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1. Introduction

The notion of singularity is basic to mathematics. In elementary algebra a singularity appears as a multiple root of a polynomial. In geometry a point on a space is non-singular if it has a tangent space whose dimension is that of the space. Both notions of singularity can be detected through the vanishing of derivatives.

Over an algebraically closed field, a variety is non-singular at a point if there exists a tangent space at the point which has the same dimension as that of the variety. More generally, a variety is non-singular at a point if its local ring is a regular local ring. A fundamental problem is to remove a singularity by simple algebraic mappings. That is, can a given variety be desingularized by a proper, birational morphism from a non-singular variety? This is always possible in all dimensions, over fields of characteristic zero. We give a complete proof of this in Chapter 6.

We also treat positive characteristic, developing the basic tools needed for this study, and giving a proof of resolution of surface singularities in positive characteristic in Chapter 7.

In Section 2.5 we discuss important open problems, such as resolution of singularities in positive characteristic and local monomialization of morphisms.

Chapter 8 gives a classification of valuations in algebraic function fields of surfaces, and a modernization of Zariski’s original proof of local uniformization for surfaces in characteristic zero.

This book has evolved out of lectures given at the University of Missouri and at the Chennai School of Mathematics. It can be used as part of a one year introductory sequence in algebraic geometry, and would provide an exciting direction after the basic notions of schemes and sheaves have been covered. A core course on resolution is covered in Chapters 2 through 6. The major ideas of resolution have been introduced by the end of Section 6.2, and after reading this far, a student will find the resolution theorems of Section 6.8 quite believable, and have a good feel for what goes into their proof.

Chapters 7 and 8 cover additional topics. These two chapters are independent, and can be chosen as possible followups to the basic material in the first 5 chapters. Chapter 7 gives a proof of resolution of singularities for surfaces in positive characteristic, and Chapter 8 gives a proof of local uniformization and resolution of singularities for algebraic surfaces. This chapter provides an introduction to valuation theory in algebraic geometry, and to the problem of local uniformization.

Chapter 10 proves foundational results on the singular locus that we need. On a first reading, I recommend that the reader simply look up the statements as needed in reading the main body of the book. Versions of almost all of these statements are much easier over algebraically closed fields of characteristic zero, and most of the results can be found in this case in standard text books in algebraic geometry.

I assume that the reader has some familiarity with algebraic geometry and commutative algebra, such as can be obtained from an introductory course on these subjects. This material is covered in books such as Atiyah Macdonald [12] or the basic sections of Eisenbud’s book [35], and the first two chapters of Hartshorne’s book on algebraic geometry [45], or Eisenbud and Harris’s book on Schemes [36].

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1.1. Notations. The notations of Hartshorne [45] will be followed, with the following differences and additions.

By a variety over a field \( K \) (or a \( K \)-variety), we will mean an open subset of an equidimensional reduced subscheme of the projective space \( \mathbf{P}^n_K \). Thus an integral variety is a “quasi-projective variety” in the classical sense. A curve is a one dimensional variety. A surface is a two dimensional variety, and a 3-fold is a three dimensional variety. A subvariety \( Y \) of a variety \( X \) is a closed subscheme of \( X \) which is a variety.

An affine ring is a reduced ring which is of finite type over a field \( K \).

If \( X \) is a variety, and \( I \) is an ideal sheaf on \( X \), we denote \( V(I) = \text{spec}(\mathcal{O}_X/I) \subset X \).

If \( Y \) is a subscheme of a variety \( X \), we denote the ideal of \( Y \) in \( X \) by \( I_Y \). If \( W_1, W_2 \) are subschemes of a variety \( X \), we will denote the scheme theoretic intersection of \( W_1 \) and \( W_2 \) by \( W_1 \cdot W_2 \). This is the subscheme \( W_1 \cdot W_2 = V(I_{W_1} + I_{W_2}) \subset W \).

A hypersurface is a codimension one subvariety of a non-singular variety.

2. Non-singularity and resolution of singularities

2.1. Newton’s method for determining the branches of a plane curve. Newton’s algorithm for solving \( f(x,y) = 0 \) by a fractional power series \( y = y(x^{r_0}) \) can be thought of as a generalization of the implicit function theorem to general analytic functions. We begin with this algorithm because of its simplicity and elegance, and because this method contains some of the most important ideas in resolution. We will see (in Section 2.5) that the algorithm immediately gives a local solution to resolution of analytic plane curve singularities, and that it can be interpreted to give a global solution to resolution of plane curve singularities (in Section 3.5). All of the proofs of resolution in this book can be viewed as generalizations of Newton’s algorithm, with the exception of the proof that curve singularities can be resolved by normalization (Theorems 2.12 and 4.3).

Suppose that \( K \) is an algebraically closed field of characteristic 0, \( K[[x,y]] \) is a ring of power series in two variables and \( f \in K[[x,y]] \) is a non-unit, such that \( x \not| f \). Write 
\[
 f = \sum_{i,j} a_{ij} x^i y^j \quad \text{with} \quad a_{ij} \in K.
\]

Let 
\[
 \text{mult} (f) = \min \{ i + j \mid a_{ij} \neq 0 \},
\]
\[
 \text{mult} (f(0,y)) = \min \{ j \mid a_{0j} \neq 0 \}.
\]

Set \( r_0 = \text{mult}(f(0,y)) \geq \text{mult}(f) \).

Set
\[
 \delta_0 = \min \left\{ \frac{i}{r_0 - j} : j < r_0 \text{ and } a_{ij} \neq 0 \right\}.
\]

\( \delta_0 = \infty \) if and only if \( f = uy^{r_0} \) where \( u \) is a unit in \( K[[x,y]] \). Suppose that \( \delta_0 < \infty \).

Then we can write
\[
 f = \sum_{i+\delta_0 j \geq \delta_0 r_0} a_{ij} x^i y^j
\]

with \( a_{0r_0} \neq 0 \) and the weighted leading form
\[
 L_{\delta_0}(x,y) = \sum_{i+\delta_0 j = \delta_0 r_0} a_{ij} x^i y^j = a_{0r_0} y^{r_0} + \text{terms of lower degree in } y
\]
has at least two non-zero terms. We can thus choose \( 0 \neq c_1 \in K \) so that
\[
L_{b_0}(1, c_1) = \sum_{i+b_0j=b_0r_0} a_{ij} c_1^j = 0.
\]
Write \( \delta_0 = \frac{p_0}{q_0} \) where \( q_0, p_0 \) are relatively prime positive integers. We make a transformation
\[
x = x_1^{p_0}, y = x_1^{p_0}(y_1 + c_1).
\]
Then
\[
f = x_1^{r_0} f_1(x_1, y_1)
\]
where
\[
f_1(x_1, y_1) = \sum_{i+b_0j=b_0r_0} a_{ij} (c_1 + y_1)^j + x_1 H(x_1, y_1).
\]
By our choice of \( c_1 \), \( f_1(0, 0) = 0 \). Set \( r_1 = \text{mult}(f_1(0, y_1)) \). We see that \( r_1 \leq r_0 \).
We have an expansion
\[
f_1 = \sum_{i} a_{ij}(1)x_1^j y_1^j.
\]
Set
\[
\delta_1 = \min \left\{ \frac{i}{r_1-j} : j < r_1 \text{ and } a_{ij}(1) \neq 0 \right\},
\]
and write \( \delta_1 = \frac{p_1}{q_1} \) with \( p_1, q_1 \) relatively prime. We can then choose \( c_2 \in K \) for \( f_1 \), in the same way that we chose \( c_1 \) for \( f \), and iterate this process, obtaining a sequence of transformations
\[
x = x_1^{p_0}, y = x_1^{p_0}(y_1 + c_1)
\]
\[
x_1 = x_2^{p_1}, y_1 = x_2^{p_1}(y_2 + c_2)
\]
\[
\vdots
\]
Either this sequence of transformations terminates after a finite number \( n \) of steps with \( \delta_n = \infty \), or we can construct an infinite sequence of transformations with \( \delta_n < \infty \) for all \( n \). This allows us to write \( y \) as a series in ascending fractional powers of \( x \).

As our first approximation, we can use our first transformation to solve for \( y \) in terms of \( x \) and \( y_1 \):
\[
y = c_1 x^{\delta_0} + y_1 x^{\delta_0}.
\]
Now the second transformation gives us
\[
y = c_1 x^{\delta_0} + c_2 x^{\delta_0 + \frac{\delta_1}{q_0}} + y_2 x^{\delta_0 + \frac{\delta_1}{q_0}}.
\]
We can iterate this procedure to get the formal fractional series
\[
y = c_1 x^{\delta_0} + c_2 x^{\delta_0 + \frac{\delta_1}{q_0}} + c_3 x^{\delta_0 + \frac{\delta_1}{q_0} + \frac{\delta_2}{q_0^2}} + \cdots
\]

**Theorem 2.1.** There exists an \( i_0 \) such that \( \delta_i \in N \) for \( i \geq i_0 \).

**Proof.** \( r_i = \text{mult}(f_i(0, y_1)) \) are monotonically decreasing, and positive for all \( i \), so it suffices to show that \( r_i = r_{i+1} \) implies \( \delta_i \in N \). Without loss of generality, we may assume that \( i = 0 \) and \( r_0 = r_1 \). \( f_1(x_1, y_1) \) is given by the expression (1). Set
\[
g(t) = f_1(0, t) = \sum_{i+b_0j=b_0r_0} a_{ij}(c_1 + t)^j.
\]
$g(t)$ has degree $r_0$. Since $r_1 = r_0$, we also have $\text{mult}(g(t)) = r_0$. Thus $g(t) = a_0 r_0 t^{r_0}$, and
\[
\sum_{i+\delta_0 j = \delta_0 r_0} a_{ij} t^j = a_0 r_0 (t - c_1)^{r_0}.
\]
In particular, since $K$ has characteristic 0, the binomial theorem shows that
\[
a_{i, r_0 - 1} \neq 0
\]
where $i$ is a natural number with $i + \delta_0 (r_0 - 1) = \delta_0 r_0$. Thus $\delta_0 \in \mathbb{N}$. □

We can thus find a natural number $m$, which we can take to be the smallest possible, and a series
\[
p(t) = \sum b_i t^i
\]
such that (3) becomes
\[
y = p(x^{\frac{1}{m}}).
\]
For $n \in \mathbb{N}$, set
\[
p_n(t) = \sum_{i=1}^n b_i t^i.
\]
Using induction, we can show that
\[
\text{mult}(f(t^m, p_n(t)) \to \infty
\]
as $n \to \infty$, and thus $f(t^m, p(t)) = 0$.
\[
y = \sum b_i x^{\frac{i}{m}}
\]
is a branch of the curve $f = 0$. This expansion is called a Puiseux series (when $r_0 = \text{mult}(f))$, in honor of Puiseux, who introduced this theory into algebraic geometry.

**Remark 2.2.** Our proof of Theorem 2.1 is not valid in positive characteristic, since we cannot conclude (4). Theorem 2.1 is in fact false over fields of positive characteristic. See Exercise 2 at the end of this section.

Suppose that $f \in K[[x, y]]$ is irreducible, and that we have found a solution $y = p(x^{\frac{1}{m}})$ to $f(x, y) = 0$. We may suppose that $m$ is the smallest natural number for which it is possible to write such a series. $y - p(x^{\frac{1}{m}})$ divides $f$ in $R_1 = K[[x^{\frac{1}{m}}, y]]$. Let $\omega$ be a primitive $m^{\text{th}}$ root of unity in $K$. Since $f$ is invariant under the $K$-algebra automorphism $\phi$ of $R_1$ determined by $x^{\frac{1}{m}} \to \omega x^{\frac{1}{m}}$ and $y \to y, y - p(\omega^j x^{\frac{1}{m}}) \mid f$ in $R_1$ for all $j$, and thus $y = p(\omega^j x^{\frac{1}{m}})$ is a solution to $f(x, y) = 0$ for all $j$. These solutions are distinct for $0 \leq j \leq m - 1$, by our choice of $m$.
\[
g = \prod_{j=0}^{m-1} \left(y - p(\omega^j x^{\frac{1}{m}})\right)
\]
is invariant under $\phi$, so $g \in K[[x, y]]$ and $g \mid f$ in $K[[x, y]]$, the ring of invariants of $R_1$ under the action of the group $\mathbb{Z}_m$ generated by $\phi$. Since $f$ is irreducible,
\[
f = u \left( \prod_{j=0}^{m-1} \left(y - p(\omega^j x^{\frac{1}{m}})\right) \right)
\]
where $u$ is a unit in $K[[x, y]]$. 
Remark 2.3. Some letters of Newton developing this idea are translated (from Latin) in Brieskorn and Knörrer’s book [16].

After we have defined non-singularity, we will return to this algorithm in (7) of Section 2.5, to see that we have actually constructed a resolution of singularities of a plane curve singularity.

Exercises

1. Construct a Puiseux series solution to \( f(x, y) = y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 = 0 \) over the complex numbers.

2. (Chevalley [20]) Prove that \( f(x, y) = xy^p - xy - 1 = 0 \) does not have a solution of the form (5) over a field of characteristic \( p > 0 \).

2.2. Smoothness and non-singularity.

Definition 2.4. Suppose that \( X \) is a scheme. \( X \) is non-singular at \( P \in X \) if \( \mathcal{O}_{X, P} \) is a regular local ring.

Recall that a local ring \( R \), with maximal ideal \( m \), is regular if the dimension of \( m/m^2 \) as a \( R/m \) vector space is equal to the Krull dimension of \( R \).

For varieties over a field, there is a related notion of smoothness.

Suppose that \( K[x_1, \ldots, x_n] \) is a polynomial ring over a field \( K \), \( f_1, \ldots, f_m \in K[x_1, \ldots, x_n] \). We define the Jacobian matrix

\[
J(f; x) = J(f_1, \ldots, f_m; x_1, \ldots, x_n) = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & & \vdots \\
\frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{pmatrix}.
\]

Let \( I = (f_1, \ldots, f_m) \) be the ideal generated by \( f_1, \ldots, f_m \), \( R = K[x_1, \ldots, x_n]/I \). Suppose that \( P \in \text{spec}(R) \) has ideal \( \mathfrak{m} \) in \( K[x_1, \ldots, x_n] \) and ideal \( \mathfrak{m} \) in \( R \). Let \( K(P) = \overline{R_{\mathfrak{m}}/\mathfrak{m}} \). We will say that \( J(f; x) \) has rank \( l \) at \( P \) if the image of the \( l \)-th fitting ideal \( I_l(J(f; x)) \) of \( I \times l \) minors of \( J(f, x) \) in \( K(P) \) is \( K(P) \) and the image of the \( s \)-th fitting ideal \( I_s(J(f; x)) \) in \( K(P) \) is \((0)\) for \( s > l \).

If \( A \) is a square matrix, we will denote the determinant of \( A \) by \(|A|\).

Definition 2.5. Suppose that \( X \) is a variety of dimension \( s \) over a field \( K \), and \( P \in X \). Suppose that \( U = \text{spec}(R) \) is an affine neighborhood of \( P \) such that \( R \cong K[x_1, \ldots, x_n]/I \) with \( I = (f_1, \ldots, f_m) \). Then \( X \) is smooth over \( K \) if \( J(f; x) \) has rank \( n - s \) at \( P \).

This definition depends only on \( P \) and \( X \) and not on any of the choices of \( U \), \( x \) or \( f \), as can be seen from a local calculation.

Theorem 2.6. Let \( K \) be a field. The set of points in a \( K \)-variety \( X \) which are smooth over \( K \) is an open set of \( X \).

Theorem 2.7. Suppose that \( X \) is a variety over a field \( K \) and \( P \in X \).

1. Suppose that \( X \) is smooth over \( K \) at \( P \). Then \( P \) is a non-singular point of \( X \).

2. Suppose that \( P \) is a non-singular point of \( X \) and \( K(P) \) is separably generated over \( K \). Then \( X \) is smooth over \( K \) at \( P \).
In the case when $K$ is algebraically closed, Theorems 2.6 and 2.7 are proven in Theorems I.5.3 and I.5.1 of [45]. Recall that an algebraically closed field is perfect. We will give the proofs of Theorems 2.6 and 2.7 for general fields in Chapter 10.

**Corollary 2.8.** Suppose that $X$ is a variety over a perfect field $K$ and $P \in X$. Then $X$ is non-singular at $P$ if and only if $X$ is smooth at $P$ over $K$.

**Proof.** This is immediate since an algebraic function field over a perfect field $K$ is always separably generated over $K$ (Theorem 13, Section 13, Chapter II, [85]). □

In the case when $X$ is an affine variety over an algebraically closed field $K$, the notion of smoothness is geometrically intuitive. Suppose that $X = V(I) = V(f_1, \ldots, f_m) \subset \mathbb{A}^n_K$ is an $s$-dimensional affine variety, where $I = (f_1, \ldots, f_m)$ is a reduced and equidimensional ideal. We interpret the closed points of $\mathbb{A}^n_K$ as the set of solutions to $f_1 = \cdots = f_m = 0$ in $K^n$. We identify a closed point $p = (a_1, \ldots, a_n) \in V(I) \subset \mathbb{A}^n_K$ with the maximal ideal $m = (x_1 - a_1, \ldots, x_n - a_n)$ of $K[x_1, \ldots, x_n]$. For $1 \leq i \leq m$,

$$f_i \equiv f_i(p) + L_i \mod m^2,$$

where

$$L_i = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(p)(x_i - a_i).$$

$f_i(p) = 0$ for all $i$ since $p$ is a point of $X$. The tangent space to $X$ in $\mathbb{A}^n_K$ at the point $p$ is

$$T_p(X) = V(L_1, \ldots, L_m) \subset \mathbb{A}^n_K.$$

We see that $\dim T_p(X) = n - \text{rank}(J(f; x)(p))$. Thus $\dim T_p(X) \geq s$ (by Remark 10.9), and $X$ is non-singular at $p$ if and only if $\dim T_p(X) = s$.

**Theorem 2.9.** Suppose that $X$ is a variety over a field $K$. Then the set of non-singular points of $X$ is an open dense set of $X$.

This theorem is proven when $K$ is algebraically closed in Corollary I.5.3 [45]. We will prove Theorem 2.9 when $K$ is perfect in Chapter 10. The general case is proven in the Corollary to Theorem 11 [78].

**Exercise**

Consider the curve $y^2 - x^3 = 0$ in $\mathbb{A}^2_K$, over an algebraically closed field $K$ of characteristic 0 or $p > 3$.

a. Show that at the point $p_1 = (1, 1)$, the tangent space $T_{p_1}(X)$ is the line $-3(x - 1) + 2(y - 1) = 0$.

b. Show that at the point $p_2 = (0, 0)$, the tangent space $T_{p_2}(X)$ is the entire plane $\mathbb{A}^2_K$.

c. Show that the curve is singular only at the origin $p_2$. 
2.3. Resolution of singularities. Suppose that $X$ is a $K$-variety, where $K$ is a field.

**Definition 2.10.** A resolution of singularities of $X$ is a proper birational morphism $\phi : Y \to X$ such that $Y$ is a non-singular variety.

A birational morphism is a morphism $\phi : Y \to X$ of varieties such that there is a dense open subset $U$ of $X$ such that $\phi^{-1}(U) \to U$ is an isomorphism. If $X$ and $Y$ are integral and $K(X)$ and $K(Y)$ are the respective function fields of $X$ and $Y$, $\phi$ is birational if and only if $\phi^* : K(X) \to K(Y)$ is an isomorphism.

A morphism of varieties $\phi : Y \to X$ is proper if for every valuation ring $V$ with morphism $\alpha : \text{spec}(V) \to X$, there is a unique morphism $\beta : \text{spec}(V) \to Y$ such that $\phi \circ \beta = \alpha$. If $X$ and $Y$ are integral, and $K(X)$ is the function field of $X$, then we only need consider valuation rings $V$ such that $K \subset V \subset K(X)$ in the definition of properness.

The geometric idea of properness is that every mapping of a "formal" curve germ into $X$ lifts uniquely to a morphism to $Y$.

One consequence of properness is that every proper map is surjective. The properness assumption rules out the possibility of "resolving" by taking the birational resolution map to be the inclusion of the open set of non-singular points into the given variety, or the mapping of the non-singular points of a partial resolution to the variety.

Birational proper morphisms of non-singular varieties (over a field of characteristic zero) can be factored by alternating sequences of blow ups and blow downs of non-singular subvarieties ([76], [10]). Locally, along a valuation, it is possible to factor a birational morphism of $n$-dimensional varieties as a product $\phi_{n-2} \circ \cdots \circ \phi_1$ where each $\phi_i$ is a product of blow ups followed by a product of blow downs ([23], [24]).

We can extend our definition of a resolution of singularities to arbitrary schemes. A reasonable category to consider is excellent (or quasi-excellent) schemes (defined in IV.7.8 [43] and on page 260 of [61]). The definition of excellence is extremely technical, but the idea is to give minimal conditions ensuring that the singular locus is preserved by natural base extensions such as completion. There are examples of non-excellent schemes which admit a resolution of singularities. Rotthaus [68] gives an example of a regular local ring $R$ of dimension 3 containing a field which is not excellent. In this case, spec($R$) is a resolution of singularities of spec($R$).

2.4. Normalization. Suppose that $R$ is an affine domain with quotient field $L$. Let $S$ be the integral closure of $R$ in $L$. Since $R$ is affine, $S$ is finite over $R$, so that $S$ is affine (c.f. Theorem 9, Chapter V [85]). We say that spec($S$) is the normalization of spec($R$).

Suppose that $V$ is a valuation ring of $L$ such that $R \subset V$. $V$ is integrally closed in $L$ (although $V$ need not be Noetherian). Thus $S \subset V$. The morphism spec($V$) → spec($R$) lifts to a morphism spec($V$) → spec($S$), so that spec($S$) → spec($R$) is proper.

If $X$ is an integral variety, we can cover $X$ by open affines spec($R_i$) with normalization spec($S_i$). The spec($S_i$) patch to a variety $Y$ called the normalization of $X$. $Y \to X$ is proper, by our local proof.

For a general (reduced but not necessarily integral) variety $X$, the normalization of $X$ is the disjoint union of the normalizations of the irreducible components of $X$.

**Theorem 2.11.** (Serre) A local ring $A$ is integrally closed if and only if $A$ is $R_1$ ($A_p$ is regular if $\text{ht}(p) \leq 1$) and $S_2$ (depth $A_p \geq 2$ if $\text{ht}(p) \geq 2$, depth $A_p = 1$ if $\text{ht}(p) = 1$).

A proof of this theorem is given in Theorem 23.8 [61].
Theorem 2.12. Suppose that $X$ is a 1-dimensional variety over a field $K$. Then the normalization of $X$ is a resolution of singularities.

Proof. Let $\overline{X}$ be the normalization of $X$. All local rings $O_{X,p}$ of points $p \in \overline{X}$ are local rings of dimension 1 which are integrally closed, so Theorem 2.11 implies they are regular.

As an example, consider the curve singularity $y^2 - x^3 = 0$ in $\mathbb{A}^2$, with affine ring $R = K[x, y]/(y^2 - x^3)$. In the quotient field $L$ of $R$ we have the relation $\left(\frac{y}{x}\right)^2 - x = 0$. Thus $\frac{y}{x}$ is integral over $R$. Since $R[\frac{y}{x}] = K[\frac{y}{x}]$ is a regular ring, it must be the integral closure of $R$ in $L$. Thus $\text{spec } (R[\frac{y}{x}]) \to \text{spec}(R)$ is a resolution of singularities.

Normalization is in general not enough to resolve singularities in dimension larger than 1. As an example, consider the surface singularity $X$ defined by $z^2 - xy = 0$ in $\mathbb{A}^3_K$, where $K$ is a field. The singular locus of $X$ is defined by the ideal $J = (z^2 - xy, y, x, 2z)$. $\sqrt{J} = (x, y, z)$, so the singular locus of $X$ is the origin in $\mathbb{A}^3$. Thus $K[x, y, z]/(z^2 - xy)$ is $R_1$. Since it is a complete intersection, it is $S_2$, so that $X$ is normal, but singular.

Kawasaki [54] has proven that under extremely mild assumptions a Noetherian scheme admits a Macaulayfication (all local rings have maximal depth). In general, Cohen-Macaulay schemes are far from being non-singular, but they do share many good homological properties with non-singular schemes.

A scheme which does not admit a resolution of singularities is given by the example of Nagata (Example 3, Appendix [63]) of a one dimensional noetherian domain $R$ whose integral closure $\overline{R}$ is not a finite $R$ module. We will show that such a ring cannot have a resolution of singularities. Suppose there exists a resolution of singularities $\phi : Y \to \text{spec}(R)$. That is, $Y$ is a regular scheme and $\phi$ is proper and birational. Let $L$ be the quotient field of $R$. $Y$ is a normal scheme by Theorem 2.11. Thus $Y$ has an open cover by affine open sets $U_1, \ldots, U_s$ such that $U_i \cong \text{spec}(T_i)$, with $T_i = R[g_{i1}, \ldots, g_{id}]$ where $g_{ij} \in L$, and each $T_i$ is integrally closed. The integrally closed subring

$$T = \Gamma(Y, O_Y) = \cap_{i=1}^s T_i$$

of $L$ is a finite $R$ module since $\phi$ is proper (Theorem III, 3.2.1 [42] or Corollary II.5.20 [45] if $\phi$ is assumed to be projective). Thus $T = \overline{R}$ which is impossible since $\overline{R}$ is not a finite $R$ module.

Exercises

1. Let $K = \mathbb{Z}_p(t)$ where $p$ is a prime, $t$ is an indeterminate. Let $R = K[x, y]/(x^p + y^p - t)$, $X = \text{spec}(R)$. Prove that $X$ is non-singular, but there are no points of $X$ which are smooth over $K$.

2. Let $K = \mathbb{Z}_p(t)$ where $p > 2$ is a prime, $t$ is an indeterminate. Let $R = K[x, y]/(x^2 + y^p - t)$, $X = \text{spec}(R)$. Prove that $X$ is non-singular, and $X$ is smooth over $K$ at every point except at the prime $(y^p - t, x)$.

2.5. Local uniformization and generalized resolution problems. Suppose that $f$ is irreducible in $K[[x, y]]$. Then the Puiseux series (5) of Section 2.1 determines an inclusion

$$R = K[[x, y]]/(f(x, y)) \to K[[t]]$$

(7)
with $x = t^m$, $y = p(t)$ the series of (5). Since the $m$ chosen in (5) of Section 2.1 is the smallest possible, $R$ and $K[[t]]$ have the same quotient field. (7) is an explicit realization of the normalization of the curve singularity germ $f(x, y) = 0$, and thus $\text{spec}(K[[t]]) \to \text{spec}(R)$ is a resolution of singularities. The quotient field $L$ of $K[[t]]$ has a valuation $\nu$ defined for non-zero $h \in L$ by

$$\nu(h(t)) = n \in \mathbb{Z}$$

if $h(t) = t^n u$ where $u$ is a unit in $K[[t]]$. This valuation can be understood in $R$ by the Puiseux series (6).

More generally, we can consider an algebraic function field $L$ over a field $K$. Suppose that $V$ is a valuation ring of $L$ containing $K$. The problem of local uniformization is to find a regular local ring, essentially of finite type over $K$ and with quotient field $L$ such that the valuation ring $V$ dominates $R$ ($R \subset V$ and the intersection of the maximal ideal of $V$ with $R$ is the maximal ideal of $R$).

If $L$ has dimension 1 over $K$ then the valuation rings $V$ of $L$ which contain $K$ are precisely the local rings of points on the unique non-singular projective curve $C$ with function field $L$ (c.f. Section I.6 [45]). The Newton method of Section 2.1 can be viewed as a solution to the local uniformization problem for complex analytic curves. If $L$ has dimension $\geq 2$ over $K$, there are many valuations rings $V$ of $L$ which are non-Noetherian (see the exercise of Section 8.1).

Zariski proved local uniformization for two dimensional function fields over an algebraically closed field of characteristic zero in [79]. He was able to patch together local solutions to prove the existence of a resolution of singularities for algebraic surfaces over an algebraically closed field of characteristic zero. He later was able to prove local uniformization for algebraic function fields of characteristic zero in [80]. This method leads to an extremely difficult patching problem in higher dimensions, which Zariski was able to solve in dimension 3 in [83], but still remains open in higher dimensions. Zariski’s proof of local uniformization can be considered as an extension of the Newton method to general valuations. In Chapter 8, we present Zariski’s proof of resolution of surface singularities through local uniformization.

Local uniformization has been proven for two dimensional function fields in positive characteristic by Abhyankar. Abhyankar has proven resolution of singularities in positive characteristic for surfaces and for three dimensional varieties [1],[3],[4].

There has recently been a resurgence of interest in local uniformization in positive characteristic ([47], [55], [71], [72]).

Some related resolution type problems are resolution of vector fields, resolution of differential forms and monomialization of morphisms.

Suppose that $X$ is a variety which is smooth over a perfect field $K$, and $D \in \text{Hom}(\Omega^1_X/K, \mathcal{O}_X)$ is a vector field. If $p \in X$ is a closed point, and $x_1, \ldots, x_n$ are regular parameters at $p$, then there is a local expression

$$D = a_1(x) \frac{\partial}{\partial x_1} + \cdots + a_n(x) \frac{\partial}{\partial x_n}$$

where $a_i(x) \in \mathcal{O}_{X,p}$. We can associate an ideal sheaf $\mathcal{I}_D$ to $D$ on $X$ by

$$\mathcal{I}_{D,p} = (a_1, \ldots, a_n)$$

for $p \in X$. 
There is an effective divisor \( F_D \) and an ideal sheaf \( J_D \) such that \( V(J_D) \) has codimension \( \geq 2 \) in \( X \) and \( \mathcal{I}_D = \mathcal{O}_X(-F_D)J_D \). The goal of resolution of vector fields is to find a proper morphism \( \pi : Y \rightarrow X \) so that if \( D' = \pi^{-1}(D) \), then the ideal \( \mathcal{I}_{D'} \subset \mathcal{O}_Y \) is a simple as possible.

This was accomplished by Seidenberg for vector fields on non-singular surfaces over an algebraically closed field of characteristic zero in [69]. The best statement that can be attained is that the order of \( J_{D'} \) at \( p \) (Definition 10.17), \( \nu_p(J_{D'}) \leq 1 \) for all \( p \in Y \). The basic invariant considered in the proof is the order \( \nu_p(J_{D'}) \). While this is an upper semi-continuous function on \( Y \), it can go up under a monoidal transform. However, we have that \( \nu_q(J_{D'}) \leq \nu_p(J_D) + 1 \) if \( \pi : Y \rightarrow X \) is the blow up of \( p \), and \( q \in \pi^{-1}(p) \). This should be compared with the classical resolution problem, where a basic result is that order cannot go up under a permissible monoidal transform (Lemma 6.4). Resolution of vector fields for smooth surfaces over a perfect field has been proven by Cano [17]. Cano has also proven a local theorem for resolution for vector fields over fields of characteristic zero [18], which implies local resolution (along a valuation) of a vector field. The statement is that we can achieve \( \nu_q(J_{D'}) \leq 1 \) (at least locally along a valuation) after a morphism \( \pi : Y \rightarrow X \). There is an analogous problem for differential forms. A recent paper on this is [19].

We can also consider resolution problems for morphisms \( f : Y \rightarrow X \) of varieties. The natural question to ask is if it is possible to perform monoidal transforms (blow ups of non-singular subvarieties) over \( X \) and \( Y \) to produce a morphism which is a monomial mapping.

**Definition 2.13.** Suppose that \( \Phi : X \rightarrow Y \) is a dominant morphism of non-singular irreducible \( K \)-varieties (where \( K \) is a field of characteristic zero). \( \Phi \) is monomial if for all \( p \in X \) there exists an étale neighborhood \( U \) of \( p \), uniformizing parameters \((x_1, \ldots, x_n)\) on \( U \), regular parameters \((y_1, \ldots, y_m)\) in \( \mathcal{O}_{Y, \Phi(p)} \), and a matrix \((a_{ij})\) of non-negative integers (which necessarily has rank \( m \)) such that

\[
\begin{align*}
y_1 &= x_1^{a_{11}} \cdots x_n^{a_{1n}} \\
\vdots \\
y_m &= x_1^{a_{m1}} \cdots x_n^{a_{mn}}
\end{align*}
\]

**Definition 2.14.** Suppose that \( \Phi : X \rightarrow Y \) is a dominant morphism of integral \( K \)-varieties. A morphism \( \Psi : X_1 \rightarrow Y_1 \) is a monomialization of \( \Phi \) [25] if there are sequences of blow ups of non-singular subvarieties \( \alpha : X_1 \rightarrow X \) and \( \beta : Y_1 \rightarrow Y \), and a morphism \( \Psi : X_1 \rightarrow Y_1 \) such that the diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\Psi} & Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Phi} & Y
\end{array}
\]

commutes, and \( \Psi \) is a monomial morphism.

If \( \Phi : X \rightarrow Y \) is a dominant morphism from a 3 dimensional variety to a surface (over an algebraically closed field of characteristic 0) then there is a monomialization of \( \Phi \) [25]. A generalized multiplicity is defined in this paper which can go up, causing a very high complexity in the proof. An extension of this result to strongly prepared morphisms from \( n \)-folds to surfaces is proven in [31]. It is not known if monomialization is true even for birational morphisms of varieties of dimension \( \geq 3 \), although it
is true locally along a valuation, from the following Theorem 2.15. Theorem 2.15 is proven when the quotient field of $S$ is finite over the quotient field of $R$ in [23]. The proof for general field extensions is in [26].

**Theorem 2.15.** (Theorem 1.1 [24], Theorem 1.1 [26]) Suppose that $R \subset S$ are regular local rings, essentially of finite type over a field $K$ of characteristic zero. Let $V$ be a valuation ring of $K$ which dominates $S$. Then there exist sequences of monoidal transforms $R \rightarrow R'$ and $S \rightarrow S'$ such that $V$ dominates $S'$, $S'$ dominates $R'$ and there are regular parameters $(x_1, \ldots, x_m)$ in $R'$, $(y_1, \ldots, y_n)$ in $S'$, units $\delta_1, \ldots, \delta_m \in S'$ and an $m \times n$ matrix $(a_{ij})$ of non-negative integers such that $\text{rank}(a_{ij}) = m$ is maximal and

$$
\begin{align*}
    x_1 &= y_1^{a_{11}} \cdots y_n^{a_{1n}} \delta_1 \\
    \vdots \\
    x_m &= y_1^{a_{m1}} \cdots y_n^{a_{mn}} \delta_m.
\end{align*}
$$

Thus (since $\text{char}(K) = 0$) there exists an etale extension $S' \rightarrow S''$ where $S''$ has regular parameters $\overline{y}_1, \ldots, \overline{y}_n$ such that $x_1, \ldots, x_m$ are pure monomials in $\overline{y}_1, \ldots, \overline{y}_n$.

The standard theorems on resolution of singularities allow one to easily find $R'$ and $S'$ such that (8) holds, but, in general, we will not have the essential condition $\text{rank}(a_{ij}) = m$. The difficulty of the problem is to achieve this condition.

This result gives very simple structure theorems for the ramification of valuations in characteristic zero function fields [33]. We discuss some of these results in Chapter 9. A generalization of monomialization in characteristic $p$ function fields of algebraic surfaces is obtained in [32] and especially in [33].

We point out that while it seems possible that Theorem 2.15 does hold in positive characteristic, there are simple examples in positive characteristic where a monomialization does not exist. The simplest example is the map of curves

$$
y = x^p + x^{p+1}
$$
in characteristic $p$.

A quasi-complete variety over a field $K$ is an integral finite type $K$-scheme which satisfies the existence part of the valuative criterion for properness (Hironaka, Chapter 0, Section 6 of [48] and Chapter 8 of [24]).

The construction of a monomialization by quasi-complete varieties follows from Theorem 2.15. Theorem 2.16 is proven for generically finite morphisms in [24] and for arbitrary morphisms in Theorem [26].

**Theorem 2.16.** (Theorem 1.2 [24], Theorem 1.2 [26]) Let $K$ be a field of characteristic zero, $\Phi : X \rightarrow Y$ a dominant morphism of proper $K$-varieties. Then there are birational morphisms of non-singular quasi-complete $K$-varieties $\alpha : X_1 \rightarrow X$ and $\beta : Y_1 \rightarrow Y$, and a monomial morphism $\Psi : X_1 \rightarrow Y_1$ such that the diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\Psi} & Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{\Phi} & Y
\end{array}
$$

commutes and $\alpha$ and $\beta$ are locally products of blow ups of non-singular subvarieties. That is, for every $z \in X_1$, there exist affine neighborhoods $V_1$ of $z$, $V$ of $x = \alpha(z)$, such that $\alpha : V_1 \rightarrow V$ is a finite product of monoidal transforms, and there exist affine
neighborhoods $W_1$ of $\Psi(z)$, $W$ of $y = \beta(\Psi(z))$, such that $\beta : W_1 \to W$ is a finite product of monoidal transforms.

A monoidal transform of a non-singular $K$-scheme $S$ is the map $T \to S$ induced by an open subset $T$ of $\text{proj}(\oplus T^n)$, where $T$ is the ideal sheaf of a non-singular subvariety of $S$.

The proof of Theorem 2.16 follows from Theorem 2.15, by patching a finite number of local solutions. The resulting schemes may not be separated.

It is an extremely interesting question to determine if a monomialization exists for all morphisms of varieties (over a field of characteristic zero). That is, the conclusions of Theorem 2.16 hold, but with the stronger conditions that $\alpha$ and $\beta$ are products of monoidal transforms on proper varieties $X_1$ and $Y_1$.

3. Curve singularities

3.1. Blowing up a point on $\mathbb{A}^2$. Suppose that $K$ is an algebraically closed field. Let

$$U_1 = \text{spec}(K[s, t]), U_2 = \text{spec}(K[u, v]) = \mathbb{A}^2_K.$$ 

Define a $K$-algebra isomorphism

$$\lambda : K[s, t] \cong K[s, t, 1_s] / K[u, v] = K[u, v, 1]$$

by $\lambda(s) = \frac{1}{v}, \lambda(t) = uv$. We define a $K$-variety $B_0$ by patching $U_1$ to $U_2$ on the open sets

$$U_2 - V(v) = \text{spec}(K[u, v, v]) \text{ and } U_1 - V(s) = \text{spec}(K[s, t, s])$$

by the isomorphism $\lambda$. The $K$-algebra homomorphisms

$$K[x, y] \to K[s, t]$$

defined by $x \mapsto st, y \mapsto t$, and

$$K[x, y] \to K[u, v]$$

defined by $x \mapsto u, y \mapsto uv$, are compatible with the isomorphism $\lambda$, so we get a morphism

$$\pi : B(p) = B_0 \to U_0 = \text{spec}(K[x, y]) = \mathbb{A}^2_K.$$ 

where $p$ denotes the origin of $U_0$. Upon localization of the above maps, we get isomorphisms

$$K[x, y]_y \cong K[s, t]_t \text{ and } K[x, y]_x \cong K[u, v]_u.$$ 

Thus $\pi : U_1 - V(t) \cong U_0 - V(y), \pi : U_2 - V(u) \cong U_0 - V(x)$, and $\pi$ is an isomorphism over $U_0 - V(x, y) = U_0 - \{p\}$.

$$\pi^{-1}(p) \cap U_1 = V(st, t) = \{V(t) \subset \text{spec}(K[s, t])\} = \text{spec}(K[s]),$$

$$\pi^{-1}(p) \cap U_2 = V(u, uv) = \{V(u) \subset \text{spec}(K[u, v])\} = \text{spec}(K[1_s]),$$

by the identification $s = \frac{1}{v}$. Thus $\pi^{-1}(p) \cong \mathbb{P}^1$. Set $E = \pi^{-1}(p)$. We have “blown up $p$” into a codimension 1 subvariety of $B(p)$, isomorphic to $\mathbb{P}^1$.

Set $R = K[x, y], m = (x, y)$.

$$U_1 = \text{spec}(K[s, t]) = \text{spec}(K[\frac{x}{y}, y]) = \text{spec}(R[\frac{x}{y}]),$$

$$U_2 = \text{spec}(K[x, \frac{y}{x}]) = \text{spec}(R[\frac{y}{x}]).$$
We see that
\[
B(p) = \text{spec}(R[\frac{y}{x}]) \cup \text{spec}(R[\frac{y}{x}]) = \text{proj}(\bigoplus_{n \geq 0} m^n).
\]

Suppose that \( q \in \pi^{-1}(p) \) is a closed point. If \( q \in U_1 \), then its associated ideal \( m_q \) is
a maximal ideal of \( R_1 = K[\frac{x}{y}, y] \) which contains \((x, y)R_1 = yR_1 \). Thus \( m_q = (y, \frac{x}{y} - \alpha) \)
for some \( \alpha \in K \). If we set \( y_1 = y, x_1 = \frac{x}{y} - \alpha \), we see that there are regular parameters
\((x_1, y_1)\) in \( \mathcal{O}_{B(p),q} \) such that
\[
x = y_1(x_1 + \alpha),
y = y_1.
\]
By a similar calculation, if \( q \in \pi^{-1}(p) \) and \( q \in U_2 \), there are regular parameters
\((x_1, y_1)\) in \( \mathcal{O}_{B(p),q} \) such that
\[
x = x_1, y = x_1(y_1 + \beta)
\]
for some \( \beta \in K \). If the constant \( \alpha \) or \( \beta \) is non-zero, then \( q \) is in both \( U_1 \) and \( U_2 \). Thus
the points in \( \pi^{-1}(p) \) can be expressed (uniquely) in one of the forms
\[
x = x_1, y = x_1(y_1 + \alpha) \text{ with } \alpha \in K, \text{ or } x = x_1 y_1, y = y_1.
\]

Since \( B(p) \) is projective over \( \text{spec}(R) \), it certainly is proper over \( \text{spec}(R) \). However,
it is illuminating to give a direct proof.

**Lemma 3.1.** \( B(p) \to \text{spec}(R) \) is proper.

**Proof.** Suppose that \( V \) is a valuation ring containing \( R \). Then \( \frac{y}{x} \) or \( \frac{x}{y} \in V \). Say
\( \frac{x}{y} \in V \). Then \( R[\frac{y}{x}] \subset V \), and we have a morphism
\[
\text{spec}(V) \to \text{spec}(R[\frac{y}{x}]) \subset B(p)
\]
which lifts the morphism \( \text{spec}(V) \to \text{spec}(R) \). \( \square \)

More generally, suppose that \( S = \text{spec}(R) \) is an affine surface over a field \( L \), and \( p \in S \) is a non-singular closed point. After possibly replacing \( S \) with an open subset \( \text{spec}(R_f) \), we may assume that the maximal ideal of \( p \) in \( R \) is \( m_p = (\pi, \gamma) \). We can then define the blow up of \( p \) in \( S \) by
\[
\pi : B(p) = \text{proj}(\bigoplus_{n \geq 0} m^n_p) \to S.
\]
We can write \( B(p) \) as the union of two affine open subsets.
\[
B(p) = \text{spec}(R[\frac{\pi}{x}]) \cup \text{spec}(R[\frac{\gamma}{x}]).
\]
\( \pi \) is an isomorphism over \( S - p \), and \( \pi^{-1}(p) \cong \mathbb{P}^1 \).

Suppose that \( S \) is a surface, and \( p \in S \) is a non-singular point, with ideal sheaf
\( m_p \subset \mathcal{O}_S \). The blow up of \( p \) in \( S \) is
\[
\pi : B(p) = \text{proj}(\bigoplus_{n \geq 0} m^n_p) \to S.
\]
\( \pi \) is an isomorphism away from \( p \), and if \( U = \text{spec}(R) \subset S \) is an affine open neighborhood of \( p \) in \( S \) such that
\[
(\pi, \gamma) = \Gamma(U, m_p) \subset R.
\]
theoretically, $\pi$

Proof. Let Lemma 3.2. (more generally, see Definition 10.17).

By $(\pi)$ of $\pi$, there is $f \in K[x, y]$ such that $V(f) = C$.

If $q \in C$ is a closed point, we will denote the corresponding maximal ideal of $K[x, y]$ by $m_q$. We set $\nu_q(C) = \max \{r \mid f \in m_q^r\}$

(more generally, see Definition 10.17).

**Lemma 3.2.** $q \in A^2_K$ is a non-singular point of $C$ if and only if $\nu_q(C) = 1$.

**Proof.** Let $R = K[x, y]$, and suppose that $m_q = (x, y)$. Let $T = R/fR$.

If $\nu_q(C) \geq 2$, then $f \in m_q^2$, and $m_qT/m_q^2T \cong m_q/m_q^2$ has dimension $2 > 1$, so that $T$ is not regular and $q$ is singular on $C$. However, if $\nu_q(C) = 1$, then we have $f \equiv \alpha x + \beta y \mod m_q^2$

where $\alpha, \beta \in K$ and at least one of $\alpha$ and $\beta$ is non-zero. Without loss of generality, we may suppose that $\beta \neq 0$. Thus

$$\bar{y} = -\frac{\alpha}{\beta} \bar{x} \mod m_q^2 + (f)$$

and $\bar{x}$ is a $K$ generator of $m_qT/m_q^2T \cong m_q/m_q^2 + (f)$. We see that $\dim_k m_qT/m_q^2T = 1$ and $q$ is non-singular on $C$.\]

**Remark 3.3.** This Lemma is also true in the situation where $q$ is a non-singular point on a non-singular surface $S$ (over a field $K$) and $C$ is a curve contained in $S$.

We need only modify the proof by replacing $R$ with the regular local ring $R = O_{S,q}$, which has regular parameters $(\bar{x}, \bar{y})$ which are $K(q)$ basis of $m_q/m_q^2$, and since $R$ is a UFD, there is $f \in R$ such that $f = 0$ is a local equation of $C$ at $q$.

Let $p$ be the origin in $A^2_K$, and suppose that $\nu_p(C) = r > 0$. Let $\pi : B(p) \to A^2_K$ be the blow up of $p$. The strict transform $\tilde{C}$ of $C$ in $B(p)$ is the Zariski closure of $\pi^{-1}(C - p) \cong C - p$ in $B(p)$. Let $E = \pi^{-1}(p)$ be the exceptional divisor. Set theoretically, $\pi^{-1}(C) = \tilde{C} \cup E$.

For some $a_{ij} \in K$, we have a finite sum

$$f = \sum_{i+j \geq r} a_{ij}x^iy^j$$

with $a_{ij} \neq 0$ for some $i, j$ with $i + j = r$. In the open subset $U_2 = \text{spec}(K[x_1, y_1])$ of $B(p)$ where

$$x = x_1, y = x_1y_1$$

$x_1 = 0$ is a local equation for $E$.

$$f = x_1^rf_1$$

where

$$f_1 = \sum_{i+j \geq r} a_{ij}x_1^{i+j-r}y_1^j.$$  

$x_1 \notin f_1$ since $\nu_p(C) = r$, so that $f_1 = 0$ is a local equation of the strict transform $\tilde{C}$ of $C$ in $U_2$.

In the open subset $U_1 = \text{spec}(K[\bar{x}_1, \bar{y}_1])$ of $B(p)$ where

$$x = \bar{x_1}\bar{y}_1, y = \bar{y}_1,$$
$y_1 = 0$ is a local equation for $E$ and

$$f = y_1 \bar{f}_1$$

where

$$\bar{f}_1 = \sum_{i+j \geq r} a_{ij} x_1^i y_1^j x^{i+j-r}$$

is a local equation of $\tilde{C}$ in $U_1$.

As a scheme, we see that $\pi^{-1}(C) = \tilde{C} + rE$. The scheme theoretic preimage of $C$ is called the total transform of $C$. Define the leading form of $f$ to be

$$L = \sum_{i+j=r} a_{ij} x^i y^j$$

Suppose that $q \in \pi^{-1}(p)$. There are regular parameters $(x_1, y_1)$ at $q$ of one of the forms

$$x = x_1, y = x_1(y_1 + \alpha) \text{ or } x = x_1, y = y_1.$$ 

In the first case we have $f_1 = \frac{\bar{f}_1}{x_1} = 0$ is a local equation of $\tilde{C}$ at $q$, where

$$f_1 = \sum_{i+j=r} a_{ij} (y_1 + \alpha)^j + x_1 \Omega$$

for some polynomial $\Omega$. Thus $\nu_q(\tilde{C}) \leq r$ and $\nu_q(\tilde{C}) = r$ implies

$$\sum_{i+j=r} a_{ij} (y_1 + \alpha)^j = a_0 y_1^r.$$ 

We then see that

$$\sum_{i+j=r} a_{ij} y_1^j = a_0 (y_1 - \alpha)^r$$

and

$$L = a_0 (y - \alpha x)^r.$$ 

In the second case we have $f_1 = \frac{\bar{f}_1}{y_1} = 0$ is a local equation of $\tilde{C}$ at $q$, where

$$f_1 = \sum_{i+j=r} a_{ij} x_1^i y_1^j + \Omega$$

for some series $\Omega$. Thus $\nu_q(\tilde{C}) \leq r$ and $\nu_q(\tilde{C}) = r$ implies

$$\sum_{i+j=r} a_{ij} x_1^i = a_r x_1^r$$

and

$$L = a_r x_1^r.$$ 

We then see that there exists a point $q \in \pi^{-1}(p)$ such that $\nu_q(\tilde{C}) = r$ precisely when $L = (ax + by)^r$ for some constants $a, b \in K$. Since $r > 0$, there is at most one point $q \in \pi^{-1}(p)$ where the multiplicity does not drop.
3.2. Completion. Suppose that $A$ is a local ring with maximal ideal $m$. A coefficient field of $A$ is a subfield $L$ of $A$ which is mapped onto $A/m$ by the quotient mapping $A \rightarrow A/m$.

A basic theorem of Cohen is that an equicharacteristic complete local ring contains a coefficient field (Theorem 27, Section 12, Chapter VIII [85]). This leads to Cohen’s structure theorem (Corollary, Section 12, Chapter VIII [85]), which shows that an equicharacteristic complete regular local ring $A$ is isomorphic to a formal power series ring over a field. In fact, if $L$ is a coefficient field of $A$, and if $(x_1, \ldots, x_n)$ is a regular system of parameters of $A$, then $A$ is the power series ring

$$A = L[[x_1, \ldots, x_n]].$$

We further remark that the completion of a local ring $R$ is a regular local ring if and only if $R$ is regular (c.f. Section 11, Chapter VIII [85]).

**Lemma 3.4.** Suppose that $S$ is a non-singular algebraic surface defined over a field $K$, $p \in S$ is a closed point, $\pi : B = B(p) \rightarrow S$ is the blow up of $p$, and suppose that $q \in \pi^{-1}(p)$ is a closed point such that $K(q)$ is separable over $K(p)$. Let $R_1 = \mathcal{O}_{S,p}$ and $R_2 = \mathcal{O}_{B,q}$, and suppose that $K_1$ is a coefficient field of $R_1$, $(x, y)$ are regular parameters in $R_1$. Then there exists a coefficient field $K_2 = K_1(\alpha)$ of $R_2$ and regular parameters $(x_1, y_1)$ of $R_2$ such that

$$\hat{\pi}^* : R_1 \rightarrow R_2$$

is the map given by

$$\sum_{i,j \geq 0} a_{ij} x^i y^j \rightarrow \sum_{i,j \geq 0} a_{ij} x_1^{i+j}(y_1 + \alpha)^j$$

where $a_{ij} \in K_1$, or $K_2 = K_1$ and

$$\hat{\pi}^* : R_1 \rightarrow R_2$$

is the map given by

$$\sum_{i,j \geq 0} a_{ij} x^i y^j \rightarrow \sum_{i,j \geq 0} a_{ij} x_1^i y_1^{i+j}$$

**Proof.** $R_1 = K_1[[x, y]]$. We have a natural homomorphism

$$\mathcal{O}_S \rightarrow \mathcal{O}_{S,p} \rightarrow R_1$$

which induces

$$q \in B(p) \times_S \text{spec}(R_1) = \text{spec}(R_1[[\frac{y}{x}]]) \cup \text{spec}(R_1[[\frac{x}{y}]]).$$

Let $m_q$ be the ideal of $q$ in $B(p) \times_S \text{spec}(R_1)$. If $q \in \text{spec}(R_1[[\frac{y}{x}]])$, then

$$K(q) \cong R_1[\frac{y}{x}] / m_q \cong K_1[\frac{y}{x}]/(f(\frac{y}{x}))$$

for some irreducible polynomial $f(\frac{y}{x})$ in the polynomial ring $K_1[[\frac{y}{x}]]$. Since $K(q)$ is separable over $K_1 \cong K(p)$, $f$ is separable. $m_q = (x, f(\frac{y}{x})) \subset R_1[[\frac{y}{x}]]$.

Let $\overline{\alpha} \in R_1[[\frac{y}{x}]][\frac{y}{x}] / m_q$ be the class of $\frac{y}{x}$. We have the residue map $\phi : R_2 \rightarrow L$ where $L = R_1[[\frac{y}{x}]] / m_q$.

There is a natural embedding of $K_1$ in $R_2$. We have a factorization of $f(t) = (t - \overline{\alpha})\overline{\gamma}(t)$ in $L[t]$ where $t - \overline{\alpha}$ and $\overline{\gamma}(t)$ are relatively prime. By Hensel’s Lemma (Theorem 17, Section 7, Chapter VIII [85]) there is $\alpha \in R_2$ such that $\phi(\alpha) = \overline{\alpha}$ and $f(t) = (t - \alpha)\gamma(t)$ in $R_2[t]$, where $\phi(\gamma(t)) = \overline{\gamma}(t)$. The subfield $K_2 = K_1[\alpha]$ of $R_2$ is
thus a coefficient field of $R_2$, and $m_q R_2 = (x, \frac{y}{x} - \alpha)$. Thus $R_2 = K_2[[x, \frac{y}{x} - \alpha]]$. Set $x_1 = x, y_1 = \frac{y}{x} - \alpha$. The inclusion

$$R_1 = K_1[[x, y]] \to R_2 = K_2[[x_1, y_1]]$$

is natural. A series

$$\sum a_{ij} x^i y^j$$

with coefficients $a_{ij} \in K_1$ maps to the series

$$\sum a_{ij} x_1^i y_1^j(y_1 + \alpha)^j.$$  

We now give an example to show that even if a regular local ring contains a field, there may not be a coefficient field of the completion of the ring containing that field. Thus the above lemma does not extend to non-perfect fields.

Let $K = \mathbb{Z}_p(t)$ where $t$ is an indeterminate. Let $R = K[x]_{(x^p-t)}$. Suppose that $\hat{R}$ has a coefficient field $L$ containing $K$. Let $\phi : \hat{R} \to \hat{R}/m$ be the residue map. Since $\phi | L$ is an isomorphism, there exists $\lambda \in L$ such that $\lambda^p = t$. Thus $(x^p - t) = (x - \lambda)^p$ in $\hat{R}$. But this is impossible since $(x^p - t)$ is a generator of $m\hat{R}$, and is thus irreducible.

**Theorem 3.5.** Suppose that $R$ is a reduced affine ring over a field $K$, and $A = R_p$ where $p$ is a prime ideal of $R$. Then the completion $\hat{A} = \hat{R}_p$ of $A$ with respect to its maximal ideal is reduced.

When $K$ is a perfect field, this is a theorem of Chevalley (Theorem 31, Section 13, Chapter VIII [85]). The general case follows from Scholie IV 7.8.3 (vii) [43].

However, the property of being a domain is not preserved under completion. A simple example is $f = y^2 - x^2 + x^3$. $f$ is irreducible in $\mathbb{C}[x, y]$, but is reducible in the completion $\mathbb{C}[[x, y]]$.

$$f = y^2 - x^2 - x^3 = (y - x\sqrt{1 + x})(y + x\sqrt{1 + x}).$$

The first part of the expansions of the two factors are

$$y - x - \frac{1}{2}x^2 + \cdots$$

and

$$y + x + \frac{1}{2}x^2 + \cdots.$$

**Lemma 3.6.** (Weierstrass Preparation Theorem) Let $K$ be a field, and suppose that $f \in K[[x_1, \ldots, x_n, y]]$ is such that

$$0 < r = \nu(f(0, \ldots, 0, y)) = \max\{ n \mid y^n \text{ divides } f(0, \ldots, 0, y) \} < \infty.$$

Then there exists a unit series $u$ in $K[[x_1, \ldots, x_n, y]]$ and non-unit series $a_i \in K[[x_1, \ldots, x_n]]$ such that

$$f = u(y^r + a_1 y^{r-1} + \cdots + a_r).$$

A proof is given in Theorem 5, Section 1, Chapter VII [85].

A concept which will be important in this book is the Tschirnhausen transformation, which generalizes the ancient notion of “completion of the square” in the solution of quadratic equations.
Definition 3.7. Suppose that \( K \) is a field of characteristic \( p \geq 0 \) and \( f \in K[[x_1, \ldots, x_n, y]] \) has an expression
\[
f = y^r + a_1 y^{r-1} + \cdots + a_r
\]
with \( a_i \in K[[x_1, \ldots, x_n]] \) and \( p = 0 \) or \( p \nmid r \). The Tschirnhausen transformation of \( f \) is the change of variables replacing \( y \) with
\[
y' = y + \frac{a_1}{r}.
\]
then has an expression
\[
f = (y')^r + b_2(y')^{r-2} + \cdots + b_r
\]
with \( b_i \in K[[x_1, \ldots, x_n]] \) for all \( i \).

Exercise
Suppose that \((R, m)\) is a complete local ring and \( L_1 \subset R \) is a field such that \( R/m \) is finite and separable over \( L_1 \). Use Hensel’s Lemma (Theorem 17, Section 7, Chapter VIII [85]) to prove that there exists a coefficient field \( L_2 \) of \( R \) containing \( L_1 \).

3.3. Blowing up a point on a non-singular surface. We define the strict transform, the total transform and \( \nu_p(C) \) analogously to the definition of Section 3.1 (More generally, see Section 4.1 and Definition 10.17).

Lemma 3.8. Suppose that \( X \) is a non-singular surface over an algebraically closed field \( K \), and \( C \) is a curve on \( X \). Suppose that \( p \in X \) and \( \nu_p(C) = r \). Let \( \pi : B(p) \to X \) be the blow up of \( p \), \( \bar{C} \) be the strict transform of \( C \), and suppose that \( q \in \pi^{-1}(p) \). Then \( \nu_q(\bar{C}) \leq r \), and if \( r > 0 \), there is at most one point \( q \in \pi^{-1}(p) \) such that \( \nu_q(\bar{C}) = r \).

Proof. Let \( f = 0 \) be a local equation of \( C \) at \( p \). If \((x, y)\) are regular parameters in \( \mathcal{O}_{X, p} \), we can write
\[
f = \sum a_{ij} x^i y^j
\]
as a series of order \( r \) in \( \mathcal{O}_{X, p} \cong K[[x, y]] \). Let
\[
L = \sum_{i+j=r} a_{ij} x^i y^j
\]
be the leading form of \( f \).
If \( q \in \pi^{-1}(p) \), then \( \mathcal{O}_{X, p} \) has regular parameters \((x_1, y_1)\) such that
\[
x = x_1, \quad y = x_1(y_1 + \alpha), \quad \text{or} \quad x = x_1y_1, \quad y = y_1.
\]
This substitution induces the inclusion
\[
K[[x, y]] = \mathcal{O}_{x, p} \to \mathcal{O}_{B(p), p} = K[[x_1, y_1]].
\]
First suppose that \((x_1, y_1)\) are defined by
\[
x = x_1, \quad y = x_1(y_1 + \alpha).
\]
Then a local equation for \( \bar{C} \) at \( q \) is \( f_1 = \frac{f}{x_1} \), which has the expansion
\[
f_1 = \sum a_{ij} x_1^{i+j-r}(y_1 + \alpha)^j + x_1 \Omega
\]
for some series \( \Omega \). We finish the proof as in Section 3.1.
Suppose that $C$ is the curve $y^2 - x^3 = 0$ in $\mathbb{A}_K^2$. The only singular point of $C$ is the origin $p$. Let $\pi : B(p) \to \mathbb{A}_K^2$ be the blow up of $p$, $\tilde{C}$ be the strict transform of $C$, $E$ be the exceptional divisor $E = \pi^{-1}(p)$.

On $U_1 = \text{spec}(K[x,y]) \subset B(p)$ we have coordinates $x_1,y_1$ with $x = x_1 y_1$, $y = y_1$. A local equation for $E$ on $U_1$ is $y_1 = 0$.

$$y^2 - x^3 = y_1^2(1 - x_1^3 y_1)$$

A local equation for $\tilde{C}$ on $U_1$ is $1 - x_1^3 y_1 = 0$, which is a unit on $E \cap U_1$, so $\tilde{C} \cap E \cap U_1 = \emptyset$.

On $U_2 = \text{spec}(K[x,\frac{x}{y}]) \subset B(p)$ we have coordinates $\bar{x}_1,\bar{y}_1$ with $x = \bar{x}_1$, $y = \bar{x}_1 \bar{y}_1$. A local equation for $E$ on $U_2$ is $x_1 = 0$.

$$y^2 - x^3 = \bar{x}_1^2(y_1^2 - \bar{x}_1)$$

A local equation for $\tilde{C}$ on $U_2$ is $\bar{y}_1^2 - \bar{x}_1 = 0$, which has order $\leq 1$ everywhere. Thus $\tilde{C}$ is non-singular, and

$$\tilde{C} \to C$$

is a resolution of singularities.

$$\Gamma(\tilde{C}, \mathcal{O}_{\tilde{C}}) = K[\bar{x}_1, \bar{y}_1]/(\bar{y}_1^2 - \bar{x}_1) = K[\bar{y}_1] = K[\frac{y}{x}].$$

This is the normalization of $K[x,y]/(y^2 - x^3)$ that we computed in Section 2.4.

As a further example, consider the curve $C$ with equation $y^2 - x^5$ in $\mathbb{A}^2$. The only singular point of $C$ is the origin $p$. Let $\pi : B(p) \to \mathbb{A}^2$ be the blow up of $p$, $E$ be the exceptional divisor, $\tilde{C}$ be the strict transform of $C$. There is only one singular point $q$ on $\tilde{C}$. Regular parameters at $q$ are $(x_1, y_1)$ where $x = x_1$, $y = x_1 y_1$. $\tilde{C}$ has the local equation

$$y_1^2 - x_1^3 = 0.$$ 

In this example $\nu_q(\tilde{C}) = \nu_q(C) = 2$, so the multiplicity has not dropped.

However, this singularity is resolved after blowing up $q$, as we calculated in the previous example.

These examples suggest an algorithm to resolve curve singularities. First blow up all singular points. If the resulting curve is not resolved, blow up the new singular points, and repeat as long as the curve is singular.

This algorithm ends after a finite number of iterations in a non-singular curve. In Sections 3.4 and 3.5 we will prove this for curves, embedded in a non-singular surface, first over an algebraically closed field $K$ of characteristic zero, and then over an arbitrary field.

3.4. Resolution of curves embedded in a non-singular surface I. In this section we consider curves embedded in a non-singular surface, over an algebraically closed field $K$ of characteristic zero, and prove that their singularities can be resolved.

