TOROIDALIZATION OF BIRATIONAL MORPHISMS OF 3-FOLDS

STEVEN DALE CUTKOSKY

Suppose that $f : X \to Y$ is a morphism of algebraic varieties, over a field $k$ of characteristic zero. The structure of such morphisms is quite rich. The simplest class of such morphisms is the toroidal morphisms. If $X$ and $Y$ are nonsingular, $f : X \to Y$ is toroidal if there are simple normal crossing divisors $D_X$ on $X$ and $D_Y$ on $Y$ such that $f^*(D_Y) = D_X$, and $f$ is locally given by monomials in appropriate etale local parameters on $X$. The precise definition of this concept is in [AK] (see also [KKMS]). We state the Definition of toroidal in 3.7. The problem of toroidalization is to determine, given a dominant morphism $f : X \to Y$, if there exists a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
$$

such that $\Phi$ and $\Psi$ are products of blow ups of nonsingular subvarieties, $X$ and $Y$ are nonsingular, and there exist simple normal crossing divisors $D_Y$ on $Y$ and $D_X = f^*(D_Y)$ on $X$ such that $f_1$ is toroidal (with respect to $D_X$ and $D_Y$). This is stated in Problem 6.2.1. of [AKMW]. Toroidalization, and related concepts, have been considered earlier in different contexts, mostly for morphisms of surfaces. Toroidalization is the strongest structure theorem which could be true for general morphisms. The concept of toroidalization fails completely in positive characteristic. A simple example is shown in [C2].

In the case when $Y$ is a curve, toroidalization follows from embedded resolution of singularities ([H]). When $X$ and $Y$ are surfaces, there are several proofs in print ([AkK], Corollary 6.2.3 [AKMW], [Mat]). They all make use of special properties of the birational geometry of surfaces. An outline of proofs of the above cases can be found in the introduction to [C2].

In [C2], the toroidalization problem is solved in the case when $X$ is a 3-fold and $Y$ is a surface. In this paper, we prove toroidalization for birational morphisms of 3-folds.

**Theorem 0.1.** Suppose that $f : X \to Y$ is a birational morphism of 3-folds which are proper over an algebraically closed field $k$ of characteristic 0. Then there exists a commutative diagram of morphisms

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
$$

where $\Phi, \Psi$ are products of blow ups of nonsingular curves and points, and there exists a simple normal crossings divisor $D_{Y_1}$ on $Y_1$ such that $D_{X_1} = f_1^{-1}(D_{Y_1})$ is a simple normal crossings divisor and $f_1$ is toroidal with respect to $D_{X_1}$ and $D_{Y_1}$.

Research partially supported by NSF.
If we relax some of the restrictions in the definition of toroidalization, there are other constructions producing a toroidal morphism $f_1$, which are valid for arbitrary dimensions of $X$ and $Y$. In [AK] it is shown that a diagram (1) can be constructed where $\Phi$ is weakened to being a modification (an arbitrary birational morphism). In [C1] and [C4], it is shown that a diagram (1) can be constructed where $\Phi$ and $\Psi$ are locally products of blow ups, but the morphisms $\Phi$, $\Psi$ and $f_1$ may not be separated. This construction is obtained by patching local solutions valid for any given valuation.

It has been shown in [AKMW] and [W1] that weak factorization of birational morphisms holds in characteristic zero, and arbitrary dimension. That is, birational morphisms of complete varieties can be factored by an alternating sequence of blow ups and blow downs of non singular subvarieties. Weak factorization of birational (toric) morphisms of toric varieties, (and of birational toroidal morphisms) has been proven by Danilov [D1] and Ewald [E] (for 3-folds), and by Wlodarczyk [W], Morelli [Mo] and Abramovich, Matsuki and Rashid [AMR] in general dimensions.

Our Theorem 0.1, when combined with weak factorization for toroidal morphisms ([AMR]), gives a new proof of weak factorization of birational morphisms of 3-folds. We point out that our proof uses an analysis of the structure as power series of local germs of a mapping, as opposed to the entirely different proof of weak factorization, using geometric invariant theory, of [AKMW] and [W1].

**Corollary 0.2.** Suppose that $f : X \rightarrow Y$ is a birational morphism of 3-folds which are proper over an algebraically closed field $k$ of characteristic zero. Then there exists a commutative diagram of morphisms factoring $f$,

$$
\begin{array}{ccccccc}
X_1 & \rightarrow & X_3 & \rightarrow & \cdots & \rightarrow & X_n & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
X & \rightarrow & X_2 & \rightarrow & X_4 & \rightarrow & \cdots & \rightarrow & Y
\end{array}
$$

where each arrow is a product of blow ups of points and nonsingular curves.

The problem of strong factorization, as proposed by Abhyankar [Ab2] and Hironaka [H], is to factor a birational morphism $f : X \rightarrow Y$ by constructing a diagram

$$
\begin{array}{ccc}
Z \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
$$

where $Z \rightarrow X$ and $Z \rightarrow Y$ factor as products of blow ups of nonsingular subvarieties. Oda [O] has proposed the analogous problem for (toric) morphisms of toric varieties.

A birational morphism $f : S \rightarrow Y$ of (nonsingular) surfaces can be directly factored by blowing up points (Zariski [Z3] and Abhyankar [Ab2]), but there are examples showing that a direct factorization is not possible in general for 3-folds (Shannon [Sh] and Sally[S]).

We also obtain as an immediate corollary the following new result, which reduces the problem of strong factorization of 3-folds to the case of toroidal morphisms.

**Corollary 0.3.** Suppose that the Oda conjecture on strong factorization of birational toroidal morphisms of 3-folds is true. Then the Abhyankar, Hironaka strong factorization conjecture of birational morphisms of complete (characteristic zero) 3-folds is true.

Abhyankar’s local factorization conjecture [Ab2], which is “strong factorization” along a valuation, follows from local monomialization (Theorem A [C1]), to reduce to a locally toroidal morphism, and local factorization for toroidal morphisms along a valuation Christensen [Ch] (for 3-folds), and Karu [K] or [CS] in general dimensions.
In the papers [C5] and [C6] we extend the results and methods of this paper to prove the following theorem which generalizes Theorem 0.1. This is the strongest possible structure theorem for dominant morphisms of 3-folds.

**Theorem 0.4.** (Theorem 1.2 [C6]) Suppose that $f : X \to Y$ is a dominant morphism of 3-folds which are proper over an algebraically closed field $k$ of characteristic 0. Further suppose that there is an equidimensional codimension 1 reduced subscheme $D_Y$ of $Y$ such that $D_Y$ contains the singular locus of $Y$, and $D_X = f^{-1}(D_Y)$ contains the singular locus of the map $f$. Then there exists a commutative diagram of morphisms

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

where $\Phi, \Psi$ are products of blow ups of nonsingular curves and points supported above $D_X$ and $D_Y$ respectively, $D_{Y_1} = \Psi^{-1}(D_Y)$ is a simple normal crossings divisor on $Y_1$, $D_{X_1} = f_1^{-1}(D_{Y_1})$ is a simple normal crossings divisor on $X_1$ and $f_1$ is toroidal with respect to $D_{Y_1}$ and $D_{X_1}$.

1. AN OUTLINE OF THE PROOF

We will say a few words about the structure of the proof of Theorem 0.1. Most of the proof of Theorem 0.1 is valid for generically finite morphisms of 3-folds. We use the fact that a birational morphism $f : X \to Y$ of complete 3-folds is an isomorphism in codimension 1 in $Y$, and has a very simple structure in codimension 2 in $Y$ (is a product of blowups of nonsingular curves above the base [Ab1] or [D]). The structure in codimension 2 is however not that much more difficult in codimension 2 for arbitrary dominant morphisms of 3-folds (in characteristic zero). We hope to develop this theme in a later paper.

If $X$ is a nonsingular variety and $D_X$ is a SNC divisor on $X$, then $D_X$ defines a toroidal structure on $X$. If $V$ is a nonsingular subvariety of $X$, which is supported on $D_X$ and makes SNCs with $D_X$, then $V$ is a possible center. Let $\Phi : X_1 \to X$ be the blow up of $V$. Then $X_1$ is nonsingular with toroidal structure $D_{X_1} = \Phi^{-1}(D_X)$.

Suppose that $f : X \to Y$ is a birational morphism of nonsingular complete 3-folds of characteristic zero, and suppose that $D_Y$ is a simple normal crossings divisor on $Y$ such that $D_X = f^*(D_Y)$ is also a simple normal crossings divisor, defining toroidal structures on $X$ and $Y$. Further suppose that the locus of points in $X$ where $f$ is not smooth is contained in $D_X$. We will refer to points where three components of $D_Y$ (or $D_X$) intersect as 3-point, points where 2-components intersect as 2-points, and the remaining points of $D_Y$ (and $D_X$) as 1-points.

We develop a series of algorithms to manipulate local germs of our mappings, expressed in terms of series and polynomials. We are required to blow up in both the domain and target.

The main result of resolution of singularities [H] tells us that we can construct a diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

where for all points $p \in X$ and $q = f(p) \in Y$ there are regular parameters $x, y, z$ at $p$ and $u, v, w$ at $q$, which contain local equations of components of $D_X$ and $D_Y$ passing
through these points, such that we have an expression

\[ u = x^a y^b z^c \gamma_1 \]
\[ v = x^d y^e z^f \gamma_2 \]
\[ w = x^g y^h z^i \gamma_3 \]  

(2)

where \( \gamma_1, \gamma_2, \gamma_3 \) are units at \( p \). The algorithms of resolution give us no information about the structure of the units \( \gamma_i \). In general, we will have that the monomials \( x^a y^b z^c, x^d y^e z^f, x^g y^h z^i \) are algebraically dependent. We can then hope to blow up nonsingular subvarieties of \( Y \), leading to new regular parameters at a point \( q' \) above \( q \) with regular parameters \( u', v', w' \) which are obtained by dividing \( u, v, w \) by each other. We will eventually obtain an expression such as \( w' = \gamma_3 - \gamma_3(p) \) by this procedure, which need not be any better than the expressions we started with for our original map \( f : X \to Y \).

In attempting to obtain a sufficiently deep understanding of map germs to prove toroidalization, we must understand information about germs such as (2), up to the level of a series expansion of the units \( \gamma_i \). In considering this problem, we are led to generalized notions of multiplicity, which can actually increase after making blow ups above \( X \). In summary, the problem of toroidalization requires new methods, which are not contained in any proofs of resolution of singularities.

In Definition 3.4 we define a prepared morphism \( f : X \to Y \) of 3-folds. This is related to the notion of a prepared morphism from a 3-fold to a surface (Definition 6.5 [C2]). The basic idea of a prepared morphism of 3-folds is that locally an appropriate projection of \( Y \) onto a surface \( S \) is a toroidal morphism. Some special care is required in handling 3-points of \( Y \). If \( f \) is prepared, by a simple local calculation involving Jacobian determinants, we have very simple expressions of the form Definition 3.1 and Lemma 3.2 at all points of \( X \). A typical case, when \( p \) is a 1-point and \( q \) is a 2-point, is

\[ u = x^a, v = x^b(\alpha + y), w = g(x, y) + x^c z \]

where \( 0 \neq \alpha \in k \) and \( g(x, y) \) is a series.

In Section 4 it is shown that we can construct from our given birational morphism \( f : X \to Y \) a commutative diagram

\[ X_1 \xrightarrow{f_1} Y_1 \]
\[ \Phi \downarrow \quad \Psi \]
\[ X \xrightarrow{f} Y \]

where \( \Phi, \Psi \) are products of blow ups of nonsingular curves and points, and there exists a SNC divisor \( D_{Y_1} \) on \( Y_1 \) such that \( D_{X_1} = f_1^{-1}(D_{Y_1}) \) is a SNC divisor and \( f_1 \) is prepared with respect to \( D_{X_1} \) and \( D_{Y_1} \). We also make \( D_{X_1} \) cuspidal for \( f_1 \) (Definition 3.8). That is, \( f_1 \) is toroidal in a neighborhood of all components of \( D_{X_1} \), which do not contain a 3-point, and in a neighborhood of all 2-curves of \( D_{X_1} \) which do not contain a 3-point.

It may appear that the local forms of a prepared morphism are very simple, and we can easily modify them to obtain a toroidal form. However, a little exploration with these local forms will reveal that the notion of being prepared is stable under blow ups of 2-curves (on \( Y \)), but is in general not stable under blow ups of points and curves which are not 2-curves (on \( Y \)). It also will quickly become apparent that it is absolutely necessary to blow up subvarieties of \( Y \)’ other than 2-curves to toroidalize. This leads to the notion of super parameters (Definition 5.5), which is necessary for all the blow ups which we consider to preserve the notion of being prepared, and for a global invariant \( \tau \) to behave well under blow ups.
To prove Theorem 0.1, we may assume that $f$ is prepared, and $D_X$ is cuspidal for $f$. These conditions are preserved throughout the proof.

We define the $\tau$-invariant of a 3-point $p \in X$ (Definition 3.9). Since $f$ is prepared, $f(p) = q$ is a 2-point or a 3-point. There are regular parameters $u, v, w$ in $O_{Y,q}$ and $x, y, z$ in $\hat{O}_{X,p}$ such that $xyz = 0$ is a local equation of $D_X$, $uv = 0$ or $uvw = 0$ is a local equation of $D_Y$ and there is an expression

$$
\begin{align*}
    &u = x^a y^b z^c, \\
    &v = x^d y^e z^f, \\
    &w = \sum_{i \geq 0} \alpha_i M_i + N
\end{align*}
$$

with $\alpha_i \in \mathbf{k}$, $M_i = x^{\alpha_i} y^b z^c$, $N = x^g y^h z^i$,

$$
\text{rank} \begin{pmatrix} a & b & c \\ d & e & f \\ a_i & b_i & c_i \end{pmatrix} = 2, \quad \det \begin{pmatrix} a & b & c \\ d & e & f \\ a_i & b_i & c_i \end{pmatrix} = 0 \text{ for all } i,
$$

$$
\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.
$$

If $q$ is a 3-point, then

$$
w = \text{unit series } N
$$

if and only if $f$ is toroidal at $p$. In this case define $\tau_f(p) = -\infty$.

Otherwise, define

$$
H_p = \mathbf{Z}(a, b, c) + \mathbf{Z}(d, e, f) + \sum_i \mathbf{Z}(a_i, b_i, c_i),
$$

$$
\begin{align*}
    &A_p = \begin{cases} 
    \mathbf{Z}(a, b, c) + \mathbf{Z}(d, e, f) + \mathbf{Z}(a_0, b_0, c_0) & \text{if } q \text{ is a 3-point (we have } w = \text{unit series } M_0) \\
    \mathbf{Z}(a, b, c) + \mathbf{Z}(d, e, f) & \text{if } q \text{ is a 2-point.}
    \end{cases}
\end{align*}
$$

Now define

$$
\tau_f(p) = |H_p/A_p|.
$$

We define

$$
\tau_f(X) = \max \{ \tau_f(p) \mid p \in X \text{ is a 3-point} \}.
$$

We show in Theorem 9.2, that $f$ is toroidal if $\tau_f(X) = -\infty$.

We have that $\tau_f(X) \geq 1$ or $\tau_f(X) = -\infty$. The proof of Theorem 0.1 is by descending induction on $\tau_f(X)$. In our proof of Theorem 0.1 we may thus assume that $\tau = \tau_f(X) \neq -\infty$ (so that $\tau \geq 1$).

**Step 1.** (Theorem 8.10) There exist sequences of blow ups of 2-curves

$$
\begin{array}{ccc}
    X_1 & \xrightarrow{f_1} & Y_1 \\
    \downarrow & & \downarrow \\
    X & \xrightarrow{f} & Y
\end{array}
$$

such that $f_1$ is prepared, $D_{X_1}$ is cuspidal for $f_1$, $\tau_f(X_1) = \tau$, and $\tau_{f_1}(p) = \tau$ implies that $f_1(p)$ is a 2-point. Theorem 8.10 is a consequence of Theorem 6.10 and the concept of 3-point relation (Definition 6.7).
**Step 2.** (Theorem 8.11) In this step we construct a commutative diagram of morphisms

\[ \begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array} \]

such that

1. \( \Phi \) and \( \Psi \) are products of blow ups of possible centers.
2. \( \tau_{f_1}(X_1) = \tau \), and if \( p \in X \) is a 3-point such that \( \tau_{f_1}(p) = \tau \) then \( f_1(p) \) is a 2-point.
3. \( D_{X_1} \) is cuspidal for \( f_1 \).
4. \( f_1 \) is \( \tau \)-very-well prepared.

Step 2 is the most difficult step technically. It is the content of Sections 7 and 8.

The definition of \( \tau \)-very-well prepared is given in Definition 7.6. It uses the concept of 2-point relation (Definition 6.6), and requires the preliminary definitions of \( \tau \)-quasi-well prepared (Definition 7.1) and \( \tau \)-well prepared (Definition 7.3).

By virtue of the result of this step, we can assume that \( f \) is \( \tau \)-very-well prepared.

We now summarize some of the properties of a \( \tau \)-very-well-prepared morphism (Definition 7.6).

There exists a finite, distinguished set of nonsingular algebraic surfaces \( \Omega(\mathcal{R}_i) \) in \( Y \), with a SNC divisor \( F_i \) on \( \Omega(\mathcal{R}_i) \) such that the intersection graph of \( F_i \) is a tree.

Suppose that \( p \in X \) is a 3-point with \( \tau_{f}(p) = \tau \) (so that \( q = f(p) \) is 2-point). Then the following conditions hold.

1. The expression (3) has the form

\[ w = \gamma M_0 \tag{4} \]

where \( \gamma \) is a unit series, \( M_0 = u^a v^b \), with \( a, b, e \in \mathbb{Z}, e > 1, \) and \( \gcd(a, b, e) = 1 \). Observe that we cannot have both \( a < 0 \) and \( b < 0 \), since \( M_0, u, v \) are all monomials in \( x, y, z \).

2. Suppose that \( V \) is the curve in \( Y \) with local equations \( u = w = 0 \) (or \( v = w = 0 \)) at \( q \). Then \( V \) is a \( \ast \)-permissible center (Definition 7.9). That is, \( V \) is a possible center for \( D_Y \) and there exists a commutative diagram of morphisms

\[ \begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array} \tag{5} \]

where \( \Psi_1 \) is the blow up of \( V \) (possibly followed by blow ups of some special 2-points), such that \( f_1 \) and \( f = \Psi_1 \circ f_1 : X_1 \to Y \) are prepared, \( \tau_{f_1}(X_1) \leq \tau \) and \( \Phi_1 \) is toroidal at 3-points \( p_1 \in (\Phi_1)^{-1}(p) \). Further, \( f_1 \) is \( \tau \)-very-well prepared.

3. There exists a surface \( \Omega(\mathcal{R}_i) \) such that

a. \( f(p) = q \in \Omega(\mathcal{R}_i) \).

b. The \( w \) of (4) gives a local equation \( w = 0 \) of \( \Omega(\mathcal{R}_i) \) at \( q \).

c. \( uv = 0 \) is a local equation of \( F_i \) (on the surface \( \Omega(\mathcal{R}_i) \)) at \( q \).

The necessity of several different surfaces \( \Omega(\mathcal{R}_i) \) arises because of the possibility that there may be several 3-points \( p_j \) with \( \tau_{f}(p_j) = \tau \) which map to \( q \), and require
different \( w \) in their expressions (4). We require that the surfaces \( \Omega(\mathcal{T}_i) \) intersect in a controlled way.

The first step in the construction of a \( \tau \)-very well prepared morphism is the construction of a morphism such that for all 3-points \( p \) with \( \tau_f(p) = \tau \), an expression (4) holds for some possibly formal \( w \).

**Step 3.** (Theorem 9.1) We construct a commutative diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \tau_{f_n}(X_n) < \tau \). By induction on \( \tau \), we then obtain the proof of Theorem 0.1.

We fix an index \( i \) of the surfaces \( \Omega(\mathcal{T}_1) \). A curve \( E \) on \( Y \) is good if it is a component of \( F_i \), and if \( j \) is such that \( E \cap \Omega(\mathcal{T}_j) \neq \emptyset \), then \( E \) is a component of \( F_j \).

In our construction we begin with \( i = 1 \), and blow up a good curve \( V \) on \( Y \), by a morphism (5). Part of the definition of \( \tau \)-very well prepared implies the existence of a good curve. Suppose that \( p \in X \) is a 3-point with \( \tau_f(X) = \tau \) and \( q = f(p) \in V \). Suppose that \( p_1 \in \Phi_i^{-1}(p) \) is a 3-point. Set \( q_1 = f_1(p_1) \). If \( V \) has local equations \( u = w = 0 \) at \( q \), then \( q_1 \) has regular parameters \( u_1, v, w_1 \) with

\[
u = u_1, w = w_1
\]

(6)

or

\[
u = u_1, w = u_1(w_1 + \alpha)
\]

(7)

and \( \alpha \in k \).

If (6) holds then \( q_1 \) is a 3-point. Since \( e > 1 \), we have

\[
\tau_{f_1}(p_1) = |H_p/A_p + M_0Z| < |H_p/A_p| = \tau.
\]

If (7) holds, then \( e > 1 \) implies \( \alpha = 0 \). Thus \( f_1 \) has the form (3), (4) at \( p_1 \) with \( (\tilde{a}, \tilde{b}, e) \) changed to \( (\tilde{a} - e, \tilde{b}, e) \). If \( V \) has local equations \( v = w = 0 \), then \( f_1 \) has the form (3), (4) at \( p_1 \) with \( (\tilde{a}, \tilde{b}, e) \) changed to \( (\tilde{a}, \tilde{b} - e, e) \).

We have SNC divisors \( \Phi_i^{-1}(F_i) \) on the surfaces \( \Phi_i^{-1}(\Omega(\mathcal{T}_1)) \). If there are no 3-points \( p_1 \) in \( X_1 \) satisfying 1, 2 and 3 of Step 2 for \( \Phi_i^{-1}(\Omega(\mathcal{T}_1)) \), then we increase \( i \) to 2.

Otherwise, there exists a good curve on \( Y_1 \) for the SNC divisor \( \Phi_i^{-1}(F_1) \) on the surface \( \Phi_i^{-1}(\Omega(\mathcal{T}_1)) \). We continue to iterate, blowing up good curves. If we always have a 3-point satisfying 1, 2 and 3 for the preimage of \( \Omega(\mathcal{T}_1) \), then we eventually obtain a form (4) with both \( \tilde{a} < 0 \) and \( \tilde{b} < 0 \) which is impossible.

We then continue this algorithm for the preimages of all of the surfaces \( \Omega(\mathcal{T}_i) \). The algorithm terminates in the construction of a morphism with a drop in \( \tau \) as desired.

The final proof of Theorem 0.1 is given after Theorem 9.2.

### 2. Notation

Throughout this paper, \( k \) will be an algebraically closed field of characteristic zero. A curve, surface or 3-fold is a quasi-projective variety over \( k \) of respective dimension 1, 2 or 3. If \( X \) is a variety, and \( p \in X \) is a nonsingular point, then regular parameters at \( p \) are regular parameters in \( \mathcal{O}_{X,p} \). Formal regular parameters at \( p \) are regular parameters in \( \mathcal{O}_{X,p} \). If \( X \) is a variety and \( V \subset X \) is a subvariety, then \( \mathcal{I}_V \subset \mathcal{O}_X \) will denote the ideal sheaf of \( V \). If \( V \) and \( W \) are subvarieties of a variety \( X \), we denote the scheme theoretic intersection \( Y = \text{spec}(\mathcal{O}_X/\mathcal{I}_V + \mathcal{I}_W) \) by \( Y = V \cdot W \).
Suppose that $a, b, c, d \in \mathbb{Q}$. Then we will write $(a, b) \leq (c, d)$ if $a \leq b$ and $c \leq d$.

A toroidal structure on a nonsingular variety $X$ is a simple normal crossing divisor (SNC divisor) $D_X$ on $X$.

We will say that a nonsingular curve $C$ which is a subvariety of a nonsingular 3-fold $X$ with toroidal structure $D_X$ makes simple normal crossings (SNCs) with $D_X$ if for all $p \in C$, there exist regular parameters $x, y, z$ at $p$ such that $x = y = 0$ are local equations of $C$, and $xyz = 0$ contains the support of $D_X$ at $p$.

