1. Statement of the problem and known results.
Throughout these lectures, unless explicitly stated otherwise, \( k \) is an algebraically closed field of characteristic zero. A variety is an open subset of an irreducible proper \( k \)-variety.

A toroidal structure on a nonsingular variety \( X \) is a SNC divisor \( D_X \). \( p \in D_X \) is an \( n \)-point if \( p \) is on (exactly) \( n \) components of \( D_X \). If \( p \in X \), regular parameters \( x_1, \ldots, x_n \) in \( \mathcal{O}_{X,p} \) (or in \( \hat{\mathcal{O}}_{X,p} \)) are permissible parameters for \( D_X \) at \( p \) if there exists \( l \) (with \( 0 \leq l \leq n \)) such that \( x_1 \cdots x_l = 0 \) is a local equation of \( D_X \) at \( p \).

A nonsingular subvariety \( V \) of \( X \) is a possible center for \( D_X \) if \( V \subset D_X \) and \( V \) makes SNCs with \( D_X \). The blow up \( \Phi : X_1 \to X \) of a possible center is called a possible blow up. \( D_{X_1} = \Phi^{-1}(D_X) \) is then a toroidal structure on \( X_1 \).

Recall that \( f : X \to Y \) is toroidal (with respect to \( D_Y \) and \( D_X \)) if \( f : (X, D_X) \to (Y, D_Y) \) is locally formally isomorphic to a morphism of toric varieties ([KKMS], [AK]).

The “toroidalization conjecture” of [AKMW] is:

**Conjecture 1.1** Suppose that \( f : X \to Y \) is a dominant morphism of nonsingular varieties. Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( \Phi, \Psi \) are products of blow ups of nonsingular centers, and there exist SNC divisors \( D_{Y_1} \) on \( Y_1 \) and \( D_{X_1} \) on \( X_1 \) such that \( f_1 \) is toroidal with respect to \( D_{Y_1} \) and \( D_{X_1} \).

A stronger version of this (which is also stated in [AKMW]) we will call the “strong toroidalization conjecture”. It is stated as follows:

**Conjecture 1.2** Suppose that \( f : X \to Y \) is a dominant morphism of nonsingular varieties. Further suppose that there is a SNC divisor \( D_Y \) on \( Y \) such that \( D_X = f^{-1}(D_Y) \) is a SNC divisor on \( X \) which contains the singular locus of the map \( f \). Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]
where $\Phi, \Psi$ are products of possible blow ups for the preimages of $D_X$ and $D_Y$ respectively, and $f_1$ is toroidal with respect to $D_{Y_1} = \Psi^{-1}(D_Y)$ and $D_{X_1} = \Phi^{-1}(D_X)$.

The characteristic zero assumption on our base field $k$ is necessary in these conjectures. The conjecture even fails in positive characteristic for morphisms of curves (where all blowups are trivial). A simple example is

$$t = x^p + x^{p+1}$$

over a field of characteristic $p$. We have that $t = x^p(1 + x)$, but $(1 + x)^{\frac{1}{p}} \notin k[[x]]$.

The case where there is an “easy” proof of the conjecture is when $Y$ is a curve.

**Theorem 1.3** Suppose that $f : X \to Y$ is a morphism from an $n$-fold to a curve. Then $f$ has a toroidalization

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\Phi} & X \\
\downarrow & \searrow & \downarrow f \\
Y & \to & Y.
\end{array}
\]

**Proof.** Let $D_Y = f(\text{sing}(f))$. Embedded resolution of hypersurface singularities implies there exists a sequence of possible blow ups $\Phi : X_1 \to X$ such that $D_{X_1} := (f \circ \Phi)^{-1}(D_Y)$ is a SNC divisor on $X_1$. Suppose that $p \in D_{X_1}$ and $q = (f \circ \Phi)(p)$. Let $t_q \in \mathcal{O}_{Y,q}$ be a regular parameter. There exist permissible parameters $x_1, \ldots, x_n$ in $\mathcal{O}_{X_1,p}$ such that

$$t_q = x_1^{a_1} \cdots x_l^{a_l} u$$

where $u \in \mathcal{O}_{X_1,p}$ is a unit. Set $\overline{x}_1 = x_1 u^{\frac{1}{a_1}} \in \hat{\mathcal{O}}_{X_1,p}$. Then

$$t_q = \overline{x}_1^{a_1} \cdots \overline{x}_l^{a_l}.$$

From now on, we will suppose that $f : X \to Y$ is a morphism of nonsingular varieties with toroidal structures $D_Y$ and $D_X = f^{-1}(D_Y)$ such that $\text{sing}(f) \subset D_X$.

The character of the toroidalization problem is completely different when $Y$ is not a curve. The essential problem is that we must then blow up above $Y$ to toroidalize.

**Example 1.4** In general, if $\dim Y \geq 2$, we must blow up above $Y$ to toroidalize.

Consider the morphism $f : X = \mathbb{A}^2 \to Y = \mathbb{A}^2$ with toroidal structures $D_Y = \{u = 0\}$ and $D_X = \{x = 0\}$, defined by

$$u = x^a$$

$$v = P(x) + x^by$$

where $P(x)$ is a polynomial.
where $P$ is a polynomial of degree less than $b$ with zero constant term. Thus $f$ is not toroidal. (To have a toroidal form, we would have to change variables to get permissible parameters related by a form $u = x^a, v = y$). Further, suppose that $\Phi: X_1 \to X$ is a sequence of blow ups of points, and $p \in X_1$ is a 1-point which maps to the origin of $X$. Then there are permissible parameters $x_1, y_1$ in $\hat{O}_{X_1, p}$ such that

$$x = x_1^m, \quad y = \sum_{i=1}^{r} \alpha_ix_1^i + x_1^ry_1.$$  

Substituting into $u$ and $v$, we find that

$$u = x_1^{qm}, \quad v = P(x_1^m) + x_1^{mb}(\sum_{i=1}^{r} \alpha_ix_1^r) + x_1^{mb+r}y_1$$

which is not toroidal.

The cases where the (strong) toroidalization conjecture is known to be true are:

1. dim($Y$) = 1, dim($X$) arbitrary.
2. dim($X$) = dim($Y$) = 2 [AkK], [CP1], [AKMW], [Mat].
3. Local monomialization (locally along a possibly non Noetherian valuation) [C1], [C2], [C5]. The full theorem is stated in Theorem 6.1 of these lectures. From 3, we infer the following theorem:

**Theorem 1.5**([C1],[C2], [C5]) *Suppose that $f: X \to Y$ is a dominant morphism of proper varieties. Then there exists a commutative diagram*

$$\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}$$

*such that $f_1$ is toroidal, and $\Phi, \Psi$ are locally products of blow ups of nonsingular centers. The morphisms $\Phi, \Psi$ and $f_1$ satisfy existence of the valuative criterion for properness, but in general uniqueness fails (so that these maps are in general not separated).

From Case 3 we also obtain the proof of “Local strong factorization” (conjectured by Abhyankar [Ab4]). Case 3 reduces the proof of the conjecture to the case of a toroidal mapping and a “toroidal” valuation. This is proven in dimension 3, by Christensen [Ch], and extended to arbitrary dimension by Karu [K]. A proof in the style of Christensen’s original proof (using determinants and elementary linear algebra) is given in [CS].