The theorem is proven by passing to the completion of the local ring of the surface at a singular point of the curve. This allows us to view a local equation of the curve as a powerseries in two variables. Our main invariant is the multiplicity of the curve at a point. We know that this multiplicity cannot go up after blowing up, so we must show that it will eventually go down, after enough blowing up.
Theorem 3.9. Suppose that $C$ is a curve on a non-singular surface $X$ over an algebraically closed field $K$ of characteristic zero. Then there exists a sequence of blow ups of points $\lambda : Y \to X$ such that the strict transform $\tilde{C}$ of $C$ on $Y$ is non-singular.

Proof. Let $r = \max(\nu_p(C) \mid p \in C)$. If $r = 1$, $C$ is non-singular, so we may assume that $r > 1$. The set $\{ p \in C \mid \nu(p) = r \}$ is a subset of the singular locus of $C$, which is a proper closed subset of the 1-dimensional variety $C$, so it is a finite set.

The proof is by induction on $r$.

We can construct a sequence of projective morphisms

$$\cdots \to X_n \xrightarrow{\pi_n} \cdots \xrightarrow{\pi_1} X_0 = X$$

(10)

where each $\pi_{n+1} : X_{n+1} \to X_n$ is the blow up of all points on the strict transform $C_n$ of $C$ which have multiplicity $r$ on $C_n$. If this sequence is finite, then there is an integer $n$ such that all points on the strict transform $C_n$ of $C$ have multiplicity $\leq r - 1$. By induction on $r$, we can repeat this process then to construct the desired morphism $Y \to X$ which induces a resolution of $C$.

We will assume that the sequence (10) is infinite, and derive a contradiction. If it is infinite, then for all $n \in \mathbb{N}$ there are closed points $p_n \in C_n$ which have multiplicity $r$ on $C_n$ and such that $\pi_{n+1}(p_{n+1}) = p_n$ for all $n$. Let $R_n = \hat{\mathcal{O}}_{X_n,p_n}$ for all $n$. We then have an infinite sequence of completions of quadratic transforms of local rings

$$R_0 \to R_1 \to \cdots \to R_n \to \cdots$$

Suppose that $(x, y)$ are regular parameters in $R_0 = \hat{\mathcal{O}}_{X_p}$, and $f = 0$ is a local equation of $C$ in $R_0 = K[[x, y]]$. $f$ is reduced by Theorem 3.5. After a linear change of variables in $(x, y)$, we may assume that $\nu(f(0, y)) = r$. By the Weierstrass preparation theorem,

$$f = u(y^r + a_1(x)y^{r-1} + \cdots + a_r(x))$$

where $u$ is a unit series. We now “complete the rth power of $f$”. Set

$$\bar{y} = y + \frac{a_1(x)}{r}$$

(11)

Then

$$y^r + a_1(x)y^{r-1} + \cdots + a_r(x) = \bar{y}^r + b_2(x)\bar{y}^{r-2} + \cdots + b_r(x).$$

(12)

for some series $b_i(x)$. We may thus assume that

$$f = y^r + a_2(x)y^{r-2} + \cdots + a_r(x).$$

(13)

Since $f$ is reduced, we must have some $a_i(x) \neq 0$. Set

$$n = \min \left\{ \frac{\text{mult}(a_i(x))}{i} \right\}$$

(14)

$n \geq 1$ since $\nu(f) = r$. $\hat{\mathcal{O}}_{X_{1,p_1}}$ has regular parameters $x_1, y_1$ such that

1. $x = x_1$, $y = x_1(y_1 + \alpha)$ with $\alpha \in K$ or
2. $x = x_1y_1$, $y = y_1$
In case 2.,

\[ f = y_1^r f_1 \]

where

\[ f_1 = 1 + \frac{a_2(x_1 y_1)}{y_1^2} + \cdots + \frac{b_r(x_1 y_1)}{y_1^r} \]

is a local equation of the strict transform \( C_1 \) of \( C \) at \( p_1 \). \( f_1 \) is a unit, so that \( \nu_{p_1}(C_1) = 0 \). Thus case 2. cannot occur. In case 1.,

\[ f = x_1^r f_1 \]

where

\[ f_1 = (y_1 + \alpha)^r + \frac{a_2(x_1)}{x_1^2} (y_1 + \alpha)^{r-2} + \cdots + \frac{a_r(x_1)}{x_1^r} \]

\( f_1 = 0 \) is a local equation at \( p_1 \) for \( C_1 \). If \( \alpha \neq 0 \), then \( \nu_{p_1}(C_1) \leq r - 1 \), so this case cannot occur.

If \( \alpha = 0 \), and \( \nu_{p_1}(C_1) = r \), then \( f_1 \) has the form of (13), with \( n \) decreased by 1.

Thus for some index \( i \) with \( i \leq n \) we have that \( p_i \) must have multiplicity \( < r \) on \( C_i \), a contradiction to the assumption that (10) has infinite length. □

The Transformation of (11) is called a Tschirnhausen transformation (Definition 3.7). The Tschirnhausen transformation finds a (formal) curve of maximal contact \( H = V(g) \) for \( C \) at \( p \). The Tschirnhausen transformation was introduced as a fundamental method in resolution of singularities by Abhyankar.

Definition 3.10. Suppose that \( C \) is a curve on a non-singular surface \( S \) over a field \( K \), and \( p \in C \). A non-singular curve \( H = V(g) \subset \text{spec}(\hat{O}_{S,p}) \) is a formal curve of maximal contact for \( C \) at \( p \) if whenever

\[ S_n \to S_{n-1} \to \cdots \to S_1 \to \text{spec}(\hat{O}_{S,p}) \]

is a sequence of blow ups of points \( p_i \in S_i \), such that the strict transform \( C_i \) of \( C \) in \( S_i \) contains \( p_i \) with \( \nu_{p_i}(C_i) = \nu_p(C) \) for \( i \leq n \), then the strict transform \( H_i \) of \( H \) in \( S_i \) contains \( p_i \).

Exercises

1. Resolve the singularities by blowing up.
   a. \( x^2 = x^4 + y^4 \)
   b. \( xy = x^6 + y^6 \)
   c. \( x^3 = y^2 + x^4 + y^4 \)
   d. \( x^2y + xy^3 = x^4 + y^4 \)
   e. \( y^2 = x^n \)
   f. \( y^4 - 2x^3y^2 - 4x^5y + x^6 - x^7 \).

2. Suppose that \( D \) is an effective divisor on a non-singular surface \( S \). That is, there exist curves \( C_i \) on \( S \), \( r_i \in \mathbb{N} \) such that \( I_D = I_{C_1} \cdots I_{C_n} \). \( D \) has simple normal crossings (SNCs) on \( S \) if for every \( p \in S \) there exist regular parameters \( (x,y) \) in \( \mathcal{O}_{S,p} \) such that if \( f = 0 \) is a local equation for \( D \) at \( p \), then \( f = \text{unit} x^a y^b \) in \( \mathcal{O}_{S,p} \).

   Find a sequence of blow ups of points making the total transform of \( y^2 - x^3 = 0 \) a SNCs divisor.
3. Suppose that $D$ is an effective divisor on a non-singular surface $S$. Show that there exists a sequence of blow ups of points 
\[ S_n \to \cdots \to S \]
such that the total transform $\pi^*(D)$ is a SNCs divisor.

3.5. **Resolution of curves embedded in a non-singular surface II.** In this section we consider curves embedded in a non-singular surface, defined over an arbitrary field $K$.

**Lemma 3.11.** Suppose that $K$ is a field, $S$ is a non-singular surface over $K$, $C$ is a curve on $S$ and $p \in C$ is a closed point. Suppose that $\pi : B = B(p) \to S$ is the blow up of $p$, $\tilde{C}$ is the strict transform of $C$ on $B$ and $q \in \pi^{-1}(p) \cap \tilde{C}$. Then 
\[ \nu_q(\tilde{C}) \leq \nu_p(C) \]
and if 
\[ \nu_q(\tilde{C}) = \nu_p(C) \]
then $K(p) = K(q)$.

**Proof.** Let $R = \hat{O}_{S,p}$, $(x, y)$ be regular parameters in $R$. Since there is a natural embedding of $\pi^{-1}(p)$ in $B \times S$ spec$(R)$,
\[ q \in B \times S \text{ spec}(R) = \text{ spec}(R[y/x]) \cup \text{ spec}(R[y/x]). \]
Without loss of generality, we may assume that $q \in \text{ spec}(R[y/x])$. Let $K_1 \cong K(p)$ be a coefficient field of $R$, $m \subset R[y/x]$ be the ideal of $q$. Since $R[y/x]/(x, y) \cong K_1[y/x]$, there exists an irreducible monic polynomial $h(t) \in K_1[t]$ such that 
\[ mR[y/x] = (x, h(y/x)). \]
Let $f \in R$ be such that $f = 0$ is a local equation of $C$, $r = \nu_p(C)$. We have an expansion in $R$,
\[ f = \sum_{i+j \geq r} a_{ij}x^iy^j \]
with $a_{ij} \in K_1$. $f = x^r f_1$ where $f_1 = 0$ is a local equation of the strict transform $\tilde{C}$ of $C$ in $\text{ spec}(R[y/x])$.
\[ f_1 = \sum_{i+j=r} a_{ij}(y/x)^j + x\Omega \]
in $R[y/x]$. Let $n = \nu_q(\tilde{C})$. Then 
\[ h(y/x)^n \text{ divides } \sum_{i+j=r} a_{ij}(y/x)^j \]
in $K_1[y/x]$.
\[ \text{deg}(\sum_{i+j=r} a_{ij}(y/x)^j) \leq r \]
in $K_1[y/x]$ implies $n \leq r$ and $\nu_q(\tilde{C}) = r$ implies $\text{deg}(h(y/x)) = 1$ so that $h = y/x - \alpha$ with $\alpha \in K_1$ and (by Lemma 3.4) there exist regular parameters $(x_1, y_1)$ in $m_q \subset \mathcal{O}_{B(p),q}$ such that 
\[ \hat{O}_{S,p} = K_1[[x, y]] \to K_1[[x_1, y_1]] = \hat{O}_{B(p),q} \]
is the natural $K_1 \cong K(p)$ algebra homomorphism

$$x = x_1, y = x_1(y_1 + \alpha).$$

\[\square\]

**Theorem 3.12.** Suppose that $C$ is a curve which is a subvariety of a non-singular surface $X$ over a field $K$. Then there exists a sequence of blow ups of points $\lambda : Y \to X$ such that the strict transform $\tilde{C}$ of $C$ on $Y$ is non-singular.

**Proof.** Let $r = \max(\nu_p(C) \mid p \in C)$. If $r = 1$, $C$ is non-singular, so we may assume that $r > 1$. The set $\{p \in C \mid \nu_p(C) = r\}$ is a subset of the singular local of $C$ which is a proper closed subset of the 1-dimensional variety $C$, so it is a finite set.

The proof is by induction on $r$.

We can construct a sequence of projective morphisms

$$\cdots \to X_n \xrightarrow{\pi_n} \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X \tag{15}$$

where each $\pi_{n+1} : X_{n+1} \to X_n$ is the blow up of all points on the strict transform $C_n$ of $C$ which have multiplicity $r$ on $C_n$. If this sequence is finite, then there is an integer $n$ such that all points on the strict transform $C_n$ of $C$ have multiplicity $\leq r - 1$. By induction on $r$, we can then repeat this process to construct the desired morphism $Y \to X$ which induces a resolution of $C$.

We will assume that the sequence (15) is infinite, and derive a contradiction. If it is infinite, then for all $n \in \mathbb{N}$ there are closed points $p_n \in C_n$ which have multiplicity $r$ on $C_n$ and such that $\pi_{n+1}(p_{n+1}) = p_n$. Let $R_n = \mathcal{O}_{X_n, p_n}$ for all $n$. We then have an infinite sequence of completions of quadratic transforms (blow ups of maximal ideals) of local rings

$$R = R_0 \to R_1 \to \cdots \to R_n \to \cdots$$

We will define

$$\delta_{p_i} \in \frac{1}{r!} \mathbb{N}$$

such that

$$\delta_{p_i} = \delta_{p_{i-1}} - 1 \tag{16}$$

for all $i \geq 1$. We can thus conclude that $p_i$ has multiplicity $< r$ on the strict transform $C_i$ of $C$ for some natural number $i \leq \delta_p + 1$. From this contradiction it will follow that (15) is a sequence of finite length.

Suppose that $f \in R_0 = \mathcal{O}_{X, p}$ is such that $f = 0$ is a local equation $C$ and $(x, y)$ are regular parameters in $R = \mathcal{O}_{X, p}$ such that $r = \text{mult}(f(0, y))$. We will call such $(x, y)$ good parameters for $f$. Let $K'$ be a coefficient field of $R$. There is an expansion

$$f = \sum_{i+j \geq r} a_{ij} x^i y^j$$

with $a_{ij} \in K'$ for all $i, j$ and $a_{0r} \neq 0$. Define

$$\delta(f ; x, y) = \min \left\{ \frac{i}{r-j} \mid j < r \text{ and } a_{ij} \neq 0 \right\}. \tag{17}$$
\[ \delta(f; x, y) \geq 1 \text{ since } (x, y) \text{ are good parameters} \]

We thus have an expression (with \( \delta = \delta(f; x, y) \))

\[
f = \sum_{i+j \delta \geq r \delta} a_{ij} x^i y^j = L_\delta + \sum_{i+j \delta > r \delta} a_{ij} x^i y^j
\]

where

\[
L_\delta = \sum_{i+j \delta = r \delta} a_{ij} x^i y^j = a_{0r} y^r + \Lambda
\] (18)

is such that \( a_{0r} \neq 0 \) and \( \Lambda \) is not zero.

Suppose that \((x, y)\) are fixed good parameters of \( f \).

Define

\[
\delta_p = \sup \{ \delta(f; x, y_1) \mid y = y_1 + \sum_{i=1}^{n} b_i x^i \text{ with } n \in \mathbb{N} \text{ and } b_i \in K' \} \in \frac{1}{r!} \mathbb{N} \cup \{ \infty \}.
\] (19)

We cannot have \( \delta_p = \infty \), since then there would exist a series

\[
y = y_1 + \sum_{i=1}^{\infty} b_i x^i
\]

such that \( \delta(f; x, y_1) = \infty \), and thus there would be a unit series \( \gamma \) in \( R \) such that

\[
f = \gamma y_1^r.
\]

But then \( r = 1 \) since \( f \) is reduced in \( R \), a contradiction. We see then that \( \delta_p \in \frac{1}{r!} \mathbb{N} \).

After possibly making a substitution

\[
y = y_1 + \sum_{i=1}^{n} b_i x^i
\]

with \( b_i \in K' \), we may assume that \( \delta_p = \delta(f; x, y) \).

Let \( \delta = \delta_p \).

Since \( \nu_{p_1}(C_1) = r \), by Lemma 3.11, we have an expression \( R_1 = K'[x_1, y_1] \), where \( R \to R_1 \) is the natural \( K' \)-algebra homomorphism such that either

\[
x = x_1, y = x_1(y_1 + \alpha)
\]

for some \( \alpha \in K' \), or

\[
x = x_1 y_1, y = y_1.
\]

We first consider the case where \( x = x_1 y_1, y = y_1 \). A local equation of \( C_1 \) (in \( \text{spec}(R_1) \)) is \( f_1 = 0 \), where

\[
f_1 = \frac{f}{y_1^r} = \sum_{i+j = r} a_{ij} x_1^i y_1^j + y_1 \Omega = a_{0r} + x_1 g + y_1 h
\]

for some series \( \Omega, g, h \in R_1 \). Thus \( f_1 \) is a unit in \( R_1 \), a contradiction.

Now consider the case where \( x = x_1, y = x_1(y_1 + \alpha) \) with \( 0 \neq \alpha \in K' \). A local equation of \( C_1 \) is

\[
f_1 = \frac{f}{x_1} = \sum_{i+j = r} a_{ij} (y_1 + \alpha)^j + x_1 \Omega
\]

for some \( \Omega \in R_1 \). Since \( \nu_{p_1}(C_1) = r \),

\[
\sum_{i+j = r} a_{ij} (y_1 + \alpha)^j = a_{0r} y_1^r.
\]
Substituting $t = y_1 + \alpha$ we have

$$\sum_{i+j=r} a_{ij}t^j = a_{0r}(t - \alpha)^r.$$  

If we now substitute $t = \frac{y}{x}$ and multiply the series by $x^r$ we obtain the leading form $L$ of $f$,

$$L = \sum_{i+j=r} a_{ij}x^iy^j = a_{0r}(y - \alpha x)^r = a_{0r}y^r + \cdots + (-1)^r a_{0r}\alpha^rx^r.$$  

Comparing with (18), we see that $r = L$ and $\delta = 1$, so that $L_\delta = L$. But we can replace $y$ with $y - \alpha x$ to increase $\delta$, a contradiction to the maximality of $\delta$. Thus $\nu_1(C_1) < r$, a contradiction.

Finally, consider the case $x = x_1, y = x_1y_1$. Set $\delta' = \delta - 1$. A local equation of $C_1$ is

$$f_1 = \frac{f}{x_1^\delta} = \sum_{i+j \geq r\delta} a_{ij}x_1^{i+j-r}y_1^j = \sum_{\tilde{r} = i+j = r\delta} a_{\tilde{r}-j+r,j}x_1^{\tilde{r}}y_1^j + \sum_{\tilde{r} = i+j > r\delta} a_{\tilde{r}-j+r,j}x_1^{\tilde{r}}y_1^j$$

where $\tilde{i} = i + j - r$. Since

$$L_{\delta'}(x_1, y_1) = \frac{1}{x_1^\delta} L_\delta(x_1, x_1y_1)$$

has at least two non-zero terms, we see that $\delta(f_1; x_1, y_1) = \delta' = \delta - 1$.

Finally, we will show that $\delta_{p_1} = \delta(f_1; x_1, y_1)$. Suppose not. Then we can make a substitution

$$y_1 = \gamma_1 - \sum b_i x_1^i = \gamma'_1 - bx_1^d + \text{ higher order terms in } x_1$$

with $0 \neq b \in K'$ such that

$$\delta(f_1; x_1, y_1') > \delta(f_1; x_1, y_1).$$

Then we have an expression

$$\sum_{\tilde{r} = i+j = r\delta} a_{\tilde{r}-j+r,j}x_1^{\tilde{r}}(y_1' - bx_1^d)^j = a_{0r}(y_1')^r + \sum_{i+j > r\delta} b_{ij}x_1^i(y_1')^j$$

so that $\delta' = d \in \mathbb{N}$, and

$$\sum_{\tilde{r} = i+j = r\delta} a_{\tilde{r}-j+r,j}x_1^{\tilde{r}}(y_1' - bx_1^d)^j = a_{0r}(y_1')^r.$$  

Thus

$$\sum_{i+j = r\delta} a_{ij}x_1^{i+j-r}y_1^j = \sum_{\tilde{r} = i+j = r\delta} a_{\tilde{r}-j+r,j}x_1^{\tilde{r}}y_1^j = a_{0r}(y_1 + bx_1^d)^r.$$  

If we now multiply these series by $x_1^\delta$ we obtain

$$L_\delta = \sum_{i+j = r\delta} a_{ij}x^iy^j = a_{0r}(y + bx^d)^r.$$  

But we can now make the substitution $y' = y - bx^d$ and see that

$$\delta(f; x, y') > \delta(f; x, y) = \delta_p,$$
a contradiction, from which we conclude that $\delta_p = \delta' = \delta_p - 1$. We can then inductively define $\delta_p$, for $i \geq 0$ by this procedure so that (16) holds. The conclusions of the Theorem now follow. \qed

Remark 3.13.  
1. This proof is a generalization of the algorithm of Section 2.1. A more general version of this, valid in arbitrary two dimensional regular local rings, can be found in [5] or [67].

2. Good parameters $(x, y)$ for $f$ which achieve $\delta_p = \delta(f; x, y)$ are such that $y = 0$ is a (formal) curve of maximal contact for $C$ at $p$ (Definition 3.10).

Exercises

1. Prove that the $\delta_p$ defined in formula (19) is equal to

$$\delta_p = \sup \{ \delta(f; x, y) \mid (x, y) \text{ are good parameters for } f \}.$$  

Thus $\delta_p$ does not depend on the initial choice of good parameters, and $\delta_p$ is an invariant of $p$.

2. Suppose that $\nu_q(C) > 1$. Let $\pi : B(p) \to X$ be the blow up of $p$, $\hat{C}$ be the strict transform of $C$. Show that there is at most one point $q \in \pi^{-1}(p)$ such that $\nu_q(\hat{C}) = \nu(p(C))$.

3. The Newton Polygon $N(f; x, y)$ is defined as follows. Let $I$ be the ideal in $R = K[[x, y]]$ generated by the monomials $x^\alpha y^\beta$ such that the coefficient $a_{\alpha\beta}$ of $x^\alpha y^\beta$ in $f(x, y) = \sum a_{\alpha\beta} x^\alpha y^\beta$ is not zero. Set

$$P(f; x, y) = \{(\alpha, \beta) \in \mathbb{Z}^2 | x^\alpha y^\beta \in I\}.$$  

Now define $N(f; x, y)$ to be the smallest convex subset of $\mathbb{R}^2$ such that $N(f; x, y)$ contains $P(f; x, y)$ and if $(\alpha, \beta) \in N(f; x, y)$, $(s, t) \in \mathbb{R}^2_+$, then $(\alpha + s, \beta + t) \in N(f; x, y)$. Now suppose that $(x, y)$ are good parameters for $f$ (so that $\nu(f(0, y)) = \nu(f) = r$). Then $(0, r) \in N(f; x, y)$. Let the slope of the steepest segment of $N(f; x, y)$ be $s(f; x, y)$. We have $0 \geq s(f; x, y) \geq -1$, since $a_{\alpha\beta} = 0$ if $\alpha + \beta < r$. Show that

$$\delta(f; x, y) = -\frac{1}{s(f; x, y)}.$$  

4. Resolution type theorems

4.1. Blow ups of ideals. Suppose that $X$ is a variety, and $\mathcal{J} \subset \mathcal{O}_X$ is an ideal sheaf. The blow up of $\mathcal{J}$ is

$$\pi : B = B(\mathcal{J}) = \text{proj}(\bigoplus_{n \geq 0} \mathcal{J}^n) \to X.$$  

$B$ is a variety and $\pi$ is proper. If $X$ is projective then $B$ is projective. $\pi$ is an isomorphism over $X - V(\mathcal{J})$, and $\mathcal{J}\mathcal{O}_B$ is a locally principal ideal sheaf.

If $U \subset X$ is an open affine subset, and $R = \Gamma(U, \mathcal{O}_X)$, $I = \Gamma(U, \mathcal{J}) = (f_1, \ldots, f_m) \subset R$, then

$$\pi^{-1}(U) = B(I) = \text{proj}(\bigoplus_{n \geq 0} I^n).$$
If $X$ is an integral scheme, we have

$$\text{proj}(\bigoplus_{n \geq 0} I^n) = \bigcup_{i=1}^m \text{spec}(R[\frac{f_1}{f_i}, \cdots, \frac{f_m}{f_i}]).$$

Suppose that $W \subset X$ is a subscheme with ideal sheaf $\mathcal{I}_W$ on $X$.
The total transform of $W$, $\pi^*(W)$ is the subscheme of $B$ with the ideal sheaf

$$\mathcal{I}_{\pi^*(W)} = \mathcal{I}_W \mathcal{O}_B.$$

Set theoretically, $\pi^*(W)$ is the preimage of $W$, $\pi^{-1}(W)$.

Let $U = X - V(\mathcal{J})$. The strict transform $\tilde{W}$ of $W$ is the Zariski closure of $\pi^{-1}(W \cap U)$ in $B(\mathcal{J})$.

**Lemma 4.1.** Suppose that $R$ is a Noetherian ring, $I, J \subset R$ are ideals and $I = q_1 \cap \cdots \cap q_m$ is a primary decomposition, with primes $p_i = \sqrt{q_i}$. Then

$$\bigcup_{n=0}^\infty (I : J^n) = \bigcap_{i \mid f \not\in p_i} q_i.$$

The proof of Lemma 4.1 follows easily from the definition of a primary ideal.

Geometrically, Lemma 4.1 says that $\bigcup_{n=0}^\infty (I : J^n)$ removes the primary components $q_i$ of $I$ such that $V(q_i) \subset V(J)$.

Thus we see that the strict transform $\tilde{W}$ of $W$ has the ideal sheaf

$$\mathcal{I}_{\tilde{W}} = \bigcup_{n=0}^\infty (\mathcal{I}_W \mathcal{O}_B : \mathcal{J}^n \mathcal{O}_B).$$

For $q \in B$,

$$\mathcal{I}_{\tilde{W},q} = \{ f \in \mathcal{O}_{B,q} \mid f \mathcal{J}^n_q \subset \mathcal{I}_W \mathcal{O}_{B,q} \text{ for some } n \geq 0 \}.$$

The strict transform has the property that

$$\tilde{W} = B(\mathcal{J} \mathcal{O}_W) = \text{proj}(\bigoplus_{n \geq 0} (\mathcal{J} \mathcal{O}_W)^n)$$

This is shown in Corollary II.7.15 [45].

**Theorem 4.2.** (Universal property of blowing up) Suppose that $\mathcal{I}$ is an ideal sheaf on a variety $V$ and $f : W \to V$ is a morphism of varieties such that $\mathcal{I} \mathcal{O}_W$ is locally principal. Then there is a unique morphism $g : W \to B(\mathcal{I})$ such that $f = \pi \circ g$.

This is proven in Proposition II.7.14 [45].

**Theorem 4.3.** Suppose that $C$ is an integral curve over a field $K$. Consider the sequence

$$\cdots \to C_n \xrightarrow{\pi} \cdots \to C_1 \xrightarrow{\pi} C$$

where $C_{n+1} \to C_n$ is obtained by blowing up the (finitely many) singular points on $C_n$. Then this sequence is finite. That is, there exists $n$ such that $C_n$ is non-singular.

**Proof.** Without loss of generality, $C = \text{spec}(R)$ is affine. Let $\overline{R}$ be the integral closure of $R$ in the function field $K(C)$ of $C$. $\overline{R}$ is a regular ring. Let $\overline{C} = \text{spec}(\overline{R})$. All ideals in $\overline{R}$ are locally principal (Proposition 9.2, page 94 [12]). By Theorem 4.2, we have a factorization

$$\overline{C} \to C_n \to \cdots \to C_1 \to C$$

for all $n$. Since $\overline{C} \to C$ is finite, $C_{i+1} \to C_i$ is finite for all $i$, and there exist affine rings $R_i$ such that $C_i = \text{spec}(R_i)$. For all $n$ we have sequences of inclusions

$$R \to R_1 \to \cdots \to R_n \to \overline{R}.$$
$R_i \neq R_{i+1}$ for all $i$ since a maximal ideal $m$ in $R_i$ is locally principal if and only if $(R_i)_m$ is a regular local ring. Since $R$ is finite over $R$, we have that (20) is finite. □

**Corollary 4.4.** Suppose that $X$ is a variety over a field $K$ and $C$ is an integral curve on $X$. Consider the sequence

$$
\cdots \to X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X
$$

where $\pi_{i+1}$ is the blow up of all points of $X$ which are singular on the strict transform $C_i$ of $C$ on $X_i$. Then this sequence is finite. That is, there exists an $n$ such that the strict transform $C_n$ of $C$ is non-singular.

**Proof.** The induced sequence

$$
\cdots \to C_n \to \cdots \to C_1 \to C
$$

of blow ups of points on the strict transform $C_i$ of $C$ is finite by Theorem 4.3. □

**Theorem 4.5.** Suppose that $V$ and $W$ are varieties and $V \to W$ is a projective birational morphism. Then $V \cong B(I)$ for some ideal sheaf $I \subset \mathcal{O}_W$.

This is proved in Proposition II.7.17 [45].

Suppose that $Y$ is a non-singular subvariety of a variety $X$. The monoidal transform of $X$ with center $Y$ is $\pi : B(I_Y) \to X$. We define the exceptional divisor of the monoidal transform $\pi$ (with center $Y$) to be the reduced divisor with the same support as $\pi^*(Y)$.

In general, we will define the exceptional locus of a proper birational morphism $\phi : V \to W$ to be the (reduced) closed subset of $V$ on which $\phi$ is not an isomorphism.

**Remark 4.6.** If $Y$ has codimension greater than or equal to 2 in $X$, then the exceptional divisor of the monoidal transform with center $Y$, $\pi : B(I_Y) \to X$ is the exceptional locus of $\pi$. However, if $X$ is singular and not locally factorial, it may be possible to blow up a non-singular codimension 1 subvariety $Y$ of $X$ to get a monoidal transform $\pi$ such that the exceptional locus of $\pi$ is a proper closed subset of the exceptional divisor of $\pi$. For an example of this kind, see the exercises at the end of this section. If $X$ is non-singular and $Y$ has codimension 1 in $X$, then $\pi$ is an isomorphism and we have defined the exceptional divisor to be $Y$. This property will be useful in our general proof of resolution in Chapter 6.

**Exercises**

1. Let $K$ be an algebraically closed field, and $X$ be the affine surface

$$
X = \text{spec}(K[x, y, z]/(xy - z^2)).
$$

Let $Y = V(x, z) \subset X$, and let $\pi : B \to X$ be the monoidal transform centered at the non-singular curve $Y$. Show that $\pi$ is a resolution of singularities. Compute the exceptional locus of $\pi$ and compute the exceptional divisor of the monoidal transform $\pi$ of $Y$.

2. Let $K$ be an algebraically closed field, and $X$ be the affine 3-fold

$$
X = \text{spec}(K[x, y, z, w]/(xy - zw)).
$$
Show that the monoidal transform $\pi : B \to X$ with center $Y$ is a resolution of singularities in each of these cases, and describe the exceptional locus and exceptional divisor (of the monoidal transform).

a. $Y = V((x, y, z, w))$

b. $Y = V((x, z))$

c. $Y = V((y, z))$.

Show that the monoidal transforms of cases b. and c. factor the monoidal transform of case c.

4.2. Resolution type theorems and corollaries. Resolution of singularities.

Suppose that $V$ is a variety. A resolution of singularities of $V$ is a proper birational morphism $\phi : W \to V$ such that $W$ is non-singular.

Principalization of ideals

Suppose that $V$ is a non-singular variety, $I \subset O_V$ is an ideal sheaf. A principalization of $I$ is a proper birational morphism $\phi : W \to V$ such that $W$ is non-singular and $IO_W$ is locally principal.

Suppose that $X$ is a non-singular variety, $I \subset O_X$ is a locally principal ideal. $I$ has Simple Normal Crossings (SNCs) at $p \in X$ if there exist regular parameters $(x_1, \ldots, x_n)$ in $O_{X,p}$ such that $I_p = x_1^{a_1} \cdots x_n^{a_n} O_{X,p}$ for some natural numbers $a_1, \ldots, a_n$.

Suppose that $D$ is an effective divisor on $X$. That is, $D = r_1 E_1 + \cdots + r_n E_n$ where $E_i$ are irreducible codimension 1 subvarities of $X$, and $r_i$ are natural numbers. $D$ has SNCs if $I_D = I_{E_1}^{r_1} \cdots I_{E_n}^{r_n}$ has SNCs.

Suppose that $W \subset X$ is a subscheme, and $D$ is a divisor on $X$. Say that $W$ has SNCs with $D$ at $p$ if

1. $W$ is non-singular at $p$.
2. $D$ is a SNC divisor at $p$.
3. There exist regular parameters $(x_1, \ldots, x_n)$ in $O_{X,p}$ and $r \leq n$ such that $I_{W,p} = (x_1, \ldots, x_r)$ (or $I_{W,p} = O_{X,p}$) and $I_{D,p} = x_1^{a_1} \cdots x_n^{a_n} O_{X,p}$ for some $a_i \in \mathbb{N}$.

Embedded resolution. Suppose that $W$ is a subvariety of a non-singular variety $X$. An embedded resolution of $W$ is a proper morphism $\pi : Y \to X$ which is a product of monoidal transforms, such that $\pi$ is an isomorphism on an open set which intersects every component of $W$ properly, the exceptional divisor of $\pi$ is a SNC divisor $D$, and the strict transform $\tilde{W}$ of $W$ has SNCs with $D$.

Resolution of indeterminacy. Suppose that $\phi : W \to V$ is a rational map of proper $K$-varieties such that $W$ is non-singular. A resolution of indeterminacy of $\phi$ is a proper non-singular $K$-variety $X$ with a birational morphism $\psi : X \to W$ and a morphism $\lambda : X \to V$ such that $\lambda = \phi \circ \psi$.

Lemma 4.7. Suppose that resolution of singularities is true for $K$-varieties of dimension $n$. Then resolution of indeterminacy is true for rational maps from $K$-varieties of dimension $n$. 
Proof. Let φ : W → V be a rational map of proper K-varieties where W is non-singular. Let U be a dense open set of W on which φ is a morphism. Let Γφ be the Zariski closure of the image in W × V of the map U → W × V defined by p → (p, φ(p)). By resolution of singularities, there is a proper birational morphism X → Γφ such that X is non-singular.

**Theorem 4.8.** Suppose that K is a perfect field, resolution of singularities is true for projective hypersurfaces over K of dimension n and principalization of ideals is true for non-singular varieties of dimension n over K. Then resolution of singularities is true for projective K-varieties of dimension n.

Proof. Suppose that V is an n dimensional projective K-variety. If V\\1 are the irreducible components of V, we have a projective birational morphism from the disjoint union of the Vi to V. Thus it suffices to assume that V is irreducible. The function field K(V) of V is a finite separable extension of a rational function field K(x1, ..., xn) (Chapter II, Theorem 30, page 104 [85]). By the theorem of the primitive element,

\[ K(V) \cong K(x_1, \ldots, x_n)[x_{n+1}]/(f). \]

We can clear the denominator of f so that

\[ f = \sum a_{i_1 \ldots i_{n+1}} x_1^{i_1} \cdots x_{n+1} \]

is irreducible in K[x1, ..., xn+1]. Let

\[ d = \max \{ i_1 + \cdots i_{n+1} \mid a_{i_1 \ldots i_{n+1}} \neq 0 \}. \]

Set

\[ F = \sum a_{i_1 \ldots i_{n+1}} x_0^{d-(i_1+\cdots+i_{n+1})} x_1^{i_1} \cdots x_{n+1}^{i_{n+1}}, \]

the homogenization of f. Let V be the variety defined by F = 0 in Pn+1. K(V) ≈ K(V) implies there is a birational rational map from V to V. That is, there is a birational morphism φ : V → V where V is a dense open subset of V. Let Γφ be the Zariski closure of \{(a, φ(a)) \mid a \in V \} in V × V. We have birational projection morphisms π1 : Γφ → V and π2 : Γφ → V. Γφ is the blow up of an ideal sheaf J on V (Theorem 4.5). By resolution of singularities for n dimensional hypersurfaces, we have a resolution of singularities f : W → V. By principalization of ideals in non-singular varieties of dimension n, we have a principalization g : W → W for J|W. By the universal property of blowing up (Theorem 4.2), we have a morphism h : W → Γφ. Hence π1 ◦ h : W → V is a resolution of singularities.

**Corollary 4.9.** Suppose that C is a projective curve over a perfect field K. Then C has a resolution of singularities.

Proof. By Theorem 3.12 resolution of singularities is true for projective plane curves over K. All ideal sheaves on a non-singular curve are locally principal since the local rings of points are Dedekind local rings.

Note that Theorem 4.3 and Corollary 4.4 are stronger results than Corollary 4.9. However, the ideas of the proofs of resolution of plane curves in Theorems 3.9 and 3.12 extend to higher dimensions, while Theorem 4.3 does not. As a consequence of embedded resolution of curve singularities on a surface, we will prove principalization
of ideals in non-singular varieties of dimension two. This simple proof is by Abhyankar (Proposition 6, [67]).

**Theorem 4.10.** Suppose that $S$ is a non-singular surface over a field $K$ and $J \subset \mathcal{O}_S$ is an ideal sheaf. Then there exists a finite sequence of blow ups of points $T \to S$ such that $\mathcal{O}_T$ is locally principal.

**Proof.** Without loss of generality $S$ is affine. Let $J = \Gamma(J, \mathcal{O}_S) = (f_1, \ldots, f_m)$. By embedded resolution of curve singularities on a surface (Exercise 3 of Section 3.4) applied to $\sqrt{f_1 \cdots f_m} \in S$, there exists a sequence of blow ups of points $\pi_1 : S_1 \to S$ such that $f_1 \cdots f_m = 0$ is a SNC divisor everywhere on $S_1$.

Let $\{p_1, \ldots, p_r\}$ be the finitely many points on $S_1$ where $J \mathcal{O}_{S_1}$ is not locally principal (they are contained in the singular points of $V(\sqrt{JO_{S_1}})$). Suppose that $p \in \{p_1, \ldots, p_r\}$. By induction on the number of generators of $J$, we may assume that $J \mathcal{O}_{S_1, p} = (f, g)$. After possibly multiplying $f$ and $g$ by units in $\mathcal{O}_{S_1, p}$, there are regular parameters $(x, y)$ in $\mathcal{O}_{S_1, p}$ such that

$$f = x^a y^b, g = x^c y^d.$$

Set $t_p = (a - c)(b - d)$. $(f, g)$ is principal if and only if $t_p \geq 0$. By our assumption, $t_p < 0$. Let $\pi_2 : S_2 \to S_1$ be the blow up of $p$, and suppose that $q \in \pi_1^{-1}(p)$. We will show that $t_q > t_p$.

The only two points $q$ where $(f, g)$ may not be principal have regular parameters $(x_1, y_1)$ such that

$$x = x_1, y = x_1 y_1 \text{ or } x = x_1 y_1, y = y_1.$$

If $x = x_1, y = x_1 y_1$, then

$$f = x_1^{a+b} y_1, g = x_1^{c+d} y_1^d,$$

$$t_q = (a + b - (c + d))(b - d) = (b - d)^2 + (a - c)(b - d) > t_p.$$

The case when $x = x_1 y_1, y = y_1$ is similar.

By induction on $\min\{t_p \mid J \mathcal{O}_{S_1, p} \text{ is not principal}\}$ we can principalize $J$ after a finite number of blow ups of points. □

## 5. Surface singularities

### 5.1. Resolution of surface singularities

In this section, we give a simple proof of resolution of surface singularities in characteristic zero.

**Theorem 5.1.** Suppose that $S$ is a projective surface over an algebraically closed field $K$ of characteristic 0. Then there exists a resolution of singularities

$$\Lambda : T \to S.$$

Theorem 5.1 is a consequence of the following Theorem 5.2, Theorem 4.8 and Theorem 4.10.

**Theorem 5.2.** Suppose that $S$ is a hypersurface of dimension 2 in a non-singular variety $V$ of dimension 3, over an algebraically closed field $K$ of characteristic 0. Then there exists a sequence of blow ups of points and non-singular curves contained in the strict transform $S_1$ of $S$

$$V_n \to V_{n-1} \to \cdots \to V_1 \to V$$

such that the the strict transform $S_n$ of $S$ on $V_n$ is non-singular.
The remainder of this section will be devoted to the proof of Theorem 5.2. Suppose that $V$ is a non-singular three dimensional variety over an algebraically closed field $K$ of characteristic 0, and $S \subset V$ is a surface.

For a natural number $t$, define

$$\text{Sing}_t(S) = \{ p \in V \mid \nu_p(S) \geq t \}.$$ 

By Theorem 10.19, $\text{Sing}_t(S)$ is Zariski closed in $V$.

Let

$$r = \max\{ t \mid \text{Sing}_t(S) \neq \emptyset \}$$

be the maximal multiplicity of points of $S$. There are two types of blow ups of non-singular subvarieties on a non-singular three dimensional variety, a blow up of a point, and a blow up of a non-singular curve.

We will first consider the blow up of a closed point $p \in V$, $\pi : B(p) \to V$. Suppose that $U = \text{spec}(R) \subset V$ is an affine open neighborhood of $p$, and $p$ has ideal $m_p = (x, y, z) \subset R$.

$$\pi^{-1}(U) = \text{proj}(\bigoplus m_p^n) = \text{spec}(R[x, y, z]) \cup \text{spec}(R[y, z]) \cup \text{spec}(R[x, y]).$$

The exceptional divisor is $E = \pi^{-1}(p) \cong \mathbb{P}^2$.

At each closed point $q \in \pi^{-1}(p)$, we have regular parameters $(x_1, y_1, z_1)$ of the following forms:

$$x = x_1, y = x_1(y_1 + \alpha), z = x_1(z_1 + \beta),$$

with $\alpha, \beta \in K$, $x_1 = 0$ a local equation of $E$, or

$$x = x_1y_1, y = y_1, z = y_1(z_1 + \alpha)$$

with $\alpha \in K$, $y_1 = 0$ a local equation of $E$, or

$$x = x_1z_1, y = y_1z_1, z = z_1,$$

$z_1 = 0$ a local equation of $E$.

We will now consider the blow up $\pi : B(C) \to V$ of a non-singular curve $C \subset V$. If $p \in V$ and $U = \text{spec}(R) \subset V$ is an open affine neighborhood of $p$ in $V$ such that $m_p = (x, y, z)$ and the ideal of $C$ is $I = (x, y)$ in $R$ then

$$\pi^{-1}(U) = \text{proj}(\bigoplus I^n) = \text{spec}(R[x]) \cup \text{spec}(R[y]).$$

$\pi^{-1}(p) \cong \mathbb{P}^1$, $\pi^{-1}(C \cap U) \cong (C \cap U) \times \mathbb{P}^1$. Let $E = \pi^{-1}(C)$ be the exceptional divisor. $E$ is a projective bundle over $C$. At each point $q \in \pi^{-1}(p)$, we have regular parameters $(x_1, y_1, z_1)$ such that:

$$x = x_1, y = x_1(y_1 + \alpha), z = z_1$$

where $\alpha \in K$, $x_1 = 0$ is a local equation of $E$, or

$$x = x_1y_1, y = y_1, z = z_1$$

where $y_1 = 0$ is a local equation of $E$.

In this section, we will analyze the blow up

$$\pi : B(W) = B(I_W) \to V$$

of a non-singular subvariety $W$ of $V$ above a closed point $p \in V$, by passing to a formal neighborhood $\text{spec}(O_{V,p})$ of $p$ and analyzing the map $\overline{\pi} : B(I_{W,p}) \to \text{spec}(O_{V,p})$. We
have a natural isomorphism \( B(\hat{I}_W) \cong B(I_W) \times V \), \( \text{spec}(\hat{O}_V) \). Observe that we have a natural identification of \( \pi^{-1}(p) \) with \( \pi^{-1}(p) \).

We say that an ideal \( I \subset \hat{O}_V \) is algebraic if there exists an ideal \( J \subset \mathcal{O}_V \), such that \( I = \hat{J} \). This is equivalent to the statement that there exists an ideal sheaf \( \mathcal{I} \subset \mathcal{O}_V \) such that \( \hat{\mathcal{I}} = I \).

If \( I \subset \hat{O}_V \) is algebraic, so that there exists an ideal sheaf \( \mathcal{I} \subset \mathcal{O}_V \) such that \( \hat{\mathcal{I}} = I \), then we can extend the blow up \( \pi : B(I) \rightarrow \text{spec}(\hat{O}_V) \) to a blow up \( \pi : B(I) \rightarrow V \).

The maximal ideal \( m_p \hat{O}_V \) is always algebraic. However, the ideal sheaf \( I \) of a non-singular (formal) curve in \( \text{spec}(\hat{O}_V) \) may not be algebraic.

One example of a formal, non-algebraic curve is

\[
I = (y - e^x) \subset C[[x, y]] = \hat{O}_{\mathbb{A}^2_{k, 0}}.
\]

A more subtle example, which could occur in the course of this section, is the ideal sheaf of the irreducible curve

\[
J = (y^2 - x^2 - x^3, z) \subset R = K[x, y, z]
\]

which we studied after Theorem 3.5. In \( \hat{R} = K[[x, y, z]] \) we have regular parameters

\[
\pi = y - x\sqrt{1 + x}, \quad \eta = y + x\sqrt{1 + x}, \quad z = z.
\]

Thus

\[
J \hat{R} = (\pi y, z) = (\pi, z) \cap (\eta, z) \subset \hat{R} = K[[\pi, \eta, z]].
\]

In this example, we may be tempted to blow up one of the two formal branches \( \pi = 0, \eta = 0 \) or \( \eta = 0, \pi = 0 \), but the resulting blown up scheme will not extend to a blow up of an ideal sheaf in \( \mathbb{A}^3_k \).

A situation which will arise in this section when we will blow up a formal curve which will actually be algebraic is given in the following Lemma.

**Lemma 5.3.** Suppose that \( V \) is a non-singular three dimensional variety, \( p \in V \) is a closed point and \( \pi : V_1 = B(p) \rightarrow V \) is the blow up of \( p \) with exceptional divisor \( E = \pi^{-1}(p) \cong \mathbb{P}^2 \). Let \( R = \mathcal{O}_{V, p} \). We have a commutative diagram of morphisms of schemes

\[
\begin{array}{ccc}
B = B(m_p \hat{R}) & \rightarrow & V_1 = B(p) \\
\downarrow \pi & & \downarrow \pi \\
\text{spec}(\hat{R}) & \rightarrow & V
\end{array}
\]

such that \( \pi^{-1}(m_p) \rightarrow \pi^{-1}(p) = E \) is an isomorphism of schemes. Suppose that \( I \subset \hat{R} \) is any ideal, and \( \hat{I} \subset \mathcal{O}_B \) is the strict transform of \( I \). Then there exists an ideal sheaf \( \mathcal{J} \) on \( V_1 \) such that \( \mathcal{J} \mathcal{O}_B = I_E \mathcal{O}_B + I \).

**Proof.** \( \hat{I}_E \) is an ideal sheaf on \( E \), so there exists an ideal sheaf \( \mathcal{J} \subset \mathcal{O}_{V_1} \) such that \( I_E \subset \mathcal{J} \) and \( \mathcal{J} / I_E \cong \hat{I}_E \). Thus \( \mathcal{J} \) has the desired property. \( \square \)

**Lemma 5.4.** Suppose that \( V \) is a non-singular three dimensional variety, \( S \subset V \) is a surface, \( C \subset \text{Sing}_p(S) \) is a non-singular curve, \( \pi : B(C) \rightarrow V \) is the blow up of \( C \), and \( \hat{S} \) is the strict transform of \( S \) in \( B(C) \). Suppose that \( p \in C \). Then \( \nu_q(\hat{S}) \leq r \) for all \( q \in \pi^{-1}(p) \), and there exists at most one point \( q \in \pi^{-1}(p) \) such that \( \nu_q(\hat{S}) = r \).

In particular, if \( E = \pi^{-1}(C) \), then either \( \text{Sing}_p(\hat{S}) \cap E \) is a non-singular curve which maps isomorphically onto \( C \) or \( \text{Sing}_p(\hat{S}) \cap E \) is a finite union of points.
Proof. By the Weierstrass preparation theorem and after a Tschirnhausen transformation (Definition 3.7), a local equation \( f = 0 \) of \( S \) in \( \hat{O}_{S,p} = K[[x,y,z]] \) has the form

\[
f = z^r + a_2(x,y)z^{r-2} + \cdots + a_r(x,y).
\]

(21)

If \( f \in (\hat{T}_{C,p})^r = \hat{T}_{C,p} \) implies \( \frac{\partial f}{\partial z} = \hat{I}_{C,p}^{r-1} \), and \( r!z = \frac{\alpha^{r-1}z}{z^r} \in \hat{T}_{C,p} \). Thus \( z \in \hat{T}_{C,p} \). After a change of variables in \( x \) and \( y \), we may assume that \( \hat{T}_{C,p} = (x,z) \). \( f \in \hat{T}_{C,p} \) implies \( x^i | a_i \) for all \( i \). If \( q \in \pi^{-1}(p) \), then \( \hat{O}_{B(C),q} \) has regular parameters \((x_1,y_1)\) such that

\[
x = x_1z_1, z = z_1
\]

or

\[
x = x_1, z = x_1(z_1 + \alpha)
\]

for some \( \alpha \in K \).

In the first case, a local equation of the strict transform of \( S \) is a unit. In the second case, the strict transform of \( S \) has a local equation

\[
f_1 = (z_1 + \alpha)^r + \frac{a_2}{x_1}(z_1 + \alpha)^{r-2} + \cdots + \frac{a_r}{x_1}.
\]

(22)

\[\nu(f_1) \leq r, \quad \nu(f_1) < r \text{ if } \alpha \neq 0. \]

Lemma 5.5. Suppose that \( p \in \text{Sing}_r(S) \) is a point, \( \pi : B(p) \to V \) is the blow up of \( p \), \( \hat{S} \) is the strict transform of \( S \) in \( B(p) \), and \( E = \pi^{-1}(p) \). Then \( \nu_q(\hat{S}) \leq r \) for all \( q \in \pi^{-1}(p) \), and either \( \text{Sing}_r(S) \cap E \) is a non-singular curve or \( \text{Sing}_r(S) \cap E \) is a finite union of points.

Proof. By the Weierstrass preparation theorem and after a Tschirnhausen transformation, a local equation \( f = 0 \) of \( S \) in \( \hat{O}_{S,p} = K[[x,y,z]] \) has the form

\[
f = z^r + a_2(x,y)z^{r-2} + \cdots + a_r(x,y).
\]

(22)

If \( q \in \pi^{-1}(p) \), and \( \nu_q(\hat{S}) \geq r \), then \( \hat{O}_{B(C),q} \) has regular parameters \((x_1,y_1)\) such that

\[
x = x_1y_1, y = y_1, z = y_1z_1
\]

or

\[
x = x_1, y = x_1(y_1 + \alpha), z = x_1z_1
\]

for some \( \alpha \in K \), and \( \nu_1(\hat{S}) = r \). Thus \( \text{Sing}_r(\hat{S}) \cap E \) is contained in the line which is the intersection of \( E \) with the strict transform of \( z = 0 \) in \( B(C) \times_V \text{spec}(\hat{O}_{V,p}) \).

Definition 5.6. \( \text{Sing}_r(S) \) has simple normal crossings (SNCs) if

1. All irreducible components of \( \text{Sing}_r(S) \) (which could be points or curves) are non-singular.
2. If \( p \) is a singular point of \( \text{Sing}_r(S) \), then there exist regular parameters \((x,y,z)\) in \( \hat{O}_{V,p} \) such that \( \hat{I}_{\text{Sing}_r(S),p} = (xy,z) \).

Lemma 5.7. Suppose that \( \text{Sing}_r(S) \) has simple normal crossings, \( W \) is a point or an irreducible curve in \( \text{Sing}_r(S) \), \( \pi : V' = B(W) \to V \) is the blow up of \( W \) and \( S' \) is the strict transform of \( S \) on \( V' \). Then \( \text{Sing}_r(S') \) has simple normal crossings.

Proof. This follows from Lemmas 5.4 and 5.5 and a simple local calculation on \( V' \).
Definition 5.8. A closed point $p \in S$ is a pregood point if in a neighborhood of $p$, $\text{Sing}_r(S)$ is either empty, a non-singular curve through $p$, or a a union of two non-singular curves intersecting transversally at $p$ (satisfying 2. of Definition 5.6).

Definition 5.9. $p \in S$ is a good point if $p$ is pregood, and if for any sequence

$$X_n \to X_{n-1} \to \cdots \to X_1 \to \text{spec}(O_{V,p})$$

of blow ups of non-singular curves in $\text{Sing}_r(S_i)$, where $S_i$ is the strict transform of $S \cap \text{spec}(O_{V,p})$ on $X_i$, then $q$ is pregood for all closed points $q \in \text{Sing}_r(S_n)$. In particular, $\text{Sing}_r(S_n)$ contains no isolated points.

A point which is not good is called bad.

Lemma 5.10. Suppose that all points of $\text{Sing}_r(S)$ are good. Then there exists a sequence of blow ups

$$V' = V_n \to \cdots \to V_1 \to V$$

of non-singular curves contained in $\text{Sing}_r(S_i)$, where $S_i$ is the strict transform of $S$ in $V_i$, such that $\text{Sing}_r(S') = \emptyset$ if $S'$ is the strict transform of $S$ on $V'$.

Proof. Suppose that $C$ is a non-singular curve in $\text{Sing}_r(S)$. Let $\pi_1: V_1 = B(C) \to V$ be the blow up of $C$, $S_i$ be the strict transform of $S$. If $\text{Sing}_r(S_1) \neq \emptyset$, we can choose another non-singular curve $C_1$ in $\text{Sing}_r(S_1)$ and blow up by $\pi_2: V_2 = B(C_1) \to V_1$. Let $S_2$ be the strict transform of $S_1$. We either reach a surface $S_n$ such that $\text{Sing}_r(S_n) = \emptyset$, or we obtain an infinite sequence of blow ups

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V$$

such that each $V_{i+1} \to V_i$ is the blow up of a curve $C_i$ in $\text{Sing}_r(S_i)$, where $S_i$ is the strict transform of $S$ on $V_i$. Each curve $C_i$ which is blown up must map onto a curve in $S$ by Lemma 5.5. Thus there exists a curve $\gamma \subset S$ such that there are infinitely many blow ups of curves mapping onto $\gamma$ in the above sequence. Let $R = O_{V,\gamma}$, a two dimensional regular local ring. $\mathcal{I}_{S,\gamma}$ is a height 1 prime ideal in this ring, and $P = \mathcal{I}_{\gamma,\gamma}$ is the maximal ideal of $R$.

$$\dim R + \text{trdeg}_K R/P = 3$$

by the dimension formula (Theorem 15.6 [61]). Thus $R/P$ has transcendence degree 1 over $K$. Let $t \in R$ be the lift of a transcendence basis of $R/P$ over $K$. $K[t] \cap P = (0)$, so the field $K(t) \subset R$. We can write $R = A_Q$ where $A$ is a finitely generated $K(t)$ algebra (which is a domain) and $Q$ is a maximal ideal in $A$. Thus $\bar{R}$ is the local ring of a non-singular point $q$ on the $K(t)$ surface $\text{spec}(A)$. $q$ is a point of multiplicity $r$ on the curve in $\text{spec}(A)$ with ideal sheaf $\mathcal{I}_{S,\gamma}$ in $R$.

The sequence

$$\cdots \to V_n \times_V \text{spec}(R) \to V_{n-1} \times_V \text{spec}(R) \to \cdots \to \text{spec}(R)$$

consists of infinitely many blow ups of points on a $K(t)$ surface, which are of multiplicity $r$ on the strict transform of the curve $\text{spec}(R/\mathcal{I}_{S,\gamma})$. But this is impossible by Theorem 3.12.

Lemma 5.11. The number of bad points on $S$ is finite.
Proof. Let

\[ B_0 = \{ \text{isolated points of } \text{Sing}_r(S) \} \cup \{ \text{singular points of } \text{Sing}_r(S) \}. \]

\( \text{Sing}_r(S) - B_0 \) is a non-singular 1 dimensional subscheme of \( V \). Let

\[ \pi_1 : V_1 = B(\text{Sing}_r(S) - B_0) \to V - B_0 \]

be its blow up. Let \( S_1 \) be the strict transform of \( S \),

\[ B_1 = \{ \text{isolated points of } \text{Sing}_r(S_1) \} \cup \{ \text{singular points of } \text{Sing}_r(S_1) \}. \]

We can iterate to construct a sequence

\[ \cdots \to (V_n - B_n) \to \cdots \to V \]

where \( \pi_n : V_n - B_n \to V - T_n \) are the induced surjective maps, with

\[ T_n = B_0 \cup \pi_1(B_1) \cup \cdots \cup \pi_n(B_n). \]

Let \( S_n \) be the strict transform of \( S \) on \( V_n \).

let \( C \subset \text{Sing}_r(S) \) be a curve. \( \text{spec}(O_{S,C}) \) is a curve singularity of multiplicity \( r \), embedded in a non-singular surface over a field of transcendence degree 1 over \( K \) (as in the proof of Lemma 5.10). \( \pi_n \) induces by base change

\[ S_n \times_S \text{spec}(O_{S,C}) \to \text{spec}(O_{S,C}) \]

which corresponds to an open subset of a sequence of blow ups of points of over \( \text{spec}(O_{S,C}) \).

So for \( n >> 0 \), \( S_n \times_S \text{spec}(O_{S,C}) \) is non-singular. Thus there are no curves in \( \text{Sing}_r(S_n) \) which dominate \( C \). For \( n >> 0 \) there are thus no curves in \( \text{Sing}_r(S_n) \) which dominate curves of \( \text{Sing}_r(S) \), so that \( \text{Sing}_r(S_n) \cap (V_n - B_n) \) is empty for large \( n \), and all bad points of \( S \) are contained in a finite set \( T_n \).

\[ \square \]

**Theorem 5.12.** Let

\[ \cdots \to V_n \to V_{n-1} \to \cdots \to V_1 \to V \quad (23) \]

be the sequence where \( \pi_n : V_n \to V_{n-1} \) is the blow up of all bad points on the strict transform \( S_{n-1} \) of \( S \). Then this sequence is finite, so that it terminates after a finite number of steps with a \( V_m \) such that all points of \( \text{Sing}_r(S_m) \) are good.

We now give the proof of Theorem 5.12.

Suppose there is an infinite sequence of the form of (23). Then there is an infinite sequence of points \( p_n \in \text{Sing}_r(S_n) \) such that \( \pi_n(p_n) = p_{n-1} \) for all \( n \). We then have an infinite sequence of homomorphisms of local rings

\[ R_0 = \hat{O}_{V,p} \to R_1 = \hat{O}_{V_1,p_1} \to \cdots \to R_n = \hat{O}_{V_n,p_n} \to \cdots \]

By the Weierstrass preparation theorem and after a Tshirnhausen transformation (Definition 3.7) there exist regular parameters \( (x, y, z) \) in \( R_0 \) such that there is a local equation \( f = 0 \) for \( S \) in \( R_0 \), of the form

\[ f = z^r + a_2(x, y)z^{r-2} + \cdots + a_r(x, y) \]

with \( \nu(a_i(x, y)) \geq i \) for all \( i \). Since \( \nu_{p_i}(S_i) = r \) for all \( i \), we have regular parameters \( (x_i, y_i, z_i) \) in \( R_i \) for all \( i \) and a local equation \( f_i = 0 \) for \( S_i \) such that

\[ x_{i-1} = x_i, y_{i-1} = y_i + \alpha_i, z_{i-1} = x_{i-1}z_{i-1} \]

with \( \alpha_i \in K \) or

\[ x_{i-1} = x_i, y_{i-1} = y_i, z_{i-1} = y_{i-1}z_{i-1} \]
and
\[ f_i = z_i^r + \tilde{a}_{2,i}(x_i, y_i)z_i^{r-2} + \cdots + \tilde{a}_{r,i}(x_i, y_i) \]
where
\[ \tilde{a}_{ji}(x_i, y_i) = \begin{cases} \frac{x_i}{y_i} & \text{if } \tilde{a}_{ji-1}(x_i, y_i) \neq 0, \\ \tilde{a}_{ji-1}(x_i, y_i) & \text{if } \tilde{a}_{ji-1}(x_i, y_i) = 0 \end{cases} \]
for all \( j \). Thus the sequence
\[ K[[x, y]] \to K[[x_1, y_1]] \to \cdots \]
is a sequence of blow ups of a maximal ideal, followed by completion. In \( K[[x_i, y_i]] \)
we have relations
\[ a_j = \gamma_{j,i} x_i^{d_j} y_i^{d_j} \tilde{a}_{j,i-1} \]
for all \( i \) and \( 2 \leq j \leq r \), where \( \gamma_{j,i} \) is a unit. By embedded resolution of curve
singularities (Exercise 3 of Section 3.4), there exists \( m_0 \) such that \( \prod_{j=2}^r a_j(x, y) = 0 \)
is a SNC divisor in \( R_i \) for all \( i \geq m_0 \). Thus for \( i \geq m_0 \) we have an expression
\[ f_i = z_i^r + \tilde{a}_{2,i}(x_i, y_i)z_i^{r-2} + \cdots + \tilde{a}_{r,i}(x_i, y_i)z_i^{r-2} + \cdots + \tilde{a}_{r,i}(x_i, y_i)z_i^{r-2} \]
where \( \tilde{a}_{ji} \) are units (or zero) in \( R_i \) and \( a_{ji} \geq j \) if \( \tilde{a}_{ji} \neq 0 \).

We observe that, by the proof of Theorem 10.19 and Remark 10.21, 
\[ \tilde{J}_{\text{Sing}_{r_i}(S_i), p_i} = \sqrt{J} \]
where \( J \) is the ideal in \( R_i \) generated by
\[ \left\{ \frac{\partial^{j+k+l} f_i}{\partial x_i^j \partial y_i^k \partial z_i^l} | 0 \leq j + k + l < r \right\}. \]

Thus \( \tilde{J}_{\text{Sing}_{r_i}(S_i), p_i} \) must be one of \( (x_i, y_i, z_i), (x_i, z_i), (y_i, z_i) \) or \( (x_i y_i, z_i) \). \( \text{Sing}_{r_i}(S_i) \)
has SNCs at \( p_i \) unless \( \sqrt{J} = (x_i y_i, z_i) \) is the completion of an ideal of an irreducible (singular) curve on \( V_i \). Let \( E = \pi^{-1}_i(p_i - 1) \cong \mathbb{P}^2 \). By construction, we have that
\( x_i = 0 \) or \( y_i = 0 \) is a local equation of \( E \) at \( p_i - 1 \). Without loss of generality, we may suppose that \( x_i = 0 \) is a local equation of \( E \). \( z_i = 0 \) is a local equation of the strict transform of \( z = 0 \) in \( V_i \times V \cup \text{spec}(R) \), so that \( (x_i, z_i) \) is the completion at \( q_i \) of the ideal sheaf of a subscheme of \( E \). Thus there is a curve \( C \) in \( V_i \) such that \( \tilde{I}_{C, p_i} = (x_i, z_i) \) by Lemma 5.3. The ideal sheaf \( J = (\mathcal{I}_{\text{Sing}_{r_i}(S_i)} : \mathcal{I}_C) \) in \( \mathcal{O}_{V_i} \) is such that \( (J \cap \mathcal{I}_C)_{p_i} \cong \mathcal{I}_{\text{Sing}_{r_i}(S_i), p_i} \), so that \( (x_i y_i, z_i) \) is not the completion of an ideal of an irreducible curve on \( V_i \), and \( \text{Sing}_{r_i}(S_i) \) has simple normal crossings at \( p_i \).

We may thus assume that \( m_0 \) is sufficiently large that \( \text{Sing}_{r_i}(S_i) \) has SNCs everywhere on \( V_i \) for \( i \geq m_0 \).

