Throughout these notes all rings will be commutative with identity. $k$ will be an algebraically closed field.

1. Preliminaries on Ring Homomorphisms

Lemma 1.1. Suppose that $\varphi : R \to S$ is a ring homomorphism with kernel $K$. Suppose that $I$ is an ideal of $R$ which is contained in $K$. Then the rule $\varphi : R/I \to S$ defined by $\varphi(x + I) = \varphi(x)$ for $x \in R$ is a well defined ring homomorphism, with kernel $K/I$.

Lemma 1.2. Suppose that $\varphi : R \to S$ is a ring homomorphism. If $I$ is an ideal in $S$, then $\varphi^{-1}(I)$ is an ideal in $R$. If $P$ is a prime ideal in $S$, then $\varphi^{-1}(P)$ is a prime ideal in $R$.

Recall that a map $\Phi : U \to V$ of sets is a 1-1 correspondence (a bijection) if and only if $\Phi$ has an inverse map; that is, a map $\Psi : V \to U$ such that $\Psi \circ \Phi = \text{id}_U$ and $\Phi \circ \Psi = \text{id}_V$.

Lemma 1.3. Let $\pi : R \to S$ be a surjective ring homomorphism, with kernel $K$.

1. Suppose that $I$ is an ideal in $S$. Then $\pi^{-1}(I)$ is an ideal in $R$ containing $K$.
2. Suppose that $J$ is an ideal in $R$ such that $J$ contains $K$. Then $\pi(J)$ is an ideal in $S$.
3. The map $I \mapsto \pi^{-1}(I)$ is a 1-1 correspondence between the set of ideals in $R$ and the set of ideals in $S$ which contain $K$. The inverse map is $J \mapsto \pi(J)$.
4. The correspondence is order preserving: for ideals $I_1, I_2$ in $S$, $I_1 \subset I_2$ if and only if $\pi^{-1}(I_1) \subset \pi^{-1}(I_2)$.
5. For an ideal $I$ in $S$, $I$ is a prime ideal if and only if $\pi^{-1}(I)$ is a prime ideal in $R$.
6. For an ideal $I$ in $S$, $I$ is a maximal ideal if and only if $\pi^{-1}(I)$ is a maximal ideal in $R$.

In the case when $S = R/K$, and $\pi : R \to R/K$ is the map $\pi(x) = x + K$ for $x \in R$, we have that $\pi(J) = J/K$ for $J$ an ideal of $R$ containing $K$.

2. Affine varieties

Affine $n$-space over $k$ is

$\mathbb{A}^n_k = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in k\}$.

An element $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is called a point. The ring of regular functions on $\mathbb{A}^n$ are the polynomial mappings

$k[\mathbb{A}^n] = \{f : \mathbb{A}^n \to \mathbb{A}^1 \mid f \in k[x_1, \ldots, x_n]\}$.

Here $k[x_1, \ldots, x_n]$ are the polynomials in the variables $x_1, \ldots, x_n$.

Theorem 2.1. (Theorem 2.19, [5]) Suppose that $L$ is an infinite field and $f \in L[x_1, \ldots, x_n]$ is a nonzero polynomial. Then there exist elements $a_1, \ldots, a_n \in L$ such that $f(a_1, \ldots, a_n) \neq 0$. 

Since an algebraically closed field is infinite, the natural surjective ring homomorphism $k[x_1, \ldots, x_n] \to k[\mathbb{A}^n]$ is an isomorphism. Thus we may identify the ring $k[\mathbb{A}^n]$ with the polynomial ring $k[x_1, \ldots, x_n]$.

The zeros of a regular function $f \in k[\mathbb{A}^n]$ are

$$Z(f) = \{ p \in \mathbb{A}^n \mid f(p) = 0 \}.$$ 

If $T \subset k[\mathbb{A}^n]$ is a subset, then the set of common zeros of the elements of $T$ is

$$Z(T) = \{ p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } p \in T \}.$$ 

A subset $W$ of $\mathbb{A}^n$ is called an algebraic set if there exists a subset $T$ of $k[\mathbb{A}^n]$ such that $W = Z(T)$.

The ideal generated by $T$ in $k[\mathbb{A}^n]$ is

$$a = \{ r_1f_1 + \cdots + r_nf_n \mid n \in \mathbb{Z}_+, f_1, \ldots, f_n \in T, r_1, \ldots, r_n \in k[\mathbb{A}^n] \}.$$ 

**Theorem 2.2.** (Hilbert’s Basis Theorem - Theorem 7.5, page 81 [2]) Suppose that $I \subset k[x_1, \ldots, x_n]$ is an ideal. Then there exists a finite set $g_1, \ldots, g_m \in I$ such that $I = (g_1, \ldots, g_m) = \{ r_1g_1 + \cdots + r_mg_m \mid r_i \in k[x_1, \ldots, x_n] \}.$

In particular, every algebraic set in $\mathbb{A}^n$ is the set of common zeros of a finite number of polynomials.

**Proposition 2.3.** Suppose that $I_1, I_2, \{ I_\alpha \}_{\alpha \in S}$ are ideals in $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$. Then

1. $Z(I_1I_2) = Z(I_1) \cup Z(I_2)$.
2. $Z(\bigcap_{\alpha \in B} I_\alpha) = \bigcap_{\alpha \in S} Z(I_\alpha)$.
3. $Z(k[\mathbb{A}^n]) = \emptyset$.
4. $\mathbb{A}^n = Z(0)$.

**Proof.** (of 1.) Suppose that $p \in Z(I_1) \cup Z(I_2)$. Then $p \in Z(I_1)$ or $p \in Z(I_2)$. Thus for every $f \in I_1$, we have $f(p) = 0$ or for every $g \in I_2$ we have that $g(p) = 0$. If $f \in I_1I_2$, then $f = \sum_{i=1}^r f_i g_i$ for some $f_1, \ldots, f_r \in I_1$ and $g_1, \ldots, g_r \in I_2$. Thus $f(p) = \sum f_i(p) g_i(p) = 0$, so that $p \in Z(I_1I_2)$.

Now suppose that $p \in Z(I_1I_2)$ and $p \notin Z(I_1)$. Then there exists $f \in I_1$ such that $f(p) \neq 0$. For any $g \in I_2$, we have $fg \in I_1I_2$ so that $f(p)g(p) = 0$. Since $f(p) \neq 0$ we have that $g(p) = 0$. Thus $p \in Z(I_2)$. \qed

Proposition 2.3 tells us that

1. The union of two algebraic sets is an algebraic set.
2. The intersection of any family of algebraic sets is an algebraic set.
3. $\emptyset$ and $\mathbb{A}^n$ are algebraic sets.

We thus have a topology on $\mathbb{A}^n$, defined by taking the closed sets to be the algebraic sets. The open sets are the complements of algebraic sets in $\mathbb{A}^n$ (any union of open sets is open, any finite intersection of open sets is open, the emptyset is open and $\mathbb{A}^n$ is open).

This topology is called the Zariski topology.

**Example 2.4.** Suppose that $I$ is a nontrivial ideal in $k[\mathbb{A}^1] = k[x]$. Then $I = (f)$ where $f = (x - \alpha_1) \cdots (x - \alpha_r)$ for some $\alpha_1, \ldots, \alpha_r \in k$. Thus $Z(I) = \{ \alpha_1, \ldots, \alpha_r \}$. The open sets in $\mathbb{A}^1$ are thus $\mathbb{A}^1$, the complement of finitely many points in $\mathbb{A}^1$ and $\emptyset$.

We see that the Zariski topology is not Hausdorff (to be Hausdorff disjoint points must have disjoint neighborhoods).
A nonempty subset $Y$ of a topological space $X$ is irreducible if it cannot be expressed as a union $Y = Y_1 \cup Y_2$ of two proper subsets, each of which is closed in $Y$ (\(\emptyset\) is not irreducible).

**Example 2.5.** $\mathbb{A}^1$ is irreducible as all proper closed subsets are finite and $\mathbb{A}^1$ is infinite.

**Definition 2.6.** An affine algebraic variety is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine variety is a quasi-affine variety. An affine algebraic set is a closed subset of $\mathbb{A}^n$. A quasi-affine algebraic set is an open subset of a closed subset of $\mathbb{A}^n$.

Given a subset $Y$ of $\mathbb{A}^n$, the ideal of $Y$ in $k[\mathbb{A}^n]$ is

$$I(Y) = \{ f \in k[\mathbb{A}^n] \mid f(p) = 0 \text{ for all } p \in Y \}.$$ 

**Theorem 2.7.** (Hilbert’s Nullstellensatz, page 85, [2]) Let $k$ be an algebraically closed field, $a$ an ideal in the polynomial ring $R = k[x_1, \ldots, x_n]$, and $f \in R$ a polynomial which vanishes at all points of $Z(a)$. Then $f^r \in a$ for some $r > 0$.

**Corollary 2.8.** Suppose that $I$ is an ideal in $k[x_1, \ldots, x_n]$. Then $I$ is a maximal ideal if and only if there exist $a_1, \ldots, a_n \in k$ such that $I = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.

The radical of an ideal $a$ in a ring $R$ is

$$\sqrt{a} = \{ f \in R \mid f^r \in a \text{ for some positive integer } r \}.$$ 

**Proposition 2.9.** The following statements hold:

a) Suppose that $Y$ is a subset of $\mathbb{A}^n$. Then $I(Y)$ is an ideal in $k[\mathbb{A}^n]$.

b) If $T_1 \subset T_2$ are subsets of $k[\mathbb{A}^n]$, then $Z(T_2) \subset Z(T_1)$.

c) If $Y_1 \subset Y_2$ are subsets of $\mathbb{A}^n$, then $I(Y_2) \subset I(Y_1)$.

d) For any two subsets $Y_1, Y_2$ of $\mathbb{A}^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

e) For any ideal $a$ of $k[\mathbb{A}^n]$, we have $I(Z(a)) = \sqrt{a}$.

f) For any subset $Y$ of $\mathbb{A}^n$, $Z(I(Y)) = \overline{Y}$, the Zariski closure of $Y$.

**Theorem 2.10.** A closed set $W \subset \mathbb{A}^n$ is irreducible if and only if $I(W)$ is a prime ideal.

**Proof.** Suppose that $W$ is irreducible and $f, g \in k[\mathbb{A}^n]$ are such that $fg \in I(W)$. Then $W \subset Z(fg) = Z(f) \cup Z(g)$. Thus $W = (Z(f) \cap W) \cup (Z(g) \cap W)$ expresses $W$ as a union of closed sets. Since $W$ is irreducible we have $W \subset Z(f)$ or $W \subset Z(g)$. Thus $f \in I(W)$ or $g \in I(W)$. We have verified that $I(W)$ is a prime ideal.

Now suppose that $W$ is not irreducible. Then $W = Z_1 \cup Z_2$ where $Z_1$ and $Z_2$ are proper subsets of $W$. $I(Z_1)$ is not a subset of $I(Z_2)$; if it were, then we would have

$$Z_2 = Z(I(Z_2)) \subset Z(I(Z_1)) = Z_1$$

by b) and f) of Proposition 2.9, which is impossible. Thus there exists $f_1 \in k[\mathbb{A}^n]$ which vanishes on $Z_1$ but not on $Z_2$. Similarly, there exists $f_2 \in k[\mathbb{A}^n]$ which vanishes on $Z_2$ and not on $Z_1$. We have $f_1 f_2 \in I(W)$, but $f_1, f_2 \notin I(W)$. Thus $I(W)$ is not a prime ideal. \(\square\)

**Definition 2.11.** A ring $R$ is Noetherian if every ascending chain of ideals

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

is stationary (there exists $n_0$ such that $I_n = I_{n_0}$ for $n \geq n_0$).

**Proposition 2.12.** (Proposition 6.3 [2]) A ring $R$ is Noetherian if and only if every ideal in $R$ is finitely generated.

**Corollary 2.13.** A polynomial ring and a quotient of a polynomial ring are Noetherian.
Corollary 2.14. Every closed set in \( \mathbb{A}^n \) is the union of finitely many irreducible ones.

Proof. Suppose that \( Z \) is an algebraic set in \( \mathbb{A}^n \) which is not the union of finitely many irreducible ones. Then \( Z = Z_1 \cup Z_2 \) where \( Z_1 \) and \( Z_2 \) are proper closed subsets of \( Z \) and either \( Z_1 \) or \( Z_2 \) is not a finite union of irreducible closed sets. By induction, we can construct an infinite chain of proper inclusions
\[
Z \supset Z_1 \supset Z_2 \supset \cdots
\]
giving an infinite chain of proper inclusions
\[
I(Z) \supset I(Z_1) \supset I(Z_2) \supset \cdots
\]
of ideals in \( k[\mathbb{A}^n] \) (by c) and f) of Proposition 2.9), a contradiction. \( \square \)

3. Regular functions and regular mappings of affine varieties

Definition 3.1. A \( k \)-algebra is a commutative ring \( R \) such that
   a) \( (R, +) \) is a \( k \)-vector space.
   b) \( (ca)b = c(ab) \) for all \( a, b \in R \), \( c \in k \)

If \( R \) is a nonzero \( k \)-algebra, then we can view \( k \) as a subring of \( R \), by identifying \( k \) with \( k1_R \).

A \( k \)-algebra \( R \) is finitely generated if \( R \) is generated by a finite number of elements as a \( k \)-algebra. If \( R \) is nonzero and is generated by \( u_1, \ldots, u_n \) as a \( k \)-algebra, and \( k[x_1, \ldots, x_n] \) is a polynomial ring, then
\[
R = \{ f(u_1, u_2, \ldots, u_n) \mid f \in k[x_1, \ldots, x_n] \} = k[u_1, \ldots, u_n].
\]

Definition 3.2. A \( k \)-algebra homomorphism \( \varphi : R \to S \) is a ring homomorphism such that \( \varphi(ca) = c\varphi(a) \) for all \( a \in R \) and \( c \in k \).

Lemma 3.3. Suppose that \( R \) is a finitely generated nonzero \( k \)-algebra, generated by \( u_1, \ldots, u_n \). Then there exists an ideal \( I \) in the polynomial ring \( k[x_1, \ldots, x_n] \) such that \( R \) is isomorphic to \( k[x_1, \ldots, x_n]/I \) as a \( k \)-algebra.

Proof. By the universal property of polynomial rings (Theorem 2.11 [5]), there exists a \( k \)-algebra homomorphism \( \varphi : k[x_1, \ldots, x_n] \to R \) defined by mapping \( x_i \) to \( u_i \) for \( 1 \leq i \leq n \), so that \( \varphi(f(x_1, \ldots, x_n)) = f(u_1, \ldots, u_n) \) for \( f \in k[x_1, \ldots, x_n] \). \( \varphi \) is surjective, so the desired isomorphism is obtained by taking \( I \) to be the kernel of \( \varphi \). \( \square \)

A ring \( R \) is reduced if whenever \( f \in R \) is such that \( f^n = 0 \) for some positive integer \( n \), we have that \( f = 0 \).

Lemma 3.4. A ring \( R \) is reduced if and only if \( \sqrt{I} = I \).

Definition 3.5. Suppose \( X \subset \mathbb{A}^n \) is a closed set. The regular functions on \( X \) are the polynomial maps on \( X \),
\[
k[X] = \{ f : X \to \mathbb{A}^1 \mid f \in k[\mathbb{A}^n] \}.
\]

We have a natural surjective \( k \)-algebra homomorphism, given by restriction, \( k[\mathbb{A}^n] \to k[X] \). \( f \in k[\mathbb{A}^n] \) is in the kernel if and only if \( f(q) = 0 \) for all \( q \in X \), which holds if and only if \( f \in I(X) \). Thus
\[
k[X] \cong k[\mathbb{A}^n]/I(X).
\]
Definition 3.6. Suppose that $X$ is an affine algebraic set. If $T \subset k[X]$ then

$$Z_X(T) = \{p \in X \mid f(p) = 0 \text{ for all } p \in T\}.$$ 

Suppose that $Y \subset X$ is a closed set. Then

$$I_X(Y) = \{f \in k[X] \mid f(p) = 0 \text{ for all } p \in Y\}.$$ 

When there is no ambiguity, we will usually write $Z(T)$ for $Z_X(T)$ and $I(Y)$ for $I_X(Y)$.

Lemma 3.7. Suppose that $X$ is a closed subset of $\mathbb{A}^n$. Let $\text{res} : k[\mathbb{A}^n] \rightarrow k[X]$ be the restriction map.

1. Suppose that $Y \subset X$. Then

$$\text{res}^{-1}(I_X(Y)) = I_{\mathbb{A}^n}(Y).$$

2. Suppose that $I$ is an ideal in $k[X]$. Then

$$Z_{\mathbb{A}^n}(\text{res}^{-1}(I)) = Z_X(I).$$

Proof. $\text{res} : k[\mathbb{A}^n] \rightarrow k[X]$ is surjective with kernel $I_{\mathbb{A}^n}(X)$. We first prove 1. Since $Y \subset X$, $f \in k[\mathbb{A}^n]$ vanishes on $Y$ if and only if the restriction $\text{res}(f)$ of $f$ to $X$ vanishes on $Y$. Thus formula 1 holds.

Now we prove 2. For $p \in Z_X(I)$ and $f \in k[\mathbb{A}^n]$, $f(p) = \text{res}(f)(p)$ since $Z_X(I) \subset X$. Thus $f \in \text{res}^{-1}(I)$ implies $f(p) = 0$ for all $p \in Z_X(I)$, so that $Z_X(I) \subset Z_{\mathbb{A}^n}(\text{res}^{-1}(I))$.

0 is an ideal, so $I_{\mathbb{A}^n}(X) \subset \text{res}^{-1}(I)$. Suppose that $p \in Z_{\mathbb{A}^n}(\text{res}^{-1}(I))$. Then $p \in Z_{\mathbb{A}^n}(I_{\mathbb{A}^n}(X)) = X$. Since $\text{res}$ is surjective, $p \in X$ and $f(p) = 0$ for all $f \in \text{res}^{-1}(I)$, we have that $g(p) = 0$ for all $g \in I$. Thus $p \in Z_X(I)$. □

Theorem 3.8. Suppose that $X$ is a closed subset of $\mathbb{A}^n$. Then the conclusions of Propositions 2.3 and 2.9 hold, with $\mathbb{A}^n$ replaced with $X$.

Proof. We will establish that e) of Proposition 2.9 holds for algebraic sets. We first establish that

$$(1) \quad \sqrt{\text{res}^{-1}(a)} = \text{res}^{-1}(\sqrt{a}).$$

To prove this, observe that

\[
\begin{align*}
f \in \text{res}^{-1}(a) & \iff \text{res}(f^n) = \text{res}(f)^n \in a \text{ for some positive integer } n \\
& \iff f^n \in \text{res}^{-1}(a) \\
& \iff f \in \sqrt{\text{res}^{-1}(a)}.
\end{align*}
\]

We have that

$$\text{res}^{-1}(I_X(Z_X(a))) = I_{\mathbb{A}^n}(Z_{\mathbb{A}^n}(\text{res}^{-1}(a))) \text{ by 1 and 2 of Lemma 3.7}$$
$$= \sqrt{\text{res}^{-1}(a)} \text{ by e) of Proposition 2.9}.$$

Thus

$$I_X(Z_X(a)) = \text{res}(\text{res}^{-1}(I_X(Z_X(a)))) = \text{res}(\sqrt{\text{res}^{-1}(a)}) = \sqrt{a}$$

by (1). □

We thus obtain a topology on a closed subset $X$ of $\mathbb{A}^n$, where the closed sets are $Z_X(I)$ for ideals $I \subset k[X]$. This topology is the restriction topology of the Zariski topology on $\mathbb{A}^n$. We call this the Zariski topology on $X$.

Theorem 3.9. Suppose that $X$ is a closed subset of $\mathbb{A}^n$. A closed set $W \subset X$ is irreducible if and only if $I_X(W)$ is a prime ideal in $k[X]$. 
Corollary 3.14. Suppose that \( \varphi : X \to \mathbb{A}^m \) is a regular map if there exist \( f_1, \ldots, f_m \in k[X] \) such that \( \varphi = (f_1, f_2, \ldots, f_m) \).

A regular map \( \varphi = (f_1, \ldots, f_m) : X \to \mathbb{A}^m \) induces a \( k \)-algebra homomorphism \( \varphi^* : k[\mathbb{A}^m] \to k[X] \) by \( \varphi^* (g) = g \circ \varphi \) for \( g \in k[\mathbb{A}^m] \). Writing \( k[\mathbb{A}^m] = k[y_1, \ldots, y_m] \), we see that \( \varphi^* \) is determined by \( \varphi^* (y_i) = f_i \) for \( 1 \leq i \leq m \); For \( g = g(y_1, \ldots, y_m) \in k[\mathbb{A}^m] \), we have \( \varphi^*(g) = g(\varphi^*(y_1), \ldots, \varphi^*(y_m)) \).

Example 3.11. Let \( C = Z(y^2 - x(x^2 - 1)) \subset \mathbb{A}^2 \). Let \( \varphi : C \to \mathbb{A}^1 \) be the projection on the first factor, so that \( \varphi(u, v) = u \) for \( u, v \in C \).

\[ \varphi^* : k[\mathbb{A}^1] = k[t] \to k[C] = k[x, y]/(y^2 - x(x^2 - 1)) \] is the \( k \)-algebra homomorphism induced by \( t \mapsto \bar{x} \). Here \( \bar{x} \) is the class of \( x \) in \( k[C] \) and \( \bar{y} \) is the class of \( y \) in \( k[C] \).

Example 3.12. Let \( \psi : \mathbb{A}^1 \to \mathbb{A}^2 \) be defined by \( \psi(s) = (s^2, s^3) \) for \( s \in \mathbb{A}^1 \).

\[ \psi^* : k[\mathbb{A}^2] = k[x, y] \to k[t] \] is the \( k \)-algebra homomorphism induced by \( x \mapsto t^2 \) and \( y \mapsto t^3 \).

Proposition 3.13. Suppose that \( X \) is a closed subset of \( \mathbb{A}^n \), \( Y \) is a closed subset of \( \mathbb{A}^m \) and \( \varphi : X \to \mathbb{A}^m \) is a regular map. Then \( \varphi(X) \subset Y \) if and only if

\[ I(Y) \subset \text{kernel } \varphi^* : k[\mathbb{A}^m] \to k[X]. \]

Proof. Let \( \varphi = (f_1, \ldots, f_m) \) where \( f_1, \ldots, f_m \in k[X] = k[\mathbb{A}^n]/I(X) \). Now \( \varphi(X) \subset Y \) holds if and only if \( h(\varphi(p)) = 0 \) for all \( h \in I(Y) \) and \( p \in X \), which holds if and only if \( \varphi^*(h) = 0 \) for all \( h \in I(Y) \), which holds if and only if \( I(Y) \subset \text{kernel } \varphi^* \).

Corollary 3.14. Suppose that \( \varphi : X \to \mathbb{A}^m \) is a regular map. Then \( \sqrt{\text{kernel } \varphi^*} = \text{kernel } \varphi^* \), and \( \varphi(X) = Z(\text{kernel } \varphi^*) \).

Proof. The fact that \( \sqrt{\text{kernel } \varphi^*} = \text{kernel } \varphi^* \) follows from the fact that \( k[\mathbb{A}^m]/\text{kernel } \varphi^* \) is isomorphic to a subring of the reduced ring \( k[X] \).

Example 3.15. A regular map may not be closed or open. Let \( \varphi : \mathbb{A}^2 \to \mathbb{A}^2 \) be defined by \( \varphi(u, v) = (u, uv) \). Then

\[ \varphi(\mathbb{A}^2) = \mathbb{A}^2 \setminus \{(0, y) \mid y \neq 0 \}. \]

Definition 3.16. Suppose that \( X \subset \mathbb{A}^n \) and \( Y \subset \mathbb{A}^m \) are closed sets. A map \( \varphi : X \to Y \) is a regular map if \( \varphi \) is the restriction of the range of a regular map \( \tilde{\varphi} : X \to \mathbb{A}^m \), such that \( \tilde{\varphi}(X) \subset Y \).

Suppose that \( \varphi : X \to Y \) is a regular map as in the definition. Let \( \pi : k[\mathbb{A}^m] = k[y_1, \ldots, y_m] \to k[Y] \) be the restriction map, which has kernel \( I(Y) \). We have that \( \tilde{\varphi}(X) \subset Y \), so \( I(Y) \subset \text{kernel } (\tilde{\varphi}^*) \) by Proposition 3.13. Thus \( \tilde{\varphi}^* \) induces a \( k \)-algebra homomorphism \( \varphi^* : k[Y] \cong k[\mathbb{A}^m]/I(Y) \to k[X] \) by Lemma 1.1, so that \( \varphi^*(f) = f \circ \varphi \) for \( f \in k[Y] \).

Thus writing \( \varphi = (f_1, \ldots, f_m) \), where \( f_1, \ldots, f_m \in k[X] \), and \( k[Y] = k[y_1, \ldots, y_m] \), where \( \bar{y}_i = \pi(y_i) \) for \( 1 \leq i \leq m \) are the restrictions of \( y_i \) to \( Y \), we have that \( f_i = \varphi^*(\bar{y}_i) = \tilde{\varphi}^*(y_i) \) for \( 1 \leq i \leq m \), and for \( g(\bar{y}_1, \ldots, \bar{y}_m) \in k[Y] \), we have that \( \varphi^*(g) = g(\varphi^*(\bar{y}_1), \ldots, \varphi^*(\bar{y}_m)) = g(f_1, \ldots, f_m) \).

Proposition 3.17. Suppose that \( \varphi : X \to Y \) is a regular map of affine algebraic sets and \( Z \subset Y \) is a closed set. Then \( \varphi^{-1}(Z) = Z(\varphi^*(I(Z))) \).
Corollary 3.18. Suppose that $X$ and $Y$ are affine algebraic sets, and $\varphi : X \to Y$ is a regular map. Then $\varphi$ is continuous.

Proposition 3.19. Suppose that $\varphi : X \to Y$ is a regular mapping of affine algebraic sets. Then $\varphi^* : k[Y] \to k[X]$ is 1-1 if and only if $\overline{\varphi(X)} = Y$.

Proof. This follows from the definition of a regular map and Corollary 3.14. \qed

Lemma 3.20. Suppose that $\varphi : X \to Y$ and $\psi : Y \to Z$ are regular mappings of affine algebraic sets. Then $\psi \circ \varphi : X \to Z$ is a regular mapping of affine algebraic sets. Further, $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

Proposition 3.21. Suppose that $X$ and $Y$ are affine algebraic sets, and $\Lambda : k[Y] \to k[X]$ is a $k$-algebra homomorphism. Then there is a unique regular mapping $\varphi : X \to Y$ such that $\varphi^* = \Lambda$.

Proof. We first prove existence. We have a closed immersion of $Y$ in $\mathbb{A}^n$, giving a surjective $k$-algebra homomorphism $\pi : k[\mathbb{A}^n] = k[y_1, \ldots, y_n] \to k[Y]$. Let $\overline{y}_i = \pi(y_i)$ for $1 \leq i \leq n$, so that $k[Y] = k[\overline{y}_1, \ldots, \overline{y}_n]$. Define a regular map $\tilde{\varphi} : X \to \mathbb{A}^n$ by $\tilde{\varphi} = (\Lambda(\overline{y}_1), \ldots, \Lambda(\overline{y}_n))$.

Proposition 3.13. Let $f : k[\mathbb{A}^n] \to k[M]$ be a regular $k$-algebra homomorphism. Then there is a unique regular mapping $\varphi : X \to Y$ such that $\varphi^* = f$.

Thus $\tilde{\varphi}(X) \subseteq Y$ by Proposition 3.13. Let $\varphi : X \to Y$ be the induced regular map. $\varphi^*$ is the homomorphism induced by $\tilde{\varphi}^*$ on the quotient $k[y_1, \ldots, y_n]/I(Y) = k[Y]$. Thus $\varphi^* = \Lambda$.

We now prove uniqueness. Suppose that $\varphi : X \to Y$ and $\Psi : X \to Y$ are regular maps such that $\varphi^* = \psi^* = \Lambda$. Suppose that $\varphi \neq \Psi$. Then there exists $p \in X$ such that $\varphi(p) \neq \psi(p)$. Let $q_1 = \varphi(p)$ and $q_2 = \psi(p)$. There exists $f \in I(q_1) \setminus I(q_2)$ since $I(q_1)$ and $I(q_2)$ are distinct maximal ideals of $k[Y]$. Thus $f(q_1) = 0$ but $f(q_2) \neq 0$. We have

$$ (\varphi^* f)(p) = f(\varphi(p)) = f(q_1) = 0 $$

but

$$ (\psi^* f)(p) = f(\psi(p)) = f(q_2) \neq 0. $$

Thus $\varphi^* \neq \psi^*$, a contradiction, so we must have that $\varphi = \psi$, and thus $\varphi$ is unique. \qed

Definition 3.22. Suppose that $X$ and $Y$ are affine algebraic sets. We say that $X$ and $Y$ are isomorphic if there are regular mappings $\varphi : X \to Y$ and $\psi : Y \to X$ such that $\psi \circ \varphi = id_X$ and $\varphi \circ \psi = id_Y$.

