AN INTRODUCTION TO SPECTRAL SEQUENCES

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1. Introduction

In these notes we give an introduction to spectral sequences. Our exposition owes much to the treatments in Eisenbud's book [2] and in Rotman's book [4].

2. Complexes and Differential Modules

Suppose that $R$ is a commutative ring with identity. A complex of $R$-modules is a sequence of $R$-module homomorphisms

\[ F : \cdots \to F_{i+1} \xrightarrow{d_{i+1}} F_i \xleftarrow{d_i} F_{i-1} \to \cdots \]

where $d^2 = 0$, or a a sequence of $R$-module homomorphisms

\[ F' : \cdots \to F_{i-1} \xrightarrow{d_{i-1}} F_i \xleftarrow{d_i} F_{i+1} \to \cdots \]

where $d^2 = 0$. The differential $d$ has degree -1 in (1) and degree +1 in (2).

We will often write $F^*$ (or $(F')^*$) to denote these complexes.

Suppose that $M$ is an $R$-module, and

\[ \cdots \to F_2 \to F_1 \to F_0 \to M \to 0 \]

is a free resolution of $M$. Suppose that $N$ is an $R$-module. Then

\[ \text{Hom}_R(F_0, N) \to \text{Hom}_R(F_1, N) \to \cdots \]

is a complex of degree 1, and

\[ \cdots \to \text{Hom}_R(N, F_1) \to \text{Hom}_R(N, F_0) \]

is a complex of degree -1.

The homology of $F$ at $F_i$ in (1) is $H_i(F) = \ker d_i / \text{im} d_{i+1}$. The homology of $F'$ at $F^i$ in (2) is $H^i(F') = \ker d_i / \text{im} d_{i-1}$.

The complex (1) (with differential of degree -1) can be converted to a complex of degree +1 by setting $F^{-i} = F_i$. For instance, (3) becomes the exact complex

\[ \cdots \to F^{-2} \to F^{-1} \to F^0 \to M \to 0 \]

of degree 1.

A differential module $(F, \varphi)$ is an $R$-module $F$ with an endomorphism $\varphi$ such that $\varphi^2 = 0$. The homology of $F$ is $H(F) = \ker \varphi / \text{im} \varphi$. The differential module $F$ is exact if $H(F) = 0$. If $(F, \varphi)$ and $(G, \psi)$ are differential modules, then a homomorphism of differentiable modules $\Lambda : F \to G$ is an $R$-module homomorphism which satisfies $\Lambda \circ \varphi = \psi \circ \Lambda$.

Suppose that $F$ is a complex (which we will assume has degree -1, although the construction with “upper indices” is valid when the degree is 1). Then $F$ can be viewed as a differential module $F$, by setting $F = \bigoplus_{i \in \mathbb{Z}} F_i$. The map $\varphi : F \to F$ is defined by the differentials $d_i : F_i \to F_{i-1}$. We have that $H_i(F) \cong \bigoplus_{i \in \mathbb{Z}} H_i(F)$. 

If
\[ 0 \to F' \xrightarrow{\alpha} F \to F'' \to 0 \]
is a short exact sequence of differential modules, then there is a connecting homomorphism \( \delta : H(F'') \to H(F') \), making
\[ H(F') \xrightarrow{\alpha} H(F) \xleftarrow{\delta} H(F'') \]
an exact triangle; that is \( \ker \alpha = \im \delta \), \( \ker \delta = \im \beta \) and \( \ker \beta = \im \alpha \). The construction of \( \delta \) is as in the construction of the connecting homomorphism of a short exact sequence of complexes. If (4) is a short exact sequence of complexes, then \( \delta \) is the direct sum of the connecting homomorphisms of the homology modules of the complexes.

3. Spectral Sequences

Definition 3.1. A Spectral sequence is a sequence of modules \( ^rE \) for \( r \geq 1 \), each with a differential \( d_r : \(^rE \to \(^rE \) satisfying \( d_r^2 = 0 \), such that \( \(^{r+1}E \cong \ker d_r / \im d_r = H(\(^rE \)) \).

Suppose that \( ^rE \) is a spectral sequence. We will define submodules
\[ 0 = ^1B \subset ^2B \subset \cdots \subset ^rB \subset \cdots \subset ^rZ \subset \cdots \subset ^2Z \subset ^1Z = \(^1E \)
\]
such that \( ^iE = ^iZ / ^iB \) for each \( i \). To do this, let \( ^1Z = ^1E \) and \( ^1B = 0 \), so that \( ^1E = ^1Z / ^1B \).

Having defined \(^1B \) and \(^1Z \) for \( i \leq r \), we define \( ^{r+1}Z \) as the kernel of the composite map
\[ ^rZ \to \(^rZ / ^rB \to \(^rE \to \(^rZ / ^rB \)
\]
and let the image of this map be \( ^{r+1}B / ^rB \). We have that
\[ ^{r+1}Z / ^{r+1}B = H(\(^rE \)) = ^{r+1}E \]
and
\[ ^{r+1}B / ^{r+1}B \subset ^{r+1}Z \subset ^rZ \]
as required. Having defined all the \(^iZ \) and \(^1B \), we set
\[ \infty Z = \cap Z_r, \ \infty B = \cup B_r. \]
We define the limit of the spectral sequence to be
\[ \infty E = \infty Z / \infty B. \]
We say that the spectral sequence collapses at \( ^rE \) if \( ^rE = \infty E \), or equivalently, the differentials \( d_i = 0 \) for \( i \geq r \).

