Determinants

Suppose that $A = (a_{ij})$ is an $n \times n$ matrix. Define $M_{i,j}$ to be the $(n - 1) \times (n - 1)$ matrix obtained from $A$ by removing the $i$-th row and $j$-th column.

The determinant of $A$ may be defined recursively in terms of determinants of smaller submatrices.

**Definition 0.1.** The determinant of $A$ is

\[ \text{Det}(A) = \begin{cases} 
  a_{11} & \text{if } n = 1 \\
  a_{11} \text{Det}(M_{11}) - a_{12} \text{Det}(M_{12}) + a_{13} \text{Det}(M_{13}) + \cdots + (-1)^{1+n} \text{Det}(M_{1n}) & \text{if } n > 1
\end{cases} \]

From the definition, we deduce the formula for $n = 2$,

\[ \text{Det} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}. \]

The determinant can be computed by expansion along any row or column.

Expansion along the $i$-th row:

\[ \text{Det}(A) = (-1)^{i+1}a_{i1}\text{Det}(M_{11}) + (-1)^{i+2}a_{i2}\text{Det}(M_{i2}) + \cdots + (-1)^{1+n}a_{in}\text{Det}(M_{in}) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\text{Det}(M_{ij}). \]

Expansion along the $j$-th column:

\[ \text{Det}(A) = (-1)^{1+j}a_{1j}\text{Det}(M_{1j}) + (-1)^{2+j}a_{2j}\text{Det}(M_{2j}) + \cdots + (-1)^{n+j}a_{nj}\text{Det}(M_{nj}) = \sum_{i=1}^{n}(-1)^{1+j}a_{ij}\text{Det}(M_{ij}). \]

Definition 0.1 is the expansion along the 1-st row.

We can rewrite the formulas in a slightly simpler form, in terms of the cofactors

\[ A_{ij} = (-1)^{i+j}\text{Det}(M_{ij}). \]

We obtain two important formulas. Suppose that $A$ is an $n \times n$ matrix. Then

1. Suppose that $A$ and $B$ are $n \times n$ matrices. Then

\[ \text{Det}(AB) = \text{Det}(A)\text{Det}(B). \]

Writing a matrix $A$ as $A = (A^1, A^2, \ldots, A^n)$ in terms of its columns allows us to think of Det as a function of $n$ elements of $\mathbb{R}^n$.

**Theorem 0.2.** The determinant satisfies the following properties

1. i) \[ \text{Det}(A^1, \ldots, A^{i-1}, A^i + B, A^{i+1}, \ldots, A^n) = \text{Det}(A^1, \ldots, A^{i-1}, A^i, A^{i+1}, \ldots, A^n) + \text{Det}(A^1, \ldots, A^{i-1}, B, A^{i+1}, \ldots, A^n) \]
   for $1 \leq i \leq n$ and $B \in \mathbb{R}^n$.

   ii) \[ \text{Det}(A^1, \ldots, A^{i-1}, cA^i, A^{i+1}, \ldots, A^n) = c\text{Det}(A^1, \ldots, A^{i-1}, A^i, A^{i+1}, \ldots, A^n) \]
   for $1 \leq i \leq n$ and $c \in \mathbb{R}$.

2. If two columns of $A$ are equal, then $\text{Det}(A) = 0$. 

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3. \( \text{Det}(I_n) = 1. \)

Det(A) is the unique function on square matrices which satisfies these three properties.

Conclusion 1. of Theorem 0.2 is the statement that Det(A) is linear in each of its columns (or Det(A) is multilinear). Conclusion 2. of Theorem 0.2 is the statement that Det(A) is alternating. Combining Theorem 0.2 and formula (1), we obtain the analogous statement of Theorem 0.2 in terms of the rows of A.

From the version of Theorem 0.2 for rows of A, we deduce the effect of an elementary row operation on a determinant.

**Theorem 0.3.** Suppose that A is an \( n \times n \) matrix. Then
4. Let \( j, k \) be integers with \( 1 \leq j \leq n \), \( 1 \leq k \leq n \) and \( j \neq k \). If the \( j \)-th row and \( k \)-th row of A are interchanged, then the determinant of the resulting matrix is \(-\text{Det}(A)\).
5. If a row of A is multiplied by a number \( c \), then the determinant of the resulting matrix is \(c\text{Det}(A)\).
6. If a scalar multiple of one row is added to another row of A, then the determinant of the resulting matrix is unchanged.

Theorem 0.3 and the observation that the determinant of an upper triangular matrix is the product of its diagonal entries, gives an efficient method of computing determinants. From Theorem 0.3, and the definition of row equivalence, we obtain the following important theorem.

**Theorem 0.4.** Suppose that A is an \( n \times n \) matrix which is row equivalent to a matrix B. Then \( \text{Det}(A) \neq 0 \) if and only if \( \text{Det}(B) \neq 0 \).

Since a square RRE form is not \( I_n \) if and only if it has a row of zeros, and the determinant of a matrix with a row of zeros has determinant zero (as can be seen from expansion of the determinant along that row) the equivalence of 2 and 3 in the following theorem now follows from 3 of Theorem 0.2 and Theorem 0.4. The other equivalences we already know.

**Theorem 0.5.** Suppose that A is an \( n \times n \) matrix. Then the following are equivalent
1. A is invertible (A is nonsingular).
2. A is row equivalent to \( I_n \).
3. \( \text{Det}(A) \neq 0 \).
4. The equation \( A(x_1, \ldots, x_n)^T = (0, \ldots, 0)^T \) has only the trivial solution.
5. Suppose that \( B \in \mathbb{R}^n \). Then \( A(x_1, \ldots, x_n)^T = B \) has a unique solution.

If the determinant \( \text{Det}(A) \neq 0 \), and \( B \in \mathbb{R}^n \), then Cramer’s rule can be used to solve a system of equations
\[ A(x_1, \ldots, x_n)^T = B. \]

Cramer’s rule tells us that the unique solution to this system is
\[ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\text{Det}(A)} \begin{pmatrix} \text{Det}(B, A^2, \ldots, A^n) \\ \text{Det}(A^1, B, A^3, \ldots, A^n) \\ \vdots \\ \text{Det}(A^1, A^2, \ldots, A^{n-1}, B) \end{pmatrix}. \]

The adjoint of the \( n \times n \) matrix A is defined to be the \( n \times n \) matrix \( \text{adj}(A) = (b_{ij}) \) where the element \( b_{ij} \) in the \( i \)-th row and \( j \)-th column of \( \text{adj}(A) \) is
\[ b_{ij} = (-1)^{i+j}\text{Det}(M_{ji}) = A_{ji}. \]
This matrix can be computed by taking the transpose of the matrix whose entry in the
ith row and jth column is $A_{ij}$.

**Theorem 0.6.** Suppose that $A$ is an $n \times n$ matrix. Then

$$A \text{adj}(A) = \text{adj}(A)A = \text{Det}(A)I_n.$$  

From equation (2) and 3 of Theorem 0.2, we see that if $A$ is invertible, then $\text{Det}(A^{-1}) = \frac{1}{\text{Det}(A)}$.

If $A$ is invertible, Theorem 0.6 gives us a determinental formula for the inverse of $A$;

$$A^{-1} = \frac{1}{\text{Det}(A)} \text{adj}(A).$$

In practice, it is much more efficient to solve systems of equations or to compute inverses using Gauss Jordan reduction than to use the above determinantal formulas. These formulas are however very important for theoretical arguments, and it is nice to know that there are such “closed formulas”.