Suppose that $X$ is a nonsingular 3-fold with toroidal structure $D_X$. If $p \in D_X$ is on the intersection of three components of $D_X$ then $p$ is called a 3-point. If $p \in D_X$ is on the intersection of two components of $D_X$ (and is not a 3-point) then $p$ is called a 2-point. If $p \in D_X$ is not a 2-point or a 3-point, then $p$ is called a 1-point. If $C$ is an irreducible component of the intersection of two components of $D_X$, then $C$ is called a 2-curve. $\Sigma(X)$ will denote the closed locus of 2-curves on $X$.

By a general point $q$ of variety $V$, we will mean a point $q$ which satisfies conditions which hold on some nontrivial open subset of $V$. The exact open condition which we require will generally be clear from context. By a general section of a coherent sheaf $F$ on a projective variety $X$, we mean the section corresponding to a general point of the $k$-linear space $\Gamma(X, F)$.

If $X$ is a variety, $k(X)$ will denote the function field of $X$. A 0-dimensional valuation $\nu$ of $k(X)$ is a valuation of $k(X)$ such that $k$ is contained in the valuation ring $V_\nu$ of $\nu$ and the residue field of $V_\nu$ is $k$. If $X$ is a projective variety which is birationally equivalent to $X$, then there exists a unique (closed) point $p_1 \in X_1$ such that $V_\nu$ dominates $O_{X_1, p_1}$. $p_1$ is called the center of $\nu$ on $X_1$. If $p \in X$ is a (closed) point, then there exists a 0-dimensional valuation $\nu$ of $k(X)$ such that $V_\nu$ dominates $O_{X, p}$ (Theorem 37, Section 16, Chapter VI [ZS]). For $a_1, \ldots, a_n \in k(X)$, $\nu(a_1), \ldots, \nu(a_n)$ are rationally independent if there exist $a_1, \ldots, a_n \in \mathbb{Z}$ which are not all zero, such that $a_1 \nu(a_1) + \cdots + a_n \nu(a_n) = 0$ (in the value group of $\nu$). Otherwise, $\nu(a_1), \ldots, \nu(a_n)$ are rationally independent.

If $f : X \to Y$ is a morphism of varieties, and $D$ is a Cartier divisor on $Y$, then $f^{-1}(D)$ will denote the reduced divisor $f^*(D)_{\text{red}}$.

If $S$ is a finite set, $|S|$ will denote the cardinality of $S$.

3. Prepared, monomial and toroidal morphisms

Throughout this section we assume that $f : X \to Y$ is a dominant morphism of nonsingular 3-folds, $D_Y$ is SNC divisor on $Y$ such that $D_X = f^{-1}(D_Y)$ is a SNC divisor, and the singular locus of $f$ is contained in $D_X$. $D_X$ and $D_Y$ define toroidal structures on $X$ and $Y$.

A possible center on a nonsingular 3-fold $X$ with toroidal structure defined by a SNC divisor $D_X$, is a point on $D_X$ or a nonsingular curve in $D_X$ which makes SNCs with $D_X$. A possible center on a nonsingular surface $S$ with toroidal structure defined by a SNC divisor $D_S$ is a point on $D_S$.

Observe that if $\Phi : X_1 \to X$ is the blow up of a possible center (a possible blow up), then $D_{X_1} = \Phi^{-1}(D_X)$ is a SNC divisor on $X_1$. Thus $D_{X_1}$ defines a toroidal structure on $X_1$. All blow ups $\Phi : X_1 \to X$ considered in this paper will be of possible centers, and we will impose the toroidal structure on $X_1$ defined by $D_{X_1} = \Phi^{-1}(D_X)$.

Suppose that $x, y, z$ are indeterminants, and $M_1, M_2, \ldots, M_r$ are Laurent monomials in $x, y, z$, so that there are expressions

$$M_i = x^{\alpha_{i1}} y^{\alpha_{i2}} z^{\alpha_{i3}}$$
with all $a_{ij}$ in $\mathbb{Z}$. We define
$$\text{rank}_{(x,y,z)}(M_1, \ldots, M_r) = n$$
if the matrix $(a_{ij})$ has rank $n$. If there is no danger of confusion, we will denote
$$\text{rank}(M_1, \ldots, M_r) = \text{rank}_{(x,y,z)}(M_1, \ldots, M_r).
$$

Suppose that $q \in Y$. We say that $u, v, w$ are (formal) permissible parameters at $q$ if $u, v, w$ are regular parameters in $\hat{O}_{Y,q}$ such that $u,v,w$ are algebraic permissible parameters if we further have that $u,v,w \in O_{Y,q}$.

**Definition 3.1.** Suppose that $u, v, w$ are (possibly formal) permissible parameters at $q \in Y$. Then $u, v$ are toroidal forms at $p \in f^{-1}(q)$ if there exist regular parameters $x, y, z$ in $\hat{O}_{X,p}$ such that
1. If $q$ is a 1-point, then $u \in O_{Y,q}$ and $u = 0$ is a local equation of $D_Y$ at $q$.
2. If $q$ is a 2-point then $u, v \in O_{Y,q}$ and $uv = 0$ is a local equation of $D_Y$ at $q$.
3. If $q$ is a 3-point then $u, v, w \in O_{Y,q}$ and $uvw = 0$ is a local equation of $D_Y$ at $q$.

$u, v, w$ are algebraic permissible parameters if we further have that $u, v, w \in O_{Y,q}$.

**Lemma 3.2.** Suppose that $q \in Y$, $p \in f^{-1}(q)$ and $u, v, w$ are permissible parameters at $q$ such that $u, v$ are toroidal forms at $p$. Then there exist permissible parameters $x, y, z$ for $u, v, w$ at $p$ such that an expression of Definition 3.1 holds for $u$ and $v$, and one of the following respective forms for $w$ holds at $p$.
1. If $q$ is a 2-point or a 3-point, $p$ is a 1-point and
$$u = x^a, v = x^b(\alpha + y) \quad (8)$$

where $0 \neq \alpha \in k$.
2. If $q$ is 2-point or a 3-point, $p$ is a 2-point and
$$u = x^a y^b, v = x^c y^d \quad (9)$$

with rank$(u, v) = 2$.
3. If $q$ is a 2-point or a 3-point, $p$ is a 2-point and
$$u = (x^a y^b)^k, v = (x^a y^b)^t(\alpha + z) \quad (10)$$

where $0 \neq \alpha \in k$, $a, b, k, t > 0$ and gcd$(a, b) = 1$.
4. If $q$ is 2-point or a 3-point, $p$ is a 3-point and
$$u = x^a y^b z^c, v = x^d y^e z^f \quad (11)$$

where rank$(u, v) = 2$.
5. If $q$ is a 1-point, $p$ is a 1-point and
$$u = x^a, v = y \quad (12)$$

6. If $q$ is a 1-point, $p$ is a 2-point and
$$u = (x^a y^b)^k, v = z \quad (13)$$

with $a, b, k > 0$ and gcd$(a, b) = 1$.

Regular parameters $x, y, z$ as in Definition 3.1 will be called permissible parameters for $u, v, w$ at $p$.

**Lemma 3.2.** Suppose that $q \in Y$, $p \in f^{-1}(q)$ and $u, v, w$ are permissible parameters at $q$ such that $u, v$ are toroidal forms at $p$. Then there exist permissible parameters $x, y, z$ for $u, v, w$ at $p$ such that an expression of Definition 3.1 holds for $u$ and $v$, and one of the following respective forms for $w$ holds at $p$.
1. If $q$ is a 2-point or a 3-point, $p$ is a 1-point, $u, v$ satisfy $(8)$ and
$$w = g(x, y) + x^c z \quad (14)$$

where $g$ is a series.
2. \(q\) is 2-point or a 3-point, \(p\) is a 2-point, \(u, v\) satisfy (9) and 
\[
\begin{align*}
  w &= g(x, y) + x^c y^f z
\end{align*}
\] 
where \(g\) is a series.

3. \(q\) is 2-point or a 3-point, \(p\) is a 2-point, \(u, v\) satisfy (10) and 
\[
\begin{align*}
  w &= g(x^a y^b, z) + x^c y^d
\end{align*}
\] 
where \(g\) is a series and \(\text{rank}(u, x^c y^d) = 2\).

4. \(q\) is 2-point or a 3-point, \(p\) is a 3-point, \(u, v\) satisfy (11) and 
\[
\begin{align*}
  w &= g(x, y, z) + N
\end{align*}
\] 
where \(g\) is a series and \(\text{rank}(u, x, y, z) = 2\).

5. \(q\) is a 1-point, \(u, v\) satisfy (12) and 
\[
\begin{align*}
  w &= g(x, y) + x^c z
\end{align*}
\] 
where \(g\) is a series.

6. \(q\) is a 1-point, \(p\) is a 2-point, \(u, v\) satisfy (13) and 
\[
\begin{align*}
  w &= g(x^a y^b, z) + x^c y^d
\end{align*}
\] 
where \(g\) is a series and \(\text{rank}(u, x^c y^d) = 2\).

Proof. Choose permissible parameters \(x, y, z\) for \(u, v, w\) at \(p\). The Lemma follows from an explicit calculation of the jacobian determinant 
\[
J = \text{Det}
\begin{pmatrix}
  \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
  \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
  \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{pmatrix},
\]

and a change of variables of \(x, y, z\). Observe that \(J = 0\) is supported on \(D_X\), since the singular locus of \(f\) is contained in \(D_X\).

We indicate the proof if (11) holds at \(p\). There exists a unit series \(\gamma\) in \(x, y, z\) and \(l, m, n \in \mathbb{N}\) such that 
\[
J = \gamma x^l y^m z^n.
\]

We compute 
\[
w = \sum c_{ijk} x^i y^j z^k
\]
with \(c_{ijk} \in k\). We compute 
\[
J = \sum c_{ijk} \text{Det}
\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  i & j & k
\end{pmatrix} x^{a+d+i-1} y^{b+e+j-1} z^{c+f+k-1} = \gamma x^l y^m z^n
\]
from which we obtain forms (11) and (17), after making a change of variables in \(x, y, z\), multiplying \(x, y, z\) by appropriate unit series. \(\square\)

Definition 3.3. Let notation be as in Lemma 3.2. If \(p \in X\) is a 3-point, we will say that permissible parameters \(u, v, w\) at \(q = f(p)\) have a monomial form at \(p\) if there exist permissible parameters \(x, y, z\) for \(u, v, w\) at \(p\) such that there is an expression 
\[
\begin{align*}
  u &= x^a y^b z^c \\
  v &= x^d y^e z^f \\
  w &= x^g y^h z^i
\end{align*}
\]
(with \(\text{rank}(u, v, w) = 3\)).

Definition 3.4. A birational morphism \(f : X \to Y\) of nonsingular 3-folds with toroidal structures determined by SNC divisors \(D_Y, D_X = f^{-1}(D_Y)\) such that the singular locus of \(f\) is contained in \(D_X\) is prepared if:
1. If \( q \in Y \) is a 3-point, \( u, v, w \) are permissible parameters at \( q \) and \( p \in f^{-1}(q) \), then \( u, v \) and \( w \) are each a unit (in \( \mathcal{O}_{X,p} \)) times a monomial in local equations of the toroidal structure \( D_X \) at \( p \). Furthermore, there exists a permutation of \( u, v, w \) such that \( u, v \) are toroidal forms at \( p \).

2. If \( q \in Y \) is a 2-point, \( u, v, w \) are permissible parameters at \( q \) and \( p \in f^{-1}(q) \), then either
   (a) \( u, v \) are toroidal forms at \( p \) or
   (b) \( p \) is a 1-point and there exist regular parameters \( x, y, z \in \mathcal{O}_{X,p} \) such that there is an expression
   \[
   u = x^a \\
v = x^c(\gamma(x, y) + x^dz) \\
w = y
   \]
   where \( \gamma \) is a unit series and \( x = 0 \) is a local equation of \( D_X \), or
   (c) \( p \) is a 2-point and there exist regular parameters \( x, y, z \) in \( \mathcal{O}_{X,p} \) such that there is an expression
   \[
   u = (x^ay^b)^k \\
v = (x^ay^b)^l(\gamma(x^ay^b, z) + x^cy^d) \\
w = z
   \]
   where \( a, b > 0 \), \( \gcd(a, b) = 1 \), \( ad - bc \neq 0 \), \( \gamma \) is a unit series and \( xy = 0 \) is a local equation of \( D_X \).

3. If \( q \in Y \) is a 1-point, and \( p \in f^{-1}(q) \), then there exist permissible parameters \( u, v, w \) at \( q \) such that \( u, v \) is a toroidal form at \( p \).

We call \( x, y, z \) in (2 (b) or 2 (c) of Definition 3.4 permissible parameters for \( u, v, w \) at \( p \).

Lemma 3.5. Suppose that \( X, Y \) are projective, \( f : X \to Y \) is birational, prepared and \( q \in Y \) is a 1-point such that \( f \) is not an isomorphism over \( q \). Then the fundamental locus of \( f \) contains a single curve \( C \) passing through \( q \) and \( C \) is nonsingular at \( q \). Furthermore, there exist algebraic permissible parameters \( u, v, w \) at \( q \) such that a form (12) or (13) of Definition 3.1 (for this fixed choice of \( u, v, w \)) holds at \( p \) for all \( p \in f^{-1}(q) \).

Proof. Since for all \( p \in f^{-1}(q) \), there exist permissible parameters at \( q \) such that a form (12) or (13) of Definition 3.1 holds at \( p \), we have that \( \dim f^{-1}(q) = 1 \). Thus if \( E' \) is an exceptional component of \( f \) such that \( q \in f(E') \), then \( f(E') \) is a curve.

Let \( D \) be the component of \( D_Y \) containing \( q \). Let \( F \) be the strict transform of \( D \) on \( X \) and let \( p \in f^{-1}(q) \cap F \). Then \( p \) must be a 2-point, and since \( f \) is birational, there exist permissible parameters \( \overline{u}, \overline{v}, \overline{w} \) at \( q \) such that there is an expression in \( \mathcal{O}_{X,p} \)
\[
\overline{u} = x^a y \\
\overline{v} = \gamma(x, y) \\
\overline{w} = x^cy^d
\]
where \( y = 0 \) is a (formal) local equation of \( F \), and \( x = 0 \) is a (formal) local equation of the other component \( E \) of \( D_X \) containing \( p \). Computing the Jacobian determinant of \( f \) at \( p \), we see that \( c = 0 \). We consider the morphism \( f^* : \mathcal{O}_{D,q} \to \mathcal{O}_{F,p} \). \( f^* : \mathcal{O}_{D,q} \to \mathcal{O}_{F,p} \) is the \( k \)-algebra homomorphism \( \hat{f}^* : k[\overline{u}, \overline{w}] \to k[[x, z]] \) given by \( \overline{u} = z \), \( \overline{w} = \phi(x^a, y, z) + x^dy^c \). Thus \( \mathcal{O}_{F,p} \) is finite over \( \mathcal{O}_{D,q} \). It follows that \( f^* : \mathcal{O}_{D,q} \to \mathcal{O}_{F,p} \) is quasi-finite, and thus \( \mathcal{O}_{F,p} \cong \mathcal{O}_{D,q} \) by Zariski's Main Theorem. In particular, \( \{ p \} = f^{-1}(q) \cap F \). We thus have that the only component of the fundamental locus
of \( f \) through \( q \) is the algebraic curve \( C = f(E) \), which has analytic local equations \( \pi = \overline{w} - \phi(\overline{w}, \overline{\pi}) = 0 \) at \( q \). Thus \( C \) is nonsingular at \( q \).

If \( E' \) is a component of the exceptional locus of \( f \) such that \( q \in f(E') \), we must have that \( f(E') = C \). Now let \( u, v, w \) be permissible parameters at \( q \) such that \( u = w = 0 \) are local equations of \( C \) at \( q \). We see that \( u, v, w \) must have a form (12) or (13) of Definition 3.1 for all \( p \in f^{-1}(q) \).

\[ \square \]

**Lemma 3.6.** Let notation be as in Definition 3.4, and suppose that \( X \) and \( Y \) are projective.

1. If \( q \in Y \) is a 2-point (or a 3-point) and for all \( p \in f^{-1}(q) \) there exist permissible parameters \( u_p, v_p, w_p \) at \( q \) which satisfy one of 2 (a) – 2 (c) of Definition 3.4 (or one of (8) – (11) of Definition 3.1) at \( p \), then any permissible parameters \( u, v, w \) at \( q \) satisfy 2 (or 1) of Definition 3.4 for all \( p \in f^{-1}(q) \).

2. If \( f : X \to Y \) is prepared and \( q \in Y \) is a 1-point, then there exist permissible parameters \( u, v, w \) at \( q \) such that for all \( p \in f^{-1}(q) \), \( u, v \) are toroidal forms in local equations at \( p \) of the toroidal structure.

**Proof.** 1 follows from the definitions, and a local calculation. The most difficult case to verify is when \( q \in Y \) is a 2-point, \( u, v, w \) are permissible parameters at \( q \) and \( p \in f^{-1}(p) \) is a 2-point.

First suppose that \( u_p, v_p, w_p \) have a form 2 (c) at \( p \),

\[
u_p = (\pi^a\overline{y}^b)^k, v_p = (\pi^a\overline{y}^b)^l(\gamma(\pi^a\overline{y}^b, \overline{\pi}) + \pi^a\overline{y}^d), w_p = \overline{z}.
\]

First observe that if \( u = v_p, v = u_p, w = w_p \), we can find regular parameters \( \pi, \overline{y}, \overline{z} \) in \( \hat{O}_{X,p} \) such that \( u, v, w \) have a form 2 (c) with respect to \( \pi, \overline{y}, \overline{z} \).

By the formal implicit function theorem, we have reduced to the case where there exist unit series \( \gamma_1, \gamma_2, \gamma_3 \) in the variables \( u, v, w \) and a series \( \lambda \) in \( u, v \) (with no constant term) such that

\[
u = \gamma_1 u_p, v = \gamma_2 v_p, w = \gamma_3(w_p + \lambda).
\]

There exist series \( a_1, a_2 \in \hat{O}_{X,p} \) such that if \( x_1 = a_1 \pi, y_1 = a_2 \overline{y}, z_1 = \lambda \), then \( x_1, y_1, z_1 \) are regular parameters in \( \hat{O}_{X,p} \) such that

\[
u = \gamma_1 u_p = (x_1^a y_1^b)^k, v = (x_1^a y_1^b)(\gamma_2 \gamma_1^{1/2} \gamma(\gamma_1^{-1/2} x_1^a y_1^b, \overline{z}) + x_1^c y_1^d), w = z_1.
\]

We have

\[
\pi^a \overline{y}^b = \gamma_1^{-1} x_1^a y_1^b
\]

and

\[
\pi^a \overline{y}^d = \gamma_2^{-1} x_1^c y_1^d.
\]

There exist series \( g, h, h_1 \) such that

\[
\overline{z} = w_p = g(u, w) + vb(u, v, w) = g(u, w) + (x_1^a y_1^b)^k h_1(x_1^a y_1^b, x_1^c y_1^d, z_1, \pi^a \overline{y}^b, \pi^a \overline{y}^d, \overline{z})
\]

and there exists a series \( h_2 \) such that

\[
v = (x_1^a y_1^b)^l h_2(x_1^a y_1^b, x_1^c y_1^d, z_1, \pi^a \overline{y}^b, \pi^a \overline{y}^d, \overline{z}).
\]

By iteration, we see that there exists a series \( g_2 \) such that

\[
\overline{z} \equiv g_2(x_1^a y_1^b, z_1) \mod (x_1^{c+1} y_1^{d+1}),
\]

and there exists a series \( h_3 \) such that

\[
v \equiv h_3(x_1^a y_1^b, z_1) \mod (x_1^{c+1} y_1^{d+1}).
\]
Substituting equations (21) and (22) into \( \gamma_2 \gamma_1^{-\frac{k}{2}} \gamma(\gamma_1^{-\frac{1}{2}} x_1^a y_1^b, z) \) in (20), we see that there exist unit series \( \gamma \) and \( \tau \) such that
\[
v = (x_1^a y_1^b) \left( \tau(y_1^b, z_1) + x_1^a y_1^b \tau_1 \right).
\]
Finally, we can find unit series \( \alpha_1, \alpha_2 \) in \( x_1, y_1, z_1 \) such that if we set
\[
x = x_1, y = y_1, z = z_1,
\]
then \( u, v, w \) have an expression of the form (2) in terms of \( x, y, z \).

Now suppose that \( u_p, v_p \) are toroidal forms at \( p \), and we have an expansion
\[
u_p = (x^p y^p)^k,
\]
\[
v_p = (x^p y^p)^l (x + z),
\]
\[
w_p = g(x^p y^p, z) + x^p y^p
\]
of type (10). If the coefficient of \( z \) in \( g \) is non zero, then we can make a change of variables, setting \( z = g(x^p y^p, z) + x^p y^p \). By consideration of the Jacobian determinant of \( u_p, v_p, w_p \), we find variables \( x, y, z \) such that \( u_p, v_p, w_p \) have an expansion of the form of (2) (c). If the coefficient of \( z \) in \( g \) is zero, then it follows that \( u, v \) are toroidal forms at \( p \).

The final case is when \( u_p, v_p \) are toroidal forms at \( p \), and we have an expansion
\[
u_p = x^p y^p,
\]
\[
v_p = x^p y^p,
\]
\[
w_p = g(x^p y^p) + x^p y^p z
\]
of type (9). In this case we have that \( u, v \) are toroidal forms of type (9).

2 follows from Lemma 3.5. \( \square \)

**Definition 3.7.** ([KKMS], [AK]) A normal variety \( X \) with a SNC divisor \( D_X \) on \( X \) is called toroidal if for every point \( p \in X \) there exists an affine toric variety \( X_\sigma \), a point \( p' \in X_\sigma \) and an isomorphism of \( k \)-algebras
\[
\hat{O}_{X,p} \cong \hat{O}_{X_\sigma,p'}
\]
such that the ideal of \( D_X \) corresponds to the ideal of \( X_\sigma - T \) (where \( T \) is the torus in \( X_\sigma \)). Such a pair \((X_\sigma, p')\) is called a local model at \( p \in X \). \( D_X \) is called a toroidal structure on \( X \).

A dominant morphism \( \Phi : X \to Y \) of toroidal varieties with SNC divisors \( D_Y \) on \( Y \) and \( D_X = \Phi^{-1}(D_Y) \) on \( X \), is called toroidal at \( p \in X \), and we will say that \( p \) is a toroidal point of \( \Phi \) if with \( q = \Phi(p) \), there exist local models \((X_\sigma, p') \) at \( p \), \((Y_q, q') \) at \( q \) and a toric morphism \( \Psi : X_\sigma \to Y_q \) such that the following diagram commutes:
\[
\begin{array}{ccc}
\hat{O}_{X,p} & \xleftarrow{\Phi^*} & \hat{O}_{X_\sigma,p'} \\
\Phi^* \uparrow & & \Psi^* \uparrow \\
\hat{O}_{Y,q} & \xleftarrow{\Psi^*} & \hat{O}_{Y_q, q'}
\end{array}
\]
\( \Phi : X \to Y \) is called toroidal (with respect to \( D_Y \) and \( D_X \)) if \( \Phi \) is toroidal at all \( p \in X \).

The following is the list of toroidal forms for a dominant morphism \( f : X \to Y \) of nonsingular 3-folds with toroidal structure \( D_Y \) and \( D_X = f^{-1}(D_X) \). Suppose that \( p \in D_X \), \( q = f(p) \in D_Y \), and \( f \) is toroidal at \( p \). Then there exist permissible parameters \( u, v, w \) at \( q \) and permissible parameters \( x, y, z \) for \( u, v, w \) at \( p \) such that one of the following forms hold:
1. \(p\) is a 3-point and \(q\) is a 3-point,
\[
\begin{align*}
  u &= x^ay^bz^c \\
  v &= x^dy^ez^f \\
  w &= x^gy^hz^i,
\end{align*}
\]
where \(a, b, d, e, f, g, h, i \in \mathbb{N}\) and
\[
\text{Det} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0.
\]

2. \(p\) is a 2-point and \(q\) is a 3-point,
\[
\begin{align*}
  u &= x^ay^b \\
  v &= x^dy^e(z + \alpha) \\
  w &= x^gy^h(z + \alpha)
\end{align*}
\]
with \(0 \neq \alpha \in \mathbb{k}\) and \(a, b, d, e, g, h \in \mathbb{N}\) satisfy \(ae - bd \neq 0\).