Local monomialization along a valuation could possibly be true in positive characteristic. It is certainly true for morphisms of curves, and for morphisms of n-folds to curves.
in dimensions where resolution of singularities is true. Good progress on this problem is made for morphisms of surfaces in [CP2].

4. \( \dim(X) = 3, \dim(Y) = 2 \) [C3] (strong toroidalization)
5. \( \dim(X) = \dim(Y) = 3, \ f \) birational [C6] (toroidalization), [C7] (strong toroidalization)

From 5, we reduce the “strong factorization” conjecture for birational morphisms of proper 3-folds to the case of toroidal morphisms, so we see that “strong factorization” of birational morphisms of proper 3-folds will follow from the Oda conjecture on “strong factorization” of toroidal varieties [O].

We further obtain a new proof of “weak factorization” of birational morphisms of proper 3-folds. Case 5 reduces “weak factorization” to the case of toroidal morphisms, which is solved in [D2] (dimension 3), [Mo], [W1], [AMR1], [AMR2] (arbitrary dimension). “Weak factorization” is proven in all dimensions (using geometric invariant theory) in [W2], [AKMW], [W3].

2. **Toroidalization of morphisms of surfaces.** In this section, we will suppose that \( f : X \to Y \) is a dominant morphism of nonsingular surfaces with toroidal structures \( D_Y \) and \( D_X = f^{-1}(D_Y) \) such that \( \text{sing}(f) \subset D_X \).

The proof that we give is from [AkK].

Suppose that \( p \in D_X \). The following are the possible toroidal forms for \( f \) at \( p \). Let \( q = f(p) \). There exist permissible parameters \( u, v \) in \( \mathcal{O}_{Y,q} \) and \( x, y \) in \( \hat{O}_{X,p} \) such that one of the following forms hold.

- \( q \) a 1-point and \( p \) a 1-point
  \[ u = x^a, v = y. \]

- \( q \) a 2-point and \( p \) a 1-point
  \[ u = x^a, v = x^b(\alpha + y) \]
  with \( 0 \neq \alpha \in k \).

- \( q \) a 2-point and \( p \) a 2-point
  \[ u = x^a y^b, v = x^c y^d \]
  with \( ad - bc \neq 0 \).

We will say that \( f \) is strongly prepared at \( p \) if one of the following forms hold.
\( q \) a 1-point

\[
(1) \quad u = x^a, v = P(x) + x^b y
\]

or \( q \) a 2-point

\[
(2) \quad u = (x^a y^b)^t, v = P(x^a y^b) + x^c y^d
\]

with \( ad - bc \neq 0, \gcd(a, b) = 1 \).

**Theorem 2.1** \( f \) is strongly prepared.

**Proof.** Suppose that \( p \in D_X \). Let \( u, v \) be permissible parameters at \( q = f(p) \), \( x, y \in \hat{O}_{X,p} \) be permissible parameters such that

\[
u = x^a
\]

if \( p \) is a 1-point, and

\[
u = (x^a y^b)^t
\]

with \( \gcd(a, b) = 1 \) if \( p \) is a 2-point.

If \( p \) is a 1-point, there exists a unit \( \delta \in \hat{O}_{X,p} \) such that

\[
u x v_y - u_y v_x = \delta x^e.
\]

If \( p \) is a 2-point, there exists a unit \( \delta \in \hat{O}_{X,p} \) such that

\[
u x v_y - u_y v_x = \delta x^e y^f.
\]

Expand \( v \) as a series \( v = \sum a_{ij} x^i y^j \) with \( a_{ij} \in k \).

If \( p \) is a 1-point, then \( ax^{a-1} v_y = \delta x^e \), from which it follows that (1) holds and \( f \) is strongly prepared at \( p \).

Suppose that \( p \) is a 2-point. Then

\[
u x v_y - u_y v_x = \sum t(a_j - b_i)a_{ij} x^{a_t + i - 1} y^{b_t + j - 1} = \delta x^e y^f.
\]

This implies that

\[
v = \sum_{a_j - b_i = 0} a_{ij} x^i y^j + \epsilon x^{e+1-at} y^{f+1-bt}
\]

where \( \epsilon \) is a unit series. It follows that (2) holds and \( f \) is strongly prepared at \( p \).
The final step of the proof of toroidalization of morphisms of surfaces is the following theorem:

**Theorem 2.2** There exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that the vertical arrows are products of (possible) blow ups of points and \( f_1 \) is toroidal.

For \( E \) a component of \( D_X \), \( p \in E \) a 1-point, define

\[ A(f, p) = \min\{b - \text{ord}_x(v)\} = A(f, E) \]

where the minimum is over permissible parameters \( u, v \) at \( q = f(p) \). Here \( \text{ord}_x(v) \) is the largest power of \( x \) which divides \( v \) in \( \mathcal{O}_{X, p} \). If \( A(f, p) > 0 \), define

\[ C(f, p) = \min\{(b - \text{ord}_x(v), \text{ord}_x(v) + a)\} = C(f, E) \]

where the minimum is in the lex order over permissible parameters \( u, v \) at \( q = f(p) \).

Now define

\[ A(f) = \max\{A(f, E) \mid E \text{ is a component of } D_X\} \]

If \( A(f) > 0 \), define

\[ C(f) = \max\{C(f, E) \mid E \text{ is a component of } D_X \text{ such that } A(f, E) > 0\} \]

**Lemma 2.3** Let \( \Psi_1 : Y_1 \to Y \) be the blow up of \( q \in D_Y \). Let \( f_1 : X \to Y_1 \) be the induced rational map.

1. Suppose that \( p \in f^{-1}(q) \) is a 1-point and \( f_1 \) is a morphism at \( p \). Then

\[ A(f_1, p) \leq A(f, p). \]

If \( A(f_1, p) = A(f, p) > 0 \), then \( (C, f_1, p) < C(f, p) \).

2. Suppose that \( p \in X \) is such that \( f_1 \) is not a morphism at \( p \). Let \( \Psi_1 : X_1 \to X \) be the blow up of \( p \) and \( E = \Psi_1^{-1}(p) \). Let \( \overline{f}_1 = f \circ \Psi_1 : X_1 \to Y \). Then

\[ A(\overline{f}_1, E) = 0 \text{ or } A(\overline{f}_1, E) < A(f). \]
By iteration of Lemma 2.3, we reduce to $A(f) = 0$ in the proof of Theorem 2.2.
We now list the prepared forms with $A(f) = 0$, with respect to suitable permissible parameters $x, y$ for $p \in D_X$ and $u, v$ for $q = f(p)$.

$q$ a 1-point and $p$ a 1-point:

$$u = x^a, \quad v = x^b(\alpha + y)$$

with $\alpha \in k$.

$q$ a 1-point and $p$ a 2-point:

$$u = x^a y^b, \quad v = x^c y^d$$

with $ad - bc \neq 0$.

$q$ a 2-point and $p$ a 1-point:

$$u = x^a, \quad v = x^b(\alpha + y)$$

with $0 \neq \alpha \in k$.

$q$ is a 2-point and $p$ is a 2-point:

$$u = x^a y^b, \quad v = x^c y^d$$

with $ad - bc \neq 0$.

