}_i) \) containing \( q_i \). Thus \( \nu_{q_{i+1}}(\mathcal{I}_{i+1}) < r \) by Lemma 3.2. We may thus assume that \( q_i \) is isolated in \( \text{Sing}_r(\mathcal{I}_i) \) for all \( i \).

Let \( I_i = (\mathcal{I}_i)_{q_i}R_i \). Suppose that \( x, y, z \) are regular parameters in \( R_0 \), such that \( V(x, y, z) \) is an approximate manifold of \( I_0 \).

For \( g \in R_0 \), we have an expansion \( g = \sum_{i+j+k \geq r} b_{ijk} x^i y^j z^k \) with \( b_{ijk} \in k \). For \( g \in I \) with \( \nu_R(g) = r \), define
\[
\gamma_{xyz}(g) = \min \left\{ \frac{k}{r-(i+j)} \mid b_{ijk} \neq 0 \text{ and } i+j < r \right\}.
\]
We have \( \gamma_{xyz}(g) \in \frac{1}{r} \mathbb{N} \cup \{ \infty \} \).

Define
\[
\gamma_{xyz}(I_0) = \min \{ \gamma_{xyz}(g) \mid g \in I_0 \text{ and } \nu_{I_0}(g) = r \}.
\]
Let \( \gamma = \gamma_{xyz}(I_0) \). For \( g \in I_0 \) such that \( \nu_{I_0}(g) = r \), define
\[
|g|_{xyz} = \sum_{(i+j)\gamma+k=r} b_{ijk} x^i y^j z^k.
\]
Define
\[
T_\gamma = \{(i,j) \mid \frac{k}{r-(i+j)} = \gamma \}
\]
where \( i+j < r \), and \( k \) is such that \( a_{ijk} \neq 0 \) for some \( g = \sum a_{ijk} x^i y^j z^k \in I_0 \) with \( \nu_{I_0}(g) = r \).

For \( g \in I_0 \) with \( \nu_{I_0}(g) = r \), let \( L(x, y, z) \) be the leading form of \( g \). Say that \( I_0 \) is solvable with respect to \( x, y, z \) if there exists \( \alpha, \beta \in k \) such that
\[
|g|_{xyz} = L(x - \alpha z^\gamma, y - \beta z^\gamma)
\]
for all $g \in I_0$ with $\nu_{R_0}(g) = r$.

The proof of Lemma 7.11 of [C] shows that there exists a change of variables

$$x_1 = x - \sum_{i=1}^{n} \alpha_i z^i, \quad y_1 = y - \sum_{i=1}^{n} \beta_i z^i$$

such that $I_0$ is not solvable with respect to $x_1, y_1, z$. We may thus assume that $I_0$ is not solvable with respect to $x, y, z$. Since $V(x, y)$ is an approximate manifold of $I_0$, $R_1$ has regular parameters $x_1, y_1, z_1$ defined by $x = x_1 z_1, y = y_1 z_1, z = z_1$. It follows from Lemma 7.13 [C] that $\gamma_{x_1 y_1 z_1}(I_1) = \gamma_{xyz}(I) - 1$, and $I_1$ is not solvable with respect to $x_1, y_1, z_1$.

As argued after the proof of Lemma 7.13 [C], the case $\gamma_{x_1 y_1 z_1}(I_1) = 1$ cannot occur, and $\gamma_{x_1 y_1 z_1}(I_1) < 1$ implies $\nu_{R_0}(I_1) < r$. We further see that $V(x_1, y_1)$ is an approximate manifold of $I_1$, since $\nu_{R_0}(I_1) = r$ and $\tau(q_1) = 2$ (by assumption). We see that after a finite number of blow ups of points, $(\nu_{R_1}(I_1), \tau(q_1))$ must drop in the lex order, a contradiction to the assumption that (18) is infinite.

Now suppose that $\tau(q) = 1$. We indicate the necessary changes in the proof of Section 7.3 [C]. We may assume that we have a sequence

$$R_0 \rightarrow R_1 \rightarrow \cdots$$

of infinite length, where $R_i = \hat{O}_{V_i, q_i}$ is the completion of the local ring of $V_i$ at $q_i$, $\nu_{q_i}(I_i) = r$ and $\tau(q_i) = 1$ for all $i$. Let $I_i = (I_i)_{q_i} R_i$. We may assume without loss of generality that $R_i \rightarrow R_{i+1}$ is not an isomorphism for all $i$.

Let $T = k[[x, y, z]]$ be a power series ring. We generalize the construction of $\Delta(g; x, y, z)$, for $g \in T$ on page 113 - 114 of [C] to an ideal $I \subset T$. For fixed $r$, and $g = \sum b_{ijk} x^i y^j z^k \in T$,

$$\Delta(g; x, y, z) = \left\{ \left( \frac{i}{r-k}, \frac{j}{r-k} \right) \in \mathbb{Q}^2 \mid k < r \text{ and } b_{ijk} \neq 0 \right\}.$$

Define

$$\Delta = \Delta(I; x, y, z) = \cup_{g \in I} \Delta(g; x, y, z).$$

Let $|\Delta| = |\Delta(I; x, y, z)|$ be the smallest convex set in $\mathbb{R}^2$ such that $\Delta \subset |\Delta|$, and $(a, b) \in |\Delta|$ implies $(a + c, b + d) \in |\Delta|$ for all $c, d \geq 0$.

We define $\alpha_{x,y,z}(I), \beta_{x,y,z}(I), \gamma_{x,y,z}(I), \delta_{x,y,z}(I)$ and $\epsilon_{x,y,z}(I)$ from $|\Delta(I; x, y, z)|$ in the same way that $\alpha_{x,y,z}(g), \beta_{x,y,z}(g), \gamma_{x,y,z}(g), \delta_{x,y,z}(g)$ and $\epsilon_{x,y,z}(g)$ are defined from $|\Delta(g; x, y, z)|$ on page 113 - 114 of [C].

Suppose that $g \in I$. We easily see that $\alpha_{xyz}(I) \leq \alpha_{xyz}(g)$, and $\alpha_{xyz}(I) = \alpha_{xyz}(g)$ implies $\beta_{xyz}(I) \leq \beta_{xyz}(g)$. We further have that $\gamma_{xyz}(I) \leq \gamma_{xyz}(g)$, and $\gamma_{xyz}(I) = \gamma_{xyz}(g)$ implies $\delta_{xyz}(I) \leq \delta_{xyz}(g)$.

Observe that (since $k$ is infinite) there exists $g \in I$ such that $\Delta(I; x, y, z) = \Delta(g; x, y, z)$.

Lemma 7.15 of [C] is true with “$g$” replaced with “$I$”.

We will say that regular parameters $x, y, z$ in $T$ are good parameters for $I$ if $g = \sum b_{ijk} x^i y^j z^k \in I$ and $\nu_{T}(g) = r$ implies $b_{000} \neq 0$.

If $\nu_{T}(I) = r$ and $x, y, z$ are regular parameters in $T$ such that $V(z)$ is an approximate manifold for $I$, then $x, y, z$ are good parameters for $I$.

Suppose that $x, y, z$ are good parameters for $I$, and $(a, b)$ is a vertex of $|\Delta| = |\Delta(I; x, y, z)|$. Define

$$S_{(a, b)} = \left\{ k \mid \left( \frac{i}{r-k}, \frac{j}{r-k} \right) = (a, b) \right\}.$$
where we consider \((i, j, k)\) such that \(k < r\) and \(b_{ijk} \neq 0\) for some \(g = \sum b_{ijk} x^i y^j z^k \in I\).

Define
\[
\{g\}_{xyz}^{ab} = b_{00r} z^r + \sum_{k \in S(a, b)} b_{a(r-k), b(r-k), k} x^{a(r-k)} y^{b(r-k)} z^k
\]
for \(g \in I\) with \(\nu_T(g) = r\).

We will say that \((a, b)\) is not prepared on \(|\Delta|\) if \(a, b\) are integers and there exists \(\eta \in k\) such that
\[
\{g\}_{xyz}^{ab} = b_{00r}(z - \eta x^a y^b)^r
\]
for all \(g \in I\) with \(\nu_T(g) = r\).

If \((a, b)\) is not prepared on \(|\Delta|\), then we can make an \((a, b)\) preparation, which is the change of parameters \(z_1 = z - \eta x^a y^b\). If all vertices of \(|\Delta|\) are prepared, say that \((I; x, y, z_1)\) is well prepared.

The proof of Lemma 7.16 of [C] (with “\(g\)” replaced with “\(I\)” in its statement) extends easily to our situation.

The generalization of Lemma 7.18 [C] in our situation is obtained by replacing “\(\Delta(g)\)” with “\(\Delta(I)\)” and inserting “for \(g \in I\) with \(\nu_T(g) = r\)” before “\(\{g\}_{xyz}^{a'b'}\) is obtained from”.

Lemma 7.20 [C] generalizes to show that we may make a change of variables \(z = z_1 + \alpha(x, y)\) to ensure that \((I; x, y, z_1)\) is well prepared. To get this statement, we must replace “\(g\)” with “\(I\)” in the statement of Lemma 7.20, and assume that \(\text{Sing}_r(I)\) has dimension < 2 (which is true for all \(I_i \subset R_i\) in (19)).

In the proof of Lemma 7.20 [C], we must replace all occurrences of “\(\Delta(g; x, y, z)\)” with “\(\Delta(I; x, y, z)\)”.