We now make the observation that we must have \( b_{ji} \neq 0 \) and \( a_{ki} \neq 0 \) for some \( j \)
and \( k \) if \( i \geq m_0 \). If we did have \( b_{ji} = 0 \) for all \( j \) (with a similar analysis if \( a_{ki} = 0 \) for all \( j \)) then, since
\[ \tilde{J}_{\text{Sing}_{r_i}(S_i), p_i} = \sqrt{J} \]
where \( J \) is the ideal in \( R_i \) generated by
\[ \left\{ \frac{\partial^{j+k+l} f_i}{\partial x_i^j \partial y_i^k \partial z_i^l} | 0 \leq j + k + l < r \right\}, \]
we have (with the condition that \( b_{ji} = 0 \) for all \( j \)) that \( \sqrt{J} = (x_i, z_i) \). Since \( \text{Sing}_{r_i}(S_i) \)
has SNCs in \( V_i \), there exists a non-singular curve \( C \subset \text{Sing}_{r_i}(S_i) \) such that \( x_i = z_i = 0 \).
Lemma 5.13. If \( \zeta_i \) respectively, for all \( i \), we have regular parameters \( (u, v, y) \) in \( \mathcal{O}_{W,q} \) such that
\[
x_i = u, \quad z_i = u(v + \beta)
\]
or
\[
x_i = uv, \quad z_i = v.
\]
If \( f' = 0 \) is a local equation of the strict transform \( S' \) of \( S_i \) in \( W \), we have that \( \nu_q(f') < r \) except possibly if \( x_i = u, z_i = uv \). Then
\[
f' = v^r + \bar{\alpha}_{21}u^{a_{21}}v^{r-2} + \cdots + \bar{\alpha}_{r1}u^{a_{r1}}v^{-r}.
\]
We see that \( f' \) is of the form of (24), with \( b_{ji} = 0 \) for all \( j \), but
\[
\min\{\frac{a_{ji}}{j} \mid 2 \leq j \leq r\}
\]
has decreased by 1. We see, using Lemma 5.4, that after a finite number of blow ups of non-singular curves in \( \text{Sing}_r \) of the strict transform of \( S \), the multiplicity must be \( < r \) at all points above \( p_i \). Thus \( p_i \) is a good point, a contradiction to our assumption. Thus \( b_{ji} \neq 0 \) and \( \bar{\alpha}_{ki} \neq 0 \) in (24) for some \( j \) and \( k \) if \( i \geq m_0 \).

If follows that, for \( i \geq m_0 \), we must have
\[
x_i = x_{i+1}, \quad y_i = y_{i+1}, \quad z_i = z_{i+1}
\]
or
\[
x_i = x_{i+1}, \quad y_i = y_{i+1}, \quad z_i = z_{i+1}.
\]
Thus in (24), we must have that for \( 2 \leq k \leq r \),
\[
a_{j,i+1} = a_{ji} + b_{ji} - j, \quad b_{j,i+1} = b_{ji} \quad \text{or}
\]
\[
a_{j,i+1} = a_{ji}, \quad b_{j,i+1} = a_{ji} + b_{ji} - j
\]
respectively, for all \( i \geq m_0 \).

Suppose that \( \zeta \in \mathbf{R} \). Then the fractional part of \( \zeta \) is
\[
\{\zeta\} = t
\]
if \( \zeta = a + t \) with \( a \in \mathbf{Z} \) and \( 0 \leq t < 1 \).

**Lemma 5.13.** For \( i \geq m_0 \), \( p_i \) is a good point on \( V_i \) if there exists \( j \) such that \( \bar{\alpha}_{ji} \neq 0 \) and
\[
1. \quad \frac{a_{ji}}{j} \leq \frac{a_{ki}}{k}, \quad \frac{b_{ji}}{j} \leq \frac{b_{ki}}{k}
\]
for all \( k \) with \( 2 \leq k \leq r \) and \( \bar{\alpha}_{ki} \neq 0 \),
\[
2. \quad \left\{ \frac{a_{ji}}{j} \right\} + \left\{ \frac{b_{ji}}{j} \right\} < 1.
\]

**Proof.** Since \( \nu_{p_i}(S_i) = r \), \( a_{ji} + b_{ji} \geq j \). Condition 2. thus implies that either \( a_{ji} \geq j \) or \( b_{ji} \geq j \). Without loss of generality, \( a_{ji} \geq j \). Then condition 1. implies that \( a_{ki} \geq k \) for all \( k \), and
\[
\hat{\mathcal{I}}_{\text{Sing}_r(S_i),p_i} \subset (x_i, z_i)R_i.
\]
Since \( \text{Sing}_r(S_i) \) has SNCs, there exists a non-singular curve \( C \subset \text{Sing}_r(S_i) \) such that \( \hat{\mathcal{I}}_{C,p_i} = (x_i, z_i) \). Let \( \pi : V' = B(C) \to V_i \) be the blow up of \( C \), \( S' \) be the strict transform of \( S \) on \( V' \). By Lemma 5.7, \( \text{Sing}_r(S') \) has SNCs. A direct local calculation
shows that there is at most one point \( q \in \pi^{-1}(p_i) \) such that \( \nu_q(S') = r \), and if such a point \( q \) exists, then \( O_{V', q} \) has regular parameters \((x', y', z')\) such that
\[
x_i = x', \quad y_i = y', \quad z_i = x'z'.
\]
A local equation \( f' = 0 \) of \( S' \) at \( q \) is
\[
f' = (z')^r + \pi_{2i}(x')^{a_{2i}-2}(y')^{b_{2i}}(z')^{r-2} + \cdots + \pi_{ri}(x')^{a_{ri}-r}(y')^{b_{ri}}.
\]
Thus we have an expression of the form (24), and conditions 1. and 2. of the statement of this Lemma hold for \( f' \). After repeating this procedure a finite number of times we will construct \( V \to V_i \) such that \( \nu_a(S) < r \) for all points \( a \) on the strict transform \( S \) of \( S \) such that \( a \) maps to \( p_i \), since \( \frac{a_{ji}}{j} + \frac{b_{ji}}{k} \) must drop by 1 every time we blow up.

Lemma 5.14. Let
\[
\delta_{j,k,i} = \left( \frac{a_{ji}}{j} - \frac{a_{ki}}{k} \right) \left( \frac{b_{ji}}{j} - \frac{b_{ki}}{k} \right).
\]
Then
\[
\delta_{j,k,i+1} \geq \delta_{j,k,i},
\]
and if \( \delta_{j,k,i} < 0 \) then
\[
\delta_{j,k,i+1} - \delta_{j,k,i} \geq \frac{1}{r^4}.
\]
Proof. We may assume that the first case of (25) holds. Then
\[
\delta_{j,k,i+1} = \delta_{j,k,i} + \left( \frac{b_{ji}}{j} - \frac{b_{ki}}{k} \right)^2.
\]
If \( \delta_{j,k,i} < 0 \) we must have \( \frac{b_{ji}}{j} - \frac{b_{ki}}{k} \neq 0 \). Thus
\[
\left( \frac{b_{ji}}{j} - \frac{b_{ki}}{k} \right)^2 \geq \frac{1}{j^2k^2} \geq \frac{1}{r^4}.
\]

Corollary 5.15. There exists \( m_1 \) such that \( i \geq m_1 \) implies condition 1. of Lemma 5.13 holds.

Proof. There exists \( m_1 \) such that \( i \geq m_1 \) implies \( \delta_{j,k,i} \geq 0 \) for all \( j, k \). Then the pairs \((\frac{a_{ji}}{k}, \frac{b_{ji}}{k})\) with \( 2 \leq k \leq r \) are totally ordered by \( \leq \), so there exists a minimal element \((\frac{a_{ji}}{j}, \frac{b_{ji}}{j})\).

Lemma 5.16. There exists an \( i' \geq m_1 \) such that we have condition 2. of Lemma 5.13,
\[
\left\{ \frac{a_{ji'}}{j} \right\} + \left\{ \frac{b_{ji'}}{j} \right\} < 1.
\]
Proof. A calculation shows that
\[
\left\{ \frac{a_{ji}}{j} \right\} + \left\{ \frac{b_{ji}}{j} \right\} \geq 1
\]
implies
\[
\left\{ \frac{a_{ji+1}}{j} \right\} + \left\{ \frac{b_{ji+1}}{j} \right\} \leq \left\{ \frac{a_{ji}}{j} \right\} + \left\{ \frac{b_{ji}}{j} \right\} - \frac{1}{r}.
With the above $i'$, $p_i'$ is a good point, since $p_i'$ satisfies the conditions of Lemma 5.13. Thus the conclusions of Theorem 5.12 must hold.

Now we give a the proof of Theorem 5.2. Let $r$ be the maximal multiplicity of points of $S$. By Theorem 5.12, there exists a finite sequence of blow ups of points $V_m \to V$ such that all points of $\text{Sing}_r(S_m)$ are good points, where $S_m$ is the strict transform of $S$ on $V_m$. By Lemma 5.10, there exists a sequence of blow ups of non-singular curves $V_n \to V_m$ such that $\text{Sing}_r(S_n) = \emptyset$, where $S_n$ is the strict transform of $S$ on $V_n$. By descending induction on $r$ we can reach the case where $\text{Sing}_2(S_n) = \emptyset$, so that $S_n$ is non-singular.

**Remark 5.17.** The resolution algorithm of this section is the good point algorithm of Abhyankar ([7], [57], [67]).

Zariski discusses early approaches to resolution in [84]. Some early proofs of resolution of surface singularities are by Albanese [11], Beppo Levi [56], Jung [53] and Walker [75]. Zariski gave several proofs of resolution of algebraic surfaces over fields of characteristic zero, including the first algebraic proof [79], which we present later in Chapter 8.

**Exercises**

1. Suppose that $S$ is a surface which is a subvariety of a non-singular three dimensional variety $V$ over an algebraically closed field $K$ of characteristic 0, and $p \in S$ is a closed point.
   a. Show that a Tschirnhausen transformation of a suitable local equation of $S$ at $p$ gives a formal hypersurface of maximal contact for $S$ at $p$.
   b. Show that if $f = 0$ is a local equation of $S$ at $p$, and $(x, y, z)$ are regular parameters at $p$ such that $\nu(f) = \nu(f(0, 0, z))$, then $\frac{\partial r - 1}{\partial z} f = 0$ is a hypersurface of maximal contact for $f = 0$ in a neighborhood of the origin.
2. Follow the algorithm to reduce the multiplicity of $f = z^r + x^a y^b = 0$ where $a + b \geq r$.

5.2. Embedded resolution of singularities. In this Section we will prove the following theorem, which is an extension of the main resolution theorem, Theorem 5.2 of the previous section.

**Theorem 5.18.** Suppose that $S$ is a surface which is a subvariety of a non-singular 3 dimensional variety $V$ over an algebraically closed field $K$ of characteristic 0, then there exists a sequence of monoidal transforms over the singular locus of $S$ such that the strict transform of $S$ is non-singular, and the total transform $\pi^*(S)$ is a SNC divisor.

**Definition 5.19.** A resolution datum $\mathcal{R} = (E_0, E_1, S, V)$ is a 4-tuple where $E_0, E_1, S$ are reduced effective divisors on the non-singular 3 dimensional variety $V$ such that $E = E_0 \cup E_1$ is a SNC divisor.

$\mathcal{R}$ is resolved at $p \in S$ if $S$ is non-singular at $p$ and $E \cup S$ has SNCs at $p$. For $r > 0$, let

$\text{Sing}_r(\mathcal{R}) = \{ p \in S \mid \nu_p(S) \geq r \text{ and } \mathcal{R} \text{ is not resolved at } p \}$.
Observe that $\text{Sing}_r(\mathcal{R}) = \text{Sing}_r(S)$ if $r > 1$, and $\text{Sing}_r(\mathcal{R})$ is a proper Zariski closed subset of $S$ for $r \geq 1$.

For $p \in S$, let
\[ \eta(p) = \text{the number of components of } E_1 \text{ containing } p. \]

We have $0 \leq \eta(p) \leq 3$. For $r, t \in \mathbb{N}$, let
\[ \text{Sing}_{r,t}(\mathcal{R}) = \{ p \in \text{Sing}_r(\mathcal{R}) \mid \eta(p) \geq t \}. \]

$\text{Sing}_{r,t}(\mathcal{R})$ is a Zariski closed subset of $\text{Sing}_r(\mathcal{R})$ (and of $S$). For the rest of this section we will fix $r = \nu(\mathcal{R}) = \nu(S, E) = \max\{ \nu_p(S) \mid p \in S, \mathcal{R} \text{ is not resolved at } p \}$, $t = \eta(\mathcal{R}) = \text{the maximum number of components of } E_1 \text{ containing a point of } \text{Sing}_r(\mathcal{R})$.

**Definition 5.20.** A permissible transform of $\mathcal{R}$ is a monoidal transform $\pi : V' \to V$ whose center is either a point of $\text{Sing}_{r,t}(\mathcal{R})$ or a non-singular curve $C \subset \text{Sing}_{r,t}(\mathcal{R})$ such that $C$ makes SNCs with $E$.

Let $F$ be the exceptional divisor of $\pi$. $\pi^*(E) \cup F$ is a SNC divisor. We define the strict transform $R'$ of $\mathcal{R}$ to be $R' = (E_0', E_1')$ where $E'$ is the strict transform of $S$, $E_0' = \pi^*((E_0)_\text{red} + F)$, $E_1' = \text{strict transform of } E_1$ if $\nu(S', \pi^*((E_0 + E_1)_\text{red} + F)) = \nu(\mathcal{R})$,
\[ E_0' = \emptyset, E_1' = \pi^*((E_0 + E_1)_\text{red} + F) \text{ if } \nu(S', \pi^*((E_0 + E_1)_\text{red} + F) < \nu(\mathcal{R}). \]

**Lemma 5.21.** With the notation of the previous definition,
\[ \nu(R') \leq \nu(\mathcal{R}) \]
and
\[ \nu(\mathcal{R}') = \nu(\mathcal{R}) \text{ implies } \eta(\mathcal{R}') \leq \eta(\mathcal{R}). \]

The proof of Lemma 5.21 is a generalization of Lemmas 5.3 and 5.4.

**Definition 5.22.** $p \in \text{Sing}_{r,t}(\mathcal{R})$ is a pregood point if

1. In a neighborhood of $p$, $\text{Sing}_{r,t}(\mathcal{R})$ is either a non-singular curve through $p$, or two non-singular curves through $p$ intersecting transversally at $p$ (satisfying 2. of Definition 5.6).
2. $\text{Sing}_{r,t}(\mathcal{R})$ makes SNCs with $E$ at $p$. That is, there exist regular parameters \{x, y, z\} in $O_{V,p}$ such that the ideal sheaf of each component of $E$ and each curve in $\text{Sing}_{r,t}(\mathcal{R})$ containing $p$ is generated by a subset of \{x, y, z\} at $p$.

$p \in S$ is a good point if $p$ is a pregood point, and if for any sequence $\pi : X_n \to X_{n-1} \to \cdots \to X_1 \to \text{spec}(O_{V,p})$ of permissible monoidal transforms centered at curves in $\text{Sing}_{r,t}(\mathcal{R}_i)$, where $\mathcal{R}_i$ is the transform of $\mathcal{R}$ on $X_i$, then $q$ is a pregood point for all $q \in \pi^{-1}(p)$.

A point $p \in \text{Sing}_{r,t}(\mathcal{R})$ is called bad if $p$ is not good.

**Lemma 5.23.** The number of bad points in $\text{Sing}_{r,t}(\mathcal{R})$ is finite.

**Lemma 5.24.** Suppose that all points of $\text{Sing}_{r,t}(\mathcal{R})$ are good. Then there exists a sequence of permissible monoidal transforms, $\pi : V' \to V$, centered at non-singular curves contained in $\text{Sing}_{r,t}(S_i)$, where $S_i$ is the strict transform of $S$ on the $i$-th monoidal transform such that if $\mathcal{R}'$ is the strict transform of $\mathcal{R}$ on $V'$ then either
\[ \nu(R') < \nu(\mathcal{R}) \]
or

\[ \nu(R') = \nu(R) \quad \text{and} \quad \eta(R') < \eta(R). \]

The proofs of the above two lemmas are similar to the proofs of Lemmas 5.11 and 5.10 respectively, with the use of embedded resolution of plane curves.

**Theorem 5.25.** Suppose that \( R \) is a resolution datum such that \( E_0 = \emptyset \). Let

\[ \ldots \to V_n \xrightarrow{\pi_4} V_{n-1} \to \ldots \xrightarrow{\pi_1} V_0 = V \tag{26} \]

be the sequence where \( \pi_n \) is the monomial transform centered at the union of bad points in \( \text{Sing}_{r,i}(R_i) \), where \( R_i \) is the transform of \( R \) on \( V_i \). Then this sequence is finite, so that it terminates in a \( V_m \) such that all points of \( \text{Sing}_{r,t}(R_m) \) are good.

**Proof.** Suppose that (26) is an infinite sequence. Then there exists an infinite sequence of points \( p_n \in V_n \) such that \( \pi_{n+1}(p_{n+1}) = p_n \) for all \( n \) and \( p_n \) is a bad point. Let \( R_n = \mathcal{O}_{V_n,p_n} \) for \( n \geq 0 \). Since the sequence (26) is infinite, its length is larger than 1, so we may assume that \( t \leq 2 \).

Let \( f = 0 \) be a local equation of \( S \) at \( p_0 \), \( g_k, 0 \leq k \leq t \) be local equations of the components of \( E_1 \) containing \( p_0 \). In \( R_0 = \mathcal{O}_{V_0,p_0} \) there are regular parameters \((x,y,z)\) such that \( \nu(f(0,0,z)) = r \) and \( \nu(g_k(0,0,z)) = 1 \) for all \( k \). This follows since for each equation this is a non-trivial Zariski open condition on linear changes of variables in a set of regular parameters. By the Weierstrass preparation theorem, after multiplying \( f \) and the \( g_k \) by units and performing a Tschirnhausen transformation on \( f \), we have expressions

\[ f = z^r + \sum_{j=2}^r a_j(x,y)z^{r-j} \]
\[ g_k = z - \phi_k(x,y), \quad (\text{or } g_k \text{ is a unit}) \tag{27} \]

in \( R_0 \). \( z = 0 \) is a local equation of a formal hypersurface of maximal contact for \( f = 0 \). Thus we have regular parameters \((x_1,y_1,z_1)\) in \( R_1 = \mathcal{O}_{V_1,p_1} \) defined by

\[ x = x_1, y = x_1(y_1 + \alpha), z = x_1z_1 \]

with \( \alpha \in K \) or

\[ x = x_1y_1, y = y_1, z = y_1z_1. \]

The strict transforms \( f_1 \) of \( f \) and \( g_{k1} \) of \( g_k \) have the form of (27) with \( a_j \) replaced by \( \frac{a_j}{x_1^j} \) (or \( \frac{a_j}{y_1^j} \)), \( \phi_k \) with \( \frac{\phi_k}{x_1} \) (or \( \frac{\phi_k}{y_1} \)). After a finite number of monoidal transforms centered at points \( p_j \) with \( \nu(f_j) = r \), we have forms in \( R_i = \mathcal{O}_{V_i,p_i} \) for the strict transforms of \( f \) and \( g_k \)

\[ f_i = z_i^r + \sum_{j=2}^r \pi_j(x_i,y_i)x_i^{a_j(y_i)}/y_i, y_i^b_j, z_i^{r-j} \]
\[ g_{ki} = z_i^r + \phi_{ki}(x_i,y_i)x_i^{c_{ki}y_i}y_i^{d_{ki}} \tag{28} \]

such that \( \pi_j, \phi_{ki} \) are either units or zero (as in the proof of Theorem 5.12). We further have that local equations of the exceptional divisor \( E_0^i \) of \( V_i \to V \) are one of \( x_i = 0 \), \( y_i = 0 \) or \( x_1y_i = 0 \). This last case can only occur if \( t < 2 \). Note that our assumption \( \eta(p_i) = t \) implies \( g_{ki} \) is not a unit in \( R_i \) for \( 1 \leq k \leq t \).

We can further assume, by an extension of the argument in the proof of Theorem 5.12, that all irreducible curves in \( \text{Sing}_{r,t}(R_i) \) are non-singular.

By Lemma 5.14, we can assume that the set

\[ T = \left\{ \left( \frac{a_{ji}}{j}, \frac{b_{ji}}{j} \right), \mid \pi_j \neq 0 \right\} \cup \{(c_{ki}, d_{ki}) \mid \phi_{ki} \neq 0 \} \]
is totally ordered. Now by Lemma 5.16 we can further assume that
\[ \left\{ \frac{a}{j_0} \right\} + \left\{ \frac{b}{j_0} \right\} < 1, \]
where \( \left( \frac{a}{j_0}, \frac{b}{j_0} \right) \) is the minimum of the \( \left( \frac{a}{j}, \frac{b}{j} \right) \). Recall that, if \( t > 0 \), \( \prod_{i=1}^t g_i = 0 \) makes SNCs with the exceptional divisor \( E_0 \) of \( V_i \to V \). These conditions imply that, after possibly interchanging the \( g_{ki} \), interchanging \( x_i, y_i \) and multiplying \( x_i, y_i \) by units in \( R_i \), that the \( g_{ki} \) can be described by one of the following cases at \( p_i \):

1. \( t = 0 \).
2. \( t = 1 \),
   a. \( g_{1i} = z_i + x_i^{\lambda}, \lambda \geq 1. \)
   b. \( g_{1i} = z_i + x_i^{\lambda}y_i^{\mu}, \lambda \geq 1, \mu \geq 1. \)
   c. \( g_{1i} = z_i \)
3. \( t = 2 \),
   a. \( g_{1i} = z_i, g_{2i} = z_i + x_i. \)
   b. \( g_{1i} = z_i + x_i, g_{2i} = z_i + \epsilon x_i, \) where \( \epsilon \) is a unit in \( R_i \) such that the residue of \( \epsilon \) in the residue field of \( R_i \) is not 1.
   c. \( g_{1i} = z_i + x_i, g_{2i} = z_i + \epsilon x_i^{\lambda}y_i^{\mu}, \) where \( \epsilon \) is a unit in \( R_i \), \( \lambda \geq 1 \) and \( \lambda + \mu \geq 2. \) If \( \mu > 0 \) we can take \( \epsilon = 1. \)

We can check explicitly by blowing up permissible curves that \( p_i \) is a good point. In this calculation, if \( t = 0 \), this follows from the proof of Theorem 5.12, recalling that \( x_i = 0, y_i = 0 \) or \( x_i y_i = 0 \) are local equations of \( E_0 \) at \( p_i \). If \( t = 1 \), we must remember that we can only blow up a curve if it lies on \( g_{1i} = 0 \), and if \( t = 2 \), we can only blow up the curve \( g_{1i} = g_{2i} = 0. \) Although we are constructing a sequence of permissible monoidal transforms over \( \text{spec}(\mathcal{O}_{V_1, p_i}) \), this is equivalent to constructing such a sequence over \( \text{spec}(\mathcal{O}_{V_1, p_i}) \), as all curves blown up in this calculation are actually algebraic, since all irreducible curves in \( \text{Sing}_{r,t}(R_i) \) are non-singular.

We have thus found a contradiction, by which we conclude that the sequence (26) has finite length.

The proof of Theorem 5.18 now follows easily from Theorem 5.25, Lemma 5.24 and descending induction on \((r, t) \in \mathbb{N} \times \mathbb{N}\), with the lexicographic order.

**Remark 5.26.** This algorithm is outlined in [67] and [57].

**Exercises**

Resolve (or at least make \( (\nu(R), \eta(R)) \) drop in the lexicographic order).

1. \( R = (0, E_1, S) \) where \( S \) has local equation \( f = z = 0, E_1 \) is the union of two components with respective local equations \( g_1 = z + xy, g_2 = z + x. \)
2. \( R = (0, E_1, S) \) where \( S \) has local equation \( f = z^3 + x^7 y^2 = 0, E_1 \) is the union of two components with respective local equations \( g_1 = z + xy, g_2 = z + y(x^2 + y^3). \)

6. **Resolution of singularities of varieties in characteristic zero**

In recent years several proofs of canonical resolution of singularities have appeared ([73], [13], [38], [39], [15], [37]). These papers have significantly simplified Hironaka’s original proof in [45]. In this chapter we prove the basic theorems of resolution of
singularities in characteristic 0. The final statements of resolution are proved in Section 6.8.

Throughout this chapter, unless explicitly stated otherwise, we will assume that all varieties are over a field \( K \) of characteristic zero. Suppose that \( X_1 \) and \( X_2 \) are subschemes of a variety \( W \). We will denote the reduced set theoretic intersection of \( X_1 \) and \( X_2 \) by \( X_1 \cap X_2 \). However, we will denote the scheme theoretic intersection of \( X_1 \) and \( X_2 \) by \( X_1 \cdot X_2 \). \( X_1 \cdot X_2 \) is the subscheme of \( W \) with ideal sheaf \( \mathcal{I}_{X_1} + \mathcal{I}_{X_2} \).

6.1. The operator \( \triangle \) and other preliminaries for resolution in arbitrary dimension.

**Definition 6.1.** Suppose that \( K \) is a field of characteristic zero, \( R = K[[x_1, \ldots, x_n]] \) is a ring of formal power series over \( K \), and \( J = (f_1, \ldots, f_r) \subset R \) is an ideal. Define

\[
\hat{\triangle}(J) = \hat{\triangle}_R(J) = (f_1, \ldots, f_r) + \left( \frac{\partial f_i}{\partial x_j} \mid 1 \leq i \leq r, 1 \leq j \leq n \right).
\]

\( \hat{\triangle}(J) \) is independent of choice of generators \( f_i \) of \( J \) and regular parameters \( x_j \) in \( R \).

**Lemma 6.2.** Suppose that \( W \) is a non-singular variety and \( J \subset \mathcal{O}_W \) is an ideal sheaf. Then there exists an ideal \( \triangle(J) = \triangle_W(J) \subset \mathcal{O}_W \) such that

\[
\triangle(J) \mathcal{O}_{W,q} = \hat{\triangle}(J)
\]

for all closed points \( q \in W \).

**Proof.** We define \( \triangle(J) \) as follows. We can cover \( W \) by affine open subsets \( U = \text{spec}(R) \) which satisfy the conclusions of Lemma 10.11, so that \( \Omega^1_{R/K} = dy_1 R \oplus \cdots \oplus dy_n R \). If \( \Gamma(U, J) = (f_1, \ldots, f_r) \), we define

\[
\Gamma(U, \triangle(J)) = (f_i, \frac{\partial f_i}{\partial y_j} \mid 1 \leq i \leq r, 1 \leq j \leq n).
\]

This definition is independent of all choices made in this expression.

Suppose that \( q \in W \) is a closed point, and \( x_1, \ldots, x_n \) are regular parameters in \( \mathcal{O}_{W,q} \). Let \( U = \text{spec}(R) \) be an affine neighborhood of \( q \) as above. Let \( A = \mathcal{O}_{W,q} \). Then \( \{dy_1, \ldots, dy_n\} \) and \( \{dx_1, \ldots, dx_n\} \) are \( A \)-bases of \( \Omega^1_{A/k} \) by Lemma 10.11 and Theorem 10.10. Thus

\[
\left| \frac{\partial y_i}{\partial x_j} \right| (q) \neq 0
\]

from which the conclusions of the Lemma follow. \( \square \)

Given \( J \subset \mathcal{O}_W \) and \( q \in W \) we define \( \nu_q(J) = \nu_q(J) \) (Definition 10.17). If \( X \) is a subvariety of \( W \) we denote \( \nu_X = \nu_{\mathcal{I}_X} \). We can define \( \triangle^r(J) \) for \( r \in \mathbb{N} \) inductively, by the formula \( \triangle^r(J) = \triangle(\triangle^{r-1}(J)) \). We define \( \triangle^0(J) = J \).

**Lemma 6.3.** Suppose that \( W \) is a non-singular variety, \( (0) \neq J \subset \mathcal{O}_W \) is an ideal sheaf, and \( q \in W \) (which is not necessarily a closed point). Then

1. \( \nu_q(J) = b > 0 \) if and only if \( \nu_q(\triangle(J)) = b - 1 \).
2. \( \nu_q(J) = b > 0 \) if and only if \( \nu_q(\triangle^{b-1}(J)) = 1 \).
3. \( \nu_q(J) \geq b > 0 \) if and only if \( q \in V(\triangle^{b-1}(J)) \).
4. \( \nu_q : W \to \mathbb{N} \) is an upper semi-continuous function.

**Proof.** The Lemma follows from Theorem 10.19. \( \square \)
If \( f : X \rightarrow I \) is an upper semi-continuous function, we define \( \max f \) to be the largest value assumed by \( f \) on \( X \), and we define a closed subset of \( X \),

\[
\max f = \{ q \in X \mid f(q) = \max f \}.
\]

We have \( \max \nu_J = V(\Delta^{b-1}(J)) \) if \( b = \max \nu_J \).

Suppose that \( X \) is a closed subset of a \( n \)-dimensional variety \( W \). We shall denote by

\[
R(1)(X) \subset X
\]

the union of irreducible components of \( X \) which have dimension \( n - 1 \).

**Lemma 6.4.** Suppose that \( W \) is a non-singular variety, \( J \subset \mathcal{O}_W \) is an ideal sheaf, \( b = \max \nu_J \) and \( Y \subset \max \nu_J \) is a non-singlar subvariety. Let \( \pi : W_1 \rightarrow W \) be the monoidal transform with center \( Y \). Let \( D \) be the exceptional divisor of \( \pi \). Then

1. There is an ideal sheaf \( J_1 \subset \mathcal{O}_{W_1} \) such that

\[
J_1 = \mathcal{I}_D^{b} J.
\]

and \( \mathcal{I}_D \not\subseteq J_1 \).

2. If \( q \in W_1 \) and \( p = \pi(q) \), then

\[
\nu_{J_1}(q) \leq \nu_J(p).
\]

3. \( \max \nu_J \geq \max \nu_{J_1} \).

**Proof.** Suppose that \( q \in W_1 \) and \( p = \pi(q) \). Let \( A \) be the Zariski closure of \( \{q\} \) in \( W_1 \) and \( B \) be the Zariski closure of \( \{p\} \) in \( W \). Then \( A \rightarrow B \) is proper. By upper semi-continuity of \( \nu_J \), there exists a closed point \( x \in B \) such that \( \nu_J(x) = \nu_J(p) \). Let \( y \in \pi^{-1}(x) \cap A \) be a closed point. By semi-continuity of \( \nu_J \), we have \( \nu_{J_1}(q) \leq \nu_J(y) \), so it suffices to prove 2. when \( q \) and \( p \) are closed points.

By Corollary 10.5, there is a regular system of parameters \( y_1, \ldots, y_n \) in \( \mathcal{O}_{W_1,p} \) such that \( y_1 = y_2 = \cdots = y_r = 0 \) are local equations of \( Y \) at \( p \). By Remark 10.18, we may assume that \( K = K(p) = K(q) \). Then we can make a linear change of variables in \( y_1, \ldots, y_n \) to assume that there are regular parameters \( \overline{y}_1, \ldots, \overline{y}_n \) in \( \mathcal{O}_{W_1,q} \) such that

\[
y_1 = \overline{y}_1, y_2 = \overline{y}_1 y_2, \ldots, y_r = \overline{y}_1 y_r, y_{r+1} = \overline{y}_{r+1}, \ldots, y_n = \overline{y}_n.
\]

\( \overline{y}_1 = 0 \) is a local equation of \( D \). By Remark 10.18, if suffices to verify the conclusions of the theorem in the complete local rings

\[
R = \hat{\mathcal{O}}_{W_1,p} = K[[y_1, \ldots, y_n]]
\]

and

\[
S = \hat{\mathcal{O}}_{W_1,q} = K[[\overline{y}_1, \ldots, \overline{y}_n]].
\]

\( Y \subset \max \nu_J \) implies \( \nu_\eta(J) = b \) where \( \eta \) is the general point of the component of \( Y \) whose closure contains \( p \). Suppose that \( f \in JR \). Let \( t = \nu_\eta(f) \geq b \). Then there is an expansion

\[
f = \sum a_{i_1, \ldots, i_n} y_1^{i_1} \cdots y_n^{i_n}
\]

in \( R \), with \( a_{i_1, \ldots, i_n} \in K \) and \( a_{i_1, \ldots, i_n} = 0 \) if \( i_1 + \cdots + i_r < t \). Further, there exists \( i_1, \ldots, i_n \) with \( i_1 + \cdots + i_n = t \) such that \( a_{i_1, \ldots, i_n} \neq 0 \).

In \( S \), we have

\[
f = \overline{y}_1^t f_1
\]
where
\[ f_1 = \sum a_{i_1,\ldots,i_n} \nu_{1}^{i_1+\cdots+i_n} \cdots \nu_{n}^{i_n} \]
is such that \( \nu_{1} \) does not divide \( f_1 \). Thus we have \( \nu_S(f_1) \leq \nu_R(f) \). In particular, we have \( \nu_S(f) \leq \nu_R(f) = b \), \( \nu_{1}^{i_1+1} \not| JS \). Thus we have that
\[ J_1 = I_{D}^{-b} J \]
satisfies 1. and
\[ \nu_{J_1}(q) \leq \nu_J(p) = b. \]

Suppose that \( W \) is a non-singular variety and \( J \subset \mathcal{O}_W \) is an ideal sheaf. Suppose that \( Y \subset W \) is a non-singular subvariety. Let \( Y_1,\ldots,Y_m \) be the distinct irreducible (connected) components of \( Y \). Let \( \pi : W_1 \to W \) be the monoidal transform with center \( Y \) and exceptional divisor \( D \). Let \( D = D_1 + \cdots + D_m \) where \( D_i \) are the distinct irreducible (connected) components of \( D \), indexed so that \( \pi(D_i) = Y_i \). Then by an argument as in the proof of Lemma 6.4, there exists an ideal sheaf \( J_1 \subset \mathcal{O}_{W_1} \) such that
\[ J\mathcal{O}_{W_1} = I_{D_1}^{i_1} \cdots I_{D_m}^{i_m} J_1 \] (30)
where \( J_1 \) is such that for \( 1 \leq i \leq m \), \( I_{D_i} \not| J_1 \) and \( c_i = \nu_{\eta_i}(J) \), with \( \eta_i \) the generic point of \( Y_i \). \( J_1 \) is called the weak transform of \( J \).

There is thus a function \( c \) on \( W_1 \) defined by \( c(q) = 0 \) if \( q \not\in D \) and \( c(q) = c_i \) if \( q \in D_i \) such that
\[ J\mathcal{O}_{W_1} = I_{D}^{-c} J_1. \] (31)

In the situation of Lemma 6.4, \( J_1 \) has the properties 1. and 2. of that Lemma. With the notation of Lemma 6.4, suppose that \( J = I_X \) is the ideal sheaf of a subvariety \( X \) of \( W \). The weak transform \( \overline{X} \) of \( X \) is the subscheme of \( W_1 \) with ideal sheaf \( J_1 \). We see that the weak transform \( \overline{X} \) of \( X \) is much easier to calculate than the strict transform \( \tilde{X} \) of \( X \). We have inclusions
\[ \tilde{X} \subset \overline{X} \subset \pi^*(X). \]

**Example 6.5.** Suppose that \( X \) is a hypersurface on a variety \( W, r = \max \nu_X \) and \( Y \subset \text{Max} \nu_X \) is a non-singular subvariety. Let \( \pi : W_1 \to W \) be the monoidal transform of \( W \) with center \( Y \) and exceptional divisor \( E \). In this case the strict transform \( \tilde{X} \) and the weak transform \( \overline{X} \) are the same scheme, and \( \pi^*(X) = \tilde{X} + rE \).

**Example 6.6.** Let \( X \subset W = \mathbb{A}^3 \) be the nodal plane curve with ideal \( I_X = (z, y^2 - x^3) \subset k[x,y,z] \). Let \( m = (x,y,z), \pi : B = B(m) \to \mathbb{A}^3 \) be the blow up of the point \( m \). The strict transform \( \tilde{X} \) of \( X \) is non-singular.

The most interesting point in \( \tilde{X} \) is the point \( q \in \pi^{-1}(m) \) with regular parameters \( (x_1,y_1,z_1) \) such that
\[ x = x_1, y = x_1y_1, z = x_1z_1, \]
\[ I_{\pi^{-1}(X),q} = (x_1z_1, x_1^2y_1 - x_1^3), \]
\[ I_{\overline{X},q} = (z_1, x_1^2 - x_1), \]
\[ I_{\tilde{X},q} = (z_1, y_1^2 - x_1). \]

In this example, \( \overline{X}, \tilde{X} \) and \( \pi^*(X) \) are all distinct.
6.2. Hypersurfaces of maximal contact and induction in resolution. Suppose that \( W \) is a non-singular variety, \( J \subset \mathcal{O}_W \) is an ideal sheaf and \( b \in \mathbb{N} \). Define

\[
\text{Sing}(J, b) = \{ q \in W \mid \nu_J(q) \geq b \}
\]

Sing\((J, b)\) is Zariski closed in \( W \) by Lemma 6.3. Suppose that \( Y \subset \text{Sing}(J, b) \) is a non-singular (but not necessarily connected) subvariety of \( W \). Let \( \pi_1 : W_1 \to W \) be the monoidal transform with center \( Y \), \( D_1 = \pi_1^{-1}(Y) \) be the exceptional divisor.

Recall (31) that the weak transform \( J_1 \) of \( J \) is defined by

\[
J_1 \mathcal{O}_{W_1} = J_{D_1} c J_1
\]

where \( c \) is locally constant on connected components of \( D_1 \). \( c \geq b \) by upper semi-continuity of \( \nu_J \).

We can now define \( J_1 \) by

\[
J_1 \mathcal{O}_{W_1} = J_{D_1}^b J_1,
\]

so that

\[
J_1 = J_{D_1}^{c-b} J_1.
\]

Suppose that

\[
W_n \xrightarrow{\pi_n} W_{n-1} \to \cdots \to W_1 \xrightarrow{\pi_1} W
\]

is a sequence of monoidal transforms such that each \( \pi_i \) is centered at a non-singular subvariety \( Y_i \subset \text{Sing}(J_i, b) \) where \( J_i \) is defined inductively by

\[
J_{i-1} \mathcal{O}_{W_i} = I_{D_i}^b J_i
\]

and \( D_i \) is the exceptional divisor of \( \pi_i \). We will say that (33) is a resolution of \((W, J, b)\) if \( \text{Sing}(J_n, b) = \emptyset \).

**Definition 6.7.** Suppose that \( r = \max \nu_J, q \in \text{Sing}(J, r) \). A non-singular codimension 1 subvariety \( H \) of an affine neighborhood \( U \) of \( q \) in \( W \) is called a hypersurface of maximal contact for \( J \) at \( q \) if

1. \( \text{Sing}(J, r) \cap U \subset H \) and
2. If

\[
W_n \to \cdots \to W_1 \to W
\]

is a sequence of monoidal transforms of the form of (33) (with \( b = r \)), then the strict transform \( H_n \) of \( H \) on \( U_n = W_n \times_W U \) is such that \( \text{Sing}(J_n, r) \cap U_n \subset H_n \).

With the assumptions of Definition 6.7, a non-singular codimension 1 subvariety \( H \) of \( U = \text{spec}(\mathcal{O}_{W,q}) \) is called a formal hypersurface of maximal contact for \( J \) at \( q \) if 1. and 2. of Definition 6.7 hold, with \( U = \text{spec}(\mathcal{O}_{W,q}) \).

If \( X \subset W \) is a codimension 1 subvariety, with \( r = \max \nu_X, q \in \max \nu_X \), then our definition of hypersurface of maximal contact for \( J = I_X \) coincides with the definition of a hypersurface of maximal contact for \( X \) given in Definition 10.20. In this case, the ideal sheaf \( J_n \) in (33) is the ideal sheaf of the strict transform \( X_n \) of \( X \) on \( W_n \).

Now suppose that \( X \subset W \) is a singular hypersurface, and \( K \) is algebraically closed. Let

\[
r = \max \{ \nu_X \} > 1
\]

be the set of points of maximal multiplicity on \( X \). Let \( q \in \text{Sing}(I_X, r) \) be a closed point of maximal multiplicity \( r \).
The basic strategy of resolution is to construct a sequence

\[ W_n \to \cdots \to W_1 \to W \]

decentralized transforms centered at non-singular subvarieties of the locus of points of multiplicity \( r \) on the strict transform of \( X \) so that we eventually reach a situation where all points of the strict transform of \( X \) have multiplicity \(< r \). We know that under such a sequence the multiplicity can never go up, and always remains \( \leq r \) (Lemma 6.4). However, getting the multiplicity to drop to less than \( r \) everywhere is much more difficult.

Notice that the desired sequence \( W_n \to W_1 \) will be a resolution of the form of (33), with \((J_i, b) = (\mathcal{I}_{X_i}, r_i)\), where \( X_i \) is the strict transform of \( X \) on \( W_i \).

Let \( q \in \operatorname{Sing}(\mathcal{I}_X, r) \) be a closed point. After Weierstrass Preparation and performing a Tschirnhausen transformation, there exist regular parameters \((x_1, \ldots, x_n, y)\) in \( R = \hat{O}_{W, q} \) such that there exists \( f \in R \) such that \( f = 0 \) is a local equation of \( X \) at \( q \), and

\[ f = y^r + \sum_{i=2}^{r} a_i(x_1, \ldots, x_n)y^{r-i}. \]  

(34)

Set \( H = V(y) \subset \operatorname{spec}(R) \). Then \( H = \operatorname{spec}(S) \) where \( S = K[[x_1, \ldots, x_n]] \).

We observe that \( H \) is a (formal) hypersurface of maximal contact for \( X \) at \( q \). The verification is as follows.

We will first show that \( H \) contains a formal neighborhood of the locus of points of maximal multiplicity on \( X \) at \( q \). Suppose that \( Z \subset \operatorname{Sing}(\mathcal{I}_X, r) \) is a subvariety containing \( q \). Let \( J = \hat{\mathcal{I}}_{Z, q} \).

Write \( J = P_1 \cap \cdots \cap P_n \) where \( P_i \) are prime ideals of the same height in \( R \). Let \( R_i = R_{P_i} \). By semi-continuity of \( \nu_X \), we have \( f \in P_i \cap R_{P_i} \) for all \( i \). Since \( \frac{\partial}{\partial y} : R_{P_i} \to R_{P_i} \), we have \( (r - 1)y = \frac{\partial}{\partial y} - 1 \in J R_{P_i} \) for all \( i \). Thus \( y \in \cap J R_{P_i} = J \) and \( Z \subset \operatorname{spec}(R) \subset H \).

Now we will verify that if \( Y \subset \operatorname{Sing}(\mathcal{I}_X, r) \) is a non-singular subvariety, \( \pi : W_1 \to W \) is the monoidal transform with center \( Y \), \( X_1 \) is the strict transform of \( X \) and \( q_1 \) is a closed point of \( X_1 \) such that \( \pi(q_1) = q \) which has maximal multiplicity \( r \), then \( q_1 \) is on the strict transform \( H_1 \) of \( H \) (over the formal neighborhood of \( q \)). Suppose that \( q_1 \) is such a point. That is, \( q_1 \in \pi^{-1}(q) \cap \operatorname{Sing}(\mathcal{I}_{X_1}, r) \). We have shown above that \( y \in \hat{\mathcal{I}}_Y \). Without changing the form of (34), we may then assume that there is \( s \leq n \) such that \( x_1 = \ldots = x_s = y = 0 \) are (formal) local equations of \( Y \) at \( q \), and there is a formal regular system of parameters \( x_1(1), \ldots, x_n(1), y(1) \) in \( \hat{O}_{W_1, q_1} \), such that

\[
\begin{align*}
x_1 &= x_1(1) \\
x_i &= x_1(1)x_i(1) \quad \text{for} \ 2 \leq i \leq s \\
y &= x_1(1)y(1) \\
x_i &= x_i(1) \quad \text{for} \ s < i \leq n.
\end{align*}
\]

(35)

Since \( Y \subset \operatorname{Sing}(\mathcal{I}_X, r) \) we have \( q_i \in (x_1, \ldots, x_n)^i \) for all \( i \). Then we have a (formal) local equation \( f_1 = 0 \) of \( X_1 \) at \( q_1 \) where

\[
\begin{align*}
f_1 &= \frac{\partial}{\partial y} = y(1)^r + \sum_{i=2}^{r} a_i(x_1(1), \ldots, x_n(1))y(1)^{r-i} \\
&= y(1)^r + \sum_{i=2}^{r} a_i(1)(x_1(1), \ldots, x_n(1))y(1)^{r-i}
\end{align*}
\]

(36)

and \( y(1) = 0 \) is a local equation of \( H_1 \) at \( q_1 \), \( x_1(1) = 0 \) is a local equation of the exceptional divisor of \( \pi \). With our assumption that \( f_1 \) also has multiplicity \( r \), \( f_1 \) has
an expression of the same form as (34), so by induction, $H$ is a (formal) hypersurface of maximal contact.

We see that (at least over a formal neighborhood of $q$) we have reduced the problem of resolution of $X$ to some sort of resolution problem on the (formal) hypersurface $H$.

We will now describe this resolution problem. Set

$$I = (a_2^{r_2}, \ldots, a_r^{r_r}) \subset S = K[[x_1, \ldots, x_n]].$$

Observe that the scheme of points of multiplicity $r$ in the formal neighborhood of $q$ on $X$ coincides with the scheme of points of multiplicity $\geq r!$ of $I$,

$$\text{Sing}(fR, r) = \text{Sing}(I, r!).$$

However, we may have $\nu_R(I) > r!$.

Let us now consider the effect of the monodial transform $\pi$ of (35) on $I$. Since $H$ is a hypersurface of maximal contact, $\pi$ certainly induces a morphism $H_1 \rightarrow H$, where $H_1$ is the strict transform of $H$. By direct verification, we see that the transform $I_1$ of $I$ (this transform is defined by (32)) satisfies

$$I_1 S_1 = \frac{1}{x_1^{r_1}} IS_1$$

where $S_1 = K[[x_1(1), \ldots, x_n(1)]] = \hat{O}_{H_1, q_1}$, and this ideal is thus, by (36), the ideal

$$I_1 S_1 = (a_2(1)^{r_2}, \ldots, a_r(1)^{r_r})$$

which is obtained from the coefficients of $f_1$, our local equation of $X_1$, in the exact same way that we obtained the ideal $I$ from our local equation $f$ of $X$. We further observe that $\nu_{S_1}(I_1 S_1) \geq r!$ if and only if $\nu_{q_1}(X_1) \geq r$. We see then that to reduce the multiplicity of the strict transform of $X$ over $q$, we are reduced to resolving a sequence of the form (33), but over a formal scheme.

The above is exactly the procedure which is followed in the proof of resolution for curves given in Section 3.4, and the proof of resolution for surfaces in Section 5.1.

When $X$ is a curve (dim $W = 2$) the induced resolution problem on the formal curve $H$ is essentially trivial, as the ideal $I$ is in fact just the ideal generated by a monomial, and we only need blow up points to divide out powers of the terms in the monomial.

When $X$ is a surface (dim $W = 3$), we were able to make use of another induction, by realizing that resolution over a general point of a curve in the locus of points of maximal multiplicity reduces to the problem of resolution of curves (over a non-closed field). In this way, we were able to reduce to a resolution problem over finitely many points in the locus of points of maximal multiplicity on $X$. The potentially worrisome problem of extending a resolution of an object over a formal germ of a hypersurface to a global proper morphism over $W$ was not a great difficulty since the blowing up of a point on the formal surface $H$ always extends trivially, and the only other kind of blow ups we had to consider were the blow ups of non-singular curves contained in the locus of maximal multiplicity. This required a little attention, but we were able to fairly easily reduce to a situation were these formal curves were always algebraic. In this case we first blew up points to reduce to the situation where the ideal $I$ was locally a monomial ideal. Then we blew up more points to make it a principal monomial ideal. At this point, we finished the resolution process by blowing up non-singular curves in the locus of maximal multiplicity on the strict transform.
of the surface $X$, which amounts to dividing out powers of the terms in the principal monomial ideal.

In Section 5.2, we extended this algorithm to obtain embedded resolution of the surface $X$. This required keeping track of the exceptional divisors which occur under the resolution process, and requiring that the centers of all monoidal transforms make SNCs with the previous exceptional divisors. We introduced the $\eta$ invariant, which counts the number of components of the exceptional locus which have existed since the multiplicity last dropped.

In this chapter we give a proof of resolution in arbitrary dimension (over fields of characteristic zero) which incorporates all of these ideas into a general induction statement. The fact that we must consider some kind of covering by local hypersurfaces which are not related in any obvious way is incorporated in the notion of General Basic Object.

The first proof of resolution in arbitrary dimension (over a field of characteristic zero) was by Hironaka [48]. The proof consists of a local proof, and a complex web of inductions to produce a global proof. The local proof uses sophisticated methods in commutative algebra to reduce the problem of reducing the multiplicity of the strict transform of an ideal after a permissible sequence of monoidal transforms to the problem of reducing the multiplicity of the strict transform of a hypersurface. The use of the Tschirnhausen transformation to find a hypersurface of maximal contact lies at the heart of this proof. The order $\nu$ and the invariant $\tau$ (of Chapter 7) are the principal invariants considered.

The proof has been significantly simplified by Hironaka and others. Hironaka outlined a proof of canonical resolution where he introduced the idea of general basic objects [51] (We define this concept in Section 6.5). Since then several accessible proofs of canonical equivariant resolution have emerged ([73], [13], [38], [39],[15],[37]).

De Jong’s theorem [34], referred to at the end of Chapter 7, can be refined to give a new proof of resolution of singularities of varieties of characteristic zero ([9], [14], [65]). The resulting theorem is not as strong as the classical results on resolution of singularities in characteristic zero (c.f. Section 6.8), as the resulting resolution $X' \to X$ may not be an isomorphism over the non-singular locus of $X$.

In the algorithm of this chapter Tschirnhausen is replaced by a method attributed to Giraud for finding algebraic hypersurfaces of maximal contact for an ideal $J$ of order $b$, by finding a hypersurface in $\Delta^{b-1}(J)$.

6.3. Pairs and basic objects.

**Definition 6.8.** A pair is $(W_0, E_0)$ where $W_0$ is a non-singular variety over a field $K$ of characteristic zero, and $E_0 = \{D_1, \ldots, D_r\}$ is an ordered collection of reduced non-singular divisors on $W_0$ such that $D_1 + \cdots + D_r$ is a reduced simple normal crossings divisor.

Suppose that $(W_0, E_0)$ is a pair, and $Y_0$ is a non-singular closed subvariety of $W_0$. $Y_0$ is a permissible center for $(W_0, E_0)$ if $Y_0$ has SNCs with $E_0$. Let $\pi : W_1 \to W$ be the monoidal transform of $W$ with center $Y_0$ and exceptional divisor $D$. By abuse of notation we identify $D_i$ ($1 \leq i \leq r$) with its strict transform on $W_1$ (which could become the empty set if $D_i$ is a union of components of $Y_0$). By a local analysis, using the regular parameters of the proof of Lemma 6.4, we see that $D_1 + \cdots + D_r + D$ is a SNC divisor on $W_1$. 
Now set $E_1 = \{D_1, \ldots, D_r, D_{r+1} = D\}$. We have thus defined a new pair $(W_1, E_1)$
called the transform of $(W_0, E_0)$ by $\pi$.

$$(W_1, E_1) \rightarrow (W_0, E_0)$$

will be called a transformation. Observe that some of the strict transforms $D_i$ may
be the empty set.

**Definition 6.9.** Suppose that $(W_0, E_0)$ is a pair where $E_0 = \{D_1, \ldots, D_r\}$, and $\tilde{W}_0$
is a non-singular subvariety of $W_0$. Define scheme theoretic intersections $\tilde{D}_i = D_i \cdot \tilde{W}_0$
for all $i$. Suppose that $\tilde{D}_i$ are non-singular for $1 \leq i \leq r$ and that $\tilde{D}_1 + \cdots + \tilde{D}_r$
is a SNC divisor. Set $\tilde{E}_0 = \{\tilde{D}_1, \ldots, \tilde{D}_r\}$. Then $(\tilde{W}_0, \tilde{E}_0)$ is a pair and

$$(\tilde{W}_0, \tilde{E}_0) \rightarrow (W_0, E_0)$$

is called an immersion of pairs.

**Definition 6.10.** A basic object is $(W_0, (J_0, b), E_0)$ where $(W_0, E_0)$ is a pair, $J_0$
is an ideal sheaf of $W_0$ which is non-zero on all connected components of $W_0$, and $b$ is
a natural number. The singular locus of $(W_0, (J_0, b), E_0)$ is

$$Sing(J_0, b) = \{q \in W_0 \mid \nu_q(J_0) \geq b\} = V(\Delta^{b-1}(J_0)) \subset W_0.$$

**Definition 6.11.** A basic object $(W_0, (J_0, b), E_0)$ is a simple basic object if

$$\nu_q(J) = b \text{ if } q \in Sing(J, b).$$

Suppose that $(W_0, (J_0, b), E_0)$ is a basic object and $Y_0$ is a non-singular closed
subvariety of $W_0$. $Y_0$ is a permissible center for $(W_0, (J_0, b), E_0)$ if $Y_0$ is permissible
for $(W_0, E_0)$ and $Y_0 \subset Sing(J_0, b)$. Suppose that $Y_0$ is a permissible center
for $(W_0, (J_0, b), E_0)$. Let $\pi : W_1 \rightarrow W_0$ be the monoidal transform of $W_0$
with center $Y_0$ and exceptional divisor $D$. Thus we have a transformation of pairs
$(W_1, E_1) \rightarrow (W_0, E_0)$. Let $\mathcal{J}_1$ be the weak transform of $J_0$ on $W_1$, so that there is a
locally constant function $c_1$ on $D$ (constant on each connected component of $D$) such
that the total transform of $J_0$ is

$$J_0\mathcal{O}_{W_1} = \mathcal{J}_1^{c_1} \mathcal{J}_1.$$

We necessarily have $c_1 \geq b$, so we can define

$$J_1 = \mathcal{J}_1^{-b} J_0 \mathcal{O}_{W_1} = \mathcal{J}_1^{c_1-b} \mathcal{J}_1.$$

We have thus defined a new basic object $(W_1, (J_1, b), E_1)$ called the transform of
$(W_0, (J_0, b), E_0)$ by $\pi$.

$$(W_1, (J_1, b), E_1) \rightarrow (W_0, (J_0, b), E_0)$$

will be called a transformation.

Observe that a transform of a basic object is a basic object, and (by Lemma 6.4)
the transform of a simple basic object is a simple basic object.

Suppose that

$$(W_k, (J_k, b), E_k) \rightarrow \cdots \rightarrow (W_0, (J_0, b), E_0)$$

is a sequence of transformations.

**Definition 6.12.** (37) is a resolution of $(W_0, (J_0, b), E_0)$ if $Sing(J_k, b) = \emptyset$. 

\[37\]
Lemma 6.15. Suppose that \( (37) \) is a sequence of transformations. Set \( \mathcal{J}_0 = J_0 \), and let \( \mathcal{J}_{i+1} \) be the weak transform of \( \mathcal{J}_i \) for \( 0 \leq i < k - 1 \). For \( 0 \leq i \leq k \) define
\[
w-\text{ord}_i : \text{Sing}(J_i, b) \rightarrow \mathbb{Z}
\]
by
\[
w-\text{ord}_i(q) = \frac{\nu_q(J_i)}{b}
\]
and define
\[
\text{ord}_i : \text{Sing}(J_i, b) \rightarrow \mathbb{Z}
\]
by
\[
\text{ord}_i(q) = \frac{\nu_q(J_i)}{b}.
\]
If we assume that \( Y_i \subset \text{Max} \) \( w-\text{ord}_i \subset \text{Sing}(J_i, b) \) for \( 1 \leq i \leq k \) in \( (37) \), then
\[
\text{max} w-\text{ord}_0 \geq \cdots \geq \text{max} w-\text{ord}_k
\]
by Lemma 6.4.

Definition 6.14. Suppose that \( (37) \) is a sequence of transformations such that \( Y_i \subset \text{Max} \) \( w-\text{ord}_i \subset \text{Sing}(J_i, b) \) for \( 0 \leq i < k \). Let \( k_0 \) be the smallest index such that
\[
\text{max} w-\text{ord}_{k_0-1} > \text{max} w-\text{ord}_{k_0} = \cdots = \text{max} w-\text{ord}_k.
\]
In particular, \( k_0 \) is defined to be 0 if \( \text{max} w-\text{ord}_0 = \cdots = \text{max} w-\text{ord}_k \). Set \( E_k = E^+_k \cup E^-_k \) where \( E^-_k \) is the ordered set of divisors \( D \) in \( E_k \) which are strict transforms of divisors of \( E_{k_0} \) and \( E^+_k = E_k - E^-_k \).

Define
\[
t_k : \text{Sing}(J_k, b) \rightarrow \mathbb{Q} \times \mathbb{Z}
\]
where \( \mathbb{Q} \times \mathbb{Z} \) has the lexicographic order by \( t_k(q) = (w-\text{ord}_k(q), \eta(q)) \),
\[
\eta(q) = \begin{cases} 
\text{Card} \{ D \in E_k \mid q \in D \} & \text{if } w-\text{ord}_k(q) < \text{max } w-\text{ord}_{k_0} \\
\text{Card} \{ D \in E^-_k \mid q \in D \} & \text{if } w-\text{ord}_k(q) = w-\text{ord}_{k_0}
\end{cases}
\]

The above definition allows us to inductively define upper semi-continuous functions \( t_0, t_1, \ldots, t_k \) on a sequence of transformations \( (37) \) such that \( Y_i \subset \text{Max} \) \( w-\text{ord}_i \subset \text{Sing}(J_i, b) \) for \( 1 \leq i \leq k \).

Lemma 6.15. Suppose that \( (37) \) is a sequence of transformations such that \( Y_i \subset \text{Max} \) \( t_i \subset \text{Max} \) \( w-\text{ord}_i \) for \( 0 \leq i \leq k \). Then
\[
\text{max } t_k \leq \cdots \leq \text{max } t_k.
\]

Lemma 6.15 follows from Lemma 6.4, and a local analysis, using the regular parameters of the proof of Lemma 6.4.

Definition 6.16. Suppose that \( (W_0, (J_0, b), E_0) \) is a basic object, and
\[
(W_k, (J_k, b), E_k) \rightarrow \cdots \rightarrow (W_0, (J_0, b), E_0)
\]
is a sequence of transformations such that \( \text{max } w-\text{ord}_k = 0 \). Then we say that
\( (W_k, (J_k, b), E_k) \) is a monomial basic object (relative to \( (38) \)).

Suppose that \( (W_k, (J_k, b), E_k) \) is a monomial basic object (relative to a sequence \( (38) \)). Let \( E_0 = \{ D_1, \ldots, D_r \} \) and let \( D_\tau \) be the exceptional divisor of \( \pi : W_i \rightarrow W_{i-1} \)
where \( \tau = i + r \). Then we have an expansion (in a neighborhood of \( \text{Sing}(J_k, b) \)) of the weak transform \( J_k \) of \( J \)
\[
J_k = \mathcal{I}^{\mathcal{D}}_{D_\tau} \cdots \mathcal{I}^{\mathcal{D}}_{D_\tau}
\]
where $d_i$ are functions on $W_k$ which are zero on $W_k - D_i$ and $d_i$ is a locally constant function on $D_i$ for $1 \leq i \leq k$.

Suppose that $B = (W_k, (J_k, b), E_k)$ is a monomial basic object (relative to (38)). We define

$$\Gamma(B) : \Sing(J_k, b) \rightarrow I_M = \mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}^N$$

by

$$\Gamma(B)(q) = (-\Gamma_1(q), \Gamma_2(q), \Gamma_3(q)),$$

where

$$\Gamma_1(q) = \min \left\{ p \text{ such that there exists } i_1, \ldots, i_p, \text{ with } r < i_1, \ldots, i_p \leq r + k, \text{ such that } d_{i_1}(q) + \cdots + d_{i_p}(q) \geq b \right\},$$

$$\Gamma_2(q) = \max \left\{ \frac{d_{i_1}(q) + \cdots + d_{i_p}(q)}{b}, \text{ with } r < i_1, \ldots, i_p \leq r + k, \text{ such that } p = \Gamma_1(q), \right\},$$

$$\Gamma_3(q) = \max \left\{ (i_1, \ldots, i_p, 0, \ldots) \text{ such that } \Gamma_2(q) = \frac{d_{i_1}(q) + \cdots + d_{i_p}(q)}{b}, \text{ with } r < i_1, \ldots, i_p \leq r + k, q \in D_{i_1} \cap \cdots \cap D_{i_p}, \right\}.$$

Observe that $\Gamma(B)$ is upper semi-continuous, and $\Max \Gamma(B)$ is non-singular.

**Lemma 6.17.** Suppose that $B = (W_k, (J_k, b), E_k)$ is a monomial basic object (relative to a sequence (38)). Then $Y_k = \Max \Gamma(B)$ is a permissible center for $B$ and if $(W_{k+1}, (J_{k+1}, b), E_{k+1}) \rightarrow (W_k, (J_k, b), E_k)$ is the transformation induced by the monoidal transform $\pi_{k+1} : W_{k+1} \rightarrow W_k$ with center $Y_k$, then $B_1 = (W_{k+1}, (J_{k+1}, b), E_{k+1})$ is a monomial basic object and

$$\max \Gamma(B) > \max \Gamma(B_1)$$

in the lexicographic order.

*Proof.* $\Max \Gamma(B)$ is certainly a permissible center. We will show that $\max \Gamma(B) > \max \Gamma(B_1)$.

We will first assume that $\max - \Gamma_1 < -1$. Let $\tilde{r} = i + r$. There exist locally constant functions $d_i$ on $W$ such that $d_i(q) = 0$ if $q \notin D_i$ and $d_i$ is locally constant on $D_i$ such that

$$J_k = \mathcal{T}_{D_{\tilde{r}}}^{d_1} \cdots \mathcal{T}_{D_{\tilde{r}}}^{d_{\tilde{r}}}.$$

Let $\mathcal{D}_{\tilde{r}}$ be the strict transform of $D_{\tilde{r}}$ on $W_{k+1}$ and let $\mathcal{D}_{\tilde{r}+1}$ be the exceptional divisor of the blow up $\pi_{k+1} : W_{k+1} \rightarrow W_k$ of $Y_k = \Max \Gamma(B)$. Then

$$J_{k+1} = \mathcal{T}_{\mathcal{D}_{\tilde{r}}}^{\mathcal{D}_{\tilde{r}}} \cdots \mathcal{T}_{\mathcal{D}_{\tilde{r}}}^{\mathcal{D}_{\tilde{r}}},$$

where for $1 \leq i \leq k$, $\mathcal{D}_{\tilde{r}}(q) = d_i(\pi_{k+1}(q))$ if $q \in \mathcal{D}_{\tilde{r}}$ and $\mathcal{D}_{\tilde{r}}(q) = 0$ if $q \notin \mathcal{D}_{\tilde{r}}$. We have

$$\mathcal{D}_{\tilde{r}+1}(q) = d_i(\pi_{k+1}(q)) + \cdots + d_{i_p}(\pi_{k+1}(q)) - b$$

if $q \in \mathcal{D}_{\tilde{r}+1}$ and $\mathcal{D}_{\tilde{r}+1}(q) = 0$ if $q \notin \mathcal{D}_{\tilde{r}+1}$.