If $E$ is a component of $D_X$ and $p \in E$ is a 1-point, define

$$I(f, p) = b - a = I(f, E).$$

Define

$$I(f) = \max\{I(f, E) \mid E \text{ is a component of } D_X\}.$$ 

**Lemma 2.4** Suppose that $A(f) = 0$. Let $\Psi_1 : Y_1 \to Y$ be the blow up of a 1-point $q \in D_Y$.
Let $f_1 : X \to Y_1$ be the induced rational map.

1. Suppose that $p \in f^{-1}(q)$ is a 1-point, $f_1$ is a morphism at $p$ and $f_1(p)$ is a 1-point. Then:
   a. If $I(f, p) > 0$ then $I(f_1, p) < I(f, p)$.
   b. If $I(f, p) \leq 0$ then $f_1$ is toroidal at $p$. 

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2. Suppose that \( p \in f^{-1}(q) \) is a 1-point and \( f_1 \) is not a morphism at \( p \). Let \( \Psi_1; X_1 \to X \) be the blow up of \( p \) and \( E = \Psi_1^{-1}(p) \). Let \( \overline{f}_1 = f \circ \Psi_1; X_1 \to Y \). Then
\[
I(f, p) < I(\overline{f}_1, E) \leq 0.
\]

Now by successive application of Lemma 2.4, we reduce to the case \( A(f) = 0 \) and \( I(f) \leq 0 \). Finally, by further application of Lemma 2.4, we prove Theorem 2.2.

3. Toroidalization of morphisms from 3-folds to surfaces. In this lecture we maintain our assumption that \( f : X \to Y \) is a dominant morphism of nonsingular varieties with toroidal structures defined by SNC divisors \( D_Y \) and \( D_X = f^{-1}(D_Y) \) such that \( \text{sing}(f) \subset D_X \).

We restrict to the case where \( \dim(Y) = 2 \). Initially, we allow \( n = \dim(X) \) to be arbitrary.

With these assumptions, we say that \( f \) is strongly prepared at \( p \in D_X \) if there exist permissible parameters \( u, v \) in \( \mathcal{O}_{Y,q} \) (where \( q = f(p) \)) and \( x_1, \ldots, x_n \) in \( \widehat{\mathcal{O}}_{X,p} \) such that \( x_1 \cdots x_l = 0 \) is a local equation of \( D_X \) at \( p \), and one of the following forms hold:

1. \( u = 0 \) is a local equation of \( D_X \),
\[
u = (x_1^{a_1} \cdots x_l^{a_l})^m, v = P(x_1^{a_1} \cdots x_l^{a_l}) + x_1^{b_1} \cdots x_l^{b_l}
\]
or
2. \( u = 0 \) is a local equation of \( D_X \),
\[
u = (x_1^{a_1} \cdots x_l^{a_l})^m, v = P(x_1^{a_1} \cdots x_l^{a_l}) + x_1^{b_1} \cdots x_l^{b_l} x_{l+1}
\]
or
3. \( uv = 0 \) is a local equation of \( D_X \),
\[
u = x_1^{a_1} \cdots x_l^{a_l-1}, v = x_2^{b_2} \cdots x_l^{b_l}.
\]

In all of these cases \( \gcd(a_1, \ldots, a_l) = 0 \) and \( P \) is a series. In Case 1 we have
\[
\text{rank} \left( \begin{array}{c} a_1 \\ b_1 \\ \vdots \\ a_l \\ b_l \end{array} \right) = 2.
\]

Recall that in the case when \( n = \dim(X) = 2 \) (and \( \dim(Y) = 2 \)), Theorem 2.1 implies that \( f \) is strongly prepared. However, the situation is much more complicated if \( X \) has higher dimension.
We will say that $f$ is prepared if conditions 1 or 2 above in the definition of strongly prepared always hold.

**Example 3.1** In general, if $n = \dim(X) \geq 3$, then $f$ is not strongly prepared.

Define a germ of a morphism $f : X \to Y$ from a 3-fold to a surface by

$$u = x^a, v = x^c F$$

where $a \geq 2$, $c \geq 0$, $r \geq 1$ and

$$F = x^rz + h(x,y)$$

where $h(x,y)$ is an arbitrary series with $h(0,0) = 0$. Define toroidal structures $D_Y = \{u = 0\}$, $D_X = \{x = 0\}$. We compute the Jacobian matrix

$$J(f) = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}
\end{pmatrix} = \begin{pmatrix}
a x^{a-1} & 0 & 0 \\
(c + r)x^{c-r-1}z + \frac{\partial x^c h}{\partial x} & x^c \frac{\partial h}{\partial y} & x^{c+r}
\end{pmatrix}.$$}

We see that the ideal of sing($f$), which is obtained from the $2 \times 2$ minors of $J(f)$, is

$$\sqrt{I_2(J(f))} = \sqrt{(x^{a+c-1} \frac{\partial h}{\partial y}, x^{a+c+r-1})} = (x).$$

Thus $\text{sing}(f) \subset D_X = f^{-1}(D_Y)$.

Theorem 2.2 for morphisms of surfaces does generalize, but the proof is much harder.

**Theorem 3.2** ([C3] if dim($X) = 3$, [CK] for arbitrary dimension) Suppose that $Y$ is a surface, and $f : X \to Y$ is strongly prepared. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
$$

such that $\Phi_1$ and $\Psi_1$ are products of possible blow ups, $f_1$ is toroidal.

At least in the case when $X$ is a 3-fold, it is possible to construct a prepared morphism.

**Theorem 3.3** ([C3]) Suppose that $X$ is a 3-fold and $Y$ is a surface. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
$$
such that $\Phi_1$ is a product of possible blow ups and $f_1$ is prepared.

Now, since prepared implies strongly prepared, Theorems 3.2 and 3.3 immediately imply

**Theorem 3.4 ([C3])** Strong toroidalization is true for dominant morphisms of 3-folds to surfaces

**Comments on the proof of Theorem 3.3**

Suppose that $p \in D_X$, $q = f(p) \in D_Y$. Then we can see from the Jacobian matrix of $f$ that there are permissible parameters $u, v$ in $O_{Y,q}$, $x, y, z$ in $\hat{O}_{X,q}$ and a series $P$ such that one of the following forms hold.

1. $p$ is a 1-point
   
   $$u = x^a, v = P(x) + x^b F_p$$
   where $x \not| F_p$, $F_p$ has no terms which are monomials in $x$.

2. $p$ is a 2-point
   
   $$u = (x^ay^b)^m, v = P(x^ay^b) + x^c y^d F_p$$
   where $\gcd(a, b) = 1$, $x \not| F_p$, $y \not| F_p$, and $x^cy^d F_p$ has no terms which are monomials in $x^ay^b$.

3. $p$ is a 3-point
   
   $$u = (x^ay^bz^c)^m, v = P(x^ay^bz^c) + x^d y^e z^f F_p$$
   where $\gcd(a, b, c) = 1$, $x \not| F_p$, $y \not| F_p$, $z \not| F_p$, and $x^dy^ez^f F_p$ has no terms which are monomials in $x^ay^bz^c$.

For $p \in D_X$, define

$$\nu(p) = \text{mult}(F_p).$$

It can be shown that $\nu(p)$ is an invariant of $p$. Set $S_r(X) = \{p \in D_X \mid \nu(p) \geq r\}$.