Proposition 3.23. Suppose that $\varphi : X \to Y$ is a regular mapping of affine algebraic sets. Then $\varphi$ is an isomorphism if and only if $\varphi^* : k[Y] \to k[X]$ is an isomorphism of $k$-algebras.

Proof. First suppose that the regular map $\varphi : X \to Y$ is an isomorphism. Then there exists a regular map $\psi : Y \to X$ such that $\psi \circ \varphi = id_X$ and $\varphi \circ \psi = id_Y$. Thus $(\psi \circ \varphi)^* = id_{k[X]}$ and $(\varphi \circ \psi)^* = id_{k[Y]}$. Now $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ and $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ by Lemma 3.20, so $\varphi^* : k[Y] \to k[X]$ is a $k$-algebra isomorphism with inverse $\psi^*$.
Now assume that \( \varphi^* : k[Y] \to k[X] \) is a \( k \)-algebra isomorphism. Let \( \Lambda : k[X] \to k[Y] \) be the \( k \)-algebra inverse of \( \varphi^* \). By Proposition 3.21, there exists a unique regular map \( \psi : Y \to X \) such that \( \psi^* = \Lambda \). Now by Lemma 3.20,
\[
(\psi \circ \varphi)^* = \varphi^* \circ \psi^* = \varphi^* \circ \Lambda = \text{id}_{k[X]}
\]
and
\[
(\varphi \circ \psi)^* = \psi^* \circ \varphi^* = \Lambda \circ \varphi^* = \text{id}_{k[Y]}.
\]
Since \( \text{id}_X^* = \text{id}_{k[X]} \), by uniqueness in Proposition 3.21, we have that \( \psi \circ \varphi = \text{id}_X \). Similarly, \( \varphi \circ \psi = \text{id}_Y \). Thus \( \varphi \) is an isomorphism. \( \square \)

**Definition 3.24.** Suppose that \( X \) is an affine algebraic set and \( t_1, \ldots, t_r \in k[X] \) are such that \( \varphi = (t_1, \ldots, t_r) \). Then \( t_1, \ldots, t_r \) are called coordinate functions on \( X \) and \( \varphi \) is called a closed immersion.

**Proposition 3.25.** Suppose that \( X \) is an affine algebraic set and \( t_1, \ldots, t_r \) are coordinate functions on \( X \). Let \( \varphi : X \to \mathbb{A}^r \) be the associated closed immersion \( \varphi = (t_1, \ldots, t_r) \), and let \( Y = \varphi(X) \). Then \( Y \) is a closed subset of \( \mathbb{A}^r \) with ideal \( I(Y) = \ker \varphi^* : k[\mathbb{A}^r] \to k[X] \) and regarding \( \varphi \) as a regular map to \( Y \), we have that \( \varphi : X \to Y \) is an isomorphism.

**Proof.** Let \( Y \) be the Zariski closure of \( Y \) in \( \mathbb{A}^r \). \( I(Y) = \ker \varphi^* \) by Corollary 3.14. Thus \( \varphi^* : k[\mathbb{A}^r] \to k[X] \) onto with kernel \( I(Y) \), so that now regarding \( \varphi \) as a regular map from \( X \) to \( Y \), we have that \( \varphi^* : k[\mathbb{A}^r] \to k[X] \) is an isomorphism. Thus \( Y = Y \) and \( \varphi : X \to Y \) is an isomorphism by Proposition 3.23. \( \square \)

Our definition of \( \mathbb{A}^n = \{ p = (a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in k \} \) gives us a particular system of coordinates, namely the coordinate functions \( x_i(p) = a_i \) for \( 1 \leq i \leq n \). If \( B = (b_{ij}) \) is an invertible \( n \times n \) matrix with coefficients in \( k \), and \( c = (c_1, \ldots, c_n) \) is a vector in \( k^n \), then \( y_i = \sum_{j=1}^n b_{ij} x_j + c_i \) for \( 1 \leq i \leq n \) defines another coordinate system \( y_1, \ldots, y_n \) on \( \mathbb{A}^n \).

**Lemma 3.26.** Suppose that \( \varphi : X \to Y \) is a regular map of affine algebraic sets and \( t_1, \ldots, t_n \) are coordinate functions on \( Y \) (giving a closed immersion of \( Y \) in \( \mathbb{A}^n \)). Suppose that \( p = (\alpha_1, \ldots, \alpha_n) \in Y \). Then \( I(p) = (t_1 - \alpha_1, \ldots, t_n - \alpha_n) \) and
\[
I(\varphi^{-1}(p)) = \sqrt{(\varphi^*(t_1) - \alpha_1, \ldots, \varphi^*(t_n) - \alpha_n)}.
\]

4. **Finite maps**

**Definition 4.1.** Suppose that \( R \) is a ring. An \( R \)-module \( M \) is an abelian group with a map \( R \times M \to M \) such that for all \( a, b \in R \) and \( x, y \in M \),
\[
\begin{align*}
\quad a(x + y) &= ax + ay \\
\quad (a + b)x &= ax + bx \\
\quad (ab)x &= a(bx) \\
\quad 1x &= x.
\end{align*}
\]

All ideals in \( R \) are \( R \)-modules. \( R^m \) for \( m \in \mathbb{Z}_+ \) is an \( R \)-module.

**Definition 4.2.** A map \( f : M \to N \) of \( R \)-modules is an \( A \)-module homomorphism if for all \( x, y \in M \) and \( a \in R \),
\[
\begin{align*}
\quad f(x + y) &= f(x) + f(y) \\
\quad f(ax) &= af(x).
\end{align*}
\]

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An $R$-module $M$ is a finitely generated $R$-module if there exists $n \in \mathbb{Z}_+$ and $f_1, \ldots, f_n \in M$ such that $M = \{r_1 f_1 + \cdots + r_n f_n \mid r_1, \ldots, r_n \in R\}$.

**Definition 4.3.** Suppose that $R$ is a subring of a ring $S$. $u \in S$ is integral over $R$ if $u$ satisfies a relation

$$u^n + a_1 u^{n-1} + \cdots + a_{n-1} u + a_n = 0$$

with $a_1, \ldots, a_n \in R$.

**Theorem 4.4.** (Proposition 5.1 [2]) Suppose that $R$ is a subring of a ring $S$, and $u \in S$. The following are equivalent:

1. $u$ is integral over $S$.
2. $R[u]$ is a finitely generated $R$-module.
3. $R[u]$ is contained in a subring $T$ of $S$ such that $T$ is a finitely generated $R$-module.

**Corollary 4.5.** Let $u_1, \ldots, u_n$ be elements of $S$ which are each integral over $R$. Then the subring $R[u_1, \ldots, u_n]$ of $S$ is a finitely generated $R$-module.

**Corollary 4.6.** Let

$$R = \{u \in S \mid u \text{ is integral over } R\}.$$

Then $R$ is a ring.

**Proof.** If $x, y \in R$ then the subring $R[x, y]$ of $S$ is a finitely generated $R$-module, by Corollary 4.5. $x + y$ and $xy$ are in $R[x, y]$ so $x + y$ and $xy$ are integral over $R$ by 3 implies 1 of Theorem 4.4. $\square$

$R$ is called the integral closure of $R$ in $S$. This construction is particularly important when $R$ is a domain and $S$ is the quotient field of $R$. In this case, $R$ is called the normalization of $R$. $R$ is said to be normal if $\overline{R} = R$. If $R$ is a domain and $S$ is a finite field extension of the quotient field of $R$, then the integral closure of $R$ in $S$ is called the normalization of $R$ in $S$. The following theorem is extremely important in algebraic geometry.

**Theorem 4.7.** (Theorem 9, page 67 [11]) Let $R$ be an integral domain which is finitely generated over a field $k$. Let $K$ be the quotient field of $R$ and let $L$ be a finite algebraic extension of $K$. Then the integral closure $R'$ of $R$ in $L$ is a finitely generated $R$-module, and is also a finitely generated $k$-algebra.

**Lemma 4.8.** Suppose that $M$ is a finitely generated $R$ module and $N$ is a submodule of $M$. Then $N$ is a finitely generated $R$-module.

We will need the following lemma.

**Lemma 4.9.** Suppose that $R$ is a subring of a ring $S$ and $S$ is a finitely generated $R$-module. Suppose that $I$ is a proper ideal in $R$. Then $IS \neq S$.

**Proof.** Assume that $IS = S$. We will derive a contradiction. There exist $n \in \mathbb{Z}_+$ and $f_1, \ldots, f_n \in S$ such that $S = Rf_1 + \cdots + Rf_n$. $IS = S$ implies $f_i \in IS = If_1 + \cdots + If_n$. Thus we have expressions

$$f_i = \sum_{j=1}^{n} \alpha_{ij} f_j$$
for $1 \leq i \leq n$, with $\alpha_{ij} \in I$. Let $A = (\alpha_{ij})$, an $n \times n$ matrix with coefficients in $I$. The system of equations (2) can be written in matrix form as

\[(I_n - A) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = 0.\]

Let $d = \det(I_n - A)$. We multiply both sides of equation (3) by the adjoint matrix $\text{adj}(I_n - A)$ and use the identity $\text{Adj}(B)B = \det(B)I_n$ for any $n \times n$ matrix $B$, to obtain $d f_i = 0$ for $1 \leq i \leq n$. Now $1 \in S$ so $1 = \sum_{i=1}^{m} r_i f_i$ for some $r_1, \ldots, r_n \in R$. Thus $d = d1 = 0$. Expanding the determinant $d$, we see that $0 = \det(I_n - A) = 1 - \alpha$ for some $\alpha \in I$. Thus $1 \in I$, a contradiction to our assumption that $I$ is a proper ideal in $E$. \hfill \Box

**Definition 4.10.** Suppose that $f : X \to Y$ is a regular map of affine varieties. $f$ is finite if $f(X)$ is dense in $Y$ (which is equivalent to $f^* : k[Y] \to k[X]$ is 1-1 by Proposition 3.19), and $k[X]$ is integral over the subring $f^*(k[Y])$.

It may sometimes be convenient to abuse notation, and identify $k[Y]$ with its isomorphic image $f^*(k[Y])$.

**Theorem 4.11.** Suppose that $f : X \to Y$ is a finite map of affine varieties. Then $f^{-1}(p)$ is a finite set for all $p \in Y$.

**Proof.** Let $t_1, \ldots, t_n$ be coordinate functions on $X$. It suffices to show that each $t_i$ assumes only finitely many values on $f^{-1}(p)$. Since $k[X]$ is integral over $k[Y]$, each $t_i$ satisfies a dependence relation

$$t_i^m + f^*(b_{m-1})t_i^{m-1} + \cdots + f^*(b_0) = 0$$

with $m \in \mathbb{Z}_+$ and $b_0, \ldots, b_{m-1} \in k[Y]$. Suppose that $q \in f^{-1}(p)$. Then

$$t_i(q)^m + b_1(p)t_i(q)^{m-1} + \cdots + b_0(p) = t_i(q)^m + f^*(b_1)(q)t_i(q)^{m-1} + \cdots + f^*(b_0)(q) = 0.$$ 

Thus $t_i(q)$ must be one of the $\leq m$ roots of this equation. \hfill \Box

**Theorem 4.12.** Suppose that $f : X \to Y$ is a finite map of affine varieties. Then $f$ is surjective.

**Proof.** Let $q \in Y$. Let $m_q = I(q)$ be the ideal of $q$ in $k[Y]$. $f^{-1}(q) = Z(f^*(m_q))$ by Proposition 3.17. Now $f^{-1}(q) = \emptyset$ if and only if $m_q k[X] = k[X]$. By Lemma 4.9, $m_q k[X]$ is a proper ideal of $k[X]$, since $m_q$ is a proper ideal of $k[Y]$. \hfill \Box

**Corollary 4.13.** A finite map $f : X \to Y$ of affine varieties is closed.

**Proof.** It suffices to verify that if $Z \subset X$ is an irreducible closed subset, then $f(Z)$ is closed in $Y$. Let $W = F(Z)$ be the closure of $f(Z)$ in $Y$. Let $\overline{f} = f[Z] : Z \to W$. $f^* : k[Y] \to k[X]$ induces the homomorphism $\overline{f}^* : k[W] = k[Y]/I(W) \to k[X]/I(Z) = k[Z]$. $\overline{f}^*$ is 1-1 by Proposition 3.19. $k[Z]$ is integral over $k[W]$ since $k[X]$ is integral over $k[Y]$. Thus $\overline{f} : Z \to W$ is a finite mapping, which is surjective by Theorem 4.12. Thus $f(Z) = W$ is closed in $Y$. \hfill \Box

Suppose that $R$ is a $k$-algebra and $y_1, \ldots, y_r$ are elements of $R$. $y_1, \ldots, y_r$ are algebraically independent over $k$ if the surjective $k$-algebra homomorphism $k[x_1, \ldots, x_r] \to k[y_1, \ldots, y_r]$ defined by mapping $x_i$ to $y_i$ for $1 \leq i \leq r$ is an isomorphism.
Corollary 4.15. Suppose that \( X \) is an affine algebraic set. Then there exists a finite map \( \varphi : X \rightarrow \mathbb{A}^r \) for some \( r \).

Proof. There exist, by Theorem 4.14, \( y_1, \ldots, y_r \in k[X] \) such that \( k[y_1, \ldots, y_r] \) is a polynomial ring and \( k[X] \) is integral over \( k[y_1, \ldots, y_r] \). Define a regular map \( \varphi : X \rightarrow \mathbb{A}^r \) by \( \varphi(p) = (y_1(p), y_2(p), \ldots, y_r(p)) \). Let \( t_1, \ldots, t_n \) be the coordinate functions on \( \mathbb{A}^r \). Then \( \varphi^* : k[\mathbb{A}^n] \rightarrow k[X] \) is the \( k \)-algebra homomorphism defined by \( \varphi^*(t_i) = y_i \) for \( 1 \leq i \leq r \). Thus \( \varphi^* \) is 1-1 and \( k[X] \) is integral over \( k[\mathbb{A}^n] \). \( \square \)

5. Dimension

Corollary 4.15 gives us an intuitive way to define the dimension of an affine variety: an affine variety \( X \) has dimension \( r \) if there is a finite map from \( X \) to \( \mathbb{A}^r \). We will take a more algebraic approach to define dimension, and then show that the intuitive approach does give the dimension (in particular, the intuitive dimension is well defined).

Definition 5.1. Suppose that \( X \) is a topological space. The dimension of \( X \), denoted \( \dim X \), is the supremum of all natural numbers \( n \) such that there exists a chain

\[
Z_0 \subset Z_1 \subset \cdots \subset Z_n
\]

of distinct irreducible closed subsets of \( X \). The dimension of an affine algebraic set or quasi-affine algebraic set is its dimension as a topological space.

Definition 5.2. The height of a prime ideal \( P \) in a ring \( R \) is the supremum of all natural numbers \( n \) such that there exists a chain

\[
P_0 \subset P_1 \subset \cdots \subset P_n = P
\]

of distinct prime ideals. The dimension of \( R \) is the supremum of the heights of all prime ideals in \( R \).

Proposition 5.3. Suppose that \( X \) is an affine algebraic set. Then the dimension of \( X \) is equal to the dimension of the ring \( k[X] \) of regular functions on \( X \).

Proof. By Theorems 3.8 and 3.9, chains

\[
Z_0 \subset Z_1 \subset \cdots \subset Z_n
\]

of distinct irreducible closed subsets of \( X \) correspond 1-1 to chains of distinct prime ideals

\[
I_X(Z_n) \subset I_X(Z_{n-1}) \subset \cdots \subset I_X(Z_0)
\]

of distinct prime ideals in \( k[X] \). \( \square \)

Suppose that \( R \) is a Noetherian ring and \( I \subset R \) is an ideal. A prime ideal \( P \) in \( R \) is called a minimal prime of \( I \) if \( I \subset P \) and if \( Q \) is a prime ideal of \( R \) such that \( I \subset Q \subset R \) then \( Q = P \). Since \( R \) is Noetherian, the ideal \( I \) has only a finite number of minimal primes \( P_1, \ldots, P_r \). We have that \( I \subset P_1 \cap \cdots \cap P_r \) and \( I = P_1 \cap \cdots \cap P_r \) if and only if \( I \) is a radical ideal (\( \sqrt{I} = I \)).

Suppose that \( X \) is a closed subset of \( \mathbb{A}^n \). Let \( V_1, \ldots, V_r \) be the irreducible components of \( X \); that is \( V_1, \ldots, V_r \) are the irreducible closed subsets of \( X \) such that \( X = V_1 \cup \cdots \cup V_r \) and we have \( V_i \not\subset V_j \) if \( i \neq j \). Then the minimal primes of \( I(X) \) are \( P_i = I(V_i) \) for
The prime ideals $\mathcal{P}_i = I_X(V_i) = P_i/I(X)$ are the minimal primes of the ring $k[X]$; that is the minimal primes of the zero ideal. We have that $\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r = (0)$ since $I(X)$ is a radical ideal.

**Proposition 5.4.** Suppose that $X$ is an affine algebraic set, and $V_1, \ldots, V_r$ are the irreducible components of $X$ (the distinct largest irreducible sets contained in $X$). Then

$$\dim X = \max\{\dim V_i\}.$$ 

**Proof.** Suppose that (4) is a chain of irreducible closed subsets of $X$. Then $Z_n$ is contained in $V_i$ for some $i$, since $Z_n$ is irreducible and $V_i$ are the irreducible components of $X$. □

**Theorem 5.5.** (Theorem A.16, [4], chapter 11, [2]) Let $A$ be a finitely generated $k$-algebra ($A$ is a quotient of a polynomial ring in finitely many variables) which is a domain. For any prime ideal $p$ in $A$ we have that

$$\text{height } p + \dim A/p = \dim A.$$ 

A chain (4) is maximal if the chain cannot be lengthened by adding an additional irreducible closed set somewhere in the chain. A chain (5) is maximal if the chain cannot be lengthened by adding an additional prime ideal somewhere in the chain.

**Corollary 5.6.** Suppose that $X$ is an affine variety. Then $k[X]$ has finite dimension. Every maximal chain of distinct prime ideals in $k[X]$ has the same finite length equal to the dimension of $k[X]$.

**Proof.** Suppose that

$$P_0 \subset P_1 \subset \cdots \subset P_n$$

is a maximal chain of distinct prime ideals. $P_n$ is a maximal ideal since every ideal is contained in a maximal ideal and a maximal ideal is a prime ideal. Thus the chain has length equal to the height of $P_n$. But $k[X]/P_n$ is a maximal ideal, so $\dim k[X] = 0$. Thus $n = \dim k[X]$ by Theorem 5.5. □

An example of a Noetherian ring which has infinite dimension is given in Example 1 of Appendix A1, page 203, of [7].

**Corollary 5.7.** Suppose that $X$ is an affine variety. Then every maximal chain of distinct irreducible closed subsets of $X$ has the same length (equal to $\dim X$).

**Proof.** Suppose that (4) is a maximal chain of distinct irreducible closed subsets of $X$. Since $X$ is irreducible, we must have that $Z_n = X$ and $Z_0$ is a point. Taking the sequence of ideals of (4), we have a maximal chain $(0) = I_X(Z_n) \subset \cdots \subset I_X(Z_0)$ of distinct prime ideals in $k[X]$. By Theorem 5.5 and Proposition 5.3, we have that $n = \dim k[X] = \dim X$. □

**Proposition 5.8.** Suppose that $X$ is an affine variety and $Y$ is a nontrivial open subset of $X$. Then $\dim X = \dim Y$.

**Proof.** Suppose that

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

is a sequence of distinct closed irreducible subsets of $Y$. Let $\overline{Z}_i$ be the Zariski closure of $Z_i$ in $X$ for $0 \leq i \leq n$. Then

$$\overline{Z}_0 \subset \overline{Z}_1 \subset \cdots \subset \overline{Z}_n$$

is a sequence of distinct closed irreducible subsets of $X$. Thus $\dim Y \leq \dim X$. In particular, $\dim Y$ is finite, so we can choose a maximal such chain. Since the chain is
maximal, \( Z_0 \) is a point and \( Z_\alpha = Y \). Now if \( W \) is an irreducible closed subset of \( X \) such that the open subset \( W \cap Y \) of \( W \) is non empty, we then have that \( W \cap Y \) is dense in \( W \). In particular, if \( A \subset B \) are irreducible closed subsets of \( X \) such that \( A \cap Y \neq \emptyset \) and \( A \cap Y = B \cap Y \), then we have that \( A = A \cap Y = B \cap Y = B \). Thus we have that (7) is a maximal chain in \( X \), and hence \( \dim Y = \dim X \) by Corollary 5.7.

Suppose that \( K \) is a field extension of the field \( k \). A set of elements \( S \) of \( K \) are algebraically independent over \( k \) if whenever \( n \in \mathbb{Z}_+, z_1, \ldots, z_n \in S \) and \( f(x_1, \ldots, x_n) \in k[x_1, \ldots, x_n] \) is a nonzero polynomial, we have that \( f(z_1, \ldots, z_n) \neq 0 \). \( S \) is a transcendence basis of \( K \) over \( k \) if the elements of \( S \) are algebraically independent over \( k \), and if \( K \) is algebraic over \( k(S) \). The cardinality of a transcendence basis of \( K \) over \( k \) is independent of choice of transcendence basis \( S \) (Theorem 25, page 99 [11]). We denote this cardinality by \( \text{trdeg}_k K \).

**Theorem 5.9.** (Theorem A.16 [4], Chapter 11 [2]) Let \( A \) be a finitely generated \( k \)-algebra which is a domain. Let \( K \) be the quotient field of \( A \). Then \( \dim A = \text{trdeg}_k K \), the transcendence degree of \( K \) over \( k \).

**Corollary 5.10.** The dimension of \( \mathbb{A}^n \) is \( \dim \mathbb{A}^n = n \).

Noether’s normalization lemma, Theorem 4.14, can be used to compute the dimension of any affine variety. We see that when \( k = \mathbb{C} \), The usual dimension of a complex variety computed using the Euclidean topology is equal to the dimension we have defined in this section, which is computed using the Zariski topology.

**Theorem 5.11.** (Krull’s Principal Ideal Theorem, Corollary 11.17, [2]) Let \( A \) be a noetherian ring, and let \( f \in A \) be an element which is neither a zero divisor nor a unit. Then every minimal prime ideal \( p \) containing \( f \) has height 1.

**Corollary 5.12.** Suppose that \( X \) is an affine variety and \( f \in k[X] \). Then

1. If \( f \) is not 0 and is not a unit in \( k[X] \) then \( Z_X(f) \cap X \) is a non empty algebraic set, all of whose irreducible components have dimension equal to \( \dim X - 1 \).
2. If \( f \) is a unit in \( k[X] \) then \( Z_X(f) = \emptyset \).
3. If \( f = 0 \) then \( Z_X(f) = X \).

**Proposition 5.13.** (Theorem 20.1 [6], Chapter 7, Section 3, [3]) A noetherian integral domain \( A \) is a unique factorization domain if and only if every prime ideal of height 1 in \( A \) is principal.

**Proposition 5.14.** Suppose that \( X \) is a variety in \( \mathbb{A}^n \). Then \( X \) has dimension \( n - 1 \) if and only if \( I(X) = (f) \) where \( f \in k[\mathbb{A}^n] = k[x_1, \ldots, x_n] \) is an irreducible polynomial.

**Proof.** Suppose that \( I(X) = (f) \) where \( f \) is irreducible. Then \( (f) \) is a prime ideal, which has height one by Theorem 5.11. Thus

\[
\dim X = \dim k[X] = \dim k[\mathbb{A}^n] - 1 = \dim \mathbb{A}^n - 1
\]

by Theorem 5.5.

Now suppose that \( X \) has dimension \( n - 1 \). Then the prime ideal \( I(X) \) has height 1 by Theorem 5.5. Since the polynomial ring \( k[\mathbb{A}^n] \) is a unique factorization domain, \( I(X) \) is a principal ideal generated by an irreducible element by Proposition 5.13.

If \( Y \) is an affine or quasi affine algebraic set contained in an affine variety \( X \), then we define the codimension of \( Y \) in \( X \) to be \( \text{codim}_X(Y) = \dim X - \dim Y \). If \( Y \) is a subvariety
of an algebraic variety $X$, then we have that \( \text{codim}_X(Y) \) is the height of the prime ideal \( I_X(Y) \) in \( k[X] \).

More generally, suppose that \( X \) is an \( n \)-dimensional affine variety and \( Y \subset X \) is an algebraic set with irreducible components \( Y_1, \ldots, Y_s \). We have

\[
\text{codim}_X(Y) = \dim(X) - \dim(Y) \\
= \dim(X) - \max\{\dim(Y_i)\} \quad \text{(by Proposition 5.4)} \\
= \min\{n - \dim(Y_i)\} \\
= \min\{\text{height } I(Y_i)\}.
\]

We will call a one dimensional affine variety a curve, and a two dimensional affine variety a surface. An \( n \)-dimensional affine variety is called an \( n \)-fold.

We see that if \( C \) is a curve, then the prime ideals in \( k[C] \) are just the maximal ideals (corresponding to the points of \( C \)) and the zero ideal (corresponding to the curve \( C \)). If \( S \) is a surface, and \( k[S] \) is a UFD, then the prime ideals in \( k[S] \) are the maximal ideals (corresponding to the points of \( S \)), principal ideals generated by an irreducible element (corresponding to the curves lying on \( S \)) and the zero ideal (corresponding to the surface \( S \)). If \( X \) is an \( n \)-fold with \( n \geq 3 \), then the prime ideals in \( k[X] \) are much more complicated. The prime ideals in \( k[A^3] \) are the maximal ideals (height 3), height 2 prime ideals, principal ideals generated by an irreducible element (height 1) and the zero ideal (height 0). The height 2 prime ideals \( p \), which correspond to curves in \( A^3 \), can be extremely complicated, although many times one has the nice case where \( p \) is generated by two elements. A height 2 prime \( p \) in \( k[A^3] \) requires at least two generators but there is no upper bound on the minimum number of generators required to generate such a prime \( p \) [1].

6. Regular functions on quasi-affine varieties

**Lemma 6.1.** Suppose that \( X \) is an affine algebraic set. Then the open sets \( D(f) = X \setminus Z(f) \) for \( f \in k[X] \) form a basis of the Zariski topology on \( X \).

**Proof.** We must show that given an open subset \( U \) of \( X \) and a point \( q \in U \), there exists \( f \in k[X] \) such that \( q \in X \setminus Z(f) \). Set \( m = I(q) \). There exists an ideal \( I \) in \( k[X] \) such that \( U = X \setminus Z(I) \). \( q \in U \) implies \( q \notin Z(I) \) which implies \( I \nsubseteq m \). Thus there exists \( f \in I \) such that \( f \notin m \). Then \( Z(I) \subset Z(f) \) implies \( X \setminus Z(f) \subset U \). \( f \notin m \) implies \( q = Z(m) \notin Z(f) \) so that \( q \in X \setminus Z(f) \). \( \square \)

The process of localization is systematically developed in [2]. We need to know the following two constructions. Suppose that \( R \) is a domain with quotient field \( K \). If \( 0 \neq f \in R \) then \( R_f \) is defined to be the following subring of \( K \):

\[
R_f = R[\frac{1}{f}] = \{ \frac{g}{f^n} \mid g \in R \text{ and } n \in \mathbb{Z}_+ \}.
\]

If \( p \) is a prime ideal in \( R \) then \( R_p \) is defined to be the following subring of \( K \):

\[
R_p = \{ \frac{f}{g} \mid f \in R \text{ and } g \in R \setminus p \}.
\]

\( R_p \) is a local ring: its unique maximal ideal is \( pR_p \).