The statements of Definition 7.22, Lemma 7.23, Definition 7.24 and Lemma 7.25 of [C] generalize to our situation, by replacing “Suppose that \(g \in T\) is reduced, \(\nu_T(g) = r\), \(\tau(g) = 1\) and \((x, y, z)\) are good parameters for \(g\)” with “Suppose that \(I \subset T\) is an ideal such that \(\nu_T(I) = r\), \(\tau(I) = 1\), \(\text{Sing}_r(I)\) has dimension < 2 and \((x, y, z)\) are good parameters for \(I\)”.

We replace line -5 of page 125 to the end of the proof with the following. Insert after “we observe that” the line “for \(g \in I\) with \(\nu_T(g) = r\)” before “\(g_1\)” with the weak transform “\(I_1\)” of \(I\), and “strict transform” with “weak transform”.

We make a similar generalization of Lemma 7.26 [C]. The last line of the proof of Lemma 7.26 must be modified as follows. Insert after “we observe that” the line “for \(g \in I\) with \(\nu_T(g) = r\) and \(g_1 = \frac{g}{r} \in I_1\) (if we perform a T3 or T1 transformation) and \(g_1 = \frac{g}{r} \in I_1\) (if we perform a T2 or T4 transformation), we have that”.

In the statements of Lemma 7.27, Lemma 7.28, Lemma 7.29 and Theorem 7.31 of [C], we again replace “\(g\)” with “\(I\)” and “\(g_1\)” with “\(I_1\)”.

In the proof of Theorem 7.31 [C], we replace “\(g\)” with “\(I\)” and “\(g_1\)” with “\(I_1\)” in the part of the proof before line -5 on page 125.

We replace line -5 of page 125 to the end of the proof with the following. Suppose that \((\alpha, \beta) = (\gamma - \delta, \delta)\). Set
\[
W = \left\{ (i, j, k) \in \mathbb{N}^3 \mid k < r \text{ and } \left( \frac{i}{r-k}, \frac{j}{r-k} \right) = (\alpha, \beta) \right\}.
\]
For \(g = \sum a_{ijk} x^i y^j z^k \in I\), with \(\nu_T(g) = r\), set
\[
F_g = \sum_{(i,j,k) \in W} a_{ijk} x^i y^j z^k.
\]
By assumption, \((\alpha, \beta)\) is prepared on \(|\Delta(I; x, y, z)|\), which implies that if \(\alpha, \beta\) are integers, then there does not exist \(\lambda \in \mathbb{k}\) such that
\
\[a_{00r}z^r + F_g = a_{00r}(z - \lambda x^\alpha y^\beta)^r\]
for all \(g \in I\) with \(\nu_T(g) = r\). Moreover,

\[F_g(x, y, z) = F_g(x, y' + \eta x, z) = \sum_{(i,j,k) \in W} \sum_{\lambda=0}^j a_{ijk}\eta^\lambda \left( i^{x+\lambda}(y')^{j-\lambda}z^k \right).\]  

By Lemma 7.21 [C], the terms in the expansion of \(g(x, y' + \eta x, z)\) contributing to \((\gamma, 0)\) in \(|\Delta(I; x, y', z)|\), where \(\gamma = \alpha + \beta\), are

\[F_{g, (\gamma, 0)} = \sum_{(i,j,k) \in W} a_{ijk}\eta^j x^i y^k \neq 0.\]

If \((I; x, y', z)\) is \((\gamma, 0)\) prepared, then \(\delta' = 0 < \beta\). Suppose that \((\gamma, 0)\) is not prepared on \(|\Delta(I; x, y', z)|\). Then \(\gamma \in \mathbb{N}\), and there exists \(\psi \in \mathbb{k}\) such that for all \(g \in I\) with \(\nu_T(g) = r\),

\[a_{00r}(z - \psi x^\gamma)^r = a_{00r}z^r + F_{g, (\gamma, 0)},\]

so that, with \(\omega = \frac{-\psi}{\eta}\), for \(0 \leq k < r\), we have for \(g \in I\) with \(\nu_T(g) = r\):

1. If \(r_{-k} \neq 0\) (in \(\mathbb{k}\)) then \(i = \alpha(r - k), j = \beta(r - k) \in \mathbb{N}\), and

\[a_{ijk} = \left( \frac{r}{r - k} \right) \omega^{r-k}.\]

2. If \(i = \alpha(r - k), j = \beta(r - k) \in \mathbb{N}\) and \(r_{-k} = 0\) (in \(\mathbb{k}\)), then \(a_{ijk} = 0\).

Suppose that \(\mathbb{k}\) has characteristic zero. By Remark 7.32 of [C], we obtain a contradiction to our assumption that \((\gamma, 0)\) is not prepared on \(|\Delta(I; x, y', z)|\). Thus \(0 = \delta' < \beta\) if \(\mathbb{k}\) has characteristic zero.

Now we consider the case where \(\mathbb{k}\) has characteristic \(p > 0\), and \(r = p^s r_0\) with \(p \nmid r_0, r_0 \geq 1\). Then (by Lemma 7.30 [C]), we have \(a_{ijk} = 0\) if \(p^s \nmid k\), for all \(g \in I\) with \(\nu_T(g) = r\). By (22), and Lemma 7.30 [C], we have that \(i = \alpha p^s, j = \beta p^s \in \mathbb{N}\), and for all \(g \in I\) with \(\nu_T(g) = r\), we have that \(a_{i,j,(r_0-1)p^s} \neq 0\).

We have an expression \(\beta p^s = ep^t\) where \(p \nmid e\). Suppose that \(t \geq s\). Then \(\beta \in \mathbb{N}\), which implies that \(\alpha = \gamma - \beta \in \mathbb{N}\), so that for all \(g \in I\) with \(\nu_T(g) = r\),

\[a_{00r}(z + \omega x^\alpha y^\beta)^r = a_{00r}z^r + F_g,\]

a contradiction, since \((\alpha, \beta)\) is by assumption prepared on \(|\Delta(I; x, y, z)|\). Thus \(t < s\). Suppose that \(e = 1\). Then \(\beta = \frac{p^{t-s}}{p^t} < 1\) and \(\alpha < 1\) (since we must have that \(\text{Sing}_p(I) = V(x, y, z)\)) which implies \(\gamma = \alpha + \beta = 1\), a contradiction to (7.15) of [C] (which is \(\alpha + \beta > 1\)). Thus \(e > 1\). Also,

\[a_{00r}z^r + F_g = a_{00r}(z^{p^s} + \omega^{p^s x^{p^s y^{p^s}}})^{r_0} = a_{00r}(z^{p^s} + \omega^{p^s x^{p^s y^{p^s}}} (y' + \eta x)^{p^s})^{r_0} = a_{00r}(z^{p^s} + \omega^{p^s x^{p^s y^{p^s}}} (y' + \eta y)^{p^s})^{r_0}.
\]

Now we make the \((\gamma, 0)\) preparation \(z = z' - \eta^2 \omega x^\gamma\) (from (21)) so that \((I; x, y', z')\) is \((\gamma, 0)\) prepared. For \(g \in I\) such that \(\nu_T(g) = r\), let \(G_g = a_{00r}z^r + F_g\). Then

\[G_g = a_{00r} \left[ (z')^{p^s} + e\omega^{p^s y^{p^s - (e-1)}(y')^{p^s x^{p^s y^{p^s}} + p^{s(e-1)} + (y')^2p^s (\Omega)} \right]^{r_0}
= a_{00r} \left[ (z')^{p^s r_0 + r_0 \left[ e\omega^{p^s y^{p^s - (e-1)}(y')^{p^s x^{p^s y^{p^s}} + p^{s(e-1)} + (y')^2p^s (\Omega)} \right] (z')^{p^s (r_0 - 1)} \right]
+ \Delta_{2}(x,y')(z')^{p^s (r_0 - 2)} + \cdots + \Delta_{r_0}(x,y')\]
for some polynomials (which depend on \( g \in I \)), \( \Omega(x, y') \), \( \Lambda_2, \ldots, \Lambda_{r_\gamma} \), where \((y')^{ip} \mid \Lambda_i \) for all \( i \). All contributions of \( S(\gamma) \cap |\Delta(I; x, y', z')| \) must come from these polynomials \( G_g \), for \( g \in I \) with \( \nu_{\gamma}(g) = r \) since \( k \) is infinite. Recall that we are assuming \((\alpha, \beta) = (\gamma - \delta, \delta)\). The term of lowest second coordinate on \( S(\gamma) \cap |\Delta(I; x, y', z')| \) is

\[
(a, b) = \left( \frac{\alpha p^s + p^d (e - 1)}{p^s} \right) \text{,}
\]

which is not in \( \mathbb{N}^2 \) since \( t < s \), and is not \((\alpha, \beta)\) since \( e > 1 \). \((a, b)\) is thus prepared on \(|\Delta(I; x, y', z')|\), and

\[
\delta' = \frac{p^d}{p^s} < \frac{p^d}{p^s} = \beta.
\]

This completes the proof of the generalization of Theorem 7.3.1 [C].

In the statement and proof of Theorem 7.34 of [C], we must replace “\( f_n \)” with the weak transform “\( I_n \)” of \( I \). Replace “there are \( f_n \in R_n \) such that \( f_n \) is a generator of \( T_{S_n, q_n} R_n \) and good parameters \((x_n, y_n, z_n)\) for \( f_n \) in \( R_n \)” in the statement of Theorem 7.34 of [C] with “there are good parameters \((x_n, y_n, z_n)\) for \( I_n = T_{S_n, q_n} R_n \) in \( R_n \).”