In Example 3.1, write

$$h(x, y) = h_0(x) + yx^m h_1(x, y)$$

where $x \not| h_1$. Suppose that $m < r$. Then we have

$$u = x^a$$

$$v = x^eh_0(x) + x^{c+m}(yh_1(x, y) + x^{r-m}z)$$

$$= P(x) + x^{c+m} F_p.$$
Thus
\[ \nu(p) = \min\{r - m + 1, 1 + \text{ord}(h_1)\}. \]
Let
\[ S_r(X) = \{ p \in D_X \mid \nu(p) \geq r \}. \]

\( S_r(X) \) is constructible but may not be Zariski closed, as is shown in the following example.

**Example 3.5** In general, \( S_r(X) \) is not Zariski closed.

Define a germ of a morphism \( f : X \to Y \) from a 3-fold to a surface at a point \( p \in X \) by
\[ u = xy, \quad v = x^2y. \]
Define toroidal structures \( D_Y = \{ uv = 0 \}, \quad D_X = \{ xy = 0 \}. \) We have \( \nu(p) = 0. \)

At 1-points \( p_1 \) on the surface \( x = 0 \), there are regular parameters \( x, y_1, z \) with \( y = y_1 + \alpha \) for some \( 0 \neq \alpha \in k. \) Set \( \overline{x} = x(y_1 + \alpha) \). We then have permissible parameters \( \overline{x}, y_1, z \) at \( p_1 \) such that
\[ u = \overline{x}, \quad v = \overline{x}^2(y_1 + \alpha)^{-1} \]
\[ = \alpha^{-1}x^2 + \overline{x}^2y_1. \]
Thus \( \nu(p_1) = 1. \)

Other important invariants in the proof are \( \gamma(p) \) and \( \tau(p) \). \( \gamma(p) \) is defined by
\[ \gamma(p) = \begin{cases} \mult F_p(0, y, z) & \text{if } p \text{ is a 1-point} \\ \mult F_p(0, 0, z) & \text{if } p \text{ is a 2-point} \end{cases} \]

Suppose that \( p \in X \) is a 1-point. We have an expression
\[ u = x^a, \quad v = P(x^a) + x^bF_p, \quad F_p = \sum_{i+j+k \geq r} a_{ijk}x^iy^jz^k \]
at \( p \), where \( \nu(p) = r \). Define
\[ \tau(p) = \max\{ j + k \mid \text{there exists } a_{ijk} \neq 0 \text{ with } i + j + k = r \}. \]

Suppose that \( p \in X \) is a 2-point. We have an expression
\[ u = (x^ay^b)^m, \quad v = P(x^a, y^b) + x^cF_p, \quad F_p = \sum_{i+j+k \geq r} a_{ijk}x^iy^jz^k \]
at \( p \), where \( \nu(p) = r \). Define

\[
\tau(p) = \max \{ k \mid \text{there exists } a_{ijk} \neq 0 \text{ with } i + j + k = r \}.
\]

\( \tau \) measures the independence of the leading form of \( F_p \) from local equations of \( D_X \).

**Lemma 3.6** \( f \) is prepared if and only if

1. If \( p \) is a 1-point then \( \nu(p) \leq 1 \).
2. If \( p \) is a 2-point then \( \gamma(p) \leq 1 \).
3. If \( p \) is a 3-point then \( \nu(p) = 0 \).

We see that Example 3.5 is prepared. It satisfies 1 and 2 of Lemma 3.6.

**Lemma 3.7** Suppose that \( \nu(p) = r \), \( \Phi : X_1 \to X \) is the blow up of \( p \) and \( p_1 \in \Phi^{-1}(p) \). Then

\[
\nu_{f \circ \Phi}(p_1) \leq r + 1.
\]

If \( \nu_{f \circ \Phi}(p_1) = r + 1 \) then there are restrictions on \( \gamma \) and \( \tau \).

**Example 3.8** \( \nu \) can increase after blowing up a point.

Recall the morphism of Example 3.5,

\[
u = xy, v = x^2y,
\]

with toroidal structures \( D_Y = \{ uv = 0 \} \), \( D_X = \{ xy = 0 \} \). We have \( \nu(p) = 0 \). Extend \( x, y \) to permissible parameters \( x, y, z \) at \( p \). Let \( \Phi : X_1 \to X \) be the blow up of \( p \). Suppose that \( p_1 \in \Phi^{-1}(p) \) is a 1-point with regular parameters \( x_1, y_1, z_1 \) defined by

\[
x = x_1, y = x_1(y_1 + \alpha), z = x_1z_1
\]

with \( 0 \neq \alpha \in k \). Then

\[
u = x_1^2(y_1 + \alpha), v = x_1^3(y_1 + \alpha).
\]

Set

\[
x_1 = x_1(y_1 + \alpha)^{\frac{1}{2}}, y_1 = (y_1 + \alpha)^{-\frac{1}{2}} - \alpha^{-\frac{1}{2}}.
\]

Then \( x_1, y_1, z_1 \) are permissible parameters at \( p_1 \), with

\[
u = x_1^2, v = \alpha^{-\frac{1}{2}}x_1^3 + x_1^3y_1.
\]
Thus $\nu(p_1) = 1$.

**Lemma 3.9** Suppose that $C \subset \overline{S}_r(X) (= \text{Zariski closure of } S_r(X))$ is a nonsingular curve which makes SNCs with $D_X$. Then either

1. $C$ is $r$-big:
   
   $$F_p \in \hat{T}_{C,p}^r \text{ for all } p \in C$$

   or

2. $C$ is $r$-small:
   
   $$F_p \in \hat{T}_{C,p}^{r-1} - \hat{T}_{C,p}^r \text{ for all } p \in C.$$ 

The invariants $\nu, \gamma$ and $\tau$ behave reasonably well under possible blow ups of such curves.

**Definition 3.10** Suppose that $r \geq 2$. $\overline{A}_r(X)$ holds if

1. $\nu(p) \leq r$ if $p \in X$ is a 1-point or a 2-point.
2. If $p \in X$ is a 1-point and $\nu(p) = r$ then $\gamma(p) = r$.
3. If $p \in X$ is a 2-point and $\nu(p) = r$, then $\tau(p) > 0$.
4. $\nu(p) \leq r - 1$ if $p \in X$ is a 3-point.

$\overline{A}_r(X)$ is stable under blow ups of points. The proof of Theorem 3.3 is by descending induction on $r$. $\overline{A}_1(X)$ holds if and only if $f$ is prepared (by Lemma 3.6). We must blow up points and curves which are $r$-big and $r$-small. There are a couple of stubborn cases which require blow ups of appropriately general curves.

4. **Toroidalization of morphisms from 3-folds to 3-folds.**

In this lecture we maintain our assumption that $f : X \to Y$ is a dominant morphism of nonsingular varieties with toroidal structures defined by SNC divisors $D_Y$ and $D_X = f^{-1}(D_Y)$ such that $\text{sing}(f) \subset D_X$.

We restrict to the case where $\dim(X) = \dim(Y) = 3$.