**Lemma 6.2.** (Proposition 3.11 [2]) The prime ideals \( b \) in \( R_p \) are in 1-1 correspondence with the prime ideals \( a \) in \( R \) which are contained in \( p \) by the maps \( a \mapsto aR_p \) and \( b \mapsto b \cap R \). The prime ideals \( b \) in \( R_f \) are in 1-1 correspondence with the prime ideal \( a \) in \( R \) which do not contain \( f \) by the maps \( a \mapsto aR_f \) and \( b \mapsto b \cap R \).
Suppose that \( X \) is an affine variety, and \( p \in X \). Let \( k(X) \) be the quotient field of \( k[X] \). \( k(X) \) is called the field of rational functions on \( X \). We have that the localization
\[
k[X]_{I(p)} = \{ \frac{f}{g} \mid f, g \in k[X] \text{ and } g(p) \neq 0 \}.
\]
These are the elements of \( k(X) \) which are defined at \( p \). Suppose that \( U \) is an open set. Define the regular functions on \( U \) to be
\[
O_X(U) = \cap_{p \in U} k[X]_{I(p)},
\]
where the intersection is over all \( p \in U \). The intersection is in \( k(X) \). For \( p \in X \), we define the local ring
\[
O_{X,p} = \cup_{p \in U} O_X(U),
\]
where the union is over all open sets \( U \) containing \( p \).

Suppose that \( U \subset V \) are open subsets of \( X \) and \( p \in V \). We then have injective restriction maps
\[
(8) \quad O_X(V) \rightarrow O_X(U)
\]
and
\[
(9) \quad O_X(V) \rightarrow O_{X,p}.
\]

**Proposition 6.3.** Suppose that \( 0 \neq f \in k[X] \). Then \( O_X(D(f)) = k[X]_f \).

**Proof.** We have \( k[X]_f = \{ \frac{g}{n} \mid g \in k[X] \text{ and } n \in \mathbb{Z}_{+} \} \). If \( \frac{g}{n} \in k[X]_f \) then \( \frac{g}{n} \in k[X]_{I(p)} \) for all \( p \in D(f) \) since \( f(p) \neq 0 \). Thus \( k[X]_f \subset O_X(D(f)) \).

Suppose that \( h \in O_X(U) \), which is a subset of \( K \). Let \( B = \{ g \in k[X] \mid gh \in k[X] \} \). If we can prove that \( f^n \in B \) for some \( n \), then we will have that \( h \in k[X]_f \), and we will be through. By assumption, if \( p \in D(f) \), then \( h \in k[X]_{I(p)} \), so there exist functions \( a, b \in k[X] \) such that \( h = \frac{a}{b} \) with \( b(p) \neq 0 \). Then \( bh = a \in k[X] \) so \( b \in B \), and \( B \) contains an element not vanishing at \( p \). Thus \( Z(B) \subset Z(f) \). We have \( f \in \sqrt{B} \) by the nullstellensatz. \( \square \)

In particular, we have that for any affine variety \( X \),
\[
(10) \quad k[X] = O_X(X)
\]

The above proposition shows that if \( U = D(f) \) for some \( f \in k[X] \), then every element of \( O_X(U) \) has the form \( \frac{a}{b} \) where \( a, b \in k[X] \) and \( b(p) \neq 0 \) for \( p \in U \). This desirable property fails in general, as follows from the following example. Let \( X = Z(xw - yz) \subset \mathbb{A}^4 \), and \( U = D(y) \cup D(w) = X \setminus Z(y, w) \). \( k[X] = k[\mathbb{A}^4]/I(X) = k[\overline{\mathbb{F}}, \overline{\mathbb{H}}, \overline{\mathbb{W}}, \overline{\mathbb{W}}] \). Let \( h \in O_X(U) \) be defined by \( h = \frac{f}{g} \) on \( D(\overline{\mathbb{F}}) \) and \( h = \frac{x}{z} \) on \( D(\overline{\mathbb{W}}) \). We have that \( \frac{x}{z} = \frac{\overline{\mathbb{F}}}{\overline{\mathbb{W}}} \) on \( D(\overline{\mathbb{F}}) \cap D(\overline{\mathbb{W}}) = X - Z(\overline{\mathbb{F}}) \) so that \( h \) is a well defined function on \( U \).

Now suppose that \( h = \frac{f}{g} \) where \( f, g \in k[X] \) and \( g \) does not vanish on \( U \). We will derive a contradiction. Let \( Z = Z_X(\overline{\mathbb{F}}, \overline{\mathbb{W}}) \). \( Z \) is a plane in \( X \) \( (k[Z] = k[x, y, z, w]/(xw - yz, y, w) \cong k[x, z]) \). We have that \( U = X \setminus Z \). Thus \( Z_X(g) \subset Z \). Suppose that \( g \) does not vanish on \( X \). Then \( \frac{x}{z} = \frac{\overline{\mathbb{F}}}{\overline{\mathbb{W}}} \) so \( \overline{\mathbb{F}}g = f\overline{\mathbb{F}} \). Now \( p = (1, 0, 0, 0) \in X \) so \( \overline{\mathbb{F}}g(p) \neq 0 \) but \( f\overline{\mathbb{F}}(p) = 0 \), which is a contradiction. Thus \( g \) is not a unit in \( k[X] \). By Krull’s principal ideal theorem (Theorem 5.11) and by Theorem 5.5 \( (X \text{ is irreducible}) \) all irreducible components of \( Z_X(g) \) have dimension \( 2 = \dim X - 1 \). Since \( Z \) is irreducible of dimension 2, we have that \( Z_X(g) = Z \). Let \( Z' = Z_X(\overline{\mathbb{F}}, \overline{\mathbb{Z}}) \), which is a plane. We have that
\[
\{(0, 0, 0, 0)\} = Z \cap Z' = Z_X(g) \cap Z'.
\]
But (again by Theorem 5.11 and Theorem 5.5) a polynomial function vanishes on an algebraic set of dimension 1 on a plane, which is a contradiction since a point has dimension 0.

**Proposition 6.4.** Suppose that \( p \in X \). Then \( \mathcal{O}_{X,p} = k[X]_{I(p)} \).

**Proof.** We have that
\[
k[X]_{I(p)} = \bigcup_{f \in k[X], f(p) \neq 0} k[X]_f = \bigcup_{p \in D(f)} \mathcal{O}_X(D(f)) = \bigcup_{p \in U} \mathcal{O}_X(U).
\]
The last equality is since the open sets \( D(f) \) are a basis for the topology of \( X \). \( \square \)

Suppose that \( X \) is an affine variety. From the Nullstellensatz, we have that there is a 1-1 correspondence between the points in \( X \) and the maximal ideals in \( k[X] \). Thus we have that
\[
k[X] = \mathcal{O}_X(X) = \cap_{p \in X} \mathcal{O}_{X,p} = \cap_{p \in X} k[X]_{I(p)} = \cap k[X]_m
\]
where the intersection is over the maximal ideals \( m \) of \( k[X] \), and we have the following important corollary:

**Corollary 6.5.** Suppose that \( U \) is a quasi-affine variety. Then \( U \) is separated; that is, if \( p, q \in U \) are distinct points then \( \mathcal{O}_{U,p} \neq \mathcal{O}_{U,q} \).

Suppose that \( Y \) is a quasi-affine variety. Then \( Y \) is an open subset of an affine variety \( X \). The regular functions on \( Y \) are \( k[Y] = \mathcal{O}_X(Y) \). A regular map from \( Y \) to \( \mathbb{A}^r \) is a map \( \varphi = (f_1, \ldots, f_r) \) where \( f_1, \ldots, f_r \in k[Y] \). Suppose that \( \varphi(Y) \subset Z \) where \( Z \) is an open subset of an irreducible closed subset of \( \mathbb{A}^n \) (a quasi-affine variety). Then \( \varphi \) induces a regular map \( \varphi : Y \to Z \). A regular map \( \varphi : Y \to Z \) is an isomorphism if there is a regular map \( \psi : Z \to Y \) such that \( \psi \circ \varphi = \text{id}_Y \) and \( \varphi \circ \psi = \text{id}_Z \).

**Proposition 6.6.** Suppose that \( X \) is an affine variety, and \( 0 \neq f \in k[X] \). Then the quasi-affine variety \( D(f) \) is isomorphic to an affine variety.

**Proof.** A choice of coordinate functions on \( X \) gives us a closed immersion \( X \subset \mathbb{A}^n \). We thus have a surjection \( k[x_1, \ldots, x_n] \to k[X] \) with kernel \( I(X) \). Let \( g \in k[x_1, \ldots, x_n] \) be a function which restricts to \( f \) on \( X \). Let \( J \) be the ideal in the polynomial ring \( k[x_1, \ldots, x_n, x_{n+1}] \) generated by \( I(X) \) and \( 1 - gx_{n+1} \). We will show that \( J \) is prime, and if \( Y \) is the affine variety \( Y = Z(J) \subset \mathbb{A}^{n+1} \), then the projection of \( \mathbb{A}^{n+1} \) onto its first \( n \) factors induces an isomorphism of \( Y \) with \( D(f) \).

\( J \) is an integral domain since
\[
k[x_1, \ldots, x_n, x_{n+1}]/J \cong [k[x_1, \ldots, x_n]/I(X)] [x_{n+1}]/(x_{n+1}f - 1) \cong k[X][\frac{1}{f}] = k[D(f)]
\]
which is an integral domain (it is a subring of the quotient field of \( k[X] \)). Thus \( J \) is prime. Now projection onto the first \( n \) factors induces a regular map \( \varphi : Y \to \mathbb{A}^n \). We have that
\[
Y = \{(a_1, \ldots, a_n, a_{n+1}) \mid (a_1, \ldots, a_n) \in X \text{ and } f(a_1, \ldots, a_n)a_{n+1} = 1 \}.
\]
Thus \( \varphi(Y) = D(f) \subset X \). In particular, we have a regular map \( \varphi : Y \to D(f) \). Now this map is \( 1 \)-1 and onto, but to show that it is an isomorphism we have to produce a regular inverse map. Let \( \pi_1, \ldots, \pi_n \) be the restrictions of \( x_1, \ldots, x_n \) to \( k[X] \). Then \( \pi_1, \ldots, \pi_n, \frac{1}{f} \in k[D(f)] \). Thus \( \psi : D(f) \to \mathbb{A}^{n+1} \) defined by \( \psi = (\pi_1, \ldots, \pi_n, \frac{1}{f}) \) is a regular map. The image of \( \psi \) is \( Y \). We thus have an induced regular map \( \psi : D(f) \to Y \). Composing the maps, we have that \( \psi \circ \varphi = \text{id}_Y \) and \( \varphi \circ \psi = \text{id}_{D(f)} \). Thus \( \varphi \) is an isomorphism. \( \square \)
Lemma 6.7. Suppose that \( U \) is a nontrivial open subset of \( X \). Then
\[
\cup_{p \in U} \mathcal{O}_{X,p} = k(X)
\]

We thus define the rational functions \( k(U) \) on a quasi-affine variety \( U \) to be \( k(X) \), where \( X \) is the Zariski closure of \( U \). We may further define \( \mathcal{O}_U(V) = \mathcal{O}_X(V) \) for an open subset \( V \) of \( U \) and \( \mathcal{O}_{U,p} = \mathcal{O}_{X,p} \) for \( p \in U \).

Suppose that \( U \) is a quasi-affine variety with field \( k(U) \) of rational functions on \( U \). A function \( f \in k(U) \) is said to be defined at a point \( p \in U \) if \( f \in \mathcal{O}_{U,p} \).

**Lemma 6.8.** Suppose that \( U \) is a quasi-affine variety and that \( f \in k(U) \). Then
\[
V = \{ p \in U \mid f \in \mathcal{O}_{U,p} \}
\]
is a dense open subset of \( U \).

**Proof.** Suppose that \( p \in V \). Since \( f \in \mathcal{O}_{X,p} \), there exist \( g, h \in k[X] \) with \( h \notin I(p) \) such that \( f = \frac{g}{h} \). For \( q \in D(h) \cap U \) we have that \( h \notin I_X(q) \), and thus \( f = \frac{g}{h} \in \mathcal{O}_{U,q} = \mathcal{O}_{V,q} \) and \( D(h) \cap U \) is an open neighborhood of \( p \) in \( V \).

\( V \) is non-empty since we can always write \( f = \frac{g}{h} \) for some \( g, h \in k[X] \) with \( h \neq 0 \). We have that \( U \cap D(h) \neq \emptyset \) since \( X \) is irreducible. Thus \( \emptyset \neq D(h) \cap U \subset V \). \( \square \)

**Lemma 6.9.** Suppose that \( U \) is a quasi affine variety and \( p \in U \). Let
\[
I_U(p) = \{ f \in k[U] \mid f(p) = 0 \}.
\]
Then \( I_U(p) \) is a prime (in fact maximal) ideal in \( k[U] \) and \( k[U]_{I_U(p)} = \mathcal{O}_{U,p} \).

**Proof.** \( U \) is an open subset of an affine variety \( \overline{U} \). We have restriction maps
\[
k[U] \rightarrow k[U] \rightarrow \mathcal{O}_{\overline{U},p} = k[\overline{U}]_{I_{\overline{U}}(p)}.
\]
\( \mathcal{O}_{\overline{U},p} \) is a local ring with maximal ideal \( m = I_{\overline{U}}(p) \mathcal{O}_{U,p} \). We have that \( m \cap k[U] = I_U(p) \) and \( m \cap k[U] = I_{\overline{U}}(p) \), so we have inclusions
\[
\mathcal{O}_{\overline{U},p} = k[U]_{I_{\overline{U}}(p)} \subset k[U]_{I_U(p)} \subset \mathcal{O}_{\overline{U},p}.
\]
\( \square \)

**Proposition 6.10.** Suppose that \( U, V \) are quasi affine varieties and \( \varphi : U \rightarrow V \) is a continuous map. Let \( \varphi^* \) be the rule \( \varphi^*(f) = f \circ \varphi \) for \( f : V \rightarrow k^1 \). Then the following are equivalent:

1. \( \varphi^* : k[V] \rightarrow k[U] \) is a k-algebra homomorphism.
2. \( \varphi^* : \mathcal{O}_V \rightarrow \mathcal{O}_U \) is a k-algebra homomorphism for all \( p \in U \).
3. \( \varphi^* : \mathcal{O}_V(W) \rightarrow \mathcal{O}_U(\varphi^{-1}(W)) \) is a k-algebra homomorphism for all open subsets \( W \) of \( V \).

**Proof.** Suppose that 1 holds, \( p \in U \) and \( \varphi(p) = q \). Then \( (\varphi^*)^{-1}(I_U(p)) = I_V(\varphi(p)) \). This follows since for \( f \in k[V] \),
\[
f \in I_V(\varphi(p)) \iff f(\varphi(p)) = 0
\]
\[
\iff \varphi^*(f)(p) = 0 \iff f \in (\varphi^*)^{-1}(I_U(p)).
\]

Thus \( \varphi^* : k[V]_{I_V(\varphi(p))} \rightarrow k[U]_{I_U(p)} \) and so \( \varphi^* : \mathcal{O}_V,\varphi(p) \rightarrow \mathcal{O}_{U,p} \) by Lemma 6.9. Thus 2 holds.

Suppose that 2 holds. Then 3 follows from the definition of regular functions. Suppose that 3 holds. Then 1 follows by taking \( W = V \). \( \square \)
Proposition 6.11. Suppose that $U$ and $V$ are quasi affine varieties and $\varphi : U \to V$ is a map. Then $\varphi$ is a regular map if and only if $\varphi$ is continuous and $\varphi^*$ satisfies the equivalent conditions of Proposition 6.10.

Proof. Suppose that $\varphi$ is regular. Then $V$ is an open subset of an affine variety $\overline{V}$ which is a closed subset of $\mathbb{A}^n$, such that the extension $\tilde{\varphi} : U \to \mathbb{A}^n$ of $\varphi$ has the form $\tilde{\varphi} = (f_1, \ldots, f_n)$ with $f_1, \ldots, f_n \in \mathcal{O}_U(U)$. We will first establish that $\tilde{\varphi}$ is continuous. Let $\{U_i\}$ be an affine cover of $U$. It suffices to show that $\varphi_i = \tilde{\varphi}|_{U_i}$ is continuous for all $i$. Now $\varphi_i^* : k[\mathbb{A}^n] \to k[U] \to k[U_i]$ is a $k$-algebra homomorphism. Since $U_i$ and $\mathbb{A}^n$ are affine, there exists a unique regular map $g_i : U_i \to \mathbb{A}^n$ such that $g_i^* = \varphi_i^*$.

Suppose that $p \in U_i$ and $q \in \mathbb{A}^n$. We have that

$$\varphi_i^*(I_{\mathbb{A}^n}(q)) \subset U_i(p) \text{ if } \varphi_i^*(f)(p) = 0 \text{ for all } f \in I_{\mathbb{A}^n}(q)$$
$$\varphi_i^*(f)(p) = 0 \text{ for all } f \in I_{\mathbb{A}^n}(q)$$
$$I_{\mathbb{A}^n}(q) \subset I_{\mathbb{A}^n}(\varphi_i(p))$$
$$I_{\mathbb{A}^n}(q) = I_{\mathbb{A}^n}(\varphi_i(p)) \text{ since } I_{\mathbb{A}^n}(q) \text{ and } I_{\mathbb{A}^n}(\varphi_i(p)) \text{ are maximal ideals}$$
$$q = \varphi_i(p) \text{ since } \mathbb{A}^n \text{ is separated.}$$

We have that $\varphi_i(p) = q$ if and only if $\varphi_i^*(I_{\mathbb{A}^n}(q)) \subset U_i(p)$ and similarly $g_i(p) = q$ if and only if $g_i^*(I_{\mathbb{A}^n}(q)) \subset U_i(p)$. Thus $\varphi_i = g_i$. Since a regular map of affine varieties is continuous, we have that $\varphi_i$ is continuous. Thus $\tilde{\varphi}$ and $\varphi$ are continuous.

Now $\tilde{\varphi}^* : \mathcal{O}_\mathbb{A}(\overline{V}) \to \mathcal{O}_U(U)$ is a $k$-algebra homomorphism since $\mathcal{O}_\mathbb{A}(\overline{V}) = k[\overline{V}] = k[\overline{x_1, \ldots, x_n}]$ and $\tilde{\varphi}(x_i) = f_i$ for all $i$. By 1 implies 2 of Proposition 6.10, applied to $\tilde{\varphi} : U \to \overline{V}$, we have that $\tilde{\varphi}^* : \mathcal{O}_\mathbb{A}(V) = \mathcal{O}_V(V) \to \mathcal{O}_U(\varphi^{-1}(V)) = \mathcal{O}_U(U)$ is a $k$-algebra homomorphism. Thus $\varphi : U \to V$ satisfies condition 1 of Proposition 6.10.

Now suppose that $\varphi : U \to V$ is continuous and $\varphi^*$ satisfies the equivalent conditions of Proposition 6.10. Then the extension $\tilde{\varphi} : U \to \mathbb{A}^n$ satisfies $\tilde{\varphi}^* : k[\mathbb{A}^n] \to k[U]$ is a $k$-algebra homomorphism. Since $\tilde{\varphi} = (\tilde{\varphi}(x_1), \ldots, \tilde{\varphi}(x_n))$, we have that $\varphi$ is a regular map. \qed

7. Rational maps of affine varieties

We begin with an important observation about regular functions, which simply follows from the fact that the restriction map (9) is an inclusion.

Proposition 7.1. Suppose that $V$ is an open subset of an affine variety $X$ and $U$ is an open subset of $V$. Suppose that $f$ and $g$ are regular functions on $V$ such that $f|U = g|U$. Then $f = g$ on $V$.

Definition 7.2. Suppose that $X$ is an affine variety. A rational map $\varphi : X \to \mathbb{A}^m$ is an $m$-tuple $\varphi = (f_1, \ldots, f_m)$ with $f_1, \ldots, f_m \in k(X)$.

The set of points of $X$ on which $f_1, \ldots, f_m$ are regular is a (nonempty) open set $W$ by Lemma 6.8. This is the set of points on which $\varphi$ is regular.

A rational map can be identified with the induced regular map from $W$ to $\mathbb{A}^m$. Suppose that $Y$ is an affine variety in $\mathbb{A}^m$ such that $\varphi(W) \subset Y$. Then $\varphi$ induces a rational map from $X$ to $Y$.

A regular map $\varphi : X \to Y$ is called dominant if $\varphi(X)$ is dense in $Y$. A rational map $\varphi : X \to Y$ is called dominant if $\varphi(W)$ is dense in $Y$ when $W$ is an open subset of $X$ on which $\varphi$ is a regular map.
Lemma 7.3. Suppose that \( \varphi : X \to Y \) is a dominant rational map of affine varieties. Then \( \varphi \) induces a 1-1 \( k \)-algebra homomorphism \( \varphi^* : k(Y) \to k(X) \) of function fields.

\[ \begin{array}{c}
\text{Proof.} \quad \text{There exists a nonempty open subset } W \text{ of } X \text{ on which } \varphi \text{ is a regular map. } W \\
\text{contains an open set } D(f) \text{ for some } f \in k[X] \text{ by Lemma 6.1. } D(f) \text{ is affine with } k[D(f)] = k[X]_f \text{ by Propositions 6.6 and 6.3. Then we have an induced } k \text{-algebra homomorphism } \\
\varphi^* : k[Y] \to k[X]_f \text{ which is 1-1 by Corollary 3.14. We thus have an induced } k \text{-algebra homomorphism of quotient fields.} \quad \square
\end{array} \]

Proposition 7.4. Suppose that \( X \) and \( Y \) are affine varieties and \( \Lambda : k(Y) \to k(X) \) is a 1-1 \( k \)-algebra homomorphism. Then there is a unique rational map \( \varphi : X \to Y \) such that \( \varphi^* = \Lambda \).

\[ \begin{array}{c}
\text{Proof.} \quad \text{Let } t_1, \ldots, t_m \text{ be coordinate functions on } X \text{ such that } k[Y] = k[t_1, \ldots, t_m]. \text{ Write } \\
\Lambda(t_i) = \frac{f_i}{g_i} \text{ with } f_i, g_i \in k[X] \text{ (and } g_i \neq 0) \text{ for } 1 \leq i \leq m. \text{ Let } g = g_1 g_2 \cdots g_m, \text{ then } \Lambda \\
\text{induces a } k \text{-algebra homomorphism } \Lambda : k[Y] \to k[X]_g. \text{ Now } k[X]_g = k[D(g)] \text{ where } D(g) \\
\text{is the affine open subset of } X, \text{ by Proposition 6.3. By Proposition 3.21, there is a unique } \\
\text{regular map } \varphi : D(g) \to Y \text{ such that } \varphi^* = \Lambda. \text{ Since a rational map of varieties is uniquely } \\
determined by the induced regular map on a nontrivial open subset, there is a unique rational map } \varphi : X \to Y \text{ inducing } \Lambda. \quad \square
\end{array} \]

Suppose that \( \alpha : X \to Y \) is a dominant rational map and \( \beta : Y \to Z \) is a rational map. Then \( \beta \circ \alpha : X \to Z \) is a rational map. We see this as follows. There exist open sets \( U \) of \( X \) on which \( \alpha \) is defined and \( V \) of \( Y \) on which \( \beta \) is defined. Since \( \varphi(U) \) is dense in \( Y \) we have that \( \varphi(U) \cap V \neq \emptyset \). Thus the open set \( (\varphi(U))^{-1}(V) \) is nonempty.

Definition 7.5. A dominant rational map \( \varphi : X \to Y \) of affine varieties is birational if there is a dominant rational map \( \psi : Y \to X \) such that \( \psi \circ \varphi = \text{id}_X \) and \( \varphi \circ \psi = \text{id}_Y \) as rational maps.

Proposition 7.6. A rational map \( \varphi : X \to Y \) is birational if and only if \( \varphi^* : k(Y) \to k(X) \) is a \( k \)-algebra isomorphism.

Theorem 7.7. A dominant rational map \( \varphi : X \to Y \) of affine varieties is birational if and only if there exist nonempty open subsets \( U \) of \( X \) and \( V \) of \( Y \) such that \( \psi : U \to V \) is a regular map which is an isomorphism.

\[ \begin{array}{c}
\text{Proof.} \quad \text{Suppose that there exist open sets } U \text{ of } X \text{ and } V \text{ of } Y \text{ such that } \varphi : U \to V \text{ is a regular map which is an isomorphism. Then there exists a regular map } \psi : V \to U \text{ such that } \\
\psi \circ \varphi = \text{id}_U \text{ and } \varphi \circ \psi = \text{id}_V. \text{ Since } \text{id}_{k[V]} = (\varphi \circ \psi)^* = \psi^* \circ \varphi^* \text{ and } \text{id}_{k[U]} = (\psi \circ \varphi)^* = \\
\varphi^* \circ \psi^*, \text{ we have that } \varphi^* : k[V] \to k[U] \text{ is an isomorphism of } k \text{-algebras, so that } \varphi^* \text{ induces } \\
an isomorphism of their quotient fields, which are respectively } k(Y) \text{ and } k(X). \quad \square
\end{array} \]
Proposition 7.8. Every affine variety $X$ is birationally equivalent to a hypersurface $Z(g) \subset \mathbb{A}^n$.

Proof. Let $\dim(X) = r$. Then the quotient field $k(X)$ is a finite algebraic extension of the rational function field $L = k(x_1, \ldots, x_r)$. By the theorem of the primitive element (Theorem 19, page 84 [11]), $k(X) \cong L[t]/f(t)$ for some irreducible polynomial $f(t) \in L[t]$. Multiplying $f(t)$ by an appropriate element $a$ of $k[x_1, \ldots, x_r]$, we obtain a primitive polynomial $g = af(t) \in k[x_1, \ldots, x_r, t]$, which is thus irreducible. The quotient field of $k[x_1, \ldots, x_r, t]/(g)$ is isomorphic to $k(X)$. Thus $X$ is birationally equivalent to $Z(g) \subset \mathbb{A}^{r+1}$ by Proposition 7.6.

8. Standard Graded Rings

In this section we summarize material from Section 2, Chapter VII of [12]. Let $T$ be the polynomial ring $T = k[x_0, x_1, \ldots, x_n]$. An element $f \in T$ is called homogeneous of degree $d$ if it is a $k$-linear combination of monomials of degree $d$. Let $T_d$ be the $k$-vector space of all homogeneous polynomials of degree $d$ (we include 0). Every polynomial $f \in T$ has a unique expression as a sum with finitely many nonzero terms $f = F_0 + F_1 + \cdots$ where $F_i$ are homogeneous of degree $i$, with $F_i = 0$ for all $i$ sufficiently large. This is equivalent to the statement that $T = \bigoplus_{i=0}^{\infty} T_i$, where $T_i$ is the $k$-vector space of homogeneous polynomials of degree $i$; that is, $T$ is a graded ring. Since $T = k[T_1] = T_0[T_1]$ is generated by elements of degree 1 as a $T_0 = k$-algebra, we say that $T$ is standard graded.

Lemma 8.1. Suppose that $I \subset T$ is an ideal. Then the following are equivalent

1. Suppose that $f \in I$ and $f = \sum F_i$ where $F_i$ is homogeneous of degree $i$. Then $F_i \in I$ for all $i$.
2. $I = \bigoplus_{i=0}^{\infty} I \cap T_i$

An ideal satisfying the conditions of Lemma 8.1 is called a homogeneous ideal. An ideal $I$ is homogeneous if and only if $I$ has a homogeneous set of generators.

Suppose that $I = \bigoplus_{i=0}^{\infty} I_i$ is a homogeneous ideal in $T$. Then $S = T/I \cong \bigoplus_{i=0}^{\infty} S_i$ where $S_i = T_i/I_i$ is a standard graded ring (elements of $S_i$ have degree $i$ and $S$ is generated by $S_1$ as a $S_0 = k$ algebra). In particular, every $f \in S$ has a unique expression as a sum with finitely many nonzero terms $f = F_0 + F_1 + \cdots$ where $F_i \in S_i$ are homogeneous of degree $i$, with $F_i = 0$ for all $i$ sufficiently large. The conclusions of Lemma 8.1 hold for ideals in $S$.

Lemma 8.2. Suppose that $I$ and $J$ are homogeneous ideals in $S$.

1. $I + J$ is a homogeneous ideal.
2. $IJ$ is a homogeneous ideal.
3. $I \cap J$ is a homogeneous ideal.
4. $\sqrt{I}$ is a homogeneous ideal.
5. If $P$ is a minimal prime ideal of $S$ containing $I$ then $P$ is homogeneous.

Suppose that $P$ is a homogeneous ideal in $S$. Then $P$ is a prime ideal if and only if it has the property that whenever $F, G \in S$ are homogeneous and $FG \in P$, then $F$ or $G$ is in $P$.

Suppose that $I$ is an ideal in the polynomial ring $T$, and $S = T/I$. Then $S = k[x_0, \ldots, x_n]$ is the $k$-algebra generated by the classes $\overline{x}_i$ of the $x_i$ in $S$. We can extend the grading of $T$ to $S$ if and only if $I$ is a homogeneous ideal. In this case, we have that $S = \bigoplus_{i=0}^{\infty} S_i$ where $S_i$ is the $k$-vector space generated by the monomials $\prod x_j^{a_j}$ where
A subset of \( P \) is a well defined subset of \( S \). This is a well defined set (independent of the set of homogeneous generators \( U \) of \( I \)).