4. Exact Couples

Definition 4.1. An exact couple is a triangle
\[ \begin{array}{c}
A \\
\gamma \downarrow & \alpha \\
E & \beta
\end{array} \]
where \( A \) and \( E \) are \( R \)-modules and \( \alpha, \beta, \gamma \) are \( R \)-module homomorphisms which are exact in the sense that \( \ker \alpha = \im \gamma \), \( \ker \gamma = \im \beta \) and \( \ker \beta = \im \alpha \).

Let \( d : E \to E \) be the composite map \( d = \beta \gamma \). Since \( \gamma \beta = 0 \), we have that \( d^2 = 0 \), so \( E \) is a differential module, with homology \( H(E) = \ker d / \im d \).
Proposition 4.2. Let (5) be an exact couple. Then there is a derived exact couple

\[
\begin{array}{c}
\alpha A \\
\gamma' \leftarrow \\
\alpha A \\
H(E)
\end{array}
\]

where

(1) \(\alpha'\) is \(\alpha\) restricted to \(\alpha A\),
(2) \(\beta'\) is \(\beta \alpha^{-1}: \alpha A \to H(E)\), taking \(\alpha a\) to the homology class of \(\beta a\) for \(a \in A\),
(3) \(\gamma'\) is the map induced by \(\gamma\) on ker \(d\) (which automatically kills \(im\ d\)).

\(\beta'\) is well defined in the Proposition 4.2, since ker \(\alpha = im \gamma\) is taken to im \(d\) by \(\beta\).

We construct the spectral sequence of the exact couple (5) by iterating (6), so that

\[
\begin{align*}
1E &= E \\
2E &= H(E) \quad \text{with differential } d_2 = \beta' \gamma' \text{ from the derived couple} \\
3E &= H(H(E)) \quad \text{from the derived couple of the derived couple} \\
&\vdots
\end{align*}
\]

Proposition 4.3. With the notation after the definition of a spectral sequence, we have

\[
\begin{align*}
r^{+1}Z &= \gamma^{-1}(im \alpha^r) \\
r^{+1}B &= \beta(ker \alpha^r).
\end{align*}
\]

Thus

\[
\begin{align*}
\infty E &= \infty Z/\infty B \\
&= \gamma^{-1}(im \alpha^r) / \beta(ker \alpha^r).
\end{align*}
\]

Proof. We prove the proposition by induction on \(r\), the case \(r = 0\) being immediate.

Assume that \(r \geq 1\), and that \(r^Z = \gamma^{-1}(im \alpha^{r-1}) \text{ and } r^B = \beta(ker \alpha^{r-1}).\) We have that

\[
r^{+1}Z = \{x \in r^Z \mid \beta \alpha^{1-r} \gamma(x) \in r^B\}.
\]

Suppose that \(x \in \gamma^{-1}(im \alpha^r).\) Then \(\gamma(x) = \alpha^r(y)\) for some \(y \in A\), so that \(x \in \gamma^{-1}(im \alpha^{r-1}) = r^Z\) and \(\beta \alpha^{1-r} \gamma(x) = \beta \alpha(y) = 0.\) Thus \(\gamma^{-1}(im \alpha^r) \subset r^{+1}Z.\)

Now suppose that \(x \in r^{+1}Z.\) Then \(\beta \alpha^{1-r} \gamma(x) \in r^B\) implies \(\alpha^{1-r} \gamma(x) = \beta(y)\) for some \(y \in ker \alpha^{r-1},\) so \(\alpha^{1-r} \gamma(x) - y \in ker \beta = im \alpha \) and \(\alpha^{1-r} \gamma(x) = y + \alpha(z)\) for some \(z \in A.\)

Thus \(\gamma(x) = \alpha^{r-1}(y) + \alpha'(z) = \alpha'(z) \in im \alpha^r,\) so that \(x \in \gamma^{-1}(im \alpha^r).\)

We have established that \(r^{+1}Z = \gamma^{-1}(im \alpha^r).\)

We have that

\[
r^{+1}B = \{\beta \alpha^{1-r} \gamma(x) \mid x \in \gamma^{-1}(im \alpha^{r-1})\}.
\]

Suppose that \(x \in \beta(ker \alpha^r).\) Then \(x = \beta(y)\) for some \(y \in ker \alpha^r.\) Thus \(\alpha^{r-1}(y) = \gamma(z)\) for some \(z \in E,\) so that \(z \in \gamma^{-1}(im \alpha^{r-1}) = r^Z.\) Now \(y = \alpha^{1-r} \gamma(z)\) implies \(x = \beta \alpha^{1-r} \gamma(z) \in r^{+1}B.\) Thus \(\beta(ker \alpha^r) \subset r^{+1}B.\)

Suppose that \(z \in r^{+1}B.\) Then \(z = \beta \alpha^{1-r} \gamma(x)\) for some \(x \in \gamma^{-1}(im \alpha^{r-1}).\) There exists \(y \in A\) such that \(\gamma(x) = \alpha^{r-1}(y).\) We have \(0 = \alpha \gamma(x) = \alpha^r(y)\) implies \(y \in ker \alpha^r.\) We have \(z = \beta \alpha^{1-r} \gamma(x) = \beta(y) \in ker \alpha^r\). Thus \(r^{+1}B \subset \beta(ker \alpha^r).\) \(\square\)

We have that

\[
d_r : r^E = r^Z/\gamma^rB \to r^Z/\gamma^rB = r^E
\]

is defined by

\[
d_r(x + r^B) = \beta \alpha^{1-r} \gamma(x) + r^B
\]

for \(x \in r^Z.\)
Suppose that \((F,d)\) is a differential module, and let \(\alpha : F \to F\) be a monomorphism. Set \(\overline{F} = F/\alpha F\). The differential \(d\) of \(F\) induces a differential on \(\overline{F}\), giving a short exact sequence of differential modules

\[
0 \to F \xrightarrow{\alpha} F \xrightarrow{\beta} \overline{F} \to 0.
\]

By the construction of (4), we have an induced exact couple

\[
\begin{array}{ccc}
H(F) & \xrightarrow{\alpha} & H(F) \\
\gamma \swarrow & & \searrow \beta \\
H(\overline{F}) & & 
\end{array}
\]

The boundary map \(\gamma\) is defined by

\[
\gamma(x) = [\alpha^{-1}d(z)]
\]

where \(z\) is a lift to \(F\) of a representative of \(x\) in \(\overline{F}\). \(\beta\) is the map on homology induced by the projection of \(F\) onto \(\overline{F}\).