3. \(p\) is a 1-point and \(q\) is a 3-point,
\[
\begin{align*}
  u &= x^a \\
  v &= x^d(y + \alpha) \\
  w &= x^g(z + \beta)
\end{align*}
\]
with \(0 \neq \alpha, \beta \in \mathbb{k}\), \(a, d, g > 0\).

4. \(p\) is a 2-point and \(q\) is a 2-point,
\[
\begin{align*}
  u &= x^ay^b \\
  v &= x^dy^e \\
  w &= z
\end{align*}
\]
with \(ae - bd \neq 0\).

5. \(p\) is a 1-point and \(q\) is a 2-point,
\[
\begin{align*}
  u &= x^a \\
  v &= x^d(y + \alpha) \\
  w &= z
\end{align*}
\]
with \(0 \neq \alpha \in \mathbb{k}\), \(a, d > 0\).

6. \(p\) is a 1-point and \(q\) is a 1-point,
\[
\begin{align*}
  u &= x^a \\
  v &= y \\
  w &= z
\end{align*}
\]
with \(a > 0\).

**Definition 3.8.** Suppose that \(f : X \to Y\) is a prepared morphism. Then \(DX\) is cuspidal for \(f\) if:

1. If \(E\) is a component of \(DX\) which does not contain a 3-point then \(f\) is toroidal in a Zariski open neighborhood of \(E\).
2. If \(C\) is a 2-curve of \(X\) which does not contain a 3-point then \(f\) is toroidal in a Zariski open neighborhood of \(C\).

**Definition 3.9.** Suppose that \(f : X \to Y\) is prepared, and \(p \in X\) is a 3-point. Suppose that \(u, v, w\) are permissible parameters at \(q = f(p)\). Then there is an expression (after possibly permuting \(u, v, w\) if \(q\) is a 3-point)
\[
\begin{align*}
  u &= x^ay^bz^c \\
  v &= x^dy^ez^f \\
  w &= \sum_{i \geq 0} \alpha_i M_i + N
\end{align*}
\] (23)
where \( x, y, z \) are permissible parameters at \( p \) for \( u, v, w \), \( \text{rank}(u, v) = 2 \), the sum in \( w \) is over (possibly infinitely many) monomials \( M_i \) in \( x, y, z \) such that \( \text{rank}(u, v, M_i) = 2 \), \( \deg(M_i) \leq \deg(M_j) \) if \( i < j \), \( \alpha_i \neq 0 \) for all \( i \). \( N \) is a monomial in \( x, y, z \) such that \( \text{rank}(u, v, N) = 3 \) and \( N \nmid M_i \) for any \( M_i \) in the series \( \sum \alpha_i M_i \).

If \( q \) is a 3-point and \( u, v, w \) is not a monomial form (at \( p \)), we necessarily have (since \( f \) is prepared) that

\[
\sum \alpha_i M_i = M_0 \gamma
\]

(24)

where \( \gamma \) is a unit series in the monomials \( \frac{M_i}{M_0} \) (in \( x, y, z \)) such that

\[
\text{rank}(u, v, M_0) = \text{rank}(u, v, \frac{M_i}{M_0}) = 2
\]

for all \( i \), and \( M_0 \mid N \).

If \( q \) is a 3-point and \( u, v, w \) have a monomial form at \( p \), so that \( w = N \), define \( \tau(p) = \tau_f(p) = -\infty \). Otherwise, define a group \( H_p = H_{f,p} \) as follows. The Laurent monomials in \( x, y, z \) form a group under multiplication. We define \( H_p = H_{f,p} \) to be the subgroup generated by \( u, v \) and the terms \( M_i \) appearing in the expansion (23). We will write the group \( H_p \) additively as:

\[
H_p = H_{f,p} = \mathbb{Z}u + \mathbb{Z}v + \sum \mathbb{Z}M_i.
\]

In \( H_p \), the expression \( p\alpha + q\beta + \sum c_i M_i \) represents the monomial \( u^{\alpha} v^{\beta} \prod M_i^{c_i} \).

Define a subgroup \( A_p \) of \( H_p \) by:

\[
A_p = A_{f,p} = \begin{cases} 
\mathbb{Z}u + \mathbb{Z}v + \mathbb{Z}M_0 & \text{if } q \text{ is 3-point} \\
\mathbb{Z}u + \mathbb{Z}v & \text{if } q \text{ is a 2-point.}
\end{cases}
\]

Define

\[
L_p = L_{f,p} = H_p/A_p, \\
\tau(p) = \tau_f(p) = |L_p|.
\]

Observe that \( \tau(p) < \infty \) in Definition 3.9, since \( H_p \) is a finitely generated abelian group, and \( H_p/A_p \) is a torsion group.

We define

\[
\tau(X) = \tau_f(X) = \max\{\tau_f(p) \mid p \in X \text{ is a 3-point}\}.
\]

**Lemma 3.10.** \( \tau_f(p) \) is independent of choice of permissible parameters \( u, v, w \) at \( q = f(p) \) and permissible parameters \( x, y, z \) at \( p \) for \( u, v, w \).

**Proof.** Suppose that \( q \in Y \) is a 3-point.

The condition \( \tau(p) = -\infty \) is independent of permuting \( u, v, w \), multiplying \( u, v, w \) by units in \( \mathcal{O}_{Y,q} \) and multiplying \( x, y, z \) by units in \( \mathcal{O}_{X,p} \) so that the conditions of (23) hold. Thus \( \tau(p) = -\infty \) is independent of choice of permissible parameters at (the 3-points) \( q \) and \( p \).

Suppose that \( u, v, w \) are permissible parameters at \( q \) and \( x, y, z \) are permissible parameters at \( p \) for \( u, v, w \) satisfying (23). Let \( \tau \) be the computation of \( \tau(p) \) for these variables.

Suppose that \( \tilde{u}, \tilde{v}, \tilde{w} \) is another set of permissible parameters at \( q \), and \( \tilde{x}, \tilde{y}, \tilde{z} \) are permissible parameters for \( \tilde{u}, \tilde{v}, \tilde{w} \) at \( p \), satisfying (23). Let \( \tau_1 \) be the computation for \( \tau(p) \) with respect to these variables. We must show that \( \tau = \tau_1 \). We may assume that \( \tau \geq 1 \) and \( \tau_1 \geq 1 \).

Since \( p \) and \( q \) are 3-points, \( \tilde{u}, \tilde{v}, \tilde{w} \) can be obtained from \( u, v, w \) by permuting the variables \( u, v, w \) and then multiplying \( u, v, w \) by unit series (in \( u, v, w \)). \( \tilde{x}, \tilde{y}, \tilde{z} \) can be
obtained from \(x, y, z\) by permuting \(x, y, z\) and then multiplying \(x, y, z\) by unit series (in \(x, y, z\)).

We then reduce to proving the following:

1. Suppose that \(\tilde{u}, \tilde{v}, \tilde{w}\) is a permutation of \(u, v, w\) such that \(\tilde{u}, \tilde{v}\) are toroidal forms at \(p\). Then there exist permissible parameters \(\tilde{x}, \tilde{y}, \tilde{z}\) at \(p\) for \(\tilde{u}, \tilde{v}, \tilde{w}\) such that a form (23) holds, and \(\tau_1 = \tau\).

2. Suppose that \(\tilde{u}, \tilde{v}, \tilde{w}\) are obtained from \(u, v, w\) by multiplying \(u, v, w\) by unit series. Then there exist permissible parameters \(\tilde{x}, \tilde{y}, \tilde{z}\) at \(p\) for \(\tilde{u}, \tilde{v}, \tilde{w}\) such that a form (23) holds, and \(\tau_1 = \tau\).

3. Suppose that \(u = \tilde{u}, v = \tilde{v}\) and \(w = \tilde{w}\) and \(x, y, z\) are two sets of permissible parameters for \(u, v, w\). Then \(\tau_1 = \tau\).

We now verify 1. The case when \(\tilde{u} = v, \tilde{v} = u, \tilde{w} = w\) is immediate. We will verify the case when \(\tilde{u} = w, \tilde{v} = v, \tilde{w} = u\).

Since the symmetric group \(S_3\) is generated by the permutations \((1, 2)\) and \((1, 3)\), the remaining cases of 1 will follow.

Since \(\tau \geq 1\), and \(\tilde{u}, \tilde{v}, \tilde{w}\) have a form (23) at \(p\), we have \(\text{rank}(v, M_0) = 2\). Since \(w, v\) are (by assumption) toroidal forms at \(p\), there exist permissible parameters \(\tilde{x}, \tilde{y}, \tilde{z}\) at \(p\) such that \(\tilde{w}, v, u\) (in this order) have an expression of the form of (23) in terms of \(\tilde{x}, \tilde{y}, \tilde{z}\). We will show that there is an isomorphism of the corresponding group \(\tilde{H}\) (computed for these variables) and the group \(H\) computed for \(u, v, w\) and \(x, y, z\) which takes the corresponding group \(\tilde{A}\) to \(A\).

With the notation of (24), we have

\[ w = M_0(\gamma + N_0) \]

where \(N_0 = \frac{N}{M_0}\) is a monomial in \(x, y, z\).

Set \(M_0 = M_0, M_i = \frac{M_i}{M_0}\) for all \(M_i\) appearing in the series \(\sum_{i \geq 1} \alpha_i M_i\). Thus \(\gamma = \alpha_0 + \sum_{i \geq 1} \alpha_i M_i\). There exist \(a_i, b_i, c_i \in N\) such that

\[ M_i = x^{a_i} y^{b_i} z^{c_i} \]

for \(0 \leq i\).

\[ H = Z u + Z v + \sum_{i \geq 0} Z M_i = Z u + Z v + \sum_{i \geq 0} Z M_i. \]

Define a finite sequence

\[ 1 = \mu(1) < \mu(2) < \cdots < \mu(\tau) \]

(for appropriate \(\tau\)) so that

\[ \sum_{i=1}^{\mu(j)} Z M_i = \sum_{i=1}^{\mu(j)} Z M_i, \]

if \(\mu(j) < n < \mu(j + 1)\) and

\[ \sum_{i=1}^{\mu(j)} Z M_i \neq \sum_{i=1}^{\mu(j+1)} Z M_i. \]
Set
\[ G_j = \sum_{i=1}^{\mu(j)} \mathbf{Z}M_i. \]

There exist \( k_i, l_i \in \mathbf{Z} \) and \( c_i \in \mathbf{N} \) with \( \gcd(c_i, k_i, l_i) = 1 \) such that \( \mathcal{M}_i' = u^{k_i}v^{l_i} \) for \( i \in \{0, \mu(1), \ldots, \mu(\tau)\} \), and there exist \( g, h, i \in \mathbf{N} \) such that \( \mathcal{N}_0 = x^g y^h z^i \). Thus \( H \) is (isomorphic to) the subgroup of \( \mathbf{Z}^d \) generated by
\[
\delta_1 = (a, b, c), \delta_2 = (d, e, f), \epsilon_0 = (a_0, b_0, c_0),
\epsilon_{\mu(1)} = (a_{\mu(1)}, b_{\mu(1)}, c_{\mu(1)}), \ldots, \epsilon_{\mu(\tau)} = (a_{\mu(\tau)}, b_{\mu(\tau)}, c_{\mu(\tau)}).
\]
\( \lambda \) is the subgroup with generators \( \delta_1, \delta_2 \) and \( \epsilon_0 \).

Since \( \text{rank}(v, M_0) = 2 \), we can make a change of variables
\[
x = \pi \lambda_1, y = \eta \lambda_2, z = \tau \lambda_3
\]
where \( \lambda_i = (\gamma + \mathcal{N}_0)^{\beta_i} \) for some \( \beta_i \in \mathbf{Q} \), so that
\[
\begin{align*}
\lambda_1^{\alpha_0} & \lambda_2^{\alpha_1} \lambda_3^{\alpha_2} = (\gamma + \mathcal{N}_0)^{-1} \\
\lambda_1^{\beta_0} & \lambda_2^{\beta_1} \lambda_3^{\beta_2} = 1 \\
\lambda_1^{\beta_0} & \lambda_2^{\beta_1} \lambda_3^{\beta_2} = (\gamma + \mathcal{N}_0)\tau
\end{align*}
\]
for some \( t \in \mathbf{Q} \). Since \( u, v, w \) are algebraically independent in \( \hat{O}_{X, \mu} \) (by Zariski’s subspace theorem, Theorem 10.14 [Ab]), we have that \( t \neq 0 \), and since \( \text{rank}(v, M_0) = 2 \),
\[
\begin{align*}
w &= \pi^0 \eta^{\beta_0} \tau^{\beta_2} \\
v &= \pi^0 \eta^{\beta_1} \tau^{\beta_2} \\
u &= \pi^0 \eta^{\beta_2} \tau^{\beta_2} (\gamma^t + \pi^{\alpha_0} \eta^{\alpha_1} \tau^{\alpha_2})
\end{align*}
\]
where \( \gamma_2 \) is a unit series in \( \mathbf{Q} \).

Set \( \tilde{M}_i = \pi^{\alpha_0} \eta^{\alpha_1} \tau^{\alpha_2} \) for \( i \geq 1 \). In the Taylor’s series expansion of \( h(\zeta) = (\alpha_0 + \zeta)^t \) we substitute \( \zeta = \sum_{j=1}^{\infty} \alpha_i \tilde{M}_i \) to see that
\[
\gamma(x, y, z)^t = \sum_{j=0}^{\infty} q_j \text{ mod } (\pi, \eta, \tau) \mathcal{N}_0
\]
where each \( q_j \) is a series in monomials of degree \( j \) in \( \{ \tilde{M}_i \mid i \geq 1 \} \), and
\[
q_0 = \alpha_0^t, q_1 = \sum_{i=1}^{\infty} \sigma_i \tilde{M}_i
\]
with
\[
\sigma_i = t\alpha_0^{t-1} \alpha_1 \beta_1 + b_1 \beta_2 + c_1 \beta_3 \alpha_i
\]
for \( i \geq 1 \). Let
\[
\omega = \sum_{j=0}^{\infty} q_j.
\]
There exists a unit series \( \gamma_3 \) such that
\[
u = \pi^{\alpha_1} \eta^{\alpha_2} (\omega + \pi^{\alpha_0} \eta^{\alpha_1} \tau^{\alpha_2} \gamma_3).
\]
We now see that the coefficient of \( \tilde{M}^{\mu(j)} \) for \( 1 \leq j \leq \tau \) in the expansion of \( \omega \) as a series in \( \pi, \eta, \tau \) is \( \sigma_{\mu(j)} \). If not, there would exist a relation \( \tilde{M}^{\mu(j)} = \tilde{M}_{i_1} \cdots \tilde{M}_{i_n} \) for some \( n > 1 \). Thus \( i_1, \ldots, i_n < \mu(j) \) and \( G^{\mu(j)} = G^{\mu(j-1)} \), a contradiction.

Since
\[
\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \neq 0,
\]
there exists a change of variables
\[ \bar{x} = \tilde{x}\phi_1, \bar{y} = \tilde{y}\phi_2, \bar{z} = \tilde{z}\phi_3 \]
where \( \phi_1, \phi_2, \phi_3 \) are unit series in \( \bar{x}, \bar{y}, \bar{z} \) such that
\[
\begin{align*}
\phi_1^{-1} \phi_2 \phi_3 &= 1 \\
\phi_1^{-1} \phi_2 \phi_3^{-1} &= 1 \\
\phi_1^{-1} \phi_2 \phi_3^{-1} &= \gamma_1^{-1}.
\end{align*}
\]
Since \( e_i(a_i, b_i, c_i) = k_i(a, b, c) + l_i(d, e, f) \) for \( i \geq 0 \), we have an expression
\[
\begin{align*}
w &= \tilde{x}^{a_0} \tilde{y}^{b_0} \tilde{z}^{c_0} \\
v &= \tilde{x}^d \tilde{y}^e \tilde{z}^f \\
u &= \tilde{x}^{a_0} \tilde{y}^{b_0} \tilde{z}^c (\omega + \tilde{x}^g \tilde{y}^h \tilde{z}^i)
\end{align*}
\]
and \( \bar{M}_i = \tilde{x}^{a_i} \tilde{y}^{b_i} \tilde{z}^{c_i} \eta_i \) for some \( e_i \)-th root of unity \( \eta_i \) in \( k \) for all \( 1 \leq i \).

\( \tilde{H} \) is thus (isomorphic to) the subgroup of \( \mathbb{Z}^3 \) with generators
\[ \bar{\delta}_1, \bar{\delta}_2, \bar{\tau}_0, \bar{\tau}_{\mu(1)}, \ldots, \bar{\tau}_{\mu(\tau)}, \]
defined by
\[ \bar{\delta}_1 = (a_0, b_0, c_0), \bar{\delta}_2 = (d, e, f), \bar{\tau}_0 = (a, b, c), \bar{\tau}_{\mu(1)} = (a_{\mu(1)}, b_{\mu(1)}, c_{\mu(1)}), \ldots, \bar{\tau}_{\mu(\tau)} = (a_{\mu(\tau)}, b_{\mu(\tau)}, c_{\mu(\tau)}). \]

\( \hat{A} \) is the subgroup of \( \tilde{H} \) generated by \( \bar{\delta}_1, \bar{\delta}_2 \) and \( \bar{\tau}_0 \). Thus, we have an isomorphism of \( H \) with \( \tilde{H} \) which takes \( A \) to \( \hat{A} \) by mapping \( \bar{\delta}_1 \) to \( \bar{\tau}_0 \), \( \bar{\delta}_2 \) to \( \bar{\tau}_0 \), \( \bar{\delta}_0 \) to \( \bar{\delta}_1 \) and \( \bar{\tau}_i \) for \( i \in \{1, \ldots, \mu(\tau)\} \).

We have thus completed the verification of 1. The verification of 2 and 3 follow from simpler calculations. We thus obtain the conclusions of the lemma when \( q \) is a 3-point.

Now assume that \( q \in Y \) is a 2-point. \( \tau(p) \) is independent of interchanging \( u \) and \( v \), multiplying \( u \) and \( v \) by unit series in \( \tilde{O}_{Y,q} \), and permuting \( x, y, z \) and multiplying \( x, y, z \) by unit series in \( \tilde{O}_{X,p} \), so that the conditions of (23) hold. If we replace \( w \) by \( w' \in \tilde{O}_{Y,q} \) so that \( u, v, w' \) are permissible parameters at \( q \), then by the formal implicit function theorem, there exists a unit series \( \alpha(u, v, w) \in \tilde{O}_{Y,q} \) and a series \( \beta(u, v) \in k[u,v] \) such that \( w = \alpha^{-1}(w' - \beta(u, v)) \).

There exists a series \( \phi \) in \( x, y, z \) such that
\[ \alpha(u, v, w) = \alpha(u, v, \sum \alpha_i M_i + N) = \alpha(u, v, \sum \alpha_i M_i) + N\phi. \]
Then
\[ w' = \beta(u, v) + \alpha(u, v, \sum \alpha_i M_i)(\sum \alpha_i M_i) + N[\alpha(u, v, \sum \alpha_i M_i) + (\sum \alpha_i M_i)\phi + N\phi]. \]
Now as in the calculation we make in the verification that \( \tau_1 = \tau \) in the case when \( q \) is a 3-point, we see that there exist permissible parameters \( \tilde{x}, \tilde{y}, \tilde{z} \) for \( u, v, w' \) at \( p \) such that \( \tau_1 = \tau \) (where \( \tau \) is computed for \( u, v, w \) and \( x, y, z \) and \( \tau' \) is computed for \( u, v, w' \) and \( \tilde{x}, \tilde{y}, \tilde{z} \) ). Thus \( \tau(p) \) is independent of choice of permissible parameters at \( q \) and \( p \) when \( q \) is a 2-point.

\[ \square \]

Lemma 3.11. Suppose that \( X \) is a nonsingular 3-fold with SNC divisor \( D_X \), defining a toroidal structure on \( X \). Suppose that \( \mathcal{I} \) is an ideal sheaf on \( X \) which is locally generated by monomials in local equations of components of \( D_X \). Then there exists a sequence of blow ups of 2-curves \( \Phi_1 : X_1 \to X \) such that \( \mathcal{I}O_{X_1} \) is an invertible ideal
sheaf. If $\mathcal{I}$ is locally generated by two equations, then $\Phi_1$ is an isomorphism away from the support of $\mathcal{I}$.

This lemma is an extension of Lemma 18.18 [C2], and is generalized to all dimensions in [G].

Proof. $X$ has a cover by affine open sets $U_1, \ldots , U_n$ such that there exist $g_{i,1}, \ldots , g_{i,\ell} \in \Gamma(U_i, \mathcal{O}_X)$ such that $g_{i,j} = 0$ are local equations in $U_i$ of irreducible components of $D_X$, and there exist $f_{i,1}, \ldots , f_{i,m(i)} \in \Gamma(U_i, \mathcal{O}_X)$ such that the $f_{i,j}$ are monomials in the $g_{i,k}$ and $\Gamma(U_i, \mathcal{I}) = (f_{i,1}, \ldots , f_{i,m(i)})$.

Let $D_{ij}$ be an effective divisor supported on the components of $D_X$ such that there is equality of divisors $D_{ij} \cap U_i = (f_{i,j}) \cap U_i$. Let $\mathcal{I}_i \subset \mathcal{O}_X$ be the ideal sheaf which is locally generated by local equations of $D_{i,1}, \ldots , D_{i,m(i)}$. By construction, $\mathcal{I}_i \mid U_i = \mathcal{I} \mid U_i$ for all $i$.

We will show that for an ideal sheaf of the form $\mathcal{I}_1$, there exists a sequence of blow ups of 2-curves, $\pi : X_1 \to X$ such that $\mathcal{I}_1 \mathcal{O}_{X_1}$ is invertible. Since $\mathcal{I}_i \mathcal{O}_{X_i}$ are locally generated by local equations of $\pi^*(D_{i,1}), \ldots , \pi^*(D_{i,m(i)})$, there exists $\pi_Z : X_2 \to X$ which is a sequence of blow ups of 2-curves such that $\mathcal{I}_1 \mathcal{O}_{X_2}$ is invertible for all $i$. Since $\mathcal{I}_i \mathcal{O}_{X_i} \mid \pi_{Z^{-1}}(U_i) = \mathcal{I} \mathcal{O}_{X_2} \mid \pi_{Z^{-1}}(U_i)$ for all $i$, $\mathcal{I} \mathcal{O}_{X_2}$ is invertible.

We may now suppose that there exists $n > 0$ and effective divisors $D_1, \ldots , D_n$ on $X$ whose supports are unions of components of $D_X$, such that $\mathcal{I}$ is locally generated by local equations of $D_1, \ldots , D_n$.

First suppose that $n = 2$. Suppose that $p \in X$ is a general point of a 2-curve. Let $x = 0, y = 0$ be local equations of the components of $D_X$ containing $p$. $x = y = 0$ are local equations of $C$ at $p$. Then there exist $a, b, c, d \in \mathbb{N}$ such that $D_1$ is defined near $p$ by the divisor of $x^ay^b$, and $D_2$ is defined near $p$ by the divisor of $x^cy^d$. Define

$$\omega(C) = \begin{cases} 
\max\{|a-c|, |b-d|, |b-d, |a-c|\} & \text{if } a-c, b-d \text{ are nonzero and have opposite signs,} \\
-\infty & \text{otherwise}
\end{cases}$$

Here the maximum is computed in the lexicographic order. We see that the stalk $\mathcal{I}_p$ is invertible if and only if $\omega(C) = -\infty$.

Further, if $\omega(C) = -\infty$ for all 2-curves $C$ of $X$, then $\mathcal{I}$ is invertible, as follows since the divisors $D_1$ and $D_2$ are given locally at a 3-point $p$ by the divisors of monomials $x^ay^b$ and $x^cy^d$ where $xyz = 0$ is a local equation of $D_X$ at $p$. $a-d$ and $b-e$ have the same signs, $a-d, c-f$ have the same signs, and $b-e, c-f$ have the same signs, so $x^ay^bz^c | x^2y^cz^l$ or $x^ay^b | x^2y^cz^l$.

Now define

$$\varpi(X) = \max\{\omega(C) \mid C \text{ is a 2-curve of } X\}.$$  

We have seen that $\mathcal{I}$ is invertible if and only if $\varpi(X) = -\infty$. Suppose that $\varpi(X) \neq -\infty$ and $C$ is a 2-curve of $X$ such that $\omega(C) = \varpi(X)$. Let $\pi : X_1 \to X$ be the blow up of $C$. Let $D_{X_1} = \pi^{-1}(D_X) = \pi^*(D_X)_{\text{red}}, D'_1 = \pi^*(D_1), D'_2 = \pi^*(D_2)$.