**Definition 9.1.** A subset \( Y \) of \( \mathbb{P}^n \) is a projective algebraic set if there exists a set \( U \) of homogeneous elements of \( S(\mathbb{P}^n) \) such that \( Y = Z(U) \).

**Proposition 9.2.** Suppose that \( I_1, I_2, \{ I_\alpha \}_{\alpha \in \Lambda} \) are homogeneous ideals in \( S(\mathbb{P}^n) \). Then

1. \( Z(I_1 I_2) = Z(I_1) \cup Z(I_2) \).
2. \( Z(\sum_{\alpha \in \Lambda} I_\alpha) = \cap_{\alpha \in \Lambda} Z(I_\alpha) \).
3. \( Z(S(\mathbb{P}^n)) = \emptyset \).
4. \( \mathbb{P}^n = Z(0) \).

Proposition 9.2 tells us that

1. The union of two algebraic sets is an algebraic set.
2. The intersection of any family of algebraic sets is an algebraic set.
3. \( \emptyset \) and \( \mathbb{P}^n \) are algebraic sets.

We thus have a topology on \( \mathbb{P}^n \), defined by taking the closed sets to be the algebraic sets. The open sets are the complements of algebraic sets in \( \mathbb{P}^n \). This topology is called the Zariski topology.

**Definition 9.3.** A projective algebraic variety is an irreducible closed subset of \( \mathbb{P}^n \). An open subset of a projective variety is a quasi-projective variety. A projective algebraic set is a closed subset of \( \mathbb{P}^n \). A quasi-projective algebraic set is an open subset of a closed subset of \( \mathbb{P}^n \).

Given a subset \( Y \) of \( \mathbb{P}^n \), the ideal \( I(Y) \) of \( Y \) in \( S(\mathbb{P}^n) \) is the ideal in \( S(\mathbb{P}^n) \) generated by the set

\[
U = \{ F \in S(\mathbb{P}^n) \mid F \text{ is homogeneous and } F(p) = 0 \text{ for all } p \in Y \}.
\]
Theorem 9.4. (Homogeneous Nullstellensatz) (Theorem 15, Section 4, Chapter VII [12])
Let $k$ be an algebraically closed field, $a$ a homogeneous ideal in the polynomial ring $R = k[x_0, \ldots, x_n]$, and $F \in R$ a homogeneous polynomial which vanishes at all points of $Z(a)$. Then $F^r \in a$ for some $r > 0$.

Proposition 9.5. Suppose that $a$ is a homogeneous ideal in the standard graded polynomial ring $T = k[x_0, \ldots, x_n] = \bigoplus_{i=0}^\infty T_i$. Then the following are equivalent:
1. $Z(a) = \emptyset$
2. $\sqrt{a}$ is either $T$ or the ideal $T_+ = \bigoplus_{d>0} T_d$
3. $T_d \subset a$ for some $d > 0$.

Proposition 9.6. The following statements hold:
(a) If $T_1 \subset T_2$ are subsets of $S(\mathbb{P}^n)$ consisting of homogeneous elements, then $Z(T_2) \subset Z(T_1)$.
(b) If $Y_1 \subset Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_2) \subset I(Y_1)$.
(c) For any two subsets $Y_1, Y_2$ of $\mathbb{P}^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
(d) If $a$ is a homogeneous ideal in $S(\mathbb{P}^n)$ with $Z(a) \neq \emptyset$, then $I(Z(a)) = \sqrt{a}$.
(e) For any subset $Y$ of $\mathbb{P}^n$, $Z(I(Y)) = \overline{Y}$, the Zariski closure of $Y$.

Suppose that $p = (a_0 : \ldots : a_n) \in \mathbb{P}^n$. Then
$$I(p) = (a_i x_j - a_j x_i \mid 0 \leq i, j \leq n).$$

Theorem 9.7. A closed set $W \subset \mathbb{P}^n$ is irreducible if and only if $I(W)$ is a prime ideal.

Proposition 9.8. Every closed set in $\mathbb{P}^n$ is the union of finitely many irreducible ones.

Suppose that $X$ is a projective variety, which is a closed subset of $\mathbb{P}^n$. We define the coordinate ring of $X$ to be $S(X) = S(\mathbb{P}^n)/I(X)$, which is a standard graded ring and a domain. Suppose that $U$ is a set of homogeneous elements of $S(X)$. Then we define $Z_X(U) = \{p \in X \mid F(p) = 0 \text{ for all } F \in U\}$.

Given a subset $Y$ of $X$, we define $I_X(Y)$ to be the ideal in $S(X)$ generated by the homogeneous elements of $S(X)$ which vanish at all points of $Y$.

The above results in this section hold with $\mathbb{P}^n$ replaced by $X$.

Suppose that $X$ is a projective algebraic set and $F \in S(X)$. We define $D(F) = X \setminus Z(F)$, which is an open set in $X$.

Lemma 9.9. Suppose that $X$ is a projective algebraic set. Then the open sets $D(F)$ for homogeneous $F \in S(X)$ forms a basis for the topology of $X$.

Theorem 9.10. Suppose that $0 \leq i \leq n$ and $D(x_i)$ is the open subset of $\mathbb{P}^n$. Then the maps $\varphi : D(x_i) \to \mathbb{A}^n$ defined by $\varphi(a_0 : \ldots : a_n) = (\frac{a_i}{a_0}, \ldots, \frac{a_n}{a_0})$ for $(a_0 : \ldots : a_n) \in D(x_i)$ and $\psi : \mathbb{A}^n \to D(x_i)$ defined by $\psi(a_1, a_2, \ldots, a_n) = (a_1 : \ldots : a_{i-1} : 1 : a_i : \ldots : a_n)$ for $(a_1, \ldots, a_n) \in \mathbb{A}^n$ are inverse homeomorphisms.

The method of this proof is by constructing a correspondence between elements of $R = k[y_1, \ldots, y_n]$ (a polynomial ring in $n$ variables) and homogeneous elements of the standard graded polynomial ring $T = k[x_0, \ldots, x_n]$. For simplicity, assume that $i = 0$. To $f(y_1, \ldots, y_n) \in R$ we associate $f^h = F(x_0, \ldots, x_r) = x_0^d f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$, where $d$ is the degree of $f$. To $F(x_0, \ldots, x_n)$ we associate $F^a = F(1, y_1, \ldots, y_n)$ (this definition is valid for all $F \in T$). One checks that $(f^h)^a = f$ for all $f \in R$ and for all homogeneous $F$, $(F^a)^h = x_0^{-m} F$ where $m$ is the highest power of $x_0$ which divides $F$. 


We extend $h$ to a map from ideals in $k[y_1, \ldots, y_n]$ to homogeneous ideals in $k[x_0, \ldots, x_n]$ by taking an ideal $I$ to the ideal $I^h$ generated by the set of homogeneous elements $\{f^h \mid f \in I\}$. A homogeneous ideal $J$ is mapped to the ideal

$$J^a = \{f^a \mid f \in J\} = \{F^a \mid F \in J \text{ is homogeneous}\}.$$

The properties which are preserved by these correspondences of ideals are worked out in detail in Section 5 of Chapter VII of [12]. The following formulas hold:

$$(I^h)^a = I \text{ for } I \text{ an ideal in } R,$$

$$(J^a)^h = \bigcup_{j=0}^\infty J : x_i^j \text{ for } J \text{ an homogeneous ideal in } T$$

where $J : x_i^j = \{f \in T \mid fx_i^j \in J\}$. We can deduce from these formulas that $a$ and $h$ give a 1-1 correspondence between prime ideals in $k[y_1, \ldots, y_n]$ and prime ideals in $k[x_0, \ldots, x_n]$ which do not contain $x_i$.

It follows that if $X$ is an affine variety, which is a closed subset of $\mathbb{A}^n$, then $\psi$ induces a homeomorphism of $X$ with $W \cap D(x_i)$, where $W$ is the projective variety $Z(I(X)^h)$, and if $W$ is a projective variety which is a closed subset of $\mathbb{P}^n$ which is not contained in $Z(x_i)$, then $\psi$ induces a homeomorphism of the closed subvariety $X = Z(I(W)^a)$ of $\mathbb{A}^n$ with $W \cap D(x_i)$.

10. REGULAR FUNCTIONS AND REGULAR MAPPINGS OF QUASI PROJECTIVE VARIETIES

Suppose that $X$ is a projective variety, which is a closed subset of $\mathbb{P}^n$. We define the function field of $X$ to be

$$k(X) = \{\frac{F}{G} \mid F, G \in S(X) \text{ are homogeneous of the same degree } d, \text{ and } G \neq 0\}.$$

$k(X)$ is a field. Suppose that $f \in k(X)$. We say that $f$ is regular at $p \in X$ if there exists an expression $f = \frac{F}{G}$, where $F, G$ are homogeneous of the same degree $d$, such that $G(p) \neq 0$.

**Lemma 10.1.** Suppose that $f \in k(X)$. Then the set

$$U = \{p \in X \mid f \text{ is regular at } p\}$$

is an open set.

**Proof.** Suppose that $p \in U$. Then there are $F, G \in S(X)$ which are homogeneous of the same degree $d$ such that $f = \frac{F}{G}$ and $G(p) \neq 0$. Then $X \setminus Z_X(G)$ is an open neighborhood of $p$ which is contained in $U$. Thus $U$ is open. \hfill $\square$

For $p \in X$, we define

$$\mathcal{O}_{X,p} = \{f \in k(X) \mid f \text{ is regular at } p\}.$$

$\mathcal{O}_{X,p}$ is a local ring. Suppose that $U$ is an open subset of $X$. Then we define

$$\mathcal{O}_X(U) = \cap_{p \in U} \mathcal{O}_{X,p}.$$

Here the intersection takes place in $k(X)$.

There are examples of quasi projective varieties $U$ such that $\mathcal{O}_U(U)$ is not a finitely generated $k$-algebra [8], and page 456 of [10].

**Definition 10.2.** Suppose that $X$ is a quasi projective variety (or a quasi affine variety) and $Y$ is a quasi projective variety (or a quasi affine variety). A regular map $\varphi : X \to Y$ is a continuous map such that for every open subset $U$ of $Y$, the rule $\varphi^*(f) = f \circ \varphi$ for $f \in \mathcal{O}_Y(U)$ gives a $k$-algebra homomorphism $\varphi^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$. 


Definition 10.2 is consistent with our earlier definitions of regular maps of affine and quasi affine varieties by Propositions 6.10 and 6.11.

**Definition 10.3.** Suppose that $X$ is a quasi projective variety (or a quasi affine variety) and $Y$ is a quasi projective variety (or a quasi affine variety). A regular map $\varphi : X \to Y$ is an isomorphism if there exists a regular map $\psi : Y \to X$ such that $\psi \circ \varphi = \text{id}_X$ and $\varphi \circ \psi = \text{id}_Y$.

**Theorem 10.4.** Suppose that $W$ is a projective variety which is a closed subset of $\mathbb{P}^n$ and $x_i$ is a homogeneous coordinate on $\mathbb{P}^n$ such that $W \cap D(x_i) \neq \emptyset$. Then $W \cap D(x_i)$ is an affine variety.

**Proof.** Without loss of generality, we may suppose that $i = 0$. Write

$$S(W) = k[x_0, \ldots, x_n]/I(W) = k[\bar{x}_0, \ldots, \bar{x}_n]$$

where $\bar{x}_i$ are the classes $x_i + I(W)$. $S(W)$ is standard graded with the $\bar{x}_i$ having degree 1. By our assumption $\bar{x}_0 \neq 0$, so $\frac{\bar{x}_i}{\bar{x}_0} \in k(W)$ for all $j$.

Let $\varphi : W \cap D(x_0) \to X$ be the homeomorphism of Theorem 9.10, where $X$ is the affine variety $X = Z(I(W)^a) \subset \mathbb{A}^n$. We have that $k[X] = k[y_1, \ldots, y_n]/I(X) = k[\bar{y}_1, \ldots, \bar{y}_n]$. For a point $p = (a_0 : a_1 : \ldots : a_n) \in W \cap D(x_0) \cap D(x_i)$ we have that $\varphi(p) = (\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0})$.

Define

$$\varphi^*(f) = f \circ \varphi = f(\frac{\bar{x}_1}{\bar{x}_0}, \ldots, \frac{\bar{x}_n}{\bar{x}_0}) = \frac{f^h}{(\bar{x}_0)^{\deg(f)}}$$

for $f \in k[X]$.

We thus have that $\varphi^*$ is a $k$-algebra homomorphism $\varphi^* : k[X] \to k(W)$ which is 1-1 (since the kernel of $\varphi^*$ is $I_X(X) = (0)$ by Theorem 9.10). We thus have an induced $k$-algebra homomorphism $\varphi^* : k(X) \to k(W)$. By (13) we have that $\varphi^*(f) \in \mathcal{O}_{W,p}$ for all $p \in W \cap D(x_0)$.

Suppose that $p = (a_0 : \ldots : a_n) \in W \cap D(x_0)$, and $h \in \mathcal{O}_{X,\varphi(p)}$. Then $h = \frac{f}{g}$ with $f, g \in k[X]$ and $g(\varphi(p)) = g(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}) \neq 0$. Since $a_0 \neq 0$, we see from (13) that $\varphi^*(g)(p) \neq 0$. Thus $\varphi^*(h) \in \mathcal{O}_{W,p}$. We thus have that $\varphi^*(\mathcal{O}_{X,\varphi(p)}) \subset \mathcal{O}_{W,p}$. Suppose that $h \in \mathcal{O}_{W,p}$. Then $h = \frac{F(\bar{x}_1, \ldots, \bar{x}_n)}{G(\bar{x}_1, \ldots, \bar{x}_n)}$ where $F, G$ are homogeneous of a common degree $d$ and $G(p) \neq 0$. We have

$$\frac{F}{G} = \frac{F}{\bar{x}_0^d} = \frac{F(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})}{G(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})} = \varphi^\ast \left( \frac{F^a}{G^a} \right).$$

Now $G^a(\varphi(p)) = \frac{G(p)}{a_0} \neq 0$ so $\frac{F^a}{G^a} \in \mathcal{O}_{X,\varphi(p)}$. We thus have that $\varphi^* : \mathcal{O}_{X,\varphi(p)} \to \mathcal{O}_{W,p}$ is an isomorphism for all $p \in W \cap D(x_0)$.

Since $\varphi$ is a homeomorphism and by the definition of $\mathcal{O}$, we have that $\varphi^* : \mathcal{O}_{\varphi^{-1}(U)} = \cap_{p \in U} \mathcal{O}_{X,\varphi(p)} \supseteq \cap_{p \in U} \mathcal{O}_{W,p} = \mathcal{O}_W(U)$ is an isomorphism for all open subsets $U \subset W \cap D(x_0)$. In particular, $\varphi$ is a regular map.

Now we have that the inverse map $\psi$ of $\varphi$ defined in Theorem 9.10 induces a map $\psi^* : k(W) \to k(X)$ which is the inverse of $\varphi^*$. By our above calculations, we then see that $\psi$ is a regular map which is an inverse to $\varphi$. Thus $\varphi$ is an isomorphism. \hfill $\square$

**Corollary 10.5.** Suppose that $X$ is a quasi affine variety. Then $X$ is isomorphic to a quasi projective variety.
Thus there exist eventually realize space of smaller dimension. Repeating this reduction at most a finite number of times, we so that

\begin{proof}
This follows from Theorems 9.10 and 10.4.
\end{proof}

We will call an open subset \( U \) of a projective variety an affine variety if \( U \) is isomorphic to an affine variety. We this identification, we have that all quasi affine varieties are quasi projective.

**Corollary 10.6.** Every point \( p \) in a quasi projective variety has an open neighborhood which is isomorphic to an affine variety.

**Proof.** Suppose that \( V \) is a quasi projective variety, and \( p \in V \) is a point. Then \( V \) is an open subset of a projective variety \( W \), which is itself a closed subset of a projective space \( \mathbb{P}^n \). By Theorem 10.4, there exists a homogeneous coordinate \( x_i \) on \( \mathbb{P}^n \) such that \( D(x_i) \cap W \) is isomorphic to an affine variety. Now by Lemma 6.1 and Proposition 6.6, there exists an affine open subset \( U \) of \( D(x_i) \cap W \) which contains \( p \) and is contained in \( V \). □

The proof of Theorem 10.4 gives us the following useful formula. Suppose that \( W \subset \mathbb{P}^n \) is a projective variety, and suppose that \( W \) is not contained in \( Z(x_i) \) Then

\[
\mathcal{O}_W(D(x_i)) = k[x_0/x_i, \ldots, x_n/x_i] \cong k[x_0, \ldots, x_n]/J
\]

where \( J = \{ f(x_0/x_i, \ldots, x_n/x_i) \mid f \in \mathcal{O}(W) \} \).

We can easily calculate the regular functions on \( \mathbb{P}^n \) now. Since \( \{ D(x_i) \mid 1 \leq i \leq n \} \) is an open cover of \( \mathbb{P}^n \),

\[
\mathcal{O}_\mathbb{P}^n(\mathbb{P}^n) = \cap_{i=0}^n \mathcal{O}_\mathbb{P}(D(x_i)) = \cap_{i=0}^n k[x_0/x_i, \ldots, x_n/x_i] = k.
\]

This statement is true for arbitrary projective varieties \( W \) (taking the intersection over the open sets \( W \cap D(x_i) \) such that \( D(x_i) \cap W \neq \emptyset \) but we need to be a little careful with the proof, as can be seen from the following example. Consider the standard graded domain \( T = \mathbb{Q}[\bar{x}_0, \bar{x}_1] = \mathbb{Q}[x_0, x_1]/x_0^2 + x_1^2 \). We compute \( L = \mathbb{Q}[\bar{x}_0, \bar{x}_1] \cap \mathbb{Q}[\bar{x}_0^2] \). We have that \( \bar{x}_0^2 = -\bar{x}_1^4 \) so \( \bar{x}_1 = -\bar{x}_0^2 \).

\[
L = \mathbb{Q}[\bar{x}_0/\bar{x}_1] \cong \mathbb{Q}[t]/(t^2 + 1) \cong \mathbb{Q}[\sqrt{-1}]
\]

which is larger than \( \mathbb{Q} \). This example shows that any proof that \( \mathcal{O}_W(W) = k \) for a projective variety \( W \) must use the assumption that \( k \) is algebraically closed.

**Theorem 10.7.** Suppose that \( W \) is a projective variety. Then the regular functions on \( W \) are \( \mathcal{O}_W(W) = k \).

**Proof.** \( W \) is a closed subset of \( \mathbb{P}^n \) for some \( n \). Let \( S(\mathbb{P}^n) = k[x_0, \ldots, x_n] \) and \( S(W) = k[\bar{x}_0, \ldots, \bar{x}_n] \). We may suppose that \( \bar{x}_i \neq 0 \) for all \( i \), for otherwise we have that \( W \subset \mathbb{Z}(x_i) \) so that \( W \) is a closed subset of \( \mathbb{Z}(x_i) \cong \mathbb{P}^{n-1} \subset \mathbb{P}^n \), and \( W \) is contained in a projective space of smaller dimension. Repeating this reduction at most a finite number of times, we eventually realize \( W \) as a closed subset of a projective space such that \( W \not\subset Z(x_i) \) for any \( i \).

\[
f \in \mathcal{O}_W(W) = \cap_{i=0}^n \mathcal{O}_W(D(x_i)) = \cap_{i=0}^n k[\bar{x}_0/\bar{x}_i, \ldots, \bar{x}_n/\bar{x}_i].
\]

Thus there exist \( N_i \in \mathbb{N} \) and homogeneous elements \( G_i \in S(W) = k[\bar{x}_0, \ldots, \bar{x}_n] \) of degree \( N_i \) such that

\[
f = \frac{G_i}{\bar{x}_i^{N_i}} \quad \text{for} \quad 0 \leq i \leq n.
\]
Let $S_i$ be the set of homogeneous forms of degree $i$ in $S(W)$ (so that $S(W) \cong \bigoplus_{i=0}^{\infty} S_i$).
We have that $x_i^N f \in S_{N_i}$ for $0 \leq i \leq n$. Suppose that $N \geq \sum N_i$. Since $S_N$ is spanned (as a $k$-vector space) by monomials of degree $N$ in $\bar{x}_0, \ldots, \bar{x}_n$, for each such monomial at least one $\bar{x}_i$ has an exponent $\geq N_i$. Thus $S_N f \subset S_{N_i}$. Iterating, we have that $S_N f^q \subset S_{N_i}$ for all $q \in \mathbb{N}$. In particular, $\bar{x}_0^N f^q \in S(W)$ for all $q > 0$. Thus the subring $S(W)[f]$ of the quotient field of $S(W)$ is contained in $x_0^{-N} S(W)$, which is a finitely generated $S(W)$ module. Thus $f$ is integral over $S(W)$ (by Theorem 4.4). Thus there exist $a_1, \ldots, a_m \in S(W)$ such that
\[ f^m + a_1 f^{m-1} + \cdots + a_m = 0. \]
Since $f$ has degree 0, we can replace the $a_i$ with their homogeneous components of degree 0 and still have a dependence relation. But the elements of degree 0 in $S(W)$ consists of the field $k$. Now $k[f]$ is a domain since it is a subring of the quotient field of $S(W)$. Thus $f \in k$ since $k$ is algebraically closed. □

**Proposition 10.8.** Suppose that $W$ is a projective variety. Then $W$ is separated (distinct points of $W$ have distinct local rings).

**Proof.** Suppose that $W$ is a closed subset of a projective space $\mathbb{P}^n$. Write $k[W] = k[x_0, \ldots, x_n]/I(W) = k[\bar{x}_0, \ldots, \bar{x}_n]$. Suppose that $p \in W$. Then $p \in D(x_i)$ for some $i$, so that since $W \cap D(x_i)$ is affine, $\mathcal{O}_{W_p}$ is the localization of $k[\bar{x}_0, \ldots, \bar{x}_n]/(\bar{x}_i)$ at a maximal ideal $m$. Let $\pi : \mathcal{O}_{W_p} \to \mathcal{O}_{W_p}/m\mathcal{O}_{W_p} \cong k$ be the residue map. Let $\pi(\bar{x}_j) = \alpha_j \in k$ for $0 \leq j \leq n$ (with $\alpha_i = 1$ since $\bar{x}_i = 1$). We have that $\pi_i(p) \neq 0$ and $\pi_j(p) = \alpha_j \pi_i(p)$. Thus $p = (\alpha_0 \cdots : \alpha_n) \in W$ is uniquely determined by the ring $\mathcal{O}_{X,p}$. □

**Proposition 10.9.** Suppose that $X$ and $Y$ are quasi projective varieties and $\varphi : X \to Y$ is a dominant regular map. Then the rule $\varphi^*(f) = f \circ \varphi$ induces a 1-1 $k$-algebra homomorphism $\varphi^* : k(Y) \to k(X)$.

**Proof.** Let $V$ be an affine open subset of $Y$, and $U$ be an affine open subset of the open subset $f^{-1}(V)$ of $X$. Then the restriction of $\varphi$ to a map of affine varieties $\varphi : U \to V$ is dominant, so the $k$-algebra homomorphism $\varphi^* : k[V] \to k[U]$ is 1-1. Taking the induced map on quotient fields, we obtain the desired homomorphism of function fields. □

**Proposition 10.10.** Suppose that $X$ and $Y$ are quasi projective varieties and $\varphi : X \to Y$ is a map. Let $\{V_i\}$ be a collection of open affine subsets covering $Y$, and $\{U_i\}$ be a collection of open subsets covering $X$, such that
1. $\varphi(U_i) \subset V_i$ for all $i$ and
2. the rule $\varphi^*(f) = f \circ \varphi$ maps $\mathcal{O}_Y(V_i)$ into $\mathcal{O}_X(U_i)$ for all $i$.

Then $\varphi$ is a regular map.

**Proof.** Suppose that $U$ is an affine subset of $U_i$. Then $\varphi^*$ induces a $k$-algebra homomorphism $\varphi^* : k[V_i] \to k[U]$ since the restriction map $\mathcal{O}_Y(U_i) \to \mathcal{O}_Y(U)$ is a $k$-algebra homomorphism. Thus we may refine our cover to assume that $U_i$ are affine for all $i$.

Let $\varphi_i : U_i \to V_i$ be the restriction of $\varphi$. Consider the $k$-algebra homomorphism $\varphi_i^* : k[V_i] \to k[U_i]$. Suppose that $p \in U_i$ and $q \in V_i$. We have that
\[ \varphi_i^*(I_{V_i}(q)) \subset I_{U_i}(p) \]
if $\varphi_i^*(f)(p) = 0$ for all $f \in I_{V_i}(q)$
if $f(\varphi_i(p)) = 0$ for all $f \in I_{V_i}(q)$
if $I_{V_i}(q) \subset I_{U_i}(\varphi_i(p))$
if $I_{V_i}\phi_i(q) = I_{V_i}(\varphi_i(p))$ since $I_{V_i}(q)$ and $I_{V_i}(\varphi_i(p))$ are maximal ideals
if $q = \varphi_i(p)$ since the affine variety $V_i$ is separated.
Now there exists a regular map \( g_i : U_i \rightarrow V_i \) such that \( g_i^* = \varphi_i^* \) (by Proposition 3.21). The calculation (15) shows that for \( p \in U_i \) and \( q \in V_i \) we have that \( g_i(p) = q \) if and only if \( g_i^*(I_{V_i}(q)) \subset I_{U_i}(p) \). Thus \( \varphi_i = g_i \) so that \( \varphi_i \) is a regular map. In particular the \( \varphi_i \) are all continuous so that \( \varphi \) is continuous.

Suppose that \( q \in Y \) and \( p \in \varphi^{-1}(q) \). Then there exist \( U_i \) and \( V_i \) such that \( p \in U_i \) and \( q \in V_i \). For \( f \in k[V_i] \),

\[
  f \in I_{V_i}(\varphi_i(p)) \quad \text{iff} \quad f(\varphi_i(p)) = 0 \\
  \text{iff} \quad \varphi_i^*(f)(p) = 0 \\
  \text{iff} \quad f \in (\varphi_i^*)^{-1}(I_{U_i}(p)).
\]

Thus \( (\varphi_i^*)^{-1}(I_{U_i}(p)) = I_{V_i}(q) \), and we have an induced \( k \)-algebra homomorphism

\[
  \varphi_i^* : \mathcal{O}_{V_i,q} = k[V_i]_{I_{V_i}(q)} \rightarrow k[U_i]_{I_{U_i}(p)} = \mathcal{O}_{U_i,p}.
\]

But this is just the statement that

\[
  \varphi^* : \mathcal{O}_{Y,q} \rightarrow \mathcal{O}_{X,p}.
\]

Thus

(16) \[
  \varphi^* : \mathcal{O}_{Y,q} \rightarrow \bigcap_{p \in \varphi^{-1}(q)} \mathcal{O}_{X,p}.
\]

Suppose that \( U \) is an open subset of \( Y \). Then

\[
  \mathcal{O}_X(\varphi^{-1}(U)) = \bigcap_{q \in U} \big( \bigcap_{p \in \varphi^{-1}(q)} \mathcal{O}_{X,p} \big).
\]

Thus by (16), we have that

\[
  \varphi^* : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\varphi^{-1}(U)).
\]

We have established that \( \varphi \) is a regular map. \( \square \)

11. Some examples of regular maps


Lemma 11.1. A composition of regular maps of quasi projective varieties is a regular map.

Proof. This follows from the definition of a regular map, since the composition of continuous functions is continuous. \( \square \)

Proposition 11.2. Suppose that \( U \) and \( V \) are quasi projective varieties.