6. Projection to Points of Small Multiplicity

In this section, we prove the following theorem.

**Theorem 6.1.** Suppose that \( K \) is an algebraic function field of dimension \( d \) over an algebraically closed field \( k \). Then there exists a a normal projective variety \( V \) such that \( K \) is the function field of \( V \), and all points of \( V \) have multiplicity \( \leq d! \).

Let \( W \) be a projective variety over \( k \) of dimension \( d \), with an embedding \( W \subset \mathbb{P}^m \). Suppose that \( q \in W \). Let \( \pi : \mathbb{P}^m \to \mathbb{P}^{m-1} \) be the rational map which is projection from the point \( q \). Let \( W_1 \) be the projective subvariety of \( \mathbb{P}^{m-1} \) which is the image of \( W \) by \( \pi \) (the closure of \( \pi(W - \{q\}) \)). Let \( \mu \) be the multiplicity of the local ring \( \mathcal{O}_{W,q} \).

**Theorem 6.2.** The following are true.

1. \( \mu \leq \deg W \).
2. Suppose that \( \mu < \deg W \). Then \( \dim W = \dim W_1 \) and

\[
[W : W_1] \deg W_1 = \deg W - \mu.
\]

3. Suppose that \( \mu = \deg W \). Then \( \dim W > \dim W_1 \) and \( W \) is a cone over \( W_1 \) with vertex \( q \).

**Proof.** Let \( \sigma : Z \to \mathbb{P}^m \) be the blow up of \( q \), with exceptional divisor \( E \). Let \( \lambda : Z \to \mathbb{P}^{m-1} \) be the morphism induced by \( \pi \circ \sigma \) (a resolution of the indeterminacy of \( \pi \)).

Let \( H_0 \) be a hyperplane of \( \mathbb{P}^m \) and let \( H_1 \) be a hyperplane of \( \mathbb{P}^{m-1} \).

We have a linear equivalence of divisors

\[
\sigma^*(H_0) - E \sim \lambda^*(H_1).
\]

Let \( \overline{W} \) be the strict transform of \( W \) on \( Z \).

By (25), and since a general hyperplane of \( \mathbb{P}^m \) does not contain \( q \), we have equality of intersection numbers

\[
\int_Z \lambda^*(H_1)^d \cdot \overline{W} = \int_Z \sigma^*(H_0)^d \cdot \overline{W} = \int_Z (-E)^d \cdot \overline{W}.
\]

By the projection formula (Proposition 2.3 [F]),

\[
\int_Z \sigma^*(H_0)^d \cdot \overline{W} = \int_{p_n} H_0^d \cdot W = \deg W.
\]

Let \( R = \mathcal{O}_{W,q} \), \( M \) be the maximal ideal of \( R \). Then the scheme-theoretic intersection of \( \overline{W} \) and \( E \) is

\[
\overline{W} \cap E = \text{Proj}(\oplus_{n \geq 0} M^n / M^{n+1}),
\]
and
\[ O_Z(-E) \otimes O_{W \cap E} \cong O_{W \cap E}(1). \]

Thus by Lemma 2.3,
\[ \int_Z(-E)^d \cdot W = -\int_Z(W \cdot E) \cdot (-E)^{d-1} = -\deg W \cap E = -\epsilon(R) = -\mu. \]

We can now rewrite (26) as
\[ (27) \quad \int_Z \lambda^*(H_1)^d \cdot W = \deg W - \mu. \]

By the projection formula, we have that
\[ \int_Z \lambda^*(H_1)^d \cdot W = \begin{cases} [W : W_1] \deg W_1 & \text{if } \dim W = \dim W_1 \\ 0 & \text{if } \dim W > \dim W_1 \end{cases} \]

Substituting into (27), we conclude that \( \mu \leq \deg W \), and \( \mu = \deg W \) if and only if \( W_1 \) has dimension \( < d \), which holds if and only if \( W \) is a cone with vertex \( q \). If \( \mu < \deg W \), we obtain (24). \( \square \)

We will say that the induced rational map \( \pi : W \to W_1 \) is a permissible projection if \( \mu < \deg W \).

Suppose that \( W \) is a projective variety. We define \( c(W) \) to be the minimum of degrees of irreducible curves on \( W \).

**Theorem 6.3.** Suppose that \( K \) is an algebraic function field of dimension \( d \) over an algebraically closed field \( k \). Then there exists a projective variety \( V_0 \), whose function field is \( K \), and an embedding of \( V_0 \) into a projective space \( \mathbb{P}^N \) so that \( V_0 \) is not contained in a hyperplane of \( \mathbb{P}^N \), and such that
\[ \deg V_0 \]

where \( c = c(V_0) \).

**Proof.** There exists a projective variety \( V \), whose function field is \( K \). Let \( H \) be a very ample divisor on \( V \).

After possibly replacing \( H \) with a high multiple of \( H \), we may assume that
\[ H^i(V, O_V(\delta H)) = 0 \]

for all \( \delta \geq 1 \) and \( i > 0 \). Thus we have
\[ h^0(V, O_V(\delta H)) = \chi(\mathcal{O}_V(\delta H)) \]

for all \( \delta \geq 1 \). \( H^0(V, O_V(\delta H)) \) gives an embedding \( \Phi_\delta \) of \( V \) into a projective space \( \mathbb{P}^{N(\delta)} \), where \( N(\delta) = h^0(V, O_V(\delta H)) - 1 \). Let \( V^\delta \) be the image of \( V \).

Let \( H_\delta \) be a hyperplane section of \( V^\delta \). By our construction, we have that \( V^\delta \) is not contained in a hyperplane of \( \mathbb{P}^{N(\delta)} \), and \( \Phi_\delta^*(H_\delta) \sim H_\delta \), so that every curve in \( V^\delta \) has degree \( \geq \delta \). Thus \( c_\delta = c(V^\delta) \geq \delta \).

We have that \( \frac{1}{d!} \deg V^\delta \) is the coefficient of \( n^d \) in the polynomial \( \chi(O_V(nH_\delta)) \).

Since by our construction, \( \chi(O_V(nH_\delta)) = \chi(O_V(\delta nH)) \), we see that \( \deg V^\delta = \delta^d \deg V \).

There exist \( a_0, a_1, \ldots, a_{d-1} \in \mathbb{Q} \) such that for all \( \delta > 0 \),
\[ N(\delta) + 1 = \chi(\mathcal{O}_V(\delta H)) = \frac{1}{d!} \delta^d \deg V + \sum_{i=0}^{d-1} a_i \delta^i . \]

Thus
\[ \frac{\deg V^\delta}{N(\delta) - d - \frac{1}{c_\delta} \deg V^\delta} = \frac{\delta^d \deg V}{\frac{\delta^d}{d!} \deg V + \sum_{i=0}^{d-1} a_i \delta^i - 1 - d - \frac{\delta^d}{c_\delta} \deg V} . \]
Since $c_3 \geq \delta$ for all $\delta$, the limit as $\delta$ goes to infinity of this expression is $d!$, and the conclusions of the theorem follow with $V_0 = V^\delta$ for $\delta$ sufficiently large.

\[\square\]

**Theorem 6.4.** Let $K$ be an algebraic function field of dimension $d$ over an algebraically closed field $k$. Let $V_0 \subset \mathbb{P}^N$ be as in the conclusions of Theorem 6.3. Then there exists a series of permissible projections

\[V_0 \xrightarrow{\pi_0} V_1 \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_{n-1}} V_n\]

such that every point on $V_n$ has multiplicity $\leq \frac{d!}{[V_0 : V_n]}$.

**Proof.** Suppose that there exists a point $q_0 \in V_0$ such that $q_0$ has multiplicity $\mu_0$ with $d! < \mu_0 < \deg V_0$. Let $\pi_0$ be the projection from $q_0$. Let $V_1$ be the image of $V_0$. If there exists a point $q_1 \in V_1$ of multiplicity $\mu_1$ with $\frac{d!}{[V_0 : V_1]} < \mu_1 < \deg V_1$, then we perform the projection $\pi_1 : V_1 \rightarrow V_2$ from the point $q_1$. After a finite number of steps this process must terminate, as the dimension of the ambient projective space drops by one after each projection. After the final projection, we have a variety $V_n$ such that if $q \in V_n$ and the multiplicity $\mu$ of $q$ on $V$ satisfies $\mu > \frac{d!}{[V_0 : V_n]}$, then $\mu = \deg V_n$.

We see that $V_n$ satisfies the conclusions of the theorem unless $V_n$ is a cone. We thus assume that $V_n$ is a cone with vertex $q$ (in the projective space $\mathbb{P}^{N-n}$), and will show that this is impossible.

By (24), we have that

\[[V_0 : V_1] \deg V_1 = \deg V_0 - \mu_0 \leq \deg V_0 - d! - 1.\]

We further have that

\[[V_1 : V_2] \deg V_2 \leq \deg V_1 - \frac{d!}{[V_0 : V_1]} - 1,\]

from which we obtain

\[[V_0 : V_2] \deg V_2 \leq [V_0 : V_1] \deg V_1 - d! - [V_0 : V_1] \leq \deg V_0 - 2d! - 2.\]

Continuing in the way, we obtain

\[[V_0 : V_n] \deg V_n \leq \deg V_0 - n(d! + 1).\]

We deduce the inequality

\[n(d! + 1) \leq \deg V_0.\]

There exists a linear subspace $L$ of $\mathbb{P}^N$ of dimension $n-1$ such that $\pi_n \circ \cdots \circ \pi_0$ is induced by the projection morphism $\Psi : \mathbb{P}^N - L \rightarrow \mathbb{P}^{N-n}$ from $L$. There exist dense open subsets $U_0 \subset V_0$ and $U_n \subset V_n$ such that $\phi = (\pi_n \circ \cdots \circ \pi_0) : U_0 \rightarrow U_n$ is a finite morphism, and $U_0 \cap L = \emptyset$ (so that $V_0 \cap \Psi^{-1}(U_n) = U_0$). Let $E = V_n - U_n$.