For $q \in \Sing(J_{k+1})$, $\Gamma(B_1)(q) = \Gamma(B)(\pi_{k+1}(q))$ if $\pi_{k+1}(q) \notin \Max \Gamma(B)$, so it suffices to prove that

$$\Gamma(B_1)(q) < \Gamma(B)(\pi_{k+1}(q))$$

if $q \in \Sing(J_{k+1})$ and $\pi_{k+1}(q) \in \Max \Gamma(B)$. Suppose that

$$\Gamma(B_1)(q) = (-p_1, w_1, (j_1, \ldots, j_{p_1}, 0, \ldots)).$$
and
\[ \Gamma(B)(\pi_{k+1}(q)) = \max \Gamma(B) = (-p, w, (i_1, \ldots, i_p, 0, \ldots)). \]

We will first verify that
\[ p_1 = \Gamma_1(q) \geq \Gamma_1(\pi_{k+1}(q)) = p. \] (39)

If \( \overline{k+1} \not\in \{j_1, \ldots, j_{p_1}\} \), we have
\[ b \leq \overline{d}_{j_1}(q) + \cdots + \overline{d}_{j_{p_1}}(q) \leq d_{j_1}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q)) \]
implies \( p_1 \geq p \).

Suppose that \( \overline{k+1} \in \{j_1, \ldots, j_{p_1}\} \), so that \( j_1 = k + 1 \). If \( p_1 < p \) there exists \( i_k \in \{i_1, \ldots, i_p\} \) such that \( i_k \not\in \{j_1, \ldots, j_{p_1}\} \).

\[
\begin{align*}
d_{j_1}(q) + \cdots + \overline{d}_{j_{p_1}}(q) &= \overline{d}_{j_2}(q) + \cdots + \overline{d}_{j_{p_1}}(q) + d_{j_1}(\pi_{k+1}(q)) + \cdots + d_{i_p}(\pi_{k+1}(q)) - b \leq (d_{j_1}(\pi_{k+1}(q)) + \cdots + d_{i_p}(\pi_{k+1}(q)) - b) + d_{j_1}(\pi_{k+1}(q)) + d_{j_2}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q)) \leq d_{i_k}(\pi_{k+1}(q)) + d_{j_2}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q)) < b \end{align*}
\]
a contradiction. Thus \( p_1 \geq p \).

Now assume that \( p_1 = p \). We will verify that \( w_1 \leq w \). If \( \overline{k+1} \not\in \{j_1, \ldots, j_{p_1}\} \) we have
\[
w_1 = \frac{\overline{d}_{j_1}(q) + \cdots + \overline{d}_{j_{p_1}}(q)}{b} \leq \frac{d_{j_1}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q))}{b} \leq w.
\]

If \( \overline{k+1} \in \{j_1, \ldots, j_{p_1}\} \), then \( j_1 = \overline{k+1} \). We must have
\[ d_{j_2}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q)) < b \]
so that
\[ w_1 b = (d_{j_1}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q)) - b) + \overline{d}_{j_2}(q) + \cdots + \overline{d}_{j_{p_1}}(q) \leq d_{j_1}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q)) + (d_{j_2}(\pi_{k+1}(q)) + \cdots + d_{j_{p_1}}(\pi_{k+1}(q)) - b) < w b.
\]
Thus \( w_1 < w \).

Assume that \( p = p_1 \) and \( w = w_1 \). We have shown that \( \overline{k+1} \not\in \{j_1, \ldots, j_{p_1}\} \) in this case. Since
\[ \overline{D}_{i_1} \cap \cdots \cap \overline{D}_{i_p} = \emptyset, \]
we must have \( (j_1, \ldots, j_{p_1}, 0, \ldots) < (i_1, \ldots, i_p, 0, \ldots) \).

It remains to verify the Lemma in the case when \( \max - \Gamma_1(B) = -1 \). This case is not difficult, and is left to the reader. \( \Box \)

6.4. **Basic objects and hypersurfaces of maximal contact.** Suppose that \( B = (W, (J, b), E) \) is a basic object and \( \{W^\lambda\}_{\lambda \in \Lambda} \) is a (finite) open cover of \( W \). We then have associated basic objects \( B^\lambda = (W^\lambda, (J^\lambda, b), E^\lambda) \) for \( \lambda \in \Lambda \) obtained by restriction to \( W^\lambda \). Observe that
\[ \text{Sing}(J^\lambda, b) = \text{Sing}(J, b) \cap W^\lambda. \]
If $B_1 = (W_1, (J_1, b), E_1) \to (W, (J, b), E)$ is a transformation, induced by a monoidal transform $\pi : W_1 \to W$, then there are associated transformations

$$B_1^\lambda = (W_1^\lambda, (J_1^\lambda, b), E_1^\lambda) \to (W^\lambda, (J^\lambda, b), E^\lambda)$$

where $W_1^\lambda = \pi^{-1}(W^\lambda)$, $J_1^\lambda = J_1 \mid W_1^\lambda$. In particular, $\{W_1^\lambda\}$ is an open cover of $W_1$.

**Definition 6.18.** Let $W_1 = W \times \mathbb{A}_k^1$ with projection $\pi : W_1 \to W$. Suppose that $(W, (J, b), E)$ is a basic object. Then there is a basic object $(W_1, (J_1, b), E_1)$ where $J_1 = J \mathcal{O}_{W_1}$. If $E = \{D_1, \ldots, D_r\}$ and $D_i = \pi^{-1}(D_i)$ for $1 \leq i \leq r$, $E_1 = \{D'_1, \ldots, D'_r\}$.

$(W_1, (J_1, b), E_1) \to (W, (J, b), E)$

is called a restriction.

**Remark 6.19.** Suppose that $(W, (J, b), E)$ is a basic object and $W_h \subset W$ is a non-singular hypersurface. Suppose that $\mathcal{I}_{W_h} \subset \Delta^{b-1}(J)$. Then $\text{Sing}(J, b) \subset V(\Delta^{b-1}(J)) \subset W_h$ and $\nu_q(\Delta^{b-1}(J)) = 1$ for any $q \in \text{Sing}(J, b)$.

Conversely, if $(W, (J, b), E)$ is a simple basic object, and $q \in \text{Sing}(J, b)$, then $\nu_q(\Delta^{b-1}(J)) = 1$, so there exists an affine neighborhood $U$ of $q$ in $W$ and a non-singular hypersurface $U_h \subset U$ such that $U \cap \text{Sing}(J, b) \subset U_h$.

**Lemma 6.20.** Suppose that

$$(W_1, (J_1, b), E_1) \to (W, (J, b), E)$$

is a transformation of basic objects. Let $D$ be the exceptional divisor of $W_1 \to W$. Then for all integers $i$ with $1 \leq i \leq b$ we have

$$\Delta^{b-i}(J) \mathcal{O}_{W_1} \subset \mathcal{I}_D$$

and

$$\frac{1}{I_D} \Delta^{b-i}(J) \mathcal{O}_{W_1} \subset \Delta^{b-i}(J_1).$$

**Proof.** Let $Y \subset \text{Sing}(J, b)$ be the center of $\pi : W_1 \to W$.

The first inclusion follows since $\nu_q(\Delta^{b-i}(J)) \geq i$ if $q \in Y$, so that $I_D$ divides $\Delta^{b-i}(J) \mathcal{O}_{W_1}$.

Now we will prove the second inclusion. The second inclusion is trivial if $i = b$, since $J \mathcal{O}_{W_1} = I_D J_1$ by definition. We will prove the second inclusion by descending induction on $i$. Suppose that the inclusion is true for some $i > 1$. Let $q \in D$ be a closed point, $p = \pi(q)$. Let $K' = K(q)$. By Lemma 10.16, it suffices to prove the inclusion on $W \times_K K'$, so we may assume that $K(q) = K$. Then there exist regular parameters $x_0, x_1, \ldots, x_n$ in $R = \mathcal{O}_{W, p}$ such that $I_{Y,p} = \langle x_0, x_1, \ldots, x_m \rangle$, with $m \leq n$, $x_0, x_1, \ldots, x_m, x_{m+1}, \ldots, x_n$, is a regular system of parameters in $R_1 = \mathcal{O}_{W_1, q}$, and $I_{D,q} = \langle x_0 \rangle$. It suffices to show that for $f \in \Delta^{b-(i-1)}(J)_p$,

$$\frac{f}{x_0^{i-1}} \in \Delta^{b-(i-1)}(J_1)_q.$$  \hspace{1cm} (40)

If $f \in \Delta^{b-i}(J)_p \subset \Delta^{b-(i-1)}(J)_p$ then the formula follows by induction, so we may assume that $f = Dg$ with $g \in \Delta^{b-i}(J)_p$. Set $D' = x_0 D$. $D'$ is a $R_1$-derivation on $R_1 = \mathcal{O}_{W_1, q}$ since $D'(\frac{x_i}{x_0}) \in \mathcal{O}_{W_1, q}$ for $1 \leq i \leq m$. By induction,

$$\frac{g}{x_0^i} \in \frac{1}{I_D} \Delta^{b-i}(J) \mathcal{O}_{W_1, q} \subset \Delta^{b-i}(J_1)_q \subset \Delta^{b-(i-1)}(J_1)_q.$$
Thus

\[ D'(\frac{g}{x_0^i}) \in \Delta^{b-1(i-1)}_{W_1}(J_1)_q. \]

\[ D'(\frac{g}{x_0}) = \frac{D(g)}{x_0^{i-1}} - iD(x_0) \frac{g}{x_0} \]

which implies

\[ \frac{f}{x_0^{i-1}} = \frac{D(g)}{x_0^{i-1}} = D'(\frac{g}{x_0}) + iD(x_0) \frac{g}{x_0} \in \Delta^{b-1(i-1)}_{W_1}(J_1)_q. \]

and (40) follows. \qed

**Lemma 6.21.** Suppose that \((W,(J,b),E)\) is a simple basic object, and \(W_h \subset W\) is a non-singular hypersurface such that \(I_{W_h} \subset \Delta^{b-1}(J)\). Suppose that

\[(W_1,(J_1,b),E_1) \rightarrow (W,(J,b),E)\]

is a transformation with center \(Y\). Let \((W_h)_1\) be the strict transform of \(W_h\) on \(W_1\). Then \(Y \subset W_h\), \((W_h)_1\) is a non-singular hypersurface in \(W_1\) and \(I_{(W_h)_1} \subset \Delta^{b-1}(J_1)\).

**Proof.** Let \(D\) be the exceptional divisor of \(\pi : W_1 \rightarrow W\). Suppose that \(q \in D\) is a closed point,

\(p = \pi(q) \in Y \subset \text{Sing}(J,b) = V(\Delta^{b-1}(J)) \subset W_h.\)

Recall that \(Y \subset \text{Sing}(J,b)\) be definition of transformation. Let \(f = 0\) be a local equation for \(W_h\) at \(p\), \(g = 0\) be a local equation of \(D\) at \(q\). Then \(\frac{f}{g} = 0\) is a local equation of \((W_h)_1\) at \(q\). \(\frac{f}{g} \in \Delta^{b-1}(J_1)_q\) by Lemma 6.20. Thus

\[ \text{Sing}(J_1,b) = V(\Delta^{b-1}(J_1)) \subset V(\frac{f}{g}). \]

\qed

**Definition 6.22.** Suppose that \(B = (W,(J,b),E)\) is a simple basic object and \(\tilde{W}\) is a non-singular closed subvariety of \(W\) such that \(I_{\tilde{W}} \subset \Delta^{b-1}(J)\). Then the coefficient ideal of \(B\) on \(\tilde{W}\) is

\[ C(J) = \sum_{i=0}^{b-1} \Delta^i(J) \frac{\omega}{\alpha} O_{\tilde{W}}. \]

**Lemma 6.23.** With the notation of Lemma 6.21 and Definition 6.22

\[ \text{Sing}(J,b) = \text{Sing}(C(J),b!) \subset W_h \subset W. \]

**Proof.** It suffices to check this at closed points \(q \in W_h\). By Lemma 10.16 we may assume that \(K(q) = K\). Then there exist regular parameters \(x_1, \ldots, x_n\) at \(q\) in \(W\) such that \(x_1 = 0\) is a local equation for \(W_h\) at \(q\). Let \(R = \hat{O}_{W,q} = K[[x_1, \ldots, x_n]],\)

\(S = \hat{O}_{W_{q},q} = K[[x_2, \ldots, x_n]]\) with natural projection \(R \rightarrow S\). It suffices to show that

\[ \nu_R(\hat{J}) \geq b \text{ if and only if } \nu_S(\hat{\Delta}^j(\hat{J}S)) \geq b - j \quad (41) \]

for \(j = 0, 1, \ldots, b - 1\). Suppose that \(f \in \hat{J} \subset R\). Write

\[ f = \sum_{i \geq 0} a_i x_1^i \]

Thus
with $a_i \in S$ for all $i$. We have $\nu_R(f) \geq b$ if and only if $\nu_S(a_j) \geq b - j$ for $j = 0, 1, \ldots, b - 1$.

Thus we have the only if direction of (41).

The ideal $\hat{\Delta}(\hat{J})$ is spanned by elements of the form

$$c_{i_1, \ldots, i_n} = \frac{\partial^{i_1 + \cdots + i_n} f}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}$$

with $f \in \hat{J}$ and $i_1 + \cdots + i_n \leq j$, and the projection of $c_{i_1, \ldots, i_n}$ in $S$ is

$$i_1! \frac{\partial^{i_2 + \cdots + i_n} a_{i_1}}{\partial x_2^{i_2} \cdots \partial x_n^{i_n}}.$$

Thus we have the only if direction of (41). \hfill \square

**Theorem 6.24.** Suppose that $(W, (J, b), E)$ is a simple basic object, and $(W^\lambda)_{\lambda \in \Lambda}$ is an open cover of $W$ such that for each $\lambda$ there is a non-singular hypersurface $W^\lambda_h \subset W^\lambda$ such that $I_{W^\lambda_h} \subset \hat{\Delta}^{\lambda - 1}(\hat{J}^\lambda)$ and $W^\lambda_h$ has simple normal crossings with $E^\lambda_h$. Let $E^\lambda_h$ be the set of non-singular reduced divisors on $W^\lambda_h$ obtained by intersection of $E$ with $W^\lambda_h$. Recall that $R(1)(Z)$ denotes the codimension 1 part of a closed reduced subscheme $Z$, as defined in (29).

1. If $R = R(1)(\text{Sing}(J, b)) \neq \emptyset$ then $R$ is non-singular, open and closed in $\text{Sing}(J, b)$, and has simple normal crossings with $E$. Furthermore, the monoidal transform of $W$ with center $R$ induces a transformation

$$(W_1, (J_1, b), E_1) \to (W, (J, b), E)$$

such that $R(1)(\text{Sing}(J_1, b)) = \emptyset$.

2. If $R(1)(\text{Sing}(J, b)) = \emptyset$, for each $\lambda \in \Lambda$, the basic object

$$(W^\lambda_h, (C(J^\lambda), b!)), E^\lambda_h)$$

has the following properties:

a. $\text{Sing}(J^\lambda, b) = \text{Sing}(C(J^\lambda), b!)$

b. Suppose that

$$(W_s, (J_s, b), E_s) \to (W, (J, b), E)$$

is a sequence of transformations and restrictions, with induced sequences for each $\lambda$

$$(W^\lambda_s, (J^\lambda_s, b), E^\lambda_s) \to (W^\lambda, (J^\lambda, b), E^\lambda).$$

Then for each $\lambda$ there is an induced sequence of transformations and restrictions

$$(W^\lambda_s, (C(J^\lambda)_s, b!), (E^\lambda_s)_s) \to (W^\lambda_h, (C(J^\lambda), b!), E^\lambda_h)$$

and a diagram where the vertical arrows are closed immersions of pairs

$$\begin{array}{cccc}
(W^\lambda_s, E^\lambda_s) & \to & \cdots & \to (W^\lambda, E^\lambda) \\
\uparrow & & & \uparrow \\
((W^\lambda_h)_s, (E^\lambda_h)_s) & \to & \cdots & \to (W^\lambda_h, E^\lambda_h)
\end{array}$$

such that

$$\text{Sing}(C(J^\lambda)_s, b!) = \text{Sing}(J^\lambda_h, b)$$
for all $i$.

**Proof.** We may assume that $W^\lambda = W$.

First suppose that $T = R(1)(\text{Sing}(J,b)) \neq \emptyset$. $T \subset \text{Sing}(J,b) \subset W_h$ implies $T$ is a union of connected components of $W_h$, so $T$ is non-singular, open and closed in $\text{Sing}(J,b)$ and has SNCs with $E$.

Suppose that $q \in T$. Let $f = 0$ be a local equation of $T$ at $q$ in $W$. Since $(W, (J,b), E)$ is a simple basic object, $J_q = f^j \mathcal{O}_{W,q}$. If $\pi_1 : W_1 \rightarrow W$ is the monoidal transform with center $T$ then $(J_1)_{q_1} = \mathcal{O}_{W,q_1}$ for all $q_1 \in W_1$ such that $\pi_1(q_1) \in T$, and necessarily, $R(1)(\text{Sing}(J_1, b)) = \emptyset$.

Now suppose that $R(1)(\text{Sing}(J,b)) = \emptyset$, and $(42)$ is a sequence of transformations and restrictions of simple basic objects.

Suppose that $(W_1, (J_1, b), E_1) \rightarrow (W, (J,b), E)$ is a transformation with center $Y_0$. By Lemma $6.21$, $Y_0 \subset W_h$, the strict transform $(W_h)_1$ of $W_h$ is a non-singular hypersurface in $W_1$ and

$$\mathcal{I}_{(W_h)_1} \subset \Delta^{b-1}(J_1).$$

Since $W_h$ has SNCs with $E$, $Y \subset W_h$ and $Y$ has SNCs with $E$, we have that $Y$ has SNCs with $E \cup W_h$. Thus $(W_h)_1$ has SNCs with $E_1$, and if $(E_h)_1 = E_1 \cdot (W_h)_1$ is the scheme theoretic intersection, we have a closed immersion of pairs

$$(W_h)_1, (E_h)_1 \rightarrow (W_1, E_1).$$

Since the case of a restriction is trivial, and each $(W_i, (J_i, b), E_i)$ is a simple basic object, we thus can conclude by induction that we have a diagram of closed immersions of pairs $(43)$ such that

$$\mathcal{I}_{(W_h)_j} \subset \Delta^{b-1}(J_j)$$

for all $j$.

We may assume that the sequence $(42)$ consists only of transformations. Let $D_{\mathcal{I}}$ be the exceptional divisor of $W_k \rightarrow W_{k-1}$. It remains to show that

$$\text{Sing}(C(J)_k, b! \mathcal{I}) = \text{Sing}(J_k, b)$$

for all $j$. By Lemma $10.16$ and Remark $10.18$ we may assume that $K$ is algebraically closed. For $k > 0$ and $0 \leq i \leq b - 1$, set

$$[\Delta^{b-i}(J)]_k = \frac{1}{\mathcal{I}_{D_{\mathcal{I}}}} [\Delta^{b-i}(J)]_{k-1} \mathcal{O}_{W_k}$$

so that

$$C(J)_k = \sum_{i=1}^{b} [\Delta^{b-i}(J)]_k^\mathcal{I} \mathcal{O}_{(W_k)_k}.$$

We will establish the following formulas $(46)$ and $(47)$ for $0 \leq k \leq s$ and $0 \leq i \leq b - 1$.

$$[\Delta^{b-i}(J)]_k \subset \Delta^{b-i}(J_k)$$

Suppose that $q \in \text{Sing}(C(J)_k, b!) \cap \mathcal{I}_{D_{\mathcal{I}}}$ is a closed point. Set $R_k = \hat{\mathcal{O}}_{W_k,q}$, $S_k = \hat{\mathcal{O}}_{(W_k)_k,q}$. Then there are regular parameters $(x_{k,1}, \ldots, x_{k,n}, z_k)$ in $R_k$ such that $\mathcal{I}_{(W_k)_k,q} = (z_k)$.
and there are generators \( \{ f_k \} \) of \( J_k R_k \) such that

\[
    f_k = \sum_{\alpha} a_{k,\alpha} z_k^\alpha
\]

with \( a_{k,\alpha} \in K[[x_{k1}, \ldots, x_{kn}]] = S_k \) such that for \( 0 \leq \alpha \leq b - 1 \)

\[
    (a_{k,\alpha})^m \in C(J)_k S_k.
\]

Assume that the formulas (46) and (47) hold. (46) implies \( C(J)_k \subset C(J_k) \), so by Lemma 6.23, \( \text{Sing}(J_k, b) \subset \text{Sing}(C(J)_k, b) \). Now (47) shows that \( \text{Sing}(C(J)_k, b! \) \subset \( \text{Sing}(J_k, b, b) \), so that \( \text{Sing}(C(J)_k, b!) = \text{Sing}(J_k, b) \) as desired.

We will now verify formulas (46) and (47). (46) is trivial if \( k = 0 \) and (47) with \( k = 0 \) follows from the proof of Lemma 6.23.

We now assume that (46) and (47) are true for \( k = t \), and prove them for \( k = t + 1 \). Since

\[
    \left[ \Delta^{b-i}(J) \right]_t \subset \Delta^{b-i}(J_t),
\]

we have

\[
    \left[ \Delta^{b-i}(J) \right]_t = \frac{1}{(D_{t+1}^D)^i} \left[ \Delta^{b-i}(J) \right]_t \subset \frac{1}{(D_{t+1}^D)^i} \Delta^{b-i}(J_t) \subset \Delta^{b-i}(J_{t+1}),
\]

where the last inclusion follows from the second formula of Lemma 6.20. We have thus verified formula (46) for \( k = t + 1 \).

Let \( q_{t+1} \in \text{Sing}(C(J_0)_t+1, b! \) be a closed point, \( q_t \) the image of \( q_{t+1} \) in \( \text{Sing}(C(J_0)_t, b! \).

By assumption, there exist regular parameters \( (x_{t1}, \ldots, x_{tn}, z_t) \) in \( R_t \), which satisfy (47). Recall that we have reduced to the assumption that \( K \) is algebraically closed. Let \( Y_t \) be the center of \( W_{t+1} \rightarrow W_t \). By (44), there exist regular parameters \( (\mathfrak{f}_1, \ldots, \mathfrak{f}_n) \) in \( S_t = K[[x_{t1}, \ldots, x_{tn}]] \subset R_t \) such that \( (\mathfrak{f}_1, \ldots, \mathfrak{f}_n, z_t) \) are regular parameters in \( R_t \), there exists \( r \) with \( r \leq n \) such that \( \mathfrak{f}_1 = \cdots = \mathfrak{f}_r = z_t = 0 \) are local equations of \( Y_t \) in \( \mathcal{O}_{W_t, q_t} \) and \( R_{t+1} = \mathcal{O}_{W_{t+1}, q_{t+1}} \) has regular parameters \( (x_{t+1,1}, \ldots, x_{t+1,n}, z_{t+1}) \) such that

\[
    \frac{x_{t+1,1}}{x_{t+1,i}} = x_{t+1,1} \quad \text{for } 2 \leq i \leq r
\]

\[
    \frac{x_{t+1,i}}{x_{t+1,i}} = x_{t+1,1} \quad \text{for } r < i \leq n.
\]

In particular, \( x_{t+1,1} = 0 \) is a local equation of the exceptional divisor \( D_{t+1} \) and \( z_{t+1} = 0 \) is a local equation of \( (W_h)_t+1 \). We have generators

\[
    f_{t+1} = \frac{f_t}{x_{t+1,1}^\alpha} = \sum_{\alpha} a_{t+1,\alpha} z_{t+1}^\alpha
\]

of \( J_{t+1} R_{t+1} = \frac{1}{x_{t+1,1}} J_t R_{t+1} \) such that

\[
    a_{t+1,\alpha} = \frac{a_{t,\alpha}}{x_{t,1}^\alpha}
\]

if \( \alpha < b \), and

\[
    (a_{t+1,\alpha})^{\frac{m}{n}} = \left( \frac{1}{(x_{t,1})^m} a_{t,\alpha} \right)^{\frac{m}{n}} \in \frac{1}{(D_{t+1}^D)^m} C(J)_t S_{t+1} = C(J)_{t+1} S_{t+1}.
\]

□
6.5. General basic objects.

Definition 6.25. Suppose that \((W_0, E_0)\) is a pair. We define a general basic object (GBO) \((\mathcal{F}_0, W_0, E_0)\) on \((W_0, E_0)\) to be a collection of pairs \((W_i, E_i)\) with closed sets \(F_i \subset W_i\) which have been constructed inductively to satisfy the following three properties:

1. \(F_0 \subset W_0\)
2. Suppose that \(F_i \subset W_i\) has been defined for the pair \((W_i, E_i)\). If \(\pi_{i+1} : (W_{i+1}, E_{i+1}) \to (W_i, E_i)\) is a restriction then \(F_{i+1} = \pi_{i+1}(F_i)\)
3. Suppose that \(F_i \subset W_i\) has been defined for the pair \((W_i, E_i)\). If \(\pi_{i+1} : (W_{i+1}, E_{i+1}) \to (W_i, E_i)\) is a transformation centered at \(Y_i \subset F_i\) then \(F_{i+1} \subset W_{i+1}\) is a closed set such that \(F_i - Y_i \cong F_{i+1} - \pi_{i+1}(Y_i)\) by the map \(\pi_{i+1}\).

The closed sets \(F_i\) will be called the closed sets associated to \((\mathcal{F}_0, W_0, E_0)\).

If

\[(W_1, E_1) \to (W_0, E_0)\]

is a transformation with center \(Y_0 \subset F_0\) then we can use the pairs \((W_i, E_i)\) and closed subsets \(F_i \subset W_i\) in Definition 6.25 which map to \((W_1, E_1)\) by a sequence of restrictions and transformations satisfying 2. and 3. to define a general basic object \((\mathcal{F}_1, W_1, E_1)\) over \((W_1, E_1)\). We will say that \(Y_0\) is a permissible center for \((\mathcal{F}_0, W_0, E_0)\) and call

\[(\mathcal{F}_1, W_1, E_1) \to (\mathcal{F}_0, W_0, E_0)\]

a transformation of general basic objects.

If

\[(W_1, E_1) \to (W_0, E_0)\]

is a restriction then we can use the pairs \((W_i, E_i)\) and closed subsets \(F_i \subset W_i\) defined in Definition 6.25 above which map to \((W_1, E_1)\) by a sequence of restrictions and transformations satisfying (2) and (3) to define a general basic object \((\mathcal{F}_1, W_1, E_1)\) over \((W_1, E_1)\). We will call

\[(\mathcal{F}_1, W_1, E_1) \to (\mathcal{F}_0, W_0, E_0)\]

a restriction of general basic objects.

A composition of restrictions and transformations of general basic objects

\[
(\mathcal{F}_r, W_r, E_r) \to \cdots \to (\mathcal{F}_0, W_0, E_0)
\]

will be called a permissible sequence for the GBO \((\mathcal{F}_0, W_0, E_0)\). (48) will be called a resolution if \(F_r = 0\).

Definition 6.26. Suppose that \((W_0, E_0)\) is a pair, \((\mathcal{F}_0, W_0, E_0)\) is a GBO over \((W_0, E_0)\), with associated closed sets \(\{F_i\}\). Further, suppose that \(\{W_0^\lambda\}_{\lambda \in \Lambda}\) is a finite open covering of \(W_0\) and \(d \in \mathbb{N}\).

Suppose that we have a collection of basic objects \(\{B_0^\lambda\}_{\lambda \in \Lambda}\) with

\[B_0^\lambda = (\tilde{W}_0^\lambda, (a_0^\lambda, b_0^\lambda), \tilde{E}_0^\lambda).\]

We will say that \(\{B_0^\lambda\}_{\lambda \in \Lambda}\) is a \(d\)-dimensional structure on the GBO \((\mathcal{F}_0, W_0, E_0)\) if for each \(\lambda \in \Lambda\)

1. we have an immersion of pairs

\[j_0^\lambda : (\tilde{W}_0^\lambda, \tilde{E}_0^\lambda) \to (W_0^\lambda, E_0^\lambda)\]

where \(\tilde{W}_0^\lambda\) has dimension \(d\).
2. If 

$$(F_r, W_r, E_r) \rightarrow \cdots \rightarrow (F_0, W_0, E_0)$$

is a permissible sequence for the GBO $(F_0, W_0, E_0)$ (with associated closed sets $F_i \subset W_i$ for $0 \leq i \leq r$), then the sequence of morphisms of pairs 

$$(W_r^\lambda, E_r^\lambda) \rightarrow \cdots \rightarrow (W_0^\lambda, E_0^\lambda)$$

induce diagrams of immersions of pairs 

$$(W_r^\lambda, E_r^\lambda) \rightarrow \cdots \rightarrow (W_0^\lambda, E_0^\lambda)$$

such that 

$$(\tilde{W}_r^\lambda, (a_r^\lambda, b_r^\lambda), \tilde{E}_r^\lambda) \rightarrow \cdots \rightarrow (\tilde{W}_0^\lambda, (a_0^\lambda, b_0^\lambda), \tilde{E}_0^\lambda)$$

is a sequence of transformations and restrictions of basic objects, and 

$$F_i^\lambda = F_i \cap W_i^\lambda = \text{Sing}(a_i^\lambda, b_i^\lambda)$$

for $0 \leq i \leq r$.

6.6. Functions on a general basic object.

**Proposition 6.27.** Let $(F_0, W_0, E_0)$ be a GBO which admits a $d$-dimensional structure, and fix notation as in Definition 6.26. Let $F_0^\lambda = \text{Sing}(a_0^\lambda, b_0^\lambda)$.

There is a function $\text{ord}^d : F_0 \rightarrow \mathbb{Q}$ such that for all $\lambda \in \Lambda$, the composition

$$F_0^\lambda \xrightarrow{j_0^\lambda} F_0 \xrightarrow{\text{ord}^d} \mathbb{Q}$$

is the function $\text{ord}^d : F_0^\lambda \rightarrow \mathbb{Q}$ defined by Definition 6.13 for $F_0^\lambda = \text{Sing}(a_0^\lambda, b_0^\lambda)$.

**Proof.** Let $\{B_0^\lambda\}_{\lambda \in \Lambda}$ where $B_0^\lambda = (\tilde{W}_0^\lambda, (a_0^\lambda, b_0^\lambda), \tilde{E}_0^\lambda)$ be the given $d$-dimensional structure on $(F_0, W_0, E_0)$. Let $E_0 = \{D_1, \ldots, D_r\}$. We have closed immersions $j_0^\lambda : \tilde{W}_0^\lambda \rightarrow W_0^\lambda$ for all $\lambda \in \Lambda$.

Suppose that $q_0 \in F_0$ is a closed point, and $\lambda, \beta \in \Lambda$ are such that there are $q_0^\lambda \in W_0^\lambda$ and $q_0^\beta \in \tilde{W}_0^\beta$ such that $j_0^\lambda(q_0^\lambda) = j_0^\beta(q_0^\beta) = q_0$. We must show that

$$\frac{\nu_{q_0}(a_0^\delta)}{b_0^\lambda} = \frac{\nu_{q_0}(a_0^\beta)}{b_0^\beta}.$$

We will prove this by expressing these numbers intrinsically, in terms of the GBO $(F_0, W_0, E_0)$, so that they are independent of $\lambda$. We will carry out the calculation for $B_0^\lambda$.

Let $(F_1, W_1, E_1) \xrightarrow{\pi_1} (F_0, W_0, E_0)$ be the restriction where $W_1 = W_0 \times A_1^\lambda$. Let $q_1 = (q_0, 0) \in W_1$. Let $L_1 = q_0 \times A_1^\lambda$. Let $L_1 \subset F_1 = \pi_1^{-1}(F_0)$.

For any integer $N > 0$ we can construct a permissible sequence of pairs

$$(W_N, E_N) \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_2} (W_1, E_1) \xrightarrow{\pi_1} (W_0, E_0)$$

where for $i \geq 1$, $W_{i+1} \rightarrow W_i$ is the blow up of a point $q_i$. $q_i$ is defined as follows. Let $L_i$ be the strict transform of $L_1$ on $W_i$, and let $D_i$ be the exceptional divisor of $\pi_i$, where $i = i + r$. Then $q_i = L_i \cap D_i$. 


We have \( q_1 \in L_1 \subset F_1 \). We must have \( L_2 \subset F_2 \) by 3. of Definition 6.25. Thus \( \pi_2 \) induces a transformation of GBOs \((\mathcal{F}_2, W_2, E_2) \to (\mathcal{F}_1, W_1, E_1)\) and \( q_2 \in F_2 \). By induction, we have for any \( N > 0 \) a permissible sequence

\[
(\mathcal{F}_N, W_N, E_N) \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_3} (\mathcal{F}_1, W_1, E_1) \xrightarrow{\pi_1} (\mathcal{F}_0, W_0, E_0)
\]

(50) induces a permissible sequence of basic objects with \( q_1^\lambda = (q_0^\lambda, 0) \) and centers \( q_i^\lambda \) for \( i \geq 1 \)

\[
(W_1^\lambda, (a_0^\lambda, b^\lambda), E_1^\lambda) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_3} (W_0^\lambda, (a_0^\lambda, b^\lambda), E_0^\lambda).
\]

(51)

Observe that \( L_i \subset \hat{W}_i^\lambda \) for all \( i \geq 1 \) so that \( q_i^\lambda \in \hat{W}_i^\lambda \) for all \( i \geq 1 \). Set \( b' = \nu_0(a_0^\lambda) \).

Suppose that \((x_1, \ldots, x_d)\) are regular parameters at \( q_0^\lambda \) in \( W_0^\lambda \). Then \((x_1, \ldots, x_d, t)\) are regular parameters at \( q_1^\lambda \) in \( \hat{W}_1^\lambda \) (where \( \mathbb{A}_k^N = \text{spec}(K[t]) \)), and thus there are regular parameters \((x_1(N), x_2(N), \ldots, x_d(N), t)\) in \( \hat{W}_N^\lambda \) at \( q_1^\lambda \) which are defined by \( x_i = x_i(N)t^{N-1} \) for \( 1 \leq i \leq d \). \( t = 0 \) is a local equation of the exceptional divisor \( \hat{D}_N^\lambda \) of \( \hat{W}_N^\lambda \rightarrow \hat{W}_{N-1}^\lambda \).

Suppose that \( f \in \mathcal{O}_{\hat{W}_N^\lambda, q_0^\lambda} \) and \( \nu_0(f) = r \). Let \( K' = K(q_0^\lambda) \). We have an expression

\[
f = \sum_{i_1, \ldots, i_d \geq r} a_{i_1, \ldots, i_d} x_1^{i_1} \cdots x_d^{i_d} \in K'[x_1, \ldots, x_d] = \mathcal{O}_{\hat{W}_N^\lambda, q_0^\lambda}
\]

where \( a_{i_1, \ldots, i_d} \in K' \). Thus in \( K'[x_1(N), \ldots, x_d(N), t] = \mathcal{O}_{\hat{W}_N^\lambda, q_0^\lambda} \) we have an expansion

\[
f = \sum_{i_1, \ldots, i_d \geq r} a_{i_1, \ldots, i_d} x_1^{i_1(N)} \cdots x_d^{i_d(N)} t^{(i_1 + \cdots + i_d)(N-1)} = t^{(N-1)r} \sum_{i_1, \ldots, i_d = r} a_{i_1, \ldots, i_d} x_1^{i_1(N)} \cdots x_d^{i_d(N)} t^{i_1 + \cdots + i_d}
\]

Thus we have an expression \( f = t^{(N-1)r} f_N \) in \( \mathcal{O}_{\hat{W}_N^\lambda, q_0^\lambda} \), where \( t \not| f_N \).

Since \( \nu_0(a_0) = b' \)

\[
a_0^\lambda \mathcal{O}_{\hat{W}_N^\lambda} = I_{\hat{D}_N^\lambda}^{(N-1)b'} K_N^\lambda
\]

where \( I_{\hat{D}_N^\lambda} \not| K_N^\lambda \). We thus have an expression

\[
a_N^\lambda = I_{\hat{D}_N^\lambda}^{(N-1)(b' - b^\lambda)} K_N^\lambda
\]

(52)

and \( \nu_0^\lambda(a_N^\lambda) \geq b^\lambda \) since \( q_{N, \lambda} \in F_N \cap \hat{W}_N^\lambda = \text{Sing}(a_N^\lambda, b^\lambda) \).

Under the closed immersion \( \hat{W}_N^\lambda \subset W_N^\lambda \) we have for \( N \geq 1 \)

\[
\hat{D}_N^\lambda = D_N \cdot \hat{W}_N^\lambda,
\]

where \( \hat{D}_N^\lambda \) has dimension \( d \) and is irreducible. Since \( F_N^\lambda \subset \hat{W}_N^\lambda \) it follows that

\[
\dim (F_N \cdot D_N) = \dim (F_N^\lambda \cdot D_N) \leq \dim \hat{D}_N^\lambda = d
\]

and the following three conditions are equivalent

1. \( \dim (F_N \cdot D_N) = \dim (F_N^\lambda \cdot D_N) = d \)
2. \( \hat{D}_N^\lambda \subset F_N^\lambda \)
3. \((N - 1)(b' - b^\lambda) \geq b^\lambda \)
Thus $b'/a' = 1$ holds if and only if $b' - b^\lambda = 0$, which holds if and only if for any $N$ \( \dim(F_N \cdot D_N) < d \). Thus the condition

$$\frac{\nu_{a_i^\lambda}(a^\lambda_0)}{b^\lambda} = \frac{b'}{b^\lambda} = 1$$

is independent of $\lambda$.

Now assume that $b' - b^\lambda > 0$. Then for $N$ sufficiently large, $(N - 1)(b - b^\lambda) \geq b^\lambda$, and $F_N \cdot D_N = \tilde{D}^\lambda_N$ is a permissible center of dimension $d$ for $(F_N, W_n, E_N)$.

We now define an extension of (50) by a sequence of transformations

$$(F_{N+S}, W_{N+S}, E_{N+S}) \xrightarrow{\pi_{N+S}^i} \cdots \xrightarrow{\pi_N^{i+1}} (F_N, W, E_N) \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_1} (F_0, W_0, E_0)$$

(53)

where $\pi_{N+i}$ is the transformation with center $Y_{N+i} = F_{N+i} \cdot \tilde{D}^\lambda_{N+i}$, and we assume that $\dim Y_{N+i} = d$ for $i = 0, 1, \ldots, S - 1$. Observe that

$$\dim Y_{N+i} = d \text{ if and only if } \tilde{D}^\lambda_{N+i} = D^\lambda_{N+i} \cdot \tilde{W}^\lambda_{N+i} \subset F_{N+i}.$$  

(54)

We thus have $Y_{N+i} = \tilde{D}^\lambda_{N+i}$. (54) induces an extension of (51) by a sequence of transformations

$$(\tilde{W}^\lambda_{N+S}, (a^\lambda_{N+S}, b^\lambda), \tilde{E}_{N+S}) \xrightarrow{\pi_{N+S}^i} \cdots \xrightarrow{\pi_N^{i+1}} (\tilde{W}^\lambda_N, (a^\lambda_N, b^\lambda), \tilde{E}_N) \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_1} (\tilde{W}^\lambda_0, (a^\lambda_0, b^\lambda), \tilde{E}_0)$$

(55)

The centers of $\pi_i$ in (54) are $\tilde{D}^\lambda_j$ for $j = N, \ldots, N + S - 1$. Thus $\tilde{W}_{N+i+1} \rightarrow \tilde{W}_{N+i}$ is the identity for $i = 0, \ldots, S - 1$.

We see then that

$$a^\lambda_{N+i} = \pi(N-1)(b'-b^\lambda)-ib^\lambda K_N^\lambda$$

for $i = 1, \ldots, S$. The statement that the extension (55) is permissible is precisely the statement that $(N - 1)(b' - b^\lambda) \geq Sb^\lambda$, which is equivalent to the statement that $\dim F_{N+i} \cdot D_{N+i} = d$ for $i = 0, 1, \ldots, S - 1$. This last statement is equivalent to

$$S \leq \ell_N = \left\lfloor \frac{(N - 1)(b' - b^\lambda)}{b^\lambda} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the greatest integer in $x$. Thus $\ell_N$ is determined both by $N$ and by the GBO $(F_0, W_0, E_0)$.

$$\lim_{N \to \infty} \frac{\ell_N}{N - 1} = \frac{b'}{b^\lambda} - 1$$

is defined in terms of the GBO, hence is independent of $\lambda$.

\[\square\]

**Theorem 6.28.** Let $(F_0, W_0, E_0)$ be a GBO which admits a $d$-dimensional structure, and fix notation as in Definition 6.26. Let $F_i$ be the closed sets associated to the GBO. Let $F_i^\lambda = \text{Sing}(a^\lambda_i, b^\lambda)$, and $\text{ord}_i^\lambda$, $t_i^\lambda$ be the functions of Definition 6.13 defined for $F_i^\lambda$.

Consider any permissible sequence of transformations

$$(F_r, W_r, E_r) \xrightarrow{\pi_r} \cdots \xrightarrow{\pi_1} (F_0, W_0, E_0).$$
1. Then for $1 \leq i \leq r$, there are functions $w$-ord${}_{i}^{d_{i}} : F_{i} \to \mathbb{Q}$ such that the composition

$$F_{i}^{\lambda} \xrightarrow{\lambda_{i}} F_{i} \xrightarrow{w$-ord${}_{i}^{d_{i}}} \mathbb{Q}$$

is the function $w$-ord${}_{i}^{\lambda} : F_{i}^{\lambda} \to \mathbb{Q}$.

2. Suppose that the centers $Y_{i}$ of $\pi_{i+1}$ are such that $Y_{i} \subset \text{Max } w$-ord${}_{i}^{d_{i}}$. Then there are functions $t_{i}^{\lambda} : F_{i} \to \mathbb{Q} \times \mathbb{Z}$ such that the composition

$$F_{i}^{\lambda} \xrightarrow{t_{i}^{\lambda}} F_{i} \xrightarrow{t_{i}} \mathbb{Q} \times \mathbb{Z}$$

is the function $t_{i}^{\lambda} : F_{i}^{\lambda} \to \mathbb{Q} \times \mathbb{Z}$.

Proof. Let $E_{0} = \{D_{1}, \ldots, D_{s}\}$. The functions $t_{i}$ are completely determined by $w$-ord${}_{i}^{d_{i}}$, with the constraint that $Y_{i} \subset \text{Max } w$-ord${}_{i}^{d_{i}}$ for all $i$, so we need only show that $w$-ord${}_{i}^{d_{i}}$ can be defined for GBOs with $d$-dimensional structure.

The functions $w$-ord${}_{i}^{d_{i}}$ are equal to ord${}_{i}^{d_{i}}$, which extend to ord${}_{0}$ on $F_{0}$ by Proposition 6.27. We thus have defined $w$-ord${}_{0} = \text{ord}_{0}$. Let $D_{i}$ be the exceptional divisor of $\pi_{i}$, where $i = i + s$, $\bar{D}_{i}^{\lambda} = W_{i}^{\lambda} \cdot D_{i}$.

For each $\lambda$ and $i$ we have expressions

$$a_{i}^{\lambda}_{j} = \prod_{D_{i}^{\lambda}} T_{D_{i}^{\lambda}} \cdot \cdots \cdot T_{D_{i}^{\lambda}} a_{i}^{\lambda}_{j}$$

where $\pi_{i}^{\lambda}$ is the weak transform of $a_{i}^{\lambda}$ on $W_{i}^{\lambda}$ and $c_{j}$ are locally constant functions on $\bar{D}_{j}^{\lambda}$. We further have that

$$a_{i}^{\lambda} = \frac{1}{T_{\bar{D}_{i}^{\lambda}}} a_{i-1}^{\lambda} \mathcal{O}_{W_{i}^{\lambda}}.$$
Suppose that $\lambda \in \Lambda$ is such that $q'_j \in W^\lambda_i$. We have associated points $q_j$ and $\eta_j$ in $\tilde{W}^\lambda_j$ for $0 \leq j \leq i$.

We thus have sequences of points

$$\{q_0, q_1, \ldots, q_i\}$$

and

$$\{q'_0, q'_1, \ldots, q'_i\}$$

with $q_j, \eta_j \in \tilde{W}^\lambda_j$ for $0 \leq j \leq i$.

From equation (58) and induction we see that the values

$$c^\lambda_j(q_j) \frac{b^\lambda_j}{b^\lambda} \text{ for } j = 1, \ldots, i (59)$$

and $w\text{-ord}^\lambda_i(q_i)$ are determined by

1. The rational numbers

$$\text{ord}^d_0(q'_0), \text{ord}^d_1(q'_1), \ldots, \text{ord}^d_i(q'_i),$$

$$\text{ord}^d_0(\eta'_0), \text{ord}^d_1(\eta'_1), \ldots, \text{ord}^d_{i-1}(\eta'_{i-1}) (60)$$

2. The functions

$$YH (q'_s, j) = \begin{cases} 1 & \text{if } q'_s \in Y_s \text{ and } Y_s \subset D_{q} \text{ locally at } q_s \\ 0 & \text{if } q'_s \not\in Y_s \text{ or } Y_s \not\subset D_{q} \text{ locally at } q_s \end{cases} (62)$$

Our theorem now follows from Proposition 6.27, since $w\text{-ord}^\lambda_i(q_i)$ is completely determined by the data (60), (61) and (62).

\section*{Proposition 6.29.}

Let $B_0 = (F_0, W_0, E_0)$ be a GBO which admits a d-dimensional structure. Fix notation as in Definition 6.26. Let $F^\lambda_i = \text{Sing}(a^\lambda_i, b^\lambda_i)$.

Consider a permissible sequence of transformations

$$B_r = (F_r, W_r, E_r) \xrightarrow{\pi_r} \ldots \xrightarrow{\pi_1} (F_0, W_0, E_0)$$

such that $\max w\text{-ord}_r = 0$. Then there are functions

$$\Gamma_r = \Gamma_r(B_r) : F_r \to I_M = \mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}^N$$

such that the composition

$$F^\lambda_r \xrightarrow{\text{def}} F_r \xrightarrow{\Gamma_r} I_M$$

is the function

$$\Gamma = \Gamma(\tilde{W}^\lambda_r, (a^\lambda_r, b^\lambda_r), E^\lambda_r) : F^\lambda_r \to I_M$$

of Definition 6.16.

\begin{proof}

Suppose that $E_0 = \{D_1, \ldots, D_s\}$. The function $\Gamma$ on the basic objects $(\tilde{W}^\lambda_r, (a^\lambda_r, b^\lambda_r), E^\lambda_r)$ depends only on the values of $\tilde{W}^\lambda_i$ for $1 \leq i \leq r$ (with the notation of the proof of Theorem 6.28). Thus the conclusions follow from (59) of the proof of Theorem 6.28.
\end{proof}
6.7. Resolution theorems for a general basic object.

Theorem 6.30. Let \((\mathcal{F}_0^d, W_0, E_0^d)\) be a GBO with associated closed sets \(\{F_i^d\}\) which admits a \(d\)-dimensional structure. Fix notation as in Definition 6.26. Let \(F_i^\lambda = \text{Sing}(a_i^\lambda, b_i^\lambda) = F_i^d \cap W_i^\lambda\).

Consider a permissible sequence of transformations
\[
(\mathcal{F}_r^d, W_r, E_r^d) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_i} (\mathcal{F}_0^d, W_0, E_0^d)
\]
such that the center \(Y_i\) of \(\pi_{i+1}\) satisfies
\[
Y_i \subset \text{Max } t_i \subset \text{Max w-ord}_i \subset F_i
\]
for \(i = 0, 1, \ldots, r - 1\) and \(\text{max w-ord}_r > 0\).

This condition implies \(\text{w-ord}_r(q) \leq \text{w-ord}_{r+1}(\pi_i(q)), t_i(q) \leq t_{i-1}(\pi_i(q))\) for \(q \in F_i\) and \(1 \leq i \leq r\). Thus
\[
\max t_0 \geq \max t_1 \geq \cdots \geq \max t_r.
\]

Let \(R(1)(\text{Max } t_r)\) be the set of points where \(\text{Max } t_r\) has dimension \(d - 1\). Then \(R(1)\) is open and closed in \(\text{Max } t_r\) and non-singular in \(W_r\).

1. If \(R(1)(\text{Max } t_r) \neq \emptyset\) then \(R(1)\) is a permissible center for \((\mathcal{F}_r^d, W_r, E_r^d)\) with associated transformation
\[
(\mathcal{F}_r^d, W_r, E_r^d) \xrightarrow{\pi_{r+1}} (\mathcal{F}_r^d, W_r, E_r^d)
\]
and either \(R(1)(\text{Max } t_{r+1}) = \emptyset\) or \(\text{max } t_r > \text{max } t_{r+1}\).

2. If \(R(1)(\text{Max } t_r) = \emptyset\), there is a GBO \((\mathcal{F}_r^{d-1}, W_r, E_r^{d-1})\) with \((d-1)\)-dimensional structure and associated closed sets \(\{F_i^{d-1}\}\) such that if
\[
(\mathcal{F}_N^{d-1}, W_N, E_N^{d-1}) \rightarrow \cdots \rightarrow (\mathcal{F}_r^{d-1}, W_r, E_r^{d-1})
\]
is a resolution \((F_r^{d-1}) = \emptyset\) then \((64)\) induces a permissible sequence of transformations
\[
(\mathcal{F}_N^d, W_N, E_N^d) \rightarrow \cdots \rightarrow (\mathcal{F}_r^d, W_r, E_r^d)
\]
such that the center \(Y_i\) of \(\pi_{i+1}\) satisfies \(Y_i \subset \text{Max } t_i \subset \text{Max w-ord}_i \subset F_i^d\) for \(r \leq i \leq N - 1\), \(\max t_r = \cdots = \max t_{N-1}\), \(\text{Max } t_i = F_i^{d-1}\) for \(i = r, \ldots, N - 1\) and one of the following holds
\begin{align*}
\text{a.} & \quad F_N^d = \emptyset \\
\text{b.} & \quad F_N^d \neq \emptyset \text{ and } \text{max } \text{w-ord}_N = 0 \\
\text{c.} & \quad F_N^d \neq \emptyset, \text{ max } \text{w-ord}_N > 0 \text{ and } \text{max } t_{N-1} > \text{max } t_N.
\end{align*}

We will now prove Theorem 6.30. Let \(E_0 = \{D_1, \ldots, D_s\}\) Let \(D_{\bar{r}}\) be the exceptional divisor of \(\pi_{\bar{r}}\) where \(\bar{r} = s + i\). Set \(b_{\bar{r}}^\lambda = b^\lambda(\text{max w-ord}_{\bar{r}}) > 0\). Let \(r_0\) be the smallest integer such that \(\text{max w-ord}_{r_0} = \text{max w-ord}_r\) in \((63)\).

There exists an open cover \(\{W_0^\lambda\}\) of \(W_0\) and a \(d\)-dimensional structure \(\{B_0^\lambda\}\) of \((\mathcal{F}_0, W_0, E_0)\) where \(B_0^\lambda = (W_0^\lambda, (a_0^\lambda, b_0^\lambda), E_0^\lambda)\) for \(\lambda \in \Lambda\). Thus for \(\lambda \in \Lambda\), we have sequences of transformations
\[
(\mathcal{F}_r^d, (a_r^\lambda, b_r^\lambda), E_r^\lambda) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_i} (\mathcal{F}_0^d, (a_0^\lambda, b_0^\lambda), E_0^\lambda)
\]
where the center $Y^\lambda_i$ of $\tilde{W}_i^\lambda$ is contained in $\text{Max } t_i$. There is an expression
$$a^\lambda_r = D^\lambda_1 \cdots D^\lambda_p.$$We now define a basic object
$$(B'_r)^\lambda = (\tilde{W}_r^\lambda, ((a'_r)^\lambda, (b')^\lambda), \tilde{E}_r^\lambda)$$
on $\tilde{W}_r^\lambda$ by
$$(a'_r)^\lambda = \left\{ \begin{array}{ll}
\pi_r^\lambda & \text{if } b^\lambda_r \geq b^\lambda \\
(\pi_r^\lambda)^{b^\lambda - b^\lambda_r} + (D^\lambda_1 \cdots D^\lambda_p)^{b^\lambda_r} & \text{if } b^\lambda_r < b^\lambda,
\end{array} \right.$$
(65)

**Lemma 6.31.** 1. $\text{Sing}(a'_r)^\lambda, (b')^\lambda = (\text{Max } w-\text{ord}_r) \cap W_r^\lambda$.
2. $(\tilde{W}_r^\lambda, ((a'_r)^\lambda, (b')^\lambda), \tilde{E}_r^\lambda)$ is a simple basic object.
3. A transformation
$$(\tilde{W}_{r+1}^\lambda, ((a'_{r+1})^\lambda, (b')^\lambda), \tilde{E}_{r+1}) \to (\tilde{W}_r^\lambda, ((a'_r)^\lambda, (b')^\lambda), \tilde{E}_r^\lambda)$$
with center $Y_r^\lambda$ induces a transformation
$$(\tilde{W}_{r+1}^\lambda, (a_{r+1}^\lambda, b_r^\lambda), \tilde{E}_{r+1}) \to (\tilde{W}_r^\lambda, (a_r^\lambda, b_r^\lambda), \tilde{E}_r^\lambda)$$
such that the center $\tilde{Y}_r^\lambda \subset \text{Max } w-\text{ord}_r \cap \tilde{W}_r^\lambda$.
4. $\text{Sing}(a'_{r+1})^\lambda, (b')^\lambda = \emptyset$ if and only if one of the following holds:
   a. $\text{max } w-\text{ord}_r > \text{max } w-\text{ord}_{r+1}$
   b. $\text{max } w-\text{ord}_r = \text{max } w-\text{ord}_{r+1}$ and $(\text{Max } w-\text{ord}_r) \cap W_r^\lambda = \emptyset$.
5. If $\text{max } w-\text{ord}_r = \text{max } w-\text{ord}_{r+1}$ then
$$\text{Sing}(a'_{r+1})^\lambda, (b')^\lambda = \text{Max } w-\text{ord}_{r+1} \cap \tilde{W}_{r+1}^\lambda$$
and $(a'_{r+1})^\lambda$ is defined in terms of $a_{r+1}^\lambda$ by (65).

**Proof.** 1. For $q \in \tilde{W}_r^\lambda$, $q \in \text{Max } w-\text{ord}_r$ if and only if $\nu_q(\pi_r^\lambda) \geq b_r^\lambda$ and $\mu_q(a^\lambda_0) \geq b_r^\lambda$. These conditions hold if and only if $\nu_q((a')^\lambda) \geq (b')^\lambda$.
2. follows since $\nu_q(\pi_r^\lambda) \leq b_r^\lambda$ for $q \in \tilde{W}_r^\lambda$.
3. is immediate from 1.
If $b^\lambda_r \geq b^\lambda$, we have
$$\pi_{r+1}^\lambda = (D^\lambda_{r+1}) {\pi_r^\lambda} = D^\lambda_{r+1} (a'_r)^\lambda \equiv (a'_{r+1})^\lambda.$$ (66)
Suppose that $b^\lambda_r < b^\lambda$. Then
$$\left(\pi_{r+1}^\lambda\right)^{b^\lambda - b_r^\lambda} + \left(D^\lambda_{r+1}\right)^{b_r^\lambda}$$
$$= (D^\lambda_{r+1}) {\pi_r^\lambda}^{b^\lambda - b_r^\lambda} + (D^\lambda_{r+1})^{b_r^\lambda}$$
(67)
where
$$c^\lambda_{r+1}(q) = \sum_{j=1}^r c_j(\eta) \nu_q(D^\lambda_j) + b_r^\lambda - b^\lambda.$$
if \( \eta \) is the generic point of the component of \( \tilde{Y}_r^\lambda \) containing \( \pi_{r+1}(q) \).

If we set \( K = \mathcal{I}_{D_1}^\lambda \cdots \mathcal{I}_{D_N}^\lambda \) (the ideal sheaf on \( \tilde{W}_r^\lambda \)) we see that (67) is equal to

\[
(\pi_r + K)\mathcal{I}_{D_{\max}}^{(\lambda - b_0)} = (a_{r+1}')^\lambda. \tag{68}
\]

Now 4. and 5. follow from 1. and equations (66), (67) and (68).

At \( q \in \max t_r \cap \tilde{W}_r^\lambda \) there exist \( N = \eta(q) \) hypersurfaces \( D_{i_1}^\lambda, \ldots, D_{i_N}^\lambda \in (\tilde{E}_r^-)^\lambda \) such that

\[q \in D_{i_1}^\lambda \cap \cdots \cap D_{i_N}^\lambda \cap \max w-\ord_r \neq \emptyset.\]

Thus there exists an affine neighborhood \( U_{\lambda,\phi} \) of \( q \) such that

\[
\max t_r \cap U_{\lambda,\phi} = D_{i_1}^\lambda \cap \cdots \cap D_{i_N}^\lambda \cap \max w-\ord_{\lambda,\phi} \tag{69}
\]

(The superscript \( \lambda, \phi \) denotes restriction to \( U_{\lambda,\phi} \)). Now define a basic object

\[(B_r')^{\lambda,\phi} = (U_r^\lambda, ((a_{r}')^{\lambda,\phi}, (b')^{\lambda,\phi}), (\tilde{E}_r^+)^{\lambda,\phi})\]

by

\[
(a_{r}')^{\lambda,\phi} = (a_r')^{\lambda,\phi} + (\mathcal{I}_{D_{i_1}}^{\lambda,\phi})(b_r')^{\lambda,\phi} + \cdots + (\mathcal{I}_{D_{i_N}}^{\lambda,\phi})(b_r')^{\lambda,\phi} \tag{70}
\]

Lemma 6.32. \hspace{1cm} 1. \( \text{Sing}(a_{r}')^{\lambda,\phi}, (b')^{\lambda,\phi} = \max t_r \cap U_{\lambda,\phi} \).

2. \( (U_r^\lambda, ((a_{r}')^{\lambda,\phi}, (b')^{\lambda,\phi}), (\tilde{E}_r^+)^{\lambda,\phi}) \) is a simple basic object.

3. A transformation

\[(U_{r+1}^\lambda, ((a_{r+1}')^{\lambda,\phi}, (b')^{\lambda,\phi}), (\tilde{E}_{r+1}^+)^{\lambda,\phi}) \rightarrow (U_r^\lambda, ((a_r')^{\lambda,\phi}, (b')^{\lambda,\phi}), (\tilde{E}_r^+)^{\lambda,\phi}) \]

with center \( Y_{r+1}^{\lambda,\phi} \) induces a transformation

\[(U_r^\lambda, (a_r', (b')^\lambda, (\tilde{E}_r^+)) \rightarrow (U_r^\lambda, (a_r', (b')^\lambda, (\tilde{E}_r^+)) \]

such that \( Y_{r+1}^{\lambda,\phi} \subset \max t_r \).

4. \( \text{Sing}(a_{r+1}')^{\lambda,\phi}, (b')^{\lambda,\phi} = \emptyset \) if and only if one of the following conditions hold:

a. \( \max t_r > \max t_{r+1} \) or

b. \( \max t_r = \max t_{r+1} \) and \( \max t_{r+1} \cap U_{r+1}^{\lambda,\phi} = \emptyset \).

5. If \( \max t_r = \max t_{r+1} \) and \( \max t_{r+1} \cap W_{r+1}^{\lambda,\phi} \neq \emptyset \), then \( \text{Sing}(a_{r+1}')^{\lambda,\phi}, (b')^{\lambda,\phi} = \max w-\ord_{r+1} \cap U_{r+1}^{\lambda,\phi} \) and \( (a_{r+1}')^{\lambda,\phi} \) is defined in terms of \( a_{r+1}' \) and \( (a_{r}')^{\lambda,\phi} \) by (63) and (65).

Proof. 1. For \( q \in U_{r}^{\lambda,\phi} \), \( q \in \max t_r \) if and only if \( q \in \max w-\ord_r \) and \( \eta(q) = N \), which hold if and only if \( \nu_q((a_r')^{\lambda,\phi}) \geq (b')^{\lambda} \) (by 1. of Lemma 6.31) and \( \nu_q(\mathcal{I}_{D_{i}}^{\lambda,\phi}) \geq 1 \) for \( 1 \leq i \leq N \) which holds if and only if \( q \in \text{Sing}((a_{r}')^{\lambda,\phi}, (b')^{\lambda,\phi}) \).

2. follows since \( \nu_q((a_{r}')^{\lambda,\phi}) \leq (b')^{\lambda,\phi} \) for \( q \in U_{r}^{\lambda,\phi} \).

3. is immediate from 1.

4. and 5. follow from 4. and 5. of Lemma 6.31, and the observation that if \( \max t_{r+1} = \max t_r \) and \( q \in \max t_{r+1} \cap U_{r+1}^{\lambda,\phi} \) then \( q \) must be on the strict transforms of the hypersurfaces \( D_{i_1}^{\lambda,\phi}, \ldots, D_{i_N}^{\lambda,\phi} \). \qed
We now observe that the conclusions of Lemmas 6.31 and 6.32, which are formulated for transformations, can be naturally formulated for restrictions also. Then it follows that the \((B^\prime)_r^\lambda,\phi\) define a GBO with \(d\)-dimensional structure on \((W_r, (E^d_r)^+)\)

\[
(F^d_r, W_r, (E^d_r)^+)
\]

with associated closed sets \(F''_r = \max t_r\).

If \(Y\) is a permissible center for (71) then \(Y\) makes simple normal crossings with \(E_r^d\). The local description (69) and (70) then implies that \(Y\) makes simple normal crossings with \(E_r\). Thus \(Y \subset \max t_r\) is a permissible center for \((F^d_r, W_r, E^d_r)\). If the induced transformations are

\[
(F^d_{r+1}, W_{r+1}, (E^d_{r+1})^+) \rightarrow (F^d_r, W_r, (E^d_r)^+)
\]

and

\[
(F^d_{r+1}, W_{r+1}, E^d_{r+1}) \rightarrow (F^d_r, W_r, E^d_r)
\]

then either \(\max t_r > \max t_{r+1}\) in which case \(F''_r = \emptyset\), or \(\max t_r = \max t_{r+1}\) and \(F''_{r+1} = \max t_{r+1}\).

We will now verify that the assumptions of Theorem 6.24 hold for the simple basic object \((U_r^\lambda,\phi, (a''_r)^\lambda,\phi, (b''_r)^\lambda,\phi, (E^+_r)^\lambda,\phi)\).

Define (by (65)) \((B'_r)^\lambda\) in the same way that we defined \((B'_r)^\lambda\). Now 5. of Lemma 6.31 implies \((B'_r)^\lambda\) is obtained by a sequence of transformations of the simple basic object \((B'_r)^\lambda\). \((E^+_r)^\lambda\) is the exceptional divisor of the product of these transformations.