The induced spectral sequence of this exact couple is called the spectral sequence of \(\alpha\) on \(F\).

We have

\[
1^E = H(\overline{F}) = \left\{ [z \in F \mid dz \in \alpha F] / \alpha F \right\} / \left\{ [dz \mid z \in F] + \alpha F/\alpha F \right\}
\]

and

\[
H(F) = \left\{ z \in F \mid dz = 0 \right\} / \alpha F.
\]

\(\alpha : H(F) \to H(F)\) is defined by

\[
\alpha(x + dF) = \alpha(x) + dF
\]

for \(x \in \left\{ z \in F \mid dz = 0 \right\}\). \(\beta : H(F) \to H(\overline{F})\) is defined by

\[
\beta(x + dF) = x + (\alpha F + dF)
\]

for \(x \in \left\{ z \in F \mid dz = 0 \right\}\). \(\gamma : H(\overline{F}) \to H(F)\) is defined by

\[
\gamma(x) = \alpha^{-1}dx + dF
\]

for \(x \in \left\{ z \in F \mid dz \in \alpha F \right\}\).

\(d_1 = \beta\gamma : 1^E \to 1^E\)

is defined by

\[
d_1(x + (\alpha F + dF)) = \alpha^{-1}dx + (\alpha F + dF)
\]

for \(x \in \left\{ z \in F \mid dz \in \alpha F \right\}\).

By Proposition 4.3, we have

\[
r+1Z = \gamma^{-1}(\text{im} \alpha^r)
\]

\[
= \left\{ z + (\alpha F + dF) \mid z \in F \text{ and } dz \in \alpha^{r+1}F \right\}
\]

and

\[
r+1B = \beta(\text{ker} \alpha^r)
\]

\[
= \left\{ z + (\alpha F + dF) \mid z \in F \text{ and } \alpha^r z \in dF \right\}
\]

\[
= \left\{ z + (\alpha F + dF) \mid z \in \alpha^{-r}dF \right\}
\]

\[
= \alpha^{-r}dF + \alpha F/\alpha F + dF.
\]
Thus
\[ r^{+1}E = r^{+1}Z/r^{+1}B = \{ z \in F \mid dz \in \alpha r^{+1}F \} + \alpha F/\alpha^{-r}dF + \alpha F \]
with differential \( d_{r+1} : r^{+1}E \to r^{+1}E \) defined by
\begin{equation}
\begin{aligned}
d_{r+1}(x + (\alpha^{-r}dF + \alpha F)) &= (\alpha^{-r-1}dx + (\alpha^{-r}dF + \alpha F)) \\
&\text{for } x \in \{ z \in F \mid dz \in \alpha r^{+1}F \}.
\end{aligned}
\end{equation}

5. Filtered Differential Modules

A filtered differential module is a differential module \((G, d)\) together with a sequence of submodules \(G^p\) for \(p \in \mathbb{Z}\) satisfying
\[ G \supset \cdots \supset G^p \supset G^{p+1} \supset \cdots \]
that are preserved by \(d\); that is, \(dG^p \subset G^p\) for all \(p\). Generally, we have that \(G^p = G\) for \(p \leq 0\).

Suppose that \(G\) is a filtered differential module. We will associate a spectral sequence to \(G\). Let \(F = \bigoplus_{p \in \mathbb{Z}} G^p\). The sum of the inclusion maps \(G^{p+1} \to G^p\) defines a map \(\alpha : F \to F\) that is a monomorphism. Its cokernel is
\[ \text{coker } \alpha = \text{gr } G = \bigoplus_{p \in \mathbb{Z}} G^p/G^{p+1}. \]

From the exact sequence of differential modules
\[ 0 \to F \xrightarrow{\alpha} F \xrightarrow{\beta} \text{gr } G \to 0 \]
we obtain the exact couple (7), with associated spectral sequence. Thus the spectral sequence of \(\alpha\) on \(F\) starts with
\[ ^1E = H(\text{gr}(G)) = \bigoplus_{p \in \mathbb{Z}} H(G^p/G^{p+1}) = \bigoplus_{p \in \mathbb{Z}} ^1E^p. \]

We have that
\begin{equation}
^1E^p = H(G^p/G^{p+1}) = \{ z \in G^p \mid dz \in G^{p+1} \}/G^{p+1} + dG^p.
\end{equation}

Further, \(H(F) \cong \bigoplus_{p \in \mathbb{Z}} H(G^p)\) and
\[ H(G^p) = \{ z \in G^p \mid dz = 0 \}/dG^p. \]

The maps \(\alpha, \beta, \gamma\) are graded; we have
\[ \gamma : H(G^p/G^{p+1}) \to H(G^{p+1}) \]
is defined by
\[ \gamma(x + (G^{p+1} + dG^p)) = dx + dG^p \]
for a representative \(x \in \{ z \in G^p \mid dz \in G^{p+1} \},\)
\[ \beta : H(G^p) \to H(G^p/G^{p+1}) \]
is defined by
\[ \beta(x + dG^p) = x + (G^{p+1} + dG^p) \]
for a representative \(x \in \{ z \in G^p \mid dz = 0 \},\)
\[ \alpha : H(G^{p+1}) \to H(G^p) \]
is defined by
\[ \alpha(x + dG^{p+1}) = x + dG^p. \]
for a representative \( z \in \{ z \in G^{p+1} \mid dz = 0 \} \).

