1. ISOMORPHISM OF GROUPS OF SMALL ORDER

Lemma 1.1. Suppose that $H$ and $G$ are groups with $|H| = |G| = n$. Suppose that $H = \{x_1, \ldots, x_n\}$ and $G = \{y_1, \ldots, y_n\}$. Then the bijective mapping $\varphi : H \rightarrow G$ defined by $\varphi(x_i) = y_i$ for $1 \leq i \leq n$ is a group homomorphism (and an isomorphism) if and only if $\varphi(x_ix_j) = y_iy_j$ for all $1 \leq i, j \leq n$; that is, if and only if substitution of $y_i$ for $x_i$ for $1 \leq i \leq n$ in the multiplication table of $x_1, x_2, \ldots, x_n$ gives the multiplication table of $y_1, y_2, \ldots, y_n$.

Let $H = \{\sigma_1 = \text{id}, \sigma_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{smallmatrix}\right), \sigma_3 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{smallmatrix}\right), \sigma_4 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{smallmatrix}\right)\}$, a subgroup of $S_4$ of order 4. The multiplication table of $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is:

$$
\begin{array}{c|cccc}
  & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\hline
\sigma_1 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\
\sigma_2 & \sigma_2 & \sigma_1 & \sigma_4 & \sigma_3 \\
\sigma_3 & \sigma_3 & \sigma_4 & \sigma_1 & \sigma_2 \\
\sigma_4 & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 \\
\end{array}
$$

Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{t_1 = (0, 0), t_2 = (1, 0), t_3 = (0, 1), t_4 = (1, 1)\}$, a group of order 4. The multiplication table of $\{t_1, t_2, t_3, t_4\}$ is:

$$
\begin{array}{c|cccc}
  & t_1 & t_2 & t_3 & t_4 \\
\hline
\ t_1 & t_1 & t_2 & t_3 & t_4 \\
\ t_2 & t_2 & t_1 & t_4 & t_3 \\
\ t_3 & t_4 & t_3 & t_1 & t_2 \\
\ t_4 & t_4 & t_3 & t_2 & t_1 \\
\end{array}
$$

$|H| = |G| = 4$ and substitution of $t_i$ for $\sigma_i$ for $1 \leq i \leq 4$ in the multiplication table of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ gives the multiplication table of $t_1, t_2, t_3, t_4$. Thus the bijective mapping $\varphi : H \rightarrow G$ defined by $\varphi(\sigma_i) = t_i$ for $1 \leq i \leq 4$ is a homomorphism and an isomorphism by Lemma 1.1. Thus $G$ and $H$ are isomorphic groups.

2. HOMOMORPHISMS OF $\mathbb{Z}$ AND $\mathbb{Z}_n$

Theorem 2.1. The following properties of homomorphisms of $\mathbb{Z}$ are true.

1. Suppose that $c \in \mathbb{Z}$. Then the mapping $\Phi_c : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\Phi_c(m) = cm$ for $m \in \mathbb{Z}$ is a group homomorphism.
2. Suppose that $\Lambda : \mathbb{Z} \rightarrow \mathbb{Z}$ is a group homomorphism. Then $\Lambda = \Phi_c$ for some $c \in \mathbb{Z}$.
3. Suppose that $c, d \in \mathbb{Z}$ and $c \neq d$. Then $\Phi_c \neq \Phi_d$. 

Theorem 2.2. The following properties of homomorphisms of 
\( \mathbb{Z}_n \) are true.

1. Suppose that \( c \in \mathbb{Z} \). Then the mapping \( \Psi_c : \mathbb{Z}_n \to \mathbb{Z}_n \) defined by \( \Psi_c([m]_n) = [cm]_n \)
   for \( m \in \mathbb{Z} \) is a well defined group homomorphism.

2. Suppose that \( \Lambda : \mathbb{Z}_n \to \mathbb{Z}_n \) is a group homomorphism. Then \( \Lambda = \Psi_c \) for some \( c \in \mathbb{Z} \).

3. Suppose that \( c, d \in \mathbb{Z} \). Then \( \Psi_c \neq \Psi_d \) if and only if \( n \mid (d - c) \).

Proof. 1. We will first show that the mapping \( \Psi_c \) is well defined. Suppose that \( a, b \in \mathbb{Z} \) and \( [a]_n = [b]_n \). Then \( b - a = qn \) for some \( q \in \mathbb{Z} \), so that \( c(b - a) = cqn \) so \( [cb]_n = [ca]_n \). Thus \( \Psi_c \) is well defined. Suppose \( [a]_n, [b]_n \in \mathbb{Z}_n \). Then
\[
\Psi_c([a]_n + [b]_n) = \Psi_c([a + b]_n) = [a + b]_n = [ca + cb]_n = [ca]_n + [cb]_n = \Psi_c([a]_n) + \Psi_c([b]_n).
\]
Thus \( \Psi_c \) is a group homomorphism.

2. Let \( c \in \mathbb{Z} \) be such that \( \Lambda([1]_n) = [c]_n \). Suppose that \( [m]_n \in \mathbb{Z}_n \). Then
\[
\Lambda([m]_n) = \Lambda(m[1]_n) = m\Lambda([1]_n) = m[c]_n = [mc]_n = \Psi_c([m]_n),
\]
where the second equality is since \( \Lambda \) is a homomorphism. Thus \( \Lambda = \Psi_c \).

3. We have that \( \Psi_c = \Psi_d \) if and only if \( \Psi_c([m]_n) = \Psi_d([m]_n) \) for all \( [m]_n \in \mathbb{Z}_n \). Since \( \Psi_c([m]_n) = m[c]_n \) and \( \Psi_d([m]_d) = m[d]_n \) for all \( m \in \mathbb{Z} \), we see that \( \Psi_c = \Psi_d \) if and only if \( [c]_n = [d]_n \), which holds if and only if \( n \mid (d - c) \). \( \Box \)

A consequence of 1 of Theorem 2.2 is that the assignment \( m[a]_n = [ma]_n \) for \( [a]_n \in \mathbb{Z}_n \)
and \( m \in \mathbb{Z} \) is well defined.