We say that $f$ is prepared (for $D_Y$ and $D_X$) if for all $q \in D_Y$ and $p \in f^{-1}(q)$ there exist permissible parameters $u, v, w$ in $\mathcal{O}_{Y,q}$ and $x, y, z$ in $\hat{\mathcal{O}}_{X,p}$ such that $u, v$ have a toroidal form in terms of $x, y, z$ (or a related condition holds).

Suppose that $f : X \to Y$ is prepared. Then $D_X$ is cuspidal for $f$ if $f$ is toroidal in a neighborhood of all components of $D_X$ which do not contain a 3-point, and in a neighborhood of all 2-curves of $D_X$ which do not contain a 3-point.
Theorem 4.1([C7]) Suppose that \( \dim(X) = \dim(Y) = 3 \). Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are products of possible blow ups, \( f_1 \) is prepared for \( D_{Y_1} = \Phi^{-1}(D_Y) \) and \( D_{X_1} = \Psi^{-1}(D_X) \), and \( D_{X_1} \) is cuspidal for \( f_1 \).

The significance of the prepared condition is that now we can read off of the Jacobian matrix of \( f \) nice local forms for \( f \). Suppose that \( f \) is prepared, \( p \in D_X \), \( q = f(p) \), and \( u, v, w \) are permissible parameters at \( p \) such that \( u, v \) are toroidal forms at \( p \). Then there are permissible parameters \( x, y, z \) in \( \hat{O}_{X,p} \) such that one of the following cases hold:

1. \( q \) is a 2-point or a 3-point, \( p \) is a 1-point and
   \[
u = x^a, v = x^b(\alpha + y), w = g(x, y) + x^c z
\]
   where \( 0 \neq \alpha \in k \) and \( g \) is a series.
2. \( q \) is 2-point or a 3-point, \( p \) is a 2-point and
   \[
u = x^a y^b, v = x^c y^d, w = g(x, y) + x^e y^f z
\]
   where \( \text{rank}((a, b), (c, d)) = 2 \) and \( g \) is a series.
3. \( q \) is a 2-point or a 3-point, \( p \) is a 2-point and
   \[
u = (x^a y^b)^k, v = (x^a y^b)^t(\alpha + z), w = g(x^a y^b, z) + x^c y^d
\]
   where \( 0 \neq \alpha \in k \), \( a, b, k, t > 0 \), \( \gcd(a, b) = 1 \), \( g \) is a series and \( \text{rank}((a, b), (c, d)) = 2 \).
4. \( q \) is a 2-point or a 3-point, \( p \) is a 3-point and
   \[
u = x^a y^b z^c, v = x^d y^e z^f, w = g(x, y, z) + N
\]
   where \( \text{rank}((a, b, c), (d, e, f)) = 2 \), \( g \) is a series in monomials \( M = x^\alpha y^\beta z^\gamma \) in \( x, y, z \) such that \( \text{rank}((a, b, c), (d, e, f), (\alpha, \beta, \gamma)) = 2 \), and \( N = x^{a'} y^{b'} z^{c'} \) is such that
   \[
   \text{rank}((a, b, c), (d, e, f), (a', b', c')) = 3.
   \]
5. \( q \) is a 1-point, \( p \) is a 1-point and
   \[
u = x^a, v = y, w = g(x, y) + x^c z
\]

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where $g$ is a series.

6. $q$ is a 1-point, $p$ is a 2-point and

$$u = (x^ay^b)^k, v = z, w = g(x^ay^b, z) + x^cy^d$$

with $a, b, k > 0$, $\gcd(a, b) = 1$, $g$ is a series and $\text{rank}((a, b), (c, d)) = 2$.

At first sight, the prepared forms for morphisms of 3-folds appear to be similar to those of prepared forms of morphisms of n-folds to surfaces, which we are able to toroidalize in Theorem 3.3. However, a little experimentation shows that the situation when the base $Y$ is a 3-fold is much more complex. The essential problem is that prepared forms are not stable under possible blow ups above $Y$ when $Y$ is a 3-fold.

However, we are able to accomplish toroidalization in the case when $f$ is birational.

**Theorem 4.2** ([C6], toroidalization; [C7], strong toroidalization) *Strong toroidalization is true for birational morphisms $f : X \to Y$ of 3-folds.*

**Outline of proof of Theorem 4.2**

By Theorem 4.1, we may assume that $f$ is prepared, and $D_X$ is cuspidal for $f$. These conditions are preserved throughout the construction.

We define the $\tau$-invariant of a 3-point $p \in X$. Since $f$ is prepared, $f(p) = q$ is a 2-point or a 3-point. There are permissible parameters $u, v, w$ in $\mathbb{O}_{Y,q}$ and $x, y, z$ in $\hat{O}_{X,p}$ such that $xyz = 0$ is a local equation of $D_X$, $uv = 0$ or $uvw = 0$ is a local equation of $D_Y$ and there is an expression

$$(3) \quad u = x^ay^bz^c, \quad v = x^dy^ez^f, \quad w = \sum_{i \geq 0} \alpha_i M_i + N$$

with $\alpha_i \in k$, $M_i = x^{ai}y^{bi}z^{ci}$, $N = x^gy^hz^i$,

$$\text{rank} \begin{pmatrix} a & b & c \\ d & e & f \\ a_i & b_i & c_i \end{pmatrix} = 2, \quad \det \begin{pmatrix} a & b & c \\ d & e & f \\ a_i & b_i & c_i \end{pmatrix} = 0 \text{ for all } i,$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.$$

If $q$ is a 3-point, then

$$w = \text{unit series } N$$
if and only if \( f \) is toroidal at \( p \). In this case define \( \tau_f(p) = -\infty \).

Otherwise, define

\[
H_p = Z(a, b, c) + Z(d, e, f) + \sum_i Z(a_i, b_i, c_i),
\]

\[
A_p = \begin{cases} 
Z(a, b, c) + Z(d, e, f) + Z(a_0, b_0, c_0) & \text{if } q \text{ is a 3-point} \\
Z(a, b, c) + Z(d, e, f) & \text{if } q \text{ is a 2-point}. 
\end{cases}
\]

Now define

\[
\tau_f(p) = |H_p/A_p|.
\]

We define

\[
\tau_f(X) = \max\{\tau_f(p) \mid p \in X \text{ is a 3-point}\}.
\]

**Theorem 4.3** Suppose that \( f \) is prepared for \( D_Y \) and \( D_X = f^{-1}(D_Y) \), and \( D_X \) is cuspidal for \( f \). Further suppose that \( \tau_f(X) = -\infty \). Then \( f \) is toroidal.

We have that \( \tau_f(X) \geq 0 \) or \( \tau_f(X) = -\infty \). The proof of Theorem 4.2 is by descending induction on \( \tau_f(X) \). In our proof of Theorem 4.2 we may thus assume that \( \tau = \tau_f(X) \neq -\infty \) (so that \( \tau \geq 0 \)).

**Step 1.** There exist sequences of blow ups of 2-curves

\[
\begin{array}{ccc}
X_1 & \overset{f_1}{\to} & Y_1 \\
\downarrow & & \downarrow \\
X & \overset{f}{\to} & Y
\end{array}
\]

such that \( f_1 \) is prepared, \( D_{X_1} \) is cuspidal for \( f_1 \), \( \tau_f(X_1) \leq \tau \), and \( \tau_{f_1}(p) = \tau \) implies that \( f_1(p) \) is a 2-point. We use the concept of 3-point relation in this step.