Since \( \alpha^{-r}(G^p) = G^{p-r} \) and \( \alpha^{r+1}(G^p) = G^{p+r+1} \), and by (9) and (10), we have

\[
    r_{z}^{+1}Z^p = \{ z + (G^{p+1} + dG^p) \in E^p \mid z \in G^p \text{ and } dz \in G^{p+r+1} \}
    = \{ z \in G^p \mid dz \in G^{p+r+1} \} + G^{p+1}/G^{p+1} + dG^p
\]

and

\[
    r_{z}^{+1}B^p = \{ z + (G^{p+1} + dG^p) \in E^p \mid z \in G^p \text{ and } z = dy \text{ for some } y \in G^{p-r} \}
    = (G^p \cap G^{p-r}) + G^{p+1}/G^{p+1} + dG^p.
\]

We thus have that

\[
    r_{z}^{i}Z^p = \{ z \in G^p \mid dz \in G^{p+r} \}/G^p \cap G^{p-r+1} + G^{p+1}.
\]

We have by (11) that \( d_r : rE^p \to rE^{p+r} \) is given by

\[
    d_r(z + (G^p \cap G^{p-r+1} + G^{p+1})) = dz + (G^{p+r} \cap G^{p+1} + G^{p+r+1})
\]

for \( z \in G^p \) such that \( dz \in G^{p+r} \).

The module \( H(G) \) is filtered by the submodules \( (H(G))^p = \text{im } H(G^p) \to H(G) \). Writing the associated graded module as \( \text{gr } H(G) = \bigoplus_{p \in \mathbb{Z}} (H(G))^p/(H(G))^{p+1} \), and writing \( K^p = \{ z \in G^p \mid dz = 0 \} \), we have exact sequences

\[
    0 \to dG \cap G^p/dG^p \to H(G^p) = K^p/dG^p \to (H(G))^p \to 0
\]

so that

\[
    (H(G))^p/(H(G))^{p+1} = K^p/(K^{p+1} + (dG \cap G^p)) = K^p + G^{p+1}/(G^p \cap dG) + G^{p+1}.
\]

since \( K^p \cap ((G^p \cap dG) + G^{p+1}) = K^{p+1} + (dG \cap G^p) \).

We have

\[
    \infty Z^p/\infty B^p = \cap_r (\{ z \in G^p \mid dz \in G^{p+r} \} + G^{p+1})/\cup_r ((G^p \cap G^{p-r}) + G^{p+1}).
\]

Writing the quotient on the right as \( M^p/N^p \), we have \( M^p \supset K^p + G^{p+1} \) and \( N^p \subset (G^p \cap dG) + G^{p+1} \). Taking the direct sum over all \( p \), we see that \( \text{gr } H(G) \) is a quotient of a submodule of \( \infty Z/\infty B \).

**Definition 5.1.** The spectral sequence of the filtered differential module \( G \) converges and for any term \( rE \) of the spectral sequence we write \( rE \Rightarrow \text{gr } H(G) \) if \( \text{gr } H(G) = \infty Z/\infty B \); that is, for each \( p \) we have

\[
    \cap_r (\{ z \in G^p \mid dz \in G^{p+r} \} + G^{p+1}) = \{ z \in G^p \mid dz = 0 \} + G^{p+1},
\]

and

\[
    \cup_r ((G^p \cap G^{p-r}) + G^{p+1}) = (G^p \cap dG) + G^{p+1}.
\]

The second of these conditions is relatively trivial; it will be satisfied as soon as \( G = \cup_p G^p \).
6. Double Complexes

**Definition 6.1.** A double complex $C$ is a commutative diagram

\[
\begin{array}{cccccccccc}
\cdots & \cdots & & & & & & & & \\
\uparrow & & & & & & & & & \uparrow \\
\cdots & d_{\text{hor}} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\uparrow & & & & & & & & & \uparrow \\
C_{i,j+1} & \rightarrow & C_{i+1,j} & \rightarrow & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & d_{\text{vert}} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\uparrow & & & & & & & & & \uparrow \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

(Extending infinitely in all four dimensions) where each row and each column is an ordinary complex.

Let $C$ be a double complex. The total complex of $C$ is the complex $\text{tot}(C)^k = \bigoplus_{i+j=k} C^{i,j}$, with differential $d : \text{tot}(C)^k \rightarrow \text{tot}(C)^{k+1}$ defined by $d(x) = (-1)^j d_{\text{hor}}(x) + d_{\text{vert}}(x)$ for $x \in C^{i,j}$.

Let $C$ be a double complex. There are two important filtrations on $\text{tot}(C)$. The first one is \{hor$G^p\}$, where $\text{hor}G^p = \bigoplus_{i+p \geq k} C^{i,j}$. The second is \{vert$G^p\}$, where $\text{vert}G^p = \bigoplus_{p+i \geq k} C^{i,j}$.

$\text{hor}G^p$ is a subcomplex of $\text{tot}(C)$, with $(\text{hor}G^k)^p = \bigoplus_{i+j=k, j \geq p} C^{i,j}$. Also, $\text{vert}G^p$ is a subcomplex of $\text{tot}(C)$, with $(\text{vert}G^k)^p = \bigoplus_{i+j=k, i \geq p} C^{i,j}$.