1. Suppose that \( U \) is an open subset of \( V \). Then the inclusion \( i : U \rightarrow V \) is a regular map.

2. Suppose that \( U \) is a closed subset of \( V \). Then the inclusion \( i : U \rightarrow V \) is a regular map.

Proof. Let \( \{V_i\} \) be an affine cover of \( V \). Let \( U_i = \varphi^{-1}(V_i) \). Then \( \{U_i\} \) is an open cover of \( U \) such that \( i(U_i) \subset V_i \) for all \( i \). In both cases 1 and 2, \( i^* \) is restriction of functions, so \( i^* : \mathcal{O}_V(V_i) \rightarrow \mathcal{O}_U(U_i) \) for all \( i \). By Proposition 10.10, \( i \) is a regular map. \( \square \)

Proposition 11.3. Suppose that \( X \) is a quasi projective variety and

\[
  \varphi = (f_1, \ldots, f_n) : X \rightarrow \mathbb{A}^n
\]

is a map. Then \( \varphi \) is a regular map if and only if \( f_i \) are regular functions on \( X \) for all \( i \).
Suppose that $X$ is a quasi projective variety, which is an open subset of a closed subset $W$ of $\mathbb{P}^n$ and $\varphi : X \to \mathbb{P}^n$ is a regular map. Let $k(W) = k[\bar{x}_0, \ldots, \bar{x}_m]$ and $k(\mathbb{P}^n) = k[y_0, \ldots, y_n]$. Suppose that $p \in X$. Then there exists an $i$ such that $p \in D(\bar{x}_i)$ and there exists a $j$ such that $\varphi(p) \in D(y_j)$. Let $V = D(y_j)$. Now $\varphi^{-1}(V)$ is an open subset of the open subset $X$ of $W$, so $\varphi^{-1}(V)$ is open in $W$. Thus $\varphi^{-1}(V) \cap D(\bar{x}_i)$ is an open subset of the affine variety $W \cap D(\bar{x}_i)$ containing $p$. There exists $h \in k[W \cap D(\bar{x}_i)]$ such that

$$p \in D(h) = W \cap D(\bar{x}_i) - Z_{W \cap D(\bar{x}_i)}(h) \subset \varphi^{-1}(V) \cap D(\bar{x}_i).$$

Let $U = D(h)$. $U$ is an affine open set, which is a neighborhood of $p$ in $X$. Consider the restriction $\varphi : U \to V$. Then $\varphi$ is a regular map of affine varieties, so on $U$, we can represent $\varphi : U \to V \cong \mathbb{A}^n$ as $\varphi = (f_1, \ldots, f_n)$ for some $f_0, \ldots, f_n \in k[U] = \mathcal{O}_X(U) \subset k(U) = k(X)$ (by Proposition 6.11). Thus we have a representation $\varphi = (f_1 : \ldots : f_{i-1} : 1 : f_i : \ldots : f_n)$ on the neighborhood $U$ of $p$.

In particular, we have shown that there exists an open neighborhood $U$ of $p$ in $X$, and regular functions $f_0, \ldots, f_n \in \mathcal{O}_X(U)$ such that

$$\varphi = (f_0 : \ldots : f_n)$$

on $U$, where there are no points on $U$ where all of the $f_i$ vanish. Suppose that $q \in X$ is another point, and $Y$ is an open neighborhood of $q$ in $X$ with regular functions $g_0, \ldots, g_n \in \mathcal{O}_X(Y)$ such that

$$\varphi = (g_0 : \ldots : g_n)$$

on $Y$, and the $g_i$ have no common zeros on $Y$. These two representations of $\varphi$ must agree on $U \cap Y$, which happens if and only if

$$f_ig_j - g_jf_i = 0 \text{ for } 0 \leq i,j \leq n$$

on $U \cap Y$.

By Proposition 10.10 (and Proposition 6.11), a collection of such representations on open sets of $X$ determines a regular mapping $\varphi : X \to \mathbb{P}^n$.

We can thus think of a regular map $\varphi : X \to \mathbb{P}^n$ as an equivalence class of expressions $(f_0 : f_1 : \ldots : f_n)$ with $f_0, \ldots, f_n \in k(X)$ and such that

$$(f_0 : f_1 : \ldots : f_n) \sim (g_0 : \ldots : g_n)$$

if and only if

$$f_ig_j - g_jf_i = 0 \text{ for } 0 \leq i,j \leq n.$$
We can thus think of a regular map \( \varphi : X \to \mathbb{P}^n \) as an equivalence class of expressions \((F_0 : F_1 : \ldots : F_n)\) with \(F_0, \ldots, F_n\) homogeneous elements of \(S(W)\) (or of \(S(\mathbb{P}^n)\)) all having the same degree, and such that
\[
(F_0 : F_1 : \ldots : F_n) \sim (G_0 : \ldots : G_n)
\]
if and only if
\[
F_iG_j - G_jF_i = 0 \text{ for } 0 \leq i, j \leq n.
\]
We further require that for each \(p \in X\) there exists a representative \((F_0 : F_1 : \ldots : F_n)\) such that some \(F_i\) does not vanish at \(p\). It is not required that the common degree of the \(F_i\) be the same as the common degree of the \(G_j\).

11.2. Linear Isomorphisms of Projective Space. Suppose that \(A = (a_{ij})\) is an invertible \((n+1) \times (n+1)\) matrix with coefficients in \(k\) (indexed as \(0 \leq i, j \leq n\)). Define homogeneous elements \(L_i\) of degree 1 in \(S = k[\mathbb{P}^n] = k[x_0, \ldots, x_n] = \mathbb{P}_1\) by \(L_i = \sum_{i=0}^{n} a_{ij}x_j\) for \(0 \leq i \leq n\). Then the \(L_i\) are a \(k\)-basis of \(S_1\) so that \(Z(L_0, \ldots, L_n) = Z(x_0, \ldots, x_n) = \emptyset\). Thus \(\varphi_A : \mathbb{P}^n \to \mathbb{P}^n\) defined by
\[
\varphi_A = (L_0 : \ldots : L_n)
\]
is a regular map. If \(B\) is another invertible \((n+1) \times (n+1)\) matrix with coefficients in \(k\), then we have that
\[
\varphi_A \circ \varphi_B = \varphi_{AB}.
\]
Thus \(\varphi_A\) is an isomorphism of \(\mathbb{P}^n\), with inverse map \(\varphi_{A^{-1}}\).

**Proposition 11.4.** Suppose that \(W\) is a projective variety which is a closed subset of \(\mathbb{P}^n\). Suppose that \(L \in S(\mathbb{P}^n)\) is a linear homogeneous form, such that \(D(L) \cap W \neq \emptyset\). Then \(D(L) \cap W\) is an affine variety.

**Proof.** Write \(L = \sum_{i=0}^{n} a_{ij}x_j\) for some nonzero \(a_{ij} \in k\), and extend the vector \((a_{00}, \ldots, a_{on})\) to a basis of \(k^{n+1}\). Arrange this basis in an \((n+1) \times (n+1)\) matrix \(A = (a_{ij})\). \(A\) is necessarily invertible. Now the isomorphism \(\varphi_A : \mathbb{P}^n \to \mathbb{P}^n\) maps \(D(L)\) to \(D(x_0)\) and \(W\) to a projective variety \(\varphi_A(W)\) which is not contained in \(Z(x_0)\). \(\varphi_A(W) \cap D(x_0)\) is an affine variety by Theorem 10.4. Thus \(W \cap D(L)\) is affine since it is isomorphic to \(\varphi_A(W) \cap D(x_0)\).

In the case that \(W \not\subset Z(L)\), composing the isomorphism \(\varphi^* : \mathcal{O}_{\varphi(W)}(D(x_0)) \cong \mathcal{O}_W(D(L))\) of the above proof with the representation of \(\mathcal{O}_{\varphi(W)}(D(x_0))\) of (14), we obtain that
\[
\mathcal{O}_W(D(L)) = k[\frac{x_0}{L}, \ldots, \frac{x_n}{L}] \cong k[\frac{x_0}{L}, \ldots, \frac{x_n}{L}]/J
\]
where \(J = \{f(\frac{x_0}{L}, \ldots, \frac{x_n}{L}) \mid f \in I(W)\}\).

11.3. The Veronese map. Suppose that \(d\) is a positive integer. Let \(x_0^d, x_0^{d-1}x_1, \ldots, x_n^d\) be the set of all monomials of degree \(d\) in \(S(\mathbb{P}^n) = k[x_0, \ldots, x_n]\). Let \(e = \binom{n+d}{d}\). There are \(e + 1\) such monomials. Since these monomials are a \(k\)-basis of \(S_d\), we have that \(Z(x_0^d, x_0^{d-1}x_1, \ldots, x_n^d) = \emptyset\). Thus we have a regular map
\[
\Lambda : \mathbb{P}^n \to \mathbb{P}^e
\]
defined by \(\Lambda = (x_0^d : x_0^{d-1}x_1 : \ldots : x_n^d)\). Let \(W\) be the closure of \(\Lambda(\mathbb{P}^n)\) in \(\mathbb{P}^e\).
We will establish that $\Lambda$ is an isomorphism of $\mathbb{P}^n$ to $W$. Let $\mathbb{P}^e$ have the homogeneous coordinates $y_{i_0i_2\ldots i_n}$ where $i_0, \ldots, i_n$ are nonnegative integers such that $i_0 + \cdots + i_n = d$. The map $\Lambda$ is defined by the equations

$$y_{i_0i_2\ldots i_n} = x_{i_0}^{i_2} \cdots x_{i_n}^{i_n}.$$ 

We certainly have that

$$y_{i_0\ldots i_n} y_{j_0\ldots j_n} - y_{k_0\ldots k_n} y_{l_0\ldots l_n} \in I(W)$$

if

$$i_0 + j_0 = k_0 + l_0, \ldots, i_n + j_n = k_n + l_n.$$ 

Further, we can check that

$$W = Z(y_{i_0\ldots i_n} y_{j_0\ldots j_n} - y_{k_0\ldots k_n} y_{l_0\ldots l_n} | i_0 + j_0 = k_0 + l_0, \ldots, i_n + j_n = k_n + l_n)$$

is the image of $\Lambda$. Suppose that $q \in W$. Then $q = \Lambda(p)$ for some $p \in \mathbb{P}^n$, so that $x_j(p) \neq 0$ for some $j$. We have that $y_{0\ldots 0d0\ldots 0} (q) = x_j^d (p) \neq 0$, where $d$ is in the $j$-th place. Thus the affine open sets $W_j = D(y_{0\ldots 0d0\ldots 0})$ of $\mathbb{P}^e$ cover $W$. Let

$$S(W) = S(\mathbb{P}^e) / I(W) = k[\{y_{i_0, \ldots, i_n}\}].$$

In $S(W)$, we have the identities

$$(20) \quad y_{i_0\ldots i_n} y_{j_0\ldots j_n} = y_{k_0\ldots k_n} y_{l_0\ldots l_n} \text{ if } i_0 + j_0 = k_0 + l_0, \ldots, i_n + j_n = k_n + l_n.$$ 

Now on each open subset $W_j = D(y_{0\ldots 0d0\ldots 0}) \cap W$ of $W$ we define a regular map

$$\Psi_j : W_j \to D(x_j) \subset \mathbb{P}^n$$

by

$$\Psi_j = (\overline{y}_{0\ldots 0(d-1)0\ldots 0} : \overline{y}_{010\ldots 0(d-1)0\ldots 0} : \cdots : \overline{y}_{00\ldots 0d0\ldots 0} : \cdots : \overline{y}_{00\ldots 0d0\ldots 1}).$$

Now we have that

$$\Psi_j \circ \Lambda = (x_0 x_j^{d-1} : x_1 x_j^{d-1} : \cdots : x_n x_j^{d-1}) = (x_0 : x_1 : \cdots : x_n) = \text{id}_{D(x_j)},$$

and the identities

$$\overline{y}_{i_1\ldots i_q} \overline{y}_{0d0\ldots 0} = \overline{y}_{010\ldots 0(d-1)0\ldots 0} \overline{y}_{10\ldots 0(d-1)0\ldots 0} \cdots \overline{y}_{0l0\ldots 0(d-1)0\ldots 1}$$

whenever $i_0 + i_1 + \cdots + i_n = d$ (which are special cases of (20)) imply that

$$\Lambda \circ \Psi_j = \text{id}_{W_j}.$$ 

Now we check, again using the identities (20), that $\Psi_j = \Psi_k$ on $W_j \cap W_k$. Thus the $\Psi_j$ patch to give a regular map $\Psi : W \to \mathbb{P}^n$ which is an inverse to $\Lambda$.

It can be shown that $I(W)$ is actually generated by the set

$$\{y_{i_0\ldots i_n} y_{j_0\ldots j_n} - y_{k_0\ldots k_n} y_{l_0\ldots l_n} | i_0 + j_0 = k_0 + l_0, \ldots, i_n + j_n = k_n + l_n\}.$$ 

In our proof, we only used the weaker statement that the radical of the ideal generated by these elements is $I(W)$.

The isomorphism $\Lambda : \mathbb{P}^n \to W$ is called the Veronese map. We can obtain the following result using this map, composed with a linear isomorphism, which generalizes Proposition 11.4.

**Proposition 11.5.** Suppose that $W$ is a projective variety which is a closed subset of $\mathbb{P}^n$. Suppose that $F \in S(\mathbb{P}^n)$ is a homogeneous form of degree $d$ such that $D(F) \cap W \neq \emptyset$. Then $D(F) \cap W$ is an affine variety.
We obtain that
\begin{equation}
\mathcal{O}_W(D(F)) = k[\frac{M}{F}] \mid M \text{ is a monomial in } x_0, \ldots, x_n \text{ of degree } d, \ F = F(x_0, \ldots, x_n)
\end{equation}
\[ \cong k[\frac{M}{F}] \mid M \text{ is a monomial in } x_0, \ldots, x_n \text{ of degree } d] / J \]
where $J = \{ \frac{G(x_0, \ldots, x_n)}{M} \mid G \in I(W) \text{ is a form of degree } d \}$.

12. Rational Maps of Projective Varieties

**Definition 12.1.** Suppose that $X$ is a projective variety. A rational map $\varphi : X \to \mathbb{P}^n$ is an equivalence class of $(n+1)$-tuples $\varphi = (f_0 : \ldots : f_n)$ with $f_0, \ldots, f_n \in k(X)$, where $(g_0 : \ldots : g_n)$ is equivalent to $(f_0 : \ldots : f_n)$ if $f_if_j - f_jg_i = 0$ for $0 \leq i, j \leq n$.

A rational map $\varphi : X \to \mathbb{P}^n$ is regular at a point $p \in X$ if there exists a representation $(f_0 : \ldots : f_n)$ of $\varphi$ such that all of the $f_i$ are regular functions at $p$ ($f_i \in \mathcal{O}_{X,p}$ for all $i$) and some $f_i(p) \neq 0$.

Let $U$ be the open set of points of $X$ on which $\varphi$ is regular. Then $\varphi : U \to \mathbb{P}^n$ is a regular map. Let $Y$ be a subvariety of $\mathbb{P}^n$ containing the Zariski closure of $\varphi(U)$. Then $\varphi : U \to Y$ is a regular map. We have an induced rational map $\varphi : X \to Y$.

Suppose that $\varphi : X \to Y$ is a rational map. Let $U$ be the open subset of $X$ such that $\varphi : U \to Y$ is a regular map. Let $A$ be an open affine subset of $Y$ such that $A \cap \varphi(U) \neq \emptyset$, and let $B$ be an open affine subset of $(\varphi(U))^{-1}(A)$. Then $\varphi$ induces a regular map of affine varieties $\varphi : B \to A$. From this reduction, we obtain that the results of Section 7 on rational maps of affine varieties are all valid for projective varieties.

It is also convenient sometimes to interpret rational maps in terms of equivalence classes of homogeneous forms $(H_0 : \ldots : H_n)$ of a common degree.

12.1. Projection from a linear subspace. A linear subspace $E$ of a projective space $\mathbb{P}^n$ is the closed subset defined by the vanishing of a set of linear homogeneous forms. Such a subvariety is isomorphic to a projective space $\mathbb{P}^d$ for some $d \leq n$. The ideal $I(E)$ is then minimally generated by a set of $n-d$ linear forms; in fact a set of linear forms $\{L_1, \ldots, L_{n-d}\}$ is a minimal set of generators of $I(E)$ if and only if $L_1, \ldots, L_{n-d}$ is a $k$-basis of the $k$-linear subspace of the homogeneous linear forms on $\mathbb{P}^n$ which vanish on $E$. We will say that $E$ has dimension $d$.

Suppose that $E$ is a $d$-dimensional linear subspace of $\mathbb{P}^n$. Let $L_1, \ldots, L_{n-d}$ be linear forms in $k[x_0, \ldots, x_n]$ which define $E$. The rational map $\varphi : \mathbb{P}^n \to \mathbb{P}^{n-d-1}$ with $\varphi = (L_1 : \ldots : L_n)$ is called the projection from $E$. The mapping is regular on the open set $\mathbb{P}^n \setminus E$.

This map can be interpreted geometrically as follows: Choose a linear subspace $F$ of $\mathbb{P}^n$ of dimension $n-d-1$ which is disjoint from $E$ (to find such an $F$, just extend $L_1, \ldots, L_{n-d}$ to a $k$-basis $L_1, \ldots, L_{n-d}, M_1, \ldots, M_{d+1}$ of $S_1$ (the vector space of linear forms in $S(\mathbb{P}^n)$), and let $F = Z(M_1, \ldots, M_{d+1})$). Suppose that $p \in \mathbb{P}^n \setminus E$. Let $G_p$ be the unique linear subspace of $\mathbb{P}^n$ of dimension $d+1$ which contains $p$ and $E$. $G_p$ intersects $F$ in a unique point. This intersection point can be identified with $\varphi(p)$.

This map depends on the choice of basis $L_1, \ldots, L_{n-d}$ of linear forms which define $E$. However, there is not a significant difference if a different basis $L_1', \ldots, L_{n-d}'$ is chosen. In this case there is a linear isomorphism $\Lambda : \mathbb{P}^{n-d} \to \mathbb{P}^{n-d}$ such that $(L_1 : \ldots : L_{n-d}) = \Lambda \circ (L_1' : \ldots : L_{n-d}')$. 

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13. Products

In this section we define the product of two varieties. First, suppose that $X$ and $Y$ are affine varieties, with $X$ a closed subset of $\mathbb{A}^m$ and $Y$ a closed subset of $\mathbb{A}^n$. It is natural to construct the product of $\mathbb{A}^m$ and $\mathbb{A}^n$ as

$$\mathbb{A}^m \times \mathbb{A}^n = \mathbb{A}^{m+n}.$$  

As sets, this gives a natural identification, and this identification makes $\mathbb{A}^m \times \mathbb{A}^n$ into an affine variety. The projections $\pi_1 : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^m$ and $\pi_2 : \mathbb{A}^m \times \mathbb{A}^n \to \mathbb{A}^n$ are regular mappings. In fact, we have that if $k[\mathbb{A}^m] = k[x_1, \ldots, x_m]$ and $k[\mathbb{A}^n] = k[y_1, \ldots, y_n]$, then $k[\mathbb{A}^m \times \mathbb{A}^n] = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$.

Now the product $X \times Y$ can be naturally identified with a subset of $\mathbb{A}^m \times \mathbb{A}^n$, and we have that $X \times Y$ is a closed subset of $\mathbb{A}^m \times \mathbb{A}^n$, as we have that $X \times Y = Z(\pi_1^*(I(X)) \cup \pi_2^*(I(Y)))$. Let $R = k[\mathbb{A}^m \times \mathbb{A}^n]$.

Proposition 13.1. $I(X)R + I(Y)R$ is a prime ideal in $R$.

Proof. We make use of properties of tensor products to prove this. We have that

$$R/(I(X)R + I(Y)R) \cong k[X] \otimes_k k[Y]$$

(Theorem 35, page 184 [11]). We have that $k(X) \otimes_k k(Y)$ is a field since $k(X)$ and $k(Y)$ are fields and $k$ is algebraically closed, by Corollary 1 to Theorem 40 on page 198 of [11]. Now Zariski and Samuel define the tensor product $k(X) \otimes_k k(Y)$ over a field $k$ by the conditions that $k(X) \otimes_k k(Y)$ contains $k(X)$ and $k(Y)$ as subrings, is generated by $k(X)$ and $k(Y)$ as a ring and $k(X)$ and $k(Y)$ are linearly disjoint over $k$ (Definition 3 page 179 [11]). Thus we have that the subring of $k(X) \otimes_k k(Y)$ generated by $k[X]$ and $k[Y]$ is a tensor product of $k[X]$ and $k[Y]$ over $k$, so that $k[X] \otimes_k k[Y]$ is naturally a subring of the field $k(X) \otimes_k k(Y)$.

Thus $X \times Y$ is an affine variety, with prime ideal $I(X \times Y) = I(X)R + I(Y)R$ in $R = k[\mathbb{A}^m \times \mathbb{A}^n]$.

Products are much more subtle over nonclosed fields, as can be seen by the following example. Let $k = \mathbb{Q}$ and let $A = \mathbb{Q}[x]/(x^2 + 1)$, and $B = \mathbb{Q}[y]/(y^2 + 1)$, which are fields. We have that

$$A \otimes_k B \cong \mathbb{Q}[x,y]/(x^2 + 1, y^2 + 1) \cong \mathbb{Q}[i][y]/(y^2 + 1) = \mathbb{Q}[i][y]/(y - i)(y + i)$$

is not a domain.

We now construct a product $\mathbb{P}^m \times \mathbb{P}^n$. As a set, we can write

$$\mathbb{P}^m \times \mathbb{P}^n = \{(a_0 : \ldots : a_m; b_0 : \ldots : b_n) \mid (a_0 : \ldots : a_m) \in \mathbb{P}^m, (b_0 : \ldots : b_n) \in \mathbb{P}^n\}.$$  

Let $S$ be a polynomial ring in two sets of variables, $S = k[x_0, \ldots, x_m, y_0, \ldots, y_n]$. We put a bigrading on $S$ by $\text{wt}(x_i) = (1,0)$ for $0 \leq i \leq m$ and $\text{wt}(y_j) = (0,1)$ for $0 \leq j \leq n$. We have

$$S = \bigoplus_{k,l} S_{k,l}$$

where $S_{k,l}$ is the $k$-vector space generated by monomials $x_0^{i_0} \cdots x_m^{i_m} y_0^{j_0} \cdots y_n^{j_n}$ where $i_0 + \cdots + i_m = k$ and $j_0 + \cdots + j_n = l$. Elements of $S_{k,l}$ are called bihomogeneous of bidegree $(k,l)$. Suppose $F \in S$ is bihomogeneous of bidegree $(k,l)$ and $(a_0 : \ldots : a_m; b_0 : \ldots : b_n) \in \mathbb{P}^m \times \mathbb{P}^n$. Suppose that $(c_0 : \ldots : c_m : d_0 : \ldots : d_n)$ is equal to $(a_0 : \ldots : a_m; b_0 : \ldots : b_n)$, so
that there exist $0 \neq \alpha \in k$ and $0 \neq \beta \in k$ such that $c_i = \alpha a_i$ for $0 \leq i \leq m$ and $d_j = \beta b_j$ for $0 \leq j \leq n$. Then

$$F(c_0, \ldots, c_m, d_0, \ldots, d_n) = \alpha^k \beta^l F(a_0, \ldots, a_m, b_0, \ldots, b_n).$$

Thus the vanishing of such a form at a point is well defined. We put a topology on the set $\mathbb{P}^m \times \mathbb{P}^n$ by taking the closed sets to be

$$Z(A) = \{(p, q) \in \mathbb{P}^m \times \mathbb{P}^n \mid F(p, q) = 0 \text{ for } F \in A\}$$

where $A$ is a set of bihomogeneous forms. We can extend this definition to bihomogeneous ideals, by considering the vanishing at a set of bihomogeneous generators.

Given a subset $Y$ of $\mathbb{P}^m \times \mathbb{P}^n$, the ideal $I(Y)$ of $Y$ in $S$ is the ideal in $S$ generated by the set

$$U = \{F \in S \mid F \text{ is bihomogeneous and } F(p, q) = 0 \text{ for all } (p, q) \in Y\}.$$

$U$ is a bihomogeneous ideal.

The irreducible closed subsets $W$ of $\mathbb{P}^m \times \mathbb{P}^n$ are characterized by the property that their ideal $I(W)$ is a prime ideal which is bihomogeneous, and does not contain either of the ideals $(x_0, \ldots, x_m)$ or $(y_0, \ldots, y_n)$. The bihomogeneous coordinate ring of the biprojective variety $W$ is $S(W) = S/I(W)$, which is a bigraded ring. The function field of $W$ is

$$k(W) = \{\frac{F}{G} \mid F, G \in S(W) \text{ are bihomogeneous of the same bidegree and } G \neq 0\}.$$

The regular functions $\mathcal{O}_{W,(p,q)}$ at a point $(p, q) \in W$ are the quotients $\frac{F}{G}$ where $F, G \in S(W)$ are bihomogeneous of the same bidegree and $G(p, q) \neq 0$. We construct regular functions on an open subset $U$ of $W$ as

$$\mathcal{O}_W(U) = \cap_{(p,q)\in U} \mathcal{O}_{W,(p,q)}.$$

We now expand our definition of regular maps (Definition 10.2) to include quasi biprojective varieties.

If $X$ is a projective variety which is a closed subset of $\mathbb{P}^m$, and $Y$ is a projective variety which is a closed subset of $\mathbb{P}^n$, then $X \times Y$ is a biprojective variety, which is a closed subset of $\mathbb{P}^m \times \mathbb{P}^n$. We have that $I(X \times Y) = I(X)S + I(Y)S$, which is a prime ideal in $S$ by the proof of 13.1. $X \times Y$ has a covering by affine charts $(X \times Y) \cap D(x_1y_i) \cong (X \cap D(x_i)) \times (Y \cap D(y_i))$ for $0 \leq i \leq m$ and $0 \leq j \leq n$.

We can represent rational mappings from $X \times Y$ to a projective space $\mathbb{P}^l$ by equivalence classes $(F_0 : \ldots : F_l)$ where the $F_i$ are bihomogeneous of the same degree. We see that the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are regular mappings.

Proposition 10.10 is valid for maps between quasi biprojective varieties; even the proof is valid in the larger setting of quasi biprojective varieties. We deduce from this that if $X, Y, Z, W$ are quasi projective varieties and $\varphi : Z \to X$ and $\psi : Z \to Y$ are regular maps then $(\varphi, \psi) : Z \to X \times Y$ is a regular map. If $\alpha : Z \to X$ and $\beta : W \to Y$ are regular maps, then $\alpha \times \beta : Z \times W \to X \times Y$ is a regular map.

We can define a coordinate ring for a product of an affine variety $X$ and a projective variety $Y$. If $x_0, \ldots, x_n$ are affine coordinates on $X$ and $y_0, \ldots, y_n$ are homogeneous coordinates on $Y$, then $S(X \times Y) = k[x_0, \ldots, x_n, y_0, \ldots, y_n] \cong k[X] \otimes_k S(Y)$. The closed subsets $W$ of $X \times Y$ have the form $W = Z(F_1, \ldots, F_m)$, where $F_1, \ldots, F_m \in S(X \times Y)$ are homogeneous with respect to the $y$ variables.
13.1. The Segre Embedding. We define the Segre embedding

$$\varphi : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^N$$

where $N = (n + 1)(m + 1) - 1$ by

$$\varphi(a_0 : \ldots : a_m; b_0 : \ldots : b_n) = (a_0 b_0 : a_0 b_1 : \ldots : a_i b_j : \ldots : a_m b_n).$$

$\varphi$ is a regular map, and it can be verified that its image is a closed sub variety of $\mathbb{P}^N$ and $\varphi$ is an isomorphism onto this image. If we take $w_{ij}$, with $0 \leq i \leq m$ and $0 \leq j \leq n$ to be the natural homogeneous coordinates on $\mathbb{P}^N$, then the image of $\varphi$ is the projective variety $W$ whose ideal $I(W)$ is generated by $\{w_{ij} w_{kl} - w_{kj} w_{il} \mid 0 \leq i, k \leq m \text{ and } 0 \leq j, l \leq n\}$.

Thus the product $X \times Y$ of two projective varieties is actually (isomorphic to) a projective variety, by the Segre embedding.

13.2. Graphs of regular maps. Suppose that $X$ and $Y$ are quasi projective varieties and $\varphi : X \to Y$ is a regular map. Then we have a regular map $\psi : X \to X \times Y$ defined by $\psi(p) = (p, \varphi(p))$ for $p \in X$. Let $\Gamma_{\varphi}$ be the image of $\psi(X)$ in $X \times Y$.

Proposition 13.2. $\Gamma_{\varphi}$ is Zariski closed in $X \times Y$.