**Step 2.** In this step we construct a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \overset{f_1}{\to} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \overset{f}{\to} & Y
\end{array}
\]

such that

1. \( \Phi \) and \( \Psi \) are products of possible blow ups.
2. \( \tau(X_1) = \tau \), and if \( p \in X \) is a 3-point such that \( \tau_f(p) = \tau \) then \( f_1(p) \) is a 2-point.
3. $D_{X_1}$ is cuspidal for $f_1$.
4. $f_1$ is $\tau$-very-well prepared.

Step 2 is the most difficult step technically.

We will not give the complete definition of $\tau$-very-well prepared, which uses the concept of 2-point relation, and requires the preliminary definitions of quasi-well prepared and well prepared.

By virtue of the result of this step, we can assume that $f$ is $\tau$-very-well prepared. We will also assume that $\tau > 0$. The case when $\tau = 0$ is actually a little easier, but the definition is a bit different.

We now summarize some of the properties of a $\tau$-very-well-prepared morphism.

There exists a finite, distinguished set of nonsingular algebraic surfaces $\Omega(\overline{R}_i)$ in $Y$, with a SNC divisor $F_i$ on $\Omega(\overline{R}_i)$ such that the intersection graph of $F_i$ is a tree.

Suppose that $p \in X$ is a 3-point with $\tau_f(p) = \tau$ (so that $q = f(p)$ is 2-point). Then the following conditions hold.

1. The expression (3) has the form

\begin{equation}
    w = \gamma M_0
\end{equation}

where $\gamma$ is a unit series, $M_0^e = u^a v^b$, with $a, b, e \in \mathbb{Z}$, $e > 1$, and $\gcd(a, b, e) = 1$. Observe that we cannot have both $a < 0$ and $b < 0$, since $M_0, u, v$ are all monomials in $x, y, z$.

2. Suppose that $V$ is the curve in $Y$ with local equations $u = w = 0$ (or $v = w = 0$) at $q$. Then $V$ is a possible center for $D_Y$ and there exists a commutative diagram of morphisms

\begin{equation}
    \begin{array}{ccc}
    X_1 & \xrightarrow{f_1} & Y_1 \\
    \Phi_1 \downarrow & & \downarrow \Psi_1 \\
    X & \xrightarrow{f} & Y
    \end{array}
\end{equation}

where $\Psi_1$ is the blow up of $V$ (possibly followed by blow ups of some special 2-points), such that $f_1$ and $\overline{f} = \Psi_1 \circ f_1 : X_1 \to Y$ are prepared, $\tau_{f_1}(X_1) \leq \tau$ and $\Phi_1$ is toroidal at 3-points $p_1 \in \Phi_1^{-1}(p)$. Further, $f_1$ is $\tau$-very-well prepared.

3. There exists a surface $\Omega(\overline{R}_i)$ such that
   a. $f(p) = q \in \Omega(\overline{R}_i)$.
   b. The $w$ of (4) gives a local equation $w = 0$ of $\Omega(\overline{R}_i)$ at $q$.
   c. $uv = 0$ is a local equation of $F_i$ (on the surface $\Omega(\overline{R}_i)$) at $q$. 

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The necessity of several different surfaces \( \Omega(\overline{R_i}) \) arises because of the possibility that there may be several 3-points \( p_j \) with \( \tau_f(p_j) = \tau \) which map to \( q \), and require different \( w \) in their expressions (4). We require that the surfaces \( \Omega(\overline{R_i}) \) intersect in a controlled way.

The first step in the construction of a \( \tau \)-very well prepared morphism is the construction of a morphism such that for all 3-points \( p \) with \( \tau_f(p) = \tau \), an expression (4) holds for some possibly formal \( w \).

**Step 3.** We construct a commutative diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

such that \( \tau_{f_n}(X_n) < \tau \). By induction on \( \tau \), we then obtain the proof of Theorem 4.2.

We fix an index \( i \) of the surfaces \( \Omega(\overline{R_i}) \). A curve \( E \) on \( Y \) is good if it is a component of \( F_i \), and if \( j \) is such that \( E \cap \Omega(\overline{R_j}) \neq \emptyset \), then \( E \) is a component of \( F_j \).

In our construction we begin with \( i = 1 \), and blow up a good curve \( V \) on \( Y \), by a morphism (5). Part of the definition of \( \tau \)-very-well prepared implies the existence of a good curve. Suppose that \( p \in X \) is a 3-point with \( \tau_f(X) = \tau \) and \( q = f(p) \in V \). Suppose that \( p_1 \in \Phi_1^{-1}(p) \) is a 3-point. Set \( q_1 = f_1(p_1) \). If \( V \) has local equations \( u = w = 0 \) at \( q \), then \( q_1 \) has regular parameters \( u_1, v, w_1 \) with

\[ (6) \quad u = u_1 w_1, w = w_1 \]

or

\[ (7) \quad u = u_1, w = u_1(w_1 + \alpha) \]

and \( \alpha \in k \).

If (6) holds then \( q_1 \) is a 3-point. Since \( e > 1 \), we have

\[ \tau_{f_1}(p_1) = |H_p/A_p + M_0\mathbf{Z}| < |H_p/A_p| = \tau. \]

If (7) holds, then \( e > 1 \) implies \( \alpha = 0 \). Thus \( f_1 \) has the form (3), (4) at \( p_1 \) with \( (\overline{a}, \overline{b}, e) \) changed to \( (\overline{a} - e, \overline{b}, e) \). If \( V \) has local equations \( v = w = 0 \), then \( f_1 \) has the form (3), (4) at \( p_1 \) with \( (\overline{a}, \overline{b}, e) \) changed to \( (\overline{a}, \overline{b} - e, e) \).

We have SNC divisors \( \Phi_1^{-1}(F_1) \) on the surfaces \( \Phi_1^{-1}(\Omega(\overline{R_i})) \). If there are no 3-points \( p_1 \) in \( X_1 \) satisfying 1, 2 and 3 of Step 2 for \( \Phi_1^{-1}(\Omega(\overline{R_i})) \), then we increase \( i \) to 2.

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Otherwise, there exists a good curve on $Y_1$ for the SNC divisor $\Phi^{-1}(F_1)$ on the surface $\Phi_1^{-1}(\Omega(\mathcal{R}_1)))$. We continue to iterate, blowing up good curves. If we always have a 3-point satisfying 1, 2 and 3 for the preimage of $\Omega(\mathcal{R}_1)$, then we eventually obtain a form (4) with both $\bar{a} < 0$ and $\bar{b} < 0$ which is impossible.

We then continue this algorithm for the preimages of all of the surfaces $\Omega(\mathcal{R}_i)$. The algorithm terminates in the construction of a morphism with a drop in $\tau$ as desired.

**Open problems.** We conclude this section with a list of open problems on toroidalization.

1. Prove (strong) toroidalization for arbitrary dominant morphisms of 3-folds.

   By Theorem 4.1 we can assume that $f$ is prepared. Much of the proof of Theorem 4.2 works in the case when $f$ is not birational.