Let \{G$^p$\} be one of these two filtrations of $\text{tot}(C)$. Let $F = \bigoplus_{p \in \mathbb{Z}} G^p$, and let $\alpha : F \rightarrow F$ be the map determined by the inclusions of $G^{p+1}$ into $G^p$. This data determines an exact couple (7)

\[
\begin{array}{ccc}
\gamma \nwarrow & & \nabla \\
H(F) & \xrightarrow{\alpha} & H(F) \\
& \searrow & H(F/\alpha(F)) \\
\end{array}
\]

Since the G$^p$ are subcomplexes of the complex $\text{tot}(C)$, (15) determines long exact cohomology sequences

\[
\cdots \rightarrow H^i(G^{p+1}) \xrightarrow{\alpha} H^i(G^p) \xrightarrow{\beta} H^i(G^p/G^{p+1}) \xrightarrow{\gamma} H^{i+1}(G^{p+1}) \rightarrow \cdots
\]

We can consider $H(F)$ and $H(F/\alpha(F))$ as bigraded modules by setting $H(F)^{p,q} = H^{p+q}(G^p)$ and $H(F/\alpha(F))^{p,q} = H^{p+q}(G^p/G^{p+1})$. The maps $\alpha$, $\beta$, $\gamma$ in (15) are then of respective bidegrees $(-1,1), (0,0)$ and $(1,0)$.

We then have that the $rE$ are bigraded, and $d_r = \beta \alpha^{-r}(r-1)\gamma$ is bihomogeneous of degree $r$ in the $p$ grading and of degree $-(r-1)$ in the $q$ grading; that is, $d_r$ is the direct sum of maps

\[d_r^{p,q} : rE^{p,q} \rightarrow rE^{p+r,q-r+1}.
\]

The differential $d_r$ goes “$r$ steps to the right and $r-1$ steps down”.

Let

\[r D^{p-r+1,q+r-1} = \alpha^{-1} H^{p+q}(G^p) = \{\text{im } H^{p+q}(G^p) \rightarrow H^{p+q}(G^{p+r+1})\}.
\]

Taking the successive derived couples of (15), we obtain long exact sequences

\[
\cdots \rightarrow r D^{p-r,2,q+r-2} \xrightarrow{\alpha} r D^{p-r+1,1,q+r-1} \beta \alpha^{-r}(r-1) \xrightarrow{\gamma} r E^{p,q} \xrightarrow{\delta} r D^{p+1,q} \rightarrow \cdots
\]
Let us consider the case when we are considering the horizontal filtration, $G^p = \text{hor} G^p$. Then (counting the total degree $p + q$ of an element of $C^{q,p}$ in the first homology group, and the relative degree $q$ of an element of $C^{q,p}$ in the second homology group) we have

$$1^{E^{p,q}} = H^{p+q}(G^p/G^{p+1}) = \{\ker d_{\text{hor}} : C^{q,p} \to C^{q+1,p} \}/\{\text{im } d_{\text{hor}} : C^{q-1,p} \to C^{q,p} \}.$$ 

is the homology of $d_{\text{hor}}$.

From our calculation (12), we have

$$1^E = \{z \in G^p \mid dz \in G^{p+1}\}/G^{p+1} + dG^p \overset{d_1}{\longrightarrow} 1^{E^{p+1}} = \{z \in G^{p+1} \mid dz \in G^{p+2}\}/G^{p+2} + dG^{p+1}$$

is defined by $d_1([x]) = [d(x)]$ for $x \in \{z \in G^p \mid dz \in G^{p+1}\}$. Suppose that $x \in C^{q,p}$, with $dx \in G^{p+1}$, represents an element of $1^{E^{p,q}}$. Then $d_{\text{hor}}(x) = 0$, so $d_1([x]) \in 1^{E^{p+1,q}}$ is represented by the element $d_{\text{vert}}(x) \in C^{q,p+1}$. We thus have that

$$2^{E^{p,q}} = \{\ker d_{\text{vert}} : 1^{E^{p,q}} \to 1^{E^{p+1,q}} \}/\{\text{im } d_{\text{vert}} : 1^{E^{p-1,q}} \to 1^{E^{p,q}} \} = H_{E_{\text{vert}}^q}^p(C^{q,*}).$$

From (14), we have the map $d_2 : 2^E \to 2^{E^{p+2}}$,

$$\{z \in G^p \mid dz \in G^{p+2}\}/G^p \cap dG^{p-1} + dG^{p+1} \overset{d_2}{\longrightarrow} 1^{E^{p+2}} = \{z \in G^{p+2} \mid dz \in G^{p+4}\}/G^{p+2} \cap dG^{p+1} + dG^{p+3},$$

is defined by $d_2([x]) = [d(x)]$ for $x \in \{z \in G^p \mid dz \in G^{p+2}\}$. Suppose that $x \in C^{q,p}$ represents an element of $2^{E^{q,p}}$. Then there exists $z' \in G^{p+1}$ such that $d(x + z') \in G^{p+2}$. We can then achieve this condition with a choice of $z' \in G^{q-1,p+1}$. We necessarily then have that $d_{\text{hor}}(x) = 0$ and $(-1)^{p+1}d_{\text{hor}}(z') = -d_{\text{vert}}(x)$, so $d_2(x) \in 1^{E^{p+1,q}}$ is represented by the element $d_{\text{vert}}(z') \in C^{q-1,p+2}$.

The map $d_2$ takes the homology class of $z$ to the homology class of $d_{\text{vert}}(z')$. For this to be zero means that $d_{\text{vert}}(z') = (-1)^{p+2}d_{\text{hor}}(z'')$ for some $z''$, and in this case $d_3$ carries the homology class of $z$ to $d_{\text{vert}}(z'')$.