Since $k$ is infinite, and by Bertini’s theorem for general hyperplane sections (Corollary 3.4.14 and 3.4.10 [FOV]), there exists a a hyperplane $\overline{H}$ of $\mathbb{P}^{N-n}$ which does not contain $q$, intersects each irreducible component of $E$ generically, and such that the scheme theoretic intersection $W = \overline{H} \cap V_n$ is a $d-1$ dimensional variety. $V_n$ is the locus of lines through $q$ and a point of $W$.

By Bertini’s theorem for general hyperplane sections, there exist hyperplanes

\[\overline{H}_1, \ldots, \overline{H}_{d-1}\]

of $\mathbb{P}^{N-n}$ such that each $\overline{H}_i$ intersects $W$ and every irreducible component of $E$ generically. In particular, the scheme-theoretic intersection $\overline{H}_1 \cap \cdots \cap \overline{H}_{d-1} \cap W$ is a reduced set of points of order $\deg V_n$ which are contained in $U_n$.

For $1 \leq i \leq d-1$, let $H_i$ be the hyperplane of $\mathbb{P}^{N-n}$ which is spanned by $q$ and the linear space $\overline{H}_i \cap \overline{H}$. By our construction, the scheme theoretic intersection
$H_1 \cap \cdots \cap H_{d-1} \cap V_n$ is the union of $s = \deg V_n$ distinct lines $L_1, \ldots, L_s$, whose generic points lie in $U_n$.

Let $H^*_1, \ldots, H^*_{d-1}$ be the hyperplanes of $\mathbb{P}^N$ such that $H^*_i \cap (\mathbb{P}^N - L) = \Psi^*(H_i)$. Since $U_0 \to U_n$ is finite, the irreducible components of

$$H^*_1 \cap \cdots \cap H^*_{d-1} \cap U_0 = \phi^{-1}(H_1 \cap \cdots \cap H_{d-1} \cap U_n)$$

are curves which dominate the irreducible components of $H^*_1 \cap \cdots \cap H^*_{d-1} \cap U_n$, so there are $\geq \deg V_n$ distinct irreducible components of $H^*_1 \cap \cdots \cap H^*_{d-1} \cap U_0$. By the weak Bézout theorem (12.3.1) of [Ab1], or Example 12.3.1 of the refined Bézout theorem of [F], we have that

(31) $\deg V_0 \geq c \deg V_n$,

where $c = c(V_0)$. We have that $V_0$ is not contained in a hyperplane of $\mathbb{P}^{N-n}$, since $V_0$ is not contained in a hyperplane section of $\mathbb{P}^N$. Thus we have the classical degree bound (Example 8.4.6 [F]),

$$\deg V_n > N - n - d.$$ 

From (31), we get that

$$(N - n - d)c < \deg V_0.$$ 

That is,

(32) $N - d - \frac{1}{c} \deg V_0 < n.$

Now from (32) and (30), we have

$$N - d - \frac{1}{c} \deg V_0 < \frac{1}{(d! + 1)} \deg V_0,$$

which contradicts the assumption of our theorem. Thus $V_n$ is not a cone. \hfill \Box

We now prove Theorem 6.1.

Let notation be as in the statement of Theorem 6.4. By Theorem 6.4, there exists a dominant rational map $V_0 \to V_n$ such that every point on $V_n$ has multiplicity $\leq \frac{d!}{\deg V_0 \cdot \deg V_n}$. Let $V$ be the normalization of $V_n$ in the function field $K$ of $V_0$, so that the function field of $V$ is $K$, and we have a finite morphism $\Psi : V \to V_n$. By Theorem 2.2, we have that every point on $V$ has multiplicity $\leq d!$.

7. Ramification

In this section we collect some basic results on ramification of local rings. For more details, the book “Ramification Theoretic Methods in Algebraic Geometry” [Ab2] by Abhyankar is an excellent reference.

Suppose that $R$ is a normal local domain with quotient field $K$, and $L$ is a finite extension of $K$. Let $T$ be the integral closure of $R$ in $L$. The finitely many local rings $S$ which are localizations of $T$ and dominate $R$ are called the extensions of $R$ to $L$.

$R$ is said to be unramified in $L$ if for every extension $S$ of $R$ to $L$ we have that $S$ is residually separable over $R$ and $M(R)S = S$.

Suppose that $X$ and $Y$ are normal integral schemes, and $f : X \to Y$ is a finite morphism. The ramification locus of $f$ is the set of points in $Y$ over which $f$ is ramified. The ramification locus is a closed subset of $X$ (c.f. Theorems 1.44 and 1.44A [Ab2]). If $X$ is nonsingular, then the theorem of the purity of the branch locus [N] tells us that if the function field of $X$ is separable over the function field of $Y$, then the ramification locus has pure codimension 1 in $X$. In particular, if $L$ is separable
over $K$ and $R$ is a regular local ring, then $R$ is unramified in $L$ if $R_Q$ is unramified in $L$ for every height one prime $Q$ of $R$.

Suppose that our local ring $R$ is an equicharacteristic complete regular local ring, with algebraically closed residue field $k$. Then $R$ is a power series ring over $k$. $T$ is itself a complete local ring. Further, if $R$ is unramified in $L$, then $R \cong T$ (as follows from Nakayama’s Lemma).

We compute the discriminant in a case which will be important in the sequel. Suppose that our local ring $R$ is a finite extension field of $K$ by Theorem 21, Section 9, Chapter V [ZS].

Suppose that $R$ is a normal local domain with quotient field $K$, $L = K(r_1, \ldots, r_n)$ is a finite extension field of $K$, and $f_i(Z) \in R[Z]$ are nonconstant monic polynomials such that $f_i(r_i) = 0$ for $1 \leq i \leq n$. Suppose that $\delta(f_i) \notin M(R)$ for $1 \leq i \leq n$. Then $R$ is unramified in $K(r_1, \ldots, r_n)$.

This is (10.17) [Ab1].

Lemma 7.1. Suppose that $R$ is a normal local domain with quotient field $K$. Let $S$ be the integral closure of $R$ in $L$. Then $S$ is a finite $R$-module, $S$ is a $d$-dimensional regular local domain, $S = R[z_1, \ldots, z_d]$, $M(S) = (z_1, \ldots, z_d)S$, and $[L : K] = n(1) \cdots n(d)$.

This is (10.20.1) [Ab1].

Suppose that $R$ is a $d$-dimensional regular local domain with $d > 0$. Let $K$ be the quotient field of $R$ and let $(y_1, \ldots, y_d)$ be a basis of $M(R)$. Suppose that $L = K(z_1, \ldots, z_d)$ is an overfield of $K$ where $z_i^{n(i)} = y_i$ for $1 \leq i \leq d$, where $n(i)$ are positive integers. Let $S$ be the integral closure of $R$ in $L$. Then $S$ is a finite $R$-module, $S$ is a $d$-dimensional regular local domain, $S = R[z_1, \ldots, z_d]$, $M(S) = (z_1, \ldots, z_d)S$, and $[L : K] = n(1) \cdots n(d)$.

8. Local uniformization of points of small multiplicity

Theorem 8.1. Let $R$ be an equicharacteristic, complete regular local domain of dimension $d \geq 1$, whose residue field is algebraically closed (so that $R$ is a power series ring over an algebraically closed field $k$). Let $p$ be the characteristic of $k$. Suppose that $L$ is a finite extension of the quotient field $K$ of $R$. Further suppose that a regular system of parameters $y_1, \ldots, y_d$ in $R$ is such that $R$ is unramified in $L$ away from $V(y_1 \cdots y_d)$, and the one dimensional regular local rings $R_{(y_i)}$ are tamely ramified in $L$ for $1 \leq i \leq d$.

Then there exists a positive integer $n$ such that $p \nmid n$, and an inclusion of fields

$$K \subset L \subset K(z_1, \ldots, z_d)$$

such that $z_i^n = y_i$ for $1 \leq i \leq d$. 

Proof. Let $T$ be the integral closure of $R$ in $L$, $H_i = R_{(y_i)}$ for $1 \leq i \leq d$. Let $H_{i,j}$ for $1 \leq j \leq u(i)$ be the distinct local rings of $T$ which dominate $H_i$ for $1 \leq i \leq d$. Let $w_{i,j}$ be the integers such that $M(H_i)H_{i,j} = M(H_{i,j})^{w_{i,j}}$. Define

$$q_{i,j} = [H_{i,j}/M(H_{i,j}) : H_i/M(H_i)].$$

Let $n$ be the least common multiple of the $w_{i,j}$. Since the $H_i$ are tamely ramified in $L$, we have that $p \nmid n$.

For $1 \leq i \leq d$, choose $z_i$ in an over field of $L$ such that $z_i^n = y_i$. Define polynomials

$$f_i(Z) = Z^n - y_i \in K[Z]$$

for $1 \leq i \leq d$.