Proof. We have an embedding of $Y$ in a projective space $\mathbb{P}^n$, as an open subset of a projective subvariety. The map $\varphi : X \to Y$ thus extends to a regular map $\tilde{\varphi} : X \to \mathbb{P}^n$ and $\Gamma_{\varphi} = \tilde{\varphi}^{-1}(\tilde{\varphi}(X))$ is a regular map. Let $\Delta_{\varphi}$ be the “diagonal” $\{(q, q) \mid q \in \mathbb{P}^n\}$. We have that $S(\mathbb{P}^n \times \mathbb{P}^n) = k[u_0, \ldots, u_n, v_0, \ldots, v_n]$ where the $u_i$ are homogeneous coordinates on the $\mathbb{P}^n$ of the first factor, and the $v_j$ are homogeneous coordinates on the $\mathbb{P}^n$ of the second factor. We have that $\Delta_{\varphi}$ is a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$ with $I(\Delta_{\varphi}) = Z(u_i v_j - u_j v_i \mid 0 \leq i, j \leq n)$. Thus $\Delta_{\varphi}$ is a closed subset of $\mathbb{P}^n \times \mathbb{P}^n$. Since the preimage of a closed set by a regular map is closed, we have that $\Gamma_{\varphi} = (\varphi \times i)^{-1}(\Delta_{\varphi})$ is closed in $X \times \mathbb{P}^n$. □

We can extend this construction to give a useful method of studying rational maps. Suppose that $X$ and $Y$ are quasi projective varieties and $\varphi : X \to Y$ is a rational map. Let $U$ be a nontrivial open subset of $X$ in which $\varphi$ is defined. Then the graph $\Gamma_{\varphi}$ of $\varphi$ is the closure of the image of the regular map $p \to (p, \varphi(p))$ from $U$ to $X \times Y$. $\Gamma_{\varphi}$ does not depend on the choice of open subset $U$ on which $\varphi$ is defined.

Proposition 13.3. Suppose that $\varphi : X \to Y$ is a rational map of quasi projective varieties. Then $\Gamma_{\varphi}$ is a quasi projective variety, and the projection $\pi_1 : \Gamma_{\varphi} \to X$ is a birational map.

Proof. It suffices to prove this in the case when $\varphi$ is itself a regular map, as $\Gamma_{\varphi}$ is the Zariski closure in $X \times Y$ of the graph $\Gamma_{\varphi}|_U$ for any dense open subset $U$ of $X$.

Now $\Gamma_{\varphi}$ is the image of the regular map $(i, \varphi) : X \to X \times Y$. Since $X$ is irreducible, its image $\Gamma_{\varphi}$ is irreducible. $\Gamma_{\varphi}$ is closed in $X \times Y$ by Proposition 13.2. The map $(i, \varphi)$ is an inverse to $\pi_1$. Since both maps are regular maps, $\pi_1$ is an isomorphism, and is thus birational. □

Theorem 13.4. Suppose that $Z$ is a closed subset of $\mathbb{P}^n \times \mathbb{A}^m$. Then the image of the second projection $\pi_2 : Z \to \mathbb{A}^m$ is Zariski closed.

Proof. Let $u_0, \ldots, u_n$ be homogeneous coordinates on $\mathbb{P}^n$ and $y_1, \ldots, y_m$ be affine coordinates on $\mathbb{A}^m$. Then $Z = I(F_1, \ldots, F_r)$ for some $F_1, \ldots, F_r \in k[u_0, \ldots, u_n, y_1, \ldots, y_m]$ where the $F_i$ are homogeneous in the $u_i$ variables. Then by “elimination theory” (Section 80, page 8 [9]) there exist $g_1, \ldots, g_r \in k[y_1, \ldots, y_r]$ such that for $(a_1, \ldots, a_m) \in \mathbb{A}^m$, the
system of equations \( F_1(u_0, \ldots, u_n, a_1, \ldots, a_r) = 0, \ldots, F_r(u_0, \ldots, u_n, a_1, \ldots, a_r) = 0 \) has nonzero solution in \((u_0, \ldots, u_n)\) if and only if \( g_1(a_1, \ldots, a_m) = \ldots = g_r(a_1, \ldots, a_m) = 0 \). Thus \( \pi_2(Z) = Z(g_1, \ldots, g_r) \) is a closed subset of \( \mathbb{A}^m \). \( \square \)

Theorem 13.4 does not hold for closed subsets of \( \mathbb{A}^n \times \mathbb{A}^m \). A simple example is to take \( Z \) to be \( Z(xy - 1) \subset \mathbb{A}^2 \cong \mathbb{A}^1 \times \mathbb{A}^1 \). The projection of \( Z \) onto the \( y \)-axis is the non closed subset \( \mathbb{A}^1 \setminus \{(0)\} \) of \( \mathbb{A}^1 \).

**Corollary 13.5.** Suppose that \( X \) is a projective variety and \( Y \) is a quasi projective variety. Then the second projection \( \pi_2 : X \times Y \to Y \) takes closed sets to closed sets.

**Theorem 13.6.** Suppose that \( \varphi : X \to Y \) is a regular map of projective varieties. Then the image of \( \varphi \) is a closed subset of \( Y \).

**Proof.** Apply the corollary to the closed subset \( \Gamma_\varphi \) of \( X \times Y \). \( \square \)

We now give another proof of Theorem 10.7.

**Corollary 13.7.** Suppose that \( X \) is a projective variety. Then \( O_X(X) = k \).

**Proof.** Suppose that \( f \in O_X(X) \). Then \( f \) is a regular map \( f : X \to \mathbb{A}^1 \). After including \( \mathbb{A}^1 \) into \( \mathbb{P}^1 \), we obtain a regular map \( f : X \to \mathbb{P}^1 \). By the theorem, we have that the image \( f(X) \) is closed in \( \mathbb{P}^1 \). Since \( f \) cannot be onto, \( f(X) \) must be a finite union of points. Since \( X \) is irreducible, \( f(X) \) is irreducible, so \( f(X) \) is a single point. Thus \( f \in k \). \( \square \)

**Corollary 13.8.** Suppose that \( X \) is a projective variety and \( \varphi : X \to \mathbb{A}^n \) is a regular map. Then \( \varphi(X) \) is a point.

**Proof.** Let \( \pi_i : \mathbb{A}^n \to \mathbb{A}^1 \) be projection onto the \( i \)-th factor. Then \( \pi_i \circ \varphi : X \to \mathbb{A}^1 \) is a regular map, so \( \pi_i \circ \varphi \) is a constant map by the previous corollary for \( 1 \leq i \leq n \). Thus \( \varphi(X) \) is a point. \( \square \)

### 14. The Blow up of an Ideal

Suppose that \( X \) is an affine variety, and \( f_0, \ldots, f_r \in k[X] \). We can define a rational map \( \Lambda_{f_0, \ldots, f_r} : X \to \mathbb{P}^r \) by \( \Lambda_{f_0, \ldots, f_r}(p) = (f_0(p) : \ldots : f_r(p)) \).

**Proposition 14.1.** Suppose that \( g_0, \ldots, g_s \in k[X] \) and \( (f_0, \ldots, f_r) = (g_0, \ldots, g_r) \) are the same ideal \( J \) in \( k[X] \). Then there is a commutative diagram of regular maps

\[
\begin{array}{ccc}
\Gamma_{\Lambda_{f_0, \ldots, f_r}} & \xrightarrow{\psi} & \Gamma_{\Lambda_{g_0, \ldots, g_s}} \\
\downarrow & & \downarrow \\
X & & X
\end{array}
\]

where the vertical arrows are the projections and \( \psi \) is an isomorphism.

**Proof.** This will follow from Theorem 14.7. In this theorem it is shown that there are graded \( k[X] \)-algebra isomorphisms

\[
S(\Gamma_{\Lambda_{f_0, \ldots, f_r}}) = S(X \times \mathbb{P}^r) / I(\Gamma_{\Lambda_{f_0, \ldots, f_r}}) \cong \bigoplus_{i \geq 0} J^i
\]

and

\[
S(\Gamma_{\Lambda_{g_0, \ldots, g_s}}) = S(X \times \mathbb{P}^s) / I(\Gamma_{\Lambda_{g_0, \ldots, g_s}}) \cong \bigoplus_{i \geq 0} J^i,
\]

which thus induce a graded \( k[X] \)-algebra isomorphism

\[
\pi : S(\Gamma_{\Lambda_{g_0, \ldots, g_s}}) \to S(\Gamma_{\Lambda_{f_0, \ldots, f_r}})
\]
Write $S(\Gamma_{a_0,\ldots,a_s}) = k[X]/[\overline{y}_0,\ldots,\overline{y}_s]$ where $\overline{y}_i$ are the restriction to $\Gamma_{a_0,\ldots,a_s}$ of the homogeneous coordinates on $\mathbb{P}^r$. Then define

$$\alpha : \Gamma_{a_0,\ldots,a_s} \rightarrow X \times \mathbb{P}^s$$

by

$$\alpha(p; a_1 : \ldots : a_r) = (p; \overline{y}(\overline{y}_0)(p; a_1 : \ldots : a_r) : \ldots : \overline{y}(\overline{y}_r)(p; a_1 : \ldots : a_r))$$

for $p \in \Gamma_{a_0,\ldots,a_s}$. $\alpha$ induces the isomorphism $\overline{\alpha}$ of coordinate rings, so $\alpha$ is an isomorphism onto $\Gamma_{a_0,\ldots,a_s}$.

**Definition 14.2.** Suppose that $X$ is an affine variety and $I \subset k[X]$ is an ideal. Suppose that $I = (f_0,\ldots,f_r)$. Let $\Lambda : X \rightarrow \mathbb{P}^n$ be the rational map induced by the regular map $\Lambda : X \setminus Z(I) \rightarrow (X \setminus Z(I)) \times \mathbb{P}^n$ defined by $\Lambda(p) = (p; f_0(p) : \ldots : f_r(p))$ for $p \in X \setminus Z(I)$. The blow up of $I$ is $B(I) = \Gamma_{\Lambda}$, with projection $\pi_1 : B(I) \rightarrow X$.

Suppose that $X \subset \mathbb{P}^n$ is a projective variety, with coordinate ring $S(X) = k[x_0,\ldots,x_n]$. Suppose that $I$ is a homogeneous ideal in $S(W)$. Each open set $D(x_i)$ is affine with regular functions $k[D(x_i)] = k[\frac{x_0}{x_i},\ldots,\frac{x_n}{x_i}]$. In $k[D(x_i)]$ we have an ideal

$$\tilde{I}(D(x_i)) = \{ f(\frac{x_0}{x_i},\ldots,\frac{x_n}{x_i}) \mid f \in I \}.$$

This definition is such that for $p \in D(x_i) \cap D(x_j)$, we have equality of ideals

$$\tilde{I}(D(x_i))|_{O_{X,p}} = \tilde{I}(D(x_j))|_{O_{X,p}}.$$

We write $\tilde{I}_p$ for this ideal in $O_{X,p}$. To every open subset $U$ of $X$ we can thus define an ideal

$$\tilde{I}(U) = \bigcap_{p \in U} \tilde{I}_p \subset \bigcap_{p \in U} O_p = O_X(U).$$

The method of the proof of Theorem 10.4 shows that we do have

$$\bigcap_{p \in D(x_i)} \tilde{I}_p = \{ f(\frac{x_0}{x_i},\ldots,\frac{x_n}{x_i}) \mid f \in I \},$$

which we initially defined to be $\tilde{I}(D(x_i))$, so our definition is consistent. If $Z(I) \neq \emptyset$, we have that $0 = \tilde{I}(X) \subset O_X(X) = k$.

**Lemma 14.3.** Suppose that $I \subset S(X)$ is a homogeneous ideal. Then there exists some $d \geq 1$ such that the ideal $J$ which is generated by the elements $I_d$ (the elements of $I$ which are homogeneous of degree $d$) satisfies $\tilde{I} = \tilde{J}$.

**Proof.** Let $F_0,\ldots,F_r$ be a homogeneous set of generators of $I$. Let $d_0,\ldots,d_r$ be their respective degrees. Suppose that $d \geq \max\{d_i\}$, and $J$ be the ideal generated by $I_d$. We will show that $\tilde{J} = \tilde{I}$. It suffices to show that $\tilde{J}(D(x_j)) = \tilde{I}(D(x_j))$ for all $j$. This follows since $x_j^{d-d_i} F_i \in J$ for all $i$, so that

$$F_i(\frac{x_0}{x_j},\ldots,\frac{x_n}{x_j}) \in \tilde{J}(D(x_j)).$$

**Definition 14.4.** Suppose that $X$ is a projective variety and $I \subset S(X) = k[x_0,\ldots,x_n]$ is a homogeneous ideal, which is generated in a single degree $d$. Suppose that $I = (F_0,\ldots,F_r)$, where $F_0,\ldots,F_r$ are homogeneous generators of degree $d$. Let $\Lambda : X \rightarrow \mathbb{P}^n$ be the rational map induced by the regular map $\Lambda : X \setminus Z(I) \rightarrow (X \setminus Z(I)) \times \mathbb{P}^n$ defined by $\Lambda(p) = (p; F_0(p) : \ldots : F_r(p))$. Let $\tilde{\Lambda} : X \rightarrow \mathbb{P}^n$ be the projective map induced by the regular map $\tilde{\Lambda} : X \setminus Z(I) \rightarrow (X \setminus Z(I)) \times \mathbb{P}^n$ defined by $\tilde{\Lambda}(p) = (p; \overline{F}_0(p) : \ldots : \overline{F}_r(p))$.
Let \( p : F_0(p) : \ldots : F_r(p) \) for \( p \in X \setminus Z(I) \). The blow up of \( I \) is \( B(I) = \Gamma_\Lambda \), with projection \( \pi_1 : B(I) \to X \).

Proposition 14.1 is also true for projective varieties \( X \) and homogeneous ideals \( I \), which are generated in a single degree, so that \( B(I) \) is independent of choice of generators of \( I \) (of the same degree \( d \)). We see that the restriction of \( \pi_1 \) to \( \pi_1^{-1}(D(x_i)) \to D(x_i) \) is the blow up of the ideal \( \tilde{I}(D(x_i)) \in k[D(x_i)] \) for \( 0 \leq i \leq m \). More generally, for any affine open subset \( U \) of \( X \), the restriction of \( \pi_1 \) to \( \pi_1^{-1}(U) \to U \) is the blow up of the ideal \( \tilde{I}(U) \) in \( k[U] \).

**Theorem 14.5.** Suppose that \( X \subset \mathbb{P}^m \) and \( Y \subset \mathbb{P}^n \) are projective varieties, and \( \varphi : Y \to X \) is a birational regular map. Then \( \varphi \) is the blow up of a homogeneous ideal in \( S(X) \) (which is generated in a single degree).

**Proof.** Let \( \psi : X \to Y \) be the inverse rational map to \( \varphi \). By Theorem 7.7, there exist open subsets \( V \) of \( Y \) and \( U \) of \( X \) such that \( \varphi : V \to U \) is an isomorphism. Define a regular map \( \gamma : Y \to X \times X \) by \( \gamma(q) = (\varphi(q), q) \) for \( q \in Y \). \( \gamma \) is an isomorphism onto its image. We have that \( \pi_1(\gamma(q)) = \varphi(q) \) for \( q \in V \).

For \( q \in V \), \( \gamma(q) = (p, \psi(p)) \) where \( p = \varphi(q) \). Since \( \varphi : V \to U \) is an isomorphism, \( \gamma : V \to \Gamma_{\psi|U} \) is an isomorphism. Thus \( \gamma(Y) \subset \Gamma_{\psi|U} \), since \( Y \) is the closure of \( V \) and \( \Gamma_{\psi|U} \) is closed. \( \gamma(Y) \) is closed in \( X \times Y \) by Theorem 13.6, and contains \( \Gamma_{\psi|U} \). Let \( \overline{\Gamma_{\psi|U}} \) be the closure of \( \Gamma_{\psi|U} \) in \( X \times Y \).

\[
\Gamma_{\psi} = \Gamma_{\psi|U} \subset \gamma(Y) \subset \Gamma_{\psi},
\]

so that \( \Gamma_{\psi} = \gamma(Y) \). We thus have a commutative diagram

\[
Y \xrightarrow{\gamma} X \xrightarrow{\varphi} \Gamma_{\psi} \xrightarrow{\pi} X,
\]

where \( \gamma \) is an isomorphism.

Choose forms \( F_0, \ldots, F_r \in S(X) \) of a common degree, so that \( (F_0 : \ldots : F_r) \) is a representative of the rational map \( \psi \). Then \( \pi : \Gamma_{\psi} \to X \) is the blowup of the ideal \( I = (F_0, \ldots, F_r) \).

Suppose that \( \Lambda : X \to Y \) is a rational map of projective varieties, with \( Y \subset \mathbb{P}^n \). Suppose that \( (F_0 : \ldots : F_n) \) represents the rational map \( \Lambda \); that is, \( F_0, \ldots, F_n \in S(X) \) have the same degree, \( U = X \setminus Z(F_0, \ldots, F_n) \neq \emptyset \), and \( \varphi|U = (F_0 : \ldots : F_n) \).

\( \Gamma_{\Lambda} \) is the closure of \( \Gamma_{\Lambda|V} \) in \( X \times Y \) for any dense open subset \( V \) of \( X \) on which \( \Lambda \) is defined (\( \Gamma_{\Lambda|V} \) is the image of \( V \) in \( V \times Y \) of the map \( p \mapsto (p, \Lambda(p)) \)). We thus have that

\[
\Gamma_{\Lambda} = \Gamma_{(F_0, \ldots, F_n)} = B(I)
\]

where \( I = (F_0, \ldots, F_n) \subset S(X) \). We obtain the statement that if \( (F_0 : \ldots : F_n) \) and \( (G_0 : \ldots : G_n) \) are two representations of \( \Lambda \), so that \( F_0, \ldots, F_n \) and \( G_0, \ldots, G_n \) are homogeneous of a common degree, \( G_0, \ldots, G_n \) are homogeneous of a common degree, and

\[
F_iG_j - F_jG_i = 0 \text{ for } 0 \leq i, j \leq n,
\]

then \( B(I) \) is isomorphic to \( B(J) \), where \( I = (F_0, \ldots, F_n) \) and \( J = (G_0, \ldots, G_n) \). In general, \( Z(I) \neq Z(J) \) for two such representations, and \( I \neq J \).
The reason this works out is that two very different ideals can have the same blow up. For instance, if X is affine, \( I \subset k[X] \) is an ideal and \( 0 \neq f \in k[X] \), then \( B(I) \) is isomorphic to \( B(fI) \). In particular, \( X \) is isomorphic to \( B(fk[X]) \) for any \( f \in k[X] \).

As an example, consider the projective variety \( A = Z(xy - zw) \subset \mathbb{P}^3 \). Let \( S(W) = k[x, y, z, w]/(xy - zw) = k[\overline{x}, \overline{y}, \overline{z}, \overline{w}] \). Consider the regular map \( \varphi : A \to \mathbb{P}^1 \) which has the representations

\[
\varphi = (x : z) = (w : y).
\]

Let \( I = (x, z) \) and \( J = (w, y) \). Since \( \varphi \) is regular, we have that \( B(\tilde{I}) \) and \( B(\tilde{J}) \) are isomorphic to \( A \). We can verify this directly by computing the blow ups directly on the affine cover \( \{D(x), D(y), D(z), D(w)\} \) of \( A \). For instance on \( D(x) \), we have

\[
k[A \cap D(x)] = k[\frac{\overline{y}}{\overline{x}}, \frac{\overline{z}}{\overline{x}}, \frac{\overline{w}}{\overline{x}}] = k[\frac{\overline{y}}{\overline{x}}, \frac{\overline{z}}{\overline{x}}, \frac{\overline{w}}{\overline{x}}]/(\frac{\overline{y}}{\overline{x}} - \frac{\overline{z} \overline{w}}{\overline{x}}).
\]

We have

\[
\tilde{I}(D(x)) = (1, \frac{\overline{y}}{\overline{x}}) = k[A \cap D(x)]
\]

and

\[
\tilde{J}(D(x)) = (\frac{\overline{w}}{\overline{x}}, \frac{\overline{y}}{\overline{x}}) = (\frac{\overline{w}}{\overline{x}}, \frac{\overline{y}}{\overline{x}}) = (\frac{\overline{w}}{\overline{x}}),
\]

and we see that both ideals are principal ideals.

**Lemma 14.6.** Suppose that \( R \) is a Noetherian ring and \( P \subset R \) is a prime ideal. Suppose that \( J, A \) are ideals in \( R \) such that \( P \not\subset J, A \subset P \) and the localizations \( A_Q = P_Q \) for \( Q \) a prime ideal in \( R \) such that \( J \not\subset Q \). Then

\[
P = A : J^\infty := \{ f \in R \mid fJ^n \subset P \text{ for some } n \geq 0 \}.
\]

**Proof.** Suppose that \( Q \) is a prime ideal of \( R \). We have that the \( R \)-modules \( (P/A)_Q \) and \( P_Q/A_Q \) are isomorphic (by Corollary 3.4 iii) \([2]\), so that \( (P/A)_Q = 0 \) if \( J \not\subset Q \). Since \( P/A \) is a finitely generated \( R \)-module (as \( R \) is noetherian) some power of \( J^n \) of \( J \) annihilates \( P/A \) (by Exercise 19 v) page 46 of \([2]\) or Corollary 2, page 106 \([3]\)). Thus \( J^nP \subset A \), so \( P \subset A : J^\infty \).

Now suppose that \( f \in R \) is such that \( fJ^n \subset P \) for some \( n \geq 0 \). Since \( P \) is a prime ideal, and there exists an element of \( J^n \) which is not in \( P \), we have that \( f \in P \). Thus \( A : J^\infty \subset P \). \( \square \)

**Theorem 14.7.** Suppose that \( X \) is an affine variety, and \( J \subset k[X] \) is an ideal. Let \( \pi : B(J) \to X \) be the blow up of \( J \). Then the coordinate ring of \( B(J) \) is

\[
S(B(J)) \cong \bigoplus_{i \geq 0} J^i
\]

as a graded \( k[X] \)-algebra. Suppose that \( J = (f_0, \ldots, f_n) \), so that \( B(J) \subset X \times \mathbb{P}^n \). Let \( R = k[X] \) and \( y_0, \ldots, y_n \) be homogeneous coordinates on \( \mathbb{P}^n \). Then

\[
\mathcal{O}_{B(J)}(D(y_i)) = R[\frac{f_0}{f_i}, \ldots, \frac{f_n}{f_i}]
\]

for \( 0 \leq i \leq n \). Let \( A = (y_i f_j - y_j f_i) \mid 0 \leq i, j \leq n \), an ideal in \( S(X \times \mathbb{P}^n) = k[X][y_0, \ldots, y_n] \). The ideal of \( B(J) \) in \( S(X \times \mathbb{P}^n) \) is

\[
I_{X \times \mathbb{P}^n}(B(J)) = A : S(X \times \mathbb{P}^n) J^\infty = \{ f \in S(X \times \mathbb{P}^n) \mid fJ^n \in A \text{ for some } n \geq 0 \}.
\]
Proof. $B(J)$ is defined to be the closure of $\varphi(X \setminus Z_X(f_0, \ldots, f_n))$ in $X \times \mathbb{P}^n$, where $\varphi(p) = (p; f_0(p) : \ldots : f_n(p))$. The coordinate ring of $X \times \mathbb{P}^n$ is $R[y_0, \ldots, y_n]$. Let $A = (y_1 f_j - y_j f_i \mid 0 \leq i, j \leq n)$, an ideal in $R[y_0, \ldots, y_n]$ which is contained in $I(\Gamma_\varphi)$.

Since $f_i$ becomes a unit in $R$, we have that $\Gamma_\varphi | D(x_i) \subset D(f_i) \times \mathbb{P}^n$ is isomorphic to $X$. Further, we calculate (using the fact that $f_i$ is a unit in $R_{f_i}$) that

$$I_{D(f_i) \times \mathbb{P}^n}((\Gamma_\varphi | D(x_i))) = A_{f_i}$$

in

$$S(D(f_i) \times \mathbb{P}^n) = R_{f_i}[y_0, \ldots, y_n] = S(X \times \mathbb{P}^n)_{f_i}.$$ 

Thus if $Q$ is a prime ideal in $S(X \times \mathbb{P}^n)$ such that $f_i \notin Q$, we have that $A_Q = I(\Gamma_\varphi)_{f_i}$.

Since this is true for $0 \leq i \leq n$, we have that $A_Q = I(\Gamma_\varphi)_{f_i}$ for $Q$ a prime ideal in $R[y_0, \ldots, y_n]$ such that $(f_1, \ldots, f_n) \not\subset Q$. By Lemma 14.6, we have that

$$A : J^\infty = I(\Gamma_\varphi).$$

Let $t$ be an indeterminate and let $P$ be the kernel of the graded $k$-algebra homomorphism

$$R[y_0, \ldots, y_n] \to R[t f_0, \ldots, t f_n] \subset R[t]$$

developing by mapping $y_j \mapsto t f_j$. $P$ is a prime ideal since $R[t]$ is a domain. We have that $A \subset P$ and for a prime ideal $Q$ in $R[y_0, \ldots, y_n]$, we have that $A_Q = P_Q$ if $J \not\subset Q$ (this follows since after localizing at such a $Q$, some $f_i$ becomes invertible). Thus by Lemma 14.6, $P = A : J^\infty = I(\Gamma_\varphi)$, and the coordinate ring of $B(J)$ is

$$R[y_0, \ldots, y_n]/P \cong R[t f_0, \ldots, t f_n] \cong \bigoplus_{i=0}^\infty J^i.$$

We have

$$\mathcal{O}_{B(J)}(B(J) \cap D(y_j)) \cong R[t f_0/t f_i, \ldots, t f_n/t f_i] = R[t f_0/t f_i, \ldots, t f_n/t f_i] \subset k(X).$$

Proposition 14.8. Suppose that $X$ is an affine variety and $J \subset k[X]$ is an ideal. Let $W$ be a closed subvariety of $X$, and let $\overline{J} = J k[W]$. Then the strict transform of $W$ in $B(J)$ is isomorphic to $B(\overline{J})$.

Proof. Let $f_0, \ldots, f_n$ be a set of generators of $J$. Let $\overline{f}_i$ be the residues of $f_i$ in $k[W]$. Let $\varphi : X \to X \times \mathbb{P}^n$ be the rational map $\varphi = i \times (f_0 : \ldots : f_n)$ and $\overline{\varphi} : W \to W \times \mathbb{P}^n$ be the rational map $\overline{\varphi} = i \times (\overline{f}_0 : \ldots : \overline{f}_n)$. We have a commutative diagram, where the vertical maps are the natural inclusions:

$$
\begin{array}{ccc}
X \setminus Z(J) & \xrightarrow{\varphi} & X \times \mathbb{P}^n \\
\uparrow & & \uparrow \\
W \setminus Z(\overline{J}) & \xrightarrow{\overline{\varphi}} & W \times \mathbb{P}^n
\end{array}
$$

Now $B(J)$ is the closure of $\varphi(X \setminus Z(J))$ in $X \times \mathbb{P}^n$, $B(\overline{J})$ is the closure of $\overline{\varphi}(W \setminus Z(\overline{J}))$ in $W \times \mathbb{P}^n$, and the strict transform of $W$ in $B(J)$ is the closure of $\varphi(W \setminus Z(J))$ in $B(J)$. The conclusions of the proposition thus follow from the above diagram. 

The blow up of a subvariety $Y$ of a projective variety $X$ is the blow up of a homogeneous ideal $I$, which has a set of generators of a common degree, such that $\overline{I}(U) = \overline{I}_X(Y)(U)$ for all affine open sets $U \subset X$. We find such an ideal by applying Lemma 14.3 to $I_X(Y)$. 

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Suppose that \( f : X \to Y \) is a birational regular map of quasi projective varieties. Let \( U \) be the largest open subset of \( Y \) on which the inverse of \( f \) is defined, and is a regular map. Suppose that \( Z \) is a subvariety of \( Y \). The strict transform of \( Z \) by \( f \) is the closure of \( f^{-1}(Z \cap U) \) in \( X \).

Let \( p \) be the origin in \( X = \mathbb{A}^2 \). Let \( m = (x_1, x_2) \) be the ideal of \( p \) in \( k[\mathbb{A}^2] = k[x_1, x_2] \). Let \( \pi : B \to X \) be the blow up of \( p \). The coordinate ring of \( B \) (as a subvariety of \( X \times \mathbb{P}^1 \)) is \( S(B) = k[\mathbb{A}^2][tx_1, tx_2] \). If \( y_0, y_1 \) are homogeneous coordinates on \( \mathbb{P}^1 \), we have that \( \{X \times D(y_0), X \times D(y_1)\} \) is an affine cover of \( X \times \mathbb{P}^1 \). Thus
\[
\{B_1 = (X \times D(y_0)) \cap B, B_2 = (X \times D(y_1)) \cap B\}
\]
is an affine cover of \( B \).

\[
k[B_1] = k[\mathbb{A}^2][\frac{tx_2}{tx_1}] = k[x_1, x_2, \frac{x_2}{x_1}] = k[x_1, \frac{x_2}{x_1}]
\]
since \( x_2 = x_1 \frac{x_2}{x_1}, x_2 \) and \( \frac{x_2}{x_1} \) are algebraically independent over \( k \), since \( x_1, x_2 \) are. Thus \( k[B_1] \) is a polynomial ring in these two variables, so that \( B_1 \) is isomorphic to \( \mathbb{A}^2 \). Similarly, we have that
\[
k[B_2] = k[x_2, \frac{x_1}{x_2}]
\]
since \( x_1 = x_2 \frac{x_1}{x_2} \) and \( B_2 \) is isomorphic to \( \mathbb{A}^2 \).