Since \((B'_r)^\lambda\) is a simple basic object, we can find an open cover \(\{V^\lambda_{r_0}\} \subset W^\lambda_{r_0}\) with non-singular hypersurfaces \((V^\lambda_{r_0})^\lambda,\psi\) in \(V^\lambda_{r_0}\) such that

\[
\mathcal{I}(V^\lambda_{r_0})^\lambda,\psi \subset \Delta^{(b')^\lambda-1}((a'_r)^\lambda,\psi).
\]

Let \((V^\lambda_{r_0})^\lambda,\psi\) be the strict transform of \((V^\lambda_{r_0})^\lambda,\psi\) in \(V^\lambda_{r_0}\). By Lemma 6.21, \((V^\lambda_{r_0})^\lambda,\psi\) is non-singular,

\[
\mathcal{I}(V^\lambda_{r_0})^\lambda,\psi \subset \Delta^{(b')^\lambda-1}((a'_r)^\lambda,\psi),
\]

and we have that \((V^\lambda_{r_0})^\lambda,\psi\) makes SNCs with \((E^+_r)^\lambda,\psi\). Let \(U^\lambda,\phi,\psi = V^\lambda,\psi \cap U^\lambda,\phi\). For fixed \(\lambda,\phi\), \(\{U^\lambda,\phi,\psi\}\) is an open cover of \(U^\lambda,\phi\). For \(q \in U^\lambda,\phi,\psi\),

\[
(I_{(V^\lambda_{r_0})^\lambda,\psi})_q \subset \Delta^{(b')^\lambda-1}((a'_r)^\lambda,\psi) \subset \Delta^{(b')^\lambda-1}((a''_r)^\lambda,\phi)_q
\]

by (70). Thus the assumptions of Theorem 6.24 hold for

\[
(B'^{\prime}_r)^\lambda,\phi = (U^\lambda,\phi, ((a''_r)^\lambda,\phi, (b''_r)^\lambda,\phi), (E^+_r)^\lambda,\phi)
\]

and \(\{U^\lambda,\phi,\psi\}\). 1. of Theorem 6.30 now follows from Theorem 6.24 applied to our GBO of (71), with \(d\)-dimensional structure \((B'^{\prime}_r)^\lambda,\phi\).

If \(R(1)(\max t_r)\) is empty, then by Theorem 6.24, applied to the GBO of (71), and its \(d\)-dimensional structure, (71) has a \((d-1)\)-dimensional structure, and we set

\[
(F^{d-1}_r, W_r, E^{d-1}_r) = (F''_r, W_r, (E^d_r)^+).
\]

Given a resolution (64), by Lemma 6.32, we have an induced sequence

\[
(F^d_{N_r}, W_N, E^d_N) \rightarrow \cdots \rightarrow (F^d_r, W_r, E^d_r)
\]

such that the conclusions of 2. of Theorem 6.30 hold.
Lemma 6.33. Let $I_1$ and $I_2$ be totally ordered sets. Suppose that
\[(W_N, E_N) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_i} (W_0, E_0)\] is a sequence of transformations of pairs together with closed sets $F_i \subset W_i$ such that the center $Y_i$ of $\pi_{i+1}$ is contained in $F_i$ for all $i$, $\pi_{i+1}(F_{i+1}) \subset F_i$ for all $i$ and $F_N = \emptyset$. Assume that for $0 \leq i \leq N$ the following conditions hold:

1. There is an upper semi-continuous function $g_i : F_i \to I_1$.
2. There is an upper semi-continuous function $g'_i : \text{Max } g_i \to I_2$.
3. $Y_i = \text{Max } g'_i$ in (72)
4. For any $q \in F_{i+1}$, with $0 \leq i \leq N - 2$,
   \[
   g_i(\pi_{i+1}(q)) \geq g_{i+1}(q) \quad \text{if } \pi_{i+1}(q) \in Y_i
   
   g_i(\pi_{i+1}(q)) = g_{i+1}(q) \quad \text{if } \pi_{i+1}(q) \notin Y_i
   \]

This condition implies that
\[
\text{max } g_0 \geq \text{max } g_1 \geq \cdots \geq \text{max } g_{N-1}
\]
and if $\text{max } g_i = \text{max } g_{i+1}$, then $\pi_{i+1} \left( \text{Max } g_{i+1} \right) \subset \text{Max } g_i$.
5. If $\text{max } g_i = \text{max } g_{i+1}$ and $q \in \text{Max } g_{i+1}$, with $0 \leq i \leq N - 2$, then
   \[
   g'_i(\pi_{i+1}(q)) > g'_{i+1}(q) \quad \text{if } \pi_{i+1}(q) \in Y_i
   
   g'_i(\pi_{i+1}(q)) = g'_{i+1}(q) \quad \text{if } \pi_{i+1}(q) \notin Y_i
   \]

Then there are upper semi-continuous functions
\[
f_i : F_i \to I_1 \times I_2
\]
defined by
\[
f_i(q) = \begin{cases} 
\left( \text{max } g_i, \text{max } g'_i \right) & \text{if } q \in Y_i \\
\text{max } f_{i+1}(q) & \text{if } q \notin Y_i 
\end{cases}
\] (73)
where in the second case we can view $q$ as a point on $W_{i+1}$ as $W_{i+1} \to W_i$ is an isomorphism in a neighborhood of $q$. For $q \in F_{i+1}$,
\[
f_i(\pi_{i+1}(q)) > f_i(q) \quad \text{if } \pi_{i+1}(q) \in Y_i
\]
\[
f_i(\pi_{i+1}(q)) = f_i(q) \quad \text{if } \pi_{i+1}(q) \notin Y_i
\]
\[
\max f_i = (\text{max } g_i, \text{max } g'_i) \quad \text{and } \text{Max } f_i = \text{Max } g'_i
\] (74)

Proof. We first show, by induction on $i$, that the $f_i$ of (73) are well defined functions. Since $F_N = \emptyset$, $Y_{N-1} = F_{N-1}$. By assumption 3.,
\[
\text{Max } g'_{N-1} = \text{Max } g_{N-1} = F_{N-1}
\]
so we can define
\[
f_{N-1}(q) = (g_{N-1}(q), g'_{N-1}(q)) = (\text{max } g_{N-1}, \text{max } g'_{N-1})
\] for $q \in F_{N-1}$.

Suppose that $q \in F_r$. If $q \in Y_r$, we define
\[
f_r(q) = (g_r(q), g'_r(q)) = (\text{max } g_r, \text{max } g'_r).
\]
Suppose that $q \notin Y_r$. As $\pi_{i+1}(F_{i+1}) \subset F_i$ for all $i$, we have an isomorphism $F_r - Y_r \cong F_{r+1} - D_{r+1}$, where $D_{r+1}$ is the reduced exceptional divisor of $W_{r+1} \to W_r$. Since $F_N = \emptyset$, we can define $f_r(q) = f_{r+1}(q)$ for $q \in F_r - Y_r$.

The properties (74) are immediate from the assumptions and (73).
It remains to prove that $f_r$ is upper semi-continuous. We prove this by descending induction on $r$. Fix $\alpha = (\alpha_1, \alpha_2) \in I_1 \times I_2$. We must prove that $\{q \in F_r \mid f_r(q) \geq \alpha\}$ is closed. If $\max f_r < \alpha$ the set is empty. If $\max f_r \geq \alpha$,

$$\{q \in F_r \mid f_r(q) \geq \alpha\} = \max g'_r \cup \pi_{r+1}(\{q' \in F_{r+1} \mid f_{r+1}(q') \geq \alpha\})$$

which is closed by upper semi-continuity of $f_{r+1}$ and properness of $\pi_{r+1}$. \(\square\)

**Theorem 6.34.** Fix an integer $d \geq 0$. There is a totally ordered set $I_d$ and functions $f_i^d$ with the following properties:

1. For each GBO $(F_0^d, W_0, E_0^d)$ with a $d$-dimensional structure and associated closed subsets $F_1^d$ there is a function

$$f_0^d : F_0^d \to I_d$$

with the property that $\max f_0^d$ is a permissible center for $(W_0, E_0^d)$.

2. If a sequence of transformations with permissible centers $Y_i$

$$(F_r^d, W_r, E_r^d) \to \cdots \to (F_0^d, W_0, E_0^d)$$

and functions $f_i^d : F_i^d \to I_d$, $i = 0, \ldots, r - 1$, have been defined with the property that $Y_i = \max f_i^d$, then there is a function $f_r^d : F_r^d \to I_d$ such that $\max f_r^d$ is permissible for $(W_r, E_r^d)$.

3. For each GBO $(F_0^d, W_0, E_0^d)$ with a $d$-dimensional structure, there is an index $N$ so that the sequence of transformations

$$(F_N^d, W_N, E_N^d) \to \cdots \to (F_0^d, W_0, E_0^d).$$

constructed by 1. and 2. is a resolution $(F_N = \emptyset)$.

4. The functions $f_i^d$ in 2. have the following properties:

a. If $q \in F_i$, with $0 \leq i \leq N - 1$ and if $q \notin Y_i$, then $f_i^d(q) = f_{i+1}^d(q)$.

b. $Y_i = \max f_i^d$ for $0 \leq i \leq N - 1$ and

$$\max f_0^d > \max f_1^d > \cdots > \max f_{N-1}^d$$

c. For $0 \leq i \leq N - 1$ the closed set $\max f_i^d$ is smooth, equidimensional and its dimension is determined by the value of $\max f_i^d$.

**Proof.** (of Theorem 6.34) The proof is by induction on $d$.

First assume that $d = 1$. Set $I_1$ to be the disjoint union $I_1 = Q \times Z \cup \{\infty\}$, where $Q \times Z$ is ordered lexicographically, and $\infty$ is the maximum of $I_1$.

Suppose that $(F_0^1, W_0, E_0^1)$ is a GBO with one dimensional structure, and with associated closed sets $F_1^1$. Define $t_0^1 = t_0^1$ ($t_0^1$ is defined by Theorem 6.28). The closed set $F_0^1$ is (locally) a codimension greater than or equal to one subset of a one dimensional subvariety of $W_0$, so $\dim F_0 = 0$. Thus $R(1)(\max t_0^1) \neq 0$, and we are in the situation of 1. of Theorem 6.30. If we perform the permissible transformation with center $\max t_0^1$,

$$(F_1^1, W_1, E_1^1) \to (F_0^1, W_0, E_0^1)$$

we have $\max t_0^1 > \max t_1^1$. We can thus define a sequence of transformations

$$(F_r^1, W_r, E_r^1) \to \cdots \to (F_0^1, W_0, E_0^1)$$

where each transformation has center $\max t_i^1$, $f_i^1 = t_i^1 : F_i^1 \to I_1$ and

$$\max t_0^1 > \cdots > \max t_r^1.$$
Since there is a natural number \( b \) such that \( \max t_1^i \in \frac{1}{b} \mathbb{Z} \times \mathbb{Z} \) for all \( i \), there is an \( r \) such that the above sequence is a resolution.

Now assume that \( d > 1 \) and that the conclusions of Theorem 6.34 hold for GBOs of dimension \( d - 1 \). Thus there is a totally ordered set \( I_{d-1} \), and functions \( f_i^{d-1} \) satisfying the conclusions of the Theorem.

Let \( I'_d \) be the disjoint union

\[
I'_d = \mathbb{Q} \times \mathbb{Z} \sqcup I_M \sqcup \{ \infty \},
\]

where \( I_M \) is the ordered set of Definition 6.16. We order \( I'_d \) as follows: \( \mathbb{Q} \times \mathbb{Z} \) has the lexicographic order. If \( \alpha \in \mathbb{Q} \times \mathbb{Z} \) and \( \beta \in I_M \) then \( \beta < \alpha \) and \( \infty \) is the maximal element of \( I'_d \). Define \( I_t = I'_d \times I_{d-1} \) with the lexicographic ordering.

Suppose that \((\mathcal{F}_0^d, W_0, E_0^d)\) is a GBO of dimension \( d \), with associated closed sets \( F_i^d \). Define \( g_0 : F_0^d \rightarrow I'_d \) by

\[
g_0(q) = t_0^d(q),
\]

and \( g_0 : \text{Max} g_0 \rightarrow I_{d-1} \) by

\[
g_0(q) = \begin{cases} 
\infty & \text{if } q \in R(1)(\text{Max } t_0^d) \\
t_0^d(q) & \text{if } q \notin R(1)(\text{Max } t_0^d)
\end{cases}
\]

where \( t_0^{d-1} \) is the function defined by the GBO of dimension \( d - 1 \) \((\mathcal{F}_0^{d-1}, W_0, E_0^{d-1})\) of 2. of Theorem 6.30.

Assume that we have now inductively defined a sequence of transformations

\[
(\mathcal{F}_r^d, W_r, E_r^d) \supseteq \cdots \supseteq (\mathcal{F}_0^d, W_0, E_0^d)
\]

and we have defined functions

\[
g_i : F_i^d \rightarrow I'_d \quad \text{and} \quad g'_i : \text{Max } g_i \rightarrow I_{d-1}
\]

satisfying the assumptions 1. - 5. of Lemma 6.33 (with \( I_1 = I'_d \) and \( I_2 = I_{d-1} \)) for \( i = 0, \ldots, r - 1 \). In particular, the center of \( \pi_{i+1} \) is \( Y_i = \text{Max } g'_i \).

If \( F_r \neq \emptyset \), define \( g_r : F_r^d \rightarrow I'_d \) by

\[
g_r(q) = \begin{cases} 
\Gamma(B_r)(q) & \text{if w-ord}^d_r(q) = 0 \\
t_r^d(q) & \text{if w-ord}^d_r(q) > 0.
\end{cases}
\]

where \( \Gamma(B_r) \) is the function defined in Proposition 6.29 for the GBO \( B_r = (\mathcal{F}_r^d, W_r, E_r^d) \).

Define \( g'_r : \text{Max } g_r \rightarrow I_{d-1} \) by

\[
g'_r(q) = \begin{cases} 
\infty & \text{if w-ord}^d_r(q) = 0 \\
\infty & \text{if } q \in R(1)(\text{Max } t_r^d) \text{ and } \text{w-ord}^d_r(q) > 0 \\
f_r^{d-1}(q) & \text{if } q \notin R(1)(\text{Max } t_r^d) \text{ and } \text{w-ord}^d_r(q) > 0
\end{cases}
\]

where \( f_r^{d-1} \) is the function defined by the GBO of dimension \( d - 1 \) \((\mathcal{F}_r^{d-1}, W_r, E_r^{d-1})\) in 2. of Theorem 6.30. Observe that if \( \text{max w-ord}^d_r > 0 \), then the center \( Y_i \) of \( \pi_{i+1} \) in (77) is defined by \( Y_i = \text{Max } g'_i \supseteq \text{Max } t_i^d \) for \( 0 \leq i \leq r - 1 \). Thus (if \( \text{max w-ord}^d_r > 0 \)) we have that (77) is a sequence of the form of (63) of Theorem 6.30.

Now Theorem 6.30, our induction assumption, and Lemma 6.17 show that the sequence (77) extends uniquely to a resolution

\[
(\mathcal{F}_N, W_N, E_N) \rightarrow \cdots \rightarrow (\mathcal{F}_0, W_0, E_0)
\]
with functions $g_i$ and $g'_i$ satisfying the assumptions 1. - 5. of Lemma 6.33. By Lemma 6.34, the functions $(g_i, g'_i)$ extend to functions

$$F^d_i \rightarrow I_d = I'_d \times I_{d-1}$$

satisfying the conclusions of Theorem 6.34.

Consider the values of the functions $f^d_i$ which we constructed at a point $q \in F_i$. $f^d_i(q)$ has $d$ coordinates, and has one of the following three types:

1. $f^d_i(q) = (t^d(q), t^{d-1}(q), \ldots, t^{d-r}(q), \infty, \cdots, \infty)$.
2. $f^d_i(q) = (t^d(q), t^{d-1}(q), \ldots, t^{d-r}(q), \Gamma(q), \infty, \cdots, \infty)$.
3. $f^d_i(q) = (t^d(q), t^{d-1}(q), \ldots, t^1(q))$.

Each coordinate is a function defined for a GBO of the corresponding dimension. $t^i(q) = (w$-ord $^i(q), \eta^i(q))$ with w-ord $^i(q) > 0$.

In case 1. $t^{d-r}$ is such that $q \in R(1)(\operatorname{Max} t^{d-r})$, that is $\operatorname{Max} t^{d-r}$ has codimension 1 in a $d - r$ dimensional GBO, and $\dim(\operatorname{Max} f^d_i) = d - r - 1$.

In case 2. w-ord $^{d-(r+1)}(q) = 0$ and $\Gamma$ is the function defined in Proposition 6.29 for a monomial GBO and $\dim(\operatorname{Max} f^d_i) = (d - r - \Gamma_1 - 1$, where $\Gamma_1$ is the first coordinate of max $\Gamma$.

In case 3. $\dim(\operatorname{Max} f^d_i) = 0$.

Thus $\operatorname{Max} f^d_i$ is non-singular and equidimensional, and 4.c. follows. □

**Theorem 6.35.** Suppose that $d$ is a positive integer. Then there are a totally ordered set $I'_d$ and functions $p^d_i$ with the following properties:

1. For each pair $(W_0, E_0)$ with $d = \dim W_0$ and ideal sheaf $J_0 \subset \mathcal{O}_{W_0}$ there is a function

$$p^d_0 : V(J_0) \rightarrow I'_d$$

with the property that $\operatorname{Max} p^d_0 \subset V(J_0)$ is permissible for $(W_0, E_0)$.

2. If a sequence of transformations of pairs with centers $Y_i \subset V(J_i)$

$$(W_r, E_r) \rightarrow \cdots \rightarrow (W_0, E_0)$$

and functions $p^d_i : V(J_i) \rightarrow I'_d$, $0 \leq i \leq r-1$ has been defined with the property that $Y_i = \operatorname{Max} p^d_i$ and $V(J_k) \neq \emptyset$, then there is a function $p^d_r : V(J_r) \rightarrow I'_d$

such that $\operatorname{Max} p^d_r$ is permissible for $(W_r, E_r)$.

3. For each pair $(W_0, E_0)$ and $J_0 \subset \mathcal{O}_{W_0}$ there is an index $N$ such that the sequence of transformations

$$(W_N, E_N) \rightarrow \cdots \rightarrow (W_0, E_0)$$

constructed inductively by 1. and 2. is such that $V(J_N) = \emptyset$. The corresponding sequence will be called a “Strong Principalization” of $J_0$.

4. Property 4. of Theorem 6.34 holds.

**Proof.** Let $J_0 = J_0$, and set $b_0 = \max \nu_{J_0}$. By Theorem 6.34, there exists a resolution

$$(W_{N_1}, (J_{N_1}, b_0), E_{N_1}) \rightarrow \cdots \rightarrow (W_0, (J_0, b_0), E_0)$$

of the simple basic object $(W_0, (J_0, b_0), E_0)$. Thus we have that $J_{N_1}$ (which is the weak transform of $J_0$ on $\mathcal{O}_{W_{N_1}}$) satisfies $b_1 = \max \nu_{J_{N_1}} < b_0$. We now can construct by Theorem 6.34 a resolution

$$(W_{N_2}, (J_{N_2}, b_1), E_{N_2}) \rightarrow \cdots \rightarrow (W_{N_1}, (J_{N_1}, b_1), E_{N_1})$$
of the simple basic object \((W_{N_i}, (J_{N_i}, b_i), E_{N_i})\).

After constructing a finite number of resolutions of simple basic objects in this way, we have a sequence of transformations of pairs
\[
(W_N, E_N) \rightarrow \cdots \rightarrow (W_0, E_0)
\]
such that the strict transform \(J_N\) of \(J_0\) on \(W_N\) is \(J_N = \mathcal{O}_{W_N}\).

We have upper semi-continuous functions
\[
f_{d_i} : \text{Max } \nu_{J_i} \rightarrow I_d
\]
satisfying the conclusions of Theorem 6.34 on the resolution sequences
\[
(W_{N_{i+1}}, (J_{N_{i+1}}, b_i), E_{N_{i+1}}) \rightarrow \cdots \rightarrow (W_N, (J_N, b_i), E_N)
\]
Now apply Lemma 6.33 to the sequence (78), with \(F_i = V(J_i), g_i = \nu_{J_i}, g'_i = f_{d_i}\). We conclude that there exist functions \(p_{d_i} : V(J_i) \rightarrow I'_{d} = \mathbb{Z} \times I_d\) with the desired properties. □

Recall our conventions on varieties in section 1.1 on Notations.

Theorem 6.36. (principalization of ideals) Suppose that \(I\) is an ideal sheaf on a non-singular variety \(W\) over a field of characteristic zero. Then there exists a sequence of monodial transforms
\[
\pi : W_1 \rightarrow W
\]
which is an isomorphism away from the closed locus of points where \(I\) is not locally principal, such that \(I\mathcal{O}_{W_1}\) is locally principal.

Proof. We can factor \(I = I_1 I_2\) where \(I_1\) is an invertible sheaf, and \(V(I_2)\) is the set of points where \(I\) is not locally principal. Then we apply Theorem 6.35 to \(J_0 = I_2\). □

Theorem 6.37. (embedded resolution of singularities) Suppose that \(X\) is an algebraic variety over a field of characteristic zero which is embedded in a non-singular variety \(W\). Then there exists a birational projective morphism
\[
\pi : W_1 \rightarrow W
\]
such that \(\pi\) is a sequence of monodial transforms, \(\pi\) is an embedded resolution of \(X\), and \(\pi\) is an isomorphism away from the singular locus of \(W\).

Proof. Set \(X_0 = X, W_0 = W, J_0 = I_{X_0} \subset \mathcal{O}_{W_0}\). Let
\[
(W_N, E_N) \rightarrow (W_0, E_0)
\]
be the strong principalization of \(J_0\) constructed in Theorem 6.35, so that the weak transform of \(J_0\) on \(W_N\) is \(J_N = \mathcal{O}_{W_N}\).

With the notation of the proofs of Theorem 6.34 and Theorem 6.35, if \(X_0\) has codimension \(r\) in the \(d\)-dimensional variety \(W_0\), we have for \(q \in \text{Reg}(X_0)\),
\[
p_{d,q}(q) = (1, (1,0), \ldots, (1,0), 0, \ldots, 0) \in I'_d
\]
where there are \(r\) copies of \((1,0)\) followed by \(d-r\) zeros. The function \(p^{d}_{q}\) is thus constant on the non-empty open set \(\text{Reg}(X_0)\) of non-singular points of \(X_0\). Let this constant be \(c \in I'_d\). By property 4. of Theorem 6.34 there exists a unique index \(r \leq N - 1\) such that \(\max p^{d}_{r} = c\). Since \(W_r \rightarrow W_0\) is an isomorphism over the dense open set \(\text{Reg}(X_0)\), the strict transform \(X_r\) of \(X_0\) must be the union of the irreducible
components of the closed set $\text{Max}_{p^d_r}$ of $W_r$. Since $\text{Max}_{p^d_r}$ is a permissible center, $X_r$ is non-singular and makes simple normal crossings with the exceptional divisor $E_r$. □

**Theorem 6.38.** (resolution of singularities) Suppose that $X$ is an algebraic variety over a field of characteristic zero. Then there is a resolution of singularities

$$\pi : X_1 \to X$$

such that $\pi$ is a projective morphism, which is an isomorphism away from the singular locus of $X$.

**Proof.** This is immediate from Theorem 6.37, after choosing an embedding of $X$ into a projective space $W$. □

**Theorem 6.39.** (resolution of indeterminacy) Suppose that $K$ is a field of characteristic zero and $\phi : W \to V$ is a rational map of projective $K$-varieties. Then there exists a projective birational morphism $\pi : W_1 \to W$ such that $W_1$ is non-singular and a morphism $\lambda : W_1 \to V$ such that $\lambda = \phi \circ \pi$. If $W$ is non-singular, then $\pi$ is a product of monoidal transforms.

**Proof.** By Theorem 6.38, there exists a resolution of singularities $\psi : W_1 \to W$. After replacing $W$ with $W_1$ and $\phi$ with $\psi \circ \phi$, we may assume that $W$ is non-singular. Let $\Gamma_\phi$ be the graph of $\phi$, with projections $\pi_1 : \Gamma_\phi \to W$ and $\pi_2 : \Gamma_\phi \to V$. As $\pi_1$ is birational and projective, there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_W$ such that $\Gamma_\phi \cong \text{Bl}(\mathcal{I})$ is the blow up of $\mathcal{I}$ (Theorem 4.5). By Theorem 6.36, there exists a principalization $\pi : W_1 \to W$ of $\mathcal{I}$. The Universal property of blowing up (Theorem 4.2) now shows that there is a morphism $\lambda : W_1 \to V$ such that $\lambda = \phi \circ \pi$. □

**Theorem 6.40.** Suppose that $\pi : Y \to X$ is a birational morphism of projective nonsingular varieties over a field $K$ of characteristic 0. Then

$$H^i(Y, \mathcal{O}_Y) \cong H^i(X, \mathcal{O}_X) \forall \ i.$$  

**Proof.** By Theorem 6.39, there exists a commutative diagram of projective morphisms

$$\begin{array}{ccc}
Z & \xrightarrow{g} & Y \\
g \downarrow & & \downarrow \pi \\
X & & 
\end{array}$$

such that $g$ is a product of blowups of nonsingular subvarieties,

$$g : Z = Z_n \xrightarrow{g_n} Z_{n-1} \xrightarrow{g_{n-1}} \ldots \xrightarrow{g_2} Z_1 \xrightarrow{g_1} Z_0 = X.$$  

We have ([60] or Lemma 2.1 [29])

$$R^ig_*\mathcal{O}_Z = \begin{cases} 0, & \text{if } i > 0 \\ \mathcal{O}_{Z_{j-1}}, & \text{if } i = 0 \end{cases}$$

Thus, by the Leray spectral sequence,

$$R^ig_*\mathcal{O}_Z = \begin{cases} 0, & \text{if } i > 0 \\ \mathcal{O}_X, & \text{if } i = 0 \end{cases}$$

and

$$g^* : H^i(X, \mathcal{O}_X) \cong H^i(Z, \mathcal{O}_Z)$$


for all $i$. Now, by considering the commutative diagram

\[
\begin{array}{ccc}
H^i(Z, \mathcal{O}_Z) & \xrightarrow{f^*} & H^i(Y, \mathcal{O}_Y) \\
g^* \searrow & & \uparrow \pi^* \\
\downarrow & & \\
H^i(X, \mathcal{O}_X)
\end{array}
\]

we conclude that $\pi^*$ is one-to-one. To show that $\pi^*$ is an isomorphism we now only need to show that $f^*$ is also one-to-one.

Resolution of indeterminancy also gives a new diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\gamma} & Z \\
\beta \searrow & \downarrow f \\
& Y
\end{array}
\]

where $\beta$ is a product of blowups of nonsingular subvarieties, so we have

\[
\beta^*: H^i(Y, \mathcal{O}_Y) \cong H^i(W, \mathcal{O}_W)
\]

for all $i$. This implies that $f^*$ is one-to-one, and the theorem is proved. \qed

For the following theorem, which is stronger than Theorem 6.38, we give a complete proof only the case of a hypersurface. In the embedded resolution constructed in Theorem 6.37, which induces the resolution of 6.38, if $X$ is not a hypersurface, there may be monodial transforms $W_{i+1} \rightarrow W_i$ whose centers are not contained in the strict transform $X_i$ of $X$, so that the induced morphism $X_{i+1} \rightarrow X_i$ of strict transforms of $X$ will in general not be a monodial transform.

**Theorem 6.41.** (A stronger theorem of resolution of singularities) Suppose that $X$ is an algebraic variety over a field of characteristic zero. Then there is a resolution of singularities

\[\pi: X_1 \rightarrow X.\]

such that $\pi$ is a sequence of monodial transforms centered in the closed sets of points of maximum multiplicity.

**Proof.** In the case of a hypersurface $X = X_0$ embedded in a non-singular variety $W = W_0$, the proof given in Theorem 6.37 actually produces a resolution of the kind asserted by this theorem, since the resolution sequence is constructed by patching together resolutions of the simple basic objects $(W_i, (\mathcal{J}_i, b), E_i)$, where $\mathcal{J}_i$ is the weak transform of the ideal sheaf $\mathcal{J}_0$ of $X_0$ in $W_0$. Since $\mathcal{J}_0$ is a locally principal ideal, and each transformation in these sequences is centered at a non-singular subvariety of $\text{Max} \mathcal{J}_i$, the weak transform $\mathcal{J}_i$ is actually the strict transform of $\mathcal{J}_0$.

For the case of a general variety $X$, we choose an embedding of $X$ in a projective space $W$. We then modify the above proof by considering the Hilbert Samuel function of $X$. It can be shown that there is a simple basic object $(W_0, (K_0, b), E_0)$ such that $\text{Max} \nu_{K_0}$ is the maximal locus of the Hilbert Samuel function. In the construction of $\rho_1^d$ of Theorem 6.35 we use $\nu_{K_0}$ in place of $\nu_{\mathcal{J}_i}$. Then the resolution of this basic object induces a sequence of monodial transformations of $X$, such that the maximum of the Hilbert Samuel function has dropped.

More details about this process are given, for instance, in [51], [13] and [73]. \qed

Exercises:
1. Follow the algorithm of this chapter to construct an embedded resolution of singularities of:
   a. The singular curve $y^2 - x^3 = 0$ in $\mathbb{A}^2$.
   b. The singular surface $z^2 - x^2y^3 = 0$ in $\mathbb{A}^3$.
   c. The singular curve $z = 0$, $y^2 - x^3 = 0$ in $\mathbb{A}^3$.
2. Give examples where $f^d_i$ achieve the 3 cases of the proof of Theorem 6.34.
3. Suppose that $K$ is a field of characteristic zero and $X$ is an integral finite type $K$-scheme. Prove that there exists a proper birational morphism $\pi : Y \to X$ such that $Y$ is non-singular.
4. Suppose that $K$ is a field of characteristic zero and $X$ is a reduced finite type $K$-scheme. Prove that there exists a proper birational morphism $\pi : Y \to X$ such that $Y$ is non-singular.
5. Let notation be as in the statement of Theorem 6.35.
   a. Suppose that $q \in V(J_0) \subset W_0$ is a closed point. Let $U = \text{spec}(\mathcal{O}_{W_0,q})$, $\overline{D}_i = D_i \cap U$ for $1 \leq i \leq r$ and $E = \{\overline{D}_1, \ldots, \overline{D}_r\}$. Consider the function $p^d_i$ of Theorem 6.35. Show that $p^d_i(q)$ depends only on the local basic object $(U, E)$ and the ideal $(J_0)_q \subset \mathcal{O}_{W_0,q}$. In particular, $p^d_i(q)$ is independent of $K$-algebra isomorphism of $\mathcal{O}_{X,q}$ which takes $J_0\mathcal{O}_{X,q}$ to $J_0\mathcal{O}_{X,q}$ and $\overline{D}_i$ to $\overline{D}_i$ for $1 \leq i \leq r$.
   b. Suppose that $\Theta : W_0 \to W_0$ is a $K$-automorphism such that $\Theta^*(J_0) = J_0$.
      Show that $p^d_i(\Theta(q)) = p^d_i(q)$ for all $q \in V(J_0)$.
   c. Suppose that $\Theta : W_0 \to W_0$ is a $K$-automorphism, and $Y \subset W_0$ is a nonsingular subvariety such that $\Theta(Y) = Y$. Let $\pi_1 : W_1 \to W$ be the monoidal transform centered at $Y$. Show that $\Theta$ extends to a $K$-automorphism $\Theta : W_1 \to W_1$ such that $\Theta(\pi_1(q)) = \pi_1(\Theta(q))$ for all $q \in W_1$ and $\Theta(D) = D$ if $D$ is the exceptional divisor of $\pi_1$.
6. Suppose that $(W_0, E_0)$ is a pair and $J_0 \subset \mathcal{O}_{W_0}$ is an ideal sheaf, with notation as in Theorem 6.35. Consider the sequence of 2. of Theorem 6.35. Suppose that $G$ is a group of $K$-automorphisms of $W_0$ such that $\Theta(D_i) = D_i$ for $1 \leq i \leq r$ and $\Theta^*(J_0) = J_0$. Show that $G$ extends to a group of $K$-automorphisms of each $W_i$ in the sequence 2. of Theorem 6.35 such that $\Theta(Y_i) = Y_i$ for all $\Theta \in G$ and each $\pi_i$ is $G$-equivariant. That is,
   $$\pi_i(\Theta(q)) = \Theta(\pi_i(q))$$
   for all $q \in W_i$. Furthermore, the functions $p^d_i$ of Theorem 6.35 satisfy $p^d_i(\Theta(q)) = p^d_i(q)$ for all $q \in W_i$.
7. (Equivariant resolution of singularities) Suppose that $X$ is a variety over a field $K$ of characteristic zero. Suppose that $G$ is a group of $K$-automorphisms of $X$. Show that there exists an equivariant resolution of singularities $\pi : X_1 \to X$ of $X$. That is, $G$ extends to a group of $K$-automorphisms of $X$ so that $\pi(\Theta(q)) = \Theta(\pi(q))$ for all $q \in X_1$. More generally, deduce equivariant versions of all the other theorems of Section 6.8 and of exercises 3 and 4.

7. RESOLUTION OF SURFACE SINGULARITIES IN POSITIVE CHARACTERISTIC

7.1. Resolution and some invariants. In this chapter we prove resolution of singularities for surfaces over an algebraically closed field of characteristic $p \geq 0$. The characteristic 0 proof in Section 5.1 depended on the existence of hypersurfaces of
maximal contact, which is false in positive characteristic (see the exercises at the end of this chapter). We prove the following theorem in this chapter.

**Theorem 7.1.** Suppose that \( S \) is a surface over an algebraically closed field \( K \) of characteristic \( p \geq 0 \). Then there exists a resolution of singularities

\[ \Lambda : T \to S. \]

Theorem 7.1 in the case when \( S \) is a hypersurface follows from induction on \( r = \max \{ t \mid \text{Sing}_t(S) \neq \emptyset \} \) in Theorems 7.6, 7.7 and 7.8. We then obtain Theorem 7.1 in general from Theorem 4.8 and Theorem 4.10.

For the remainder of this chapter we will assume that \( K \) is an algebraically closed field of characteristic \( p \geq 0 \), \( V \) is a non-singular 3 dimensional variety over \( K \), and \( S \) is a surface in \( V \).

It follows from Theorem 10.19 that for \( t \in \mathbb{N} \),

\[ \text{Sing}_t(S) = \{ p \in S \mid \nu_p(S) \geq t \} \]

is Zariski closed.

Let

\[ r = \max \{ t \mid \text{Sing}_t(S) \neq \emptyset \}. \]

Suppose that \( p \in \text{Sing}_r(S) \) is a closed point, \( f = 0 \) is a local equation of \( S \) at \( p \), and \((x, y, z)\) are regular parameters at \( p \) in \( V \) (or in \( \hat{O}_{V, p} \)). There is an expansion

\[ f = \sum_{i+j+k \geq r} a_{ijk} x^i y^j z^k \]

with \( a_{ijk} \in K \) in \( \hat{O}_{V, p} = K[[x, y, z]] \). The leading form of \( f \) is defined to be

\[ L(x, y, z) = \sum_{i+j+k=r} a_{ijk} x^i y^j z^k. \]

We define a new invariant, \( \tau(p) \), to be the dimension of the smallest linear subspace \( M \) of the \( K \)-subspace spanned by \( x, y \) and \( z \) in \( K[x, y, z] \) such that \( L \in k[M] \). This subspace is uniquely determined. This dimension is in fact independent of choice of regular parameters \((x, y, z) \) at \( p \) (or in \( \hat{O}_{V, p} \)). If \( x, y, z \) are regular parameters in \( O_{V, p} \), we will call the subvariety \( N = V(M) \) of \( \text{spec}(O_{V, p}) \) to be an “approximate manifold” to \( S \) at \( p \). If \( x, y, z \) are regular parameters in \( \hat{O}_{V, p} \), we call \( N = V(M) \subset \text{spec}(\hat{O}_{V, p}) \) a (formal) approximate manifold to \( S \) at \( p \). \( M \) is dependent of our choice of regular parameters at \( p \). Observe that

\[ 1 \leq \tau(q) \leq 3. \]

If there is a non-singular curve \( C \subset \text{Sing}_r(S) \) such that \( q \in C \), then \( \tau(q) \leq 2 \).

**Example 7.2.** Let \( f = x^2 + y^2 + z^2 + x^5 \). If \( \text{char}(K) \neq 2 \), then \( \tau = 3 \) and \( x = y = z = 0 \) are local equations of the approximate manifold at the origin.

However, if \( \text{char}(K) = 2 \), then \( \tau = 1 \) and \( x + y + z = 0 \) is a local equation of an approximate manifold at the origin.

**Example 7.3.** Let \( f = y^2 + 2xy + x^2 + z^2 + z^5 \), \( \text{char}(K) > 2 \). \( L = (y + x)^2 + z^2 \) so that \( \tau = 2 \) and \( x + y = z = 0 \) are local equations of an approximate manifold at the origin.
Lemma 7.4. Suppose that $q \in S$ is a closed point such that $\nu_q(S) = r$, $\tau(q) = n$ (with $1 \leq n \leq 3$), $L_1 = \cdots = L_n = 0$ are local equations at $q$ of a (possibly formal) approximate manifold $N$ of $S$ at $q$. Let $\pi : V_1 \to V$ be the blow up of $q$, $E = \pi^{-1}(q)$ be the exceptional divisor of $\pi$, $S_1$ be the strict transform of $S$ on $V_1$, $N_1$ be the strict transform of $N$ on $V_1$. Suppose that $q_1 \in \pi^{-1}(q)$. Then $\nu_{q_1}(S_1) \leq \nu_q(S)$, and if $\nu_{q_1}(S_1) = r$, then $\tau(q_1) \leq \tau(q_1)$ and $q_1 \in N_1 \cap E$.

We further have that $N_1 \cap E = \emptyset$ if $\tau(q) = 3$, $N_1 \cap E$ is a point if $\tau(q) = 2$ and $N_1 \cap E$ is a line if $\tau(q) = 1$.

Proof. Let $f = 0$ be a local equation of $S$ at $q$, $(x, y, z)$ be regular parameters at $p$ such that

1. $x = y = z = 0$ are local equations of $N$ if $\tau(q) = 3$,
2. $y = z = 0$ are local equations of $N$ if $\tau(q) = 2$,
3. $z = 0$ is a local equation of $N$ if $\tau(q) = 1$.

In $\hat{\mathcal{O}}_{V_1, p} = K[[x, y, z]]$, write

$$f = \sum a_{ijk} x^i y^j z^k = L + G$$

where $L$ is the leading form of degree $r$, and $G$ has order $> r$. $q_1$ has regular parameters $(x_1, y_1, z_1)$ such that one of the following cases hold:

1. $x = x_1, y = x_1(y_1 + \alpha), z = x_1(z_1 + \beta)$ with $\alpha, \beta \in K$
2. $x = x_1 y_1, y = y_1, z = y_1(z_1 + \beta)$ with $\beta \in K$
3. $x = x_1 z_1, y = y_1 z_1, z = z_1$

Suppose that the first case holds. $S_1$ has a local equation $f_1 = 0$ at $q_1$ such that

$$f_1 = L(1, y_1 + \alpha, z_1 + \beta) + x_1 \Omega.$$

Thus $\nu_{q_1}(f_1) \leq r$, and $\nu_{q_1}(f_1) = r$ implies

$$L(1, y_1 + \alpha, z_1 + \beta) = \sum_{i+j+k=r} a_{ijk} (y_1 + \alpha)^j (z_1 + \beta)^k = \sum_{j+k=r} b_{jk} y_1^j z_1^k$$

for some $b_{jk} \in k$. Thus

$$L(1, \frac{y}{x}, \frac{z}{x}) = \sum_{j+k=r} \left( \frac{y}{x} - \alpha \right)^j \left( \frac{z}{x} - \beta \right)^k$$

implies $V(y - \alpha x, z - \beta x) \subset N$. We conclude that $\tau(q) \leq 2$ if $\nu_{q_1}(S_1) = \nu_q(S)$.

Suppose that $\tau(q) = 2$ and $\nu_{q_1}(S_1) = \nu_q(S)$. Then we must have that $\alpha = \beta = 0$, so that $q_1 \in N_1$. We further have

$$f_1 = L(y_1, z_1) + x_1 \Omega$$

so that $\tau(q_1) \geq \tau(q)$.

If $\tau(q) = 1$, then $L = cz^r$ for some (non-zero) $c \in K$, and we must have $\beta = 0$, so that $q_1 \in N_1$. We further have

$$f_1 = L(z_1) + x_1 \Omega$$

so that $\tau(q_1) \geq \tau(q)$.

There is a similar analysis in cases 2 and 3. □
Lemma 7.5. Suppose that $C \subset \text{Sing}_r(S)$ is a non-singular curve and $q \in C$ is a closed point. Let $\pi : V_1 \to V$ be the blow up of $C$, $E = \pi^{-1}(C)$ be the exceptional divisor, $S_1$ be the strict transform of $S$ on $V_1$. Suppose that $(x, y, z)$ are regular parameters in $\mathcal{O}_{V,q}$ (or $\hat{\mathcal{O}}_{V,q}$) such that $x = y = 0$ are local equations of $C$ at $q$, and $x = 0$ or $x = y = 0$ are local equations of a (possible formal) approximate manifold $N$ of $S$ at $q$. Let $N_1$ be the strict transform of $N$ on $V_1$. Suppose that $q_1 \in \pi^{-1}(q)$. Then $\nu_{q_1}(S_1) \leq r$, and if $\nu_{q_1}(S_1) = r$ then $\tau(q) \leq \tau(q_1)$ and $q_1 \in N_1 \cap E$.

Furthermore, $N_1 \cap E = \emptyset$ if $\tau(q) = 2$ and $N_1 \cap E$ is a curve which maps isomorphically onto $\text{spec}(\mathcal{O}_{C,q})$ (or $\text{spec}(\hat{\mathcal{O}}_{C,q})$ in the case where $N$ is formal) if $\tau(q) = 1$.

We omit the proof of Lemma 7.5 as it is similar to Lemma 7.4.

Theorem 7.6. There exists a sequence of blow ups of points in $\text{Sing}_r(S_i)$

$$V_n \to V_{n-1} \to \cdots \to V,$$

where $S_i$ is the strict transform of $S$ on $V_i$, so that all curves in $\text{Sing}_r(S_n)$ are non-singular.

Proof. This is possible by Corollary 4.4 and since (by Lemma 7.4) the blow up of a point in $\text{Sing}_r(S_i)$ can introduce at most one new curve into $\text{Sing}_r(S_{i+1})$, which must be non-singular. \hfill \Box

Theorem 7.7. Suppose that all curves in $\text{Sing}_r(S)$ are non-singular, and

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V$$

is a sequence of blow ups of non-singular curves in $\text{Sing}_r(S_i)$ where $S_i$ is the strict transform of $S$ on $V_i$. Then $\text{Sing}_r(S_i)$ is a union of non-singular curves and a finite number of points for all $i$. Further, this sequence is finite. That is, there exists an $n$ such that $\text{Sing}_r(S_n)$ is a finite set.

Proof. If $\pi : V_1 \to V$ is the blow up of the non-singular curve $C$ in $\text{Sing}_r(S)$, and $C_1 \subset \text{Sing}_r(S_1) \cap \pi^{-1}(C)$ is a curve, then by Lemma 7.5, $C_1$ is non-singular and maps isomorphically onto $C$. Furthermore, the strict transform of a non-singular curve must be non-singular, so $\text{Sing}_r(S_i)$ must be a union of non-singular curves, and a finite number of points for all $i$.

Suppose that we can construct an infinite sequence

$$\cdots \to V_n \to V_{n-1} \to \cdots \to V$$

by blowing up non-singular curves in $\text{Sing}_r(S_i)$. Then there exist curves $C_i \subset \text{Sing}_r(V_i)$ for all $i$ such that $C_{i+1} \to C_i$ are dominant morphisms for all $i$, and

$$\nu_{\mathcal{O}_{V_i,C_i}}(I_{S_i,C_i}) = r$$

for all $i$. We have an infinite sequence

$$\mathcal{O}_{V,C} \to \mathcal{O}_{V_1,C_1} \to \cdots \to \mathcal{O}_{V_n,C_n} \to \cdots$$

of local rings, where, after possibly reindexing to remove the maps which are the identity, each homomorphism is a localization of the blow up of the maximal ideal of a 2 dimensional regular local ring. As explained in the characteristic zero proof of resolution of surface singularities (Lemma 5.10), we can view the $\mathcal{O}_{S_i,C_i}$ as local
rings of a closed point of a curve on a non-singular surface defined over a field of
transcendence degree 1 over \( K \). By Theorem 3.12 (or Corollary 4.4).

\[ \nu_{\mathcal{O}_V(n_\nu, c_n)}(J_{S_n, c_n}) < r \]

for large \( n \), a contradiction. \( \square \)

We can now state the main resolution theorem.

**Theorem 7.8.** Suppose that \( \text{Sing}_r(S) \) is a finite set. Consider a sequence of blow ups

\[ \cdots \to V_n \to \cdots \to V_2 \to V_1 \to V \]  (81)

where \( V_{i+1} \to V_i \) is the blow up of a curve in \( \text{Sing}_r(S_i) \) if such a curve exists, and \( V_{i+1} \to V_i \) is the blow up of a point in \( \text{Sing}_r(S_i) \) otherwise. Let \( S_i \) be the strict transform of \( S \) on \( V_i \). Then \( \text{Sing}_r(S_i) \) is a union of non-singular curves and a finite number of points for all \( i \). Further, this sequence is finite. That is, there exists \( V_n \) such that \( \text{Sing}_r(S_n) = \emptyset \).

Theorem 7.8 is an immediate consequence of Theorem 7.9 below, and Theorems 7.6 and 7.7.

**Theorem 7.9.** Let the assumptions be as in the statement of Theorem 7.8, and suppose that \( q \in \text{Sing}_r(S) \) is a closed point. Then there exists an \( n \) such that all closed points \( q_n \in S_n \) such that \( q_n \) maps to \( q \) and \( \nu_{q_n}(S_n) = r \) satisfy \( \tau(q_n) > \tau(q) \).

The proof of this Theorem is very simple if \( \tau(q) = 3 \). By Lemma 7.4 the multiplicity must drop at all points above \( q \) in the blow up of \( q \). We will prove Theorem 7.9 in the case \( \tau(q) = 2 \) in Section 7.2 and prove Theorem 7.9 in the case \( \tau(q) = 1 \) in Section 7.3.

### 7.2. \( \tau(q) = 2 \)

In this section we prove Theorem 7.9 in the case that \( \tau(q) = 2 \).

Suppose that \( \tau(q) = 2 \), and that there exists a sequence (81) of infinite length. We can then choose an infinite sequence of closed points \( q_i \in \text{Sing}_r(S_i) \) such that \( q_{i+1} \) maps to \( q_i \) for all \( i \). By Lemma 7.4, we have \( \tau(q_i) = 2 \) for all \( i \) and that each \( q_i \) is isolated in \( \text{Sing}_r(S_i) \).

Thus each map \( V_{i+1} \to V_i \) is either the blow up of the unique point \( q_i \in \text{Sing}_r(S_i) \) which maps to \( q \) or \( V_{i+1} \to V_i \) is an isomorphism over \( q \). After reindexing, we may assume that each map is the blow up of a point \( q_i \) such that \( q_{i+1} \) maps to \( q_i \) for all \( i \). Let \( R_i = \hat{\mathcal{O}}_{V_i,q_i} \). We have a sequence

\[ R_0 \to R_1 \to \cdots \to R_i \to \cdots \]  (82)

of infinite length, where each \( R_{i+1} \) is the completion of a local ring of a closed point of the blow up of the maximal ideal of \( \text{spec}(R_i) \), and \( \nu_{q_i}(S_i) = r \) and \( \tau(q_i) = 2 \) for all \( i \). We will derive a contradiction. Since \( \tau(q) = 2 \), there exist regular parameters \((x, y, z)\) in \( R_0 \) such that a local equation \( f = 0 \) of \( S \) in \( R_0 \) has the form

\[ f = \sum_{i+j+k \geq r} a_{ijk} x^i y^j z^k \]

and the leading form of \( f \) is

\[ L(x, y) = \sum_{i+j=r} a_{ij} x^i y^j. \]
For an arbitrary series
\[ g = \sum b_{ijk} x^i y^j z^k \in R_0, \]
define
\[ \gamma_{xyz}(g) = \min \left\{ \frac{k}{r - (i + j)} \mid b_{ijk} \neq 0 \text{ and } i + j < r \right\} \in \frac{1}{r} \mathbb{N} \cup \{ \infty \} \]
with the convention that \( \gamma_{xyz}(g) = \infty \) if and only if \( b_{ijk} = 0 \) whenever \( i + j < r \). We have \( \gamma_{xyz}(g) < 1 \) if and only if \( \nu_{R_0}(g) < r \). Furthermore, \( \gamma_{xyz}(g) = \infty \) if and only if \( g \in (x, y)^r \). Set \( T = \gamma_{xyz}(g) \). Define
\[ [g]_{xyz} = \sum_{(i+j) \gamma+k=1} b_{ijk} x^i y^j z^k. \] (83)

There is an expansion
\[ g = [g]_{xyz} + \sum_{(i+j) \gamma+k>r} b_{ijk} x^i y^j z^k. \] (84)

Returning to our series \( f \) which is a local equation of \( S \), set \( \gamma = \gamma_{xyz}(f) \),
\[ T_\gamma = \left\{ (i, j) \mid \frac{k}{r - (i + j)} = \gamma, i + j < r, \text{ for some } k \text{ such that } a_{ijk} \neq 0 \right\}. \]
Then
\[ [f]_{xyz} = L + \sum_{(i,j) \in T_\gamma} a_{i,j,\gamma(r-i-j)} x^i y^j z^{\gamma(r-i-j)}. \] (85)

**Definition 7.10.** \([f]_{xyz}\) is solvable if \( \gamma \in \mathbb{N} \) and there exists \( \alpha, \beta \in K \) such that
\[ [f]_{xyz} = L(x - \alpha z^\gamma, y - \beta z^\gamma). \]

**Lemma 7.11.** There exists a change of variables
\[ x_1 = x - \sum_{i=1}^{n} \alpha_i z^i, y_1 = y - \sum_{i=1}^{n} \beta_i z^i \]
such that \([f]_{xyz}\) is not solvable.

**Remark 7.12.** We can always choose \((x, y, z)\) that are regular parameters in \( O_{V_0,q_0} \). Then \((x, y, z)\) are also regular parameters in \( O_{V_0,q_0} \). However, since we are only blowing up maximal ideals in (81), we may work with any formal system of regular parameters \((x, y, z)\) in \( R_0 \) (see Lemma 5.3).

**Proof.** (of Lemma 7.11) If there doesn’t exist such a change of variables then there exists an infinite sequence of changes of variables for \( n \in \mathbb{N} \)
\[ x_n = x - \sum_{i=1}^{n} \alpha_i z^i, y_n = y - \sum_{i=1}^{n} \beta_i z^i \]
such that \( \gamma_n \in \mathbb{N} \) for all \( n \), \( \gamma_{n+1} = \gamma_{x_n y_n z}(f) \), and \( \gamma_{n+1} > \gamma_n \) for all \( n \). Let
\[ x_\infty = x - \sum_{i=1}^{\infty} \alpha_i z^i, y_\infty = y - \sum_{i=1}^{\infty} \beta_i z^i. \]

Thus
\[ \gamma_\infty = \gamma_{x_\infty y_\infty z}(f) = \infty. \]
and
\[ f \in (x - \sum_{i=1}^{\infty} \alpha_i z^i, y - \sum_{i=1}^{\infty} \beta_i z^i)^r \subset R_0. \]

But by construction of \( V_0, p \) is isolated in \( \text{Sing}_r(S) \). If \( I \subset \mathcal{O}_{V_0} \) is the reduced ideal defining \( \text{Sing}_r(S) \) at \( q \), then by Remark 10.21, \( \hat{I} = IR_0 \) is the reduced ideal whose support is the locus where \( f \) has multiplicity \( r \) in \( \text{spec}(R_0) \). Thus \((x, y, z) = \hat{I} \subset (x - \sum \alpha_i z^i, y - \sum \beta_i z^i)\), which is impossible.

We may thus assume that \([f]_{xyz} \) is not solvable. Since the leading form of \( f \) is \( L(x, y) \) and \( \tau(p) = 2 \), \( R_1 \) has regular parameters \( (x_1, y_1, z_1) \) defined by
\[ x = x_1 z_1, y = y_1 z_1, z = z_1. \quad (86) \]

Let \( f_1 = \frac{f}{x} = 0 \) be a local equation of the strict transform of \( f \) in \( R_1 \).

**Lemma 7.13.**
1. \( \gamma_{x_1 y_1 z}(f_1) = \gamma_{xyz}(f) - 1 \).
2. \([f_1]_{x_1 y_1 z} = \frac{1}{x^r}[f]_{xyz} \) so \([f]_{xyz} \) not solvable implies \([f_1]_{xyz} \) is not solvable.
3. If \( \gamma_{x_1 y_1 z}(f_1) > 1 \), then the leading form of \( f_1 \) is \( L(x_1, y_1) = \frac{1}{x^r} L(x, y) \).

**Proof.** The Lemma follows from substitution of \((86)\) in \((83), (84)\) and \((85)\). \( \square \)

We claim that the case \( \gamma_{x_1 y_1 z}(f_1) = 1 \) cannot occur (with our assumption that \( \nu_q(S_1) = r \) and \( \tau(q_1) = 2 \)). Suppose that it does. Then the leading form of \( f_1 \) is \([f_1]_{x_1 y_1 z} \) and there exists a form \( \Psi \) of degree \( r \) and \( \pi, \beta, \tau, d, \tau, f \in K \) such that
\[ [f_1]_{x_1 y_1 z} = \Psi(\pi x_1 + \beta y_1 + \tau z, d x_1 + \tau y_1 + f z). \]

We have an expression
\[ L(x_1, y_1) + z \Omega = \Psi(\pi x_1 + \beta y_1 + \tau z, d x_1 + \tau y_1 + f z) \]
so that
\[ L(x, y) = \Psi(\pi x + \beta y, d x + \tau y). \]

We have \( \omega \pi - \beta d \neq 0 \) since \( \tau(q) = 2 \). By Lemma 7.14 below, \([f_1]_{x_1 y_1 z} \) is thus solvable, a contradiction to 2. of Lemma 7.13.

Since \( \gamma_{x_1 y_1 z}(f_1) < 1 \) implies \( \nu_{R_1}(f_1) < r \), it now follows from Lemma 7.13 that after a finite sequence of blow ups \( R_0 \to R_i \), where \( R_i \) has regular parameters \( (x_i, y_i, z_i) \) with
\[ x = x_i z_i^1, y = y_i z_i^1, z = z_i \]
by 3. of Lemma 7.13, we reach a reduction \( \nu_q(S_i) < r \) or \( \nu_q(S_i) = r \), \( \tau(q_i) = 3 \), a contradiction to our assumption that \((82)\) has infinite length. Thus Theorem 7.9 has been proven when \( \tau(q) = 2 \).

**Lemma 7.14.** Let \( \Phi = \Phi(x, y), \Psi = \Psi(u, v) \) be forms of degree \( r \) over \( K \) and suppose that
\[ \Phi(x, y) = \Psi(\pi x + \beta y, d x + \tau y) \]
for some \( \pi, \beta, d, \tau \in K \) with \( \omega \pi - \beta d \neq 0 \). Then for all \( \pi, \tau, f \in K \) there exist \( \alpha, \beta \in K \) such that
\[ \Phi(x + \alpha z, y + \beta z) = \Psi(\pi x + \beta y + \tau z, d x + \tau y + f z). \]
Lemma 7.15. Let \( \tau \) be the line through \((a, b, c, d)\) with slope \(-1\), \(V\) the subvariety of \(\mathbb{P}^3\) defined by \((a, b, c, d)\) and \(\nu_{q_n}(S_n) = r, \tau(q_n) = 1\) for all \(n\). After possibly reindexing, we may assume that for all \(n, q_n\) is on the subvariety of \(V\) which is blownup under \(V_{n+1} \to V_n\). Let \(R_n = \mathcal{O}_{V_{n}, q_n}\) for \(n \geq 0\). We then have an infinite sequence

\[ R = R_0 \to R_1 \to \cdots \to R_n \to \cdots \tag{87} \]

Let \(J_i \subset \mathcal{O}_{V_{i}, q_i}\) be the reduced ideal defining \(\text{Sing}_r(S_i)\) at \(q_i\). Then by Lemmas 7.4 and 7.5, \(J_i\) is either the maximal ideal \(m_i\) of \(\mathcal{O}_{V_{i}, q_i}\) or a regular height 2 prime ideal \(p_i\) in \(\mathcal{O}_{V_{i}, q_i}\). Then the multiplicity \(r\) locus of \(\mathcal{O}_{S_i, q_i}\) (which is a quotient of \(R_i\) by a principal ideal) is defined by the reduced ideal \(J_i = J_iR_i\) by Remark 10.21. \(J_i = \hat{m}_i = m_iR_i\) or \(J_i = \hat{p}_i = p_iR_i\), a regular prime in \(R_i\). By the construction of the sequence (81), we see that \(\mathcal{O}_{V_{i+1}, q_{i+1}}\) is a local ring of the blow up of \(m_i\) if \(J_i = \hat{m}_i\) and a local ring of the blow up of \(p_i\) otherwise. Thus \(R_{i+1}\) is the completion of a local ring of the blow up of \(\hat{m}_i\) or \(\hat{p}_i\).

We are thus free to work with formal parameters and equations (which define the ideal \(\hat{I}_{S_i, q_i} = \hat{I}_{S_i, q_i}R_i\) in the \(R_i\), since the ideals \(\hat{m}_i\) and \(\hat{p}_i\) are determined in \(R_i\) by the multiplicity \(r\) locus of \(\text{Sing}_r(\mathcal{O}_{S_i, q_i})\). Suppose that \(T = K[[x, y, z]]\) is a power series ring, \(r \in \mathbb{N}\) and

\[ g = \sum b_{ijk}x^iy^jz^k \in T. \]

We can construct a polygon in the following way.

Define

\[ \Delta = \Delta(g; x, y, z) = \left\{ \left( \frac{i}{r - k}, \frac{j}{r - k} \right) \in \mathbb{Q}^2 \mid k < r \text{ and } b_{ijk} \neq 0 \right\}. \]

Let \(|\Delta|\) be the smallest convex set in \(\mathbb{R}^2\) such that \(\Delta \subset |\Delta|\) and \((a, b) \in |\Delta|\) implies \((a + c, b + d) \in |\Delta|\) for all \(c, d \geq 0\). For \(a \in \mathbb{R}\), let \(S(a)\) be the line through \((a, 0)\) with slope -1, \(V(a)\) be the vertical line through \((a, 0)\).

Suppose that \(|\Delta| \neq \emptyset\). We define \(\alpha_{xyz}(g)\) to be the smallest \(a\) appearing in any \((a, b) \in |\Delta|\), \(\beta_{xyz}(g)\) to be the smallest \(b\) such that \((\alpha_{xyz}(g), b) \in |\Delta|\). Let \(\gamma_{xyz}(g)\) be the smallest number \(\gamma\) such that \(S(\gamma) \cap |\Delta| \neq \emptyset\) and let \(\delta_{xyz}(g)\) be such that \((\gamma_{xyz}(g) - \delta_{xyz}(g), \delta_{xyz}(g))\) is the lowest point on \(S(\gamma_{xyz}(g)) \cap |\Delta|\). \((\alpha_{xyz}(g), \beta_{xyz}(g))\) and \((\gamma_{xyz}(g) - \delta_{xyz}(g), \delta_{xyz}(g))\) are vertices of \(|\Delta|\). Define \(\epsilon_{xyz}(g)\) to be the absolute value of the largest slope of a line through \((\alpha_{xyz}(g), \beta_{xyz}(g))\) such that no points of \(|\Delta|)\) are points of \(|\Delta|\), which lie below it.

Lemma 7.15. The vertices of \(|\Delta|\) are points of \(\Delta\), which lie on the lattice \(\frac{1}{r} \mathbb{Z} \times \frac{1}{r} \mathbb{Z}\).
2. \( \nu_R(g) < r \) holds if and only if \( |\Delta| \) contains a point on \( S(c) \) with \( c < 1 \) which holds if and only if there is a vertex \((a, b)\) with \( a + b < 1 \).

3. \( \alpha_{xyz}(g) < 1 \) if and only if \( g \notin (x, z)^r \).

4. A vertex of \( |\Delta| \) lies below the line \( b = 1 \) if and only if \( g \notin (y, z)^r \).

**Proof.** The proof of Lemma 7.15 follows directly from the definitions. \( \square \)

Suppose that \( g \in T \) is such that \( \nu_T(g) = r \). Let

\[
L = \sum_{i+j+k=r} b_{ijk} x^i y^j z^k
\]

be the leading form of \( g \). \((x, y, z)\) will be called good parameters for \( g \) if \( b_{00r} = 1 \).

We now suppose that \( \nu_T(g) = r \), and \((x, y, z)\) are good parameters for \( g \). Let \( \Delta = \Delta(g; x, y, z) \) and suppose that \((a, b)\) is a vertex of \( |\Delta| \). Let

\[
S(a, b) = \{ k | \left( \frac{i}{r-k}, \frac{j}{r-k} \right) = (a, b) \text{ and } b_{ijk} \neq 0 \}.
\]

Define

\[
\{g\}_{xy}^{ab} = z^r + \sum_{k \in S(a, b)} b_{a(r-k), b(r-k), k} x^{a(r-k)} y^{b(r-k)} z^k.
\]

We will say that \( \{g\}_{xy}^{ab} \) is solvable (or that the vertex \((a, b)\) is not prepared on \( |\Delta(g; x, y, z)| \)) if \( a, b \) are integers and

\[
\{g\}_{xy}^{ab} = (z - \eta x^a y^b)^r
\]

for some \( \eta \in K \). We will say that the vertex \((a, b)\) is prepared on \( |\Delta(g; x, y, z)| \) if \( \{g\}_{xy}^{ab} \) is not solvable. If \( \{g\}_{xy}^{ab} \) is solvable, we can make an \((a, b)\) preparation, which is the change of parameters

\[
z_1 = z - \eta x^a y^b.
\]

If all vertices \((a, b)\) of \( |\Delta| \) are prepared, then we say that \( g; x, y, z \) is well prepared.

**Lemma 7.16.** Suppose that \( T \) is a power series ring over \( K \), \( g \in T \) is such that \( \nu_T(g) = r \), \( \tau(g) = 1 \), \((x, y, z)\) are good parameters of \( g \), and \( g; x, y, z \) is well prepared. Then \( z = 0 \) is an approximate manifold of \( g = 0 \).