Define fields $K^* = K(z_1, \ldots, z_d) \subset L^* = L(z_1, \ldots, z_d)$. Let $R^*$ be the integral closure of $R^*$ in $K^*$, $T^*$ be the integral closure of $R$ in $L^*$.

Suppose that $V^*$ is a one dimensional localization of $R^*$, and that $W^*$ is a one dimensional localization of $T^*$ which dominates $V^*$. We will show that (eq 16)

$$(35) \quad W^*$$

is residually separable algebraic over $V^*$ and $M(V^*)W^* = M(W^*)$.

$W^*$ dominates a one dimensional localization $W$ of $T$ and dominates a one dimensional localization $V$ of $R$.

First suppose that $V \neq H_i$ for any $i$. Since $y_1, \ldots, y_d$ is a unit in $V$, and thus also in $W$, we have by (33) that the discriminant $\delta(f_i) \notin M(W)$ for $1 \leq i \leq d$. By Lemma 7.1, $W$ is unramified in $L^*$. Since $V$ is unramified in $L$, we then have that (35) holds.

Now suppose that $V = H_i$ for some $i$. Then $W = H_{i,j}$ for some $j$. Choose $x \in W$ such that $xW = M(W)$. There exists a unit $x' \in W$ such that $y_i = x'x^{v_{i,j}}$. Let $z = z_i^{\frac{1}{w_{i,j}}} x^{-1} \in L$. Define a polynomial

$$f(x) = Z^{w_{i,j}} - x' \in \mathbb{L}[Z].$$

We have that $f(z) = 0$ and $\delta(f) \notin M(W)$ (by (33)). Further, $\delta(f_j) \notin M(W)$ if $j \neq i$. Lemma 7.1 thus implies that $W$ is unramified in $L' = L(z_1, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_d) \subset L^*$.

Let $T'$ be the integral closure of $T$ in $L'$, and let $W'$ be the local ring of $T'$ such that $W^*$ dominates $L'$. We have that $M(W') = xW'$ and $W'$ is residually separable algebraic over $W$. Further, $L' = L'(z_i)$ and $z_i^{\frac{1}{w_{i,j}}} = xz$. Since $z$ is a unit in $W$, $M(W') = (xz)W'$. By Lemma 7.2, we have that $W^*/M(W^*) = W'/M(W')$ and $M(W^*) = z_iW^*$. It follows that (35) holds.

Since we have verified that $V^*$ is unramified in $L^*$ for all one dimensional local rings of $R^*$, and $L^*$ is separable over $K^*$, by the purity of the branch locus $[N]$, we have that $R^*$ is unramified in $L^*$. Since $R^*$ is complete with algebraically closed residue field, $T^* = R^*$. As $L^*$ is the quotient field of $T^*$, we have that $L^* = K^*$. \qed

Corollary 8.2. Let assumptions be as in Theorem 8.1. Then there exists a subgroup $G$ of $Gal(K(z_1, \ldots, z_d)/K) \cong \mathbb{Z}_n^d$ such that $L = K(z_1, \ldots, z_d)^G$. Let $A$ be the integral closure of $R$ in $K(z_1, \ldots, z_d)$. Then $A = k[[z_1, \ldots, z_d]]$, and the integral closure $T$ of $R$ in $L$ is $T = A^G$. In particular, there exist elements $s_i \in T$ such that $T$ is generated by $s_1, \ldots, s_m$ as an $R$ module, and there exist natural numbers $a(i, j)$ such that

$$s_j^n = y_1^{a(1,j)} y_2^{a(2,j)} \cdots y_d^{a(d,j)}$$

for $1 \leq j \leq m$.

Proof. By its construction, and Lemma 7.2, $K(z_1, \ldots, z_d)$ is Galois over $K$ with Abelian Galois group isomorphic to $\mathbb{Z}_n^d$, and $A = k[[z_1, \ldots, z_n]]$. $K(z_1, \ldots, z_d)$ is thus Galois over $L$. Let $H = Gal(K(z_1, \ldots, z_d)/K)$ and $G = Gal(K(z_1, \ldots, z_d)/L) \subset H$.

Elements of $H$ act on $K(z_1, \ldots, z_d)$ by multiplying $z_i$ by an $n$-th root of unity. $A^G$ is integrally closed, is finite over $R$ and has quotient field $L$. Thus $T = A^G$. The conclusions of the corollary follow. \qed
Lemma 8.3. Let $R$ be a $d$-dimensional regular local domain with $d \geq 2$, which is essentially of finite type over an algebraically closed field $k$ with $R/M(R) \cong k$. Suppose that $y_1, \ldots, y_d$ are regular parameters in $R$ and $f \in R$ is such that $f = y_1^{a_1} \cdots y_d^{a_d}$ for some natural numbers $a_1, \ldots, a_d$. Further suppose that $e$ is a positive integer. Let $\pi$ denote the remainder modulo $e$ of an integer $n$. Let $V$ be a valuation ring of the quotient field $K$ of $R$ which contains $k$, such that $V/M(V) = k$, and which dominates $R$. Then there exists a sequence of monoidal transforms

$$R = R_1 \rightarrow R_2 \rightarrow \cdots \rightarrow R_n$$

along $V$ such that there exist regular parameters $x_1, \ldots, x_d$ in the regular local ring $R_n$, a unit $e \in R_n$ and natural numbers $b_1, \ldots, b_d$ such that

$$f = ex_1^{b_1} \cdots x_d^{b_d}$$

with $\overline{b}_1 + \cdots + \overline{b}_d < e$.

Proof. The proof is by induction on $\overline{a}_1 + \cdots + \overline{a}_d$. Suppose that $\overline{a}_1 + \cdots + \overline{a}_d \geq e$, and the assertion of the lemma is true for all values of $\overline{a}_1 + \cdots + \overline{a}_d$ smaller than the given value. There exists a unique integer $t$ with $2 \leq t \leq d$ such that for every subsequence $1 \leq i(1) < \cdots < i(t-1) \leq d$ we have $\overline{a}_{i(1)} + \cdots + \overline{a}_{i(t-1)} < e$, but for some subsequence $1 \leq i(1) < \cdots < i(t) \leq d$ we have $\overline{a}_{i(1)} + \cdots + \overline{a}_{i(t)} \geq e$. After possibly reindexing $y_1, \ldots, y_d$ we may assume that $\overline{a}_1 + \cdots + \overline{a}_t \geq e$. After a further reindexing, we may assume that $\nu(y_{i+1}) = \nu(y_{i+1})$ for all $i \leq t$. Let

$$R_1 = R[\frac{y_2}{y_1}, \frac{y_3}{y_1}, \ldots, \frac{y_t}{y_1}]_{M_1}$$

where $M_1$ is the contraction of the maximal ideal of $V$ to $R[\frac{y_2}{y_1}, \frac{y_3}{y_1}, \ldots, \frac{y_t}{y_1}]$. There exist $\lambda_i \in k$ for $2 \leq i \leq t$ such that $M(R_1) = (y_1', \ldots, y_d')$, where

$$y_i = \begin{cases} 
  y_i' & \text{if } i = 1 \\
  y_i' (y_i' + \lambda_i) & \text{if } 1 < i \leq t \\
  y_i' & \text{if } i > t 
\end{cases}$$

We then have that

$$f = \epsilon'(y_1'^{a'_1}(y_2'^{a'_2}) \cdots (y_d')^{a'_d})$$

where

$$a'_i = \begin{cases} 
  a_1 + a_2 + \cdots + a_t & \text{if } i = 1 \\
  0 & \text{if } 2 \leq i \leq t \text{ and } \lambda_i \neq 0 \\
  a_i & \text{otherwise}
\end{cases}$$

and $\epsilon'$ is a unit in $R_1$. Since $\overline{a}_1 + \cdots + \overline{a}_i - \overline{a}_i < e$, $\overline{a}_i < e$ and $\overline{a}_1 + \cdots + \overline{a}_t \geq e$, we see that

$$\overline{a}_1' = \overline{a}_1 + \cdots + \overline{a}_t - \overline{e} < \overline{a}_1,$$

and hence

$$\overline{a}_1' + \cdots + \overline{a}_d < \overline{a}_1 + \cdots + \overline{a}_d.$$
In summary, we have

Proof. By Theorem 6.1, there exists a normal local ring $S'$ of $L$ which is dominated by $V$, and such that $e(S') < p$ if $k$ has positive characteristic $p$. Suppose that $e(S) > 1$. There exists a system of parameters $x_1, x_2, x_3$ in $S$ such that $e((x_1, x_2, x_3)S) = e(S)$ by Theorem 2.1. Let $K = k(x_1, x_2, x_3)$ and $R = k[x_1, x_2, x_3][x_1, x_2, x_3]$. $R$ is a 3 dimensional regular local ring which is dominated by $S$ (by Corollary 1 to Theorem 12, Section 9, Chapter VIII [ZS]). There exist $r_1, \ldots, r_n$ in the integral closure of $R$ in $L$ such that $L = K(r_1, \ldots, r_n)$. For $1 \leq j \leq n$, there exist nonconstant polynomials $f_j(Z) \in R[Z]$ such that $f_j(r_j) = 0$. We further have that $r_j \in S$ for all $j$ (by Zariski’s main theorem). Let $f = \prod_{j=1}^n \delta(f_j) \in R$ (where $\delta$ is the discriminant).

By Theorem 1.2, there exists a sequence of monoidal transforms $R \rightarrow R_0$ along $V$ such that $f = 0$ is a SNC divisor on the spectrum of the regular local ring $R_0$. Let $S_0$ be the normal local ring which is a local ring of the integral closure of $R_0$ in $L$, and is dominated by $V$.