We can define the function $\omega$ for 2-curves on $X_1$, relative to $D'_1$ and $D'_2$, and define $\varpi(X_1)$.

By a local calculation (as shown in the proof of Lemma 18.18 [C2]) we see that $\omega(C_1) < \varpi(X)$ if $C_1$ is a 2-curve which is contained in the exceptional divisor of $\pi$.

Suppose that $C_1, \ldots , C_r$ are the 2-curves $C$ on $X$ such that $\omega(C) = \varpi(X)$. We obtain a reduction $\varpi(X_1) < \varpi(X)$ after blowing up (the strict transforms of) these $r$ curves. By induction on $\varpi(X)$, we must obtain that $\mathcal{I} \mathcal{O}_{X_2}$ is invertible after an appropriate sequence of blow ups of 2-curves $X_2 \to X$.

Now suppose that $\mathcal{I}$ is locally generated by local equations of $D_1, \ldots , D_n$ (with $n > 2$). Let $\mathcal{I}_1 \subset \mathcal{I}$ be the ideal sheaf which is locally generated by local equations
of $D_1$ and $D_2$. We have seen that there exists a sequence of blow ups of 2-curves
\( \pi_1 : X_1 \to X \) such that \( \mathcal{I}_1 \mathcal{O}_{X_1} \) is invertible. Thus there exists a
divisor \( \overline{D} \) on \( X_1 \) whose support is a union of components of \( D_{X_1} \) such that \( \mathcal{I}_1 \mathcal{O}_{X_1} \) is locally generated by a local equation of \( \overline{D} \).

Let \( \overline{D}_i = \pi_i^*(D_i) \) for \( 3 \leq i \leq n \). Then \( \mathcal{I}_1 \mathcal{O}_{X_1} \) is locally generated by local equations of the \( n - 1 \) divisors \( \overline{D}_2, \overline{D}_3, \ldots, \overline{D}_n \). By induction, there exists a sequence of blow ups of 2-curves \( X_2 \to X \) such that \( \mathcal{I}_1 \mathcal{O}_{X_2} \) is invertible.

\[ \square \]

**Lemma 3.12.** Suppose that \( f : X \to Y \) is a prepared morphism, and \( \overline{C} \) is a 2-curve in \( Y \). then there exists a sequence of blowups of 2-curves \( \Phi : X_1 \to X \) such that \( \mathcal{I}_C \mathcal{O}_{X_1} \) is invertible and \( \Phi \) is an isomorphism over \( f^{-1}(Y - \overline{C}) \).

**Proof.** The Lemma is a consequence of Lemma 3.11. \[ \square \]

**Lemma 3.13.** Suppose that \( X \) is a nonsingular 3-fold with SNC divisor \( D_X \), defining a
toroidal structure on \( X \). Suppose that \( \mathcal{I} \) is an ideal sheaf on \( X \) which is locally
generated by monomials in local equations of components of \( D_X \). Then there exists
a sequence of blow ups of 2-curves and 3-points \( \Phi_1 : X_1 \to X \) such that \( \mathcal{I}_1 \mathcal{O}_{X_1} \) is an
invertible ideal sheaf and \( \Phi_1 \) is an isomorphism away from the support of \( \mathcal{I} \).

The proof of this lemma follows from the proof of principalization of ideals, as in
[BEV] or [BrM] (cf. the proof of Theorem 6.3 [C3]) in the case when the ideal to be
principalized is locally generated by monomials in the toroidal structure.

4. Preparation

The following theorem is Theorem 19.11 [C2], with the additional conclusions that
all 2-curves of \( X_2 \) contain a 3-point, and all components of \( D_{X_2} \) contain a 3-point.

**Theorem 4.1.** Suppose that \( \Phi : X \to S \) is a dominant morphism from a nonsingular
3-fold \( X \) to a nonsingular surface \( S \) and \( D_S \) is a SNC divisor on \( S \) such that \( D_X = \Phi^{-1}(D_S) \) is a SNC divisor which contains the singular locus of \( \Phi \). Further suppose
that every component of \( D_X \) contains a 3-point and every 2-curve of \( X \) contains a 3-point.

1. Then there exists a sequence of blow ups of possible centers \( \alpha_1 : X_1 \to X \) such that
   (a) The fundamental locus of \( \alpha_1 \) is contained in the union of irreducible
   components \( E \) of \( D_X \) such that \( E \) contains a point \( p \) such that \( \Phi \) is not
   prepared at \( p \) (Definition 6.5 [C2]).
   (b) \( \Phi_1 = \Phi \circ \alpha_1 : X_1 \to S \) is prepared (Definition 6.5 [C2])
   (c) Each 2-curve of \( X_1 \) contains a 3-point and each component of \( D_{X_1} = \alpha_1^{-1}(D_X) \) contains a 3-point.

2. Further, there exist sequences of blow ups of possible centers, \( \alpha_2 : X_2 \to X_1 \)
   and \( \beta : S_1 \to S \) such that:
   (a) There is a commutative diagram
       \[
       \begin{array}{ccc}
       X_2 & \xrightarrow{\Phi_2} & S_1 \\
       \downarrow \alpha_2 & & \downarrow \beta \\
       X_1 & \xrightarrow{\Phi_1} & S \\
       \end{array}
       \]
       such that \( \Phi_2 \) is toroidal,
   (b) \( \beta \) is an isomorphism away from \( \beta^{-1}(\Phi_1(Z)) \) where \( Z \) is the locus where
       \( \Phi_1 \) is not toroidal.
   (c) \( \alpha_2 \) is an isomorphism away from \( \alpha_2^{-1}(\Phi_1^{-1}(\Phi_1(Z))) \)
(d) Each 2-curve of $X_2$ contains a 3-point and each component of $D_{X_2}$ contains a 3-point.

Proof. For the proof we need only make some small modifications in the proof of Theorem 19.11 [C2].

We first prove 1 of the theorem. By Lemma 6.2 [C2], there exists a commutative diagram

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\alpha_0} & S \\
\downarrow & & \\
X & \xrightarrow{\Phi_0} & S
\end{array}
$$

such that $\Phi_0 : X_0 \to S$ is a weakly prepared morphism (Definition 6.1 [C2]), and the fundamental locus of $\alpha_0$ is contained in the locus where $\Phi$ is not weakly prepared, (which is contained in the locus where $\Phi$ is not prepared). It is not necessary to blow up points on $S$ since $D_S, D_X$ are SNC divisors. Let $D_{X_0} = \alpha_0^{-1}(D_X)$. By further blowing up of points in the exceptional locus of $\alpha_0$, we may assume that all components of $D_{X_0}$ contain a 3-point, and all 2-curves of $X_0$ contain a 3-point.

By Theorem 17.2 [C2] there exists a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\alpha_1} & S \\
\downarrow & & \\
X_0 & \xrightarrow{\Phi_0} & S
\end{array}
$$

such that $\Phi_1$ is prepared. The algorithm consists of a sequence

$$
X_1 = Y_n \xrightarrow{\tau_n} \cdots \xrightarrow{\tau_2} Y_1 \xrightarrow{\tau_1} Y_0 = X_0
$$

of blow ups of points and nonsingular curves which are possible centers; that is make SNCs with the preimage $D_{Y_i}$ of $D_{X_0}$ and are contained in a component $E$ of $D_Y$ such that $E$ contains a point which is not prepared for $\Phi_0 \circ \alpha_1 \circ \cdots \circ \alpha_i : Y_i \to S$.

If $p \in Y_i$ is prepared for $\Phi_i$, then all points of $\tau_{i+1}^{-1}(p)$ are prepared for $\Phi_{i+1}$. Thus conditions (a) and (b) of 1 hold.

The strict transforms on $X_1$ of all components of $D_X$ must contain a 3-point and if $C$ is a 2-curve of $X$ which is not contained in the fundamental locus of $\alpha_1 = \alpha_0 \circ \tau$, then its strict transform on $X_2$ must also contain a 3-point. Thus any components of $D_{X_2}$ which do not contain a 3-point, and 2-curves of $X_2$ which do not contain a 3-point, are contained in the exceptional locus of $\alpha_1$.

To complete the proof of 1 of the theorem, we need only show that if $p \in X_1$ is a point in the exceptional locus of $\alpha_1$, and $\alpha^* : X \to X_1$ is the blow up of $p$, then $\Phi_1 \circ \alpha^* : X \to S$ is prepared. This can directly be seen by substituting local equations for the blow up of a point into (17)–(20) of Definition 6.5 [C2]. We can thus construct $\alpha_1 : X_1 \to X$ such that $\Phi_1 : X_1 \to S$ is prepared and all conditions of 1 hold.

We now verify 2 of the theorem. We first observe that $\Phi_1$ (from the conclusions of part 1 of this theorem) is strongly prepared (Definition 18.1 [C2]). We now examine the monomialization algorithm of Chapter 18 [C2] and the toroidalization algorithm of Chapter 19 [C2], applied to $\Phi_1 : X_1 \to S$.

The monomialization algorithm of Theorem 18.19 and Theorem 18.21 [C2] consists in constructing a commutative diagram

$$
\begin{array}{ccc}
\tilde{X}_2 & \xrightarrow{\tilde{\Phi}_2} & X_1 \\
\downarrow & \downarrow & \downarrow \\
\tilde{S}_1 & \xrightarrow{\tilde{\Psi}_1} & S
\end{array}
$$
such that $\tilde{\Phi}_2$ is monomial (all points of $\tilde{X}_2$ are good for $\tilde{\Phi}_2$ as defined in Definition 18.5 [C2]).

(25) has a factorization

$$
\begin{align*}
\tilde{X}_2 &= Z_t \xleftarrow{\delta_t} \cdots \rightarrow Z_1 \xleftarrow{\delta_1} Z_0 = X_1 \\
\Phi_2 &\downarrow \Omega_1 \xleftarrow{\delta_1} \cdots \rightarrow \Omega_1 \xleftarrow{\delta_0} \Phi_1 \downarrow (26)
\end{align*}
$$

where each $\Omega_i$ is strongly prepared, each $\delta_{i+1}$ is the blow up of a point $q_i$ such that $\Omega^{-1}_i(q_i)$ contains a point $p_i$ at which $\Omega_i$ is not monomial and $\epsilon_{i+1}$ is a sequence of blow ups of curves which are exceptional to such $q_i$. This step is accomplished by performing the algorithms of Lemmas 18.16, 18.17 and 18.18 of [C2]. We will make a minor modification in the algorithm of Theorem 18.19, which will ensure that all 2-curves of $\tilde{X}_2$ contain a 3-point, and all components of $D_{\tilde{X}_2}$ contain a 3-point.

Each map $Z_{i+1} \rightarrow Z_i$ has a factorization

$$
Z_{i+1} = Z_m \xrightarrow{\lambda_m} Z_{m-1} \rightarrow \cdots \rightarrow \lambda_1 Z_0 = Z_i (27)
$$

where each $\lambda_{i+1}$ is a blow up of a 2-curve or of a curve $C_j$ which contains a 1-point, makes SNCs with the preimage $D_{\tilde{Z}_j}$ of $D_X$ on $\tilde{Z}_j$, and is contained in a component of $D_{\tilde{Z}_j}$. To construct (27) we successively apply Lemmas 18.16, 18.17, 18.18 of [C2].

The algorithms of Lemma 18.16 and Lemmas 18.18 [C2] consist of a sequence of blow ups of 2-curves and the condition that all 2-curves contain a 3-point and all components of $D_{\tilde{Z}_j}$ contain a 3-point is preserved by this condition.

The algorithm of Lemma 18.17 [C2] consists of a sequence of blow ups of curves $\lambda_{j+1} : \tilde{Z}_{j+1} \rightarrow \tilde{Z}_j$ of $C_j \subset D_{\tilde{Z}_j}$ which are not 2-curves, and are contained in the locus where $m_{q_j} O_{\tilde{Z}_j}$ is not invertible. Let $p \in C_j$ be a general point, so that $p$ is a 1-point. There exist permissible parameters $(u, v)$ at $q_i$ and regular parameters $x, y, z$ in $O_{\tilde{Z}_j, p}$ such that a form (185) of Lemma 18.12 [C2] holds,

$$
u = x^k, v = x^c y (28)$$

with $c < k$, $x = 0$ is a local equation of $D_{\tilde{Z}_j}$ at $p$, and $x = y = 0$ are local equations of $C_j$. $u = v = 0$ are local equations of $q$ in $T_i$. The exceptional divisor of $\lambda_{j+1}$ contains a 2-curve which is a section over $C_j$. At the two point $p_1 \in \lambda_j^{-1}(p)$, we have regular parameters $x_1, y_1, z_1$ in $O_{\tilde{Z}_{j+1}, p}$ such that

$$
u = x_1^k y_1, v = x_1^c y_1^{c+1}. (29)$$

If $C_j$ contains a 2-point then all components of $D_{\tilde{Z}_{j+1}}$ contain a 3-point, and all 2-curves of $\tilde{Z}_{j+1}$ contain a 3-point.

Suppose that $C_j$ does not contain a 2-point. Then $u, v$ have an expression of the form (28) at all points $p \in C_j$.

If $C_j$ does not contain a 2-point, we modify the algorithm of Lemma 18.17 [C2], inserting an extra step here, by performing the blow up $\lambda_j' : \tilde{Z}_{j+1} \rightarrow \tilde{Z}_{j+1}$ of the point $p_1$. In this case, all points $p \in C_j$ are general points, and we may choose $p = \lambda_j^{-1}(p_1)$ to be any point of $C_j$ which is convenient. We will make use of this observation in the proof of Theorem 4.5. Points $p_2$ above $p_1$ have regular parameters $(x_2, y_2, z_2)$ such that

$$
x_1 = x_2, y_1 = x_2(y_2 + \alpha), z_1 = x_2(z_2 + \beta) (30)$$

with $\alpha, \beta \in k$,

$$
x_1 = x_2 y_2, y_1 = y_2, z_1 = y_2(z_2 + \beta) (31)$$
with \( \beta \in k \), or
\[
x_1 = x_2 z_2, y_1 = y_2 z_2, z_1 = z_2
\]
(32)

Substituting (30) into (29) we have
\[
u = x_2^{2k} (y_2 + \alpha)^k, \quad v = x_2^{2c+1} (y_2 + \alpha)^{c+1}.
\]
(33)

Thus if \( \alpha \neq 0 \) we have a good point of the form (183) of [C2] and \( m_\eta, O_{\hat{Z}_{j+1}, p_2} \) is invertible. If \( \alpha = 0 \), then
\[
u = x_2^k y_2^k, \quad v = x_2^{c+1} y_2^{c+1}
\]
which is a good point of the form (179) of [C2] and is a form (187) of [C2] if \( m_\eta, O_{\hat{Z}_{j+1}, p_2} \) is not invertible.

Under substitution of (31) into (29), we see that
\[
u = x_2^{k} y_2^k, \quad v = x_2^{c+1} y_2^{c+1}
\]
which is a good point of the form (179) of [C2] and is a form (187) of [C2] if \( m_\eta, O_{\hat{Z}_{j+1}, p_2} \) is not invertible.

Under substitution of (32) into (29), we obtain
\[
u = x_2^{k} y_2^{c+1} z_2^{c+1}, \quad v = x_2^{c+1} y_2^{c+1} z_2^{c+1}
\]
which is a good point of the form (193) of [C2] if \( m_\eta, O_{\hat{Z}_{j+1}, p_2} \) is not invertible.

Observe that the locus of points in \((\lambda'_{j+1})^{-1}(p_1)\) where \( m_\eta, O_{\hat{Z}_{j+1}, p} \) is not invertible is a union of 2-curves.

We now continue the algorithm as in the the proof of Lemma 18.17 [C2]. As the invariant \( \Omega(C_j) = k - c \) of Lemma 18.17 [C2] which is decreased in the algorithm of Lemma 18.17 [C2] is computed at generic points of curves \( C_j \) (which contain a 1-point) and for which \( m_\eta, O_{\hat{Z}, p} \) is not invertible, these invariants are not affected by inserting these new blow ups of points \( \lambda'_{j+1} \) into (27). Thus the conclusions of Theorem 18.19 [C2] will hold, for the modified \( \tilde{X}_2 \to \tilde{S}_1 \), but we may further assume that each 2-curve of \( \tilde{X}_2 \) contains a 3-point and each component of \( D_{\tilde{X}_2} \) contains a 3-point.

Theorem 19.9 [C2] and Theorem 19.10 [C2] imply there exists a commutative diagram
\[
\begin{array}{ccc}
\tilde{X}_3 & \xrightarrow{\pi_3} & \tilde{X}_2 \\
\Phi_3 \downarrow & & \downarrow \Phi_2 \\
\tilde{S}_2 & \xrightarrow{\psi_2} & \tilde{S}_1
\end{array}
\]
(34)

such that \( \tilde{X}_3 \to \tilde{S}_2 \) is toroidal, and 2 (b), 2 (c) of the conclusions of the theorem hold. We will indicate how we can modify the proof slightly to ensure that 2 (d) of the conclusions of the theorem holds for \( \tilde{X}_3 \to \tilde{S}_2 \).

The algorithm of Theorem 19.9 [C2] consists of a sequence of blow ups of curves above \( \tilde{X}_2 \) and finitely many blow ups of points over \( \tilde{S}_1 \).

In the algorithm, we first construct a diagram
\[
\begin{array}{ccc}
\tilde{X}_3' & \xrightarrow{\pi_3'} & \tilde{X}_2 \\
\tilde{\Phi}_3' \downarrow & & \downarrow \tilde{\Phi}_2 \\
\tilde{S}_2' & \xrightarrow{\psi_2'} & \tilde{S}_1
\end{array}
\]
(35)

which has a factorization by a diagram of the form (26), so that a global invariant \( I(\tilde{\Phi}_3') \leq 0 \) (this invariant is defined on page 227 of [C2]). The factorizations (27) of the morphisms of (26) consist of a sequence of blow ups of curves, using first Lemma...
18.17 [C2] to blow up curves which contain a 1-point and are possible centers (make SNCs with \(D_{\mathbb{Z}_j}\) and are contained in a component of \(D_{\mathbb{Z}_j}\)) and then Lemma 18.18 [C2] to blow up 2-curves.

If a 2-curve is blown up, then the condition that all 2-curves contains a 3-point is preserved.

Suppose that a curve \(C_j\) is blown up which contains a 1-point by \(\lambda_{j+1} : \mathbb{Z}_{j+1} \to \mathbb{Z}_j\) (in (27)). This is analyzed in Lemma 19.8 [C2]. Let \(p \in C_j\) to be a general point. Then a form (185) of [C2] (as in (28) of our analysis of monomialization) holds at \(p\), and if \(p_1 \in \lambda_{j+1}^{-1}(p)\) is the 2-point, then a form (29) holds at \(p_1\).

Assuming that \(C_j\) does not contain a 2-point. We now modify the algorithm of Theorem 19.9 [C2] by blowing up the point \(p_1\). Let \(\lambda'_{j+1} : \mathbb{Z}'_{j+1} \to \mathbb{Z}_{j+1}\) be this map. Let \(E\) be the exceptional divisor of \(\lambda'_{j+1}\). We see (from (33)) that a form

\[
u = \pi_2^{2k}, v = \pi_2^{2c+1}(\pi + \bar{y}_2)
\]

with \(\pi \neq 0\) holds at a general point \(p_2\) of \(E\). Let

\[
\tilde{\Omega}_j = \Omega_i \circ \lambda_1 \circ \ldots \circ \lambda_{j+1} \circ \lambda'_{j+1}.
\]

We have

\[
I(\tilde{\Omega}_j, E) = (2c + 1) - 2k = 2(c - k) + 1 < 0\quad (36)
\]

since \(c < k\). We may thus continue the algorithm of Theorem 19.9 [C2]. We modify (35) by adding in these blow ups, \(\lambda'_{j+1}\), to achieve the reduction \(I(\tilde{\Phi}_3', E) \leq 0\) for all components \(E\) of \(D_{\tilde{X}_3}\) which contain a 1-point mapping to a 1-point, and so that all components of \(D_{\tilde{X}_3}\) contain a 3-point, and all 2-curves of \(\tilde{X}_3\) contain a 3-point.

The algorithm of Theorem 19.10 [C2] consists of a sequence of blow ups of curves over \(\tilde{X}_3\) and points over \(S_2\). We construct a commutative diagram

\[
\begin{array}{ccc}
\tilde{X}_3 & \xrightarrow{\pi_3'} & \tilde{X}'_3 \\
\Phi_3 & \downarrow & \Phi_3' \\
S_2 & \xrightarrow{\psi_2'} & S_2'
\end{array}
\]

such that \(\tilde{X}_3\) is toroidal, which has a factorization by a diagram of the form (26). The factorization (27) of the morphisms in (26) consists of a sequence of blow ups of curves \(C_j\) which are possible centers (make SNCs with \(D_{\mathbb{Z}_j}\) and are contained in \(D_{\mathbb{Z}_j}\)), using Lemma 18.17 [C2].

Suppose that a curve \(C_j\) is blown up by \(\lambda_{j+1} : \mathbb{Z}_{j+1} \to \mathbb{Z}_j\) in equation (27). If \(C_j\) contains a 2-point then (assuming that all components of \(D_{\mathbb{Z}_j}\) contain a 3-point and all 2-curves of \(\mathbb{Z}_j\) contain a 3-point) all components of \(D_{\mathbb{Z}_{j+1}}\) contain a 3-point, and all 2-curves of \(\mathbb{Z}_{j+1}\) contain a 3-point.

Suppose that \(C_j\) does not contain a 2-point. Then \(u, v\) have an expression of the form (28) at all \(p \in C_j\). Let \(p \in C_j\). If \(p_1 \in \lambda_{j+1}^{-1}(p)\) is the 2-point, then (29) holds at \(p_1\). We now modify the algorithm of Theorem 19.10 [C2] by blowing up the 2-point \(p_1\). Let \(\lambda'_{j+1} : \mathbb{Z}'_{j+1} \to \mathbb{Z}_{j+1}\) be this map. Let \(E\) be the exceptional divisor of \(\lambda'_{j+1}\). Let

\[
\overline{\Omega}_j = \Omega_i \circ \lambda_1 \circ \ldots \circ \lambda_{j+1} \circ \lambda'_{j+1}.
\]

We have \(I(\overline{\Omega}_j, E) < 0\) (as shown in (36)). Now by Lemma 19.6 [C2] we can continue the algorithm of Theorem 19.10 [C2] to achieve the conclusions of Theorem 19.10 [C2], with the conclusions 2 (a) - 2 (d) of the conclusions of this theorem.
**Lemma 4.2.** Suppose that $f : X \to Y$ is a birational projective morphism of nonsingular 3-folds with toroidal structure, defined by SNC divisors $D_Y$ and $D_X = f^{-1}(D_Y)$, there exists $q \in D_Y$ such that there exist uniformizing parameters $u, v, w$ on $Y$ (an etale morphism $Y \to \text{spec}(k[u, v, w])$) such that $u = v = w = 0$ are equations of $q$ in $Y$,

$$D_Y = \{uv = 0\},$$

and the fundamental locus of $f$ is $C_1 \cup C_2$ where $u = w = 0$ are equations of $C_1$ in $Y$, $v = w = 0$ are equations of $C_2$ in $Y$. Suppose that $u = 0$, $v = 0$ are integral surfaces in $Y$, and $C_1, C_2$ are irreducible. Let $\pi : Y \to S = \text{spec}(k[u, v])$ be the projection. Let $\eta = \pi(q)$ and $\gamma = \pi^{-1}(\eta) \subset Y$. Let $D_S = \{uv = 0\}$, a SNC divisor on $S$. Assume that $g = \pi \circ f : X \to S$ is toroidal away from $f^{-1}(q)$, and prepared away from $f^{-1}(\gamma)$.

Then there exists a sequence of blow ups

$$X_n \xrightarrow{\Psi_n} X_{n-1} \to \cdots \xrightarrow{\Psi_1} X$$

where each $\Psi_i : X_i \to X_{i-1}$ is the blow up of a possible center (a point or a nonsingular curve contained in $D_{X_{i-1}} = \Phi_{i-1}^{-1}(D_{X_{i-2}})$ which makes SNCs with $D_{X_{i-1}}$) which is supported over $f^{-1}(q)$ such that if $F$ is a component of $D_X$ which dominates a component of $D_Y$ (or dominates $C_1$ or $C_2$), and $F$ is the strict transform of $F$ on $X_n$, then $X_n \to S$ is toroidal in a neighborhood of $F_n$, and $X_n \to S$ is prepared on $F_n$ away from the strict transform of $\gamma$. Further, $X_n \to S$ is prepared away from the preimage of $\gamma$, and is toroidal away from the preimage of $q$.