**Proof.** \( \nu_T(g) = r \) implies all vertices \((a, b)\) of \( |\Delta(g; x, y, z)| \) satisfy \( a + b \geq 1 \). The leading form of \( g \) is

\[
L = \sum_{i+j+k=r} b_{ijk} x^i y^j z^k
\]

where \( b_{00r} = 1 \). \( \tau(g) = 1 \) implies there exists a linear form \( \pi x + \tilde{b} y + \tau z \) such that

\[
(\pi x + \tilde{b} y + \tau z)^r = L.
\]

As \( b_{00r} = 1 \), we must have \( \tau = 1 \). If \( \pi \neq 0 \), then \((1, 0)\) is a vertex of \( |\Delta(g; x, y, z)| \), which is not prepared since

\[
\{g\}_{xy}^{(1,0)} = (z + \pi x)^r.
\]

Thus \( \pi = 0 \). If \( \tilde{b} \neq 0 \), then \((0, 1)\) is a vertex of \( |\Delta(g; x, y, z)| \) which is not prepared since

\[
\{g\}_{xy}^{(0,1)} = (z + \tilde{b} y)^r.
\]

Thus \( L = z^r \) and \( z = 0 \) is an approximate manifold of \( g = 0 \). \( \square \)
Lemma 7.17. Consider the terms in the expansion
\[ h = \sum_{\lambda=0}^{k} \eta^k y^k \binom{k}{\lambda} x^{i+(k-\lambda)a} y^{j+(k-\lambda)b} z_1^\lambda \]  
(88)
obtained by substituting \( z_1 = z - \eta x^a y^b \) into the monomial \( x^i y^j z^k \). Define a projection for \((a,b,c) \in \mathbb{N}^3\) such that \( c < r \),
\[ \pi(a,b,c) = \left( \frac{a}{r-c}, \frac{b}{r-c} \right). \]

Then

1. Suppose that \( k < r \). Then the exponents of monomials in (88) with non-zero coefficients project into the line segment joining \((a,b)\) to \((\frac{i}{r-k}, \frac{j}{r-k})\).
   a. If \((a,b) = (\frac{i}{r-k}, \frac{j}{r-k})\), then all these monomials project to \((a,b)\).
   b. If \((a,b) \neq (\frac{i}{r-k}, \frac{j}{r-k})\), then \( x^i y^j z_1^k \) is the unique monomial in (88) which projects onto \((\frac{i}{r-k}, \frac{j}{r-k})\). No monomial in (88) projects to \((a,b)\).

2. Suppose that \( r \leq k \), and \((i,j,k) \neq (0,0,r)\). Then all exponents in (88) with non-zero coefficients and \( z_1 \) exponent less that \( r \) project into \( (a,b) + \mathbb{Q} \mathbb{Z} \) - \{(a,b)\}.

3. Suppose that \((i,j,k) = (0,0,r)\). Then all exponents in (88) with non-zero coefficients and \( z_1 \) exponent less than \( r \) project to \((a,b)\).

Proof. If \( k \neq r \), we have
\[ \frac{i+(k-\lambda)a}{r-\lambda} = a + \frac{(r-k)}{r-\lambda} \left( \frac{i}{r-k} - a \right) \]
\[ \frac{j+(k-\lambda)b}{r-\lambda} = b + \frac{(r-k)}{r-\lambda} \left( \frac{j}{r-k} - b \right). \]
Suppose that \( k < r \). Since \( 0 \leq \lambda \leq k \), we have \( 0 < \frac{r-k}{r-\lambda} \leq 1 \). Thus
\[ \left( \frac{i + (k-\lambda)a}{r-\lambda}, \frac{j + (k-\lambda)b}{r-\lambda} \right) \]
is on the line segment joining \((a,b)\) to \((\frac{i}{r-k}, \frac{j}{r-k})\) and 1. follows.

If \( k > r \) and \( 0 \leq \lambda < r \leq k \), then \( \frac{r-k}{r-\lambda} < 0 \), \( \frac{i}{r-k} - a < 0 \) and \( \frac{j}{r-k} - b < 0 \). Thus 2. follows if \( r \neq k \).

Suppose that \( k = r \) and \( 0 \leq \lambda < r \). Then
\[ \left( \frac{i + (k-\lambda)a}{r-\lambda}, \frac{j + (k-\lambda)b}{r-\lambda} \right) = \left( \frac{i}{r-\lambda} + a, \frac{j}{r-\lambda} + b \right) \]
and the last case of 2. and 3. follow. \( \square \)

We deduce from this Lemma that

Lemma 7.18. Suppose that \( z_1 = z - \eta x^a y^b \) is an \((a,b)\) preparation. Then
(1) \( | \Delta(g; x, y, z_1) | \subseteq | \Delta(g; x, y, z) | - \{(a,b)\} \).
(2) If \((a',b')\) is another vertex of \( | \Delta(g; x, y, z) | \), then \((a',b')\) is a vertex of
\[ | \Delta(g; x, y, z_1) | \]
and \( \{y \}_{x,y,z_1} \) is obtained from \( \{y \}_{x,y,z} \) by substituting \( z_1 \) for \( z \).
Example 7.19. It is not always possible to well prepare after a finite number of vertex preparations. Consider over a field of characteristic 0 (or \( p > r \))

\[
g_m = y(y-x)(y-2x) \cdots (y-rx) + (z-x^m + x^{m+1} - \cdots)^r
\]

with \( m \geq 2 \). The vertices of \( | \Delta(g;x,y,z) | \) are \((0,1+\frac{1}{r})\), \((1,\frac{1}{r})\) and \((m,0)\). We can remove the vertex \((m,0)\) by the \((m,0)\) preparation \( z_1 = z-x^m \). Thus

\[
g = y(y-x)(y-2x) \cdots (y-rx) + (z-x^2 + x^3 - \cdots)^r
\]
can only be well prepared by the formal substitution

\[
z_1 = z - \sum_{i=2}^{\infty} (-1)^i x^i.
\]

Lemma 7.20. Suppose that \( g \) is reduced, \( \nu_T(g) = r \), \( \tau(g) = 1 \) and \((x,y,z)\) are good parameters for \( g \). Then there is a formal series \( \phi(x,y) \in K[[x,y]] \) such that under the substitution \( z = z_1 + \phi(x,y) \), \((x,y,z_1)\) are good parameters for \( g \) and \((g;x,y,z_1)\) is well prepared.

\[
\begin{align*}
\text{Proof.} \quad & \text{Let } v_L \text{ be the lowest vertex of } | \Delta(g;x,y,z) |. \quad \text{Let } h(v_L) \text{ be the second coordinate of } v_L. \quad \text{Set } b = h(v_L). \quad \text{If } v_L \text{ is not prepared, make a } v_L \text{ preparation } z_1 = z-\eta x^a y^b \\
& \text{where } (a,b) = v_L \text{ to remove } v_L \text{ in } | \Delta(g;x,y,z) |. \quad \text{Let } v_{L_1} \text{ be the lowest vertex of } | \Delta(g;x,y,z_1) |. \quad \text{If } v_{L_1} \text{ is not prepared and } h(v_{L_1}) = h(v_L) \text{ we can again prepare } v_{L_1} \text{ by a } v_{L_1} \text{ preparation } z_2 = z - \eta_1 x^b y^b. \quad \text{We can iterate this procedure to either achieve } | \Delta(g;x,y,z_n) | \text{ such that the lowest vertex } v_{L_n} \text{ is solvable or } h(v_{L_n}) > h(v_L), \text{ or we can construct an infinite sequence of } v_{L_n} \text{ preparations (with } h(v_{L_n}) = b \text{ for all } n) \\
& z_{n+1} = z_n - \eta_n x^a y^b
\end{align*}
\]

where \((a_n,b) = v_{L_n} \text{ such that the lowest vertex } v_{L_n} \text{ of } | \Delta(g;x,y,z_n) | \text{ is not prepared and } h(v_{L_n}) = h(v_L) \text{ for all } n. \quad \text{Since } a_{n+1} > a_n \text{ for all } n, \text{ we can then make the formal substitution } z' = z - \sum_{i=0}^{\infty} \eta_i x^{a_i} y^b, \text{ to get } | \Delta(g;x,y,z') | \text{ whose lowest vertex } v_{L'} \text{ satisfies } h(v_{L'}) > h(v_L).

In summary, there exists a series \( \Phi'(x) \) such that if we set \( z' = z - \sum h(v_{L_i}) \Phi'(x) \), and \( v_{L'} \) is the lowest vertex of \( | \Delta(g;x,y,z') | \) then either \( v_{L'} \) is prepared or \( h(v_{L'}) > h(v_L) \).

By iterating this procedure, we construct a series \( \Phi(x,y) \) such that if \( z = z - \Phi(x,y) \), then either the lowest vertex \( v_T \) of \( | \Delta(g;x,y,z) | \) is prepared, or \( | \Delta(g;x,y,z) | \) = \( \emptyset \). This last case only occurs if \( g = \text{unit } z' \), and thus cannot occur since \( g \) is assumed to be reduced.

We can thus assume that \( v_L \) is prepared. We can now apply the same procedure to \( v_T \), the highest vertex of \( | \Delta(g;x,y,z) | \), to reduce to the case where \( v_T \) and \( v_L \) are both prepared. Then after a finite number of preparations we find a change of variables \( z' = z - \Phi(x,y) \) such that \( (g;x,y,z') \) is well prepared. \( \square \)

We will also consider change of variables of the form

\[
y_1 = y - \eta x^n
\]

for \( \eta \in K, n \) a positive integer, which we will call translations.
Lemma 7.21. Consider the expansion
\[ h = \sum_{\lambda=0}^{j} \eta^{j-\lambda} \left( \frac{j}{\lambda} \right) x^{i+(j-\lambda)n} y_1^\lambda z^k \] (89)

obtained by substituting \( y_1 = y - \eta x^n \) into the monomial \( x^i y^j z^k \). Consider the projection for \((a,b,c) \in \mathbb{N}^3\) such that \( c < r \) defined by
\[ \pi(a,b,c) = \left( \frac{a}{r-c}, \frac{b}{r-c} \right). \]

Suppose that \( k < r \). Set \((a,b) = (\frac{1}{r-\epsilon}, \frac{j}{r-\epsilon})\). Then \( x^i y_1^j z^k \) is the unique monomial in (89) whose coefficients projects onto \((a,b)\). All other monomials in (89) with non-zero coefficient project to points below \((a,b)\) on the line through \((a,b)\) with slope \(-\frac{1}{n}\).

Proof. The slope of the line through the point \((a,b)\) and
\[ \left( \frac{i + (j-\lambda)n}{r-\lambda}, \frac{\lambda}{r-\lambda} \right) \]
is
\[ \frac{j-\lambda}{i - (i + (j-\lambda)n)} = -\frac{1}{n}. \]
Since \( \frac{\lambda}{r-\lambda} < \frac{j}{r-\lambda} \) for \( \lambda < j \), the Lemma follows. \( \square \)

Definition 7.22. Suppose that \( g \in T \) is reduced, \( \nu_T(g) = r \), \( \tau(g) = 1 \) and \((x,y,z)\) are good parameters for \( g \). Let \( \alpha = \alpha_{xyz}(g), \beta = \beta_{xyz}(g), \gamma = \gamma_{xyz}(g), \delta = \delta_{xyz}(g), \epsilon = \epsilon_{xyz}(g) \). \((g; x, y, z)\) will be said to be very well prepared if it is well prepared and one of the following conditions holds.

1. \((\gamma - \delta, \delta) \neq (\alpha, \beta)\) and if we make a translation \( y_1 = y - \eta x, \) with subsequent well preparation \( z_1 = z - \Phi(x, y), \) then \( \alpha_{xyz_1}(g) = \alpha, \beta_{xyz_1}(g) = \beta, \gamma_{xyz_1}(g) = \gamma \) and \( \delta_{xyz_1}(g) \leq \delta. \)

2. \((\gamma - \delta, \delta) = (\alpha, \beta)\) and one of the following cases hold:
   a. \( \epsilon = 0 \)
   b. \( \epsilon \neq 0 \) and \( \frac{1}{\epsilon} \) is not an integer
   c. \( \epsilon \neq 0 \) and \( n = \frac{1}{\epsilon} \) is a (positive) integer and for any \( \eta \in K, \) if \( y_1 = y - \eta x^n \) is a translation, with subsequent well preparation \( z_1 = z - \Phi(x, y), \) then \( \epsilon_{xyz_1}(g) = \epsilon. \) Further, if \((c, d)\) is the lowest point on the line through \((\alpha, \beta)\) with slope \(-\epsilon\) in \( |\Delta(g; x, y, z_1)| \) and \((c_1, d_1)\) is the lowest point on this line in \( |\Delta(g; x, y_1, z_1)| \), then \( d_1 \leq d. \)

Lemma 7.23. Suppose that \( g \in T \) is reduced, \( \nu_T(g) = r, \tau(g) = 1 \) and \((x,y,z)\) are good parameters for \( g \). Then there are formal substitutions
\[ z_1 = z - \phi(x, y), y_1 = y - \psi(x) \]
where \( \phi(x, y), \psi(x) \) are series such that \((g; x, y_1, z_1)\) is very well prepared.
Proof. By Lemma 7.20, we may suppose that \((g; x, y, z)\) is well prepared. Let \(\alpha = \alpha_{xyz}(g)\), \(\beta = \beta_{xyz}(g)\), \(\gamma = \gamma_{xyz}(g)\), \(\delta = \delta_{xyz}(g)\), \(\epsilon = \epsilon_{xyz}(g)\). Let \((\gamma - t, t)\) be the highest vertex on \(\Delta(g; x, y, z)\) on the line \(a + b = \gamma\). Let \(\eta \in K\) and consider the translation \(y_1 = y - \eta x\). By Lemma 7.21, \((\gamma - t, t)\) and \((\alpha, \beta)\) are prepared vertices of \(|\Delta(g; x, y_1, z)|\). Thus if \(z_1 = z - \Phi(x, y)\) is a well preparation of \((g; x, y_1, z)\), then \((\gamma - t, t)\) and \((\alpha, \beta)\) are vertices of \(|\Delta(g; x, y_1, z_1)|\).

We can thus choose \(\eta \in K\) so that after the translation \(y_1 = y - \eta x\), and a subsequent well preparation \(z_1 = z - \Phi(x, y)\), we have \(\alpha_{x,y_1,z_1} = \alpha\), \(\beta_{x,y_1,z_1} = \beta\), \(\gamma_{x,y_1,z_1} = \gamma\) and \(\delta_{x,y_1,z_1} \geq \delta\) is a maximum for substitutions \(y_1 = y - \eta x\), \(z_1 = z - \Phi(x, y)\) as above. If \((\gamma - \delta_{x,y_1,z_1}, \delta_{x,y_1,z_1}) \neq (\alpha, \beta)\), then after replacing \((x, y, z)\) with \((x, y_1, z_1)\), we have achieved the condition 1. of the Lemma, so that \((g; x, y, z)\) is very well prepared.

We now assume that \((g; x, y, z)\) is well prepared, and \((\alpha, \beta) = (\gamma - \delta, \delta)\). If \(\epsilon = 0\) or \(\frac{1}{\epsilon}\) is not an integer then \((g; x, y, z)\) is very well prepared.

Suppose that \(n = \frac{1}{\epsilon}\) is an integer. We then choose \(\eta \in K\) such that with the translation \(y_1 = y - \eta x^n\), and subsequent well preparation, we maximize \(d\) for points \((c, d)\) of the line through \((\alpha, \beta)\) with slope \(-\epsilon\) on the boundary of \(|\Delta(f; x, y_1, z)|\). By Lemma 7.21 \(\alpha, \beta\) and \(\gamma\) are not changed. If we now have that \(d \neq \beta\), we are very well prepared.

If this process does not end after a finite number of iterations, then there exists an infinite sequence of changes of variables (translations followed by well preparations)

\[y_i = y_{i+1} + \eta_{i+1} x^{n_{i+1}}, \ z_i = z_{i+1} + \phi_i(x, y_{i+1})\]

such that \(n_{i+1} > n_i\) for all \(i\). Given \(n \in \mathbb{N}\), there exists \(\sigma(n) \in \mathbb{N}\) such that \(\sigma(n) > \sigma(n - 1)\) for all \(n\), and \(i \geq \sigma(n)\) implies all vertices \((a, b)\) of \(|\Delta(g; x, y_i, z_i)|\) below \((\alpha, \beta)\) have \(a \geq n\). This last condition follows since all vertices must lie in the lattice \(\frac{1}{\epsilon} \mathbb{Z} \times \frac{1}{\epsilon} \mathbb{Z}\), and there are only finitely many points common to this lattice and the region \(0 \leq a \leq n, \ 0 \leq b \leq \beta\). Thus \(x^n i \phi_i(x, y_i)\) if \(i \geq \sigma(n)\). Let

\[\Psi_i(x, y) = \sum_{j=1}^{i} \phi_j(x, y - \sum_{k=1}^{j} \eta_k x^{n_k}),\]

so that \(z_i = z - \Psi_i(x, y)\) for all \(i\). \(\{\Psi_i(x, y)\}\) is a Cauchy sequence in \(K[[x, y]]\). Thus \(z' = z - \sum_{i=1}^{\infty} \phi_i(x, y_i)\) is a well defined series in \(K[[x, y, z]]\). Set \(y' = y - \sum_{i=1}^{\infty} \eta_i x^{n_i}\). \(|\Delta(g; x, y', z')|\) then has the single vertex \((\alpha, \beta)\), and \((g; x, y', z')\) is thus very well prepared.

**Definition 7.24.** Suppose that \(g \in T\), \(g\) is reduced, \(\nu_T(g) = r\), \(\tau(g) = 1\), and \((x, y, z)\) are good parameters for \(g\). We consider 4 types of monoidal transforms \(T \to T_1\), where \(T_1\) is the completion of the local ring of a monoidal transform of \(T\), and \(T_1\) has regular parameters \((x_1, y_1, z_1)\) related to the regular parameters \((x, y, z)\) of \(T\) by one of the following rules.

**T1 Singr(g) = V(x, y, z),**

\[x = x_1, y = x_1(y_1 + \eta), z = x_1z_1,\]

with \(\eta \in K\). Then \(g_1 = \frac{2}{x_1}\) is the strict transform of \(g\) in \(T_1\), and if \(\nu_{T_1}(g_1) = r\) and \(\tau(g_1) = 1\), then \((x_1, y_1, z_1)\) are good parameters for \(g_1\).
T2  \( \text{Sing}_r(g) = V(x, y, z), \)

\[ x = x_1 y_1, y = y_1, z = y_1 z_1. \]

Then \( g_1 = \frac{a}{y_1} \) is the strict transform of \( g \) in \( T_1 \), and if \( \nu_1(g_1) = r \) and \( \tau(g_1) = 1 \), then \((x_1, y_1, z_1)\) are good parameters for \( g_1 \).

T3  \( \text{Sing}_r(g) = V(x, z), \)

\[ x = x_1, y = y_1, z = x_1 z_1. \]

Then \( g_1 = \frac{a}{x_1} \) is the strict transform of \( g \) in \( T_1 \), and if \( \nu_1(g_1) = r \) and \( \tau(g_1) = 1 \), then \((x_1, y_1, z_1)\) are good parameters for \( g_1 \).

T4  \( \text{Sing}_r(g) = V(y, z), \)

\[ x = x_1, y = y_1, z = y_1 z_1. \]

Then \( g_1 = \frac{a}{y_1} \) is the strict transform of \( g \) in \( T_1 \), and if \( \nu_1(g_1) = r \) and \( \tau(g_1) = 1 \), then \((x_1, y_1, z_1)\) are good parameters for \( g_1 \).

**Lemma 7.25.** With the assumptions of Definition 7.24, suppose that \( g \in T \) is reduced, \( \nu_1(g) = r \), \( \tau(g) = 1 \), \((x, y, z)\) are good parameters for \( g \), \((x, y, z)\) and \((x_1, y_1, z_1)\) are related by a transformation of one of the above types \( T_1 - T_4 \), and \( \nu_1(g_1) = r_1 \), \( \tau(g_1) = 1 \). Then there is a 1-1 correspondence

\[ \sigma : \Delta(g, x, y, z) \to \Delta(g_1, x_1, y_1, z_1) \]

defined by

1. \( \sigma(a, b) = (a + b - 1, b) \) if the transformation is a \( T_1 \) with \( \eta = 0 \).
2. \( \sigma(a, b) = (a, a + b - 1) \) if the transformation is a \( T_2 \).
3. \( \sigma(a, b) = (a - 1, b) \) if the transformation is a \( T_3 \).
4. \( \sigma(a, b) = (a, b - 1) \) if the transformation is a \( T_4 \).

The proof of Lemma 7.25 is straightforward, and is left to the reader.

**Lemma 7.26.** In each of the four cases of the preceding lemma, if \( \sigma(a, b) = (a_1, b_1) \) is a vertex of \( \mid \Delta(g_1; x_1, y_1, z_1) \mid \), then \((a, b)\) is a vertex of \( \mid \Delta(g; x, y, z) \mid \), and if \((g; x, y, z)\) is \((a, b)\) prepared then \((g_1; x_1, y_1, z_1)\) is \((a_1, b_1)\) prepared. In particular, \((g_1; x_1, y_1, z_1)\) is well prepared if \((g; x, y, z)\) is well prepared.

**Proof.** In all cases, \( \sigma \) (when extended to \( \mathbb{R}^3 \)) takes line segments to line segments and interior points of \( \mid \Delta(g; x, y, z) \mid \) to interior points of \( \mid \Delta(g_1; x_1, y_1, z_1) \mid \). Thus the boundary of \( \sigma(\mid \Delta(g; x, y, z) \mid) \) is the union of the image by \( \sigma \) of the line segments on the boundary of \( \sigma(\mid \Delta(g; x, y, z) \mid) \). If \((a_1, b_1)\) is a vertex of \( \sigma(\mid \Delta(g; x, y, z) \mid) \), it must then necessarily be \( \sigma(a, b) \) with \((a, b)\) a vertex of \( \sigma(\mid \Delta(g; x, y, z) \mid) \).

To see that \((g_1; x_1, y_1, z_1)\) is \((a_1, b_1)\) prepared if \((g; x, y, z)\) is \((a, b)\) prepared, we observe that

\[ \{g_1\}^{a_1, b_1}_{x_1, y_1, z_1} = \begin{cases} \frac{1}{\eta b} \{g\}^{a, b}_{x, y, z} & \text{if a T3 transformation or a T1 with } \eta = 0 \\ \frac{1}{\eta a} \{g\}^{a, b}_{x, y, z} & \text{if a T2 or T4 transformation} \end{cases} \]

\[ \square \]

**Lemma 7.27.** Suppose that assumptions are as in Definition 7.24 and \((x, y, z)\) and \((x_1, y_1, z_1)\) are related by a \( T_3 \) transformation. Suppose that \( \nu_1(g_1) = r \) and \( \tau(g_1) = 1 \). If \((g; x, y, z)\) is very well prepared, then \((g_1; x_1, y_1, z_1)\) is very well prepared.

\[ \beta_{x_1 y_1 z_1} (g_1) = \beta_{xyz} (g), \delta_{x_1 y_1 z_1} (g_1) = \delta_{xyz} (g), \epsilon_{x_1 y_1 z_1} (g_1) = \epsilon_{xyz} (g) \]
and
\[ \alpha_{x_1 y_1 z_1} (g_1) = \alpha_{xyz} (g) - 1, \gamma_{x_1 y_1 z_1} (g_1) = \gamma_{xyz} (g) - 1. \]
We further have \( \text{Sing}_r (g_1) \subset V (x_1, z_1) \).

**Proof.** Well preparation is preserved by Lemma 7.26. Suppose that \((g; x, y, z)\) is very well prepared and
\[ (\alpha_{xyz}, \beta_{xyz}) \neq (\gamma_{xyz} - \delta_{xyz}, \delta_{xyz}). \]
Let \( \gamma = \gamma_{xyz} (g) \), \( \gamma_1 = \gamma_{xyz} (g_1) = \gamma - 1 \). The terms in \( g \) which contribute to the line \( S(\gamma) \) are
\[
\sum_{i + j + \gamma k = r \gamma} b_{ijk} x^i y^j z^k. \tag{90}
\]
They are transformed to
\[
\sum_{(i + k - r) + j + \gamma_1 k = r \gamma_1} b_{ijk} x_1^{i + k - r} y_1^j z_1^k \tag{91}
\]
which are the terms in \( g_1 \) which contribute to the line \( S(\gamma_1) \). We have
\[ \delta_{xyz} (g) = \delta_{x_1 y_1 z_1} (g_1) = \min \{ j \middle| \frac{j}{r - k} \in \mathbb{Z}, b_{ijk} \neq 0 \text{ in } (90) \}. \]
If a translation \( y = \gamma + \eta x \) transforms (90) into
\[
\sum_{i + j + \gamma k = r \gamma} c_{ijk} x^i \gamma^j z^k, \tag{92}
\]
then the translation \( y_1 = \gamma_1 + \eta x \) transforms (91) into
\[
\sum_{(i + k - r) + j + \gamma_1 k = r \gamma_1} c_{ijk} x_1^{i + k - r} \gamma_1^j z_1^k. \tag{93}
\]
Since (92) and (93) are the terms which contribute to \( \delta_{xyz} (g) \) and \( \delta_{x_1 y_1 z_1} (g_1) \), we have \( \delta_{xyz} (g) = \delta_{x_1 y_1 z_1} (g_1) \) and \((g; x, y, z)\) well prepared implies \((g_1; x_1, y_1, z_1)\) well prepared.

We can make a similar argument if
\[ (\alpha_{xyz}, \beta_{xyz}) = (\gamma_{xyz} - \delta_{xyz}, \delta_{xyz}). \]

The statement \( \text{Sing}_r (g_1) \subset V (x_1, z_1) \) follows since \( \text{Sing}_r (g) = V (x, y, z) \), so that \( \text{Sing}_r (g_1) \subset V (x_1) \), the exceptional divisor of \( T_3 \), and \( \text{Sing}_r (g_1) \subset V (z_1) \) since \( z = 0 \) is an approximate manifold of \( g = 0 \) (Lemmas 7.16 and 7.5).

**Lemma 7.28.** Suppose that assumptions are as in Definition 7.24, \((x, y, z)\) and \((x_1, y_1, z_1)\) are related by a \( T_1 \) transformation with \( \eta = 0 \), \((g; x, y, z)\) is very well prepared.
prepared and \( \nu \), \( \tau(g_1) = 1 \). Let \((g_1; x_1, y', z')\) be a very well preparation of \((g_1; x_1, y_1, z_1)\). Let

\[
\alpha = \alpha_{xyz}(g), \beta = \beta_{xyz}(g), \gamma = \gamma_{xyz}(g), \delta = \delta_{xyz}(g), \epsilon = \epsilon_{xyz}(g),
\]

\[
\alpha_1 = \alpha_{x_1,y,z}(g_1), \beta_1 = \beta_{x_1,y,z}(g_1), \gamma_1 = \gamma_{x_1,y,z}(g_1), \delta_1 = \delta_{x_1,y,z}(g_1), \epsilon_1 = \epsilon_{x_1,y,z}(g_1).
\]

Then

\[
(\beta_1, \delta_1, \frac{1}{\epsilon_1}, \alpha_1) < (\beta, \delta, \frac{1}{\epsilon}, \alpha)
\]

in the Lexicographical order, with the convention that if \( \epsilon = 0 \), then \( \frac{1}{\epsilon} = \infty \). If \( \beta_1 = \beta \), \( \delta_1 = \delta \) and \( \epsilon \neq 0 \), then \( \frac{1}{\epsilon_1} = \frac{1}{\epsilon} - 1 \).

We further have \( \text{Sing}_1(g_1) \subset V(x_1, z') \).

**Proof.** Let \( \sigma : \Delta(g;x,y,z) \to \Delta(g_1; x_1,y_1, z_1) \) be the 1-1 correspondence of Lemma 7.25 1., which is defined by \( \sigma(a,b) = (a+b-1, b) \). \( \sigma \) transforms lines of slope \( m \neq -1 \) to lines of slope \( \frac{m}{m+1} \), and transforms lines of slope \(-1\) to vertical lines.

First suppose that \(-\epsilon \leq -1\). Then \( (\alpha, \beta) \neq (\gamma - \delta, \delta) \), and we are in case 1. of Definition 7.22. Then the line segment through \((\alpha, \beta)\) and \((\gamma - \delta, \delta)\) (which has slope \(-1\)) is transformed to a segment with positive slope, or a vertical line. Thus \( \beta_{x_1,y_1,z_1} = \delta < \beta \). Since \((g;x_1,y_1,z_1)\) is well prepared, after very well preparation we have \( \beta_1 < \beta \).

Suppose that \(-1 < -\epsilon \leq -\frac{1}{2} \). We have \((\alpha, \beta) = (\gamma - \delta, \delta) \). Then the line segment through \((\alpha, \beta)\) and \((c,d)\) (with the notation of Definition 7.22) is transformed to the line segment through \((\alpha + \beta - 1, \beta)\) and \((c + d - 1, d)\) which has slope

\[
-\epsilon_{x_1,y_1,z_1} = \frac{-\epsilon}{-\epsilon + 1} < -1.
\]

Thus

\[
(\alpha_{x_1,y_1,z_1}, \beta_{x_1,y_1,z_1}) = (\alpha + \beta - 1, \beta)
\]

and

\[
(\gamma_{x_1,y_1,z_1}(g_1) - \delta_{x_1,y_1,z_1}(g_1), \delta_{x_1,y_1,z_1}(g_1)) \neq (\alpha_{x_1,y_1,z_1}(g_1), \beta_{x_1,y_1,z_1}(g_1)).
\]

Let \((\gamma_{x_1,y_1,z_1}(g_1) - t, t)\) be the highest point on the line \( S(\gamma_{x_1,y_1,z_1}(g_1)) \) in \( |\Delta(g_1; x_1, y_1, z_1)| \). By Lemma 7.26, \((g_1; x_1, y_1, z_1)\) is well prepared. Thus (by Lemma 7.21) very well preparation does not affect the vertices \((\alpha_{x_1,y_1,z_1}(g_1), \beta_{x_1,y_1,z_1}(g_1))\) and \((\gamma_{x_1,y_1,z_1}(g_1) - t, t)\) of \( |\Delta(g_1; x_1, y_1, z_1)| \) which are below the horizontal line of points whose second coordinate is \( d \). Thus \( \beta_1 = \beta \) and \( \delta_1 \leq t \leq d < \delta \).

Suppose that \(-\frac{1}{2} < -\epsilon < 0 \). In this case we have \((\alpha, \beta) = (\gamma - \delta, \delta) \). Then the line segment through \((\alpha, \beta)\) and \((c,d)\) (with the notation of Definition 7.22) is transformed to the line segment through \((\alpha + \beta - 1, \beta)\) and \((c + d - 1, d)\) which has slope

\[
-\epsilon_{x_1,y_1,z_1} = \frac{-\epsilon}{-\epsilon + 1},
\]

so that \(-1 < -\epsilon_{x_1,y_1,z_1} < 0 \). Thus

\[
(\alpha_{x_1,y_1,z_1}, \beta_{x_1,y_1,z_1}) = (\gamma_{x_1,y_1,z_1} - \delta_{x_1,y_1,z_1}, \delta_{x_1,y_1,z_1}) = (\alpha + \beta - 1, \beta),
\]

and \((c_1, d) = (c + d - 1, d)\) is the lowest point on the line through \((\alpha_{x_1,y_1,z_1}, \beta_{x_1,y_1,z_1})\) with slope \(-\epsilon_{x_1,y_1,z_1}\) which is on \( |\Delta(g_1; x_1, y_1, z_1)| \). By Lemma 7.26, \((g_1; x_1, y_1, z_1)\) is well
Lemma 7.30. Suppose that \( p \) is a prime, \( s \) is a non-negative integer and \( r_0 \) is a positive integer such that \( p \nmid r_0 \). Let \( r = r_0 p^s \). Then

1. \( \binom{\lambda}{r} \equiv 0 \mod p \), if \( p^s \nmid \lambda \), \( 0 \leq \lambda \leq r \) is an integer.
2. \( \binom{\lambda p^s}{r} \equiv \binom{r_0}{\lambda} \mod p \), if \( 0 \leq \lambda \leq r_0 \) is an integer.

Proof. Compare the expansions over \( \mathbb{Z}_p \) of \( (x + y)^r = (x^{p^s} + y^{p^s})^{r_0} \). □
Apply the translation $y' = y - \eta x$ and well prepare by some substitution $z' = z - \Phi(x, y')$. This does not change $(\alpha, \beta)$ or $\gamma$. Set $\delta' = \delta_{x, y', z'}(g)$.

First assume that $\delta' < \beta$. We have regular parameters $(x_1, y_1, z_1)$ in $T_1$ such that $x = x_1$, $y' = x_1 y_1$, $z' = x_1 z_1$ by Lemma 7.16 so we have a 1-1 correspondence (by Lemma 7.25 1.)

$$\sigma : \Delta(g; x, y, z') \rightarrow \Delta(g_1; x_1, y_1, z_1)$$
defined by $\sigma(a, b) = (a+b-1, b)$. $(g_1; x_1, y_1, z_1)$ is well prepared (by Lemma 7.26). We are essentially in case 1 ($-1 \leq \eta < 1$) of the proof of Lemma 7.28 (this case only requires well preparation) so that $(\alpha_1, \beta_1) = \sigma(\gamma - \delta', \delta')$ and $\beta_1 = \delta' < \beta$. After very well preparation, we thus have regular parameter $x_1, y_1', z_1'$ in $T_1$ such that $(g_1; x_1, y_1', z_1')$ is very well prepared and $\beta_{x_1, y_1', z_1'}(g_1) < \beta_{x, y, z}(g)$. We have thus reduced the proof to showing that $\delta' < \beta$. If $(\alpha, \beta) \neq (\gamma - \delta, \delta)$, we have $\delta' < \delta < \beta$ since $(g; x, y, z)$ is very well prepared.

Let $g = \sum a_{ijk} x^i y^j z^k$. Suppose that $(\alpha, \beta) = (\gamma - \delta, \delta)$. Set

$$W = \{(i, j, k) \in \mathbb{N}^3 \mid k < r \text{ and } (i/r, j/r) = (\alpha, \beta)\},$$

$$F = \sum_{(i, j, k) \in W} a_{ijk} x^i y^j z^k.$$ 

By assumption, $z' + F$ is not solvable.

$$F(x, y, z) = F(x, y + \eta x, z) = \sum_{(i, j, k) \in W} \sum_{\lambda=0}^j a_{ijk} \eta^\lambda \binom{j}{\lambda} x^i y^j (y')^j - \lambda z^k.$$ 

By Lemma 7.21, the terms in the expansion of $g(x, y' + \eta x, z)$ contributing to $(\gamma, 0)$ in $|\Delta(g; x, y', z)|$, where $\gamma = \alpha + \beta$, are

$$F(\gamma, 0) = \sum_{(i, j, k) \in W} a_{ijk} \eta^j x^i y^j z^k \neq 0.$$ 

If $(g; x, y', z)$ is $(\gamma, 0)$ prepared, then $\delta' = 0 < \beta$. Suppose that

$$(g)_{x, y', z}^{\gamma_0} = z' + F(\gamma, 0)$$
is solvable. Then $\gamma \in \mathbb{N}$ and there exists $\psi \in K$ such that

$$(z - \psi x^\gamma)^r = z' + F(\gamma, 0)$$
so that, with $\omega = \frac{z'}{\eta^r} \in K$, for $0 \leq k < r$, we have
1. If \( \left( \frac{r}{r-k} \right) \neq 0 \) (in \( K \)) then \( i = \alpha(r-k), j = \beta(r-k) \in N \) and
\[
a_{ijk} = \left( \frac{r}{r-k} \right) \omega^{r-k}.
\] (97)

2. If \( i = \alpha(r-k), j = \beta(r-k) \in N \) and \( \left( \frac{r}{r-k} \right) = 0 \) (in \( K \)) then \( a_{ijk} = 0 \).

Thus (by Lemma 7.30) \( a_{ijk} = 0 \) if \( p^s \nmid k \).

Suppose that \( K \) has characteristic 0. By Remark 7.32, we obtain a contradiction to our assumption that \( \{ g \}^{(\gamma,0)}_{x,y,z} \) is solvable. Thus \( 0 = \delta' < \beta \) if \( K \) has characteristic 0.

Now we consider the case where \( K \) has characteristic \( p > 0 \), and \( r = p^s r_0 \) with \( p \nmid r_0, r_0 \geq 1 \). By (97) and Lemma 7.30, we have \( i = \alpha p^s, j = \beta p^s \in N \) and \( a_{i,j,(r_0-1)p^s} \neq 0 \).

We have an expression \( \beta p^s = e p' \), where \( p \nmid e \). Suppose that \( t \geq s \). Then \( \beta \in N \) which implies \( \alpha = \gamma - \beta \in N \), so that
\[
(z + \omega x^\alpha y^\beta)^{r^0} = z^{r^0} + F
\]
a contradiction, since \( z^{r^0} + F \) is by assumption not solvable. Thus \( t < s \). Suppose that \( e = 1 \). Then \( \beta = p^{t-s} < 1 \) and \( \alpha < 1 \) (since we must have Sing_\( e \)(\( g \)) = \( V(x,y,z) \)) implies \( \gamma = \alpha + \beta = 1 \), a contradiction to (94). Thus \( e > 1 \).

\[
z^{r^0} + F = (z^{p^s} + \omega x^{\alpha p^s} y^{\beta p^s})^{r_0} = (z^{p^s} + \omega x^{\alpha p^s} ((y^\prime + \eta x)^{\beta p^s})^{r_0} = (z^{p^s} + \omega x^{\alpha p^s} [((y^\prime)^{p^s} + \eta p^s x^{p^s}])^{r_0}
\]

Now make the \((\gamma,0)\) preparation \( z = z^\prime - \eta^s \omega x^\gamma \) (from (96)) so that \( (g; x,y^\prime, z^\prime) \) is \((\gamma,0)\) prepared.

\[
z^{r^0} + F = (z^{p^s} + e \omega x^{\alpha p^s} (y^\prime)^{p^s-x^{\alpha p^s}} + p^s (e-1) + (y^\prime)^{2p^s} \Omega)^{r_0} = (z^{p^s})^{r_0} + r_0 + [e \omega x^{\alpha p^s} (y^\prime)^{p^s-x^{\alpha p^s}} + p^s (e-1) + (y^\prime)^{2p^s} \Omega] z^{p^s} r_0 - 2) + \cdots + \Lambda_{r_0}(x,y)
\]

for some polynomials \( \Omega(x,y), \Lambda_2, \ldots, \Lambda_{r_0} \) where \( (y^\prime)^{ip^s} \mid \Lambda_i \) for all \( i \). Since all contributions to \( S(\gamma) \cap \Delta(g; x,y^\prime, z^\prime) \) must come from this polynomial, (recall that we are assuming \( (\alpha, \beta) = (\gamma - \delta, \delta) \)) the term of lowest second coordinate on \( S(\gamma) \cap \Delta(g; x,y^\prime, z^\prime) \) is
\[
(a,b) = \left( \frac{\alpha p^s + p^s (e-1) + p^s}{p^s} \right)
\]
which is not in \( N^2 \) since \( t < s \), and is not \((\alpha, \beta)\) since \( e > 1 \). Thus \( (g; x,y^\prime, z^\prime) \) is not \((a,b)\) solvable, and
\[
\delta' = \frac{p^e}{p^s} < \frac{ep^s}{p^s} = \beta.
\]

\[ \square \]

**Remark 7.32.** In Theorem 7.31 suppose that \( K \) has characteristic 0. From (97) we see that if \( \{ g \}^{(\gamma,0)}_{x,y,z} \) is solvable, then for \( 0 \leq k \leq r-1 \) there exists \( i,j \in N \) with
$i = \alpha(r - k), \ j = \beta(r - k)$ so that $\alpha, \beta \in \mathbb{N}$. Comparing (97) with the binomial expansion

$$(z + \omega x^\alpha y^\beta)^r = \sum_{i=0}^{r} \binom{r}{i} (\omega x^\alpha y^\beta)^{r-i} z^i = z^r + F$$

we see that $z^r + F$ is solvable, a contradiction. Thus $(\gamma, 0)$ is not solvable on $|\Delta(g; x', y', z)|$ if $K$ has characteristic 0, and after well preparation, $(\gamma, 0)$ is a vertex of $|\Delta(g; x, y, z')|$

**Example 7.33.** With the notation of Theorem 7.31, it is possible to have $(\gamma, 0)$ solvable in $|\Delta(g; x, y, z)|$ if $K$ has positive characteristic, as is shown by the following simple example. Suppose that the characteristic $p$ of $K$ is greater than 5.

Let $g = z^p + xyp^{2-1} + x^p - 1y^2$.

The origin is isolated in $\text{Sing}_p(g = 0)$, and $(g; x, y, z)$ is very well prepared. The vertices of $|\Delta(g; x, y, z)|$ are $(\alpha, \beta) = (\gamma - \delta, p\delta) = (\frac{1}{p}, p - \frac{1}{p})$ and $(p - \frac{1}{p}, \frac{2}{p})$. Set $y = y' + \eta x$ with $0 \neq \eta \in K$.

$$g = z^p + (xyp^{2-1} + x^p - 1)(y' + \eta x)^2$$

$$= z^p + \left(\sum_{i=0}^{p^2-1} \binom{p^2-1}{i} \eta^{p^2-1-i} x^{p^2-2-i}(y')^i \right) + \eta^2 x^{p^2+1} + 2\eta y' x^{p^2} + x^{p^2-1}(y')^2$$

The vertices of $|\Delta(g; x, y', z')|$ are $(\alpha, \beta) = (\frac{1}{p}, p - \frac{1}{p})$ and $(\gamma, 0) = (p, 0)$. However, $(\gamma, 0)$ is solvable. Set $z' = z + \eta x^{p^{2-2}} x^p$.

$$g = (z')^p + \left(\sum_{i=1}^{p^2-1} \binom{p^2-1}{i} \eta^{p^2-1-i} x^{p^2-2-i}(y')^i \right) + \eta^2 x^{p^2+1} + 2\eta y' x^{p^2} + x^{p^2-1}(y')^2$$

Since $\binom{p^2-1}{1} \equiv -1 \mod p$, $\delta_1 = \delta_{xy'}(g) = \frac{1}{p}$ and the vertices of $|\Delta(g; x, y', z')|$ are $(\alpha, \beta) = (\frac{1}{p}, p - \frac{1}{p})$, $(\gamma_1 - \delta_1, \delta_1) = (p - \frac{1}{p}, 1 - \frac{1}{p})$ and $(p + \frac{1}{p}, 0)$.

**Theorem 7.34.** For all $n \in \mathbb{N}$, there are $f_n \in \mathcal{R}_n$ such that $f_n$ is a generator of $\mathcal{I}_n$, and good parameters $(x_n, y_n, z_n)$ for $f_n$ in $\mathcal{R}_n$ such that if

$$\beta_n = \beta_{x_ny_nz_n}(f_n), \epsilon_n = \epsilon_{x_ny_nz_n}(f_n), \alpha_n = \alpha_{x_ny_nz_n}(f_n), \delta_n = \delta_{x_ny_nz_n}(f_n)$$

$$\sigma(n) = (\beta_n, \delta_n, \frac{1}{\epsilon_n}, \alpha_n)$$

then $\sigma(n+1) < \sigma(n)$ for all $n$ in the lexicographical order in $\frac{1}{r_1}\mathbb{N} \times \frac{1}{r_2}\mathbb{N} \times (\mathbb{Q} \cup \{\infty\}) \times \frac{1}{r_1}\mathbb{N}$. If $\beta_{n+1} = \beta_n$, $\delta_{n+1} = \delta_n$ and $\frac{1}{\epsilon_n} \neq \infty$, then

$$\frac{1}{\epsilon_{n+1}} = \frac{1}{\epsilon_n} - 1.$$
Let $f$ be a generator of $\mathcal{I}_{S,s}R_0$. We first choose (possibly formal) regular parameters $(x, y, z)$ in $R$ which are good for $f = 0$, such that $|\Delta(f; x, y, z)|$ is very well prepared. By Lemma 7.16, $z = 0$ is an approximate mainfold for $f = 0$. Let $\alpha = \alpha_{x,y,z}(f)$, $\beta = \beta_{x,y,z}(f)$.

By assumption, $q$ is isolated in $\text{Sing}_r(f)$.

Suppose that $f_i$ and regular parameters $(x_i, y_i, z_i)$ in $R_i$ have been defined for $i \leq n$ as specified above.

If $\text{Sing}_r(f_n) = V(y_n, z_n)$, then $R_n \to R_{n+1}$ must be a T4 transformation, 

$$x_n = x_{n+1}', y_n = y_{n+1}', z_n = y_{n+1}', z_{n+1}' + 1.$$ 

Thus $(f_{n+1}; x_{n+1}', y_{n+1}', z_{n+1}')$ is prepared by Lemma 7.29. Further $\text{Sing}_r(f_{n+1}) \subset V(y_{n+1}', z_{n+1}')$. If $\text{Sing}_r(f_{n+1}) = V(x_{n+1}', y_{n+1}', z_{n+1}')$, make a change of variables to very well prepare. Otherwise, set $y_{n+1} = y_{n+1}', z_{n+1} = z_{n+1}'$. We have $\beta_{n+1} < \beta_n$ by Lemma 7.29.

If $\text{Sing}_r(f_n) = V(x_n, z_n)$ then $R_n \to R_{n+1}$ must be a T3 transformation 

$$x_n = x_{n+1}, y_n = y_{n+1}, z_n = x_{n+1}z_{n+1},$$ 

and $(f_n; x_n, y_n, z_n)$ is very well prepared. Thus $(f_{n+1}; x_{n+1}, y_{n+1}, z_{n+1})$ is also very well prepared.

$$(\beta_{n+1}, \delta_{n+1}, \frac{1}{\epsilon_{n+1}}, \alpha_{n+1}) < (\beta_n, \delta_n, \frac{1}{\epsilon_n}, \alpha_n)$$

by Lemma 7.27. Further $\text{Sing}_r(f_{n+1}) \subset V(x_{n+1}, z_{n+1})$.

Now suppose that $\text{Sing}_r(f_n) = V(x_n, y_n, z_n)$. Then $(f; x_n, y_n, z_n)$ is very well prepared, and $R_{n+1}$ must be a T1 or T2 transformation of $R_n$.

If $R_{n+1}$ is a T2 transformation of $R_n$, 

$$x_n = x_{n+1}y_{n+1}', y_n = y_{n+1}', z_n = z_{n+1}' + 1,$$

then $(f_{n+1}; x_{n+1}, y_{n+1}', z_{n+1}')$ is well prepared with $\beta_{n+1} < \beta_n$ by Lemma 7.29. Further, $\text{Sing}_r(f_{n+1}) \subset V(y_{n+1}', z_{n+1}')$. If $\text{Sing}_r(f_{n+1}) = V(x_{n+1}, y_{n+1}', z_{n+1}')$, then make a further change of variables to very well prepare. Otherwise, set $y_{n+1} = y_{n+1}', z_{n+1} = z_{n+1}'$.

If $R_{n+1}$ is a T1 transformation of $R_n$, 

$$x_n = x_{n+1}, y_n = x_{n+1}(y_{n+1} + \eta), z_n = x_{n+1}z_{n+1},$$

then $\text{Sing}_r(f_{n+1}) \subset V(x_{n+1}, z_{n+1}')$ by Lemma 7.5.

If $\beta_n \neq 0$ and $\eta \neq 0$ in the T1 transformation relating $R_n$ and $R_{n+1}$, we can change variables to $y_{n+1}$, $z_{n+1}$ to very well prepare, with $\beta_{n+1} < \beta_n$ and $\text{Sing}_r(f_{n+1}) \subset V(x_{n+1}, z_{n+1})$ by Theorem 7.31.

If $\eta = 0$ in the T1 transformation of $R_n$, and $(x_{n+1}, y_{n+1}, z_{n+1})$ are regular parameters of $R_{n+1}$ obtained from $(x_{n+1}, y_{n+1}', z_{n+1}')$ by very well preparation, then $(f_{n+1}; x_{n+1}, y_{n+1}, z_{n+1})$ is very well prepared,

$$(\beta_{n+1}, \delta_{n+1}, \frac{1}{\epsilon_{n+1}}, \alpha_{n+1}) < (\beta_n, \delta_n, \frac{1}{\epsilon_n}, \alpha_n)$$

and $\text{Sing}_r(f_{n+1}) \subset V(x_{n+1}, z_{n+1})$ by Lemma 7.28.

If $\beta_n = 0$ we can make a translation $y_n' = y_n + \eta x_n$, and have that $(g_n; x_n, y_n', z_n)$ is very well prepared, and $\beta_n, \gamma_n, \delta_n, \epsilon_n, \alpha_n$ are unchanged. Thus we are in the case of $\eta = 0$. 

\square
7.4. **Remarks and further discussion.** The first proof of resolution of singularities of surfaces in positive characteristic was given by Abhyankar ([1], [3]). The proof used valuation theoretic methods to prove local uniformization (defined in Section 2.5).

The most general proof of resolution of surfaces was given by Lipman [58]. This lucid article gives a self contained geometric proof.

The algorithm we give in this chapter is sketched in lectures by Hironaka [49]. These notes are very rough, and require correction at some points.

This proof of resolution of surfaces which are hypersurfaces over an algebraically closed field (Theorem 7.1) extends to give a proof of resolution of excellent surfaces. Some general remarks about the proof are given in the final 3 pages of Hironaka’s notes [49]. The first part of our proof of Theorem 7.1 generalizes to the case of an excellent surface $S$, and we reduce to proving a generalization of Theorem 7.9, when $V = \text{spec}(R)$ where $R$ is a regular local ring with maximal ideal $m$ and $S = \text{spec}(R/I)$ is a two dimensional equidimensional local ring with $\text{Sing}_r(R/I) = V(m)$.

There are two main new points which must be considered. The first point is to find a generalization of the polygon $|\Delta(f; x, y, z)|$ to the case where $\hat{O}_{S,q} \cong R/I$ is as above. This is accomplished by the general theory of characterestic polyhedra [50].

The second point is to extend the theory to the case of non-closed residue field. This is accomplished in [21].

We will now restrict to the case when our surface $S$ is a hypersurface over an arbitrary field $K$. The proof of Theorem 7.9 in the case when $\tau(q) = 3$ is the same, and if $\tau(q) = 2$, the proof is really the same, since we must then have equality of residue fields $K(q_i) = K(q_{i+1})$ for all $i$ in the sequence (82), under the assumption that $\tau(q_i) = 2, \nu_q(S_i) = r$ for all $i$. This is the analogue of our proof of resolution of curves embedded in a non-singular surface, over an arbitrary field.

In the case when $\tau(q) = 1$, some extra care is required. It is possible to have non-trivial extensions of residue fields $k(q_i) \rightarrow k(q_{i+1})$ in the sequence (87). However, if $k(q_{i+1}) \neq k(q_i)$, we must have $\beta_{i+1} < \beta_i$ (This is proven in the theorem of page 26 [21]).

Thus $(\beta_i, \delta_i, \frac{1}{\epsilon_i}, \alpha_i)$ drops after every transformation in (87) and we see that we must eventually have $\nu$ or $\tau$ drop.

The essential difficulty in resolution in positive characteristic, as opposed to characteristic zero, is the lack of existence of a hypersurface of maximal contact (see the exercises at the end of this section).

Abhyankar is able to finesse this difficulty to prove resolution for 3 dimensional varieties $X$, over an algebraically closed field $K$ of characteristic $p > 5$ (13.1, [4]). We reduce to proving local uniformization, by a patching argument (9.1.7, [4]).

The starting point of the proof of local uniformization is to use a projection argument (12.4.3, 12.4.4 [4]) to reduce the problem to the case of a point on a normal variety of multiplicity $\leq 3! = 6$. This projection method can be traced to Albanese. A simple exposition of this method is given on page 200 of [57]. We will sketch the proof of resolution, in the case of a hypersurface singularity of multiplicity $\leq 6$. We then have that a local equation of this singularity is

$$w^r + a_{r-1}(x, y, z)w^{r-1} + \cdots + a_0(x, y, z) = 0$$ (98)
Since $r \leq 6 < \text{Char}(K)$, $\frac{1}{r} \in K$, and we can perform a Tschirnhausen transformation to reduce to the case

$$w^r + a_{r-2}(x,y,z)w^{r-2} + \cdots + a_0(x,y,z) = 0.$$  

Now $w = 0$ is a hypersurface of maximal contact, and we have a good theorem for embedded resolution of 2 dimensional hypersurfaces, so we reduce to the case where each $a_i$ is a monomial in $x, y, z$ times a unit. Now we have reduced to a combinatorial problem, which can be solved in a characteristic free way.

A resolution in the case where (98) has degree $p = \text{Char} K$ is found by Cossart [22]. The proof is extraordinarily difficult.

Suppose that $X$ is a variety over a field $K$. In [34], de Jong has shown that there is a dominant proper morphism of $K$-varieties $X' \to X$ such that $\text{dim } X' = \text{dim } X$ and $X'$ is non-singular. If $K$ is perfect, then the finite extension of function fields $K(X) \to K(X')$ is separable. This is weaker than a resolution of singularities since the map is in general not birational (the extension of function fields $K(X') \to K(X)$ is finite). This proof relies on sophisticated methods in the theory of moduli spaces of curves.

Exercises

1. (Narasimhan [64]) Let $K$ be an algebraically closed field of characteristic 2, and let $X$ be the 3-fold in $\mathbb{A}^4_K$ with equation

$$f = w^2 + xy^3 + yz^3 + zx^7 = 0.$$  

a. Show that the maximal multiplicity of a singular point on $X$ is 2, and the locus of singular points on $X$ is the monomial curve $C$ with local equations

$$y^3 + zx^6 = 0, \quad xy^2 + z^3 = 0, \quad yz^2 + x^7 = 0, \quad w^2 + zx^7 = 0$$

which has the parameterization $t \to (t^7, t^{19}, t^{15}, t^{32})$. Thus $C$ has embedding dimension 4 at the origin, so there cannot exist a hypersurface (or formal hypersurface) of maximal contact for $X$ at the origin.

b. Resolve the singularities of $X$.

c. Resolve the singularities of $S$.

2. (Hauser [46]) Let $K$ be an algebraically closed field of characteristic 2, and let $S$ be the surface in $\mathbb{A}^2_K$ with equation $f = x^2 + y^2 z + y^2 z^4 + z^7 = 0$.

a. Show that the maximal multiplicity of points on $S$ is 2, and the singular locus of $S$ is defined by the singular curve $y^2 + z^3 = 0, x + yz^2 = 0$.

b. Suppose that $X$ is a hypersurface on a non-singular variety $W$, and $p \in X$ is a point in the locus of maximal multiplicity $r$ of $X$. A hypersurface $H$ through $p$ is said to have permanent contact with $X$ if under any sequence of blow ups $\pi : W_1 \to W$ of $W$, with non-singular centers, contained in the locus of points of multiplicity $r$ on the strict transform of $X$, the strict transform of $H$ contains all points of the intersection of the strict transform of $X$ with multiplicity $r$ which are in the fiber $\pi^{-1}(p)$. Show that there does not exist a non-singular hypersurface of permanent contact at the origin for the above surface $S$. Conclude that a hypersurface of maximal contact does not exist for $S$.

c. Resolve the singularities of $S$. 
8. Local Uniformization and Resolution of Surface Singularities

In this chapter we present a modern proof of resolution of surface singularities through local uniformization of valuations in the spirit of Zariski [79] and [81]. An introduction to this approach was given earlier in Section 2.5.


Let $L$ be an algebraic function field of dimension two over an algebraically closed field $K$ of characteristic zero. That is, $L$ is a finite extension of a rational function field in two variables over $K$. A valuation of the function field $L$ is a valuation $\nu$ of the field $L$ which is trivial on $K$. $\nu$ is a homomorphism $\nu : L^\ast \to \Gamma$ from the multiplicative group of $L$ onto an ordered abelian group $\Gamma$ such that

1. $\nu(ab) = \nu(a) + \nu(b)$ for $a, b \in L^\ast$
2. $\nu(a + b) \geq \min \{\nu(a), \nu(b)\}$ for $a, b \in L^\ast$
3. $\nu(c) = 0$ for $0 \neq c \in K$

We formally extend $\nu$ to $L$ by setting $\nu(0) = \infty$. We will assume throughout this section that $\nu$ is non-trivial, that is, $\Gamma \neq 0$.

Some basic references to the valuation theory of algebraic function fields are [85] and [3].

A basic invariant of a valuation is its rank. A subgroup $\Gamma'$ of $\Gamma$ is said to be isolated if it has the property that if $0 \leq \alpha \in \Gamma'$ and $\beta \in \Gamma$ is such that $0 \leq \beta < \alpha$, then $\beta \in \Gamma'$. The isolated subgroups of $\Gamma$ form a single chain of subgroups. The length of this chain is the rank of the valuation. Since $L$ is a two-dimensional algebraic function field, we have that rank $(\nu) \leq 2$ (c.f. Corollary to the Lemma of Section 10, Chapter VI of [85]). If $\Gamma$ is of rank 1 then $\Gamma$ is embeddable as a subgroup of $R$ (c.f. page 45 of Section 10, Chapter VI [85]). If $\Gamma$ is of rank 2, then there is a non-trivial isolated subgroup $\Gamma_1 \subset \Gamma$ such that $\Gamma_1$ is embeddable as a subgroup of $R$ and $\Gamma/\Gamma_1$ is embeddable as a subgroup of $R$. If these subgroups of $R$ are discrete, then $\nu$ is called a discrete valuation. In particular, a discrete valuation (in an algebraic function field of dimension two) can be of rank 1 or of rank 2. The ordered chain of prime ideals $p$ of $V$ correspond to the isolated subgroups $\Gamma'$ of $\Gamma$ by

$$P = \{f \in V \mid \nu(f) \in \Gamma - \Gamma'\} \cup \{0\}$$

(c.f. Theorem 15, Section 10, Chapter VI [85]).

In our analysis, we will find a rational function field of two variables $L'$ over which $L$ is finite, and consider the restricted valuation $\nu' = \nu \mid L'$. We will consider $L' = K(x, y)$ where $x, y \in L$ are algebraically independent elements such that $\nu(x), \nu(y)$ are non-negative. Let $[L' : L] = r$. Suppose that $w \in L$ and suppose that

$$F(x, y, w) = A_0(x, y)w^r + \cdots + A_r(x, y) = 0$$

is the irreducible relation satisfied by $w$, with $A_i \in K[x, y]$ for all $i$. Since $\nu(F) = \infty$, there must be two distinct terms $A_i w^{r-i}, A_j w^{r-j}$ of the same value, where we can assume that $i < j$. Thus $(j - i)\nu(w) = \nu(A_i/A_j)$, and we have that $r!\Gamma' \subset \Gamma$. Thus $\Gamma$ and $\Gamma'$ have the same rank, and $\Gamma$ is discrete if and only if $\Gamma'$ is discrete.

Associated to the valuation $\nu$ is the valuation ring

$$V = \{f \in L \mid \nu(f) \geq 0\}.$$
V has a unique maximal ideal

\[ m_V = \{ f \in V \mid \nu(f) > 0 \} . \]

The residue field \( K(V) = V/m_V \) of \( V \) contains \( K \) and has transcendence degree \( r^* \leq r \) over \( K \). \( \nu \) is called an \( r^* \)-dimensional valuation. In our case (\( \nu \) non-trivial) we have \( r^* = 0 \) or 1.

If \( R \) is local ring contained in \( L \), we will say that \( \nu \) dominates \( R \) if \( R \subseteq V \) and \( m_V \cap R \) is the maximal ideal of \( R \). If \( S \) is a subring of \( V \) we will say that the center of \( \nu \) on \( S \) is the prime ideal \( m_V \cap S \). We will say that \( R \) is an algebraic local ring of \( L \) if \( R \) is essentially of finite type over \( K \) (\( R \) is a localization of a finite type \( K \)-algebra) and \( R \) has quotient field \( L \).

We again consider our algebraic extension \( L' = K(x,y) \rightarrow L \). The valuation ring of \( \nu' \) is \( V' = V \cap L \), with maximal ideal \( m_{V'} = m_V \cap L \). The residue field \( K(V') \) is thus a subfield of \( K(V) \). Suppose that \( w^* \in K(V) \), and let \( w \) be a lift of \( w^* \) to \( V \). We have an irreducible equation of algebraic dependence (99) of \( w \) over \( K[x,y] \). Let \( A_h \) be such that

\[ \nu(A_h) = \min \{ \nu(A_1), \ldots, \nu(A_r) \} . \]

We may assume that \( h \neq r \), because if \( \nu(A_r) < \nu(A_i) \) for \( i < r \), then we would have \( \nu(A_r) < \nu(A_i w^{r-i}) \) for \( 1 \leq i \leq r-1 \) since \( \nu(w) \geq 0 \), so that \( \nu(F) = \nu(A_r) < \infty \), which is impossible. \( \nu(A_h^i) \geq 0 \) for \( 0 \leq i \leq r \), so that \( A_h^i \in V' \) for all \( i \). Let \( a_i^* \) be the corresponding residues in \( K(V') \). We have a relation

\[ a_r^*(w^*)^r + \cdots + a_{r-1}(w^*)^r-h+1 + (w^*)^r-h + a_{r+1}(w^*)^r-h-1 + \cdots + a_r^* = 0 . \]

Since \( r-h > 0 \), \( w^* \) is algebraic over \( K(V') \). Thus \( \nu \) and \( \nu' = \nu \mid L' \) have the same dimension.

We will now give a classification of the various types of valuations which occur on \( L \).

**one-dimensional valuations**

Choose \( x,y \in L \) such that \( x,y \) are algebraically independent over \( K \), \( \nu(x) = 0 \), \( \nu(y) > 0 \) and the residue of \( x \) in \( K(V) \) is transcendental over \( K \). Let \( L' = K(x,y) \), and let \( \nu' = \nu \mid L' \) with valuation ring \( V' = V \cap L' \). By construction, if \( m_{V'} \) is the maximal ideal of \( V' \), there exists \( c \in K \) such that \( m_{V'} \cap K[x,y] \subseteq (x-c,y) \). Suppose (if possible) that \( p = m_{V'} \cap K[x,y] = (x-c,y) \). Then the residue of \( x \) in \( K(V) \) is \( c \), a contradiction. Thus there exists an irreducible \( f \in R = K[x,y] \) such that \( p = (f) \). It follows that the valuation ring \( V' \) is \( R_p \) (c.f. Theorem 9, Section 5, Chapter VI [85]) and the value group of \( \nu' \) is \( Z \). The valuation ring \( V \) is a localization of the integral closure of the local Dedekind domain \( V' \) in \( L \). Thus \( V \) is itself a local Dedekind domain which is a localization of a ring of finite type over \( K \). In particular, \( V \) is itself a regular algebraic local ring of \( L \).