Let $K^*, K^*_0, L^*, L^*_0$ be the respective quotient fields of the respective completions with respect to their maximal ideals $R^*, R^*_0, S^*, S^*_0$ of $R, R_0, S, S_0$. By Theorem 2.4, we may identify $K^*, K^*_0, L^*, L^*_0$ with subfields of $L_0$. We have $L^* = K^*(L)$ and $L^*_0 = K^*_0(L)$.

We will now show that Theorem 8.1 applies to $R_0^*$ in the field $L_0^*$. By Theorem 2.2, we have

$$e(S_0) \leq e(M(R_0) s_0) = e(M(R_0^*) s_0^*) = e(R_0^*)[L_0^* : K_0^*]$$

$$= [L_0^* : K_0^*] \leq [L^* : K^*]$$

$$= e(M(R^*))[L^* : K^*] = e(M(R^*) S^*) = e(S^*) = e(S).$$

In summary, we have

$$e(S_0) \leq [L_0^* : K_0^*] \leq [L^* : K^*] = e(S).$$

We have that $L_0^* = K_0^*(r_1, \ldots, r_n)$.

Since $f = 0$ is a SNC divisor on the spectrum of $R_0$, there exists a system of regular parameters $y_1, y_2, y_3$ in $R_0$, a unit $e \in R_0$ and natural numbers $c_1, c_2, c_3$ such that $f = c_1 y_1^{c_2} y_2^{c_3} y_3^{c_3}$. Suppose that $Q$ is a height one prime of $R_0^*$ which does not contain $y_1 y_2 y_3$. Since $f = \prod_{j=1}^n \delta(f_j)$, we have that $\delta(f_j)$ are units in $(R_0^*)_Q$ for $1 \leq i \leq n$, so that $(R_0^*)_Q$ is unramified in $L_0^*$ by Lemma 7.1. Thus $R_0^*$ is unramified in $L_0^*$ away from $V(y_1 y_2 y_3)$.

By (34) and the fact that $[L_0^* : K_0^*] < p$ by (36) (if $k$ has positive characteristic $p$), we see that for $1 \leq i \leq 3$, $(R_0^*)_Q$ is tamely ramified in $L_0^*$. Thus the conclusions of Theorem 8.1 hold for the inclusion $K_0^* \subset L_0^*$. Now by Corollary 8.2, there exists $s \in L_0^*$ and a prime number $q$ which is not equal to $p$, such that $g(Z) = Z^q - y_1 a_1 y_2 a_2 y_3 a_3 \in K_0^*[Z]$ is the minimal polynomial of $s$ over $K_0^*$ for some natural numbers $a_1, a_2, a_3$. Thus, $[K_0^*(s) : K_0^*] = q$. $q[L_0^* : K_0^*(s)] = [L_0^* : K_0^*] = e(S)$ implies that

$$[L_0^* : K_0^*(s)] \leq \frac{e(S)}{q}.$$
their maximal ideals $R_1^\ast$, $S_1^\ast$ of $R_1$ and $S_1$. We have

$$[L_1^\ast : K_1^\ast(s)] \leq [L_0^\ast : K_0^\ast(s)] \leq \frac{e(S)}{q}.$$  

Let $F$ be the local ring of the integral closure of $R_1^\ast$ in $K_1^\ast(s)$ which is dominated by $S_1^\ast$. By Theorem 2.2, we have

$$e(S_1) \leq e(M(F)S_1^\ast) = e(F)[L_1^\ast : K_1^\ast(s)] \leq \left(\frac{e(F)}{q}\right) e(S).$$

We will now show that $e(F) < q$, from which it will follow that $e(S_1) < e(S)$. We will then have achieved a reduction in the multiplicity of $S$ from which the conclusions of the theorem will follow from induction.

Let $x = s(z_1^m z_2^m z_3^m)^{-1}$. We have $x^q = (x^s)^q z_1^{b_1} z_2^{b_2} z_3^{b_3} \in R_1^\ast$. Thus $x^q$ has order $\nu_{R_1^\ast}(x^q)$ less that $q$ in $R_1^\ast$ and $K_1^\ast(s) = K_1^\ast(x)$.

First assume that $x^q$ is a unit in $R_1^\ast$. This is equivalent to $b_1 = b_2 = b_3 = 0$. Thus $R_1^\ast$ is unramified in $K_1^\ast(s)$, so that $F \cong R_1^\ast$ and $e(F) = 1 < q$.

Now assume that $x^q$ has positive order in $R_1^\ast$. We have

$$e(F) \leq [K_1^\ast(x) : K_1^\ast] \leq q.$$  

Assume $[K_1^\ast(x) : K_1^\ast] = q$. Then $Z^F = x^q$ is the minimal polynomial of $x$ over $K_1^\ast$.

Let $F' = R_1^\ast(x) \subset F$. $F'$ and $F$ have the same quotient field, and $F$ is finite over $F'$, so $e(F) \leq e(F')$.

Since

$$F' \cong K[[z_1, z_2, z_3, z]]/(z^q - e^s z_1^{b_1} z_2^{b_2} z_3^{b_3}),$$

we have $e(F') = b_1 + b_2 + b_3 < q$. \qed

9. Patching

**Theorem 9.1.** Suppose that $K$ is a 3-dimensional algebraic function field over an algebraically closed field $k$. Let $N$ be a set of valuation rings of $K$ which contain $k$ and whose residue fields are $k$. Suppose that $V_0$ and $V_1$ are normal projective varieties with function field $K$ such that each element of $N$ dominates a regular point on $V_0$ or $V_1$. Then there exists a normal projective variety $W$, with birational morphisms to $V_0$ and $V_1$, such that each element of $N$ dominates a nonsingular point of $W$.

**Proof.** Let $Z \subset V_0 \times_k V_1$ be the graph of the birational map from $V_0$ to $V_1$. The projection from $Z$ to $V_0$ is a projective birational morphism. Hence $Z$ is the blow up of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{V_0}$. Let $U_0 \subset V_0$ be the dense open set of nonsingular points in $V_0$. By Theorem 1.3, there exists a sequence of monoidal transforms $W_0 \to U_0$ such that $\mathcal{I}_W$ is invertible. $W_0 \to U_0$ is the blow up of an ideal sheaf $\mathcal{J}$ on $U_0$. Let $\mathcal{J}'$ be an ideal sheaf on $V_0$ which extends $\mathcal{J}$, and let $\pi_1 : X_1 \to V_0$ be the blow up of $\mathcal{J}'$. $\pi_1^{-1}(U_0) \cong W_0$ and $\mathcal{I}_1 = \mathcal{I}_W$ is an ideal sheaf on $X_1$ which is invertible on $W_0$. Let $V_2 \to X_1$ be the normalization of the blow up of $\mathcal{I}_1$, with induced projective morphism $\pi_2 : V_2 \to V_0$. Let $U_2$ be the nonsingular locus of $V_2$. We have that $W_0 \cong \pi_2^{-1}(U_0) \subset U_2$ is nonsingular, and by the universal property of blowing up, there exists a projective morphism $\alpha : V_2 \to V_1$. Let $U_1$ be the nonsingular locus of $V_1$, and $F_{\alpha}$ be the Zariski closure in $V_1$ of the fundamental locus of $\alpha^{-1} \mid U_1$.

Since $\alpha : V_2 \to V_1$ is projective and birational, $V_2$ is the blow up of an ideal sheaf $\mathcal{K}$ in $\mathcal{O}_{V_1}$.

Let $W_1 \to U_1$ be the principalization of $K\mathcal{O}_{U_1}$ of Theorem 1.3. We have a factorization

$$W_1 = X_n \to X_{n-1} \to \cdots \to X_0 = U_1$$
where each map $X_i \to X_{i-1}$ is the blow up of a point or nonsingular curve in the locus where $K\mathcal{O}_{X_{i-1}}$ is not invertible.

Since $W_1 \to U_1$ is a projective birational morphism, there exists an ideal sheaf $\mathcal{H}$ in $\mathcal{O}_{U_1}$ such that $W_1$ is the blow up of $\mathcal{H}$. Let $\mathcal{H}'$ be an ideal sheaf in $\mathcal{O}_{V_1}$ which extends $\mathcal{H}$ and such that the support of $\mathcal{O}_{V_1}/\mathcal{H}'$ is the Zariski closure of the support of $\mathcal{O}_{U_1}/\mathcal{H}$, which is equal to $F_s$. Let $\beta : V_3 \to V_1$ be the normalization of the blow up of $\mathcal{H}'$. Let $U_3$ be the nonsingular locus of $V_3$. We have that $W_1 = \beta^{-1}(U_1) \subset U_3$. Let $\gamma : V_3 \to V_2$ be the induced birational map, and

$$L = \left\{ p_2 \in V_2 \mid p_2 \text{ is the center of some } \nu \in N \text{ such that } \text{the center of } \nu \text{ on } V_1 \text{ is a singular point} \right\}.$$  

We have by assumption that $L \subset U_2$. Suppose that $p_2 \in L$. Let $p_1 = \alpha(p_2)$. Further suppose that there exists a curve $\Gamma_2$ in $V_2$ such that $p_2 \in V_2$ and $\Gamma_2$ is in the fundamental locus of $\gamma^{-1}$. Then $\alpha(\Gamma_2) \subset F_s$. Now suppose that $\alpha(\Gamma_2) = \Gamma_1$ is a curve. Let $\eta_1$ be the generic point of $\Gamma_1$ and $\eta_2$ be the generic point of $\Gamma_2$. By [Ab3], the birational local homomorphism $\mathcal{O}_{V_1,\eta_1} \to \mathcal{O}_{V_2,\eta_2}$ of two dimensional regular local rings factors uniquely as a sequence

$$(39) \quad \mathcal{O}_{V_1,\eta_1} = R_0 \to R_1 \to \cdots \to R_m = \mathcal{O}_{V_2,\eta_2}$$

of blow ups of regular local rings at the maximal ideal, followed by localization at a maximal ideal. Further, by the universal property of $\mathcal{K}$, we have that $\mathcal{K}_{\eta_1}R_i$ is invertible, but $\mathcal{K}_{\eta_1}R_i$ is not invertible for $i < m$.

