Matrix Theory, Exam 2.

Name:

You may not use calculators on this test. You also may not use determinants. Your answers must be justified to receive credit.

1. (15 points) Suppose that $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear mapping, such that

$$L \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \text{ and } L \left( \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$ 

Compute $L \left( \begin{pmatrix} 6 \\ 4 \end{pmatrix} \right)$.

2. (15 points) Let

$$\beta = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$ 

$\beta$ is a basis of $\mathbb{R}^3$ (you do not need to show this). Determine the vector $\vec{v} \in \mathbb{R}^3$ which has the coordinate vector

$$(\vec{v})_\beta = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$
3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined by

$$L \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \left( \begin{array}{c} -3x_1 + 2x_2 \\ -15x_1 + 8x_2 \end{array} \right).$$

a) (15 points) Let

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

be the standard basis of $\mathbb{R}^2$. Compute the matrix $M_\beta^\beta(L)$ of $L$. 
b) (15 points) Let

$$\beta' = \left\{ \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

which is a basis of $\mathbb{R}^2$. In this part of Problem 3, compute the matrix $M_{\beta'}^\beta(L)$, answering the following questions.

i) Circle the Roman numeral of the Algorithm from the “Algorithm List” at the end of this test which you will use in this problem.

$$I \quad II \quad III \quad IV \quad V \quad VI$$

ii) Write down the appropriate matrix to apply your selected algorithm from Part b i) to, transform this matrix into RRE form, and apply the algorithm.

iii) Write down what the matrix $M_{\beta'}^\beta(L)$ is, as found in your calculation in Part b ii).
4. Let $L : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ be the linear mapping defined by

$$
L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 + x_4 + x_5 \\ 3x_1 + 3x_4 + 6x_5 \\ 2x_1 + 2x_4 + 3x_5 \\ 4x_1 + 4x_4 + 8x_5 \end{pmatrix}.
$$

a) (10 points) Let $\beta_1$ be the standard basis of $\mathbb{R}^5$ and $\beta_2$ be the standard basis of $\mathbb{R}^4$
Compute the matrix $A = M_{\beta_2}^{\beta_1} (L)$ of $L$. 
b) (15 points) In this part of Problem 4, compute a basis of the Kernel of $L$, answering the following questions.

i) Circle the Roman numeral of the Algorithm from the "Algorithm List" at the end of this test which you will use.

   \[ I \quad II \quad III \quad IV \quad V \quad VI \]

ii) Using the matrix $A$ which you found in Part a), write down the appropriate matrix to apply your selected algorithm from Part b i) to, transform this matrix into RRE form, and apply the algorithm.

iii) Write down the basis of the Kernel of $L$ which you found in Part b ii).
5. (15 points) Find a subset of the polynomials \{1 + x + x^3, 2 + 2x + 2x^3, 3x, 1 + 4x + x^3\} which is a basis of their span in \( P_4 \), answering the following questions.

i) Circle the Roman numeral of the Algorithm from the “Algorithm List” at the end of this test which you will use in this problem.

\[ I \quad II \quad III \quad IV \quad V \quad VI \]

ii) Write down the “standard basis” \( \beta^* \) of \( P_4 \) which you will use in this problem.

iii) Write down the appropriate matrix to apply your selected algorithm from Part i) to, transform this matrix into RRE form, and apply the algorithm.

iv) Write down the basis (of the span of the given polynomials) which you have computed from Part iii).
Algorithm List

I. Computation of a basis of the span of a set of row vectors. Suppose that 
v_1, \ldots, v_m \in \mathbb{R}_n. Transform the \( m \times n \) matrix 
\[
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_m 
\end{pmatrix}
\]
into a reduced row echelon form \( B \). The nonzero rows of \( B \) form a basis of \( \text{Span}\{v_1, \ldots, v_m\} \).

II. Computation of a subset of a set of column vectors which is a basis of the span of the set. Suppose that \( w_1, \ldots, w_n \in \mathbb{R}^m \). Transform the \( m \times n \) matrix 
\[
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_m 
\end{pmatrix}
\]
into a reduced row echelon form \( B \). Let \( \sigma(1) < \sigma(2) < \cdots < \sigma(r) \) be the indices of the columns \( B^1 \) of \( B \) which contain a leading 1. Then \( \{w_{\sigma(1)}, \ldots, w_{\sigma(r)}\} \) is a basis of \( \text{Span}\{w_1, w_2, \ldots, w_n\} \).

III. Extension of a set of linearly independent row vectors to a basis of \( \mathbb{R}_n \). Suppose that \( w_1, \ldots, w_m \in \mathbb{R}_n \) are linearly independent. Let \( \{e_1, \ldots, e_n\} \) be the standard basis of \( \mathbb{R}_n \). Transform the \( m \times n \) matrix 
\[
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_m 
\end{pmatrix}
\]
into a reduced row echelon form \( B \). Let \( \sigma(1) < \sigma(2) < \cdots < \sigma(n-m) \) be the indexes of the columns of \( B \) which do not contain a leading 1. Then \( \{w_1, \ldots, w_m, e_{\sigma(1)}, \ldots, e_{\sigma(n-m)}\} \) is a basis of \( F_n \).

IV. Computation of a basis of the solution space of a homogeneous system of equations. Let \( A = (a_{ij}) \) be an \( m \times n \) matrix, and \( X = (x_i) \) be a \( n \times 1 \) matrix of indeterminates. Let \( N(A) \) be the null space of the matrix \( A \) (the subspace of \( \mathbb{R}^n \) of all \( X \in \mathbb{R}^n \) such that \( AX = 0_m \)). A basis for \( N(A) \) can be found by solving the system \( AX = 0_m \) using Gaussian elimination to find the general solution, putting the general solution into a column vector and expanding with indeterminate coefficients. The vectors in this expansion are a basis of \( N(A) \). \( N(A) \) is also called the "solution space of the system of equations \( AX = 0_m \)."

V. The transition matrix between bases. Suppose that \( V \) is a vector space, and 
\( \beta = \{v_1, \ldots, v_n\}, \beta' = \{w_1, \ldots, w_n\} \) are bases of \( V \).
If \( \beta^* \) is a (standard) basis of \( V \), then the \( n \times 2n \) matrix 
\[
\begin{pmatrix}
(w_1)_{\beta^*}, (w_2)_{\beta^*}, \ldots, (w_n)_{\beta^*} \\
(v_1)_{\beta^*}, \ldots, (v_n)_{\beta^*}
\end{pmatrix}
\]
is transformed by elementary row operations into the reduced row echelon form \( (I_n, M_{\beta^*}^{\beta}) \).
VI. The matrix of a linear mapping. Suppose that $V$ and $W$ are vector spaces, $F : V \to W$ is a linear mapping, $\beta = \{v_1, \ldots, v_n\}$ is a basis of $V$ and $\beta' = \{w_1, \ldots, w_m\}$ is a basis of $W$.

Let $\beta^*$ be a standard basis of $W$. The $m \times (m + n)$ matrix

$\begin{pmatrix}
(w_1)_{\beta^*}, (w_2)_{\beta^*}, \ldots, (w_m)_{\beta^*}, (F(v_1))_{\beta^*}, \ldots, (F(v_n))_{\beta^*}
\end{pmatrix}$

is transformed by elementary row operations into the reduced row echelon form $(I_m, M_\beta^*(F))$. 