Further assume that every irreducible component of $D_X$ contains a 3-point and every 2-curve of $D_X$ contains a 3-point. Then every irreducible component of $D_{X_n} = (\Psi_1 \circ \cdots \circ \Psi_n)^{-1}(D_X)$ contains a 3-point and every 2-curve of $D_{X_n}$ contains a 3-point.

**Proof.** After possibly blowing up points and curves over $X$ which are supported over $f^{-1}(q)$, we may assume that $f^{-1}(q)$ is a divisor, and if $F$ is a component of $D_X$ which dominates a component $E$ of $D_Y$, and $L$ is an exceptional component of $f$ which intersects $F$, then $f(L) \subset E$. Let $E_1$ be the component of $D_Y$ with local equation $u = 0$, $E_2$ be the component of $D_Y$ with local equation $v = 0$. We may further assume that if $F$ is a component of $D_X$ which dominates $C_1$ (respectively $C_2$) and $L$ is an exceptional component of $f$ which intersects $F$, then $f(L) \subset E_1$ (respectively $f(L) \subset E_2$). Finally, if every irreducible component of $D_X$ contains a 3-point and every 2-curve of $D_X$ contains a 3-point, we may assume that this condition is preserved. Let $G = f^{-1}(q)$.

Suppose that $F$ is a component of $D_X$ which dominates a component $E$ of $D_Y$. Without loss of generality, $E$ has the equation $u = 0$. By assumption, $g$ is toroidal at points of $F - f^{-1}(q)$. Suppose that $p \in F \cap f^{-1}(q)$. Then $p$ must be a 2-point or a 3-point. Recall that $uv = 0$ is an equation of the SNC divisor $D_X$ on $X$.

If $p \in F \cap f^{-1}(q)$ is a 2-point then there exist regular parameters $x, y, z$ at $p$ such that $xyz = 0$ is a local equation of $D_X$, $x = 0$ is a local equation of $F$ and there is an expression

$$u = xy^g \lambda_1, v = y^c \lambda_2$$

where $\lambda_1, \lambda_2$ are units, $g > 0$ and $c > 0$.

Thus $g$ is toroidal and prepared in a neighborhood of $p$.

If $p \in F \cap f^{-1}(q)$ is a 3-point and $p$ is not on the strict transform of the component $E'$ of $D_Y$ with local equation $v = 0$, then there exist regular parameters $x, y, z$ at $p$ such that $xyz = 0$ is a local equation of $D_X$, $x = 0$ is a local equation of $F$, and there is an expression

$$u = xy^b z^c \lambda_1, v = y^d z^c \lambda_2$$
where \( \gamma_1, \gamma_2 \) are units, \( b, c > 0 \) and \( d + e > 0 \). Thus \( g \) is toroidal and prepared in a neighborhood of \( p \).

If \( p \in F \cap f^{-1}(q) \) is a point on the strict transform on \( X \) of the component \( E' \) of \( D_Y \) with local equation \( v = 0 \), then \( p \) is a 3-point and there is an expression
\[
\begin{align*}
    u &= xz^a \lambda_1 \\
    v &= yz^b \lambda_2
\end{align*}
\]  
(39)

at \( p \) where \( a, b > 0 \), \( \lambda_1, \lambda_2 \) are units, \( x = y = 0 \) are local equations of the strict transform \( \gamma_1 \) of \( \gamma = \pi^{-1}(q) \), and \( g \) is toroidal in a neighborhood of \( p \).

Suppose that \( F \) is a component of \( D_X \) which dominates \( C_1 \). By assumption, \( g \) is toroidal at points of \( F - f^{-1}(q) \), and on points of the strict transform of a component of \( D_Y \).

Suppose that \( p \in F \cap f^{-1}(q) \) is not on the strict transform of a component of \( D_Y \).

If \( p \) is a 2-point, then there exist regular parameters \( x, y, z \) at \( p \) and unit series \( \lambda_1, \lambda_2 \) such that
\[
\begin{align*}
    u &= xz^a \lambda_1 \\
    v &= yz^b \lambda_2,
\end{align*}
\]
\( x = 0 \) is a local equation of \( F \), \( y, z = 0 \) are local equations of exceptional components of \( D_X \) which maps into \( E_1 \), so that \( a, b, c > 0 \). Thus \( g \) is toroidal and prepared at \( p \).

If \( p \) is a 3-point, then there exist regular parameters \( x, y, z \) at \( p \) such that
\[
\begin{align*}
    u &= x^a y^b z^c \lambda_1 \\
    v &= y^d z^e \lambda_2
\end{align*}
\]
where \( x = 0 \) is a local equation of \( F \), \( y, z = 0 \) are local equations of exceptional components of \( D_X \) which map into \( E_1 \), \( d + e > 0 \) and \( \lambda_1, \lambda_2 \) are unit series. Thus \( a, b, c > 0 \) and \( f \) is toroidal and prepared at \( p \).

The same analysis applies if \( F \) is a component of \( D_X \) which dominates \( C_2 \).

\[\square\]

**Lemma 4.3.** Suppose that \( f : X \to Y \) is a birational morphism of nonsingular projective 3-folds. Then there exists a commutative diagram
\[
\begin{array}{ccc}
    X_1 & \xrightarrow{f_1} & Y_1 \\
    \downarrow & & \downarrow \\
    X & \xrightarrow{f} & Y
\end{array}
\]

where the vertical arrows are products of blow ups of nonsingular subvarieties such that the (reduced) fundamental locus \( \Gamma \) of \( f_1 \) is a union of nonsingular curves and points such that two curves of \( \Gamma \) intersect in at most one point, and this intersection is transversal (the two curves have distinct tangent directions). Further, the intersection of any three curves of \( \Gamma \) is empty.

**Proof.** Let \( S \) be a reduced (but not necessarily irreducible) surface in \( Y \) containing the fundamental locus of \( f \). By the standard theorems of resolution of singularities ([H], Section 6.8 [C3]), there exists a commutative diagram
\[
\begin{array}{ccc}
    X_1 & \xrightarrow{f_1} & Y_1 \\
    \Phi_1 \downarrow & & \downarrow \Psi_1 \\
    X & \xrightarrow{f} & Y
\end{array}
\]

where \( \Phi_1 \) and \( \Psi_1 \) are products of blow ups of points and nonsingular curves, such that \( \Phi_1^{-1}(S) \) is a divisor whose irreducible components are nonsingular, which necessarily contains the fundamental curve \( \Gamma_1 \) of \( f_1 \) (the reduced 1-dimensional scheme consisting of the 1-dimensional components of the fundamental locus of \( f_1 \)). Let \( S_1, \ldots, S_n \) be the irreducible components of \( \Phi_1^{-1}(S) \).
Let $H$ be a hyperplane section of $Y_1$. Let $q \in \Gamma_1$ be a singular point. Let $m_q \subset \mathcal{O}_{Y_1,q}$ be the ideal sheaf of the point $q$, $R = \mathcal{O}_{Y_1,q}$, $m_q = m_q R$ be the maximal ideal of $R$.

Suppose that $\bar{\alpha}, \bar{\beta} \in m_q/m_q^2$ are linearly independent over $k$. There exist regular parameters $u,v,w$ in $\mathcal{O}_{Y_1,q}$ such that $\bar{\alpha} = [u], \bar{\beta} = [v] \in m_q/m_q^2$. Let $\mathcal{I}_{\bar{\alpha}} \subset \mathcal{O}_{Y_1}$ be the ideal sheaf defined by

$$\mathcal{I}_{\bar{\alpha},p} = \begin{cases} \mathcal{O}_{Y_1,p} \left( u \right) + m_q^2 & \text{if } p \neq q \\ \mathcal{O}_{Y_1,p} \left( v \right) + m_q^2 & \text{if } p = q. \end{cases}$$

Let $Y_{\bar{\alpha}}$ be the blow up of $\mathcal{I}_{\bar{\alpha}}$, with projection $\pi_{\bar{\alpha}} : Y_{\bar{\alpha}} \to Y_1$. Let $\mathcal{L}_{\bar{\alpha}} = \mathcal{I}_{\bar{\alpha}} \mathcal{O}_{Y_{\bar{\alpha}}}$. Then $\mathcal{M}_{m}^{\bar{\alpha}} = \pi_{\bar{\alpha}}^* \mathcal{O}_{Y_1}(mH) \otimes \mathcal{L}_{\bar{\alpha}}$ is very ample for $m \gg 0$. $\mathcal{I}_{\bar{\alpha},p}$ is a complete ideal, so

$$\mathcal{N}_{m}^{\bar{\alpha}} = \mathcal{O}_{Y_1}(mH) \otimes \mathcal{I}_{\bar{\alpha}} \cong \left( \pi_{\bar{\alpha}} \right)_*(\mathcal{M}_{m}^{\bar{\alpha}})$$

and

$$\Gamma(Y_{\bar{\alpha}}, \mathcal{M}_{m}^{\bar{\alpha}}) = \Gamma(Y_{1}, \mathcal{N}_{m}^{\bar{\alpha}}).$$

Since $\mathcal{M}_{m}^{\bar{\alpha}}$ is generated by global sections, the divisor of a general section of $\Gamma(Y_{\bar{\alpha}}, \mathcal{M}_{m}^{\bar{\alpha}})$ is irreducible and nonsingular away from $\pi_{\bar{\alpha}}^{-1}(q)$ by Bertini’s theorem, (cf. Theorems 7.18 and 7.19 [I]). Thus the divisor of a general section of $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\alpha}})$ is irreducible and is nonsingular away from $q$. Consider the exact sequence

$$0 \to m_q^2 \to \mathcal{I}_{\bar{\alpha}} \to \mathcal{I}_{\bar{\alpha}}/m_q^2 \to 0.$$

Tensoring with $\mathcal{O}_{Y_1}(mH)$ for $m \gg 0$, we see that there is a surjection

$$\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\alpha}}) \to \mathcal{I}_{\bar{\alpha}}/m_q^2 \cong k.$$

Thus a general section $\sigma \in \Gamma(Y_1, \mathcal{N}_{m}^{\bar{\alpha}})$ is such that

$$\sigma \equiv \lambda_1 u \mod m_q^2$$

for some $0 \neq \lambda_1 \in k$. In particular, the divisor $D_\sigma$ of a general section $\sigma$ is irreducible, nonsingular and $u = 0$ is a local equation of its tangent space at $q$.

In an analogous way, we can define an ideal sheaf $\mathcal{I}_{\bar{\beta}} \subset \mathcal{O}_{Y_1}$ by

$$\mathcal{I}_{\bar{\beta},p} = \begin{cases} \mathcal{O}_{Y_1,p} \left( u \right) + m_q^2 & \text{if } p \neq q \\ \mathcal{O}_{Y_1,p} \left( v \right) + m_q^2 & \text{if } p = q. \end{cases}$$

and show that for $m \gg 0$, if $\mathcal{N}_{m}^{\bar{\beta}} = \mathcal{O}_{Y_1}(mH) \otimes \mathcal{I}_{\bar{\beta}}$, the divisor $D_\tau$ of a general section $\tau$ of $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\beta}})$ is irreducible, nonsingular, and $v = 0$ is a local equation of its tangent space at $q$.

Since the base locus of the linear system of $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\beta}})$ is the point $q$, the divisor $D_\sigma$ of a general section $\sigma$ of $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\beta}})$ intersects $\Gamma_1$ at $q$ and at finitely many other points. Since the base locus of the linear system $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\beta}})$ is $q$, if $D_\sigma$ is the divisor of a general section $\tau$ of $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\beta}})$, (for appropriate $m' \geq m$) then $\Gamma_1$ intersects the scheme $D_\sigma \cdot D_\tau$ at the point $q$ only, and by Bertini’s theorem, $\gamma = D_\sigma \cdot D_\tau$ is an irreducible curve which is nonsingular away from $q$. Since $\sigma \equiv \lambda_1 u \mod m_q^2$ and $\tau \equiv \lambda_2 v \mod m_q^2$ for some $0 \neq \lambda_1, \lambda_2 \in k$, we have that $\gamma$ is nonsingular at $q$, and $u = v = 0$ are local equations of the tangent space to $\gamma$ at $q$. By Bertini’s theorem applied to the surfaces $S_1, \ldots, S_n$, we see that the divisor $D_\sigma$ of a general section $\sigma$ of $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\beta}})$ intersects the $S_i$ transversally away from $q$ (the tangent spaces of $D_\sigma$ and $S_i$ intersect in a line), and a further application of Bertini’s theorem shows that the divisor $D_\tau$ of a general section $\tau$ of $\Gamma(Y_1, \mathcal{N}_{m}^{\bar{\beta}})$ is such that $\gamma = D_\sigma \cdot D_\tau$ intersects each $S_i$ transversally away from $q$. 
In summary, for \( m' \gg m \gg 0 \) there exist \( \sigma_1 \in \Gamma(Y_1, \mathcal{O}_{Y_1}(mH)), \) and \( \sigma_2 \in \Gamma(Y_1, \mathcal{O}_{Y_1}(m'H)) \), with respective divisors \( H_1 \) and \( H_2 \) such that \( \gamma = H_1 \cdot H_2 \) is a nonsingular curve which intersects \( \Gamma_1 \) in the point \( q \) only, \( \gamma \) intersects each \( S_i \) transversally at all points other than \( q \), and for some nonzero \( \lambda_1, \lambda_2 \in k, \) \( \sigma_1 \) has image \( \lambda_1 \tilde{\alpha} \) and \( \sigma_2 \) has image \( \lambda_2 \tilde{\beta} \) in \( m_q/m_q^2 \) under the natural maps \( \Gamma(Y_1, \mathcal{O}_{Y_1}(mH)) \otimes m_q \to m_q/m_q^2 \), and \( \Gamma(Y_1, \mathcal{O}_{Y_1}(m'H) \otimes m_q) \to m_q/m_q^2 \). Let \( f_i = 0 \) be a local equation of \( S_i \) at \( q \) for all \( i \) such that \( q \in S_i \). We have a surjection \( m_q/m_q^2 \to \mathcal{m}_q/\mathcal{m}_q^2 \) and \( \dim_k \mathcal{m}_q/\mathcal{m}_q^2 > 1 \) since \( q \) is a singular point of \( \Gamma_1 \). Choose \( \tilde{\alpha}, \tilde{\beta} \in m_q/m_q^2 \) so that the \( k \)-span of \( \tilde{\alpha} \) and \( \tilde{\beta} \) does not contain the class of \( f_i \) for any \( S_i \) containing \( q \), and so that the images of \( \tilde{\alpha} \) and \( \tilde{\beta} \) in \( \mathcal{m}_q/\mathcal{m}_q^2 \) are linearly independent over \( k \). Now choose \( H_1, H_2 \) and \( \gamma = H_1 \cdot H_2 \) as above.

Let \( \Psi_2 : Y_2 \to Y_1 \) be the blow up of \( \gamma \). Let \( \Gamma_1 \) be the strict transform of \( \Gamma_1 \) on \( Y_2 \), \( S_i \) for \( 1 \leq i \leq n \) be the strict transforms of \( S_i \) on \( Y_2 \). We can assume that \( u = 0 \) is a local equation of \( H_1 \) at \( q \), \( v = 0 \) is a local equation of \( H_2 \) at \( q \). By assumption, \( u, v, f_i \) is a regular system of parameters in \( \mathcal{O}_{Y_1,q} \) for all \( i \) such that \( q \in S_i \), so \( \gamma \) intersects \( S_i \) transversally at \( q \). Thus for all \( i \) such that \( q \in S_i \), \( \overline{S}_i \) is the blow up of \( (\text{the ideal sheaf of } q) \) and a finite number of other points on \( S_i \), which are disjoint from \( \Gamma_1 \). In particular, each \( \overline{S}_i \) is nonsingular.

Let \( C_1, \ldots, C_m \) be the irreducible components of \( \Gamma_1 \) containing \( q \) and let \( K_i \) be the function field of \( C_i \) for \( 1 \leq i \leq m \). Let \( \overline{K} = K_1 \oplus \cdots \oplus K_m \). Since \( \Gamma_1 \) is reduced, we have natural inclusions

\[
R \to \overline{A} = A_1 \oplus \cdots \oplus A_m \to \overline{K}
\]

where \( A_i \) is the integral closure of \( \mathcal{O}_{C_i,q} \) in \( K_i \). \( \overline{A} \) is finite over \( R \), since \( \overline{A} \) is the normalization of \( R \). Let \( s(q) \) be the length of \( \overline{A} \) as an \( R \)-module.

Suppose that \( q_1 \in \Gamma_1 \cap \Psi_2^{-1}(q) \). Let \( R_1 = \mathcal{O}_{Y_1,q_1} \). \( R_1 \) is a local ring of the blow up of \( (u,v)R \) in \( K \). \( (u,v)R \) is not a principal ideal in \( R \), since \( \tilde{\alpha}, \tilde{\beta} \) are linearly independent in \( \mathcal{m}_q/\mathcal{m}_q^2 \), and by construction, \( \mathcal{m}_q/R_1 \) is principal. Thus we have inclusions

\[
R \to R_1 \to \overline{K}
\]

with \( R \neq R_1 \). Since \( R_1 \) is finite over \( R \), we have an inclusion \( R_1 \subset \overline{A} \), and \( \overline{A} \) has length \( s_1 < s(q) \) as an \( R_1 \)-module.

The strict transform \( \gamma' \) of \( \gamma \) on \( X_1 \) is necessarily a nonsingular curve. Let \( \Phi_2' : X_2' \to X_1 \) be the blow up of \( \gamma' \). \( \mathcal{I}_2 \mathcal{O}_{X_2'} \) is invertible, except possibly over \( q \). There exists a sequence of blow ups \( X_2 \to X_2' \) supported over \( q \) such that \( \mathcal{I}_2 \mathcal{O}_{X_2} \) is invertible. Thus we have a natural morphism \( f_2 : X_2 \to Y_2 \). The fundamental curve of \( f_2 \) is contained in the union of the strict transform \( \Gamma_1' \) of \( \Gamma_1 \) and the nonsingular curve \( l_2 = \Psi_2^{-1}(q) \). \( l_2 \) lies on all surfaces \( \overline{S}_i \) such that \( S_i \) contains \( q \).

By induction on

\[
\max\{s(q) \mid q \in \Gamma_1 \text{ is a singular point}\}
\]

we can iterate the blowups of such curves \( \gamma \) until we construct a commutative diagram

\[
\begin{array}{ccc}
X_3 & \xrightarrow{\Phi_3} & Y_3 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{\Phi_2} & Y_1
\end{array}
\]

such that

1. If \( \Gamma' \) is the strict transform of \( \Gamma_1 \) on \( Y_3 \) then \( \Gamma' \) is a disjoint union of nonsingular curves.
2. The strict transform \( S_i' \) of \( S_i \) on \( Y_3 \) is nonsingular for all \( i \).
The fundamental curve $\Gamma_3$ of $f_4$ is the union of $\Gamma'$ and a union of curves which are contained in the exceptional loci of the morphisms $\lambda_i = (\Psi_i \mid S'_i) : S'_i \to S_i$ for $1 \leq i \leq n$.

For $q \in \Gamma_3$, and $S_i$ such that $q \in S_i$, $\Psi_i^{-1}(q) = \lambda_i^{-1}(q)$ is a SNC divisor on $S'_i$. Thus $\Gamma_3$ can only fail to satisfy the conclusions of the theorem at a finite number of points $q'$ such that $q'$ is contained in a (unique) component $C_i'$ of $\Gamma'$ which is contained in some $S'_i$, and there exists a neighborhood $U$ of $q'$ in $Y_3$ such that $\Gamma_3 \cap U$ is a union of components of $$(\lambda_i^{-1}(q) \cup C_i') \cap U,$$

which is a divisor on the surface $U \cap S'_i$. Now we can choose a nonsingular curve $\tilde{\gamma}$ on $Y_3$ which intersects $\Gamma_3$ at $q'$ only, and intersects the surfaces $S'_1, \ldots, S'_n$ transversally. Let $\Psi_4 : Y_4 \to Y_3$ be the blow up of $\tilde{\gamma}$, $X'_4 \to X_3$ be the blow up of the strict transform of $\tilde{\gamma}$ on $X_3$, and let $X_4 \to X'_4$ be a principalization of $T_4 \mathcal{O}_{X'_4}$ (obtained by blowing up points and nonsingular curves) which is an isomorphism away from points above $q'$. Thus we have a commutative diagram

$$
\begin{array}{ccc}
X_4 & \xrightarrow{f_4} & Y_4 \\
\Phi_4 \downarrow & & \downarrow \Psi_4 \\
X_3 & \xrightarrow{f_3} & Y_3
\end{array}
$$

such that the fundamental curve of $f_4$ is contained in the union of the strict transform of $\Gamma_3$ on $Y_4$ and the curve $l_4 = \Psi_4^{-1}(q')$.

Let $S''_i$ be the strict transform of $S'_i$ on $Y_4$ for $1 \leq i \leq n$. For $i$ such that $q' \in S''_i$, $S''_i \to S'_i$ is the blow up of $q'$ on $S'_i$, with exceptional divisor $l_4$. Thus by embedded resolution of plane curve singularities (cf. Section 3.4, Exercise 3.13 [C3]), after a finite number of blow ups of such curves $\tilde{\gamma}$ we obtain a diagram

$$
\begin{array}{ccc}
X_5 & \xrightarrow{f_5} & Y_5 \\
\downarrow & & \downarrow \\
X_4 & \xrightarrow{f_4} & Y_4
\end{array}
$$

such that the fundamental locus of $f_5$ satisfies the conclusions of this lemma. \hfill \Box

**Remark 4.4.** In the conclusions of Lemma 4.3, we can assume that the fundamental locus of $f_1$ has no isolated points. To see this, we make the following construction. Let $A$ be the isolated points in the fundamental locus of $f_1$. Let $\gamma$ be a general curve on $Y_1$ through $A$ (an intersection of two general hypersurface sections through $A$). Then $\gamma \cap \Gamma = A$. Let $\Phi_2 : X_2 \to X_1$ be the blow up of the strict transform of $\gamma$ on $X_1$. The fundamental locus of the resulting map $X_2 \to Y_1$ is $\gamma \cup \Gamma_1$, which satisfies the conclusions of Lemma 4.3, and has no isolated points.

**Theorem 4.5.** Suppose that $f : X \to Y$ is a birational morphism of nonsingular projective 3-folds and the fundamental locus $\Gamma$ of $f$ is a union of nonsingular curves such that two curves of $\Gamma$ intersect in most one point, and this intersection is transversal. Further assume that the intersection of any three curves of $\Gamma$ is empty. Then there exists a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

such that the vertical arrows are products of blow ups of points and nonsingular curves, $X_1, Y_1$ have toroidal structures $D_{Y_1}, D_{X_1} = f_1^{-1}(D_X)$, $f_1$ is prepared (Definition
3.4), every 2-curve of $X_1$ contains a 3-point and every component of $D_{X_1}$ contains a 3-point.

Proof. Let $\{C_1, \ldots, C_n\}$ be the irreducible components of $\Gamma$.

Let $H$ be a hyperplane section of $Y$. Whenever $j \neq i$ and $C_i \cap C_j$ is nonempty, let $C_i \cap C_j = \{q_{ij}\}$. For $m \gg 0$, and $1 \leq i \leq n$, let $H_i$ be divisors of general sections $\sigma_i$ of $\Gamma(Y, \mathcal{I}_{C_i} \otimes \mathcal{O}_Y(mH))$. By the arguments using Bertini’s theorem of Lemma 4.3, we conclude the following:

1. Each $H_i$ is a nonsingular irreducible surface and $\bar{D}_Y = H_1 + \cdots + H_n$ is a SNC divisor on $Y$.
2. For $i \neq j$, if $C_i \cap C_j \neq \emptyset$, then $H_i$ intersects $C_j$ transversally at $q_{ij}$ plus a sum of general points of $C_j$. If $C_i \cap C_j = \emptyset$, then $H_i$ intersects $C_j$ transversally at a sum of general points of $C_j$.
3. $H_i \cap H_j \cap H_k$ is disjoint from $\Gamma$ for $i, j, k$ distinct.