**zero-dimensional valuations of rank 2**

Let \( \Gamma_1 \) be the non-trivial isolated subgroup of \( \Gamma \), with corresponding prime ideal \( p \subset V \). The composition of \( \nu \) with the order perserving homomorphism \( \Gamma \rightarrow \Gamma/\Gamma_1 \) defines a valuation \( \nu_1 \) of \( L \) of rank 1. Let \( V_1 \) be the valuation ring of \( \nu_1 \). \( V_1 \) consists of
the elements of $K$ whose value by $\nu$ is in $\Gamma_1$ or is positive. Hence $V_1 = V_p$. $\nu$ induces a valuation $\nu_2$ of $K(V_p)$ with value group $\Gamma_1$, where $\nu_2(f) = \nu(f^*)$ if $f^* \in V_p$ is a lift of $f$ to $V_p$. Since $K(V_p)$ has a non-trivial valuation, which vanishes on $K$, $K(V_p)$ cannot be an algebraic extension of $K$. Thus $K(V_p)$ has positive transcendence degree over $K$, so it must have transcendence degree 1. Let $x^* \in K(V_p)$ be such that $x^*$ is transcendental over $K$. Let $x$ be a lift of $x^*$ to $V_p$. Then $K(x) \subset V_p$, so that $\nu_1$ is trivial on $K(x)$, and $\nu_1$ is a valuation of the one-dimensional algebraic function field $L$ over $K(x)$.

Let $V_2 \subset K(V_p)$ be the valuation ring of $\nu_2$, and let $\pi : V_p \to K(V_p)$ be the residue map. Then $V = \pi^{-1}(V_2)$. $\nu_1$ is a 1 dimensional valuation of $L$, so $V_p$ is an algebraic local ring which is a local Dedekind domain. The residue field $K(V_p)$ is an algebraic function field of dimension 1 over $K$, so $V_2$ is also a local Dedekind domain. Thus $\Gamma$ is a discrete group. Let $w$ be a regular parameter in $V_p$.

Let $w$ be a regular parameter in $V_p$. Given any non-zero element $f \in L$, we have a unique expression $f = w^mg$ where $m = \nu_1(f) \in \mathbb{Z}$, and $g$ is a unit in $V_p$. Let $g^*$ be the residue of $g$ in $K(V_p)$. Let $n = \nu_2(g^*) \in \mathbb{Z}$. Then, up to an order preserving isomorphism of $\Gamma$, $\nu(f) = (m, n) \in \Gamma = \mathbb{Z}^2$, where $\mathbb{Z}^2$ has the lexicographic order.

**discrete zero-dimensional valuations of rank 1.**

Let $x \in L$ be such that $\nu(x) = 1$. If $y \in L$ and $\nu(y) = n_1$, then there exists a unique $c \in K$ such that $n_2 = \nu(y - c_1x^{n_1}) > \nu(y)$. If we iterate this construction infinitely many times, we have a uniquely determined Laurent series expansion

$$c_1x^{n_1} + c_2x^{n_2} + \ldots$$

with $\nu(y - c_1x^{n_1} - \cdots - c_mx^{n_m}) = n_{m+1} > n_m$ for all $m$. This construction defines an embedding of $K$ algebras $L \subset K \ll x \gg$, where $K \ll x \gg$ is the field of Laurent series in $x$. The order valuation on $K \ll x \gg$ restricts to the valuation $\nu$ on $L$.

**non-discrete zero-dimensional valuations of rank 1**

We subdivide this case by the rational rank of $\nu$. The rational rank is the dimension of $\Gamma \otimes \mathbb{Q}$ as a rational vector space. We may normalize $\nu$ so that $\Gamma$ is an ordered subgroup of $\mathbb{R}$ and $1 \in \Gamma$.

Suppose that $\nu$ has rank larger than 1. Choose $x, y \in L$ such that $\nu(x) = 1$ and $\nu(y) = \tau$ is a positive irrational number. Suppose that $F(x, y) = \sum a_{ij}x^iy^j$ is a polynomial in $x$ and $y$ with coefficients in $K$. $\nu(a_{ij}x^iy^j) = i + j\tau$ for any term with $a_{ij} \neq 0$, so all the monomials in the expansion of $F$ have distinct values. It follows that $\nu(F) = \min \{i + j\tau \mid a_{ij} \neq 0\}$. We conclude that $x$ and $y$ are algebraically independent, since a relation $F(x, y) = 0$ has infinite value. Furthermore, if $L_1 = K(x, y)$ and $\nu_1 = \nu \mid L_1$, we have that the value group $\Gamma_1$ of $\nu_1$ is $\mathbb{Z} + \mathbb{Z}\tau$. Since $\Gamma/\Gamma_1$ is a finite group, we have that $\Gamma$ is a group of the same type. In particular, $\Gamma$ has rational rank 2.

Now suppose that $\nu$ has rational rank 1. We can normalize $\Gamma$ so that it is an ordered subgroup of $\mathbb{Q}$, whose denominators are not bounded, as $\Gamma$ is not discrete. In Example 3, Section 15, Chapter VI [85], examples are given of two dimensional algebraic function fields with value group equal to any given subgroup of the rationals.
There is a nice interpretation of this kind of valuation in terms of certain “formal” Puiseux series (c.f. MacLane and Shilling [59]) where the denominators of the rational exponents in the Puiseux series expansion are not bounded.

Exercise

Let $K$ be a field, $R = K[x, y]$ be a polynomial ring and $L$ be the quotient field of $R$.

a. Show that the rule $\nu(x) = 1$, $\nu(y) = \pi$ defines a $K$-valuation on $L$ which satisfies

$$\nu(f) = \min\{i + j\pi \mid a_{ij} \neq 0\}$$

if $f = \sum a_{ij}x^iy^j \in K[x, y]$.

b. Compute the rank, rational rank and value group $\Gamma$ of $\nu$.

c. Let $V$ be the valuation ring of $\nu$. Suppose that $\tau \in \Gamma$. Show that

$$I_\tau = \{f \in V \mid \nu(f) \geq \tau\}$$

is an ideal of $V$.

d. Show that $V$ is not a Noetherian ring.

8.2. Local uniformization of algebraic function fields of surfaces. In this section, we prove the following local uniformization theorem.

**Theorem 8.1.** Suppose that $K$ is an algebraically closed field of characteristic zero, $L$ is a two dimensional algebraic function field over $K$ and $\nu$ is a valuation of $L$. Then there exists a regular local ring $A$ which is essentially of finite type over $K$ with quotient field $L$ such that $\nu$ dominates $A$.

Our proof is by a case by case analysis of the different types of valuations of $L$ which were classified in the previous section. If $\nu$ is 1 dimensional, we have that $R = V$ satisfies the conclusions of the theorem. We must prove the theorem in the case when $\nu$ has dimension zero, and one of the following three cases hold: $\nu$ has rational rank 1, $\nu$ has rank 1 and rational rank 1, $\nu$ has rank 2. The case when $\nu$ has rank 1 includes the cases when $\nu$ is discrete and $\nu$ is not discrete. We will follow the notation of the preceding section.

**valuations of rational rank 1 and dimension 0**

**Lemma 8.2.** Suppose that $S = K[x_1, \ldots, x_n]$ is a polynomial ring, $\nu$ is a rank 1 valuation with valuation ring $V$ and maximal ideal $m_V$ which contains $S$ such that $K(V) = K$ and $m_V \cap S = (x_1, \ldots, x_n)$. Suppose that there exist $f_i \in S$ for $i \in \mathbb{N}$ such that

$$\nu(f_1) < \nu(f_2) < \cdots < \nu(f_n) < \cdots$$

is a strictly increasing sequence. Then $\lim_{i \to \infty} \nu(f_i) = \infty$.

**Proof.** Suppose that $\rho > 0$ is a positive real number. We will show that there are only finitely many real numbers less than $\rho$ which are the values of elements of $S$. Let $\tau = \min \{\nu(f) \mid f \in (x_1, \ldots, x_n)\}$. $\tau$ is well defined since $S$ is Noetherian. In fact, if there where not a minimum, we would have an infinite descending sequence of positive real numbers

$$\tau_1 > \tau_2 > \cdots \tau_1 > \cdots$$
and elements \( f_i \in (x_1, \ldots, x_n) \) such that \( \nu(f_i) = \tau_i \). Let \( I_{\tau_i} = \{ f \in S \mid \nu(f) \geq \tau_i \} \). \( I_{\tau_i} \) are distinct ideals of \( S \) which form a strictly ascending chain

\[
I_{\tau_1} \subset I_{\tau_2} \subset \cdots \subset I_{\tau_n} \subset \cdots
\]

which is impossible.

Choose a positive integer \( r \) such that \( r > \frac{\rho}{\tau} \). If \( f \in S \) is such that \( \nu(f) < \rho \) then write \( f = g + h \) with \( h \in (x_1, \ldots, x_n)^r \) and \( \deg(g) < r \). \( \nu(h) > \rho \) implies \( \nu(g) = \nu(f) \). This implies that \( f \) is a finite number of elements \( f_1, \ldots, f_m \in \tau \) (the monomials of degree < \( r \)) such that if \( 0 < \lambda < \rho \) is the value of an element of \( S \), then there exist \( c_1, \ldots, c_m \in K \) such that \( h_1 f_1 + \cdots + c_m f_m = \lambda \).

We will replace the \( f_i \) with appropriate linear combinations of the \( f_i \) so that

\[
\nu(f_1) < \nu(f_2) < \cdots < \nu(f_m).
\]

By induction, it suffices to make a linear change in the \( f_i \) so that

\[
\nu(f_1) < \nu(f_i) \text{ if } i > 1.
\]

After reindexing the \( f_i \), we can suppose that there exists an integer \( l \geq 1 \) such that

\[
\nu(f_1) = \nu(f_2) = \cdots = \nu(f_l)
\]

and

\[
\nu(f_{i+1}) > \nu(f_i) \text{ if } l < i.
\]

\[
\nu\left( \frac{f_i}{f_j} \right) = 0 \text{ for } 2 \leq i \leq l \text{ implies } \frac{f_i}{f_j} \in V \text{ and there exist } c_i \in K(V) = K \text{ such that } \frac{f_i}{f_j} \text{ has residue } c_i \text{ in } K(V). \text{ Then } \frac{f_i}{f_j} - c_i \in m_V \text{ implies } \nu\left( \frac{f_i}{f_j} - c_i \right) > 0, \text{ so that } \nu(f_i - c_i f_j) > \nu(f_j) \text{ for } 2 \leq i \leq l. \text{ After replacing } f_i \text{ with } f_i - c_i f_j \text{ for } 2 \leq i \leq l, \text{ we have that (100) is satisfied.}
\]

**Theorem 8.3.** Suppose that \( S = K[x, y] \) is a polynomial ring over an algebraically closed field \( K \) of characteristic zero, \( \nu \) is a rational rank 1 valuation of \( K(x, y) \) with valuation ring \( V \) and maximal ideal \( m_V \), \( K(V) = V/m_V = K \) such that \( S \subset V \) and the center of \( V \) on \( S \) is the maximal ideal \((x, y)\). Further suppose that \( f \in K[x, y] \) is given. Then there exists a birational extension \( K[x, y] \to K[x', y'] \) \((K[x, y] \text{ and } K[x', y'] \text{ have a common quotient field})\) such that \( K[x', y'] \subset V \), the center of \( \nu \) on \( K[x', y'] \) is \((x', y')\) and \( f = (x')^l \delta \) where \( l \) is a non-negative integer and \( \delta \in K[x', y'] \) is not in \((x', y')\).

**Proof.** Set \( r = \nu \left( f(0, 0) \right) \). We have \( 0 \leq r \leq \infty \). If \( r = 0 \) we are done, so suppose that \( r > 0 \). We have an expansion

\[
f = \sum_{i=1}^{d} x^{\alpha_i} \gamma_i(x) y^{\beta_i} + \sum_{i>d} x^{\alpha_i} \gamma_i(x) y^{\beta_i}
\]

where the first sum is over terms with minimal value \( \rho = \nu(x^{\alpha_1} \gamma_1(x) y^{\beta_1}) \) for \( 1 \leq i \leq d \), \( \gamma_i(x) \in K[x] \) are polynomials with non-zero constant term, \( \beta_1 < \cdots < \beta_d < \beta_{d+1} \). If \( i > d \), \( \nu(x^{\alpha_i} \gamma_i(x) y^{\beta_i}) > \rho \) if \( i > d \).

Set \( \frac{\nu(a)}{\nu(y)} = \frac{a}{b} \) with \( a, b \in \mathbb{N} \), \( (a, b) = 1 \). There exist non-negative integers \( a', b' \) such that \( ab' - ba' = 1 \). \( \nu(x^{\alpha_i}) = 0 \) implies there exists \( 0 \neq c \in K(V) = K \) such that \( \nu(x^{\alpha_i} - c) > 0 \). Set

\[
x = x^\sigma_1 (y_1 + c)^a' \quad \text{and} \quad y = x^\sigma_1 (y_1 + c)^b'.
\]
We have that
\[ x_1 = x^{b'}y^{-a'}, \quad y_1 + c = x^{-b}y^a \]
so that \( \nu(y_1) > 0 \) and
\[ \nu(x_1) = \frac{\nu(y)}{b}[ab' - a'b] = \frac{\nu(y)}{b} > 0. \]
Set \( \alpha = \alpha_1a + \beta_1b \). We have that \( \alpha_ia + \beta_ib = \alpha \) for \( 1 \leq i \leq d \). Thus there exist \( e_i \in \mathbb{Z} \) such that
\[ \left( \begin{array}{cc} a & b \\ a' & b' \end{array} \right) \left( \begin{array}{c} \alpha_i - \alpha_1 \\ \beta_i - \beta_1 \end{array} \right) = \left( \begin{array}{c} 0 \\ e_i \end{array} \right) \]
for \( 1 \leq i \leq d \). By Cramer’s rule, we have
\[ \beta_i - \beta_1 = \text{Det} \left( \begin{array}{cc} a & 0 \\ a' & e_i \end{array} \right) = ae_i. \]
Thus \( e_i = \frac{\beta_i - \beta_1}{a} \).

We have a factorization \( f = x_1^n f_1(x_1, y_1) \) where
\[ f_1(x_1, y_1) = (y_1 + c)^{\alpha_1b'} \sum_{i=1}^{d} \gamma_i(y_1 + c)^{\beta_i} + x_1 \Omega \]
is the strict transform of \( f \) in the birational extension \( K[x_1, y_1] \) of \( K[x, y] \). \( \text{ord}(f, 0, y) = r \) implies there exists an \( i \) such that \( \alpha_i = 0, \beta_i = r \). Thus \( \nu(z^r) \geq \rho \) and \( \beta_i \leq r \) for \( 1 \leq i \leq d \). We have
\[ r_1 = \text{ord} f_1(0, y_1) \leq \frac{\beta_d - \beta_1}{a} \leq r. \]
If \( r_1 < r \) then we have a reduction, and the theorem follows from induction on \( r \), since the conclusions of the theorem hold in \( K[x, y] \) if \( r = 0 \). We thus assume that \( r_1 = r \). Then \( \beta_d = r, \beta_1 = 0, a = 1 \) and \( \alpha_d = 0 \). Further, there is an expression
\[ \sum_{i=1}^{d} \gamma_i(0)(y_1 + c)^{\beta_i} = \gamma_d(0)y_1^r. \]
Let \( u \) be an indeterminate, and set
\[ g(u) = \sum_{i=1}^{d} \gamma_i(0)u^{\beta_i} = \gamma_d(0)(u - c)^r. \]
The binomial theorem implies that \( \beta_{d-1} = r - 1 \) and thus
\[ \nu(y) = \nu(x^{\alpha_d - 1}). \]
Let \( 0 \neq \lambda \in k(V) = k \) be the residue of \( \frac{y^r}{\nu(y^r)} \). Then \( \nu(y - \lambda x^{\alpha_d - 1}) > \nu(y) \). Set \( y' = y - \lambda x^{\alpha_d - 1} \) and \( f'(x, y') = f(x, y) \). We have \( \text{ord} f'(0, y') = \text{ord} f(0, y) = r \).

Replacing \( y \) with \( y' \) and \( f \) with \( f' \) and iterating the above procedure after \( (101) \), we either achieve a reduction \( \text{ord} f_1(0, y_1) < r \) or we find that there exists \( a'(x) \in K[x] \) such that \( \nu(y' - a'(x)) > \nu(y') \). We further have that \( \nu(y') \leq \nu(f) \). Set \( y'' = y' - a'(x) \) and repeat the algorithm following \( (101) \) until we either reach a reduction \( \text{ord} f_1(0, y_1) < r \) or show that there exist polynomials \( a^{(i)}(x) \in K[x] \) for \( i \in \mathbb{N} \) such that we have an infinite increasing bounded sequence
\[ \nu(a(x)) < \nu(a'(x)) < \cdots \]
By Lemma 8.2, this is impossible. Thus there is a polynomial \( q(x) \in K[x] \) such that after replacing \( y \) with \( y - q(x) \), and following the algorithm after (101), we achieve a reduction \( r_1 < r \) in \( K[x_1, y_1] \). By induction on \( r \), we can achieve the conclusions of the theorem.

We now give the proof of Theorem 8.1 in the case of this subsection, that is, \( \nu \) has dimension 0 and rational rank 1.

Let \( x, y \in L \) be algebraically independent over \( K \) and of positive value. Let \( w \) be a primitive element of \( L \) over \( K(x, y) \) of positive value. Let \( R = K[x, y, w] \). \( R \subset V \), the center of \( \nu \) on \( R \) is the maximal ideal \( (x, y, w) \), and \( R \) has quotient field \( L \). Let \( z \) be algebraically independent over \( K(x, y) \), and let \( f(x, y, z) \) in the polynomial ring \( K[x, y, z] \) be the irreducible polynomial such that \( R \cong K[x, y, z]/(f) \). After possibly making a change of variables, replacing \( x \) with \( x + \beta z \) and \( y \) with \( y + \gamma z \) for suitable \( \beta, \gamma \in K \), we may assume that

\[
r = \text{ord } f(0, 0, z) < \infty.
\]

If \( r = 1 \), then \( R_{mv \cap R} \) is a regular local ring which is dominated by \( \nu \), so the conclusions of the theorem are satisfied. We thus suppose that \( r > 1 \). Let \( \pi : K[x, y, z] \to K[x, y, w] \) be the natural surjection. For \( g \in K[x, y, z] \), we will denote \( \nu(\pi(g)) \) by \( \nu(g) \). Write

\[
f(x, y, z) = \sum_{i=1}^{d} a_i(x, y)z^{\sigma_i} + \sum_{i>d} a_i(x, y)z^{\sigma_i},
\]

where \( a_i(x, y) \in K[x, y] \) for all \( i \), \( a_i(x, y)z^{\sigma_i} \) are the terms of minimal value \( \rho = \nu(a_i(x, y)z^{\sigma_i}) \) for \( 1 \leq i \leq d, \nu(a_i(x, y)z^{\sigma_i}) > \rho \) for \( i > d \) and \( \sigma_1 < \sigma_2 < \cdots < \sigma_d \).

By Theorem 8.3, there exists a birational extension \( K[x, y] \to K[x_1, y_1] \) with \( K[x_1, y_1] \subset V \) and \( mv \cap K[x_1, y_1] = (x_1, y_1) \) such that

\[
f = \sum_{i=1}^{d} x_1^{\lambda_i} \pi_i(x_1, y_1)z^{\sigma_i} + \sum_{i>d} x_1^{\lambda_i} \pi_i(x_1, y_1)z^{\sigma_i},
\]

where \( \pi_i(0, 0) \neq 0 \) for all \( i \). \( R_1 = K[x_1, y_1, z_1]/(f) \) is a domain with quotient field \( L \), \( R \to R_1 \) is birational, \( R_1 \subset V \) and \( mv \cap R_1 = (x_1, y_1, z) \). Write \( \nu_{(x_1)} = \frac{s}{t} \) with \( s, t \) relatively prime positive integers. Let \( s', t' \) be positive integers such that \( s't - st' = 1 \). Set

\[
x_1 = x_2^t(z_2 + c_2)^{t'}, y_1 = y_2, z = x_2^s(z_2 + c_2)^{s'}
\]

where \( 0 \neq c_2 \in K \) is determined by the condition \( \nu(z_2) > 0 \). Let \( \alpha = \lambda_1 t + \sigma_1 s \). We have \( \alpha = \lambda_i t + \sigma_i s \) for \( 1 \leq i \leq d \). We have (as in the proof of Theorem 8.3) that \( \frac{\sigma_d - \alpha}{t} \) is a non-negative integer for \( 1 \leq i \leq d \) and \( f = x_2^s f_2(x_2, y_2, z_2) \) where

\[
f_2(x_2, y_2, z_2) = (x_2 + c_2)^{t\lambda_1 + s\sigma_1} \left( \sum_{i=1}^{d} \pi_i(z_2 + c_2)^{\frac{\sigma_i - \sigma_1}{t}} \right) + x_2 \Omega
\]

is the strict transform of \( f \) in \( k[x_2, y_2, z_2] \). Set \( R_2 = K[x_2, y_2, z_2]/(f_2) \). \( R_1 \to R_2 \) is birational, \( R_2 \subset V \) and \( mv \cap R_2 = (x_2, y_2, z_2) \).

Set

\[
r_1 = \text{ord } f_2(0, 0, z_2) \leq \frac{\sigma_d - \sigma_1}{t} \leq r.
\]
If \( r_1 < r \) we have a reduction, so we may assume that \( r_1 = r \). Then we have \( \sigma_d = r \), \( \sigma_1 = 0 \), \( t = 1 \) and \( a_d(0,0) \neq 0 \). We further have a relation

\[
\sum_{i=1}^{d} \alpha_i(0,0)(z_2 + c_2)^{\sigma_i} = \alpha_d(0,0)z_2^r.
\]

Set

\[
g(u) = \sum_{i=1}^{d} \alpha_i(0,0)u^{\sigma_i} = \alpha_d(0,0)(u - c_2)^r.
\]

The binomial theorem shows that \( \sigma_{d-1} = r - 1 \) and \( \nu(z) = \nu(a_{d-1}(x, y)) \). There exists \( c \in k \) such that \( \nu(z - ca_{d-1}(x, y)) > \nu(z) \). \( ra_dz^{r-1} \) is a minimal value term of \( \frac{\partial f}{\partial z} \) so that \( \nu(\frac{\partial f}{\partial z}) \geq \nu(z) \). Set \( z' = z - ca_{d-1}(x, y) \), \( f'(x, y, z') = f(x, y, z) \). We have

\[
\nu(z' - a'(x, y)) > \nu(z') \quad \text{and} \quad \nu(z') \leq \nu(\frac{\partial f'}{\partial z'}) = \nu(\frac{\partial f}{\partial z}).
\]

Set \( z'' = z' - a'(x, y) \) and repeat the algorithm following (102). If we do reach a reduction \( r_i < r \) after a finite number of iterations, there exist polynomials \( a_i(x, y) \in K[x, y] \) such that \( \nu(z'' - a'(x, y)) < \nu(z') \) and

\[
\nu(a(x, y)) < \nu(a'(x, y)) < \cdots
\]

is an increasing bounded sequence of real numbers. By Lemma 8.2 this is impossible.

Thus there exists a polynomial \( a(x, y) \in K[x, y] \) such that after replacing \( z \) with \( z - a(x, y) \), the algorithm following (102) results in a reduction \( r_1 < r \). We then repeat the algorithm, so that by induction on \( r \), we construct a birational extension of local rings \( R \to R_1 \) so that \( R_1 \) is a regular local ring which is dominated by \( V \).

**valuations of rank 1, rational rank 2 and dimension 0**

After normalizing \( \nu \), we have that its value group is \( \mathbb{Z} + \tau \mathbb{Z} \) where \( \tau \) is a positive irrational number. Choose \( x, y \in L \) such that \( \nu(x) = 1 \) and \( \nu(y) = \tau \). By the analysis of the preceding case (rational rank 1 and dimension 0), \( x \) and \( y \) are algebraically independent over \( K \). Let \( w \) be a primitive element of \( L \) over \( K(x, y) \) which has positive value. There exists an irreducible polynomial \( f(x, y, z) \) in the polynomial ring \( K[x, y, z] \) such that \( T = K[x, y, w] \cong K[x, y, z]/(f) \) and \( mV \cap T = \{x, y, w\} \). After possibly replacing \( x \) with \( x + \beta z^m \) and \( y \) with \( y + \gamma z^m \) where \( m \) is sufficiently large that \( m\nu(w) > \max \{\nu(x), \nu(y)\} \), and \( \beta, \gamma \in K \) are suitably chosen, we may assume that

\[
r = \text{ord} (f(0,0, z)) < \infty.
\]

If \( r = 1 \), \( T_{mV \cap T} \) is a regular local ring, so we assume that \( r > 1 \). Let \( \pi : K[x, y, z] \to K[x, y, w] \) be the natural surjection. For \( g \in K[x, y, z] \), we will denote \( \nu(\pi(g)) \) by \( \nu(g) \). Write

\[
f = \sum_{i=1}^{d} a_i x^{\alpha_i} y^{\beta_i} z^{\gamma_i} + \sum_{i>d} a_i x^{\alpha_i} y^{\beta_i} z^{\gamma_i}.
\]
where all \( a_i \in K \) are non-zero, \( a_i x^{\alpha_i} y^{\beta_i} z^{\gamma_i} \) for \( 1 \leq i \leq d \) are the minimal value terms, and \( \gamma_1 < \gamma_2 < \cdots < \gamma_d \).

There exist integers \( s,t \) such that \( \nu(z) = s + \tau t \). There exist integers \( M,N \) such that
\[
\nu(x^{\alpha_i} y^{\beta_i} z^{\gamma_i}) = M + N \tau
\]
for \( 1 \leq i \leq d \). We thus have equalities
\[
M = \alpha_1 + s\gamma_1 = \cdots = \alpha_d + s\gamma_d \\
N = \beta_1 + t\gamma_1 = \cdots = \beta_d + t\gamma_d.
\]
(103)

We further have
\[
(\alpha_i + s\gamma_i) + (\beta_i + t\gamma_i)\tau > M + N \tau
\]
if \( i > d \). Since \( \ord(f(0,0,z)) = r \) and \( a_d x^{\alpha_d} y^{\beta_d} z^{\gamma_d} \) is a minimal value term, we have \( \gamma_d \leq r \). We expand \( \tau \) into a continued fraction:
\[
\tau = h_1 + \frac{1}{h_2 + \frac{1}{h_3 + \cdots}}.
\]

Let \( \frac{f_j}{g_j} \) be the convergent fractions of \( \tau \) (c.f. Section 10.2 [44]). Since \( \lim \frac{f_j}{g_j} = \tau \), we have
\[
(\alpha_i + s\gamma_i) + (\beta_i + t\gamma_i)\frac{f_j}{g_j} > M + N \frac{f_j}{g_j}
\]
for \( j = p - 1 \) and \( j = p \) whenever \( i > d \) and \( p \) is sufficiently large. We further have that \( s g_{p} + t f_{p} > 0 \) and \( s g_{p-1} + t f_{p-1} > 0 \) for \( p \) sufficiently large. Since \( \nu(x^{s} y^{t}) = \nu(z) \), there exists \( c \in K = K(V) \) such that \( \nu(x^{s} y^{t} - c) > 0 \). Consider the extension
\[
K[x,y,z] \to K[x_1,y_1,z_1]
\]
(104)
where
\[
x = x_1^{g_p} y_1^{g_{p-1}}, y = x_1^{f_p} y_1^{f_{p-1}}, \\
z = x^{s} y^{t} (z_1 + c) = x_1^{s g_p + t f_p} y_1^{s g_{p-1} + t f_{p-1}} (z_1 + c).
\]
(105)

We have \( f_{p-1} g_{p} - f_{p} g_{p-1} = \epsilon = \pm 1 \) (c.f. Theorem 150 [44]) and \( \epsilon, -\tau + \frac{f_{p-1}}{g_{p-1}} \) and \( \tau - \frac{f_{p}}{g_{p}} \) have the same signs (c.f. Theorem 154 [44]). From (105), we see that
\[
x_1 = \frac{x^{f_{p-1}}}{y^{g_{p-1}}}, y_1 = \frac{y^{g_p}}{x^{f_p}}
\]
from which we deduce that (104) is birational, and
\[
\epsilon \nu(x_1) = g_{p-1} (\frac{f_{p-1}}{g_{p-1}} - \tau), \epsilon \nu(y_1) = g_p (\tau - \frac{f_p}{g_p}).
\]
Hence \( \nu(x_1), \nu(y_1), \nu(z_1) \) are all \( > 0 \). We have
\[
f(x,y,z) = x_1^{M g_p + N f_p} y_1^{M g_{p-1} + N f_{p-1}} f_1(x_1,y_1,z_1)
\]
where
\[
f_1 = (z_1 + c)^{\gamma_1} \left( \sum_{i=1}^{d} a_i (z_1 + c)^{\gamma_i - \gamma_1} \right) + x_1 y_1 H
\]
is the strict transform of \( f \) in \( K[x_1,y_1,z_1] \). We thus have constructed a birational extension \( T \to T_1 = K[x_1,y_1,z_1]/(f_1) \) such that \( T_1 \subset V \) and \( m\nu \cap T_1 = (x_1,y_1,z_1) \).
We have \( f_1(0, 0, 0) = 0 \), so \( u = c \) is a root of 
\[
\phi(u) = \sum_{i=1}^{d} a_i u^{\gamma_i} = 0.
\]
Let \( r_1 = \text{ord} f_1(0, 0, z_1) \). Then \( r_1 \leq \gamma_d - \gamma_1 \leq r \). If \( r_1 < r \) we have obtained a reduction, so we may assume that \( r_1 = r \). Thus \( \gamma_1 = 0, \gamma_d = r, \alpha_d = \beta_d = 0 \) (since \( \gamma_i \leq r \) for \( 1 \leq i \leq d \)) and 
\[
\phi(u) = a_d(u - c)^r.
\]
Hence \( \gamma_{d-1} = r - 1 \) and \( a_{d-1} \neq 0 \), by the binomial theorem. We have \( \nu(z) = \alpha_{d-1} + \beta_{d-1} \tau \). Our conclusion is that there exist non-negative integers \( m \) and \( n \) such that \( \nu(z) = m + n \tau \). There exists \( c \in K \) such that 
\[
\nu(z - cx^m y^n) > \nu(z).
\]
We make a change of variables in \( K[x, y, z] \), replacing \( z \) with \( z' = z - cx^m y^n \), and let \( f'(x, y, z') = f(x, y, z) \). We have \( \nu(z') > \nu(z) \) and \( \text{ord } f'(0, 0, z') = r \). Since \( ra_d z'^{-1} \) is a minimal value term of \( \partial f \), we have \( \nu(z) \leq \nu(\partial f \partial z') \). We further have \( \frac{\partial f}{\partial z'} = \frac{\partial f}{\partial z} \).

We now apply a transformation of the kind (105) with respect to the new variables \( x, y, z' \). If we do not achieve a reduction \( r_1 < r \), we have a relation \( \nu(z') = m_1 + n_1 \tau \) with \( m_1, n_1 \) non-negative integers, and we have 
\[
\nu(z) < \nu(z') \leq \nu(\frac{\partial f}{\partial z'}).
\]
We have that there exists \( c_2 \in K \) such that 
\[
\nu(z' - c_2 x^{m_1} y^{n_1}) > \nu(z').
\]
Set \( z'' = z' - c_2 x^{m_1} y^{n_1} \). We now iterate the procedure starting with the transformation (105). If we do not achieve a reduction \( r_1 < r \) after a finite number of iterations, we construct an infinite bounded sequence (since \( \frac{\partial f}{\partial z'} \neq 0 \) in \( T \)) 
\[
m + n \tau < m_1 + n_1 \tau < \cdots < m_i + n_i \tau < \cdots
\]
where \( m_i, n_i \) are non-negative integers, which is impossible. We thus must achieve a reduction \( r_1 < r \) after a finite number of iterations. By induction on \( r \), we can construct a birational extension \( T \to T_1 \) such that \( T_1 \subset V \) and \( (T_1)_{m \tau \cap T} \) is a regular local ring.

**valuations of rank 2 and dimension 0**

Let \( 0 \subset p \subset m_V \) be the distinct prime ideals of \( V \). Let \( x, y \in L \) be such that \( x, y \) are a transcendence basis of \( L \) over \( K \), \( \nu(x) \) and \( \nu(y) \) are positive and the residue of \( x \) in \( K(V_p) \) is a transcendence basis of \( K(V_p) \) over \( K \). Let \( w \) be a primitive element of \( L \) over \( K(x, y) \) such that \( \nu(w) > 0 \). Let \( T = K[x, y, w] \subset V \). Let \( f(x, y, z) \) be an irreducible element in the polynomial ring \( K[x, y, z] \) such that 
\[
K[x, y, z]/(f) \cong K[x, y, w].
\]
By our construction, the quotient field of \( T \) is \( L \) and \( \text{trdeg}_K(T/p \cap T) = 1 \). Thus \( p \cap T \) is a prime ideal of a curve on the surface \( \text{spec}(T) \).

Set \( R_0 = R_{m \tau \cap T} \). Let \( C_0 \) be the curve in \( \text{spec}(T) \) with ideal \( p \cap R_0 \) in \( R_0 \). Corollary 4.4 implies there exists a proper birational morphism \( \pi : X \to \text{spec}(T) \) which is a
sequence of blow ups of points such that the strict transform \( \overline{C}_0 \) of \( C_0 \) on \( X \) is non-singular. Let \( X_1 \to X \) be the blow up of \( \overline{C}_0 \). Since \( X_1 \to \text{spec}(T) \) is proper, there exists a unique point \( a_1 \in X_1 \) such that \( V \) dominates \( R_1 = \mathcal{O}_{X_1, a_1} \). \( R_1 \) is a quotient of a 3 dimensional regular local ring (as explained in Lemma 5.4). Let \( C_1 \) be the curve in \( X_1 \) with ideal \( p \cap R_1 \) in \( R_1 \). \((R_1)_{p \cap R_1} \) dominates \( R_{p \cap R} \) and \((R_1)_{p \cap R_1} \) is a local ring of a point on the blow up of the maximal ideal of \( R_{p \cap R} \). We can iterate this procedure to construct a sequence of local rings (with quotient field \( L \))

\[
R_0 \to R_1 \to \cdots \to R_i \to \cdots
\]
such that \( V \) dominates \( R_i \) for all \( i \). \( R_i \) is quotient of a regular local ring of dimension 3 and \((R_{i+1})_{p \cap R_{i+1}} \) is a local ring of the blow up of the maximal ideal of \((R_i)_{p \cap R_i} \) for all \( i \). Furthermore, each \( R_i \) is a localization of a quotient of a polynomial ring in 3 variables.

Since \((R_0)_{p \cap R_0} \) can be considered as a local ring of a point on a plane curve over the field \( K(x) \) (as is explained in the proof of Lemma 5.10), by Theorem 3.12 (or Corollary 4.4) there exists an \( i \) such that \((R_i)_{p \cap R_i} \) is a regular local ring.

After replacing the \( T \) in (106) with a suitable affine ring, we can now assume that \( T_{p \cap T} \) is a regular local ring. If \( T_{m \cap T} \) is a regular local ring we are done, so suppose that \( T_{m \cap T} \) is not a regular local ring. Let \( \mathfrak{p} = \pi^{-1}(p \cap T) \) where \( \pi : K[x, y, z] \to T \) is the natural surjection. If \( g \in K[x, y, z] \) is a polynomial, we will denote \( \nu(\pi(g)) \) by \( \nu(g) \). Let \( I \subset K[x, y, z] \) be the ideal \( I = (f, \partial f / \partial x, \partial f / \partial y, \partial f / \partial z) \) defining the singular locus of \( T \). Since \( I \not\subset \mathfrak{p} \) (as \( T_{p \cap T} \) is a regular local ring) one of \( \pi(\partial f / \partial x), \pi(\partial f / \partial y), \pi(\partial f / \partial z) \not\in p \cap T \). Thus

\[
\min\{\nu(\partial f / \partial x), \nu(\partial f / \partial y), \nu(\partial f / \partial z)\} = (0, n)
\]

for some \( n \in \mathbb{N} \). Write

\[
f = F_r + F_{r+1} + \cdots + F_m
\]

where \( F_i \) is a form of degree \( i \) in \( x, y, z \) of degree \( i \) and \( r = \text{ord}(f) \). After possibly reindexing the \( x, y, z \) we may assume that \( 0 < \nu(x) \leq \nu(y) \leq \nu(z) \).

Since \( K(V) = K \), there exist \( c, d \in k \) (which could be 0) such that \( \nu(y - cx) > \nu(x) \) and \( \nu(z - dx) > \nu(x) \). Thus we have a birational transformation \( K[x, y, z] \to K[x_1, y_1, z_1] \) (a quadratic transformation) where

\[
x_1 = x, y_1 = y - cx, z_1 = z - dx,
\]

with inverse

\[
x = x_1, y = x_1(y_1 + c), z = x_1(z_1 + d).
\]

In \( K[x_1, y_1, z_1] \) we have \( f(x, y, z) = x_1^r f_1(x_1, y_1, z_1) \) where

\[
f_1(x_1, y_1, z_1) = F_r(1, y_1 + c, z_1 + d) + x_1 F_{r+1}(1, y_1 + c, z_1 + d) + \cdots + x_1^{m-r} F_m(1, y_1 + c, z_1 + d)
\]

is the strict transform of \( f \). Let \( T_1 = K[x_1, y_1, z_1] / (f_1) \). \( T \to T_1 \) is birational, \( T_1 \subset V \) and \( m \cap T_1 = (x_1, y_1, z_1) \). We calculate

\[
\begin{align*}
\frac{\partial f}{\partial x} &= x_1^{r-1} (x_1 \frac{\partial f_1}{\partial x_1} - (y_1 + c) \frac{\partial f_1}{\partial y_1} - (z_1 + d) \frac{\partial f_1}{\partial z_1} + r f_1) \\
\frac{\partial f}{\partial y} &= x_1^{r-1} \frac{\partial f_1}{\partial y_1} \\
\frac{\partial f}{\partial z} &= x_1^{r-1} \frac{\partial f_1}{\partial z_1}
\end{align*}
\]
Hence
\[ \min \{ \nu\left( \frac{\partial f}{\partial x} \right), \nu\left( \frac{\partial f}{\partial y} \right), \nu\left( \frac{\partial f}{\partial z} \right) \} \geq (r - 1)\nu(x_1) + \min \{ \nu\left( \frac{\partial f_1}{\partial x_1} \right), \nu\left( \frac{\partial f_1}{\partial y_1} \right), \nu\left( \frac{\partial f_1}{\partial z_1} \right) \}. \]

Since we have assumed that \( T_{mV \cap T} \) is not regular, we have \( r > 1 \). We further have that \( \nu(x_1) > 0 \), so we conclude that
\[ \min \{ \nu\left( \frac{\partial f_1}{\partial x_1} \right), \nu\left( \frac{\partial f_1}{\partial y_1} \right), \nu\left( \frac{\partial f_1}{\partial z_1} \right) \} = (0, n_1) < (0, n). \]

Let \( r_1 = \text{ord} (f_1) \leq r \). \((T_1)_{mV \cap T_1}\) is a regular local ring if and only if \( r_1 = 1 \). Thus we may assume that \( r_1 > 1 \).

We now apply to \( T_1 \) a new quadratic transform, to get
\[ \min \{ \nu\left( \frac{\partial f_2}{\partial x_2} \right), \nu\left( \frac{\partial f_2}{\partial y_2} \right), \nu\left( \frac{\partial f_2}{\partial z_2} \right) \} = (0, n_2) < (0, n_1). \]

Thus after a finite number of quadratic transforms \( T \rightarrow T_1 \), we construct a \( T_1 \) such that \((T_1)_{mV \cap T_1}\) is a regular local ring which is dominated by \( V \).

**Remark 8.4.** Zariski proves Local Uniformization for arbitrary characteristic zero algebraic function fields in [80].

8.3. **Resolving systems and the Zariski-Riemann manifold.** Let \( L/K \) be an algebraic function field. A projective model of \( L \) is a projective \( K \)-variety whose function field is \( L \).

Let \( \Sigma(L) \) be the set of valuation rings of \( L \). If \( X \) is a projective model of \( L \) then there is a natural morphism \( \pi_X : \Sigma(L) \rightarrow X \) defined by \( \pi_X(V) = p \) if \( p \) is the (unique) point of \( X \) whose local ring is dominated by \( V \). We will say that \( p \) is the center of \( X \). There is a topology on \( \Sigma(L) \) where a basis for the topology are the open sets \( \pi_X^{-1}(U) \) for an open set \( U \) of a projective model \( X \) of \( L \). Zariski shows that \( \Sigma(L) \) is quasi-compact (every open cover has a finite subcover) in [82] and Section 17, Chapter VI [85]. \( \Sigma(L) \) is called the Zariski-Riemann manifold of \( L \).

Let \( N \) be a set of zero-dimensional valuation rings of \( L \). A resolving system of \( N \) is a finite collection \( \{ S_1, \ldots, S_r \} \) of projective models of \( L \) such that for each \( V \in N \), the center of \( V \) on at least one of the \( S_i \) is a non-singular point.

**Theorem 8.5.** Suppose that \( L \) is a two-dimensional algebraic function field over an algebraically closed field \( K \) of characteristic zero. Then there exists a resolving system for the set of all zero-dimensional valuation rings of \( L \).

**Proof.** By local uniformization (Theorem 8.1), for each valuation \( W \) of \( L \) there exists a projective model \( X_W \) of \( L \) such that the center of \( W \) on \( X_W \) is non-singular. Let \( U_W \) be an open neighborhood of the center of \( W \) in \( X_W \) which is non-singular, and let \( A_W = \pi_W^{-1}(U_W) \), an open neighborhood of \( W \) in \( \Sigma(L) \). The set \( \{ A_W \mid W \in \Sigma(L) \} \) is an open cover of \( \Sigma(L) \). Since \( \Sigma(L) \) is quasi-compact, there is a finite subcover \( \{ A_{W_1}, \ldots, A_{W_s} \} \) of \( \Sigma(L) \). \( X_{W_1}, \ldots, X_{W_s} \) is then a resolving system for the set of all zero-dimensional valuation rings of \( L \). \( \square \)

**Lemma 8.6.** Suppose that \( L \) is a two-dimensional algebraic function field over an algebraically closed field \( K \) of characteristic zero, and
\[ R_0 \subset R_1 \subset \cdots \subset R_s \subset \cdots \]
is a sequence of distinct normal two dimensional algebraic local rings of $L$. Let $\Omega = \cup R_i$. If every one dimensional valuation ring $V$ of $L$ which contains $\Omega$ intersects $R_i$ in a height one prime for some $i$, then $\Omega$ is the valuation ring of a zero-dimensional valuation of $L$.

**Proof.** $\Omega$ is by construction integrally closed in $L$. Thus $\Omega$ is the intersection of the valuation rings of $L$ containing $\Omega$ (c.f. Corollary to Theorem 8, Section 5, Chapter VI [85]). Observe that $\Omega \neq L$. We see this as follows. If $m_i$ is the maximal ideal of $R_i$, then $m = \cup m_i$ is a non-zero ideal of $\Omega$. Suppose that $1 \in m$. Then $1 \in m_i$ for some $i$, which is impossible.

We will show that $\Omega$ is an intersection of zero-dimensional valuation rings. To prove this, it suffices to show that if $V$ is a one-dimensional valuation ring containing $\Omega$, then there exists a zero-dimensional valuation ring $W$ such that $V$ is a localization of $W$ and $\Omega \subset W$. Since $V$ is one-dimensional, $V$ is a local Dedekind domain and is an algebraic local ring of $L$. Let $K(V)$ be the residue field of $V$. Let $m_V$ be the maximal ideal of $V$, and let $p_i = R_i \cap m_V$. By assumption, there exists an index $j$ such that $i \geq j$ implies that $p_i$ is a height one prime in $R_i$. $(R_i)_{p_i}$ is a normal local ring which is dominated by $V$. Since both $(R_i)_{p_i}$ and $V$ are local Dedekind domains, and they have the common quotient field $L$, we have that $(R_i)_{p_i} = V$ for $i \geq j$ (c.f. Theorem 9, Section 5, Chapter VI [85]). Thus the residue field $K_i$ of $(R_i)_{p_i}$ is $K(V)$ for $i \geq j$. The one dimensional algebraic local rings $S_i = R_i/p_i$ for $i \geq j$ have quotient field $K(V)$ and $S_{i+1}$ dominates $S_i$. $\cup S_i$ is not $K(V)$ by the same argument we used to show that $\Omega$ is not $L$. Thus there exists a valuation ring $V'$ of $K(V)$ which dominates $\cup S_i$. Let $\pi : V \rightarrow K(V)$ be the residue map. $W = \pi^{-1}(V')$ is a zero dimensional valuation ring of $V$ and $W$ contains $\Omega$.

Since we have established that $\Omega$ is the intersection of zero-dimensional valuation rings, we need only show that $\Omega$ is contained in two distinct zero-dimensional valuation rings. Suppose that $\Omega$ is contained in two distinct zero-dimensional valuations, $V_1$ and $V_2$.

We will first construct a projective surface $Y$ with function field $L$ such that $V_1$ and $V_2$ dominate (unique) distinct points $q_1$ and $q_2$ of $Y$. Let $x, y$ be a transcendence basis of $L$ over $K$. Let $\nu_1, \nu_2$ be valuations of $L$ whose respective valuation rings are $V_1$ and $V_2$. If $\nu_1(x)$ and $\nu_2(x)$ are both $\leq 0$, replace $x$ with $\frac{1}{x}$. Suppose that $\nu_1(x) \geq 0$ and $\nu_2(x) < 0$. Let $c \in K$ be such that the residue of $x$ in the residue field $K(V_1) \cong K$ of $V_1$ is not $c$. After replacing $x$ with $x - c$, we have that $\nu_1(x) = 0$ and $\nu_2(x) < 0$. We can now replace $x$ with $\frac{1}{x}$ to get that $\nu_1(x) \geq 0$ and $\nu_2(x) \geq 0$. In this way, after possibly changing $x$ and $y$, we can assume that $x$ and $y$ are contained in the valuation rings $V_1$ and $V_2$. Since $V_1$ and $V_2$ are distinct zero-dimensional valuation rings, there exists $f \in V_1$ such that $f \notin V_2$ and $g \in V_2$ such that $g \notin V_1$ (c.f. Theorem 3, Section 3, Chapter VI [85]). As above, we can assume that $\nu_1(f) = 0$, $\nu_2(f) < 0$, $\nu_1(g) < 0$ and $\nu_2(g) = 0$. Let $a = \frac{1}{f}$, $b = \frac{g}{b}$, $A$ be the integral closure of $K[x, y, a, b]$ in $L$. By construction, $A \subset V_1 \cap V_2$. Let $m_1 = A \cap m_{V_1}$, $m_2 = A \cap m_{V_2}$. We have that $a \in m_2$ but $a \notin m_1$ and $b \in m_1$ but $b \notin m_2$. Thus $m_1 \neq m_2$. We now let $Y$ be the normalization of a projective closure of $\text{spec}(A)$. $Y$ has the desired properties.

Since $R_i$ is essentially of finite type over $K$, it is a localization of a finitely generated $K$-algebra $B_i$ which has quotient field $L$. Let $X_i$ be the normalization of a projective closure of $\text{spec}(B_i)$. Since $R_i$ is integrally closed, the projective surface $X_i$ has a point $a_i$ with associated local ring $O_{X_i, a_i} = R_i$. 


There is a natural birational map \( T_i : X_i \to Y \). We will now establish that \( T_i \) cannot be a morphism at \( a_i \). If \( T_i \) where a morphism at \( a_i \), there would be a unique point \( b_i \) on \( Y \) such that \( R_i = \mathcal{O}_{X_i, a_i} \) dominates \( \mathcal{O}_{Y, b_i} \). Since \( V_1 \) and \( V_2 \) both dominate \( \mathcal{O}_{X_i, a_i} \), \( V_1 \) and \( V_2 \) both dominate \( \mathcal{O}_{Y, b_i} \), a contradiction.

By Zariski’s main theorem (c.f. Theorem V.5.2 [45]) there exists a one-dimensional valuation ring \( W \) of \( L \) such that \( W \) dominates \( \mathcal{O}_{X_i, a_i} \) and \( W \) dominates the local ring of the generic point of a curve on \( Y \). Let \( \Gamma_i \) be the set of one-dimensional valuation rings \( W \) of \( K \) such that \( W \) dominates \( \mathcal{O}_{X_i, a_i} \) and \( W \) dominates the local ring of the generic point of a curve on \( Y \). \( \Gamma_i \) is a finite set since the valuation rings \( W \) of \( \Gamma_i \) are in 1-1 correspondence with the curves contained in \( \pi_1^{-1}(a_i) \), where \( \pi_1 : \Gamma_i \to X_i \) is the projection of the graph of \( T_i \) onto \( X_i \). Since \( R_{i+1} = \mathcal{O}_{X_{i+1}, a_{i+1}} \) dominates \( R_i = \mathcal{O}_{X_i, a_i} \), \( \Gamma_i+1 \subset \Gamma_i \) for all \( i \). Since each \( \Gamma_i \) is a non-empty finite set, there exists a one-dimensional valuation ring \( W \) of \( L \) such that \( W \) dominates \( R_i \) for all \( i \), a contradiction.

**Theorem 8.7.** Suppose that \( L \) is a two-dimensional algebraic function field over an algebraically closed field \( K \) of characteristic zero, \( R \) is a two-dimensional normal algebraic local ring with quotient field \( L \) and that

\[
R \to R_1 \to R_2 \to \ldots \to R_i \to \ldots
\]

is a sequence of normal local rings such that \( R_{i+1} \) is obtained from \( R_i \) by blowing up the maximal ideal, normalizing, and localizing at a maximal ideal so that \( R_{i+1} \) dominates \( R_i \). Then \( \Omega = \bigcup R_i \) is a zero-dimensional valuation ring of \( L \).

**Proof.** By Lemma 8.6, we are reduced to showing that there does not exist a one-dimensional valuation ring \( V \) of \( L \) such that \( \Omega \) is contained in \( V \) and \( m_V \cap R_i \) is the maximal ideal for all \( i \).

Suppose that \( V \) is a one-dimensional valuation ring of \( L \) which contains \( \Omega \). Let \( \nu \) be a valuation of \( L \) whose associated valuation ring is \( V \). Since \( V \) is one-dimensional, there exists \( \alpha \in V \) such that the residue of \( \alpha \) in \( K(V) \) is transcendental over \( K \).

Let \( m_1 \) be the maximal ideal of \( R_i \). Assume that \( m_V \cap R_i \) is the maximal ideal \( m_0 \) of \( R \). Write \( \alpha = \frac{\xi}{\eta} \) with \( \xi, \eta \in R \). \( \alpha \notin R \) since \( R/m_0 \cong K \). Hence \( \eta \in m_0 \) and also \( \xi \in m_0 \) since \( \nu(\alpha) \geq 0 \). Since \( R_1 \) is a local ring of the normalization of the blow up of \( m_0 \), \( m_0 R_1 \) is a principal ideal. In fact if \( \xi \in m_0 \) is such that \( \nu(\xi) = \min \{\nu(f) \mid f \in m_0\} \), we have that \( R_1 = m_0 R_1 \). Thus \( \xi_1 = \frac{\xi}{\eta} \in R_1 \) and \( \eta_1 = \frac{\xi}{\eta} \in R_1 \).

If we assume that \( m_V \cap R_1 = m_1 \), we have that \( \xi_1, \eta_1 \in m_1 \), \( \alpha = \frac{\xi_2}{\eta_2} \) with \( \xi_2, \eta_2 \in R_2 \) and

\[
\nu(\xi) > \nu(\xi_1) > 0, \nu(\xi_1) > \nu(\xi_2)
\]

\[
\nu(\eta) > \nu(\eta_1) > 0, \nu(\eta_1) > \nu(\eta_2).
\]

More generally, if we assume that \( m_i = m_V \cap R \) for \( i = 1, \ldots, n \), we can find elements \( \xi_i, \eta_i \in R_i \) for \( 1 \leq i \leq n \) such that \( \alpha = \frac{\xi_i}{\eta_i} \) and

\[
\nu(\xi) > \nu(\xi_1) > \cdots > \nu(\xi_n) > 0
\]

\[
\nu(\eta) > \nu(\eta_1) > \cdots > \nu(\eta_n) > 0.
\]

Since the value group of \( V \) is \( \mathbb{Z} \), it follows that \( n \leq \min \{\nu(\xi), \nu(\eta)\} \}. Hence for all \( i \) sufficiently large \( m_V \cap R_i \) is not the maximal ideal. \( m_V \cap R_i \) must then be a height one prime in \( R_i \), for otherwise we would have \( m_V \cap R_i = (0) \), which would imply that \( V = L \), against our assumption. \( \square \)
Remark 8.8. The conclusions of this theorem are false in dimension 3 (Shannon [70]).

Theorem 8.9. Suppose that \( S, S' \) are normal projective surfaces over an algebraically closed field \( K \) of characteristic zero and \( T \) is a birational map from \( S \) to \( S' \). Let \( S_1 \to S \) be the projective morphism defined by taking the normalization of the blow up of the finitely many points of \( T \) where \( T \) is not a morphism. If the induced birational map \( T_1 \) from \( S_1 \) to \( S' \) is not a morphism, let \( S_2 \to S_1 \) be the normalization of the blow up of the finitely many fundamental points of \( T_1 \). We then iterate to produce a sequence of birational morphisms of surfaces

\[
S \leftarrow S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_i \leftarrow \cdots
\]

This sequence must be of finite length, so that the induced rational map \( S_i \to S' \) is a morphism for all \( i \) sufficiently large.

Proof. Since \( S_i \) is a normal surface, the set of points where \( T_i \) is not a morphism is finite (c.f. Lemma 5.1 [45]). Suppose that no \( T_i : S_i \to S' \) is a morphism. Then there exists a sequence of points \( p_i \in S_i \) for \( i \in \mathbb{N} \) such that \( p_{i+1} \) maps to \( p_i \) and \( T_i \) is not a morphism at \( p_i \) for all \( i \). Let \( R_i = \mathcal{O}_{S_i, p_i} \). The birational maps \( T_i \) give an identification of the function fields of the \( S_i \) and \( S' \) with a common field \( L \). By Theorem 8.7 \( \Omega = \bigcup R_i \) is a zero-dimensional valuation ring. Hence there exists a point \( q \in S' \) such that \( \Omega \) dominates \( \mathcal{O}_{S', q} \). \( \mathcal{O}_{S', q} \) is a localization of a \( K \)-algebra \( B = K[f_1, \ldots, f_r] \) for some \( f_1, \ldots, f_r \in L \). Since \( B \subset \Omega \) there exists some \( i \) such that \( B \subset R_i \). Since \( \Omega \) dominates \( R_i \) we necessarily have that \( R_i \) dominates \( \mathcal{O}_{S', q} \). Thus \( T_i \) is a morphism at \( p_i \), a contradiction to our assumption. Thus our sequence must be finite. \( \square \)

Theorem 8.10. Suppose that \( S \) and \( S' \) are projective surfaces over an algebraically closed field \( K \) of characteristic zero which form a resolving system for a set of zero-dimensional valuations \( N \). Then there exists a projective surface \( S^* \) which is a resolving system for \( N \).

Proof. By Theorem 8.9, there exists a birational morphism \( \overline{S} \to S \) obtained by a sequence of normalizations of the blow ups of all points where the birational map to \( S' \) is not defined. Now we construct a birational morphism \( \overline{S} \to S' \), applying the algorithm of Theorem 8.9 by only blowing up the points where the birational map to \( \overline{S} \) is not defined and which are nonsingular points. The algorithm produces a birational map \( \overline{S} \to S \) which is a morphism at all non-singular points of \( \overline{S} \).

Let \( S^* \) be the graph of the birational map from \( \overline{S} \) to \( \overline{S} \), with natural projections \( \pi_1 : S^* \to \overline{S} \) and \( \pi_2 : S^* \to \overline{S} \). We will show that \( S^* \) is a resolving system for \( N \). Suppose that \( V \in N \). Let \( \overline{p}, \overline{p}', \overline{p}^* \) and \( p^* \) be the centers of \( V \) on the respective projective surfaces \( \overline{S}, S', \overline{S}' \) and \( S^* \).

First suppose that \( \overline{p}' \) is a singular point. Then \( \overline{p} \) must be nonsingular, as \( \{\overline{S}, \overline{S}'\} \) is a resolving system for \( N \). The birational map from \( \overline{S} \) to \( \overline{S}' \) is a morphism at \( \overline{p} \), and \( \pi_1 \) is an isomorphism at \( \overline{p} \). Thus the center of \( V \) on \( S^* \) is a nonsingular point.

Now suppose that \( \overline{p}' \) is a nonsingular point. Then \( \overline{p}' \) is a nonsingular point of \( \overline{S} \) and the birational map from \( \overline{S} \) to \( \overline{S} \) is a morphism at \( \overline{p}' \). Thus \( \pi_2 \) is an isomorphism at \( \overline{p}' \) and the center of \( V \) on \( S^* \) is a nonsingular point. \( \square \)
Theorem 8.11. Suppose that $L$ is a two-dimensional algebraic function field over an algebraically closed field of characteristic zero. Then there exists a nonsingular projective surface $S$ with function field $L$.

Proof. The Theorem follows from Theorem 8.5 and Theorem 8.9, by induction on $r$ applied to a resolving system $\{S_1, \ldots, S_r\}$ for the set of zero-dimensional valuation rings of $L$. □

Remark 8.12. Zariski proves the generalization of Theorem 8.10 to dimension 3 in [83], and deduces resolution of singularities for characteristic zero 3-folds from his proof of local uniformization for characteristic zero algebraic function fields [80]. Abhyankar proves local uniformization in dimension three and characteristic $p > 5$, from which he deduces resolution of singularities for 3-folds of characteristic $p > 5$ [4].

It is a very interesting and evidently extremely difficult problem to generalize Theorem 8.10 to dimension $\geq 4$.

Exercises
1. Prove that all the birational extensions constructed in Section 8.2 are products of blow ups of points and nonsingular curves.
2. Identify where characteristic zero is used in the proofs of this chapter. All but one of the cases of Section 8.2 extend without great difficulty to characteristic $p > 0$. Some care is required to ensure that $K(x,y) \to L$ is separable. In [1], Abhyankar accomplishes this and gives a very different proof in the remaining case to prove local uniformization in characteristic $p > 0$ for algebraic surfaces.

9. Ramification of valuations and simultaneous resolution of singularities

Suppose that $L$ is an algebraic function field over a field $K$, and $\text{trdeg}_KL < \infty$ is arbitrary. We will use the notation on valuation rings of Section 8.1.

Suppose that $R$ is a local ring contained in $L$. We will say that $R$ is algebraic (or an algebraic local ring of $L$) if $L$ is the quotient field of $R$ and if $R$ is essentially of finite type over $K$. Let $K(S)$ denotes the residue field of a ring $S$ containing $K$ with a unique maximal ideal. We will also denote the maximal ideal of a ring $S$ containing a unique maximal ideal by $m_S$.

Suppose that $L \to L^*$ is a finite separable extension of algebraic function fields over $K$, and $V^*$ is a valuation ring of $L^*/K$ associated to a valuation $\nu^*$ with value group $\Gamma^*$. Then the restriction $\nu = \nu^* | L$ of $\nu^*$ to $L$ is a valuation of $L/K$ with valuation ring $V = L \cap V^*$. Let $\Gamma$ be the value group of $\nu$.

There is a commutative diagram

$$
\begin{array}{ccc}
L & \to & L^* \\
\uparrow & & \uparrow \\
V = L \cap V^* & \to & V^*
\end{array}
$$

There are associated invariants (c.f. page 53, Section 11, Chapter VI [85]) namely, the reduced ramification index of $\nu^*$ relative to $\nu$,

$$e = [\Gamma^*: \Gamma] < \infty.$$
and the relative degree of $\nu^*$ with respect to $\nu$,

$$f = [K(V^*) : K(V)].$$

The classical case is when $V^*$ is an algebraic local Dedekind domain. In this case $V$ is also an algebraic local Dedekind domain. If $m_{V^*} = (y)$ and $m_V = (x)$, then there is a unit $\gamma \in V^*$ such that

$$x = \gamma y^e.$$

Since we thus have

$$\Gamma^*/\Gamma \cong \mathbb{Z}/e\mathbb{Z},$$

we see that $e$ is the reduced ramification index.

Still assuming that $V^*$ is an algebraic local Dedekind domain, if we further assume that $K = K(V^*)$ is algebraically closed, and $(e, \text{char}(K)) = 1$, then we have that the induced homomorphism on completions with respect to the respective maximal ideals

$$\hat{V} \rightarrow \hat{V}^*$$

is given by the natural inclusion of $K$ algebras

$$K[[x]] \rightarrow K[[\overline{y}]],$$

where $\overline{y} = \sqrt[e]{y}$ and $x = \overline{y}^e$. We thus have a natural action of $\Gamma^*/\Gamma$ on $\hat{V}^*$ (a generator multiplies $\overline{y}$ by a primitive $e$-th root of unity) and the ring of invariants by this action is

$$\hat{V} \cong (\hat{V}^*)^{\Gamma^*/\Gamma}.$$

In this chapter we find analogs of these results for general valuation rings. The basic approach is to find algebraic local rings $R$ of $L$ and $S$ of $L^*$ such that $V^*$ dominates $S$ ($S$ is contained in $V^*$ and $m_{V^*} \cap S = m_S$) and $V$ dominates $R$ such that the ramification theory of $V \rightarrow V^*$ is captured in

$$R \rightarrow S,$$  \hspace{1cm} (107)

and we have a theory for $R \rightarrow S$ comparable to that of the above analysis of $V \rightarrow V^*$ in the special case when $V^*$ is an algebraic local Dedekind domain.

This should be compared with Zariski’s Local Uniformization Theorem [80] (also Section 2.5 and Chapter 8 of this book). It is proven in [80] that if $\text{char}(K) = 0$ and $V$ is a valuation ring of an algebraic function field $L/K$, then $V$ dominates an algebraic regular local ring of $L$. The case when $\text{trdeg}_K L = 2$ is proven in Section 8.2.

We first consider the following diagram:

$$\begin{array}{ccc} L & \rightarrow & L^* \\ \uparrow & & \uparrow \\ S & & \end{array}$$

(108)

where $L^*/L$ is a finite separable extension of algebraic function fields over $L$, and $S$ is a normal algebraic local ring of $L^*$. We say that $S$ lies above an algebraic local ring $R$ of $L$ if $S$ is a localization at a maximal ideal of the integral closure of $R$ in $L^*$.

Abhyankar and Heinzer have given examples of diagrams (108) where there does not exist a local ring $R$ of $L$ such that $S$ lies above $R$ (c.f. [33]). To construct an extension of the kind (107), we must at least be able to find an $S$ dominated by $V^*$ such that $S$ lies over an algebraic local ring $R$ of $L$. To do this, we consider the concept of a monoidal transform.
Suppose that $V$ is a valuation ring of the algebraic function field $L/K$, and suppose that $R$ is an algebraic local ring of $L$ such that $V$ dominates $R$. Suppose that $p \subset R$ is a regular prime; that is, $R/p$ is a regular local ring. If $f \in p$ is an element of minimal value then $R[f]$ is contained in $V$, and if $Q = m_V \cap R[p]$, then $R_1 = R[p/Q]$ is an algebraic local ring of $L$ which is dominated by $V$. We say that $R \to R_1$ is a monoidal transform along $V$. $R_1$ is the local ring of the blow up of spec$(R)$ at the nonsingular subscheme $V(p)$. If $R$ is a regular local ring then $R_1$ is a regular local ring. In this case there exists a regular system of parameters $(x_1, \ldots, x_n)$ in $R$ such that if height$(p) = r$, then $R_1 = R[p/x_1, \ldots, x_r]Q$.