**Theorem 6.2.** Associated with the double complex $C$ are two spectral sequences $\{^r_{\text{hor}} E \}$ and $\{^r_{\text{vert}} E \}$, corresponding, respectively, to the horizontal and vertical filtrations of $\text{tot}(C)$. The $1^E$ terms are bigraded by the components given by

$$1_{\text{hor}}^{E^{p,q}} = H^q(C^{*,p}), \quad 1_{\text{vert}}^{E^{p,q}} = H^q(C^{p,*}).$$

If $C^{i,j} = 0$ for all $i < 0$ or for all $j > 0$, then the horizontal spectral sequence converges, so that $\lim_{\text{hor}}^\infty E = gr_{\text{hor}} H(\text{tot}(C))$.

Symmetrically, if $C^{i,j} = 0$ for all $i > 0$ or for all $j < 0$, then the vertical spectral sequence converges.

**Proof.** We have

$$1_{\text{hor}}^{E^{p,q}} = H(\text{hor} G^p/\text{hor} G^{p+1})^{p,q} = H^{p+q}(gr_{\text{tot}}(C)^p) = H^q(C^{*,p}).$$

The case for the vertical spectral sequence is similar.

The proof of convergence uses the bigrading. We will prove convergence for the horizontal spectral sequence. The proof for the vertical spectral sequence is similar. Let $\{G^p\}$ be the horizontal filtration of $\text{tot}(C)$. We must show that

$$\bigcap_{r}(\{z \in G^p \mid dz \in G^{p+r}\} + G^{p+1}) = \{z \in G^p \mid dz = 0\} + G^{p+1},$$

and

$$\bigcup_{r}((G^p \cap dG^{p-r}) + G^{p+1}) = (G^p \cap dG) + G^{p+1}.$$
Since \( \text{tot}(C) = \cup_p G^p \), the second condition is trivially satisfied. The first condition means that if \( z \in G^p \) and for each \( r \) there is an element \( y_r \in G^{p+1} \) such that \( d(z - y_r) \equiv 0 \mod G^{p+r} \), then there is an element \( y \in G^{p+1} \) such that \( d(z - y) = 0 \). Breaking \( z \) up into a sum of elements of the same total degree, we may assume that \( z \in (G^q)^p = \bigoplus_{i+j=q,j \geq p} C_{i,j} \) for some \( q \). We can then assume (by possibly replacing \( y_r \) with its part which is homogeneous of total degree \( q \)) that

\[
d(z - y_r) \in (G^{q+1})^{p+r} = \bigoplus_{i+j=q+1,j \geq p+r} C_{i,j}.
\]

If \( C_{i,j} = 0 \) for \( j > 0 \), then \( (G^{q+1})^{p+r} = 0 \) for \( r > -p \). If on the other hand, \( C_{i,j} = 0 \) for \( i < 0 \), then \( (G^{q+1})^{p+r} = 0 \) if \( r > q + 1 - p \). Thus in either case \( d(z - y_r) = 0 \) for suitable \( r \), and we may take \( y = y_r \) for this value of \( r \).

\[\square\]

7. Balanced Tor

Suppose that \( M \) and \( N \) are \( R \)-modules. We will show that \( \text{Tor}_i^R(M, N) \) can be computed from a free resolution of either \( M \) or \( N \). Let

\[ P : \cdots \rightarrow P_i \xrightarrow{\psi_i} P_{i-1} \rightarrow \cdots \rightarrow P_0 \]

and

\[ Q : \cdots \rightarrow Q_i \xrightarrow{\psi_i} Q_{i-1} \rightarrow \cdots \rightarrow Q_0 \]

be free resolutions of \( M \) and \( N \) respectively.

We will show that

\[ H(P \otimes_R N) \cong H(\text{tot}(P \otimes_R Q)) \cong H(M \otimes_R Q) \]

as \( R \)-modules. Since \( \text{Tor}_i^R(M, N) \) computed from a free resolution of \( M \) is the first of these, and \( \text{Tor}_i^R(M, N) \) computed from a free resolution of \( N \) is the last, this will suffice.

Let \( \text{vert}E \) be the vertical spectral sequence associated with the third quadrant double complex \( F = P \otimes_R Q \), which may be written with upper indices, using the convention that \( P_i = P_{-i} \) and \( Q^j = Q_{-j} \).

\[
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
P \otimes Q : & & & & & & \\
& & & & & & \\
\cdots \rightarrow P^i \otimes Q^{j+1} & \xrightarrow{\varphi_i \otimes 1} & P^{i+1} \otimes Q^{j+1} & \rightarrow \cdots \\
& 1 \otimes \psi & \uparrow & 1 \otimes \psi & \\
& \cdots \rightarrow P^i \otimes Q^j & \xrightarrow{\varphi_i \otimes 1} & P^{i+1} \otimes Q^j & \rightarrow \cdots \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