Let $\bar{D}_X = f^{-1}(\bar{D}_Y)$. Away from $f^{-1}(\Gamma)$, $\bar{D}_X$ is a SNC divisor and $f$ is prepared (Definition 3.4). Suppose that $\eta \in C_j$ is a general point. Then $f^{-1}(H_j)$ is a SNC divisor over $\eta$ and $f$ has fiber dimension 1 over $\eta$. Further, $f$ is a product of blow ups of sections over $C_j$ above $\eta$ which make SNCs with the preimage of $H_j$ (by [Ab] or [D], since $f$ is birational and $\eta \in C_j$ is a general point). If $u = 0$ is a local equation of $H_j$ at $\eta$ and $v = 0$ is a local equation of a nonsingular surface transversal to $C_j$ at $\eta$ then $u, v$ are toroidal forms (Definition 3.1) at all points of $f^{-1}(\eta)$. In fact we have a form

\[ u = x^a, v = y \quad (40) \]

or

\[ u = x^a y^b, v = z \quad (41) \]

at all points $p \in f^{-1}(\eta)$.

If $\eta \in H_i$ for some $i \neq j$ (and $\eta \neq q_{ij}$), then we can take $v = 0$ to be a local equation of $H_i$ at $\eta$.

Thus $f^{-1}(\bar{D}_Y)$ is a SNC divisor except possibly over $A = \{q_{ij}\}$ and over a finite number of 1-points $B = \{q_k\}$ of $\bar{D}_Y$ (contained in $\Gamma$). After possibly extending $B$ by adding a finite number of points which are 1-points of $\bar{D}_Y$, we have that $f$ is prepared away from the points of $A \cup B$.

Index $B$ as $B = \{q_{n+1}, \ldots, q_r\}$. For $q_i \in B$, let $H_i$ be the divisor of a general section $\sigma_i \in \Gamma(Y, \mathcal{I}_{q_i} \otimes \mathcal{O}_Y(mH))$. For $i \geq n + 1$, $H_i$ intersects $\Gamma$ at $q_i$ plus a sum of general points of $C_1, \ldots, C_n$ (we can make our initial choice of $m$ so that this property holds),

\[ D_Y = H_1 + \cdots + H_r = \bar{D}_Y + H_{n+1} + \cdots + H_r \]

is a SNC divisor on $Y$, and $D_X = f^{-1}(D_Y)$ is a SNC divisor on $X$, except possibly over points of $A \cup B$. Thus after blowing up points and nonsingular curves supported above $f^{-1}(A \cup B)$, we may assume that $D_X = f^{-1}(D_Y)$ is a SNC divisor, and every irreducible component of $D_X$ contains a 3-point, every 2-curve of $D_X$ contains a 3-point.

Observe that the points where $f$ is not prepared are intersection points $H_i \cdot H_j \cdot \Gamma$ for $i \neq j$. We may assume that $r \geq 3$. For $i \neq j$ let $\Gamma_{ij} = H_i \cdot H_j$. $\Gamma_{ij}$ are nonsingular irreducible curves (by Bertini’s theorem).

We now apply for $i = 1$ and $j = 2$ a general construction that we will iterate for all $i < j$. We will assume that $n \geq 2$ and $C_1 \cap C_2 \neq \emptyset$. The case when $C_1 \cap C_2 = \emptyset$ (or $n = 1$) is simpler. Let $\gamma = \Gamma_{12}, q = q_{12}, D_{12} = \Gamma_{12} \cap (C_1 \cup C_2) = \Gamma_{12} \cap \Gamma$. 


$H_1, H_2, H_3$ are the divisors of sections $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(Y, \mathcal{O}_Y(mH))$ which define a rational map $\pi : Y \to \mathbb{P}^2$ by $Q \mapsto (\sigma_1 : \sigma_2 : \sigma_3)(Q)$ for closed points $Q \in Y$. Let $U = Y - (\cup_{j \geq 2} H_j)$, an affine neighborhood of $q$ in $Y$ on which $\pi$ is a morphism. Let

$$f_1 = \frac{\sigma_1}{\sigma_3}, f_2 = \frac{\sigma_2}{\sigma_3} \in \Gamma(U, \mathcal{O}_Y).$$

$$\pi : U \to \mathbb{A}^2$$

is defined by the inclusion of $k$-algebras $k[u, v] \to \Gamma(U, \mathcal{O}_Y)$ given by

$$u = f_1, v = f_2.$$

Thus there exists an affine neighborhood $\bar{U}$ of $q$ such that $u, v, w$ and $\bar{f}$ form a regular system of parameters at all points of $\bar{\gamma}$. Thus $\pi : U \to \mathbb{A}^2$ is smooth in a neighborhood of $\bar{\gamma}$. Since $C_1$ and $C_2$ intersect $\bar{\gamma}$ transversally, there exists an open neighborhood $\bar{U}$ of $\bar{\gamma}$ in $U$ such that

$$\pi : \bar{U} \to \mathbb{A}^2$$

is smooth and $\pi | C_1 \cap \bar{U}$, $\pi | C_2 \cap \bar{U}$ are unramified. $\mathbb{A}^2$ has toroidal structure $uv = 0$, and $\bar{U}$ has toroidal structure $D \cap \bar{U}$ which is defined by $f_1, f_2 = 0$. Let $X = f^{-1}(\bar{U})$, $\bar{f} = f | X$, $g = \pi \circ \bar{f} : X \to \mathbb{A}^2$. $u, v - v(g(p))$ must have a form (40) or (41) at points $p$ of $X$ above $C_1 - q$, and $v, u - u(g(p))$ have a form (40) or (41) at points $p$ of $X$ above $C_2 - q$, so $g$ is toroidal away from $\beta = f^{-1}(q)$ and is prepared away from $f^{-1}(\gamma)$.

Let $R = \mathcal{O}_{Y,q}$ with maximal ideal $m$, $I_{C_1} = \mathcal{I}_{C_1,q}$, $I_{C_2} = \mathcal{I}_{C_2,q}$. $I_{C_1}$ has generators $u, w_1$, $I_{C_2}$ has generators $v, w_2$ where $u, v, w_1$ and $u, v, w_2$ are bases of $m/m^2$ since $H_2$ intersects $C_1$ transversally at $q$ and $H_1$ intersects $C_2$ transversally at $q$. By the formal implicit function theorem in $R$, $w_2 = \phi(w_1 - \psi(u, v))$ where $\phi$ is a unit series, and $\psi$ is a series.

$$\hat{I}_{C_2} = (v, w_1 - \psi(u, v)) = (v, w_1 - \psi(u, 0)).$$

Set $\hat{w} = w_1 - \psi(u, 0)$. $\hat{I}_{C_1} = (u, \hat{w})$ and $\hat{I}_{C_2} = (v, \hat{w})$. Thus

$$\hat{I}_{C_1} \cap \hat{I}_{C_2} = \hat{I}_{C_1} \cap \hat{I}_{C_2} = (uv, \hat{w}).$$

There exists $w \in I_{C_1} \cap I_{C_2}$ such that $w \equiv \hat{w} \mod m^2 R$. We have $I_{C_1} \cap I_{C_2} = (uv, w)$. Thus there exists an affine neighborhood $\bar{U}_1$ of $q$ in $\bar{U}$ and $w \in \Gamma(\bar{U}_1, \mathcal{O}_Y)$ such that $u, v, w$ are uniformizing parameters in $\bar{U}_1$, $u = w = 0$ are equations of $C_1$, and $v = w = 0$ are local equations of $C_2$ in $\bar{U}_1$.

After possibly blowing up points supported above $q$, we may suppose that every irreducible component of $D_X$ contains a 3-point and every 2-curve of $D_X$ contains a 3-point.

By Lemma 4.2 (applied to $f^{-1}(\bar{U}_1) \to \bar{U}_1$, and extending trivially to $X \to \bar{U}$), there exists a commutative diagram

$$\begin{array}{ccc}
\mathbb{A}^2 & \xrightarrow{g_0} & \mathbb{A}^2 \\
\uparrow g & & \\
X & \xrightarrow{\phi_0} & \bar{X}
\end{array}$$

such that $\phi_0$ is an isomorphism away from $f^{-1}(q)$, $g_0$ is a toroidal in a neighborhood of the strict transform of $D_Y$ on $\bar{X}_0$, and is toroidal in a neighborhood of all components of $D_X$ which dominate $C_1$ or $C_2$, and is prepared in a neighborhood of all components of $D_X$ which dominate a component of $D_Y$ or dominate $C_1$ or $C_2$, away from the
strict transform of $\tau$ on $\mathcal{X}_0$. Further, every irreducible component of $D_{\mathcal{X}_0} = \Phi_0^{-1}(D_{\mathcal{X}})$ contains a 3-point and every 2-curve of $D_{\mathcal{X}_0}$ contains a 3-point.

We now apply the algorithm of the proof of Lemma 4.2 to the other points of $D_{12}$ (the conclusions of the lemma hold if $C_1 = \emptyset$ or $C_2 = \emptyset$). We construct $\Phi'_0 : \mathcal{X}'_0 \to \mathcal{X}_0$ such that

1. $\Phi_0 \circ \Phi'_0$ is an isomorphism away from $(\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$.
2. $g'_0 = g_0 \circ \Phi'_0$ is toroidal in a neighborhood of the strict transform of $D_Y$ on $\mathcal{X}'_0$, and in a neighborhood of all components of $D_{\mathcal{X}_0}$ which dominate $C_1$ or $C_2$ and is toroidal away from $(\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$.
3. $g'_0$ is prepared in a neighborhood of all components of $D_{\mathcal{X}_0}$ which dominate a component of $D_\Sigma$ or dominate $C_1$ or $C_2$, away from the strict transform $\gamma_1$ of $\tau$ on $\mathcal{X}'_0$.
4. $D_{\mathcal{X}_0} = (\Phi'_0)^{-1}(D_{\mathcal{X}'_0}) = (\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_Y)$ and $(f \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$ are SNC divisors.
5. Every irreducible component of $D_{\mathcal{X}_0}$ contains a 3-point and every 2-curve of $D_{\mathcal{X}_0}$ contains a 3-point.

Let $\Phi_1 : \mathcal{X}_1 \to \mathcal{X}'_0$ be the blow up of the strict transform $\gamma_1$ of $\tau$ on $\mathcal{X}'_0$. Let $E_1 = \Phi_1^{-1}(\gamma_1)$. Since $\gamma_1$ is a 2-curve of $D_{\mathcal{X}_0}$, we have that every irreducible component of $D_{\mathcal{X}_1} = \Phi_0^{-1}(D_{\mathcal{X}_0})$ contains a 3-point and every 2-curve of $D_{\mathcal{X}_0}$ contains a 3-point.

If $p \in \gamma_1 - (\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$ then $u, v$ are part of a regular system of parameters at $p$, and $u = v = 0$ are local equations of $\gamma_1$ on $\mathcal{X}'_0$ at $p$. If $p \in \gamma_1 \cap (\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$, then $u, v$ have a toroidal form of the type of equation (39) at $p$,

$$u = x^a, v = y^b$$

(44)

where $a, b > 0$, $x = y = 0$ are (formal) local equations of $\gamma_1$, $xyz = 0$ is a local equation of $D_{\mathcal{X}_0}$.

Suppose that $p' \in E_1 \cap (\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$. Then $p = \Phi_1(p') \in \gamma_1 \cap (\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$, and $u, v$ have a toroidal form (44) at $p$. Thus $p'$ has (formal) regular parameters $x_1, y_1, z_1$ such that

$$x = x_1, y = x_1(y_1 + \alpha), z = z_1$$

for some $\alpha \in k$, or

$$x = x_1y_1, y = y_1, z = z_1.$$ 

Substituting into (44), we see that $g_1 = g'_0 \circ \Phi_1 : \mathcal{X}_1 \to \mathcal{A}^2$ is toroidal and prepared in a neighborhood of the strict transform of $D_Y$ and in a neighborhood of $E_1$. Furthermore, $g_1$ is prepared and toroidal in a neighborhood of all components $D_{\mathcal{X}_1}$ which dominate $C_1$ or $C_2$ and $g_1$ is prepared and toroidal away from $(\mathcal{f} \circ \Phi_0 \circ \Phi'_0)^{-1}(D_{12})$.

By 1 of Theorem 4.1 there exists a morphism $\Phi_2 : \mathcal{X}_2 \to \mathcal{X}_1$ such that $\Phi_2$ is an isomorphism away from $(\mathcal{f} \circ \Phi_0 \circ \Phi'_0 \circ \Phi_1 \circ \Phi_2)^{-1}(D_{12})$, $g_2 = g_1 \circ \Phi_2 : \mathcal{X}_2 \to \mathcal{A}^2$ is prepared, all 2-curves of $\mathcal{X}_2$ contain a 3-point, all components of $D_{\mathcal{X}_2}$ contain a 3-point.

Now by 2 of Theorem 4.1, there exists a commutative diagram

$$\begin{array}{ccc}
\mathcal{X}_3 & \xrightarrow{g_3} & S_1 \\
\Phi_3 \downarrow & & \downarrow \Phi_1 \\
\mathcal{X}_2 & \xrightarrow{g_2} & \mathcal{A}^2
\end{array}$$
such that $\Phi_3$ is an isomorphism away from $(g_3 \circ \Phi_3)^{-1}(\overline{\gamma})$, $\Psi_1$ is an isomorphism away from $\overline{\gamma}$ and $g_3$ is toroidal. We further have that all 2-curves of $X_3$ contain a 3-point and all components of $D_{X_3}$ contain a 3-point.

In the sequences (27) of the proof of Theorem 4.1, we only insert point blow ups $\lambda_{j+1} : Z_{j+1} \to Z_j$ at a point above the curve $C_j \subset Z_j$ if $C_j$ contains no 2-points. In this case, we are free to blow up the 2-point $p_1$ above any point $p \in C_j$ that we wish. We can thus assume that $p_1$ lies above $\beta_3 = (f \circ \Phi_0 \circ \Phi_2)^{-1}(D_{12})$.

$\Psi_1 : S_1 \to \mathbb{A}^2$ is the blow up of an ideal sheaf $I \subset \mathcal{O}_{\mathbb{A}^2}$. We have (by the algorithm of 2 of Theorem 4.1) that $\Phi_3$ is the blow up of $\mathcal{I}_C$ away from $\beta_3$.

We can assume that $\Psi_1$ is nontrivial so that $m_\sigma \mathcal{I}_C$ is an invertible ideal sheaf (where $m_\sigma = (u, v) \subset k[u, v]$).

Let $\overline{Y}_3 = \overline{U} \times_{\mathbb{A}^2} S_1$. $\overline{Y}_3$ is obtained from $\overline{U}$ by blowing up sections over $\sigma = \pi^{-1}(\overline{\gamma})$, which make SNCs with the preimage of $D_{\overline{\gamma}}$. Let $\Psi_3 : \overline{Y}_3 \to U$ be the first projection. $\overline{Y}_3$ is nonsingular and $D_{\overline{Y}_3} = \overline{Y}_3^{-1}(D_U)$ is a SNC divisor. Let $\overline{\Phi} = \overline{\Phi}_0 \circ \cdots \circ \overline{\Phi}_3 : \overline{X}_3 \to \overline{X}$.

There is a natural birational morphism $\overline{f}_3 : \overline{X}_3 \to \overline{Y}_3$ where $\overline{f}_3 = (f \circ \overline{\Phi}) \times g_3$. By our construction, $\overline{f}_3$ is an isomorphism away from $(\overline{f} \circ \overline{\Phi})^{-1}(C_1 \cup C_2)$. Recall that $\overline{\Phi}_1 : \overline{X}_1 \to \overline{X}_0$ is the blow up of $m_\sigma \mathcal{I}_C$ away from $(\overline{f} \circ \overline{\Phi}_0 \circ \overline{\Phi}_3)^{-1}(C_1 \cup C_2)$.

Now consider the commutative diagrams

\[\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\uparrow & & \uparrow \\
\overline{X} & \xrightarrow{\overline{f}} & \overline{Y}
\end{array}\]

where the vertical arrows are the natural inclusions, and

\[\begin{array}{ccc}
X_3 & \xrightarrow{\Phi} & X \\
\overline{f}_3 & & \overline{f} \\
\overline{Y}_3 & \xrightarrow{\overline{\Phi}_3} & U.
\end{array}\]  \hspace{1cm} (45)

$\overline{Y}_3$ is constructed by blowing up sections over $\sigma$ (which make SNCs with the preimage of $D_{\overline{f}_3}$). $\overline{X}_3 - \overline{f}_3^{-1}(D_{12})$, $\overline{f}$ is constructed by blowing up the isomorphic sections over $\sigma$ and $\overline{f}_3 \mid \overline{X}_3 - \overline{\Phi}_3^{-1}(\overline{f}^{-1}(C_1 \cup C_2))$ is an isomorphism.

Recall that $f$ is an isomorphism over $H_j \cap \gamma$ for $j > 2$, and $(\cup_{j>2} H_j) \cap \gamma = \gamma - \overline{\gamma}$.

There exists a factorization of the above diagram (45) over $U - D_{12}$ by a commutative diagram

\[\begin{array}{ccc}
\hat{X}_m & \xrightarrow{\Omega_m} & \cdots & \xrightarrow{\Omega_2} & \hat{X}_1 & \xrightarrow{\Omega_1} & \hat{X} \\
\downarrow & & & & \downarrow & & \downarrow \\
\hat{Y}_m & \xrightarrow{\Lambda_m} & \cdots & \xrightarrow{\Lambda_2} & \hat{Y}_1 & \xrightarrow{\Lambda_1} & \hat{Y}
\end{array}\]  \hspace{1cm} (46)

where $\hat{Y} = U - D_{12}$, $\hat{X} = X - f^{-1}(D_{12})$, $\hat{Y}_m = \overline{Y}_3 - \overline{\Phi}_3^{-1}(D_{12})$, $\hat{X}_m = \overline{X}_3 - \overline{\Phi}_3^{-1}(f^{-1}(D_{12}))$. The vertical arrows are isomorphisms away from the preimage of $C_1 \cup C_2$, each map $\hat{Y}_{i+1} \to \hat{Y}_i$ is the blow up of a section $\gamma_i$ over $\gamma = \gamma \cap \hat{Y}$ which makes SNCs with the toroidal structure ($\gamma_i$ is a possible blow up).
We can thus extend (46) to a commutative diagram of projective morphisms

\[
\begin{array}{c}
\tilde{X}_3 \xrightarrow{f_3} X - f^{-1}(D_{12}) \\
\downarrow \quad \downarrow \\
\tilde{Y}_3 \xrightarrow{\Psi_3} Y - D_{12}
\end{array}
\]  

\hspace{1cm} (47)

such that the vertical arrows are isomorphisms away from \( \Gamma \), and the horizontal arrows are isomorphisms over \( Y - \gamma \), by performing the blow up \( \tilde{Y}_1 \to Y - D_{12} \) of \( \gamma \cap (Y - D_{12}) \), then blowing up points over \( H_j \cap \gamma \) for \( j > 2 \) to ensure that the closure of \( \tilde{\gamma}_1 \) in \( Y_1 \) makes SNCs with the toroidal structure, and continue to construct \( \tilde{Y}_3 \to Y - D_{12} \) which extends \( \tilde{Y}_m \to \tilde{Y} \). Now we can perform the corresponding point blow ups over \( X \) \(( f \) is an isomorphism over \( H_j \cap \gamma \) for \( j > 2 \)) and blow ups of the closures of \( \tilde{\gamma}_i \) to get \( \tilde{X}_3 \to X - f^{-1}(D_{12}) \) which extends \( \tilde{X}_m \to \tilde{X} \). Patching (45) and (47) we have a commutative diagram

\[
\begin{array}{c}
X_3 \xrightarrow{f_3} Y_3 \\
\Phi_3 \downarrow \quad \downarrow \Psi_3 \\
X \xrightarrow{\pi} Y
\end{array}
\]  

\hspace{1cm} (48)

of projective morphisms where \( \Phi_3 \) and \( \Psi_3 \) are products of blow ups of possible centers (points and nonsingular curves which make SNCs with the preimages of \( D_X \) and \( D_Y \)) such that \( f_3 \) is prepared in a neighborhood of \( \Psi^{-1}_3(\gamma) \), \( \Psi_3 \) is an isomorphism away from \( \gamma \), \( \Phi_3 \) is an isomorphism away from \( f^{-1}(\gamma) \), and \( f_3 \) is an isomorphism away from \( \Psi^{-1}_3(\Gamma) \). We further have that every 2-curve of \( X_3 \) contains a 3-point and every component of \( D_X \) contains a 3-point.

We now iterate this construction for the other pairs \( H_i, H_j \) with \( 1 \leq i < j \leq r \), first continuing the procedure for \( H_1 \) and \( H_3 \). We will indicate this next step.

Let \( H'_1 = \Psi^{-1}_3(H_1) \) for \( 1 \leq i \leq r \). \( H'_1, H'_3, H'_2 \) are divisors of sections \( \sigma'_1, \sigma'_3, \sigma'_2 \in H^0(Y_3, \mathcal{O}_{Y_3}(m\Psi^{-1}_3(H))) = H^0(Y, \mathcal{O}_Y(mH)). \) Let \( U' = Y_3 - (\cup_{j \notin \{1, 3\}} H'_j) \). Since \( \Psi_3 \) is an isomorphism away from \( \gamma = \Gamma_{12} \subset H_2, U' \cong Y - (\cup_{j \notin \{1, 3\}} H_j) \). Thus \( U' \) is affine and we have a morphism \( \pi' : U' \to \mathbb{A}^2 \) defined by the inclusion of \( \mathbb{k}[u, v] \to \Gamma(U', \mathcal{O}_{U'}) \) given by

\[ u = f_1 = \frac{\sigma'_1}{\sigma'_2}, v = f_2 = \frac{\sigma'_3}{\sigma'_2}. \]

Let \( \gamma' \) be the strict transform of \( \Gamma_{13} \) in \( Y_3 \) and let \( \vec{\gamma}' = \gamma' \cap U' \). Since \( \Gamma_{13} \cap \Gamma \cap H_i = \emptyset \) for \( i \notin \{1, 3\} \), and the fundamental locus of \( f_3 \) is contained in \( \Psi^{-1}_3(\Gamma) \), \( f_3 \) is an isomorphism over \( \gamma' - \vec{\gamma}' \).

As in the construction of (45), there exists an open neighborhood \( \bar{U}' \) of \( \gamma' \) in \( U' \) such that \( \pi' : \bar{U}' \to \mathbb{A}^2 \) is smooth, and enjoys the properties of the morphism \( \pi : \bar{U} \to \mathbb{A}^2 \) constructed in (43). Thus there exists a commutative diagram

\[
\begin{array}{c}
X'_3 \xrightarrow{\varphi'} \bar{X}' \\
\downarrow \quad \downarrow \varphi' \\
Y'_3 \xrightarrow{\psi'} \bar{U'}
\end{array}
\]
having the properties of (45) (where $X' = f_3^{-1}(U')$, $f' = f_3 | X$). We can thus (as in the construction of (48)) construct a commutative diagram

\[
\begin{array}{ccc}
X_4 & \xrightarrow{f_4} & Y_4 \\
\Phi_4 \downarrow & & \downarrow \Psi_4 \\
X_3 & \xrightarrow{f_3} & Y_3 \\
\Phi_3 \downarrow & & \downarrow \Psi_3 \\
X & \xrightarrow{f} & Y
\end{array}
\]

where $\Phi_4$ and $\Psi_4$ are products of blow ups of points and nonsingular curves which are possible centers (make SNCs with the preimages of $D_X$ and $D_Y$), such that $f_4$ is prepared in a neighborhood of $(\Psi_3 \circ \Psi_4)^{-1}(\Gamma_{12} \cup \Gamma_{13})$, $\Psi_3 \circ \Psi_4$ is an isomorphism away from $\Gamma_{12} \cup \Gamma_{13}$, $\Phi_3 \circ \Phi_4$ is an isomorphism away from $f^{-1}(\Gamma_{12} \cup \Gamma_{13})$ and $f_4$ is an isomorphism away from $(\Psi_3 \circ \Psi_4)^{-1}(\Gamma)$. We further have that every 2-curve of $X_4$ contains a 3-point and every component of $D_X$ contains a 3-point.