The main tool we use for our analysis is the Local Monomialization Theorem 2.15. With the notation of (107) and (108), consider a diagram

\[
\begin{array}{ccc}
L & \rightarrow & L^* \\
\uparrow & & \uparrow \\
V = V^* \cap L & \rightarrow & V^* \\
\uparrow & & \uparrow \\
S^* & & \\
\end{array}
\]

(109)

where $S^*$ is an algebraic local ring of $L^*$.

The natural question to consider is Local Simultaneous Resolution; that is, does there exist a diagram (constructed from (109))

\[
\begin{array}{ccc}
V & \rightarrow & V^* \\
\uparrow & & \uparrow \\
R & \rightarrow & S \\
\uparrow & & \uparrow \\
S^* & & \\
\end{array}
\]

such that $S^* \to S$ is a sequence of monoidal transforms, $S$ is a regular local ring, and there exists a regular local ring $R$ such that $S$ lies above $R$?

The answer to this question is no, even when $\text{trdeg}_K(L^*) = 2$, as was shown by Abhyankar in [2]. However, it follows from a refinement in Theorem 4.8 [33] (strong monomialization) of Theorem 2.15 that if $K$ has characteristic zero, $\text{trdeg}_K(L^*)$ is arbitrary and $V^*$ has rational rank 1, then Local Simultaneous Resolution is true, since in this case (8) becomes

\[
x_1 = \delta_1 y_1^{a_1}, x_2 = y_2, \ldots, x_n = y_n,
\]

and $S$ lies over $R_0$.

The natural next question to consider is Weak Simultaneous Local Resolution; that is, does there exist a diagram (constructed from (109))

\[
\begin{array}{ccc}
V & \rightarrow & V^* \\
\uparrow & & \uparrow \\
R & \rightarrow & S \\
\uparrow & & \uparrow \\
S^* & & \\
\end{array}
\]

such that $S^* \to S$ is a sequence of monoidal transforms, $S$ is a regular local ring, and there exists a normal local ring $R$ such that $S$ lies above $R$?
This was conjectured by Abhyankar on page 144 \[8\] (and is implicit in \[1\]). If \(\text{trdeg}_K(L^*) = 2\), the answer to this question is yes, as was shown by Abhyankar in \[1\], \[3\]. If \(\text{trdeg}_K(L^*)\) is arbitrary, and \(K\) has characteristic zero, then the answer to this question is yes, as we prove in \[27\] and \[33\]. This is a simple corollary of Local Monomialization. In fact, Weak Simultaneous Local Resolution follows from the following theorem, which will be of use in our analysis of ramification.

**Theorem 9.1.** (Theorem 4.2 \[33\]) Let \(K\) be a field of characteristic zero, \(L\) an algebraic function field over \(K\), \(L^*\) a finite algebraic extension of \(L\), \(\nu^*\) a valuation of \(L^*/K\), with valuation ring \(V^*\). Suppose that \(S^*\) is an algebraic local ring of \(L^*\) which is dominated by \(\nu^*\) and \(R^*\) is an algebraic local ring of \(L\) which is dominated by \(S^*\). Then there exists a commutative diagram

\[
\begin{array}{ccc}
R_0 & \to & R \\
\uparrow & & \uparrow \\
R^* & \to & S^*
\end{array}
\]

(110)

where \(S^* \to S\) and \(R^* \to R_0\) are sequences of monoidal transforms along \(\nu^*\) such that \(R_0 \to S\) have regular parameters of the form of the conclusions of Theorem 2.15, \(R\) is a normal algebraic local ring of \(L\) with toric singularities which is the localization of the blowup of an ideal in \(R_0\), and the regular local ring \(S\) is the localization at a maximal ideal of the integral closure of \(R\) in \(L^*\).

**Proof.** By resolution of singularities \[48\] (c.f. Theorem 2.6, Theorem 2.9 \[24\]), we first reduce to the case where \(R^*\) and \(S^*\) are regular, and then construct, by the Local Monomialization Theorem, Theorem 2.15, a sequence of monoidal transforms along \(\nu^*\)

\[
\begin{array}{ccc}
R_0 & \to & S \\
\uparrow & & \uparrow \\
R^* & \to & S^*
\end{array}
\]

(111)

so that \(R_0\) is a regular local ring with regular parameters \((x_1, \ldots, x_n)\), \(S\) is a regular local ring with regular parameters \((y_1, \ldots, y_n)\), there are units \(\delta_1, \ldots, \delta_n\) in \(S\), and a matrix of natural numbers \(A = (a_{ij})\) with nonzero determinant \(d\) such that

\[x_i = \delta_i y_1^{a_{i1}} \cdots y_n^{a_{in}}\]

for \(1 \leq i \leq n\). After possibly reindexing the \(y_i\) we may assume that \(d > 0\). Let \((b_{ij})\) be the adjoint matrix of \(A\). Set

\[f_i = \prod_{j=1}^{n} x_j^{b_{ij}} = \left(\prod_{j=1}^{n} \delta_j^{b_{ij}}\right)y_i^{d}
\]

for \(1 \leq i \leq n\). Let \(R\) be the integral closure of \(R_0[f_1, \ldots, f_n]\) in \(L\), localized at the center of \(\nu^*\). Since \(\sqrt{m_R S} = m_S\), Zariski’s Main Theorem (10.9 \[4\]) shows that \(R\) is a normal algebraic local ring of \(L\) such that \(S\) lies above \(R\).

We thus have a sequence of the form (110). \(\square\)

In the theory of resolution of singularities a modified version of the Weak Simultaneous Resolution Conjecture is important. This is called Global Weak Simultaneous Resolution.
Suppose that \( f : X \to Y \) is a proper, generically finite morphism of \( k \)-varieties. Does there exist a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( f_1 \) is finite, \( X_1 \) and \( Y_1 \) are complete \( k \)-varieties such that \( X_1 \) is nonsingular, \( Y_1 \) is normal and the vertical arrows are birational?

This question has been posed by Abhyankar (with the further conditions that \( Y_1 \to Y \) is a sequence of blowups of nonsingular subvarieties and \( Y, X \) are projective) explicitly on page 144 of [8] and implicitly in the paper [2].

We answer this question in the negative, in an example constructed in Theorem 3.1 of [28]. The example is of a generically finite morphism \( X \to Y \) of nonsingular, projective surfaces, defined over an algebraically closed field \( K \) of characteristic not equal to 2. This example also gives a counterexample to the principle that a geometric statement that is true locally along all valuations should be true globally.

We now state our main theorem on ramification of arbitrary valuations.

**Theorem 9.2.** (Theorem 6.1 [33]) Suppose that char\((K) = 0\) and we are given a diagram of the form of (109)

\[
\begin{array}{ccc}
L & \to & L^* \\
\downarrow & & \downarrow \\
V = V^* \cap L & \to & V^*
\end{array}
\]

\( S^* \). Let \( \overline{K} \) be an algebraic closure of \( K(V^*) \). Then there exists a regular algebraic local ring \( R_1 \) of \( L \) such that if \( R_0 \) is a regular algebraic local ring of \( L \) which contains \( R_1 \) such that

\[
R_0 \to R \to S \subset V^* \\
\uparrow \\
S^*
\]

is a diagram satisfying the conclusions of Theorem 9.1, then

1. There is an isomorphism \( \mathbb{Z}^n/A\mathbb{Z}^n \cong \Gamma^*/\Gamma \) given by
   \[ (b_1, \ldots, b_n) \mapsto b_1\nu^*(y_1) + \cdots + b_n\nu^*(y_n). \]
2. \( \Gamma^*/\Gamma \) acts on \( \hat{S} \hat{\otimes}_{K(S)} \overline{K} \) and the invariant ring is
   \[ \hat{R} \hat{\otimes}_{K(R)} \overline{K} \cong (\hat{S} \hat{\otimes}_{K(S)} \overline{K})^{\Gamma^*/\Gamma}. \]
3. The reduced ramification index is
   \[ e = |\Gamma^*/\Gamma| = |\text{Det}(A)|. \]
4. The relative degree is
   \[ f = [K(V^*) : K(V)] = [K(S) : K(R)]. \]
It is shown in [33] and [30] that we can realize the valuation rings \( V \) and \( V^* \) as directed unions of local rings of the form of \( R \) and \( S \) in the above Theorem, and the unions of the valuation rings is a Galois extension with Galois group \( \Gamma^*/\Gamma \).

We now give an overview of the proof of Theorem 9.2, referring to [33] for details. \( \nu^* \) is our valuation of \( L^* \) with valuation ring \( V^* \), and \( \nu = \nu^* \mid L \) is the restriction of \( \nu^* \) to \( K \). In the statement of the Theorem \( \Gamma^* \) is the value group of \( \nu^* \) and \( \Gamma \) is the value group of \( \nu \). By assumption, we have regular parameters \((x_1, \ldots, x_n) \) in \( R_0 \) and \((y_1, \ldots, y_n) \) in \( S \) which satisfy the equations

\[
x_i = \delta_i \prod_{j=1}^{n} y_j^{a_{ij}} \quad \text{for } 1 \leq i \leq n.
\]

of (8). We have relations

\[
\nu(x_i) = \sum_{j=1}^{n} a_{ij} \nu^*(y_j) \in \Gamma
\]

for \( 1 \leq i \leq n \). Thus there is a group homomorphism

\[
Z^n / A^* Z^n \to \Gamma^*/\Gamma
\]

defined by 1. To show that this homomorphism is an isomorphism we need an extension of Local Monomialization (proven in Theorem 4.9 [33]). 3. is immediate from 1.

\( R_0 \to S \) is monomial, and \( R_0 \to R \) is a weighted blowup. Lemma 4.4 of [33] shows that if \( M^* \) is the quotient field of \( \hat{S} \otimes_{K(S)} K \) and \( M \) is the quotient field of \( \hat{R} \otimes_{K(R)} K \), then \( M^*/M \) is Galois with Galois group \( \text{Gal}(M^*/M) \cong Z^n/A^* Z^n \cong Z^n/A^* Z^n \). Furthermore, \( (\hat{S} \otimes_{K(S)} K) Z^n/A^* Z^n \cong \hat{R} \otimes_{K(R)} K \).

We see that \( \hat{R} \otimes_{K(R)} K \) is a quotient singularity, by a group whose invariant factors are determined by \( \Gamma^*/\Gamma \). This observation leads to the counterexample to global weak simultaneous resolution stated earlier in this section. This example is written up in detail in [28]. We will give a very brief outline of the construction. With the notation of (112), if a valuation \( \nu \) of \( K(Y) \) has at least two distinct extensions \( \nu_1^* \) and \( \nu_2^* \) to \( K(X) \) which have nonisomorphic quotients \( \Gamma_1^*/\Gamma \) and \( \Gamma_2^*/\Gamma \), then this presents an obstruction to the existence of a map \( X_1 \to Y_1 \) (with the notation of (112)) which is finite, and \( X_1 \) is nonsingular. If there were such a map, we would have points \( p_1 \) and \( p_2 \) on \( X_1 \), whose local rings \( R_1 \) and \( R_2 \) are regular and are dominated by \( \nu_1^* \) and \( \nu_2^* \) respectively, and a point \( p \) on \( Y_1 \) whose local ring \( R \) is dominated by \( p \). If we choose our valuation \( \nu \) carefully, we can ensure that \( \hat{R} \) is a quotient of \( \hat{R}_1 \) by \( \Gamma_1^*/\Gamma \) and \( \hat{R} \) is also a quotient of \( \hat{R}_2 \) by \( \Gamma_2^*/\Gamma \). We can in fact choose the extended valuations so that we can conclude that not only are these quotient groups not isomorphic, but the invariant rings are not isomorphic, which is a contradiction.

An interesting question is if the conclusions of Local Monomialization (Theorem 2.15) hold if the ground field \( K \) has positive characteristic. This is certainly true if \( \text{trdeg}_K(L^*) = 1 \).

Now suppose that \( \text{trdeg}_K(L^*) = 2 \) and \( K \) is algebraically closed of positive characteristic. Strong monomialization (a refinement in Theorem 4.8 [33] of Theorem 2.15) is true if the value group \( \Gamma^* \) of \( V^* \) is a finitely generated group (Theorem 7.3 [33]) or if \( V^*/V \) is defectless (Theorem 7.35 [33]). The defect is an invariant which is trivial.
in characteristic zero, but is a power of \( p \) in an extension of characteristic \( p \) function fields.

Theorem 7.38 [33] gives an example showing that Strong Monomialization fails if \( \text{trdeg}_K(L^*) = 2 \) and \( \text{char}(K) > 0 \). Of course the value group is not finitely generated and there is defect in this example. This example does satisfy the less restrictive conclusions of Local Monomialization (Theorem 2.15).

It follows from Strong Monomialization in characteristic zero (a refinement of Theorem 2.15) that Local Simultaneous Resolution is true (for arbitrary \( \text{trdeg}_K \)) if \( K \) has characteristic zero and rational rank \( \nu^* = 1 \). We consider this condition when \( \text{trdeg}_K(L^*) = 2 \) and \( K \) is algebraically closed of positive characteristic. In Theorem 7.33 [33] it is shown that in many cases Local Simultaneous Resolution does hold if \( \text{trdeg}_K(L^*) = 2 \), \( \text{char}(K) > 0 \) and the value group is not finitely generated. For instance, Local Simultaneous Resolution holds if \( \Gamma \) is not \( p \)-divisible. The example of Theorem 7.38 [33] discussed above does in fact satisfy Local Simultaneous Resolution, although it does not satisfy Strong Monomialization.

10. Smoothness and non-singularity II

10.1. Proofs of the basic theorems. Suppose that \( P \in \mathbb{A}^n_K = \text{spec}(K[x_1, \ldots, x_n]) \) has height \( n - r \). Let \( m_P \subset O_{\mathbb{A}^n_P} \) be the ideal of \( P \). The differential

\[
d : O_{\mathbb{A}^n} \to \Omega^1_{\mathbb{A}^n} = O_{\mathbb{A}^n}dx_1 \oplus \cdots \oplus O_{\mathbb{A}^n}dx_n
\]

is defined by

\[
d(u) = \frac{\partial u}{\partial x_1}dx_1 + \cdots + \frac{\partial u}{\partial x_n}dx_n.
\]

\( d \) induces a linear transformation of \( K(P) \)-vector spaces

\[
d_P : m_P/(m_P)^2 \to \Omega^1_{\mathbb{A}^n,P} \otimes K(P)
\]

Define \( D(P) \) to be the image of \( d_P \). Since \( O_{\mathbb{A}^n,P} \) is a regular local ring, \( \dim_{K(P)} m_P/m_P^2 = n - r \). Thus \( \dim_{K(P)} D(P) \leq n - r \), and \( \dim_{K(P)} D(P) = n - r \) if and only if

\[
d_P : m_P/(m_P)^2 \to D(P)
\]

is a \( K(P) \)-linear isomorphism.

**Theorem 10.1.** Suppose that \( P \in \mathbb{A}^n_K = \text{spec}(K[x_1, \ldots, x_n]) \) is a closed point. Then \( \dim_{K(P)} D(P) = n \) if and only if \( K(P) \) is separable over \( K \).

**Proof.** Suppose that \( P \in \text{spec}(K[x_1, \ldots, x_n]) \) is a closed point. Let \( m_P \subset O_{\mathbb{A}^n,P} \) be the ideal of \( P \). Let \( \alpha_i \) be the image of \( x_i \) in \( K(P) \) for \( 1 \leq i \leq n \). Let \( f_i(z_1) \in K[z_1] \) be the minimal polynomial of \( \alpha_i \) over \( K \). We can then inductively define \( f_i(z_1, \ldots, z_i) \) in the polynomial ring \( K[z_1, \ldots, z_i] \) so that \( f_i(\alpha_1, \ldots, \alpha_{i-1}, z_i) \) is the minimal polynomial of \( \alpha_i \) over \( K(\alpha_1, \ldots, \alpha_{i-1}) \) for \( 2 \leq i \leq n \). Let \( I \) be the ideal

\[(f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n)) \subset O_{\mathbb{A}^n,P}.
\]

By construction \( I \subset m_P \) and \( K[x_1, \ldots, x_n]/I \cong K(P) \), so we have \( I = m_P \). Thus \( f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n) \) is a regular system of parameters in \( O_{\mathbb{A}^n,P} \), and the images of \( f_1(x_1), f_2(x_1, x_2), \ldots, f_n(x_1, \ldots, x_n) \) in \( m_P/m_P^2 \) form a \( K(P) \)-basis. Thus \( \dim_{K(P)} D(P) = n \) if and only if \( d_P(f_1), \ldots, d_P(f_n) \) are linearly independent.
over $K(P)$. This holds if and only if $J(f; x)$ has rank $n$ at $P$, which holds if and only if the determinant

$$|J(f; x)| = \prod_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

is not zero in $K(P)$. This condition holds if and only if $K(\alpha_1, \ldots, \alpha_i)$ is separable over $K(\alpha_1, \ldots, \alpha_{i-1})$ for $1 \leq i \leq n$, which is true if and only if $K(P)$ is separable over $K$.

Corollary 10.2. Suppose that $P \in A^n_K = \text{spec}(K[x_1, \ldots, x_n])$ is a closed point such that $K(P)$ is separable over $K$. Let $m_P \subset O_{A^n, P}$ be the ideal of $P$, and suppose that $u_1, \ldots, u_n \in m_P$. Then $u_1, \ldots, u_n$ is a regular system of parameters in $O_{A^n, P}$ if and only if $J(u; x)$ has rank $n$ at $P$.

Proof. This follows since $d_P : m_P/m_P^2 \to D(P)$ is an isomorphism if and only if $K(P)$ is separable over $K$.

Lemma 10.3. Suppose that $A$ is a regular local ring of dimension $b$ with maximal ideal $m$, $p \subset A$ is a prime such that $\text{dim}(A/p) = a$. Then

$$\text{dim}_{A/m}(p + m^2/m^2) \leq b - a$$

and $\text{dim}_{A/m}(p + m^2/m^2) = b - a$ if and only if $A/p$ is a regular local ring.

Proof. Let $A' = A/p, m' = mA'$. $k = A/m \cong A'/m'$. There is a short exact sequence of $k$-vector spaces

$$0 \to p/p \cap m^2 \cong p + m^2/m^2 \to m/m^2 \to m'/(m')^2 = m/(p + m^2) \to 0.$$  

(113)

The Lemma now follows since

$$\text{dim}_k m'/(m')^2 \geq a$$

and $A'$ is regular if and only if $\text{dim}_k m'/(m')^2 = a$ (Corollary 11.5 [12]).

We now give two related results, Lemma 10.4 and Corollary 10.5.

Lemma 10.4. Suppose that $A$ is a regular local ring of dimension $n$ with maximal ideal $m$, $p \subset A$ is a prime such that $A/p$ is regular, $\text{dim}(A/p) = n - r$. Then there exist regular parameters $u_1, \ldots, u_n$ in $A$ such that $p = (u_1, \ldots, u_r)$ and $u_{r+1}, \ldots, u_n$ map to regular parameters in $A/p$.

Proof. Consider the exact sequence (113). Since $A$ and $A'$ are regular, there exists regular parameters $u_1, \ldots, u_n$ in $A$ such that $u_{r+1}, \ldots, u_n$ map to regular parameters in $A'$, and $u_1, \ldots, u_r$ are in $p$. $(u_1, \ldots, u_r)$ is thus a prime ideal of height $r$ contained in $p$. Thus $p = (u_1, \ldots, u_r)$.

Corollary 10.5. Suppose that $Y$ is a subvariety of dimension $t$ of a variety $X$ of dimension $n$. Suppose that $q \in Y$ is a nonsingular closed point of both $Y$ and $X$. Then there exist regular parameters $u_1, \ldots, u_n$ in $O_{X, q}$ such that $I_{Y, q} = (u_1, \ldots, u_{n-t})$, and $u_{n-t+1}, \ldots, u_n$ map to regular parameters in $O_{Y, q}$. If $v_{n-t+1}, \ldots, v_n$ are regular parameters in $O_{Y, q}$, then there exist regular parameters $u_1, \ldots, u_n$ as above such that $u_i$ maps to $v_i$ for $n - t + 1 \leq i \leq n$. 

Lemma 10.6. Let \( I = (f_1, \ldots, f_m) \subset K[x_1, \ldots, x_n] \) be an ideal. Suppose that \( P \in V(I) \) and \( J(f; x) \) has rank \( n \) at \( P \). Then \( K(P) \) is separable algebraic over \( K \) and \( n \) of the polynomials \( f_1, \ldots, f_m \) form a system of regular parameters in \( \mathcal{O}_{\mathbb{A}_K^n, P} \).

Proof. Let \( \overline{m} \) be the ideal of \( P \) in \( K[x_1, \ldots, x_n] \). After possibly reindexing the \( f_i \), we may assume that \( |J(f_1, \ldots, f_n; x)| \not\in \overline{m} \). By the nullstellensatz, \( \overline{m} \) is the intersection of all maximal ideals \( \overline{n} \) of \( K[x_1, \ldots, x_n] \) containing \( m \). Thus there exists a closed point \( Q = A_K^n \) with maximal ideal \( \overline{n} \) containing \( m \) such that \( |J(f_1, \ldots, f_n; x)| \not\in \overline{n} \), so that \( J(f_1, \ldots, f_n; x) \) has rank \( n \) at \( \overline{n} \). We thus have \( \dim_{K(Q)} D(Q) = n \), so that \( K(Q) \) is separable over \( K \) and \( f_1, \ldots, f_n \) is a regular system of parameters in \( \mathcal{O}_{\mathbb{A}_K^n, Q} \). By Theorem 10.1 and Corollary 10.2. Since \( I \subset \overline{m} \subset \overline{n} \), we have \( P = Q \).

Lemma 10.7. Suppose that \( P \in \mathbb{A}_K^n = \text{spec}(K[x_1, \ldots, x_n]) \) with ideal \( m \subset K[x_1, \ldots, x_n] \) is such that \( m \) has height \( n-r \) and \( t_1, \ldots, t_{n-r} \in \overline{m} \subset \mathcal{O}_{\mathbb{A}_K^n, P} \). Let \( \alpha_i \) be the image of \( x_i \) in \( K(P) \) for \( 1 \leq i \leq r \). Then the determinant \( |J(t_1, \ldots, t_{n-r}; x_{r+1}, \ldots, x_n)| \) is nonzero at \( P \) if and only if \( \alpha_1, \ldots, \alpha_r \) is a separating transcendence basis of \( K(P) \) over \( K \) and \( t_1, \ldots, t_{n-r} \) is a regular system of parameters in \( \mathcal{O}_{\mathbb{A}_K^n, P} \).

Proof. Suppose that \( |J(t_1, \ldots, t_{n-r}; x_{r+1}, \ldots, x_n)| \) is nonzero at \( P \). There exist \( s_0, s_1, \ldots, s_{n-r} \in K[x_1, \ldots, x_n] \) such that \( s_i \in K[x_1, \ldots, x_n] \) for all \( i \), \( s_0 \not\in \overline{m} \) and \( s_i = t_i \) for \( 1 \leq i \leq n-r \). We then have that \( |J(s_1, \ldots, s_{n-r}; x_{r+1}, \ldots, x_n)| \) is nonzero at \( P \). Let \( K^* = K(\alpha_1, \ldots, \alpha_r) \), \( \overline{n} \) be the kernel of \( K^*[x_{r+1}, \ldots, x_n] \to K(P) \), and let \( s_i^* \) for \( 1 \leq i \leq n-r \) be the image of \( s_i \) under \( K[x_1, \ldots, x_n] \to K^*[x_{r+1}, \ldots, x_n] \). \( \overline{n} \) is the ideal of a point \( Q \in \mathbb{A}_K^{n-r} \). We have \( K(P) = K^*(Q) \) and \( |J(s_1^*, \ldots, s_{n-r}^*; x_{r+1}, \ldots, x_n)| \neq 0 \) at \( Q \). By Lemma 10.6 \( K^*(Q) \) is separable algebraic over \( K^* \) and \( (s_1^*, \ldots, s_{n-r}^*) \) is a regular system of parameters in \( \mathcal{O}_{\mathbb{A}_K^{n-r}, Q} = K^*[x_{r+1}, \ldots, x_n]|_{\overline{n}} \). Since \( \text{trdeg}_K K(P) = r \), \( \alpha_1, \ldots, \alpha_r \) is a separating transcendence basis of \( K(P) \) over \( K \). Thus \( K^* \subset K[x_1, \ldots, x_n]|_{\overline{n}} \) and \( K[x_1, \ldots, x_n]|_{\overline{n}} = K^*[x_{r+1}, \ldots, x_n]|_{\overline{n}} \), so that \( t_1, \ldots, t_{n-r} \) is a regular system of parameters in \( \mathcal{O}_{\mathbb{A}_K^n, P} = K[x_1, \ldots, x_n]|_{\overline{n}} \).

Now suppose that \( \alpha_1, \ldots, \alpha_r \) is a separating transcendence basis of \( K(P) \) over \( K \) and \( t_1, \ldots, t_{n-r} \) is a regular system of parameters in \( \mathcal{O}_{\mathbb{A}_K^n, P} = K[x_1, \ldots, x_n]|_{\overline{n}} \). Let \( K^* = K(\alpha_1, \ldots, \alpha_r) \), \( \overline{n} \) be the kernel of \( K^*[x_{r+1}, \ldots, x_n] \to K(P) \). \( \overline{n} \) is the ideal of a point \( Q \in \mathbb{A}_K^{n-r} \). Then \( K^*(Q) = K(P) \) and \( K^*(Q) \) is separable over \( K^* \).

We have a natural inclusion \( K^* \subset K[x_1, \ldots, x_n]|_{\overline{n}} \). Thus
\[
K[x_1, \ldots, x_n]|_{\overline{n}} = K^*[x_{r+1}, \ldots, x_n]|_{\overline{n}}
\]
and \( t_1, \ldots, t_{n-r} \) is a regular system of parameters in \( K^*[x_{r+1}, \ldots, x_n]|_{\overline{n}} \). By Corollary 10.2, \( |J(t_1, \ldots, t_{n-r}; x_{r+1}, \ldots, x_n)| \) is \( \neq 0 \) at \( Q \), and thus is \( \neq 0 \) at \( P \).

Theorem 10.8. Suppose that \( P \in \mathbb{A}_K^n = \text{spec}(K[x_1, \ldots, x_n]) \) and \( t_1, \ldots, t_{n-r} \) are regular parameters in \( \mathcal{O}_{\mathbb{A}_K^n, P} \). Then \( \text{rank}(J(t; x)) = n-r \) at \( P \) if and only if \( K(P) \) is separably generated over \( K \).

Proof. Let \( \alpha_i \), \( 1 \leq i \leq n \) be the images of \( x_i \) in \( K(P) \), \( \overline{m} \) be the ideal of \( P \) in \( K[x_1, \ldots, x_n] \).
Suppose that \( \text{rank}(J(t; x)) = n - r \) at \( P \). After possibly reindexing the \( x_i \) and \( t_j \), we have \(|J(t_1, \ldots, t_{n-r}; x_{r+1}, \ldots, x_n)| \neq 0 \) at \( P \). Now \( K(P) \) is separably generated over \( K \) by Lemma 10.7.

Suppose that \( K(P) \) is separably generated over \( K \). Let \( \zeta_1, \ldots, \zeta_r \) be a separating transcendence basis of \( K(P) \) over \( K \). There exists \( \Psi_j(x) \in K[x_1, \ldots, x_n] \) for \( 0 \leq j \leq r \) with \( \Psi_0(x) \notin \mathfrak{m} \) such that \( \zeta_j = \Psi_j(\alpha) \). Let \( \mathfrak{m} \) be the kernel of the \( K \)-algebra homomorphism \( K[x_1, \ldots, x_{n+r}] \to K(P) \) defined by \( x_i \mapsto \alpha_i \) for \( 1 \leq i \leq n \), \( x_i \mapsto \zeta_{i-n} \) if \( n < i \). \( \mathfrak{m} \) is the ideal of a point \( Q \in \mathbb{A}^{n+r}_K \). Let \( \Phi_0(x), \Phi_1(x), \ldots, \Phi_{n-r}(x) \in K[x_1, \ldots, x_n] \) be such that \( \Phi_0 \notin \mathfrak{m} \) and

\[
t_i = \frac{\Phi_i(x)}{\Phi_0(x)} \text{ for } 1 \leq i \leq n - r.
\]

Let \( \Phi_{n-r+j}(x) = x_{n+j} \Phi_0(x) - \Psi_j(x) \) for \( 1 \leq j \leq r \). Let \( I \subset K[x_1, \ldots, x_{n+r}] \) be the ideal generated by \( \Phi_1, \ldots, \Phi_n \). \( I \subset \mathfrak{m} \). We will show that \( \Phi_1, \ldots, \Phi_n \) are in fact a regular system of parameters in \( \mathcal{O}_{\mathbb{A}^{n+r}_K, Q} = K[x_1, \ldots, x_{n+r}] \). It suffices to show that if \( F(x) \in \mathfrak{m} \), then there exists \( A(x) \in K[x_1, \ldots, x_{n+r}] - \mathfrak{m} \) such that \( AF \in I \). Given such an \( F \), we have an expansion

\[
\Psi_0(x)^s F(x) = \sum_{j=1}^{r} B_j(x) \Phi_{n-r+j}(x) + G(x_1, x_2, \ldots, x_n)
\]

where \( B_j(x) \in K[x_1, \ldots, x_{n+r}] \) and \( s \) is a nonnegative integer. \( F \in \mathfrak{m} \) implies \( G \in \mathfrak{m} \). Thus there exists an expansion

\[
G = \sum_{i=1}^{n-r} f_i t_i
\]

with \( f_i \in K[x_1, \ldots, x_n][\mathfrak{m}] \), and there exist \( h \in K[x_1, \ldots, x_n] - \{\mathfrak{m}\} \), \( C_i \in K[x_1, \ldots, x_n] \) such that

\[
hG = \sum_{i=1}^{r} C_i \Phi_i(x)
\]

and \( h \Psi_0(x)^s F(x) \in I \).

Since \( \Phi_1, \ldots, \Phi_n \) is a regular system of parameters in \( K[x_1, \ldots, x_{n+r}] \) and \( \zeta_1, \ldots, \zeta_r \) is a separating transcendence basis of \( K(Q) \) over \( K \), \(|J(\Phi_1, \ldots, \Phi_n; x_1, \ldots, x_n)| \) is nonzero at \( Q \) by Lemma 10.7. Thus \( J(\Phi_1, \ldots, \Phi_n; x_1, \ldots, x_n) \) is of rank \( n - r \) at \( P \), and by Lemma 10.7, \( \alpha_1, \ldots, \alpha_n \) must contain a separating transcendence basis of \( K(P) \) over \( K \).

**Proof of Theorem 2.7.**

Let \( s = \dim X \). There exists an affine neighborhood \( U = \text{spec}(R) \) of \( P \) in \( X \) such that \( R = K[x_1, \ldots, x_n]/I \), \( I = (f_1, \ldots, f_m) \). Let \( m_P \in \text{spec}(K[x_1, \ldots, x_n]) \) be the ideal of \( P \) in \( \mathbb{A}^n_K \). Suppose that \( \dim (K[x_1, \ldots, x_n]/m_P) = t \).

Suppose that \( K(P) \) is separably generated over \( K \) and \( P \) is a nonsingular point of \( X \). By Theorem 10.8, \( d_P : m_P/m_P^2 \to D(P) \) is an isomorphism. By Lemma 10.3, \( \dim_{K(P)} d_P(I) = n - s \) (take \( A = \mathcal{O}_{\mathbb{A}^n_K, P}, p = I_{m_P} \) in the statement of the Lemma, so that \( b = n - t, a = s - t \)) which is equivalent to the condition that \( J(f; x) \) has rank \( n - s \) at \( P \).
Suppose that $X$ is smooth at $P$ so that $J(f; x)$ has rank $n - s$ at $P$. Then 
$\dim_{K(P)}d_P(I) = n - s$ so that $\dim_{K(P)}I + m_P/m_P^2 \geq n - s$. Thus $P$ is a nonsingular point of $X$ by Lemma 10.3.

**Remark 10.9.** With the notation of the proof of Theorem 2.7, for any point $P \in V(I)$, we have 
$\dim_{K(P)} = n - s$ so that $\dim_{K(P)}I + m_P/m_P^2 \geq n - s$. Thus $P$ is a nonsingular point of $X$ by Lemma 10.3.

**Proof of Theorem 2.6**

It suffices to prove that the locus of smooth points of $X$ lying on an open affine subset $U = \text{spec}(R)$ of $X$ of the form of Definition 2.5 is open. Let $A = I_{n-2}(J(f; x))$. 
$p \in U - V(A)$ if and only if $J(f; x)$ has rank greater than or equal to $n - s$ at $p$, which holds if and only if $J(f; x)$ has rank $n - s$ by Remark 10.9.

**Proof of Theorem 2.9 when $K$ is perfect**

Suppose that $K$ is perfect. The openness of the set of nonsingular points of $X$ follows from Theorem 2.6 and Corollary 2.8.

Let $\eta \in X$ be the generic point of an irreducible component of $X$. Then $\mathcal{O}_{X, \eta}$ is a field which is a regular local ring. Thus $\eta$ is a nonsingular point. We conclude that the nonsingular points of $X$ are dense in $X$.

10.2. **Non-singularity and uniformizing parameters.** Consider the sheaf of differentials $\Omega^1_{X/K}$. With the notation of Definition 2.5, we have, by the “second fundamental exact sequence” (Theorem 25.2 [61]), a presentation 
$$(K[x_1, \ldots, x_n])^m J(f, x) \rightarrow \Omega^1_{K[x_1, \ldots, x_n]/K} \otimes R \rightarrow \Omega^1_{R/k} \rightarrow 0$$

**Theorem 10.10.** Suppose that $X$ is a variety of dimension $r$ over a field $K$, and $P \in X$ is a closed point such that $X$ is nonsingular at $P$ and $K(P)$ is separable over $K$.

1. Suppose that $y_1, \ldots, y_r$ are regular parameters in $\mathcal{O}_{X, P}$. Then 
$$\Omega^1_{X/K, P} = dy_1 \mathcal{O}_{X, P} \oplus \cdots \oplus dy_r \mathcal{O}_{X, P}.$$ 

2. Let $\mathfrak{m}$ be the ideal of $P$ in $\mathcal{O}_{X, P}$. Suppose that $f_1, \ldots, f_r \in \mathfrak{m} \subset \mathcal{O}_{X, P}$ and 
$$df_1, \ldots, df_r$$ generate $\Omega^1_{X/K, P}$ as an $\mathcal{O}_{X, P}$ module. Then $f_1, \ldots, f_r$ are regular parameters in $\mathcal{O}_{X, P}$.

**Proof.** We can restrict to an affine neighborhood of $X$, and assume that 
$$X = \text{spec}(K[x_1, \ldots, x_n]/I).$$

The conclusions of 1. of the Theorem follow from Theorem 10.8 when $X = \mathbb{A}^n_K$.

In the general case of 1., by Corollary 10.5, there exist regular parameters $(\overline{y}_1, \ldots, \overline{y}_n)$ in $A = K[x_1, \ldots, x_n]/\overline{m}$ (where $\overline{m}$ is the ideal of $P$ in $\text{spec}(K[x_1, \ldots, x_n])$ such that 
$$IA = (\overline{y}_{r+1}, \ldots, \overline{y}_n)$$ and $(\overline{y}_1, \ldots, \overline{y}_r)$ map to the regular parameters $(y_1, \ldots, y_r)$ in
Proposition 10.11. Suppose that $P$ is a variety of dimension $r$ over a perfect field $K$. Suppose that $P \in X$ is a closed point such that $X$ is nonsingular at $P$, and $y_1, \ldots, y_r$ are regular parameters in $\mathcal{O}_{X,P}$. Then there exists an affine neighborhood $U = \text{spec}(R)$ of $P$ in $X$ such that $y_1, \ldots, y_r \in R$, the natural inclusion $S = K[y_1, \ldots, y_r] \rightarrow R$ induces a morphism $\pi : U \rightarrow \mathbf{A}^r_K$ such that

$$\Omega^1_{R/K} = dy_1R \oplus \cdots \oplus dy_rR, \quad (114)$$

whose derivation $\text{Der}_K(R, R) = \text{Hom}_R(\Omega^1_{R/K}, R) = \frac{\partial}{\partial y_1}R \oplus \cdots \oplus \frac{\partial}{\partial y_r}R$.

Let $w_i = \sum_{j=1}^r a_{ij} dy_j$ for $1 \leq i \leq r$. By Nakayama’s Lemma we have that the $w_i$ generate $\Omega^1_{X/K,P}$ as an $R$-module. Thus $| (a_{ij}) | \notin \mathfrak{m}$ and $(a_{ij})$ is invertible over $R$, so that $f_1, \ldots, f_r$ generate $\mathfrak{m}$.

Lemma 10.11. Suppose that $X$ is a variety of dimension $r$ over a perfect field $K$. Suppose that $P \in X$ is a closed point such that $X$ is nonsingular at $P$, and $y_1, \ldots, y_r$ are regular parameters in $\mathcal{O}_{X,P}$. Then there exists an affine neighborhood $U = \text{spec}(R)$ of $P$ in $X$ such that $y_1, \ldots, y_r \in R$, the natural inclusion $S = K[y_1, \ldots, y_r] \rightarrow R$ induces a morphism $\pi : U \rightarrow \mathbf{A}^r_K$ such that

$$\Omega^1_{R/K} = dy_1R \oplus \cdots \oplus dy_rR, \quad (114)$$

Let $w_i = \sum_{j=1}^r a_{ij} dy_j$ for $1 \leq i \leq r$. By Nakayama’s Lemma we have that the $w_i$ generate $\Omega^1_{X/K,P}$ as an $R$-module. Thus $| (a_{ij}) | \notin \mathfrak{m}$ and $(a_{ij})$ is invertible over $R$, so that $f_1, \ldots, f_r$ generate $\mathfrak{m}$.

**Proof.** By Theorem 10.10, there exists an affine neighborhood $U = \text{spec}(R)$ of $P$ in $X$ such that $y_1, \ldots, y_r \in R$ and $dy_1, \ldots, dy_r$ is a free basis of $\varpi^1_{R/K}$. Consider the natural inclusion $S = K[y_1, \ldots, y_r] \rightarrow R$, with induced morphism $\pi : \text{spec}(R) \rightarrow \text{spec}(S)$. Suppose that $a \in U$ is a closed point with maximal ideal $\mathfrak{m}$ in $R$ and $b = \pi(a)$, with maximal ideal $\mathfrak{p}$ in $S$. Suppose that $(x_1, \ldots, x_r)$ are regular parameters in $B = S_\mathfrak{p}$. Let $A = R_{\mathfrak{m}}$. By construction, $\varpi^1_{S/K}$ is generated by $dy_1, \ldots, dy_r$ as an $S$ module. Thus $\varpi^1_{A/K}$ is generated by $\varpi^1_{B/K}$ as an $A$ module. By 1. of Theorem 10.10, $dx_1, \ldots, dx_r$ generate $\varpi^1_{B/K}$ as a $B$ module, and hence they generate $\varpi^1_{A/K}$ as an $A$ module. By 2. of Theorem 10.10, $x_1, \ldots, x_r$ is a regular system of parameters in $A$. \qed
10.3. Higher derivations. Suppose that $K$ is a field and $A$ is a $K$-algebra. A higher derivation of $A$ over $K$ of length $m$ is a sequence

$$D = (D_0, D_1, \ldots, D_m)$$

if $m < \infty$, and a sequence

$$D = (D_0, D_1, \ldots)$$

if $m = \infty$, of $K$-linear maps $D_i: A \to A$ such that $D_0 = \text{id}$ and

$$D_i(xy) = \sum_{j=1}^{i} D_j(x)D_{j-i}(y)$$

(115)

for $1 \leq i \leq m$ and $x, y \in A$.

Let $t$ be an indeterminate. $D = (D_0, D_1, \ldots)$ is a higher derivation of $A$ over $K$ (of length $\infty$) precisely when the map $E_t: A \to A[[t]]$ defined by

$$E_t(f) = \sum_{i=0}^{\infty} D_i(f)t^i$$

is a $K$-algebra homomorphism with $D_0(f) = f$, $D = (D_0, D_1, \ldots, D_m)$ is a higher derivation of $A$ over $K$ of length $m$ precisely when the map $E_t: A \to A[t]/(t^{m+1})$ defined by

$$E_t(f) = \sum_{i=0}^{m} D_i(f)t^i$$

is a $K$-algebra homomorphism with $D_0(f) = f$.

Example 10.12. Suppose that $A = K[y_1, \ldots, y_r]$ is a polynomial ring over a field $K$. For $1 \leq i \leq r$, let $\epsilon_j$ be the vector in $\mathbb{N}^r$ with a 1 in the $j$th coefficient and 0 everywhere else. For $f = \sum a_{i_1, \ldots, i_r} y_1^{i_1} \cdots y_r^{i_r} \in A$ and $m \in \mathbb{N}$ define

$$D_{me_j}(f) = \sum a_{i_1, \ldots, i_r} \binom{i_j}{m} y_1^{i_1} \cdots y_j^{i_j - m} \cdots y_r^{i_r}.$$ 

Then $D^*_j = (D_0^*, D_1^*, D_2^*, \ldots)$ is a higher derivation of $A$ over $K$ of length $\infty$ for $1 \leq j \leq r$. Define

$$D^*_{i_1, \ldots, i_r} = D^*_{i_r, \epsilon_r} \circ D^*_{i_{r-1}, \epsilon_{r-1}} \circ \cdots \circ D^*_{i_1, \epsilon_1}$$

for $(i_1, \ldots, i_r) \in \mathbb{N}^r$. $D^*_{i_1, \ldots, i_r}$ are called Hasse-Schmidt derivations [52]. If $K$ has characteristic zero,

$$D^*_{i_1, \ldots, i_r} = \frac{1}{i_1! i_2! \cdots i_r!} \frac{\partial^{i_1 + \cdots + i_r}}{\partial y_1^{i_1} \cdots \partial y_r^{i_r}}.$$

Lemma 10.13. Suppose that $X$ is a variety of dimension $r$ over a perfect field $K$. Suppose that $P \in X$ is a closed point such that $X$ is nonsingular at $P$, $y_1, \ldots, y_r$ are regular parameters in $\mathcal{O}_{X,P}$, and $U = \text{spec}(R)$ is an affine neighborhood of $P$ in $X$ such that the conclusions of Lemma 10.11 hold. Then there are differential operators $D_{i_1, \ldots, i_r}$ on $R$ which are uniquely determined on $R$ by the differential operators $D^*_{i_1, \ldots, i_r}$ on $S$ of Example 10.12, such that if $f \in \mathcal{O}_{X,P}$, and

$$f = \sum a_{i_1, \ldots, i_r} y_1^{i_1} \cdots y_r^{i_r}$$

in $\mathcal{O}_{X,P} \cong K(P)[[y_1, \ldots, y_r]]$, where we identify $K(P)$ with the coefficient field of $\mathcal{O}_{X,P}$ containing $K$. Then $a_{i_1, \ldots, i_r} = D_{i_1, \ldots, i_r}(f)(P)$ is the residue of $D_{i_1, \ldots, i_r}(f)$ in $K(P)$ for all indices $i_1, \ldots, i_r$.

Proof. If $K$ has characteristic zero we can take

$$D_{i_1, \ldots, i_r} = \frac{1}{i_1! \cdots i_r!} \frac{\partial^{i_1 + \cdots + i_r}}{\partial y_1^{i_1} \cdots \partial y_r^{i_r}}.$$ 

(116)
Suppose that \( K \) has characteristic \( p \geq 0 \). Consider the inclusion of \( K \)-algebras
\[
S = K[y_1, \ldots, y_r] \to R
\]
of Lemma 10.11, with induced morphism
\[
\pi : U = \text{spec}(R) \to V = \text{spec}(S).
\]
Let \( D_j^* \) for \( 1 \leq j \leq r \) be the higher order derivation of \( S \) over \( K \) defined in Example 10.12. \( \Omega_R^1 / S = 0 \) by 1. of Theorem 10.10, (114) and the “first fundamental exact sequence” (Theorem 25.1 [61]). Thus \( \pi \) is non-ramified by Corollary IV.17.4.2 [43]. By Theorem IV.18.10.1 [43] and Proposition IV.6.15.6 [43] \( \pi \) is equal.

We will now prove that \( D_j^* \) extends uniquely to a higher derivation \( \hat{D}_j \) of \( R \) over \( K \) for \( 1 \leq j \leq r \). It suffices to prove that \((D_0^*, D_{\epsilon_j}^*, \ldots, D_{m_{\epsilon_j}}^*)\) extends to a higher derivation of \( R \) over \( K \) for all \( m \). The extension of \( D_0^* = id \) to \( D_0 = id \) is immediate. Suppose that an extension \((D_0, D_1, \ldots, D_m)\) of \((D_0^*, D_{\epsilon_j}^*, \ldots, D_{m_{\epsilon_j}}^*)\) has been constructed. Then we have a commutative diagram of \( K \)-algebra homomorphisms
\[
\begin{array}{ccc}
R & \xrightarrow{g} & R[t]/(t^{m+1}) \\
\uparrow & & \uparrow \\
S & \xleftarrow{f} & R[t]/(t^{m+2})
\end{array}
\]
where
\[
g(a) = \sum_{i=0}^{m} D_{\epsilon_j}(a)t^i,
\]
\[
f(a) = \sum_{i=0}^{m+1} D_{m_{\epsilon_j}}(a)t^i.
\]
Since \((t^{m+1})R[t]/(t^{m+2})\) is an ideal of square zero, and \( S \to R \) is formally etale (Definition IV.17.3.1 [43] and Definition 17.1.1 [43]) there exists a unique \( K \)-algebra homomorphism \( h \) making a commutative diagram
\[
\begin{array}{ccc}
R & \xrightarrow{g} & R[t]/(t^{m+1}) \\
\uparrow & \xrightarrow{h} & \uparrow \\
S & \xleftarrow{f} & R[t]/(t^{m+2})
\end{array}
\]
Thus there is a unique extension of \((D_0^*, D_{\epsilon_j}^*, \ldots, D_{m_{\epsilon_j}}^*)\) to a higher derivation of \( R \).

Set \( D_{i_1, \ldots, i_r} = D_{i_1}\epsilon_j \circ \cdots \circ D_{i_r} \).

Let \( A = \mathcal{O}_{X, P} \). Since \( K(P) \) is separable over \( K \), and by the exercise of Section 3.2, we can identify the residue field \( K(P) \) of \( A \) with the coefficient field of \( \hat{A} \) which contains \( K \). Thus we have natural inclusions
\[
K[y_1, \ldots, y_r] \subset A \subset \hat{A} \cong K(P)[[y_1, \ldots, y_r]]
\]
We will now show that \( D_{\epsilon_j} \) extends uniquely to a higher derivation of \( \hat{A} \) over \( K \) for \( 1 \leq j \leq r \). Let \( m \) be the maximal ideal of \( A \). Let \( t \) be an indeterminate. \( \hat{A}[[t]] \) is a complete local ring with maximal ideal \( n = m\hat{A}[[t]] + t\hat{A}[[t]] \). Define \( E_t : A \to \hat{A}[[t]] \) by \( E_t(f) = \sum_{n=0}^{\infty} D_{n\epsilon_j}(f)t^n \). \( E_t \) is a \( K \)-algebra homomorphism since \( D_{\epsilon_j} \) is a higher derivation. \( E_t \) is continuous in the \( m \)-adic topology, as \( E_t(m^n) \subset n^a \) for all \( a \). Thus
Remark 10.14. Suppose that the assumptions of Lemma 10.13 hold, and that $K$ is an algebraically closed field. Then the following properties hold.

1. Suppose that $Q \in U$ is a closed point and $a$ is the corresponding maximal ideal in $R$. Then $a = (y_1, \ldots, y_r)$ where $y_i = y_i - \alpha_i$ with $\alpha_i = y_i(Q) \in K$ for $1 \leq i \leq r$, and $K[[y_1, \ldots, y_r]] = \hat{R}_a = \hat{\mathcal{O}}_{U,Q}$.

2. The differential operators $D_{i_1, \ldots, i_r}$ on $R$ are such that if $f \in R$, and $a$ is a maximal ideal in $R$, then with the notation of $1$,

$$f = \sum \alpha_{i_1, \ldots, i_r} \bar{y}_1^{i_1} \cdots \bar{y}_r^{i_r}$$

in $K[[y_1, \ldots, y_r]] = \hat{R}_a$, where $\alpha_{i_1, \ldots, i_r} = D_{i_1, \ldots, i_r}(f)(Q)$ is the residue of $D_{i_1, \ldots, i_r}(f)$ in $K(Q)$ for all indices $i_1, \ldots, i_r$.

Proof. The ideal of $\pi(Q)$ in $S$ is $(y_1 - \alpha_1, \ldots, y_r - \alpha_r)$ with $\alpha_i = y_i(Q) \in K$. Let $\bar{y}_i = y_i - \alpha_i$ for $1 \leq i \leq r$. $a = (\bar{y}_1, \ldots, \bar{y}_r)$ by Theorem 10.10 2, and 1. follows. The derivations $\frac{\partial}{\partial \bar{y}_1}, \ldots, \frac{\partial}{\partial \bar{y}_r}$ are exactly the derivations $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_r}$ on $S$ of Example 10.12, so they extend to the differential operators $D_{i_1, \ldots, i_r}$ on $S$ of Example 10.12. 2. now follows from Lemma 10.13 applied to $Q$ and $\bar{y}_1, \ldots, \bar{y}_r$. \hfill \Box

Lemma 10.15. Let notation be as in Lemmas 10.11 and 10.13. In particular we have a fixed closed point $P \in U = \text{spec}(R)$ and differential operators $D_{i_1, \ldots, i_r}$ on $R$. Let $I \subset R$ be an ideal. Define an ideal $J_{m,U} \subset R$ by

$$J_{m,U} = \{ D_{i_1, \ldots, i_r}(f) \mid f \in I \text{ and } i_1 + \cdots + i_r < m \}.$$

Suppose that $Q \in U$ is a closed point, with corresponding ideal $m_Q$ in $R$, and suppose that $(z_1, \ldots, z_r)$ is a regular system of parameters in $m_Q$.

Let $\overline{D}_{i_1, \ldots, i_r}$ be the differential operators constructed on $\mathcal{O}_{W,Q}$ by the construction of Lemmas 10.11 and 10.13. Then

$$(J_{m,U})_{m_Q} = \{ \overline{D}_{i_1, \ldots, i_r}(f) \mid f \in I \text{ and } i_1 + \cdots + i_r < m \}.$$

Proof. When $K$ has characteristic zero, this follows directly from differentiation and (116).

Assume that $K$ has characteristic zero. By Lemma 10.13 and Theorem IV.16.11.2 [43], $U$ is differentiably smooth over $\text{spec}(K)$ and $\{ D_{i_1, \ldots, i_r} \mid i_1, \ldots, i_r < m \}$ is an $R$-basis of the differential operators of order less than $m$ on $R$. Since $\{dz_1, \ldots, dz_r\}$ is a basis of $\Omega^1_{R_{m_Q}/K}$ (by Lemma 10.11), $\{ \overline{D}_{i_1, \ldots, i_r} \mid i_1, \ldots, i_r < m \}$ is an $R_{m_Q}$-basis of the differential operators of order less than $m$ on $R_{m_Q}$ by Theorem IV.16.11.2 [43]. The Lemma follows. \hfill \Box

10.4. Upper semi-continuity of $\nu_q(I)$.

Lemma 10.16. Suppose that $W$ is a nonsingular variety over a perfect field $K$ and $K'$ is an algebraic field extension of $K$. Let $W' = W \times_K K'$, with projection $\pi_1 : W' \to W$. Then $W'$ is a nonsingular variety over $K'$. Suppose that $q \in W$ is a closed point. Then there exists a closed point $q' \in W'$ such that $\pi_1(q') = q$. Furthermore,
Corollary 10.2, determinant

Proof. The fact that $W'$ is nonsingular follows from Theorem 2.7 and the definition of smoothness. Let $q' \in W'$ be a closed point such that $\pi_1(q') = q$.

We first prove 1. Suppose that $U = \text{spec}(R)$ is an affine neighborhood of $q$ in $W$. There is a presentation $R = K[x_1, \ldots, x_n]/I$. We further have that $\pi_1^{-1}(U) = \text{spec}(R')$, with $R' = K'[x_1, \ldots, x_n]/I(K'[x_1, \ldots, x_n])$. Let $\overline{\pi}$ be the ideal of $q'$ in $K'[x_1, \ldots, x_n]$. By the exact sequence of Lemma 10.3, there exists a regular system of parameters $\overline{f}_1, \ldots, \overline{f}_n$ in $K[x_1, \ldots, x_n]$, such that $I_{\overline{\pi}} = (\overline{f}_1, \ldots, \overline{f}_n)$ and $\overline{f}_i$ maps to $f_i$ for $1 \leq i \leq r$. By Corollary 10.2, the determinant $|J(\overline{f}_1, \ldots, \overline{f}_n)|$ is nonzero at $q$, so it is nonzero at $q'$. By Corollary 10.2, $\overline{f}_1, \ldots, \overline{f}_n$ is a regular system of parameters in $K'[x_1, \ldots, x_n]$. Thus $f_1, \ldots, f_r$ is a regular system of parameters in $R_{\overline{\pi}} = \mathcal{O}_{W', q'}$.

We will now prove 2. Since $\overline{\pi}$ maps to a maximal ideal of $\mathcal{O}_{W, q} \otimes_K K'$,

$\mathcal{O}_{W', q'} = (\mathcal{O}_{W, q} \otimes_K K')^{\overline{\pi}}$ (‘$\overline{\pi}$’ denotes completion with respect to $\overline{\pi}$). Thus $\mathcal{O}_{W', q'}$ is faithfully flat over $\mathcal{O}_{W, q}$, and $(I \mathcal{O}_{W', q'}) \cap \mathcal{O}_{W, q} = I$ (by Theorem 7.5 (ii) [61]).

Suppose that $(R, m)$ is a regular local ring containing a field $K$ of characteristic zero. Let $K'$ be the residue field of $R$. Suppose that $J \subset R$ is an ideal. We define the order of $J$ in $R$ to be

$$\nu_R(J) = \max \{ b \mid J \subset m^b \}.$$ If $J$ is a locally principal ideal, then the order of $J$ is its multiplicity. However, order and multiplicity are different in general.

**Definition 10.17.** Suppose that $q$ is a point on a variety $W$ and $J \subset \mathcal{O}_W$ is an ideal sheaf. We denote

$$\nu_q(J) = \nu_{\mathcal{O}_W}(J \mathcal{O}_{W, q}).$$

If $X \subset W$ is a subvariety, we denote

$$\nu_q(X) = \nu_q(\mathcal{I}_X).$$

**Remark 10.18.** Let notations be as in Lemma 10.16 (except we allow $K$ to be an arbitrary, not necessarily perfect field). Observe that if $q \in W$ and $q' \in W'$ are such that $\pi(q') = q$ and $J \subset \mathcal{O}_W$ is an ideal sheaf, then

$$\nu_{\mathcal{O}_{W', q'}}(J_q) = \nu_{\mathcal{O}_{W', q'}}(J_q \mathcal{O}_{W', q'}) = \nu_{\mathcal{O}_{W', q'}}(J_q \mathcal{O}_{W', q'}) = \nu_{\mathcal{O}_{W', q'}}(J_q \mathcal{O}_{W', q'}).$$

Suppose that $X$ is a noetherian topological space and $I$ is a totally ordered set. $f : X \to I$ is said to be an upper semi-continuous function if for any $\alpha \in I$,

$$\{ q \in X \mid f(q) \geq \alpha \}$$

is a closed subset of $X$.

Suppose that $X$ is a subvariety of a nonsingular variety $W$. Define

$$\nu_X : W \to \mathbb{N}$$

by $\nu_X(q) = \nu_q(X)$ for $q \in W$. The order $\nu_q(X)$ is defined in Definition 10.17.
**Theorem 10.19.** Suppose that $K$ is a perfect field, $W$ is a nonsingular variety over $K$ and $I$ is an ideal sheaf on $W$. Then

$$\nu_q(I) : W \to \mathbb{N}$$

is an upper semi-continuous function.

**Proof.** Suppose that $m \in \mathbb{N}$. Suppose that $U = \text{spec}(R)$ is an open affine subset of $W$ such that the conclusions of Lemma 10.11 hold on $W$. Set $I = \Gamma(U, I)$.

With the notations of Lemmas 10.11 and 10.13, define an ideal $J_{m,U} \subset R$ by

$$J_{m,U} = \{D_{i_1,\ldots,i_r}(g) \mid g \in I \text{ and } i_1 + \cdots + i_r < m\}.$$  

By Lemma 10.15, this definition is independent of choice of $y_1, \ldots, y_r$ satisfying the conclusions of Lemma 10.11, so our definition of $J_{m,U}$ determines a sheaf of ideals $J_m$ on $W$.

We will show that

$$V(J_m) = \{\eta \in W \mid \nu_\eta(I) \geq m\},$$

from which upper semi-continuity of $\nu_\eta(I)$ follows.

Suppose that $\eta \in W$ is a point. Let $Y$ be the closure of $\{\eta\}$ in $W$, and suppose that $P \in Y$ is a closed point such that $Y$ is nonsingular at $P$.

By Corollary 10.5, Lemma 10.11 and Lemma 10.15, there exists an affine neighborhood $U = \text{spec}(R)$ of $P$ in $W$ such that there are $y_1, \ldots, y_r \in R$ with the properties that $y_1 = \ldots = y_t = 0$ (with $t \leq r$) are local equations of $Y$ in $U$ and $y_1, \ldots, y_r$ satisfy the conclusions of Lemma 10.11.

For $g \in I_P$, consider the expansion

$$g = \sum a_{i_1,\ldots,i_r} y_1^{i_1} \cdots y_r^{i_r}$$

in $\mathcal{O}_{W,P} \cong K(P)[[y_1, \ldots, y_r]]$, where we have identified $K(P)$ with the coefficient field of $\mathcal{O}_{W,P}$ containing $K$ and (with the notation of Lemma 10.13)

$$a_{i_1,\ldots,i_r} = D_{i_1,\ldots,i_r}(g)(P).$$

$\eta \in V(J_m)$ is equivalent to $D_{i_1,\ldots,i_r}(g) \in (y_1, \ldots, y_r)$ whenever $g \in I_P$ and $i_1 + \cdots + i_r < m$, which is equivalent to

$$a_{i_1,\ldots,i_r} = D_{i_1,\ldots,i_r}(g)(P) = 0$$

if $g \in I_P$ and $i_1 + \cdots + i_t < m$, which holds if and only if

$$g \in \mathcal{I}_{Y,P}^m \cap \mathcal{O}_{W,P} = \mathcal{I}_{Y,P}^m$$

for all $g \in I_P$, where the last equality is by Lemma 10.16. Localizing $\mathcal{O}_{W,P}$ at $\eta$, we see that $\nu_\eta(I) \geq m$ if and only if $\eta \in V(J_m)$. \qed

**Definition 10.20.** Suppose that $W$ is a nonsingular variety over a perfect field $K$ and $X \subset W$ is a subvariety.

For $b \in \mathbb{N}$, define

$$\text{Sing}_b(X) = \{q \in W \mid \nu_q(W) \geq b\}.$$  

$\text{Sing}_b(X)$ is a closed subset of $W$ by Theorem 10.19. Let

$$r = \max\{\nu_X(q) \mid q \in W\}.$$  

Suppose that $q \in \text{Sing}_r(X)$. A subvariety $H$ of an affine neighborhood $U$ of $q$ in $W$ is called a hypersurface of maximal contact for $X$ at $q$ if
Remark 10.21. Suppose that \( W \) is a nonsingular variety over a perfect field \( K \) and \( X \subset W \) is a subvariety, \( q \in W \). Let \( T = \hat{O}_{W,q} \), and define
\[
\text{Sing}_b(\hat{O}_{X,q}) = \{ P \in \text{spec}(T) \mid \nu(\mathcal{I}_{X,q}T_P) \geq b \}.
\]
Then \( \text{Sing}_b(\hat{O}_{X,q}) \) is a reduced ideal whose support is \( \text{Sing}_b(X) \), then \( J_qT \) is a reduced ideal whose support is \( \text{Sing}_b(\hat{O}_{X,q}) \).

Proof. Let \( U = \text{spec}(R) \) be an affine neighborhood of \( q \in W \) satisfying the conclusions of Lemma 10.13 (with \( p = q \)), and Let notation be as in Lemma 19.13 (with \( p = q \)). With the further notation of Theorem 10.19 (and \( \mathcal{I} = \mathcal{I}_X \)) we see that \( \Gamma(U,J) = \sqrt{J_{b,U}} \). \( J_qT \) is a reduced ideal by Theorem 3.5.

Suppose that \( P \in \text{Sing}_b(\hat{O}_{X,q}) \). Then \( \nu_{T_P}(g) \geq b \) for all \( g \in \mathcal{I}_{X,q} \). Thus \( g \in P^{(b)} \subset T \), and there exists \( h \in T - P \) such that \( gh \in P^b \). Now induction in the formula (115) shows that \( D_{i_1,\ldots,i_r}(gh) \in P^{b-i_1-\cdots-i_r} \) and \( D_{i_1,\ldots,i_r}(g) \in P^{b-i_1-\cdots-i_r} \subset P \) if \( i_1 + \cdots + i_r < b \). Thus \( J_qT \subset P \).

Now suppose that \( P \in \text{spec}(T) \) and \( J_qT \subset P \). Let \( S = O_{W,q} \). If \( J_q = P_1 \cap \cdots \cap P_m \) is a primary decomposition, then all primes \( P_i \) are minimal since \( J_q \) is reduced. By Theorem 3.5 there are primes \( P_i \) in \( T \) and a function \( \sigma(i) \) such that \( P_iT = P_i \cap \cdots \cap P_{\sigma(i)} \) is a minimal primary decomposition. Thus \( J_qT_{P_{ij}} = P_{ij}T_{P_{ij}} = P_{ij}T_{P_{ij}} \) for all \( i,j \).

We have \( \cap_{i,j} P_{ij} \subset P \) so \( P_{ij} \subset P \) for some \( i,j \). \( \mathcal{I}_{X,q} \subset P_{ij}S_{P_i} \) implies \( \mathcal{I}_{X,q} \subset P_{ij}T_{P_{ij}} \) implies \( \mathcal{I}_{X,q} \subset P_{ij}T_{P_{ij}} \). Thus \( \nu_{T_P}(\mathcal{I}_{X,q}T_P) \geq b \) and \( P \in \text{Sing}_b(\hat{O}_{X,q}) \).

\[ \square \]

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