We have

\[
\text{vert}E_{i,j} = \text{vert}E_{-i,-j} = H_{-j}(P^i \otimes_R Q^*_i) = H_j(P_i \otimes_R Q_*).
\]

\[
\cdots \rightarrow P_i \otimes Q_j \xrightarrow{1 \otimes \psi_j} P_i \otimes Q_{j-1} \rightarrow \cdots \rightarrow P_i \otimes Q_0 \rightarrow P_i \otimes N \rightarrow 0
\]

is exact since \( P_i \) is free. Thus \( H_j(P_i \otimes_R Q_*) = 0 \) for \( j > 0 \). Tensoring the short exact sequence

\[
0 \rightarrow \varphi_1(Q_1) \rightarrow Q_0 \rightarrow H_0(Q_*) \cong N \rightarrow 0
\]
with $P_i$, we obtain the short exact sequence
\[ 0 \rightarrow P_i \otimes \varphi_1(Q_1) \rightarrow P_i \otimes Q_0 \rightarrow P_i \otimes H_0(Q_*) \cong P_i \otimes N \rightarrow 0, \]
from which we compute
\[ H_0(P_i \otimes_R Q_*) \cong P_i \otimes_R N. \]
Thus the only nonzero $1_{\text{vert}}E_{i,j}$ terms are those with $j = 0$. The differential $d_1$ is induced by $d_{\text{hor}} = \varphi \otimes 1$. Thus $1_{\text{vert}}E$ is the complex $P \otimes_R N$. Thus $1_{\text{vert}}E$ is the complex $P \otimes_R N$ and
\[ 2_{\text{vert}}E_{i,j} = 2_{\text{vert}}E_{-i,-j} = \begin{cases} H_i(P \otimes_R N) & \text{for } j = 0 \\ 0 & \text{for } j \neq 0. \end{cases} \]
We have
\[ d_2^{i,-j} : 2_{\text{vert}}E_{-i,-j} \rightarrow 2_{\text{vert}}E_{-i+2,-j-1} \]
so $d_2 = 0$. We further calculate
\[ 3_{\text{vert}}E_{-i,-j} = (\ker 2_{\text{vert}}E_{-i,-j} d_3^{2_{\text{vert}}E_{-i+2,-j-1}})/(\text{im } 2_{\text{vert}}E_{-i-2,-j+1} d_3^{2_{\text{vert}}E_{-i,-j}}) \]
so that $3_{\text{vert}}E_{-i,-j} = 0$ if $j \neq 0$. Since
\[ d_3^{i,-j} : 3_{\text{vert}}E_{-i,-j} \rightarrow 3_{\text{vert}}E_{-i+3,-j-2} \]
we have that $d_3 = 0$. Continuing in this way, we calculate that $d_k = 0$ for $k \geq 2$.

It follows that the spectral sequence degenerates at $2E$; that is $\infty_{\text{vert}}E = 2_{\text{vert}}E$. Since all nonzero terms have $j = 0$,
\[ \bigoplus_{i+j=k} \infty_{\text{vert}}E_{i,j} = \infty_{\text{vert}}E_{k,0}, \]
and the filtration of $H(\text{tot}(P \otimes_R Q))$ has only one nonzero piece. Thus we get
\[ H(P \otimes_R N) = \text{gr}_{\text{vert}}(H(\text{tot}(P \otimes_R Q))) = H(\text{tot}(P \otimes_R Q)). \]
By symmetry (calculating the horizontal spectral sequence of $P \otimes_R N$) we get
\[ H(\text{tot}(P \otimes_R Q)) = H(M \otimes_R Q). \]

8. Exact sequence of terms of low degree

**Proposition 8.1.** If $C^{i,j}$ is a third quadrant double complex ($C^{i,j} = 0$ if $i > 0$ or $j > 0$), then writing $E$ for $\text{vert}E$, we have

a. $H^0(\text{tot}(C)) \cong H^0_{\text{hor}}H^0_{\text{vert}}(C^{*,*})$.

b. There is a 5-term exact sequence
\[ H^{-2}(\text{tot}(C)) \rightarrow H^{-2}_{\text{hor}}H^0_{\text{vert}}(C^{*,*}) \xrightarrow{d_3} H^0_{\text{hor}}H^{-1}_{\text{vert}}(C^{*,*}) \rightarrow H^{-1}(\text{tot}(C)) \rightarrow H^{-1}_{\text{hor}}H^0_{\text{vert}}(C^{*,*}) \rightarrow 0 \]

**Proof.** We first prove a. We have an exact sequence (17)
\[ 2D^{0,0} \rightarrow 2D^{-1,1} \rightarrow 2E^{0,0} \rightarrow 2D^{0,0}. \]
Since $G^p = 0$ for $p > 0$, we compute that $2D^{0,0} = 2D^{1,0} = 0$.

Hence $2D^{-1,1} = \text{im } H^0(G^0) \rightarrow H^0(G^{-1}) = C^{*0,0}/d(C^{*0,-1}) + d(C^{*-1,0}) = H^0(\text{tot}(C))$.

Since $2E^{0,0} = H^0_{\text{hor}}H^0_{\text{vert}}(C^{*,*})$, a follows.

We now prove b. From (17), we have exact sequences
\[ 2D^{-3,1} \rightarrow 2E^{-2,0} \rightarrow 2D^{-2,0} \rightarrow 2D^{-2,1} \rightarrow 2E^{-1,0} \rightarrow 2D^{1,0} \]
\[ 2D^{0,-1} \rightarrow 2D^{-1,0} \xrightarrow{d_3} 2E^{0,-1} \rightarrow 2D^{1,-1} \]

We calculate that these sequences are
\[ H^{-2}(\text{tot}(C)) \to H_{\text{hor}}^{-2}H^0_{\text{vert}}(C^{*,*}) \xrightarrow{\sim} 2D^{-1,0} \to H^{-1}(\text{tot}(C)) \to H_{\text{hor}}^{-1}H^0_{\text{vert}}(C^{*,*}) \to 0 \]
\[ 0 \to 2D^{-1,0} \xrightarrow{\beta \alpha^{-1}} H_{\text{hor}}^0H_{\text{vert}}^{-1}(C^{*,*}) \to 0 \]

We obtain the exact sequence b by patching these exact sequences, using the fact that \( d_2 = \beta \alpha^{-1} \gamma \).

\[ \square \]

The following proposition has a similar proof.