Let \( \pi_1 = \pi|B_1 \) and \( \pi_2 = \pi|B_2 \). We have that \( \pi_1(a_1, a_2) = (a_1, a_1a_2) \) for \( (a_1, a_2) \in \mathbb{A}^2 \) and \( \pi_2(b_1, b_2) = (b_1b_2, b_1) \) for \( (b_1, b_2) \in \mathbb{A}^2 \). Let \( E = \pi^{-1}(p) \). \( E = Z(x_1, x_2) \), so \( E \cap B_1 \) has the equation \( x_1 = 0 \) and \( E \cap B_2 \) has the equation \( x_2 = 0 \).

\( E = \pi^{-1}(p) \) is the algebraic set \( Z_B(x_1, x_2) \subset B \subset X \times \mathbb{P}^1 \). We compute
\[
S(B)/(x_1, x_2)S(B) = (k[\mathbb{A}^2][tx_1, tx_2])/(x_1, x_2) \\
\cong ([k[\mathbb{A}^2]/(x_1, x_2)][tx_1, tx_2]) \\
= k[tx_1, tx_2].
\]

Since \( tx_1 \) and \( tx_2 \) are algebraically independent over \( k \), this ring is isomorphic to a graded polynomial ring in two variables, which is the coordinate ring of \( \mathbb{P}^1 \). Since \( E \subset \{p\} \times \mathbb{P}^1 \), we have that \( E = \{p\} \times \mathbb{P}^1 \), with \( I(E) = (x_1, x_2) \).

Suppose that \( C \subset \mathbb{A}^2 \) is a curve which contains \( p \). Let \( f(x_1, x_2) \in k[\mathbb{A}^2] \) be an equation of \( C \). Write
\[
f = \sum_{i+j \geq r} a_{ij} x_1^i x_2^j
\]
where \( a_{ij} \in k \) and some \( a_{ij} \neq 0 \) with \( i + j = r \) (\( r \) is the order of \( f \)). Let \( \tilde{C} \) be the strict transform of \( C \) in \( B \). In \( B_1 \), we have that
\[
f = \sum a_{ij} x_1^{i+j} \left(\frac{x_2}{x_1}\right)^j = x_1^r f_1(x_1, \frac{x_2}{x_1})
\]
where
\[
f_1 = \sum a_{ij} x_1^{i+j-r} \left(\frac{x_2}{x_1}\right)^j
\]
is an equation of \( \tilde{C} \cap B_1 \). Similarly,
\[
f_2 = \sum a_{ij} \left(\frac{x_1}{x_2}\right)^i x_2^{i+j-r}
\]
is an equation of \( \tilde{C} \cap B_2 \).

In this particular example, we have the desirable condition that \( I(B) = (x_1y_1 - x_2y_0) = A \), so \( I(B) \) is actually generated by the obvious relations \( A \).
15. Finite Maps of Quasi Projective Varieties

15.1. Affine and Finite Maps.

Definition 15.1. Suppose that $X$ and $Y$ are quasi projective varieties and $\varphi : X \to Y$ is a regular map.

1. $\varphi$ is affine if for every $q \in Y$ there exists an affine neighborhood $U$ of $q$ in $Y$ such that $\varphi^{-1}(U)$ is an affine open subset of $X$.
2. $\varphi$ is finite if for every $q \in Y$ there exists an affine neighborhood $U$ of $q$ in $Y$ such that $\varphi^{-1}(U)$ is an affine open subset of $X$ and $\varphi : \varphi^{-1}(U) \to U$ is a finite map of affine varieties (as defined in Definition 4.10).

It is not so difficult to verify directly that if $X$ and $Y$ are affine varieties, and $\varphi : X \to Y$ is a finite map of quasi projective varieties, then $\varphi$ is a finite map of affine varieties (as defined in Definition 4.10). Certainly this follows from the much more general statement of Theorem 15.5 below.

Lemma 15.2. Suppose that $X$ is an affine variety. Then $X$ is quasi compact (every open subcover has a finite subcover).

Lemma 15.3. Suppose that $A$ is a domain and $f_1, \ldots, f_n \in A$ are such that the ideal $(f_1, \ldots, f_n) = A$. Suppose that $N$ is a positive integer. Then $(f_1^N, \ldots, f_n^N) = A$.

Lemma 15.4. Suppose that $A$ is a domain which is a subring of a domain $B$, and there exist $f_1, \ldots, f_n \in A$ such that

1. The ideal $(f_1, \ldots, f_n) = A$ and
2. The localization $B_{f_i}$ is a finitely generated $A_{f_i}$-algebra for all $i$.

Then $B$ is a finitely generated $A$-algebra.

Further suppose that $B_{f_i}$ is a finitely generated $A_{f_i}$-module for all $i$. Then $B$ is a finitely generated $A$-module.

Proof. By assumption, there exist $r_i \in \mathbb{Z}_+$ for $1 \leq i \leq n$ and $z_{i1}, \ldots, z_{ir_i} \in B_{f_i}$ for $1 \leq i \leq n$ such that $B_{f_i} = A_{f_i}[z_{i1}, \ldots, z_{ir_i}]$.

After possibly multiplying the $z_{ij}$ by a positive power of $f_i$, we may assume that $z_{ij} \in B$ for all $i, j$. Let $C = A[\{z_{ij}\}]$.

$C$ is a finitely generated $A$-algebra which is a subring of $B$. We will show that $B = C$.

Suppose that $b \in B$. Then $b \in B_{f_i}$ implies there are polynomials $g_i \in A_{f_i}[x_1, \ldots, x_{r_i}]$ such that $b = g_i(z_{i1}, \ldots, z_{ir_i})$ for $1 \leq i \leq n$. Since the polynomials $g_i$ have only a finite number of nonzero coefficients, which are in $A_{f_i}$, there exists a positive integer $N$ such that $f_i^N g_i \in A[x_1, \ldots, x_{r_i}]$ for $1 \leq i \leq n$. Thus

$$f_i^N b = f_i^N(g_i(z_{i1}, \ldots, z_{ir_i})) \in A[z_{i1}, \ldots, z_{ir_i}] \subset C$$

for all $i$. By Lemma 15.3, there exist $c_i \in A$ such that $\sum c_i f_i^N = 1$. Thus

$$b = (\sum c_i f_i^N) b = \sum c_i f_i^N b \in C.$$

The proof that $C$ is finite over $A$ if the $B_{f_i}$ are finite over $A_{f_i}$ is a variation of the above argument. \qed

Theorem 15.5. Suppose that $\varphi : X \to Y$ is a regular map of quasi projective varieties.
1. Suppose that \( \varphi \) is affine. Suppose that \( U \) is an affine open subset of \( Y \). Then \( V = \varphi^{-1}(U) \) is an affine open subset of \( X \).

2. Suppose that \( \varphi \) is finite. Suppose that \( U \) is an affine open subset of \( Y \). Then \( V = \varphi^{-1}(U) \) is an affine open subset of \( X \), and the restriction of \( \varphi \) to a regular map from \( V \) to \( U \) is a finite map of affine varieties.

**Proof.** Let \( A = k[U] = \mathcal{O}_Y(U) \). Suppose that \( q \in U \) and let \( V \) be an affine neighborhood of \( q \) in \( Y \) such that \( \varphi^{-1}(V) \) is affine. Then there exist \( f \in k[Y] \subset A \) such that \( q \in D(f) \subset U \cap V \). Since \( f \) is open in this form, there exist \( q \in \mathcal{D}(f) = A \) and \( \varphi^{-1}(\mathcal{D}(f)) \) is affine for all \( i \). Thus \( Z_U(f_1, \ldots, f_n) = \emptyset \), so \( \sqrt{(f_1, \ldots, f_n)} = I(Z_U(f_1, \ldots, f_n)) = A \). Thus

\[
\text{(22)} \quad (f_1, \ldots, f_n) = A.
\]

Let \( V_i = \varphi^{-1}(D(f_i)) \) for \( 1 \leq i \leq n \). The \( V_i \) are an affine cover of \( V \). Let \( B = \mathcal{O}_Y(V) \). \( \varphi \) gives us a \( k \)-algebra homomorphism \( \varphi^* : A \to B \subset k(X) \). Let \( B_i = \mathcal{O}_Y(V_i) = k[V_i] \) for \( 1 \leq i \leq n \). The restriction of \( \varphi \) to \( V_i \) gives 1-1 \( k \)-algebra homomorphisms \( \varphi^*: k[D(f_i)] = A_{f_i} \to B_i = k[V_i] \subset k(X) \), realizing \( B_i \) as finitely generated \( A_{f_i} \)-algebras (\( B_i \) are finitely generated \( k \)-algebras since the \( V_i \) are affine). Now \( V_i \cap V_j \) is precisely the open subset \( D(\varphi^*(f_j)) \) of \( V \), so for all \( i, j \), \( V_i \cap V_j \) is affine with regular functions

\[
k[V_i \cap V_j] = (B_i)_{\varphi^*(f_j)} = (B_j)_{\varphi^*(f_i)} \subset k(X)
\]

by Propositions 6.3 and 6.6.

Since \( \varphi^*(f_j) \) does not vanish on \( V_j \), \( \varphi^*(f_j) \) is a unit in \( B_j \), so \( (B_j)_{\varphi^*(f_j)} = B_j \), and

\[
B_j \subset (B_j)_{\varphi^*(f_i)} = (B_i)_{\varphi^*(f_j)}
\]

for all \( j \). Now

\[
B = \mathcal{O}_Y(V) = \cap_{i=1}^n \mathcal{O}_Y(V_i) = \cap_{i=1}^n B_i.
\]

We compute

\[
B_{\varphi^*(f_j)} = (\cap_{i=1}^n B_i)_{\varphi^*(f_j)} = (\cap_{i=1}^n (B_i)_{\varphi^*(f_i)}) = B_j.
\]

By Lemma 15.4, \( B \) is a finitely generated \( A \)-algebra, and since \( A \) is a finitely generated \( k \)-algebra, \( B \) is a finitely generated \( k \)-algebra. Thus there exists an affine variety \( Z \) such that \( k[Z] = B \). Let \( t_1, \ldots, t_m \in B \) generate \( B \) as a \( k \)-algebra (the \( t_i \) are coordinate functions on \( Z \). Since \( B = \mathcal{O}_X(V) \), \( \alpha = (t_1, \ldots, t_m) \) induces a regular map \( \alpha : Z \to V \). Now \( \alpha_i = \alpha \mid V_i \) induce isomorphisms \( \alpha_i : V_i \to D_Z(\varphi^*(f_i)) \) of affine varieties for all \( i \), since \( \alpha_i^* \) induces an isomorphism of regular functions. We may thus define isomorphic regular maps \( \psi_i : D_Z(\varphi^*(f_i)) \to V_i \) by requiring that \( (\psi_i)^* = (\alpha_i^*)^{-1} \). The \( \psi_i \) patch to give a continuous map \( \psi : V \to Z \) which is a regular map by Proposition 10.10. Since \( \psi \) is an inverse to \( \alpha \), we have that \( V \cong Z \) is an affine variety.

It is now not difficult to verify that \( V \to U \) is a finite map of affine varieties, if \( X \to Y \) is a finite map of quasi projective varieties.

\[\square\]

### 15.2. Finite Mappings.

**Theorem 15.6.** Suppose that \( X \) and \( Y \) are quasi projective varieties and \( \varphi : X \to Y \) is a finite regular map. Then \( \varphi \) is a closed map and \( \varphi \) is surjective.
Proof. It suffices to prove this statement for the members of an affine cover \( \{ V_i \} \) of \( Y \), and the maps \( \varphi : U_i \to V_i \) where \( U_i = \varphi^{-1}(V_i) \). We either choose the \( V_i \) so that \( \varphi^{-1}(U_i) \) are affine with \( \varphi : U_i \to V_i \) affine, or we pick an arbitrary affine cover \( \{ V_i \} \) and apply Theorem 15.5 to get this statement. We obtain that \( \varphi : U_i \to V_i \) is a closed mapping by Corollary 4.13. \( \square \)

Theorem 15.7. Suppose that \( X \) and \( Y \) are quasi-projective varieties and \( \varphi : X \to Y \) is a dominant regular map. Then \( \varphi(X) \) contains an open set of \( Y \).

Proof. It suffices to prove this for the map between an affine open subset \( V \) of \( Y \) and the restriction of \( \varphi \) to an affine open subset \( U \) contained in the preimage of \( U \). Thus we may assume that \( X \) and \( Y \) are affine. Let \( r \) be the transcendence degree of \( k(X) \) over \( k(Y) \). Let \( u_1, \ldots, u_r \in k[X] \) be such that \( u_1, \ldots, u_r \) is a transcendence basis of \( k(X) \) over \( k(Y) \). Then

\[
k[Y] \subset k[Y]_1 \subset \ldots \subset k[Y]_r = k[Y \times \mathbb{A}^r] \subset k[X].
\]

Thus \( \varphi \) factors as the composition \( \varphi = g \circ h \) where \( h : X \to Y \times \mathbb{A}^r \) and \( g : Y \times \mathbb{A}^r \to Y \) is the projection onto the second factor.

Every element \( v \in k[X] \) is algebraic over \( k(Y \times \mathbb{A}^r) \). Hence there exists for it an element \( a \in k[Y \times \mathbb{A}^r] \) such that \( av \) is integral over \( k[Y \times \mathbb{A}^r] \). Let \( v_1, \ldots, v_m \) be coordinate functions on \( X \) (so that \( k[X] = k[v_1, \ldots, v_m] \)). For each \( v_i \) choose \( a_i \in k[Y \times \mathbb{A}^r] \) such that \( a_iv_i \) is integral over \( k[Y \times \mathbb{A}^r] \). Let \( F = a_1 \cdots a_m \). Then \( [X]_{h^*(F)} \) is integral over \( k[Y \times \mathbb{A}^r]_F \), so that \( h : D(h^*(F)) = D(F) \) is finite. Thus \( D(F) \subset h(X) \) by Theorem 15.6. It remains to show that \( g(D(F)) \) contains a set that is open in \( Y \).

We have an expression

\[
F = \sum f_{i_1, \ldots, i_r} u_1^{i_1} \cdots u_r^{i_r} \in [Y \times \mathbb{A}^r] = [Y][u_1, \ldots, u_r]
\]

with \( f_{i_1, \ldots, i_r} \in k[Y] \). If \( p \in Y \) and some \( f_{i_1, \ldots, i_r}(p) \neq 0 \), then there exists a point \( q \in \mathbb{A}^r \) such that \( F(p, q) \neq 0 \) (by the Nullstellensatz). Thus \( \cup D(f_{i_1, \ldots, i_r}) = Y \setminus Z(\{ f_{i_1, \ldots, i_r} \}) \subset g(D(F)) \). \( \square \)

Theorem 15.8. Suppose that \( X \) is a projective variety which is a closed subset of a projective space \( \mathbb{P}^n \), and \( X \subset \mathbb{P}^n \setminus E \) where \( E \) is a dimensional linear subspace. Then the projection \( \pi : X \to \mathbb{P}^{n-d-1} \) from \( E \) determines a finite map \( X \to \pi(X) \).

Proof. Let \( y_0, \ldots, y_{n-d-1} \) be homogeneous coordinates on \( \mathbb{P}^{n-d-1} \) and let \( L_0, \ldots, L_{n-d-1} \) be a basis of the vector space of linear forms vanishing \( E \). Define \( \pi \) by the formula \( \pi = (L_0 : \ldots : L_{n-d-1}) \). \( \pi \) is a regular map on \( X \) since \( E \cap X = \emptyset \), so the forms \( L_0, \ldots, L_{n-d-1} \) do not vanish simultaneously on \( X \).

Let \( U_i = \pi^{-1}(D(y_i)) \cap X = D(L_i) \cap X \). \( U_i \) is thus an affine open subset of \( X \). We will show that for all \( i \) such that \( U_i \neq \emptyset \), \( U_i \to \pi(X) \cap D(y_i) \) is a finite map. \( \pi(X) \) is a closed subset of \( \mathbb{P}^{n-d-1} \) by Theorem 13.6. Hence \( \pi(X) \) is a projective variety and \( \pi(X) \cap D(y_i) \) is an affine open subset of \( \pi(X) \).

Every function \( g \in k[U_i] \) is the restriction of a form \( \frac{G}{L_1 \cdots L_{n-d}} \) where \( m \) is the degree of the homogeneous form \( G \in S(\mathbb{P}^n) \) by formula (14). Let \( z_0, \ldots, z_{n-d} \) be homogeneous coordinates on \( \mathbb{P}^{n-d} \), and define a rational map \( \pi_1 = (L_0^m : \ldots : L_{n-d}^m) : G \) from \( \mathbb{P}^n \) to \( \mathbb{P}^{n-d} \). \( \pi_1 \) induces a regular map of \( X \) and its image \( \pi_1(X) \) is closed in \( \mathbb{P}^{n-d} \) by Theorem 13.6. Let \( F_1, \ldots, F_s \) be a set of generators of \( I(\pi_1(X)) \subset S(\mathbb{P}^{n-d}) \). As \( X \cap E = \emptyset \), the forms \( L_0, \ldots, L_{n-d-1} \) do not vanish simultaneously on \( X \). Thus the point \( (0 : \ldots : 0 : 1) \) is not contained in \( \pi_1(X) \), so that

\[
Z_{\mathbb{P}^{n-d}}(z_0, \ldots, z_{n-d-1}, F_1, \ldots, F_s) = \emptyset.
\]
By Proposition 9.5, we have that $T_i \subset (z_0, \ldots, z_{n-d-1}, F_1, \ldots, F_s)$ for some $l > 0$, where $T_i$ is the vector space of forms of homogeneous forms of degree $l$ on $\mathbb{P}^{n-d}$. In particular, we have an expression

$$z^l_{n-d} = \sum_{j=0}^{n-d-1} z_j H_j + \sum_{j=1}^{s} F_j P_j$$

where $H_j, P_j \in S(\mathbb{P}^{n-d})$ are polynomials. Denoting by $H^{(q)}$ the homogeneous component of $H$ of degree $q$, we have that

$$\Phi(z_0, \ldots, z_{n-d}) = \sum_{j=0}^{n-d-1} z_j H_j^{(l-1)}.$$ 

we have that $\Phi \in I(\pi_1(X))$. The homogeneous polynomial $\Phi$ has degree $l$, and as a polynomial in $z_{n-d}$ it has the leading coefficient 1, so that it has an expression

$$\Phi = \sum_{j=0}^{n-d-1} A_{l-j}(z_0, \ldots, z_{n-d-1}) z_j^{l-1}.$$ 

Substitution of the defining formulas $\pi_1^+(z_i) = L_i^m$ for $0 \leq i \leq n-d-1$ and $\pi_1^+(z_{n-d}) = G$ induces a $k$-algebra homomorphism $\pi_1^+: S(\mathbb{P}^{n-d}) \to S(\mathbb{P}^n)$. Since the $F_i$ vanish on $\pi_1(X)$, we have that $\pi_1^+(F_i) \in I(X)$. We thus have that

$$\pi_1^+(\Phi) = \Phi(L_0^m, \ldots, L_{n-d-1}^m, G) \in I(X)$$

is a homogeneous form of degree $lm$ in $S(\mathbb{P}^n)$. Dividing this form by $L_i^{ml}$, we obtain a relation

$$\left(\frac{G}{L_i^m}\right)^l + \sum_{j=0}^{l-1} A_{l-j}(x_0^{m}, \ldots, 1, \ldots, x_{n-d-1}^{m}) \left(\frac{G}{L_i^m}\right)^j \in I(X \cap D(L_i)) \subset k[D(L_i)],$$

where $x_r = \frac{y_r}{y_1}$ are coordinates on $D(y_i) \cong \mathbb{A}^{n-d-1}$.

Since $k[\pi(X) \cap D(y_i)] = k[D(y_i)]/I(\pi(X) \cap D(y_i)) = k[\frac{y_0}{y_1}, \ldots, \frac{y_{n-d-1}}{y_n}]/I(\pi(X) \cap D(y_i))$ and $k[X \cap D(L_i)] = k[D(L_i)]/I(X \cap D(L_i))$, and $g$ is the residue of $\frac{G}{L_i^m}$ in $k[X \cap D(L_i)]$, we obtain the desired dependence relation.

**Remark 15.9.** Looking back at the proof, we see that we have also proved the following theorem. Writing the coordinate ring of $X$ as $S(X) = S(\mathbb{P}^n)/I(X)$ and the coordinate ring of $\pi(X)$ as $S(\pi(X)) = k[\overline{\mathbb{g}_0}, \ldots, \overline{\mathbb{g}_m}] = S(\mathbb{P}^m)/I(\pi(X))$, where $m = n-d-1$, we showed that the 1-1 graded $k$-algebra homomorphism

$$\varphi^*: S(\pi(X)) \to S(X)$$

defined by $\varphi^*(\overline{\mathbb{g}_i}) = L_i$ for $0 \leq i \leq m$ makes $S(X)$ an integral extension of $S(\pi(X))$. By applying this theorem to a Veronese embedding of $X$, or modifying the proof using formula (21) instead of (14), we obtain the following generalization.

**Theorem 15.10.** Let $F_0, \ldots, F_s$ be linearly independent forms of degree $m$ on a $\mathbb{P}^n$ that do not vanish simultaneously on a closed subvariety $X \subset \mathbb{P}^n$. Then $\varphi = (F_0 : \ldots : F_s)$ determines a finite mapping $X \to \varphi(X)$. 44
Corollary 15.11. Suppose that $X$ is a projective variety. Then there exists a finite map $\varphi : X \to \mathbb{P}^n$ onto a projective space.

Proof. $X$ is a closed subset of a projective space $\mathbb{P}^n$. If $X \neq \mathbb{P}^n$, choose a point $p \in \mathbb{P}^n \setminus X$ and let $\pi : X \to \mathbb{P}^n$ be the projection from $p$. $X \to \pi(X)$ is a finite map and $\pi(X)$ is a projective variety which is a closed subset of $\mathbb{P}^{n-1}$. We continue until the image of $X$ is the whole ambient projective space. A composition of finite maps is finite so the resulting map is finite. \qed

Corollary 15.12. (Projective Nullstellensatz) Suppose that $R$ is the coordinate ring of a projective variety. Then there exist linear forms $L_0, \ldots, L_m$ in $R$ such that the graded $k$-algebra homomorphism

$$\varphi^* : k[x_0, \ldots, x_m] \to R$$

is an integral extension, where $k[x_0, \ldots, x_m]$ is a polynomial ring and $\varphi^*(x_i) = L_i$ for $0 \leq i \leq m$.

Proof. This statement follows from Corollary 15.11 and Remark 15.9. \qed

15.3. Construction of the Normalization. Suppose that $R$ is an integrally closed ring and $S$ is a multiplicative set. Then the localization $S^{-1}R$ is integrally closed.

Definition 15.13. Suppose that $X$ is a quasi projective variety, and $p \in X$. $p$ is a normal point of $X$ if $\mathcal{O}_{X,p}$ is integrally closed. $X$ is normal if all points of $X$ are normal points of $X$.

Proposition 15.14. Suppose that $X$ is a normal quasi projective variety. Then $\mathcal{O}_X(X)$ is integrally closed.

Proof. Suppose that $f \in k(X)$ is integral over $\mathcal{O}_X(X)$. Then for all $p \in X$, $f$ is integral over $\mathcal{O}_{X,p}$, so that $f \in \mathcal{O}_{X,p}$. Thus

$$f \in \cap_{p \in X} \mathcal{O}_{X,p} = \mathcal{O}_X(X).$$

\qed

Theorem 15.15. Suppose $X$ is quasi projective variety and $\Lambda : k(X) \to L$ is a $k$-algebra homomorphisms of fields, such that $L$ is a finite extension of $k(X)$. Then there is a unique normal quasi projective variety $Y$ with function field $k(Y) = L$, and a finite regular map $\pi : Y \to X$ such that $\pi^* : k(X) \to k(Y)$ is the homomorphism $\Lambda$.

If $X$ is affine then $Y$ is affine. If $X$ is projective then $Y$ is projective.

We first prove uniqueness. Suppose that $\pi : Y \to X$ and $\pi' : Y' \to X$ each satisfy the conclusions of the Theorem. Suppose that $p \in X$ and that $U$ is an affine neighborhood of $p$ in $X$. Then $V = \pi^{-1}(U)$ is an affine open subset of $Y$ since $\pi$ is finite. $k[V]$ is integrally closed in $L$ and is finite over $k[U]$. Thus $k[V]$ is the integral closure of $k[U]$ in $L$. We thus have that $k[V'] = k[V]$ where $V' = (\pi')^{-1}(U)$, so that the identity map is an isomorphism of the affine varieties $V$ and $V'$. Since this holds for an affine cover of $X$, we have that $Y = Y'$.

We now prove existence for an affine variety $X$. Let $R$ be the integral closure of $k[X]$ in $L$. Then $R$ is a finitely generated $k$-algebra (by Theorem 4.7) which is a domain, so that $R = k[Y]$ for some affine variety $Y$. The inclusion $\Lambda : k[X] \to k[Y]$ induces a finite regular map $Y \to X$.

We now prove existence for a projective variety $X$, from which existence for a quasi projective variety follows. We begin with some preliminaries on graded rings.
Suppose that $A = \bigoplus_{i=0}^{\infty} A_i$ is a graded ring, which is a finitely generated $A_0 = k$ algebra. Then we can write $A = k[\overline{x}_1, \ldots, \overline{x}_n]$ where $\overline{x}_i \in A_{d_i}$ for some $d_i$; that is $\overline{x}_i$ has degree $d_i$. We can put a grading on the polynomial ring $U = k[x_1, \ldots, x_n]$ be letting the degree of $x_i$ be $d_i$. Let $U_d$ be the $k$-subspace of $U$ spanned by the monomials $x_1^{a_1} \cdots x_n^{a_n}$ of degree $d$; that is $a_1d_1 + \cdots + a_nd_n = d$. Then $U$ is a graded ring with $U = \bigoplus_{i=0}^{\infty} U_i$. The $k$-algebra homomorphism $\varphi : U \to A$ defined by $\varphi(f(x_1, \ldots, x_n)) = f(\overline{x}_1, \ldots, \overline{x}_n)$ is graded; that is if $F \in U_d$ then $\varphi(F) \in A_d$. The kernel $P$ of $\varphi$ is a weighted homogeneous prime ideal. That is, $P$ is generated by elements which are contained in the $U_i$. Writing $P = \bigoplus_{i=0}^{\infty} P_i$, we have that $U/P \cong \bigoplus_{i=0}^{\infty} U_i/P_i$. We have a graded isomorphism $U/P \cong A$.

For a graded ring $A$ and $d \in \mathbb{Z}_+$, we define a graded ring $A^{(d)}$ by $A^{(d)} = \bigoplus_{i=0}^{\infty} A_i^{(d)}$, where $A_i^{(d)} = A_{id}$.

**Lemma 15.16.** Suppose that $A = \bigoplus_{i=0}^{\infty} A_i$ is a graded ring, which is a finitely generated $A_0 = k$ algebra. Then there exists $d \in \mathbb{Z}_+$ such that $A^{(d)}$ is a standard graded $k$-algebra (generated by $A_1^{(d)}$).

We introduce the following convenient way of representing the direct sum $A \cong \bigoplus_{i=0}^{\infty} A_i$ concretely.

**Lemma 15.17.** Let $t$ be an indeterminate. Then $A$ is isomorphic as a graded ring to the subring $\bigoplus_{i=0}^{\infty} A_it^i$ of the polynomial ring $A[t]$, where $A[t]$ has the grading given by $\deg(f) = 0$ for $f \in A$ and $\deg(t) = 1$.

Now suppose that $X \subset \mathbb{P}^n$ is a projective variety, with homogeneous coordinate ring

$$R = S(X) = k[x_0, \ldots, x_n]/P = k[\overline{x}_0, \ldots, \overline{x}_n].$$

Let $\alpha \in R_1$ be a nonzero element, and let $\Sigma \subset R$ be the multiplicative system of non zero homogeneous elements. Then the localization $R\Sigma$ is graded.