2. Suppose that $f : X \to Y$ is a dominant morphism from an $n$-fold to a surface $Y$.

   Prove that there exists a commutative diagram

   $\begin{array}{ccc}
   X_1 & \xrightarrow{f_1} & Y \\
   \downarrow{\Phi_1} & \searrow & \\
   X & \xrightarrow{f} & Y
   \end{array}$

   such that $\Phi_1$ is a product of possible blow ups and $f_1$ is (strongly) prepared. Recall that Theorem 3.2 now implies that $f$ can be toroidalized.

3. Prove the toroidalization conjecture in all dimensions.

5. **Valuations in Algebraic Geometry.** Suppose that $K$ is an algebraic function field of dimension (transcendence degree $n$) over a ground field $k$. A valuation of $K$ is a homomorphism

   $$\nu : K^\times \to \Gamma_\nu$$

   of the multiplicative group of $K$ onto a totally ordered abelian group, such that $\nu(a) = 0$ if $a \in k^\times$. We formally extend $\nu$ to $K$ by defining $\nu(0) = \infty$. Associated to a valuation $\nu$ we have a valuation ring

   $$V_\nu = \{f \in K \mid \nu(f) \geq 0\}.$$

   $V_\nu$ is a local ring with maximal ideal $m_\nu = \{f \in K \mid \nu(f) > 0\}$. $V_\nu$ contains $k$.

   If $A \subset B$ are local rings with respective maximal ideals $m_A$ and $m_B$, we say that $B$ dominates $A$ if $m_B \cap A = m_A$.

   The connection of valuation theory to algebraic geometry is explained by the following lemma.
Lemma 5.1 Suppose that $X$ is a projective variety with function field $K = k(X)$, and $\nu$ is a valuation of $k$. Then there exists a unique point $p \in X$ such that $V_\nu$ dominates the local ring $\mathcal{O}_{X,p}$.

This lemma is of course true (by definition) on any proper $k$-variety. The point $p$ (which may not be closed) is called the center of $\nu$ on $X$.

We define a locally ringed space $\Sigma_K$, which we call the Zariski-Riemann manifold of $K$. The points of $\Sigma_K$ are the valuation rings of $K$. We define a topology on $\Sigma_K$ by taking basic open sets to be

$$\overline{U} = \{ V \in \Sigma_K \mid V \text{ dominates } \mathcal{O}_{X,p} \text{ for some } p \in U \},$$

where $U$ is an open subset of a proper $k$-variety $X$, with function field $k(X) = K$.

The local ring $\mathcal{O}_{\Sigma_K,p}$ of a point $p = V_p \in \Sigma_K$ is the valuation ring $V_p$. For $\overline{U} \subset \Sigma_K$ an open set, we define

$$\Gamma(\overline{U}, \mathcal{O}_{\Sigma_K}) = \cap_{V_p \in \overline{U}} V_p.$$

Theorem 5.2 Suppose that $X$ is a proper variety with function field $K$. Then the mapping $\pi_X : \Sigma_K \to X$ defined by $\pi_X(V) = p$ if $V$ dominates $\mathcal{O}_{X,p}$ is continuous.

If $K$ has dimension 1, then the valuation rings of $K$ are local Dedekind domains which are essentially of finite type over $k$. Thus, when $K$ has dimension 1, $\Sigma_K$ is the (unique) nonsingular projective curve with function field $K$.

However, if $K$ has dimension $\geq 2$, then there are valuation rings $V$ of $K$ which are not Noetherian. Still, $\Sigma_K$ is in fact always quasi-compact ([Z2], [ZS]).

There are three main invariants associated to a valuation ring $V$ ([Z1], [ZS]). They are:

1. The dimension of $V$ is the transcendence degree of $V/m_V$ over $k$ (this number is always finite, although in general $V/m_V$ is not a finitely generated extension of $k$).
2. The rank of $V$ is the length $n$ of the sequence of prime ideals

$$0 = p_1 \subset \cdots \subset p_n = m_V$$

in $V$. This is also the number of isolated subgroups of $\Gamma_V$.
3. The rational rank of $V$ is the dimension of the vector space $\Gamma_V \otimes \mathbb{Q}$. The rational rank is always finite.
We will denote the respective invariants by \( \dim(V) \), \( \text{rank}(V) \) and \( \text{rrank}(V) \). We have \( ([\text{Ab}3], [\text{ZS}] ) \)

\[
\dim(V) + \text{rrank}(V) \leq \text{trdeg}_k(K)
\]

and

\[
\text{rank}(V) \leq \text{rrank}(V).
\]

Valuation rings in dimension 2.

There are 4 types of valuation rings in dimension 2 \( ([\text{Z1}], [\text{MS}], [\text{ZS}], [\text{C4}] ) \). They are:

1. \( V \) is one dimensional. \( \Gamma_V = \mathbb{Z} \), \( \dim(V) = 1 \) and \( \text{rrank}(V) = \text{rank}(V) = 1 \).
2. \( V \) is discrete, zero dimensional of rank 1. \( \Gamma_V = \mathbb{Z} \), \( \dim(V) = 1 \) and \( \text{rrank}(V) = \text{rank}(V) = 1 \).
3. \( V \) is discrete, zero dimensional of rank 2. \( \Gamma_V = \mathbb{Z} \oplus \mathbb{Z} \) with the lex order, \( \dim(V) = 0 \) and \( \text{rrank}(V) = \text{rank}(V) = 2 \).
4. \( V \) is non-discrete of dimension zero. We have \( \dim(V) = 0 \), \( \text{rank}(V) = 1 \), and \( \text{rrank}(V) = 1 \) or 2.

We now give characteristic examples of these types. All valuations of \( K \) can be obtained by these constructions. Suppose that \( X \) is a surface with \( k(X) = K \) and \( p \in X \) is a nonsingular (closed) point. Let \( x, y \) be regular parameters in \( A = \mathcal{O}_{X,p} \). For simplicity, we assume that \( k \) is algebraically closed.

1. \( V \) is one dimensional. \( V = \mathcal{O}_{X,E} \) where \( X \) is a normal surface with \( k(X) = K \), and \( E \) is a codimension 1 subvariety.
2. \( V \) is discrete, zero dimensional of rank 1. Embed \( A \) into a power series ring \( k[[t]] \), by mapping \( x \) to \( t \) and \( y \) to a power series \( P(t) \) which is transcendental over \( k[t] \). Then \( V = k[[t]] \cap K \).
3. \( V \) is discrete, zero dimensional of rank 2. For \( f \in A \), we can factor \( f = x^n g(x, y) \) in \( \hat{A} = k[[x, y]] \), so that \( x \nmid g \). Define \( \nu(f) = (n, \text{ord} g(0, y)) \).
4. \( V \) is non-discrete of dimension zero.
   a. \( \text{rrank}(V) = 2 \). Choose \( \tau \in \mathbb{R} \) which is irrational. Define \( \nu(x) = 1, \nu(y) = \tau \). If \( f \in A \), expand

\[
f = \sum a_{ij}x^i y^j
\]
in $k[[x, y]]$ where $a_{ij} \in k$. Define

$$\nu(f) = \min\{i + \tau j \mid a_{ij} \neq 0\}.$$  

Since $\tau$ is irrational, there is a unique monomial in $f$ which attains this minimum. This property implies that $\nu$ is a valuation. The value group of $\nu$ is the ordered subgroup $\mathbb{Z} + \mathbb{Z} \tau$ of $\mathbb{R}$.

b. $\text{rrank}(V) = 1$. This is the really interesting case. Let $S$ be the field of “formal” series

$$f = \sum_{\rho} a_{\alpha,\rho} t^{\alpha_{\rho}},$$

where $\alpha_{\rho} \in \mathbb{R}$ increase monotonically with $\rho$, $a_{\alpha,\rho} \in k$ are nonzero and the sum is over all ordinal numbers $\rho \leq \sigma$ for some fixed ordinal number $\sigma$. $S$ has a valuation defined by $\nu(g(t)) = \text{ord}(g(t))$. We embed $A$ in $S$ by mapping $x$ to $t$ and $y$ to some $P(t) \in S$. For $f \in A$, $\nu(f) = \text{ord}(f(t, P(t))$. Any subgroup of $\mathbb{Q}$ can be obtained as a value group $\Gamma_V$ by this construction.