**Proposition 8.2.** If \( C^{i,j} \) is a first quadrant double complex (\( C^{i,j} = 0 \) if \( i < 0 \) or \( j < 0 \)), then writing \( E \) for \( \text{vert} E \), we have

a. \( H^0(\text{tot}(C)) \cong H^0_{\text{hor}}H^0_{\text{vert}}(C^{*,*}) \).

b. There is a 5-term exact sequence

\[ 0 \to H^1_{\text{hor}}H^0_{\text{vert}}(C^{*,*}) \to H^1(\text{tot}(C)) \to H^0_{\text{hor}}H^1_{\text{vert}}(C^{*,*}) \xrightarrow{d_2} H^2_{\text{hor}}H^0_{\text{vert}}(C^{*,*}) \to H^2(\text{tot}(C)) \]

### 9. A Rework of the Theory for Homology

In the above treatment, we considered double complexes where the differentials \( d_{\text{hor}} \) and \( d_{\text{vert}} \) have degree 1. Thus we developed our theory in terms of cohomology. We gave a procedure for converting problems in homology (\( d_{\text{hor}} \) and \( d_{\text{vert}} \) have degree -1) into problems in cohomology. For instance, the proof of balanced Tor used this method. However, the whole development of the spectral sequence of a double complex can be developed directly in terms of homology, and some developments of spectral sequences do this. The theory is exactly the same, but with different indexing. We indicate here how things change. The treatment in [4], which is the most detailed of the references, is in terms of homology, as is the treatment in [3]. The treatments in [2] and [1] are in terms of cohomology.

To begin with, a double complex is now a commutative diagram

\[
\begin{array}{cccccc}
& & \vdots & & \vdots & \\
& & \downarrow & & \downarrow & \\
\cdots & \leftarrow C^{i,j+1} & \xrightarrow{d_{\text{hor}}} & C^{i+1,j+1} & \leftarrow \cdots \\
& \downarrow d_{\text{vert}} & & \downarrow d_{\text{vert}} & \\
\cdots & \leftarrow C^{i,j} & \xrightarrow{d_{\text{hor}}} & C^{i+1,j} & \leftarrow \cdots \\
& & \vdots & & \vdots & \\
\end{array}
\]

(Extending infinitely in all four dimensions) where each row and each column is an ordinary complex.

The total complex of \( C \) is now the complex \( \text{tot}(C) \) defined by \( \text{tot}(C)^k = \bigoplus_{i+j=k} C^{i,j} \), with differential \( \delta : \text{tot}(C)^k \to \text{tot}(C)^{k-1} \) defined by \( \delta(x) = (-1)^j d_{\text{hor}}(x) + d_{\text{vert}}(x) \) for \( x \in C^{i,j} \).

Let \( C \) be a double complex. The two filtrations on \( \text{tot}(C) \) are \( \{ \text{hor} G^p \} \), where \( \text{hor} G^p = \bigoplus_{j \leq p} C^{i,j} \), and \( \{ \text{vert} G^p \} \), where \( \text{vert} G^p = \bigoplus_{i \leq p} C^{i,j} \).

\( \text{hor} G^p \) is a subcomplex of \( \text{tot}(C) \), with \( (\text{hor} G^p)^k = \bigoplus_{i+j=k, j \leq p} C^{i,j} \). Also, \( \text{vert} G^p \) is a subcomplex of \( \text{tot}(C) \), with \( (\text{vert} G^p)^k = \bigoplus_{i+j=k, i \leq p} C^{i,j} \).
Let \( \{G^p\} \) be one of these two filtrations of \( \text{tot}(C) \). Let \( F = \bigoplus_{p \in \mathbb{Z}} G^p \), and let \( \alpha : F \to F \) be the map determined by the inclusions of \( G^{p-1} \) into \( G^p \). This data determines an exact couple (7)

\[
H(F) \xrightarrow{\alpha} H(F) \\
\gamma \searrow \ \\ H(F/\alpha(F)) \xrightarrow{\beta}
\]

Since the \( G^p \) are subcomplexes of the complex \( \text{tot}(C) \), (18) determines long exact homology sequences

\[
\cdots \to H_i(G^{p-1}) \xrightarrow{\alpha} H_i(G^p) \xrightarrow{\beta} H_i(G^p/G^{p-1}) \xrightarrow{\gamma} H_{i-1}(G^p) \to \cdots
\]

We can consider \( H(F) \) and \( H(F/\alpha(F)) \) as bigraded modules by setting \( H(F)_{p,q} = H(p+q(G^p)) \) and \( H(F/\alpha(F))_{p,q} = H(p+q(G^p/G^{p-1})) \). The maps \( \alpha, \beta, \gamma \) in (18) are then of respective bidegrees \((1,-1)\), \((0,0)\) and \((-1,0)\).

We then have that the \( r \) are bigraded, and \( d_r \) is bihomogeneous of degree \(-r\) in the \( p \) grading and of degree \( r - 1 \) in the \( q \) grading; that is, \( d_r \) is the direct sum of maps

\[
d_r : rE_{p,q} \rightarrow rE_{p-r,q+r-1}.
\]

The differential \( d_r \) goes “\( r \) steps to the right and \( r - 1 \) steps down”.

Let

\[
rD_{p+r-1,q-r+1} = D_{p+r-1,q-r+1} = \{ \text{im} \ H_{p+q}(G^p) \to H_{p+q}(G^{p+r-1}) \}.
\]

Taking the successive derived couples of (18), we obtain short exact sequences

\[
\cdots \to rD_{p+r-2,q-r+2} \xrightarrow{\alpha} rD_{p+r-1,q-r+1} \xrightarrow{\beta \alpha^{-(r-1)}} rE_{p,q} \xrightarrow{\gamma} rD_{p-1,q} \to \cdots
\]

References