We can continue in this way to construct, by induction, a commutative diagram

\[
\begin{array}{ccc}
X_5 & \xrightarrow{f_5} & Y_5 \\
\Phi_5 \downarrow & & \downarrow \Psi_5 \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that $f_5$ is prepared in a neighborhood of $\Psi_5^{-1}(\bigcup_{i \neq j} \Gamma_{ij})$, $\Psi_5$ is an isomorphism away from $\bigcup_{i \neq j} \Gamma_{ij}$, $\Phi_5$ is an isomorphism away from $f^{-1}(\bigcup_{i \neq j} \Gamma_{ij})$ and $f_5$ is an isomorphism away from $\Psi_5^{-1}(\Gamma)$. Thus $f_5$ is prepared.

\[\square\]

5. First Properties of Prepared Morphisms

In this section, we suppose that $f : X \to Y$ is a birational morphism of nonsingular projective 3-folds, with toroidal structures $D_Y$ and $D_X = f^{-1}(D_Y)$ such that $D_Y$ contains the fundamental locus of $f$.

**Lemma 5.1.** Suppose that $f : X \to Y$ is prepared, $C \subset Y$ is a curve such that $C$ is contained in the fundamental locus of $f$ and $C$ contains a 1-point. Let $\Psi_1 : Y_1 \to Y$ be the blow up of $C$, and let $q \in C$ be a 1-point. Then there exists an affine neighborhood $Y$ of $q$ such that $C \cap Y$ is nonsingular, and if $Y_1 = \Psi_1^{-1}(Y)$ and $\overline{X} = f^{-1}(Y)$, then there exists a factorization $f_1 : \overline{X} \to Y_1$ such that $\Psi_1 \circ f_1 = f | \overline{X}$. Further, $f_1$ is prepared.

**Proof.** Since $f$ is birational, $f^{-1}(q)$ has dimension 1 and $C$ is the only component of the fundamental locus of $f$ through $q$ (by Lemma 3.5), there exists a neighborhood $Y$ of $q$ and a factorization

\[f_1 = \Psi_1^{-1} \circ f : \overline{X} = f^{-1}(Y) \to Y_1 = \Psi_1^{-1}(Y)\]

of the blow up $\Psi_1$ of $C$ by a morphism $f_1$ (by [D]). We will show that $f_1$ is prepared. As shown in the proof of Lemma 3.5, there exist permissible parameters $(u, v, w)$ at $q$ such that for all $p \in f^{-1}(q)$, $u, v$ are toroidal forms at $p$ and $u = w = 0$ are local equations of $C$ at $q$. Thus if $p \in f^{-1}(q)$ we have permissible parameters $x, y, z$ for $u, v, w$ at $p$ such that if $p$ is a 1-point,

\[
\begin{align*}
u &= x^a \\
v &= y \\
w &= \lambda(x, y) + x^c z,
\end{align*}
\]
and if $p$ is a 2-point,
\[ u = (x^a y^b)^k \\
   v = z \\
   w = \lambda(x^a y^b, z) + x^c y^d \] (50)
with $ad - bc \neq 0$. Further, $(u, w) \hat{\mathcal{O}}_{X, p}$ is invertible. In case (49), if $u \mid w$, we have that $f_1(p)$ is a 1-point, and there exists $\alpha \in k$ such that if
\[ u_1 = u, v_1 = v, w_1 = \frac{w}{u} - \alpha, \]
then $u_1, v_1, w_1$ are regular parameters at $f_1(p)$ and there exists a series $\overline{\lambda}$ such that
\[ u_1 = x^a \\
v_1 = y \\
w_1 = \lambda(x, y) + x^{c-a} z \] (51)
of the form (12) of Definition 3.1 at $p$. If $w \mid u$ (and $u \not\mid w$) in Case (49), we have that $w = x^n \gamma$, where $\gamma$ is a unit series and $n < a$. Thus after computing a Taylor series expansion of an appropriate root of $\gamma$, we see that (by a similar argument to Case 1.1 of Lemma 57 of [C2]) that there exist regular parameters $\overline{x}, \overline{y}, \overline{z}$ at $p$ and a unit series $\overline{\lambda}$ such that
\[ u = \overline{x}^n(\overline{\lambda}(\overline{x}, \overline{y}) + \overline{x}^{-n} \overline{z}) \\
v = y \\
w = \overline{x}^n. \]
Thus $f_1(p) = q_1$ is a 2-point which has regular parameters
\[ w_1 = w, u_1 = \frac{u}{w}, v_1 = v \]
so that $u_1 w_1 = 0$ is a local equation of $D_{\gamma_1}$, and
\[ w_1 = \overline{x}^n \\
u_1 = \overline{x}^{-n}(\overline{\lambda}(\overline{x}, \overline{y}) + \overline{x}^{-n} \overline{z}) \\
v_1 = y \] (52)
which has the form 2 (b) of Definition 3.4.

In case (50), if $u \mid w$, we have that $f_1(p)$ is a 1-point, and there exists $\alpha \in k$ such that if
\[ u_1 = u, v_1 = v, w_1 = \frac{w}{u} - \alpha, \]
then $u_1, v_1, w_1$ are regular parameters at $f_1(p)$ and there exists a series $\overline{\lambda}$ such that
\[ u_1 = (x^a y^b)^k \\
v_1 = z \\
w_1 = \lambda(x^a y^b, z) + x^{c-a} y^{d-bk} \] (53)
of the form (13) of Definition 3.1 at $p$. If $w \mid u$ and $u \not\mid w$ in case (50), we either have an expression
\[ w = (x^a y^b)^l(\gamma(x^a y^b, z) + x^c y^d) \] (54)
at $p$ with $l < k$ and $\gamma$ is a unit series, or we have an expression (after possibly making a change of variables in $x$ and $y$, multiplying $x$ and $y$ by appropriate unit series)
\[ w = x^c y^d \] (55)
with $ad - bc \neq 0$. Suppose that (54) holds. Then there exist regular parameters $\pi, \eta, z$ at $p$ such that

$$
\begin{align*}
  u &= (x^a y^b)^k (\tau(x^a y^b, z) + x^c y^d) \\
  v &= z \\
  w &= (x^a y^b)^l
\end{align*}
$$

for some unit series $\tau$, and $ad - bc \neq 0$. Thus $f_1(p) = q_1$ is a 2-point which has regular parameters

$$
  w_1 = w, u_1 = \frac{u}{w}, v_1 = v
$$

such that $u_1 w_1 = 0$ is a local equation of $D_{Y_1}$ and

$$
\begin{align*}
  u &= (x^a y^b)^l \\
  u_1 &= (x^a y^b)^k - l (\tau(x^a y^b, z) + x^c y^d) \\
  v_1 &= z
\end{align*}
$$

which has the form 2 (c) of Definition 3.4.

If (55) holds, $f_1(p) = q_1$ is a 2-point which has regular parameters

$$
  w_1 = w, u_1 = \frac{u}{w}, v_1 = v
$$

such that $u_1 w_1 = 0$ is a local equation of $D_{Y_1}$ and

$$
\begin{align*}
  u &= x^c y^d \\
  u_1 &= x^{a - c} y^{b - d} \\
  v_1 &= z
\end{align*}
$$

which has the form (9) of Definition 3.1.

Comparing equations (51), (52), (53), (56) and (57), we see that after possibly replacing $Y$ with a smaller affine neighborhood of $q$, $f_1 : X \to Y_1$ is prepared.

Lemma 5.2. Suppose that $f : X \to Y$ is prepared and $C \subset Y$ is a 2-curve. Then there exists a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y \\
\end{array}
$$

where $\Psi_1 : Y_1 \to Y$ is the blow up of $C$, $\Phi_1 : X_1 \to X$ is a product of blow ups of 2-curves, $\Phi_1$ is an isomorphism above $f^{-1}(Y - C)$ and $f_1$ is prepared. If $p_1 \in X_1$ is a 3-point, so that $p = \Phi_1(p_1)$ is necessarily also a 3-point, we have that

$$
\tau_{f_1}(p_1) = \tau_f(p).
$$

Further, if $D_X$ is cuspidal for $f$, then $D_{X_1}$ is cuspidal for $f_1$.

Proof. Let $\Psi_1 : Y_1 \to Y$ be the blow up of $C$. By Lemma 3.12, there exists a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y \\
\end{array}
$$

such that $\Phi_1 : X_1 \to X$ is a sequence of blow ups of 2-curves, $\Phi_1$ is an isomorphism above $f^{-1}(Y - C)$ and $f_1$ is a morphism.

We will show that $f_1 : X_1 \to Y_1$ is prepared.

Suppose that $q \in C \subset Y$ and $p \in (f \circ \Phi_1)^{-1}(q)$. Let $q_1 = f_1(p), p' = \Phi_1(p)$.

Suppose that $q$ is a 2-point
Let \( u, v, w \) be permissible parameters at \( q \). Suppose that \( p' \) satisfies 2 (a) of Definition 3.4. If \( q_1 \) is a 2-point, which has permissible parameters \( u_1, v_1, w_1 \) defined by

\[
u = u_1, v = u_1 v_1, w = w_1\]

or

\[
u = u_1 v_1, v = v_1, w = w_1,\]

then \( u_1, v_1 \) are toroidal forms at \( p \). If \( q_1 \) is a 1-point, which has permissible parameters \( u_1, v_1, w_1 \) defined by

\[
u = u_1, v = u_1 (v_1 + \alpha), w = w_1,\]

with \( 0 \neq \alpha \), then \( u_1, v_1 \) are toroidal forms at \( p \).

Further, if \( p \) is a 3-point, then \( p' \) is a 3-point and \( q_1 \) is a 2-point. Thus \( \tau_f(p') \geq 1 \). After possibly interchanging \( u \) and \( v \), we have that \( u, v, w \) have a form (23) at \( p' \) in terms of permissible parameters \( x, y, z \) for \( u, v, w \) at \( p' \), there are \( u_1, v_1 \in O_{v_1,q_1} \) such that \( u_1, v_1, w \) are permissible parameters at \( q_1 \), with \( u = u_1, v = u_1 v_1 \) and there are permissible parameters \( x_1, y_1, z_1 \) in \( O_{X_1,p} \) for \( u_1, v_1, w \) such that

\[
\begin{align*}
x &= x_1^{a_{11}} y_1^{a_{21}} z_1^{a_{31}} \\
y &= x_1^{a_{21}} y_1^{a_{22}} z_1^{a_{23}} \\
z &= x_1^{a_{31}} y_1^{a_{22}} z_1^{a_{23}}
\end{align*}
\]

where \( a_{ij} \) are natural numbers with \( \text{Det}(a_{ij}) = \pm 1 \). Thus \( Z u + Z v = Z u_1 + Z v_1, H_{f_1,p} = H_{f \circ \Phi_1,p} = H_{f,p'}, A_{f_1,p} = A_{f \circ \Phi_1,p} = A_{f,p'}, L_{f_1,p} = L_{f \circ \Phi_1,p} = L_{f,p'} \) and \( \tau_{f_1}(p) = \tau_{f \circ \Phi_1}(p) = \tau_f(p') \).

Now suppose that \( u, v, w \) satisfy 2 (b) of Definition 3.4 at \( p' \). Then \( \Phi_1 \) is an isomorphism near \( p = p' \) since \( p' \) is a 1-point. If permissible parameters at \( q_1 \) are \( u_1, v_1, w \) with \( u = u_1, v = u_1 v_1, w = w_1 \), then \( u_1, v_1, w_1 \) have the form 2 (b) at \( p \) also. If \( u = u_1 v_1, v = v_1 \) then we can interchange \( u \) and \( v \) in 2 (b) and make an appropriate change of permissible parameters at \( p \) to get a form 2 (b) for \( u_1, v_1, w_1 \) at \( p \). If \( u = u_1, v = u_1 (v_1 + \alpha) \), with \( \alpha \neq 0 \), then \( q_1 \) is a 1-point, and \( u_1, w_1 \) are toroidal forms at \( p \).

Suppose that \( u, v, w \) satisfy 2 (c) of Definition 3.4 at \( p' \). Then there are regular parameters \( x_1, y_1, z_1 \) in \( O_{X_1,p} \) defined by

\[
x = x_1^\alpha y_1, y = x_1^\alpha y_1, z = z_1
\]

with \( \overline{\alpha} \overline{d} - \overline{\alpha} \overline{e} = \pm 1 \) or

\[
x = x_1^\alpha (y_1 + \alpha), y = x_1^\alpha (y_1 + \alpha), z = z_1
\]

with \( \overline{\alpha} \overline{d} - \overline{\alpha} \overline{e} = \pm 1 \) and \( 0 \neq \alpha \in k \).

If (58) holds, then \( u, v, w \) satisfy 2 (c) of Definition 3.4 in \( O_{X_1,p} \). If \( q_1 \) is a 2-point, which has regular parameters \( u_1, v_1, w_1 \) defined by

\[
u = u_1 v_1, v = v_1, w = w_1
\]

or

\[
u = u_1 v_1, v = u_1 v_1, w = w_1
\]

then at \( p \) there is an expression of \( u_1, v_1, w_1 \) of the form 2 (c) of Definition 3.4. If \( q_1 \) is a 1-point, which has regular parameters \( u_1, v_1, w_1 \) defined by

\[
u = u_1 v_1, v = u_1 (v_1 + \alpha), w = w_1
\]

with \( 0 \neq \alpha \in k \), then \( u_1, w_1 \) are toroidal forms at \( p \).
If (59) holds, then $p$ is a 1-point and there exist $\overline{f}_1, \overline{v}_1 \in \overline{O}_{X_1, p}$ such that $\overline{f}_1, \overline{v}_1, z$ are regular parameters in $\overline{O}_{X_1, p}$ and

$$x^a y^b = \overline{f}_1, x^c y^d = \overline{v}_1 (\overline{y}_1 + \beta)$$

for some $0 \neq \beta \in k$, and positive integers $s, t, u, v, w$ thus satisfy 2 (b) of Definition 3.4 at $p$. If $q_1$ is a 2-point, which has regular parameters $u_1, v_1, w_1$ defined by (60) or (61), then at $p$ there is an expression of $u_1, v_1, w_1$ of the form 2 (b) of Definition 3.4. If $q_1$ is a 1-point, which has permissible parameters $u_1, v_1, w_1$ defined by (62), then $q_1$ is a 1-point and $u,v,w$ are toroidal forms at $p$.

**Suppose that $q$ is a 1-point.** Let $u, v, w$ be permissible parameters at $q$ (satisfying 3 of Definition 3.4). In this case, $\Psi_1$ is an isomorphism near $q_1$, so we can identify $q$ with $q_1$. $u, v, w$ thus have a form (12) or (13) of Definition 3.1 at $p'$. If (12) holds at $p'$, then $p = p'$, so $u, v, w$ have a form (12) at $p$. Suppose that (13) holds at $p'$. Then $p$ has regular parameters defined by (58) or (59), and we see that $u, v, w$ have a form (12) or (13) at $p$.

**Suppose that $q$ is a 3-point.** Let $u, v, w$ be permissible parameters at $q$. Then after possibly permuting $u, v$ and $w$, we have that $u, v, w$ have one of the forms (8) - (11) of Definition 3.1 at $p'$, and thus $u, v, w$ also have one of the forms (8) - (11) of Definition 3.1 at $p$.

First assume $q_1$ is a 3-point. Further assume that $p$ is a 3-point. Then $u, v, w$ have the form of (23) at $p$, and have a form (11) of Definition 3.1 at $p$. Furthermore, if $\tau_f(p') = -\infty$, then $\tau_{f_{\Phi_1}}(p) = -\infty$, and if $\tau_f(p) \geq 1$, then $\tau_{f_{\Phi_1}}(p) \geq 1$ and $H_{f, p'} = H_{f_{\Phi_1}, p'}$, $A_{f, p'} = A_{f_{\Phi_1}, p}$ and $\tau_f(p') = \tau_{f_{\Phi_1}}(p)$. After possibly interchanging $u$ and $v$, $q_1$ has permissible parameters $u_1, v_1, w_1$ such that one of the following equations (64), (65) or (66) hold:

$$u = u_1, v = u v_1, w = w_1$$

or

$$u = u_1, v = v_1, w = u w_1$$

or

$$u = u_1 w_1, v = v_1, w = w_1.$$  

Assume that (64) or (65) holds. Then $u_1, v_1, w_1$ have a form (11) of Definition 3.1 at $p$. If $\tau_f(p') = -\infty$, then $\tau_{f_1}(p) = -\infty$, and if $\tau_f(p') \geq 1$, then

$$H_{f_{\Phi_1}, p} = H_{f_1, p},$$

$A_{f_{\Phi_1}, p} = A_{f, p}$ and $\tau_{f_1}(p) = \tau_{f_{\Phi_1}}(p) = \tau_f(p')$.

Assume that (66) holds. If $\tau_f(p') = -\infty$, then we can interchange $u$ and $w$ to obtain the case (64), which we have already analyzed, so we may assume that $\tau_f(p') \geq 1$. In (23), we have at $p$ an expression $w = M_0(\gamma + \overline{N}_0)$ where $M_0$ is a monomial and

$$\gamma = \alpha_0 + \sum_{i \geq 1} \alpha_i \overline{M}_i$$

is such that

$$\text{rank}(u, v, \overline{M}_i) = \text{rank}(u, v, M_0) = 2$$

for $1 \leq i$ and

$$\text{rank}(u, v, \overline{N}_0) = 3.$$

Thus

$$H_{f_{\Phi_1}, p} = Zu + Zv + ZM_0 + \sum_{i \geq 1} Z\overline{M}_i = H_{f, p'}.$$
and

\[ A_{f \circ \Phi_1, p} = Zu + Zv + ZM_0 = A_{f, p'} \].

If \( \text{rank}(v, M_0) = 2 \), then \( w, v \) is a toroidal form at \( p \), so we have, after a change of variables in \( x, y, z \) at \( p \), an expression of \( w, v, u \) of the form of (23), and (by Lemma 3.10) we obtain the same calculation of \( H_{f \circ \Phi_1, p} \) and \( A_{f \circ \Phi_1, p} \) for these new parameters. Thus (66) has been transformed into (65), from which it follows that \( w_1, v_1, u_1 \) have a form (11) of Definition 3.1 at \( p \) and \( \tau_{f_1}(p) = \tau_f(p') \).

If \( \text{rank}(v, M_0) = 1 \), then \( \text{rank}(w, u) = 2 \) and \( w, u \) is a toroidal form at \( p \). As in the above paragraph, we change variables to obtain a form (64), from which it follows that \( w_1, u_1, v_1 \) have a form (11) of Definition 3.1 at \( p \) and \( \tau_{f_1}(p) = \tau_f(p') \).

Suppose that \( q_1 \) is a 3-point and \( p \) is a 2-point. Then, after possibly permuting \( u, v, w \), we have that \( u, v, w \) are such that \( u, v \) are toroidal forms of type (9) or type (10) of Definition 3.1 at \( p \), and \( w = M \gamma \) where \( M \) is a monomial in \( x, y, \) and \( \gamma \) is a unit series.

First assume that \( u, v \) are of type (9) of Definition 3.1 at \( p \). After possibly interchanging \( u \) and \( v \), \( q_1 \) has permissible parameters \( u_1, v_1, w_1 \) of one of the forms (64) (65) or (66). In any of these cases, we have an expression

\[
\begin{align*}
  u_1 &= M_1 \gamma_1, \\
  v_1 &= M_2 \gamma_2, \\
  w_1 &= M_3 \gamma_3
\end{align*}
\]

where \( \gamma_1, \gamma_2, \gamma_3 \) are unit series and \( M_1, M_2, M_3 \) are monomials in \( x, y \) with

\[
\text{rank}(M_1, M_2, M_3) = 2.
\]

Thus (after possibly interchanging \( u_1, v_1, w_1 \)) \( u_1, v_1, w_1 \) have a form (9) of Definition 3.1 at \( p \).

Now assume that \( u, v \) is of type (10) of Definition 3.1 at \( p \) (and there does not exist a permutation of \( u, v, w \) such that \( u, v \) are of type (9) at \( p \)). We continue to assume that \( q_1 \) is a 3-point. Then there are permissible parameters \( x, y, z \) at \( p \) such that

\[
\begin{align*}
  u &= (x^a y^b)^k, \\
  v &= (x^a y^b)^l(\alpha + z), \\
  w &= (x^a y^b)^m[\gamma(x^a y^b, z) + x^c y^d]
\end{align*}
\]

where \( \gamma \) is a unit series and \( ad - bc \neq 0 \).

If \( u = v = 0 \) are local equations for \( C \) at \( q \) then after possibly permuting \( u \) and \( v \), we may assume that \( q_1 \) has permissible parameters \( u_1, v_1, w_1 \) defined by

\[
\begin{align*}
  u &= u_1, \\
  v &= v_1, \\
  w &= w_1.
\end{align*}
\]

Thus \( u_1, v_1, w_1 \) have a form (10) of Definition 3.1 at \( p \). Otherwise, we can assume, after possibly interchanging \( u \) and \( v \) that \( u = w = 0 \) are local equations for \( C \). If permissible parameters are defined at \( q_1 \) by

\[
\begin{align*}
  u &= u_1, \\
  v &= v_1, \\
  w &= u_1 w_1,
\end{align*}
\]

then \( u_1, v_1, w_1 \) have a form (10) of Definition 3.1 at \( p \).

The remaining possibility is that \( u, v, w \) are permissible parameters at \( q_1 \), where

\[
\begin{align*}
  u &= u_1 w_1, \\
  v &= v_1, \\
  w &= w_1.
\end{align*}
\]

Then

\[
\begin{align*}
  u_1 &= (x^a y^b)^{k-l}[\gamma + x^c y^d]^{-1}, \\
  v_1 &= (x^a y^b)^l(\alpha + z), \\
  w_1 &= (x^a y^b)^m[\gamma + x^c y^d].
\end{align*}
\]

Define new regular parameters \( \overline{x}, \overline{y}, z \) at \( p \) by \( x = \overline{x} \lambda_x, y = \overline{y} \lambda_y \), where \( \lambda_x, \lambda_y \) are unit series such that

\[
x^a y^b = (\gamma + x^c y^d) \overline{x}^{-1} \overline{y}^{-1}.
\]
Then
\[ u_1 = (x^a y^b z^c)^{k-l} (\gamma + x^e y^d)^{-\frac{k}{l}} \]
\[ v_1 = (x^a y^b)^{l} (\gamma + x^e y^d)^{\frac{k}{l}} (\alpha + z) \]
\[ w_1 = (x^a y^b)^l. \]

If
\[ \frac{\partial \gamma}{\partial z}(0,0,0) \neq 0, \]
then \( u_1, u \) have a form (10) of Definition 3.1 at \( p \). If
\[ \frac{\partial \gamma}{\partial z}(0,0,0) = 0, \]
then \( u_1, v_1 \) have a form (10) of Definition 3.1 at \( p \).

There is a similar analysis (to the case when (10) holds at \( p \)) if \( p \) is a 1-point (and \( q_1 \) is a 3-point). After possibly permuting \( u, v, w \), we find permissible parameters \( u_1, v_1, w_1 \) at \( q \) such that one of the forms (64) – (66) hold. As in the above paragraph, we see that (after possibly interchanging \( u_1, v_1, w_1 \)) \( u_1, v_1, w_1 \) have a form (8) of Definition 3.1 at \( p \).

Still assuming that \( q \) is a 3-point, assume that \( q_1 \) is a 2-point. If \( p \) is a 3-point then \( p' \) is also a 3-point and there are permissible parameters \( x, y, z \) for \( u, v, w \) such that
\[ u = x^a y^b z^c \gamma_1 \]
\[ v = x^d y^e z^f \gamma_2 \]
\[ w = x^l y^m z^n \gamma_3 \]
where \( \gamma_1, \gamma_2, \gamma_3 \) are unit series, and
\[ \text{rank} \begin{pmatrix} a & b & c \\ d & e & f \\ l & m & n \end{pmatrix} \geq 2. \]

After possibly permuting \( u, v \) and \( w \), we may assume that \( q_1 \) has permissible parameters \( u_1, v_1, w_1 \) defined by
\[ u = u_1, v = v_1, w = u_1(w_1 + \alpha) \] (67)
with \( \alpha \neq 0 \). Thus \((a, b, c) = (l, m, n)\), so that
\[ \text{rank} \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = 2 \]
and \( \tau_{f_1}(p) = \tau_{f c \Phi_1}(p) = \tau_f(p') \geq 1 \). We thus have that \( u_1, v_1 \) are toroidal forms at \( p \), of type (11) of Definition 3.1 and \( \tau_{f_1}(p) = \tau_f(p') \).