**Local uniformization.**

An algebraic local ring $R$ of an algebraic function field $K$ is the local ring $\mathcal{O}_{X, p}$ of a point on a variety $X$ with $k(X) = K$. A monoidal transform of an algebraic local ring $R$ in an inclusion $R \subset R_1$ where $R_1 = \mathcal{O}_{X_1, p_1}$ is a local ring of the blow up $\pi_1 : X_1 \to X$ of a nonsingular subvariety of $X$, and $\pi_1(p_1) = p$. If $V$ is a valuation ring which dominates $R$, then there is a unique point $p_1 \in X_1$ whose local ring $R_1$ is dominated by $V$ ($p_1$ is the center of $V$ on $X_1$). We say that $R \to R_1$ is a monoidal transform along $V$.

**Theorem 5.3** ([Z3]) Suppose that $k$ is a field of characteristic zero, $K$ is an algebraic function field over $k$, and $\nu$ is a valuation of $K$ which dominates an algebraic local ring $R$ of $K$. Then there exists a sequence of monoidal transforms $R \to R'$ along $\nu$ such that $R'$ is a regular local ring.

The first proof of resolution in dimension 3 used local uniformization.

**Corollary 5.4** ([Z1], [Z4]) Resolution of singularities is true in characteristic zero for varieties of dimension $\leq 3$.

[Z4] was published in 1944, about 20 years before Hironaka’s characteristic zero proof [H] of resolution in all dimensions.
The first proof of resolution of surfaces and of resolution of 3-folds in positive characteristic was by local uniformization ([Ab1], [Ab2]). There has recently been progress on local uniformization in positive characteristic (including [Kuhl2], [Sp], [T]).

6. Local Monomialization.

Suppose that $R \subset S$ is a local homomorphism of local rings essentially of finite type over a field $k$, and that $V$ is a valuation ring of the quotient field $K$ of $S$, such that $V$ dominates $S$. Then we can ask if there are sequences of moniodal transforms $R \to R'$ and $S \to S'$ along $V$ such that $V$ dominates $S'$, $S'$ dominates $R'$ and $R' \to S'$ is a “monomial mapping”,

$$
\begin{array}{c}
R' \rightarrow S' \subset V \\
\uparrow \quad \uparrow \\
R \rightarrow S
\end{array}
$$

We completely answer this question in the affirmative when $k$ is a field of characteristic zero.

**Theorem 6.1** ([C1], [C3], [C5]) Suppose that $k$ is a field of characteristic zero, $K \to K^*$ is a (possibly transcendental) extension of algebraic function fields over $k$, and that $\nu^*$ is a valuation of $K^*$ which is trivial on $k$. Further suppose that $R$ is an algebraic local ring of $K$ and $S$ is an algebraic local ring of $K^*$ such that $S$ dominates $R$ and $\nu^*$ dominates $S$. Then there exist sequences of monoidal transforms $R \to R'$ and $S \to S'$ along $\nu^*$ such that $R'$ and $S'$ are regular local rings, $S'$ dominates $R'$, there exist regular parameters $(y_1, \ldots, y_n)$ in $S'$, $(x_1, \ldots, x_m)$ in $R'$, units $\delta_1, \ldots, \delta_m \in S'$ and an $m \times n$ matrix $(c_{ij})$ of nonnegative integers such that $(c_{ij})$ has rank $m$, and

$$
x_i = \prod_{j=1}^n y_j^{c_{ij}} \delta_i
$$

for $1 \leq i \leq m$.

We make a few comments and observations.

1. In the case when $K = k$, so that $R$ is just the field $k$, Theorem 6.1 specializes to the local uniformization theorem Theorem 5.3.

2. Since $k$ has characteristic zero in Theorem 6.1, and $(c_{ij})$ has rank $m$, we can obtain a toroidal form of $R' \to S'$, by choosing regular parameters $\overline{y}_1, \ldots, \overline{y}_n$ in $\hat{S}'$ so that

$$
x_i = \prod_{j=1}^n \overline{y}_j^{c_{ij}}.
$$
3. The question of the existence of a diagram (8) makes sense over fields of positive characteristic. Certainly it is true when $R$ has dimension 1, and $S$ has dimension $\leq 2$ (or in any dimension of $S$ where good resolution theorems hold).

4. Some applications of Theorem 6.1 were given in Section 1 to local toroidalization and local factorization. Applications to the ramification theory of general valuations are given in [CP2], and [CG].

5. The proof of Theorem 6.1 actually gives a very special form to the matrix $(c_{ij})$ which depends on the rank and rational rank of $\nu$ and $\nu^*$, which we call “strong local monomialization”.

For simplicity, assume that $K^*$ is finite over $K$. Let $r = \text{rank } \nu = \text{rank } \nu^*$, $s = \text{rrank } \nu = \text{rrank } \nu^*$. In the valuation ring $V^*$ of $\nu^*$, let

$$0 = P_0 \subset \cdots \subset P_r = m_V$$

be the chain of prime ideals. Then $V_{P_i}/(P_{i-1})_{P_i}$ for $1 \leq i \leq r$ are rank 1 valuation rings of rational rank $s_i$, where $s_1 + \cdots + s_r = s$. These are the “composite” valuation rings of $V$.

The (square) matrix $C = (c_{ij})$ has the block form

$$C = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & A_r \end{pmatrix}.$$  

Each $A_i$ has the block form

$$A_i = \begin{pmatrix} B_i \\ & I \end{pmatrix}$$

where $B_i$ is an $s_i \times s_i$ matrix and $I$ is an appropriate identity matrix.

6. The first case where local monomialization along a valuation is open is when $K$ and $K^*$ have dimension 2 over an algebraically closed field $k$ of positive characteristic. Local monomialization for all of the cases of valuations in the classification given in Section 5 can be proven to be true fairly easily, except for the last case 4 b, where $V$ is a nondiscrete valuation ring of rank 1. This case is studied in [CP2]. An example is given where “strong monomialization” (stated in 5. above) fails. Good local forms are constructed along valuations in general, which are shown to give strong monomialization for defectless ([ZS], [Kuhl1]) extensions.

The less restrictive question of local monomialization along nondiscrete valuations in an extension $K^*/K$ with defect remains open.
Bibliography.