By our construction of the sequence (38),

$$V_3 \times_{V_1} \text{Spec}(\mathcal{O}_{V_1,\eta_1}) \to \text{Spec}(\mathcal{O}_{V_1,\eta_1})$$

can be factored as a product of blow ups of points over $\eta_1$ (which are generic points of curves which dominate $\Gamma_1$), at which the extension of the ideal sheaf $\mathcal{K}$ is not invertible. Comparing with (39), we see that $V_3 \to V_2$ is an isomorphism in a neighborhood of $\eta_2$, a contradiction to our assumption that $\Gamma_2$ is in the fundamental locus of $\gamma^{-1}$. Thus $\alpha(\Gamma_2) = p_1$, which is a singular point of $F_s$.

It follows that there exists an open subset $\overline{U}_2$ of $V_2$, which is contained in the nonsingular locus $U_2$ of $V_2$, such that $L \subset \overline{U}_2$, and the fundamental locus $G$ of the birational map $\overline{U}_2 \to V_3$ is a union of curves containing points of $L$ which contract by $\alpha$ to points in the set of points $\alpha(L)$, and of some isolated points in $L$. $G$ is closed in $V_2$.

Let $Z_2 \subset \overline{U}_2 \times V_3$ be the graph of the birational map from $\overline{U}_2$ to $V_3$. There exists an ideal sheaf $\mathcal{J}$ on $\overline{U}_2$ such that $Z_2$ is the blow up of $\mathcal{J}$, and the support of $\mathcal{O}_{\overline{U}_2}/\mathcal{J}$ is $G$. By Theorem 1.3, there exists a sequence of monoidal transforms $W_2 \to \overline{U}_2$ such that $\mathcal{J}\mathcal{O}_{W_2}$ is invertible, and $W_2 \to \overline{U}_2$ is an isomorphism away from $G$.

Let $\mathcal{J}'$ be an extension of $\mathcal{J}$ to $V_2$ such that the support of $\mathcal{O}_{V_2}/\mathcal{J}'$ is the Zariski closure of $G$ in $V_2$. Let $\delta : V_4 \to V_2$ be the normalization of the blow up of $\mathcal{J}'$. Finally, let $V_5$ be the normalization of the graph of the birational map $V_4 \to V_3$. We will show that every element of $N$ has a nonsingular center on $V_5$.

Suppose that $\nu \in N$.

Let $p_1, p_2, p_3, p_4, p_5$ be the respective centers of $\nu$ on $V_1, V_2, V_3, V_4, V_5$.

Suppose that $p_1 \notin F_s$. Then

$$\mathcal{O}_{V_5, p_5} = \mathcal{O}_{V_1, p_1} \subset \mathcal{O}_{V_2, p_2} = \mathcal{O}_{V_4, p_4}$$

since $\alpha(G) \subset F_s$ implies $p_2 \notin G$. If $p_1$ is a singular point of $V_1$, then $\nu \in N$ implies $p_4$ is a nonsingular point on $V_4$. If $p_1$ is a nonsingular point of $V_1$, then $\mathcal{O}_{V_1, p_1} = \mathcal{O}_{V_2, p_2}$, so $p_4$ is a nonsingular point on $V_4$. Since $V_4 \to V_3$ is a morphism at $p_4$ and $p_4$ is a
nonsingular point of $V_4$, we have that $O_{V_5,p_5} \cong O_{V_4,p_4}$, and $p_5$ is a nonsingular point of $V_5$.

Suppose that $p_1 \in F_*$ is a nonsingular point of $V_1$. Then $\gamma$ is a morphism at the nonsingular point $p_3$, and

$$O_{V_4,p_4} = O_{V_2,p_2} \subset O_{V_3,p_3}.$$ 

Thus $O_{V_5,p_5} \cong O_{V_3,p_3}$ and $p_5$ is a nonsingular point of $V_5$.

Suppose that $p_1 \in F_*$ is a singular point of $V_1$ Then $p_2 \in L \subset U_2$. Thus $O_{V_3,p_3} \subset O_{V_4,p_4}$ and $p_4$ is a nonsingular point on $V_4$. It follows that $O_{V_5,p_5} \cong O_{V_4,p_4}$ and $p_5$ is a nonsingular point of $V_5$.

Corollary 9.2. Suppose that $K$ is a 3-dimensional algebraic function field over an algebraically closed field $k$. Let $N_{\mathfrak{p}}$ be the set of all valuation rings of $K$ which contain $k$ and whose residue field is $k$. Suppose that $\{V_1, \ldots, V_n\}$ are normal projective varieties with function field $K$ such that each element of $N_{\mathfrak{p}}$ dominates a nonsingular point on some $V_i$, with $1 \leq i \leq n$. Then there exists a nonsingular projective variety $W$ such that the function field of $W$ is $K$.

Proof. The proof is immediate from induction on the statement of Theorem 9.1. □

10. Resolution of Singularities

We first prove the existence of nonsingular models.

Theorem 10.1. Suppose that $k$ is an algebraically closed field of characteristic $\neq 2, 3$ or 5, and $K$ is a 3-dimensional algebraic function field over $k$. Then there exists a nonsingular projective variety $V$ whose function field is $K$.

Proof. Suppose that $V$ is a valuation ring of $K$ which contains $k$. We will first show that there exists a normal projective variety $W_V$ whose function field is $K$, such that the center $p$ of $V$ on $W_V$ is a nonsingular (not necessarily closed) point of $W_V$.

If the residue field of $V$ is $k$, then this follows from Theorem 8.4. Suppose that the residue field of $V$ strictly contains $k$, so that it is a transcendental extension. There exists a valuation ring $V^*$ of $K$ which contains $k$, and whose residue field is $k$, and such that $V$ is a localization of $V^*$ at a prime ideal (as follows from the construction of composite valuations on page 57 [Ab2]). By Theorem 8.4, there exists a normal projective variety $W_{V^*}$, with function field $K$, such that the center of $V^*$ on $W_{V^*}$ is at a nonsingular point $p^*$. Thus $O_{W_{V^*},p^*}$ is a regular local ring. The center of $V$ on $W_{V^*}$ is thus a point $p$ which is a nonsingular point of $W_{V^*}$, since the local ring of $p$ is a localization of $O_{W_{V^*},p^*}$.

By quasi-compactness of the Zariski-Riemann manifold (Theorem 40, Section 17, Chapter VI [ZS]), there exists a finite set of normal projective varieties $\{V_1, \ldots, V_n\}$ with function field $K$, such that the center of every valuation of $V$ is at a nonsingular point of some $V_i$. Now the existence of a nonsingular projective variety with function field $K$ follows from Corollary 9.2. □

We now prove the existence of a resolution of singularities, but do not require it to be an isomorphism above the nonsingular locus of $V$.

Theorem 10.2. Suppose that $V$ is a projective variety of dimension 3 over an algebraically closed field $k$ of characteristic $\neq 2, 3$ or 5. Then there exists a nonsingular projective variety $W$ and a birational morphism $\phi : W \to V$. 
Proof. By Theorem 10.1, there exists a nonsingular projective variety $W$ whose function field is $K$. Let $\Phi : V \to W$ be the birational map between $V$ and $W$, induced by the equality of their function fields with $K$. Let $\Gamma_\Phi \subset V \times_k W$ be the graph of $\Phi$. Projection onto the second factor is a birational projective morphism $\pi_2 : \Gamma_\Phi \to W$. Thus $\pi_2$ is the blow up of a sheaf of ideals $I$ on $W$. By Theorem 1.3, there exists a sequence of monoidal transforms $W_1 \to W$ such that $IO_{W_1}$ is invertible. Thus there is a projective birational morphism $W_1 \to V$. □

11. A Stronger Resolution Theorem

In this section we prove Theorem 1.1. We require the following lemma.

Lemma 11.1. Suppose that $V$ is a projective variety over a field $k$, and $C \subset V$ is a curve such that the generic point of $C$ is contained in the nonsingular locus of $V$. Then there exists a sequence of blow ups of closed points

$$V_n \to V_{n-1} \to \cdots \to V_1 \to V$$

such that the strict transform $C_n$ of $C$ in $V_n$ is contained in the nonsingular locus of $V_n$.

Proof. By Corollary 4.4 [C], there exists a sequence of blow ups of points $V_n \to V$ such that the strict transform $C_n$ of $C$ in $V_n$ is nonsingular. We may thus assume that $C$ is nonsingular from the outset.

Suppose that $p$ is a point in the intersection of $C$ and the singular locus of $V$. Let

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V_1 \to V_0 = V$$

be the sequence obtained by first blowing up $p$, then blowing up the point $p_1$ on the strict transform $C_1$ of $C$ above $p$, and then iterating this construction.