**Lemma 15.18.** There is an isomorphism of graded rings $R\Sigma \cong k(X)[\alpha]$.

**Proof.** In our representation $R \cong \bigoplus_{i=0}^{\infty} R_it^i$, we have that a homogeneous element $\beta \in R\Sigma$ of degree $d$ has an expression $\beta = \frac{\alpha t^i}{f\alpha^d}$ where $\alpha \in R_i$ and $f \in R_m$ and $i - m = d$. Thus the elements of $R\Sigma$ of degree $0$ are exactly the elements of $k(X)$. If $\beta$ has degree $d > 0$, then we have

$$\beta = (\alpha t)^d \frac{a_i t^i}{m+d F_{\alpha^d}}.$$

where $\frac{a_i t^i}{m+d F_{\alpha^d}}$ has degree $0$. Thus $\frac{a_i t^i}{F_{\alpha^d}} \in k(X)$, and we have that $(R\Sigma)_d \cong k(X)(\alpha t)^d$. In particular $R\Sigma = k(X)[\alpha]$. \qed

Let $S$ be the integral closure of $R \cong k[R_1t]$ in the ring $L[\alpha] \cong L[\alpha t]$. Suppose that $f \in L[\alpha t]$ is integral over $R$. Then writing $f = \sum a_i t^i$ where each $a_i \in L[\alpha]$ has degree $i$, and remembering that $t$ is an indeterminate, we calculate that each $a_i$ is homogeneous over $R$, and has a dependence relation

$$a_i^n + f_1a_i^{m-1} + \cdots + f_m = 0 \quad (23)$$

where $f_j \in R_{ij}$.

Thus $S$ is a graded subring of $L[\alpha]$. Since $R$ is finitely generated over $k$, and $L[\alpha]$ is a finitely generated $k$-algebra, we have that the integral closure of $S$ in the field $L(\alpha)$ is a finite $R$ module by Theorem 4.7. Thus the submodule $S$ is a finitely generated $R$-module by Lemma 4.8.
Although $S$ is graded, it may be that $S$ is not generated in degree 1. By Lemma 15.16, there exists $d \in \mathbb{Z}_+$ such that $S^{(d)}$ is generated in degree 1.

We have that $R^{(d)} = R \cap k(X)[\alpha^d]$. We will show that $S^{(d)}$ is the integral closure of $R^{(d)}$ in $L[\alpha^d]$. If $x \in L[\alpha^d]$ is integral over $R^{(d)}$, then as an element of $L[\alpha]$, it is integral over $R$. Thus $x \in S \cap L[\alpha^d] = S^{(d)}$. If $x \in S^{(d)}$, then $x$ is integral over $R$ and since we have a homogeneous equation of integral dependence (23), where all of the coefficients lie in $R^{(d)}$, it is integral over $R^{(d)}$.

Choosing a basis of $S^{(d)}_1$, we have an isomorphism

$$S^{(d)} \cong k[y_0, \ldots, y_m]/P^* = k[y_0, \ldots, y_m]$$

where $P^*$ is a homogeneous prime ideal, and the $y_i$ all have degree 1. Let $Y \subset \mathbb{P}^m$ be the projective variety $Y = Z(P^*)$. We have $S(Y) \cong S^{(d)}$.

We will show that $Y$ is normal. $S^{(d)}$ is integrally closed in $L[\alpha^d]$. Since $L[\alpha^d]$ is isomorphic to a polynomial ring over a field, it is integrally closed in its quotient field. Hence $S^{(d)}$ is integrally closed in its quotient field. Thus the localization $S^{(d)}_{y_i}$ is integrally closed. By (19),

$$O_Y(D(y_i)) = \{ \frac{f}{y_i^m} \mid m \in \mathbb{Z}_+ \text{ and } f \in S^{(d)}_m \}$$

which is the set of elements of $S^{(d)}_{y_i}$ of degree 0. Thus it is the intersection $S^{(d)}_{y_i} \cap L$ taken within $L[\alpha^d]$, which is integrally closed in $L$. The localization $S^{(d)}_{y_i}$ is integrally closed since $S^{(d)}$ is. Since the local ring of every point of $Y$ is a localization of one of the normal local rings $O_Y(D(y_i))$, all of these local rings are integrally closed, so that $Y$ is normal.

By the Veronese map $\varphi : \mathbb{P}^n \to \mathbb{P}^e$ where $e = \binom{n+d}{n}$, we have an isomorphism of $X$ with a closed subset of $\mathbb{P}^e$, so that the coordinate ring of $\varphi(X)$ satisfies $S(\varphi(X)) \cong R^{(d)}$. Choosing a basis of $R^{(d)}_1$, we have an isomorphism $R^{(d)} \cong k[z_0, \ldots, z_l]/P' = k[z_0, \ldots, z_l]$ where $P'$ is a homogeneous prime ideal, and the $z_i$ all have degree 1.

Our graded inclusion $G^{(d)} \subset S^{(d)}$ gives us an expression

$$z_i = \sum_j a_{ij} y_i$$

with $a_{ij} \in k$ for all $i, j$.

Let $L_i = \sum_j a_{ij} y_i$. We now show that $Z(L_1, \ldots, L_l) \cap Y = \emptyset$. Suppose that $p \in Z(L_1, \ldots, L_l) \cap Y$. Since $S^{(d)}$ is integral over $R^{(d)}$, for $0 \leq i \leq m$ we have that $\overline{y}_i$ is integral over $R^{(d)}$, by a homogeneous relation. Thus we have equations

$$y_i^n + b_{i1}(L_0, \ldots, L_l) y_i^{n-1} + \cdots + b_{in}(L_0, \ldots, L_l) \in P^*$$

where $b_{ij}$ are homogeneous polynomials of degree $j$. Evaluating at $p$, we obtain that $y_i^n(p) = 0$, so that $y_i(p) = 0$ for all $i$, which is impossible. Thus $Z(L_1, \ldots, L_l) \cap Y = \emptyset$.

By Theorem 15.8, the rational map $\pi = (L_0 : \ldots : L_l)$ from $Y$ to $\varphi(X) \cong X$ is a finite morphism. By our construction, the induced map $\pi^* : k(X) \to k(Y)$ is $\Lambda$.

16. Dimension of quasi projective algebraic sets

Suppose that $X$ is a quasi projective algebraic set. We define the dimension of $X$ to be its dimension as a topological space (Definition 5.1). The following is proved in the same way as Proposition 5.4.
**Proposition 16.1.** Suppose that $X$ is a quasi projective algebraic set, and $V_1, \ldots, V_n$ are its irreducible components. Then $\dim X = \max \{ \dim V_i \}$.

**Theorem 16.2.** Suppose that $X$ is a projective variety. Then

1. $\dim X = \text{trdeg}_k(X)$.
2. Any maximal chain of distinct irreducible closed subsets of $X$ has length $n = \dim X$.
3. Suppose that $U$ is a dense open subset of $X$. Then $\dim U = \dim X$.

**Proof.** We have $W \subset \mathbb{P}^n$. Let $x_0, \ldots, x_n$ be homogeneous coordinates on $\mathbb{P}^n$. Suppose that

$$W_0 \subset W_1 \subset \cdots \subset W_n$$

is a chain of distinct irreducible closed subsets of $X$. There exists an open set $D(x_i)$ such that $W_0 \cap D(x_i) \neq \emptyset$. Then

$$W_0 \cap D(x_i) \subset \cdots \subset W_n \cap D(x_i)$$

is a chain of distinct closed subsets of the affine variety $U = W \cap D(x_i)$. Thus $n \leq \dim U = \text{trdeg}_k(X)$.

Suppose that

$$Y_0 \subset \cdots \subset Y_m$$

is a chain of maximal length of irreducible closed subsets of $U$, so that $m = \text{trdeg}_k(X)$ by Proposition 5.3 and Theorem 5.9. Then the Zariski closures $\overline{Y_i}$ of the the $Y_i$ in $X$ are distinct, so that

$$\overline{Y}_0 \subset \cdots \subset \overline{Y}_n$$

is a chain of distinct irreducible closed subsets of $X$. Thus if (24) is a maximal chain, then so is (25). Thus $n \geq \text{trdeg}_k(X)$. We have that the length $n$ of all maximal chains of distinct closed irreducible subsets of $X$ of $n = \text{trdeg}_k(X)$.

Now the proof that all nontrivial opens subsets of $X$ have the same dimension as $X$ follows from the proof of Proposition 5.8. \hfill \Box

**Theorem 16.3.** Suppose that $W \subset \mathbb{P}^n$ is a projective variety of dimension $\geq 1$, and $F \in S(\mathbb{P}^n)$ is a form which is not contained in $I(W)$. Then $W \cap Z(F) \neq \emptyset$ and all irreducible components of $Z(F) \cap W$ have dimension $\dim W - 1$.

**Proof.** Suppose that $X$ is an irreducible component of $W \cap Z(F)$. Then there exists an open subset $D(x_i)$ of $\mathbb{P}^n$ such that $X \cap D(x_i) \neq \emptyset$. Let $d$ be the degree of $F$. $\frac{F}{x_i} \in \mathcal{O}_{\mathbb{P}^n}(D(x_i))$ and $X \cap D(x_i)$ is an irreducible component of $Z_{D(x_i)}(\frac{F}{x_i}) \cap (W \cap D(x_i))$. Since $\frac{F}{x_i}$ does not restrict to the zero element on $W \cap D(x_i)$, we have that $X \cap D(x_i)$ has dimension $\dim X - 1$ by Theorem 5.12.

Suppose that $Z(F) \cap W = \emptyset$. Then by Theorem 15.10, the $\varphi = (F)$ induces a finite regular map from $W$ to $\mathbb{P}^0$, which is a point, so that $k(\mathbb{P}^0) = k$. Since $\varphi$ is finite, $k(W)$ is a finite field extension of $k(\mathbb{P}^0)$, so that $k(W) = k$ and $\dim W = \text{trdeg}_k(k(W) = 0$. Thus $\dim W = 0$.

\hfill \Box

**Corollary 16.4.** Suppose that $W \subset \mathbb{P}^n$ is a projective algebraic set, and $F_1, \ldots, F_r \in S(\mathbb{P}^n)$ are forms (of degree $> 0$). Then

$$\dim Z(F_1, \ldots, F_r) \cap W \geq \dim W - r.$$
If \( r \leq \dim W \) then \( Z(F_1, \ldots, F_r) \cap W \neq \emptyset \) (The dimension of \( \emptyset \) is -1).

**Corollary 16.5.** Suppose that \( W \) is a quasi projective algebraic set, and \( f_1, \ldots, f_r \in \mathcal{O}_W(W) \). Suppose that \( Z_W(f_1, \ldots, f_r) \neq \emptyset \). Then
\[
\dim Z_W(f_1, \ldots, f_r) \geq \dim W - r.
\]

16.1. The Theorem on Dimension of Fibers.

**Lemma 16.6.** Suppose that \( p_1, \ldots, p_s, q_1, \ldots, q_r \in \mathbb{P}^n \) for some \( s, r \) and \( n \). Then there exists a homogeneous form \( F \in S(\mathbb{P}^n) \) such that \( F(p_1) = \cdots = F(p_s) = 0 \) and \( F(q_i) \neq 0 \) for \( 1 \leq i \leq r \).

**Proposition 16.7.** Suppose that \( X \) is a quasi projective variety of dimension \( m \geq 1 \) and \( p \in X \). Then there exists an affine neighborhood \( U \) of \( p \) in \( X \) and \( f_1, \ldots, f_m \in \mathcal{O}_X(U) \) such that \( Z_U(f_1, \ldots, f_m) = \{p\} \).

**Proof.** \( X \) is an open subset of a projective variety \( W \subset \mathbb{P}^n \). Choose a point \( q_1 \in W \setminus \{p\} \).

By Lemma 16.6, there exists a form \( F_1 \in S(\mathbb{P}^n) \) such that \( F_1(q) \neq 0 \) and \( F_1(p) = 0 \). Let \( X_1 = Z(F_1) \cap W \). By Theorem 16.3, \( X_1 = X_{1,1} \cup \cdots \cup X_{1,r} \) is a union of irreducible components each of dimension \( m - 1 \). At least one of the components necessarily contains \( p \). If \( m > 1 \), we continue, choosing points \( q_i \in X_{1,i} \) for \( 1 \leq i \leq r \), none of which are equal to \( p \). By Lemma 16.6, there exists a form \( F_2 \in S(\mathbb{P}^n) \) such that \( F_2(q_i) \neq 0 \) for \( 1 \leq i \leq r \), and \( F_2(p) = 0 \). By Theorem 16.3, for each \( i \), \( Z(F_2) \cap X_{1,i} = X_{2,i,1} \cup \cdots \cup X_{2,i,s_i} \) is a union of irreducible components each of dimension \( m - 2 \). Thus \( Z(F_1, F_2) \cap W = \cup X_{2,i,j} \) is a union of irreducible components of dimension \( m - 2 \), at least one of which contains \( p \). Continuing by induction, we find homogeneous forms \( F_1, \ldots, F_m \in S(\mathbb{P}^n) \) such that \( Z(F_1, \ldots, F_m) \cap W \) is a zero dimensional algebraic set which contains \( p \). Thus \( Z(F_1, \ldots, F_m) \cap W = \{a_0, a_1, \ldots, a_t\} \) for some points \( a_0 = p, a_1, \ldots, a_t \in W \). Now by Lemma 16.6, there exists a form \( G \in S(\mathbb{P}^n) \) such that \( G(a_i) = 0 \) for \( 1 \leq i \leq t \) and \( G(p) \neq 0 \). Let \( L \) be a linear form on \( \mathbb{P}^n \) such that \( L(p) \neq 0 \). Let \( d_i \) be the degree of \( F_i \) and \( e \) be the degree of \( G \). Then
\[
f_1 = \frac{F_1}{L^{d_1}}, \ldots, f_m = \frac{F_m}{L^{d_m}}, g = \frac{G}{L^e} \in \mathcal{O}_{\mathbb{P}^n}(D(L)).
\]

Let \( V \) be an affine neighborhood of \( p \) in \( X \) such that \( V \subset X \cap D(L) \). Then \( Z(f_1, \ldots, f_m) \cap V \subset \{a_0, \ldots, a_t\} \). Let \( U \) be an affine neighborhood of \( p \) in \( (V \setminus Z(g)) \cap X \). Then \( Z_U(f_1, \ldots, f_m) \cap X = \{p\} \).

**Theorem 16.8.** Let \( \varphi : X \to Y \) be a dominant regular map between quasi projective varieties. Let \( \dim X = n \) and \( \dim Y = m \). Then \( m \leq n \) and
1. \( \dim \varphi^{-1}(p) \geq n - m \) for any \( p \in \varphi(X) \).
2. There exists a nonempty open subset \( U \subset X \) such that \( \dim \varphi^{-1}(p) = n - m \) for \( p \in U \).

**Proof.** We prove 1. The conclusion of 1 is local in \( Y \), so we can replace \( Y \) with an open neighborhood \( U \) of \( p \) in \( Y \) and \( X \) with \( \varphi^{-1}(U) \). By Proposition 16.7, we may assume that there exist \( f_1, \ldots, f_m \in k[Y] \) so that \( Z_Y(f_1, \ldots, f_m) = \{p\} \). Then the equations \( \varphi^*(f_1) = \cdots = \varphi^*(f_m) = 0 \) define \( \varphi^{-1}(p) \) in \( X \). By Corollary 16.5, \( \dim \varphi^{-1}(p) \geq n - m \).

Now we prove 2. We may replace \( Y \) with an affine open subset \( W \) and \( X \) by an affine open subset \( V \subset \varphi^{-1}(W) \). Since \( V \) is dense in \( \varphi^{-1}(W) \) and \( \varphi \) is dominant, \( \varphi(V) \) is dense in \( W \). Hence \( \varphi \) determines an inclusion \( \varphi^* : k[W] \to k[V] \), hence an inclusion \( k(W) = k(Y) \subset k(V) = k(X) \). Let \( S = k[W] \). Consider the subring \( R \) of \( k(V) \) generated
by $k(W)$ and $k[V]$. This is a domain which is a finitely generated $k(W)$-algebra. Further, the quotient ring of $R$ is $k(V)$. Now $k(W)$ is not algebraically closed, but Noether’s normalization lemma does not need this assumption. By Noether’s normalization lemma (Theorem 4.14) we have that there exist $t_1, \ldots, t_r$ in $R$ such that $t_1, \ldots, t_r$ are algebraically independent over $k(W)$, and $R$ is integral over the polynomial ring $k(W)[t_1, \ldots, t_r]$. We may assume, after multiplying by an element of $k[W]$, that $t_1, \ldots, t_r \in k[V]$. Since the quotient field of $R$ is $k(V)$, we have that
\[
\dim Y = n - m.
\]

Now consider the subring $S[t_1, \ldots, t_r]$ of $k[V]$. $S[t_1, \ldots, t_r]$ is a polynomial ring over $S$, so $S[t_1, \ldots, t_r] = k[W \times \mathbb{A}^r]$, and we have a factorization of $\varphi$ by
\[
V \xrightarrow{\pi} W \times \mathbb{A}^r \xrightarrow{\psi} W.
\]

$k[V]$ is a finitely generated $k$-algebra, so it is a finitely generated $S[t_1, \ldots, t_r]$-algebra, say generated by $v_1, \ldots, v_l$ as a $S[t_1, \ldots, t_r]$-algebra. Since $R$ is integral over $k(W)[t_1, \ldots, t_r]$, there exist polynomials $F_i(x)$ in the indeterminate $x$,
\[
F_i(x) = x^{d_i} + P_i,1(t_1, \ldots, t_r)x^{n-1} + \cdots + P_i,r(t_1, \ldots, t_r)
\]
where the $P_i,j$ are polynomials with coefficients in $k(W)$, such that
\[
F_i(v_i) = 0 \text{ for } 1 \leq i \leq l.
\]

Let $g \in S = k[W]$ be a common denominator of the polynomials $P_{ij}$. Then $P_{ij} \in (S_g)[t_1, \ldots, t_r]$ for all $i, j$. Thus $k[V]_g$ is finite over $(S_g)[t_1, \ldots, t_r]$.

Let $U = D(g) \subset W$. We then have a factorization of $\varphi$ restricted to $U$ as
\[
\varphi^{-1}(U) \xrightarrow{\pi} U \times \mathbb{A}^r \xrightarrow{\psi} U
\]
where $U$ is affine with regular functions $k[U] = k[W]_g = S_g$ and $\varphi^{-1}(U)$ is affine with regular functions $k[\varphi^{-1}(U)] = k[V]_g$. For $g \in U$, we have that $\psi^{-1}(y) = \{y\} \times \mathbb{A}^r$ has dimension $r$. Suppose that $A$ is an irreducible closed subset of $\varphi^{-1}(U)$ which maps into $\{y\} \times \mathbb{A}^r$. Then the restriction of $\varphi$ from $A$ to $\pi(A)$ is finite, so that the extension $k(A)$ of $k(\pi(A))$ is an algebraic extension. Thus $\dim A = \dim \pi(A)$. Since $\pi(A)$ is a subvariety of $\{y\} \times \mathbb{A}^r \cong \mathbb{A}^r$, we have that $\dim A \leq r = n - m$.

Thus $\dim \varphi^{-1}(y) \leq n - m$. By part 1 of this problem, $\dim \varphi^{-1}(y) = n - m$ for $y \in U$.

**Corollary 16.9.** Suppose that $\varphi : X \to Y$ is a dominant regular map between quasi projective varieties. Then the sets
\[
Y_k = \{ p \in Y \mid \dim \varphi^{-1}(p) \geq k \}
\]
are closed in $Y$.

**Proof.** Let $\dim X = n$ and $\dim Y = m$. By Theorem 16.8, $Y_{n-m} = Y$, and there exists a proper closed subset $Y'$ of $Y$ such that $Y_k \subset Y'$ if $k > n - m$. If $Z_i$ are the irreducible components of $Y'$, and $\varphi_i : \varphi^{-1}(Z_i) \to Z_i$, the restrictions of $\varphi$, then $\dim Z_i < \dim Y$, and we can prove the corollary by induction on $\dim Y$. □
17. Nonsingularity

17.1. The Tangent Space. Suppose that \( p = (b_1, \ldots, b_n) \in \mathbb{A}^n \). Let \( \pi_i = x_i - b_i \) for \( 1 \leq i \leq n \). Since translation by \( p \) is an isomorphism of \( \mathbb{A}^n \), we have that \( k[\mathbb{A}^n] = k[x_1, \ldots, x_n] = k[\pi_1, \ldots, \pi_n] \) is a polynomial ring. Suppose that \( f \in k[\mathbb{A}^n] \). Then \( f \) has a unique expansion

\[
 f = \sum a_{i_1, \ldots, i_n} (x_1 - b_1)^{i_1} \cdots (x_n - b_n)^{i_n}
\]

with \( a_{i_1, \ldots, i_n} \in k \). If \( f(p) = 0 \), we have that \( a_{0, \ldots, 0} = 0 \), and

\[
 f \equiv L_p(f) \mod I(p)^2,
\]

where

\[
 L_p(f) = a_{1,0,\ldots,0} (x_1 - b_1) + \cdots + a_{0,\ldots,0,1} (x_n - b_n) \equiv \frac{\partial f}{\partial x_1}(p)(x_1 - b_1) + \cdots + \frac{\partial f}{\partial x_n}(p)(x_n - b_n)
\]

(28)

Definition 17.1. (Extrinsic definition of tangent space) Suppose that \( X \) is an affine variety, which is a closed subvariety of \( \mathbb{A}^n \), and that \( p \in X \). The tangent space to \( X \) at \( p \) is the linear subvariety \( T_p(X) \) of \( \mathbb{A}^n \), defined by

\[
 T_p(X) = Z(L_p(f) \mid f \in I(X)).
\]

If \( I = (f_1, \ldots, f_r) \), then \( I(T_p(X)) = (L_p(f_1), \ldots, L_p(f_r)) \).

Lemma 17.2. Suppose that \( R \) is a ring, \( m \) is a maximal ideal of \( R \), and \( N \) is an \( R \) module, such that \( m^a N = 0 \) for some positive integer \( a \). Then \( N_m \cong N \).

Proof. Suppose that \( f \in R \setminus \{m\} \). We will prove that for any \( r \in \mathbb{Z}_+ \), there exists \( e \in R \) such that \( f e \equiv 1 \mod m^r \). Taking \( r = a \), we then have that \( N_m \cong N \).

\( R/m \cong k \) is a field, and the residue of \( f \) in \( R/m \) is nonzero. Thus for any \( h \in R \), there exists \( g \in R \) such that \( fg \equiv h \mod m \). Taking \( h = 1 \), we get that there exists \( e_0 \in R \) such that \( fe_0 \equiv 1 \mod m \).

Suppose that we have found \( e \in R \) such that \( fe \equiv 1 \mod m^r \). Let \( x_1, \ldots, x_n \) be the set of generators of \( m \). There exists \( h_i \in R \) such that \( fe - 1 = \sum \sum_{i_1 + \cdots + i_n = r} h_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \).

There exist \( g_{i_1, \ldots, i_n} \in R \) such that \( fg_{i_1, \ldots, i_n} \equiv h_{i_1, \ldots, i_n} \mod m \). Thus

\[
 \sum_{i_1 + \cdots + i_n = r} f g_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \equiv \sum_{i_1 + \cdots + i_n = r} h_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \mod m^{r+1}.
\]

Set \( e' = e + \sum g_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n} \) to get \( ge' \equiv 1 \mod m^{r+1} \). \( \square \)

Let \( V \) be the \( n \)-dimensional \( k \)-vector space which is spanned by \( \pi_1, \ldots, \pi_n \) in \( k[\mathbb{A}^n] \).

For \( f \in I(X) \), we have that \( L_p(f) \in V \). Let \( W \) be the subspace of \( V \) spanned by \( \{L_p(f) \mid f \in I(X)\} \).

Let \( \mathfrak{m} \) be the maximal ideal of the local ring \( \mathcal{O}_{X,p} \). By Lemma 17.2,

\[
 \mathfrak{m}/\mathfrak{m}^2 \cong \mathcal{I}(p)/(I(p)^2 + I(X)) \cong I(p)/(I(p)^2 + I(T_p(X))) \cong V/W.
\]

We can naturally identify the set of points of \( \mathbb{A}^n \) with the dual vector space \( \text{Hom}_k(V, k) \), by associating to \( q \in \mathbb{A}^n \) the linear map \( L \mapsto L(q) \) for \( L \in V \). Now

\[
 \text{Hom}_k(V/W, k) = \{ \varphi \in \text{Hom}_k(V, k) \mid \varphi(W) = 0 \}
\]

\[
 = \{ q \in \mathbb{A}^n \mid L_p(f)(q) = 0 \text{ for all } f \in I(X) \}
\]

\[
 = T_p(X).
\]

This gives us the following alternate definition of the tangent space.
Definition 17.3. (Intrinsic definition of tangent space) Suppose that $X$ is a quasi projective variety, and that $p \in X$. The tangent space to $X$ at $p$ is the $k$-vector space $T_p(X)$ defined by

$$T_p(X) = \text{Hom}_k(m/m^2, k),$$

where $m$ is the maximal ideal of $\mathcal{O}_{X,p}$.

17.2. Nonsingularity and the singular locus.

Theorem 17.4. (Corollary 11.15, page 121 [2]) Suppose that $R$ is a Noetherian local ring, with maximal ideal $m$ and residue field $k = R/m$. Then $\dim_k m/m^2 \geq \dim R$.

Corollary 17.5. Suppose that $X$ is a quasi projective variety and $p \in X$. Then $\dim_k T_p(X) \geq \dim X$.

Definition 17.6. A Noetherian local ring $R$ is a regular local ring if $\dim_k m/m^2 = \dim R$.

Definition 17.7. A point $p$ of a quasi projective variety $X$ is a nonsingular point of $X$ if $\dim_k T_p(X) = \dim R$.

Proposition 17.8. A point $p$ of a quasi projective variety $X$ is a nonsingular point of $X$ if and only if $\mathcal{O}_{X,p}$ is a regular local ring.

Proposition 17.9. Suppose that $X$ is an affine variety of dimension $r$, which is a closed subvariety of $\mathbb{A}^n$, and $f_1, \ldots, f_t \in k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$ are a set of generators of $I(X)$. Suppose that $p \in X$. Then

$$\dim_k T_p(X) = n - s$$

where $s$ is the rank of the $t \times n$ matrix

$$A = \left( \frac{\partial f_i}{\partial x_j}(p) \right).$$

In particular, $s \leq n - r$, and $p$ is a nonsingular point of $X$ if and only if $s = n - r$.

Proof. Let $s$ be the rank of $A$. Going back to our analysis of $T_p(X)$, we have that $\mathbb{A}_1, \ldots, \mathbb{A}_n$ is a $k$-basis of $V$, and $W$ is the subspace of $V$ spanned by $\{L_p(f_1), \ldots, L_p(f_t)\}$. This subspace has dimension equal to the rank of $A$ by (28). Since $T_p(X)$ and $V/W$ are $k$-vector spaces of the same dimension, we have that $\dim_k T_p(X) = n - s$.

Corollary 17.10. Suppose that $X$ is a quasi projective variety. Then the set of nonsingular points of $X$ is an open subset of $X$.

Theorem 17.11. Suppose that $X$ is a quasi projective variety. Then the set of nonsingular points of $X$ is a dense open subset of $X$.

Proof. By Proposition 7.8, $X$ is birational to a hypersurface $Z(f) \subset \mathbb{A}^n$, where $f$ is irreducible in $k[x_1, \ldots, x_n]$. Since the nonsingular locus is open, any nontrivial open subset of a variety is dense, and birational varieties have isomorphic open subsets (by Theorem 7.7), we need only show that the nonsingular locus of $Z(f)$ is nontrivial. Thus we may assume that $X = Z(f)$. Suppose that every point of $X$ is singular. Then $Z(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) = Z(f)$, so $\frac{\partial f}{\partial x_i} \in I(X) = (f)$ for all $i$. Since $\deg(\frac{\partial f}{\partial x_i}) < \deg(f)$ for all $i$ (here the degree of a polynomial is the largest total degree of a monomial appearing in the polynomial), the only way this is possible is if $\frac{\partial f}{\partial x_i} = 0$ for all $i$. If characteristic $k$ is zero, this implies that $f \in k$, which is impossible. If $k$ has positive characteristic, then $f$ must be a polynomial in $x_1^p, \ldots, x_n^p$ with coefficients in $k$. Since the $p$-th roots of these coefficients are in $k$ (as $k$ is algebraically closed) we have that $f = g^p$ for some $g \in k[x_1, \ldots, x_n]$, contradicting the fact that $f$ is irreducible. \qed
References