We have a projective embedding of $V$ into a projective space $X$ over $k$. The sequence (41) is obtained by constructing the corresponding sequence of blow ups of points

$$\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$$

and taking the strict transform $V_n$ of $V$ in $X_n$.

Let $R_n = O_{X_n,p_n}$ for $n \geq 0$. We have a sequence of homomorphisms of local rings

$$R = R_0 \to R_1 \to \cdots \to R_n \to \cdots$$

Let

$$I_n = I_{V_n,p_n}, I_{C_n} = I_{C_n,p_n},$$

$m_n$ be the maximal ideal of $R_n$ for $n \geq 0$.

There exist $s \leq d$ and regular parameters $x, y_1, \ldots, y_s, \ldots, y_d$ in $R$ such that $y_1 = \cdots = y_d = 0$ are local equations of $C$ in $X$, $y_1, \ldots, y_s \in I$ and the classes of $y_1, \ldots, y_n$ are an $R/m$ basis of $I/m^2 \cap I$.

If $y_1, \ldots, y_s$ do not generate $I$, then there exists $f \in I - (y_1, \ldots, y_s)$ such that $f \notin I_{C}^2$. This can be seen as follows. Since the generic point of $C$ is in the nonsingular locus of $V$, there exists a basis $y_1, \ldots, y_s, h_1, \ldots, h_t$ of $IR_{IC}$ which extends to a regular system of parameters of $R_{IC}$. Thus

$$h_1 = \sum_{i=1}^{d} \lambda_i y_i$$
with \( \lambda_i \in R_{I_C} \) and some \( \lambda_i \) is a unit in \( R_{I_C} \), with \( s + 1 \leq i \leq d \). Without loss of generality, \( i = d \). For \( 1 \leq i \leq d \), express

\[
\lambda_i = \frac{f_i}{g_i}
\]

with \( f_i, g_i \in R \) and \( g_i \not\in I_C \). We have \( f_d \not\in I_C \).

\[
(\prod_{j=1}^{d} g_j)h_1 = \sum_{i=1}^{d} f_i(\prod_{j \neq i} g_j)y_i \in I_{R_{I_C}} \cap R = I.
\]

Thus

\[
\sum_{i=1}^{d} f_i(\prod_{j \neq i} g_j)y_i \in I - I_{I_C}^2.
\]

There are regular parameters \( x, y_1(n), \ldots, y_d(n) \) in \( R_n \) defined by \( y_i = y_i(n)x^n \) for \( 1 \leq i \leq d \). We have that \( I_{C_n} = (y_1(1), \ldots, y_d(n)) \). Observe that the residue field of \( R_n \) is the residue field of \( R \) for all \( n \geq 0 \).

After subtracting an appropriate element of \( (y_1, \ldots, y_s) \) from \( f \), and possibly permuting \( y_{s+1}, \ldots, y_d \), we have an expansion of \( f \) in \( \hat{R} \),

\[
f = \sum_{i=1}^{d} a_i(x)y_i + \sum_{i_1 + \cdots + i_d \geq 2} a_{i_1, \ldots, i_d}(x)y_1^{i_1} \cdots y_d^{i_d} = \sum_{i=1}^{d} x^m \overline{a}_i(x)y_i + \sum_{i_1 + \cdots + i_d \geq 2} a_{i_1, \ldots, i_d}(x)y_1^{i_1} \cdots y_d^{i_d}
\]

where \( s < t \leq d \), \( \overline{a}_i(x) \) are unit series and \( m_i = \text{ord } a_i(x) \) for \( t \leq i \leq d \). We may assume that

\[
m_d = \min\{m_1, \ldots, m_d\}.
\]

For \( n > m_d \), we have an expression

\[
f = x^{m_d+n}\sum_{i=1}^{d} x^{m_i - m_d} \overline{a}_i(x)y_i(n) + x\Omega
\]

for some \( \Omega \in \hat{R}_n \). Let \( K \) be the quotient field of \( R \). Then

\[
\frac{f}{x^{m_d+n}} \in \hat{R}_n \cap K = R_n
\]

is in \( I_n \), as \( x = 0 \) is a local equation of the exceptional divisor of \( X_n \rightarrow X_{n-1} \), and \( x \) is thus not in \( I_n \).

\[
y_1(n), \ldots, y_s(n), \frac{f}{x^{m_d+n}} \in I_n
\]

extend to a regular system of parameters in \( R_n \). Iterating this procedure, we achieve that for large \( n \), \( I_n \) is generated by part of a regular system of parameters in \( R_n \). Thus \( p_n \) is in the nonsingular locus of \( V_n \).

We repeat this procedure for each of the finitely many points of \( C \) which are in the singular locus of \( V \), to construct a sequence (40) satisfying the conclusions of the lemma.

\[\square\]

We now prove Theorem 1.1.

By Theorem 10.2, there exists a projective, birational morphism \( \Phi : W \rightarrow V \) such that \( W \) is nonsingular. Let \( U \subseteq V \) be the open subset consisting of the nonsingular points of \( V \). Let \( G \) be the Zariski closure in \( V \) of the fundamental locus of the birational map \( \Phi^{-1} : U \rightarrow W \). \( \Phi \) is the blow up of an ideal sheaf \( \mathcal{I} \) of \( V \).
By Lemma 11.1, there exists a sequence of blow ups of points, after which we perform a normalization, $\beta : V_1 \to V$ such that $\beta$ is an isomorphism above $U$, and the strict transforms of all curves of $G$ in $V_1$ are contained in the nonsingular locus of $V_1$. $\beta$ is the blow up of an ideal sheaf $J$ of $V$.

By Theorem 1.3, applied to the ideal sheaf $J^3$, there exists a sequence of blow ups of points and nonsingular curves $\gamma : W_1 \to W$ such that $J^3$ is locally principal, so that $W_1$ is nonsingular, $\Psi : W_1 \to V_1$ is a morphism, and $W_1 \to W$ is an isomorphism above $\Phi^{-1}(U)$.

Let $C_1, \ldots, C_r$ be the curves in $V_1$ which are in the fundamental locus of the birational map $\Psi^{-1} : V_1 \to W_1$, are disjoint from $\beta^{-1}(U)$, and intersect the strict transform on $V_1$ of a curve of $G$. $\beta$ maps each $C_i$ into the singular locus of $V$. Since each $C_i$ intersects the nonsingular locus of $V_1$, by the Zariski - Abhyankar factorization theorem [Ab3], $W_1 \to V_1$ factors as a sequence of blow ups of nonsingular curves above $V_1$ in a neighborhood of the generic point of each $C_i$.

We now construct a birational morphism $V_2 \to V_1$ by blowing up a sequence of points on the intersection points of the strict transform of the $C_i$ and the strict transform of a curve in $G$, so that the strict transforms of the $C_i$ are disjoint from the strict transforms of the curves in $G$.

We may now construct a sequence of blow ups of curves (which may possibly be singular) $V_3 \to V_2$ which dominate the $C_i$, so that the birational map $W_1 \to V_3$ is an isomorphism above the generic points of each $C_i$. We may replace $V_3$ with its normalization. Since the $C_i$ contract to the singular locus of $V$, $\delta : V_3 \to V$ is an isomorphism over the nonsingular locus of $V$. $V_3 \to V_1$ is the blow up of an ideal sheaf $K$ of $V_1$. By Theorem 1.3, applied to the ideal sheaf $K^3$, we may construct a birational morphism $W_2 \to W_1$ such that $W_2$ is nonsingular, and $W_2 \to W_1$ is an isomorphism over the complement of a finite number of points of the $C_i$ in $V_1$.

The induced morphism $\alpha : W_2 \to V_3$ is the blow up of an ideal sheaf $L$ of $V_3$.

Let $H$ be the connected component of the fundamental locus of the birational map $\alpha^{-1} : V_3 \to W_2$ containing the strict transform of $G$.

Let $\epsilon : V_3 \to V$ be the induced morphism. By our construction, there exists a Zariski open subset $U^*$ of $V_3$ such that $U^*$ is contained in the nonsingular locus of $V_3$, $\epsilon^{-1}(U) \subset U^*$, $\epsilon$ is an isomorphism above $U$, and the intersection of $U^*$ and the fundamental locus of $\alpha^{-1}$ is $H$.

Since $V_3$ is normal, there exists an effective Weil divisor $D$ on $V_3$ and an ideal sheaf $M_1$ on $V_3$ such that the support of $O_{V_3}/M$ has dimension $\leq 1$, and $L = O_{V_3}(-D) \cap M_1$.

Since the support of $O_{V_3}/M_1|U^*$ is $H$, which is Zariski closed in $V_3$, we may define an ideal sheaf $M$ on $V_3$ by $M|U^* = O_{V_3}|U^*$, and $M|(V_3 - H) = M_1|(V_3 - H)$. Let $L_1 = O_{V_3}(-D) \cap M$. Then $L_1|U^*$ is a Cartier divisor, and $L_1|(V_3 - H) = L|(V_3 - H)$.

Let $V_4 \to V_3$ be the blow up of $L_1$. By our construction, $V_4 \to V_3$ is an isomorphism above points of $\epsilon^{-1}(U)$, so that $V_4 \to V$ is an isomorphism above the nonsingular locus of $V$. Since the birational map $V_4 \to V_3$ is an isomorphism above the complement of $H$ in $V_3$, $V_4$ is nonsingular. Thus $V_4 \to V$ is a resolution as desired.

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