We have a similar analysis if \( q_1 \) is a 2-point, \( p \) is a 2-point (\( q \) is a 3-point), and \( u, v, w \) satisfy (9) of Definition 3.1 at \( p \). Then \( q_1 \) has permissible parameters \( u_1, v_1, w_1 \) satisfying (67), and \( u_1, v_1 \) are toroidal forms of type (9) of Definition 3.1 at \( p \).

Now assume that \( p \) is a 2-point, \( q_1 \) is a 2-point, (\( q \) is a 3-point) and \( u, v, w \) satisfy (10) of Definition 3.1 at \( p \) and (9) of Definition 3.1 does not hold at \( p \) for any permutation of \( u, v, w \). Then we have permissible parameters \( x, y, z \) at \( p \) such that
\[ u = (x^a y^b)^l \]
\[ v = (x^a y^b)^l(\beta + z) \]
\[ w = (x^a y^b)^m(\gamma(x^a y^b, z) + x^e y^d) \]
where \( \gamma \) is a unit series and \( \beta \neq 0 \). If \( u = w = 0 \) are local equations of \( C \) at \( q \), then \( q_1 \) has regular parameters \( u_1, v_1, w_1 \) defined by
\[ u = u_1, v = v_1, w = u_1(w_1 + \alpha) \] (68)
with \( \alpha \neq 0 \). Thus \( u_1, v_1 \) are toroidal forms of type (10) of Definition 3.1 at \( p \).
If \( u = v = 0 \) are local equations of \( C \) at \( q \), then \( q_1 \) has permissible parameters \( u_1, v_1, w_1 \) defined by

\[
\begin{align*}
  u &= u_1, v = u_1(v_1 + \beta), w = w_1 \\
\end{align*}
\]

with \( \beta \neq 0 \) and we thus have \( t = l \). We have

\[
\begin{align*}
  u_1 &= (x_1^a y_1^b)^l \\
  w_1 &= (x_1^a y_1^b)^m (\gamma + x_1^c y_1^d) \\
  v_1 &= z
\end{align*}
\]

\( u_1, w_1, v_1 \) have the form 2 (c) of Definition 3.4 at \( p \).

There is a similar analysis in the case when \( (q) \) is a 3-point \( q_1 \) is a 2-point and \( p \) is a 1-point. Then (after possibly permuting \( u, v, w \)) \( u, v \) satisfy (8) of Definition 3.1 at \( p \). If \( u = w = 0 \) are local equations of \( C \) at \( q \), then \( q_1 \) has regular parameters \( u_1, v_1, w_1 \) defined by (68), and \( u_1, v_1 \) are toroidal forms of type (8) of Definition 3.1 at \( p \). If \( u = v = 0 \) are local equations of \( C \) at \( q \), then \( q_1 \) has regular parameters \( u_1, v_1, w_1 \) defined by (69), and \( u_1, w_1, v_1 \) have the form 2 (b) of Definition 3.4 at \( p \).

Comparing the above expressions, we see that \( f_1 \) is prepared.

\[ \square \]

**Remark 5.3.** Suppose that \( f : X \to Y \) is prepared and \( \Phi_1 : X_1 \to X \) is either the blow up of a 3-point or the blow up of a 2-curve. Then \( f_1 = f \circ \Phi_1 : X_1 \to Y \) is prepared. If \( p_1 \in X_1 \) is a 3-point then \( p = \Phi_1(p_1) \) is a 3-point and \( \tau_1(p_1) = \tau_1(p) \). If \( D_X \) is cuspidal for \( f \), then \( D_{X_1} \) is cuspidal for \( f_1 \).

The remark follows by substitution of local forms of \( \Phi_1 \) into local forms (Definition 3.4) of the prepared morphism \( f \).

**Lemma 5.4.** Suppose that \( f : X \to Y \) is prepared, \( \Omega \) is a set of 2-points of \( Y \), and we have assigned to each \( q_0 \in \Omega \) permissible parameters \( u = u_{q_0}, v = v_{q_0}, w = w_{q_0} \) at \( q_0 \). Then there exists a commutative diagram

\[
\begin{array}{ccc}
  X_1 & \xrightarrow{f_1} & Y_1 \\
  \Phi \downarrow & & \downarrow \Psi \\
  X & \xrightarrow{f} & Y
\end{array}
\]

such that

1. \( f_1 \) is prepared.
2. \( \Phi, \Psi \) are products of blow ups of 2-curves.
3. Let

\[
\Omega_1 = \left\{ q_1 \in Y_1 \text{ such that } q_1 \text{ is a 2-point and } q_1 = f_1(p_1) \right\}
\]

for some 3-point \( p_1 \in (f \circ \Phi)^{-1}(\Omega) \).

Suppose that \( q_1 \in \Omega_1 \) with \( q_0 = \Psi(q_1) \in \Omega \). Then there exist permissible parameters \( u_1, v_1, w_1 \) at \( q_1 \) such that

\[
\begin{align*}
  u_{q_0} &= u_1^\pi v_1^\beta w_1^\gamma \\
  v_{q_0} &= u_1^\alpha v_1^\beta w_1^\gamma \\
  w_{q_0} &= w_1
\end{align*}
\]

for some \( \pi, \beta, \gamma, \delta \in \mathbb{N} \) with \( \pi \alpha - \beta \delta = \pm 1 \), and if \( p_1 \in f_1^{-1}(q_1) \) is a 3-point, then there exist permissible parameters \( x, y, z \) at \( p_1 \) for \( u_1, v_1, w_1 \) such that

\[
\begin{align*}
  u_1 &= x_1^{a_1} y_1^{b_1} z_1^{c_1} \\
  v_1 &= x_1^{a_1} y_1^{b_1} z_1^{c_1} \\
  w_1 &= \gamma_1 + N_1
\end{align*}
\]
where \( N_1 = x_1^{g_1} y_1^{h_1} z_1^{i_1}, \) with \( \text{rank}(u_1, v_1, N_1) = 3, \) \( \gamma_1 = \sum \alpha_i M_i \) where \( \alpha_i \in k \) and each \( M_i \) is a monomial in \( x_1, y_1, z_1 \) such that there are expressions
\[
M_i^{e_i} = u_1^{a_i} v_1^{b_i},
\]
with \( a_i, b_i, e_i \in \mathbb{N} \) and \( \gcd(a_i, b_i, e_i) = 1 \) for all \( i \). Further, there is a bound \( r \in \mathbb{N} \) such that \( e_i \leq r \) for all \( M_i \) in expressions (72).

4. If \( D_X \) is cuspidal for \( f \), then \( D_X \) is cuspidal for \( f_1 \).

5. \( \Phi \) is an isomorphism above \( f^{-1}(Y - \Sigma(Y)) \).

**Proof.** Suppose that \( q_0 \in \Omega \). Let the 3-points in \( f^{-1}(q_0) \) be \( \{p_1, \ldots, p_t\} \). Each \( p_i \) has permissible parameters \( x, y, z \) such that there is an expression of the form (23),
\[
\begin{align*}
u_{q_0} &= x^a y^b z^c, \\
v_{q_0} &= x^d y^e z^f, \\
w_{q_0} &= \gamma + N,
\end{align*}
\]
with \( a, b, e \in \mathbb{Z} \), \( e > 0 \).

We construct an infinite commutative diagram of morphisms
\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\Phi_n & \downarrow & \Psi_n \\
\vdots & & \vdots \\
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 & \downarrow & \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]
(75)
as follows. Order the 2-curves of \( Y \), and let \( \Psi_1 : Y_1 \rightarrow Y \) be the blow up of the 2-curve \( C \) of smallest order. Then construct (by Lemma 5.2) a commutative diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 & \downarrow & \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]
(76)
where \( f_1 \) is prepared, \( \Phi_1 \) is a product of blow up of 2-curves and \( \Phi_1 \) is an isomorphism above \( f^{-1}(Y - C) \). Order the 2-curves of \( Y_1 \) so that the 2-curves contained in the exceptional divisor of \( \Psi_1 \) have larger order than the order of the (strict transforms of the) 2-curves of \( Y \).

Let \( \Psi_2 : Y_2 \rightarrow Y_1 \) be the blow up of the 2-curve \( C_1 \) on \( Y_1 \) of smallest order, and construct a commutative diagram
\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & Y_2 \\
\Phi_2 & \downarrow & \Psi_2 \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]
as in (76). We now iterate to construct (75). Let \( \Phi_n = \Psi_1 \circ \cdots \circ \Psi_n : Y_n \rightarrow Y \), \( \Phi_n = \Phi_1 \circ \cdots \circ \Phi_n : X_n \rightarrow X \).

Let \( \nu \) be a 0-dimensional valuation of \( k(X) \). Let \( p_n \) be the center of \( \nu \) on \( X_n \), \( q_n = f_n(p_n) \). We will say that \( \nu \) is resolved on \( X_n \) if one of the following holds:
1. $\Psi_n(q_n) \not\in \Omega$ or
2. $\Psi_n(q_n) = q_0 \in \Omega$ and
   (a) $p_n$ is not a 3-point or
   (b) $p_n$ is a 3-point such that a form (71) holds for $p_n$ and $q_n = f_n(p_n)$ so that (72) holds.

Observe that if $\nu$ is resolved on $X_n$, then there exists a neighborhood $U$ of the center of $\nu$ in $X_n$ such that if $\omega$ is a 0-dimensional valuation of $k(X)$ whose center is in $U$, then $\omega$ is resolved on $X_n$, and if $n' > n$, then $\nu$ is resolved on $X_{n'}$.

We will now show that for any 0-dimensional valuation $\nu$ of $k(X)$, there exists $n \in \mathbb{N}$ such that $\nu$ is resolved on $X_n$.

If the center of $\nu$ on $Y$ is not in $\Omega$ or if the center of $\nu$ on $X$ is not a 3-point, then $\nu$ is resolved on $X$, so we may assume that the center of $\nu$ on $Y$ is $q_0 \in \Omega$ and the center of $\nu$ on $X$ is a 3-point $p$.

Suppose that $\nu(u_q)$ and $\nu(v_q)$ are rationally dependent. Then there exists $n$ such that the center of $\nu$ on $Y_n$ is a 1-point. Thus $\nu$ is resolved on $X_n$.

Suppose that $\nu(u_q)$ and $\nu(v_q)$ are rationally independent. At the center $p$ of $\nu$ on $X$,

$$u = u_q, v = v_q, w = w_q$$

have an expression (23). We may identify $\nu$ with an extension of $\nu$ to the quotient field of $\mathcal{O}_{X,p}$ which dominates $\mathcal{O}_{X,p}$. We have

$$u = x^a y^b z^c, v = x^d y^e z^f,$$

and

$$M_i = u^{k_i} v^{l_i} = x^{a_i} y^{b_i} z^{c_i}$$

with $k_i, l_i \in \mathbb{Z}$, $e_i > 0, a_i, b_i, c_i \in \mathbb{N}$. Thus, for all $i$, $(k_i, l_i) \in \sigma$, where

$$\sigma = \{(k, l) \in \mathbb{Q}^2 \mid ka + ld \geq 0, kb + le \geq 0, kc + lf \geq 0\}.$$

Since

$$\text{rank} \left( \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right) = 2,$$

$\sigma$ is a rational polyhedral cone which contains no nonzero linear subspaces, and is contained in the (irrational) half space

$$\{(k, l) \mid k\nu(u) + l\nu(v) \geq 0\}.$$

Let $\lambda_1 = (m_1, m_2), \lambda_2 = (n_1, n_2)$ be integral vectors such that $\sigma = \mathbb{Q} + \lambda_1 + \mathbb{Q} + \lambda_2$. Since $\lambda_1, \lambda_2$ are rational points in $\sigma$, we have $\nu(u^{m_1} v^{m_2}) > 0$ and $\nu(u^{n_1} v^{n_2}) > 0$.

Since $\nu(u)$ and $\nu(v)$ are rationally independent, there exists (by Theorem 2.7 [C1]) a sequence of quadratic transforms $k[u, v] \rightarrow k[u_1, v_1]$ such that the center of $\nu$ on $k[u_1, v_1]$ is $(u_1, v_1)$, there is an expression

$$u = u_1^{\tilde{r}_1} v_1^{\tilde{r}_1}, v = u_1^{\tilde{r}_1} v_1^{\tilde{r}_1}$$

for some $\tilde{r}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{N}$ with $\tilde{r} \tilde{d} - \tilde{b} \tilde{c} = \pm 1$, and $u^{m_1} v^{m_2}, u^{n_1} v^{n_2} \in k[u_1, v_1]$. Thus there exists a rational polyhedral cone $\sigma_1 \subset \mathbb{Q}^2$ containing $\lambda_1$ and $\lambda_2$ such that $k[u_1, v_1] = k[\sigma_1 \cap \mathbb{Z}^2]$. We thus have $M_i \subset k[u_1, v_1]$ for all $i$, so that for all $i$, $M_i$ is a monomial in $u_1$ and $v_1$.

There exists $n$ such that the center of $\nu$ on $Y_n$ has permissible parameters $u_1, v_1, w_1$ where

$$u = u_1^{\tilde{r}_1} v_1^{\tilde{r}_1}, v = u_1^{\tilde{r}_1} v_1^{\tilde{r}_1}, w = w_1.$$

We thus have $\tilde{a}_i, \tilde{b}_i \in \mathbb{N}$ such that

(77)
for all $i$. Let $p_n$ be the center of $\nu$ on $X_n$. If $p_n$ is not a 3-point, then $\nu$ is resolved on $X_n$.

If $p_n$ is a 3-point, then $u_1, v_1, w_1$ (defined by (77)) are permissible parameters at $f_n(p_n)$. There exist permissible parameters $x_1, y_1, z_1$ at $p_n$ for $u_1, v_1, w_1$ defined by

$$
\begin{align*}
x &= a_{11}^{\alpha_1} y_1^{\alpha_2} z_1^{\alpha_3} \\
y &= a_{21}^{\beta_1} y_1^{\beta_2} z_1^{\beta_3} \\
z &= a_{31}^{\gamma_1} y_1^{\gamma_2} z_1^{\gamma_3}
\end{align*}
$$

where $a_{ij} \in \mathbb{N}$, and $\text{Det}(a_{ij}) = \pm 1$. Substituting into the expression (23) of $u, v, w$ at $p$, we have expressions

$$
\begin{align*}
u_1 &= ax_1^{\tilde{a}} b_1^{\tilde{b}} c_1^{\tilde{c}} \\
v_1 &= ax_1^{\tilde{d}} b_1^{\tilde{e}} c_1^{\tilde{f}} \\
w &= ax_1^{\tilde{g}} b_1^{\tilde{h}} c_1^{\tilde{i}}
\end{align*}
$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{g}, \tilde{h}, \tilde{i}$ are arbitrary parameters.

Let $\vec{v}_1 = (\tilde{a}, \tilde{b}, \tilde{c})$, $\vec{v}_2 = (\tilde{d}, \tilde{e}, \tilde{f})$, $\sigma_2 = Q_+ \vec{v}_1 + Q_+ \vec{v}_2 \subset \mathbb{Q}^2$.

By Gordon’s Lemma, (Proposition 1 [F]) $\sigma_2 \cap \mathbb{Z}^3$ is a finitely generated semi group. Let $\vec{w}_1, \ldots, \vec{w}_n \in \sigma_2 \cap \mathbb{Z}^3$ be generators. There exists $0 \neq r \in \mathbb{N}$ and $\delta_j, \epsilon_j \in \mathbb{N}$ such that

$$
\vec{w}_j = \frac{\delta_j}{r} \vec{v}_1 + \frac{\epsilon_j}{r} \vec{v}_2
$$

for $1 \leq j \leq n$. Since the exponents of

$$
M_i = u_1^{\tilde{a}_i} v_1^{\tilde{b}_i}
$$

are in $\sigma_2 \cap \mathbb{Z}^3$ for all $i$, we have an expression

$$
M_i = u_1^{\tilde{a}_i} v_1^{\tilde{b}_i}
$$

with $\tilde{a}_i, \tilde{b}_i \in \mathbb{N}$ for all $M_i$ appearing in the expansion (23) of $w$. Thus $\nu$ is resolved on $X_n$.

By compactness of the Zariski-Riemann manifold of $X ([Z1])$, there exist finitely many $X_i, 1 \leq i \leq t$, such that the center of any 0-dimensional valuation $\nu$ of $k(X)$ is resolved on some $X_i$. Thus $X_t \rightarrow Y_t$ satisfies the conclusions of the Lemma.

\[ \square \]

**Definition 5.5.** Suppose that $f : X \rightarrow Y$ is prepared, and $q \in Y$ is a 2-point. Permissible parameters $u, v, w$ at $q$ are super parameters for $f$ at $q$ if at all $p \in f^{-1}(q)$, there exist permissible parameters $x, y, z$ for $u, v, w$ at $p$ such that we have one of the forms:

1. $p$ is a 1-point

$$
\begin{align*}
u &= x^\alpha \\
v &= x^\beta (\alpha + y) \\
w &= x^\gamma (x, y) + x^\delta (z + \beta)
\end{align*}
$$

where $\gamma$ is a unit series (or zero), $0 \neq \alpha \in k$ and $\beta \in k$. 


2. \( p \) is a 2-point of the form of (9) of Definition 3.1
\[
\begin{align*}
    u &= x^a y^b \\
    v &= x^c y^d \\
    w &= x^e y^f \gamma(x,y) + x^g y^h(z + \beta)
\end{align*}
\]
where \( ad - bc \neq 0 \), \( \gamma \) is a unit series (or zero), and \( \beta \in k \).

3. \( p \) is a 2-point of the form of (10) of Definition 3.1
\[
\begin{align*}
    u &= (x^a y^b)^k \\
    v &= (x^a y^b)^l(\alpha + z) \\
    w &= (x^a y^b)^m(\gamma(x^a y^b, z) + x^c y^d)
\end{align*}
\]
where \( 0 \neq \alpha \in k \), \( ad - bc \neq 0 \) and \( \gamma \) is a unit series (or zero).

4. \( p \) is a 3-point
\[
\begin{align*}
    u &= x^a y^b z^c \\
    v &= x^d y^e z^f \\
    w &= x^g y^h z^{i} \gamma + x^j y^k z^l
\end{align*}
\]
where \( \text{rank}(u,v,x^j y^k z^l) = 3 \), \( \text{rank}(u,v,x^g y^h z^i) = 2 \) and \( \gamma \) is a unit series in monomials \( M \) such that \( \text{rank}(u,v,M) = 2 \) (or \( \gamma \) is zero).

Lemma 5.6. Suppose that \( f : X \to Y \) is prepared, \( q \in Y \) is a 2-point and \( u,v,w \) are permissible parameters at \( q \). Then there exists a commutative diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\tag{79}
\]
such that

1. \( f_1 \) is prepared.
2. \( \Phi_1 \) is a product of blow ups of 2-curves and 3-points such that \( \Phi_1 \) is an isomorphism above \( f^{-1}(Y - \Sigma(Y)) \). \( \Psi_1 \) is a product of blow ups of 2-curves.
3. Suppose that \( q_1 \in \Psi_1^{-1}(q) \) is a 2-point, so that \( q_1 \) has permissible parameters \( u_1, v_1, w_1 \) defined by
\[
\begin{align*}
    u &= u_1^a v_1^b \\
    v &= u_1^c v_1^d \\
    w &= w_1
\end{align*}
\tag{80}
\]
for some \( a, b, c, d \in \mathbb{N} \) with \( ad - bc \neq 0 \). Then \( u_1, v_1, w_1 \) are super parameters at \( q_1 \).

4. If \( D_X \) is cuspidal for \( f \), then \( D_{X_1} \) is cuspidal for \( f_1 \).
5. Suppose that \( p_1 \in X_1 \) is a 3-point. Then \( p = \Phi_1(p) \) is a 3-point and \( \tau_{f_1}(p_1) = \tau_{f}(p) \).

Proof. By Lemma 5.2 and Remark 5.3, any diagram (79) satisfying 2 satisfies 1, 4 and 5. Further, if \( p \in f^{-1}(q) \), \( u,v,w \) are super parameters at \( p, p_1 \in \Phi_1^{-1}(p) \) is such that \( f_1(p_1) = q_1 \) is a 2-point, then the permissible parameters \( u_1, v_1, w_1 \) of (80) at \( q_1 \) are super parameters at \( p_1 \).

Step 1. We will show that there exists a sequence of blow ups of 2-curves and 3-points \( \Phi_1 : X_1 \to X \) such that \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \) and \( u,v,w \) are super parameters at all 3-points \( p \in (f \circ \Phi_1)^{-1}(q) \).
Suppose that $p \in f^{-1}(q)$ is a 3-point, so that there exist permissible parameters $x, y, z \in \mathcal{O}_{X,p}$ at $p$ for $u, v, w$ such that

\begin{align}
\begin{aligned}
u &= x^ay^bz^c \\
u &= x^dy^ez^f \\
u &= g(x, y, z) + N
\end{aligned}
\end{align}

of the form of (11) of Definition 3.1 and (17) of Lemma 3.2. There exist regular parameters $\pi, \gamma, \tau$ in $\mathcal{O}_{X,p}$, and unit series $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}_{X,p}$ such that

$$x = \pi \lambda_1, y = \gamma \lambda_2, z = \tau \lambda_3.$$ 

$\pi \gamma \tau = 0$ is a local equation of $D_X$ at $p$.

There is an expression $q = \sum \alpha_i M_i$ where $\alpha_i \in k$ and $M_i = x^{a_i} y^{b_i} z^{c_i}$ are monomials in $x, y, z$ such that $\text{rank}(u, v, M_i) = 2$. Let $IP$ be the ideal in $\mathcal{O}_{X,p}$ generated by the $\pi \gamma \tau$ for $\alpha_i, \beta_i, \gamma_i$ appearing in some $M_i$. There exists an $\eta$ such that

$$IP = (\pi^{a_0} \gamma^{b_0} \tau^{c_0}, \pi^{a_1} \gamma^{b_1} \tau^{c_1}, \ldots, \pi^{a_i} \gamma^{b_i} \tau^{c_i}).$$

We have relations $M_i^{c_i} = u^{a_i} v^{\beta_i}$ with $e_i, \alpha_i, \beta_i \in \mathbb{Z}$ and $e_i > 0$ for all $i$. Thus for $a \in \text{spec} \mathcal{O}_{X,p}$, $(IP)_a$ is principal if $u$ or $v$ is not in $a$.

By Lemma 3.13, there exists a sequence of blow ups of 2-curves and 3-points $\Phi_1 : X_1 \to X$ such that $\Phi_1$ is an isomorphism over $f^{-1}(X - \Sigma(Y))$ and $IP \mathcal{O}_{X_1,p_1}$ is invertible for all 3-points $p \in X$ such that $f(p) = q$ and $p_1 \in \Phi_1^{-1}(p)$. $f \circ \Phi_1 : X_1 \to X$ is prepared by Remark 5.3.

Suppose that $p_1 \in X_1$ is a 3-point. Then $p = \Phi_1(p_1)$ is also a 3-point, and $\tau_{f \circ \Phi_1}(p_1) = \tau_f(p)$ (by Remark 5.3). Let notation be as in (81). There exist permissible parameters $x_1, y_1, z_1$ for the permissible parameters $u, v, w$ at $p_1$ such that $x_1, y_1, z_1$ are defined by

$$x = x_1^{a_{11}} y_1^{a_{21}} z_1^{a_{31}},$$

$$y = x_1^{a_{12}} y_1^{a_{22}} z_1^{a_{32}},$$

$$z = x_1^{a_{13}} y_1^{a_{23}} z_1^{a_{33}},$$

where $a_{ij} \in \mathbb{N}$, $\text{Det}(a_{ij}) = \pm 1$. Thus all of the $M_i$ and $N$ are distinct monomials when expanded in the variables $x_1, y_1, z_1$, so that $g(x, y, z)$ is a monomial in $x_1, y_1, z_1$ times a unit series (in $x_1, y_1, z_1$). Thus $u, v, w$ are super parameters at $p_1$.

**Step 2.** We will show that there exists a sequence of blow ups of 2-curves $\Phi_2 : X_2 \to X_1$, where $\Phi_2 : X_1 \to X$ is the map constructed in Step 1, such that $\Phi_2$ is an isomorphism over $(f \circ \Phi_1)^{-1}(Y - \Sigma(y))$ and $u, v, w$ are super parameters at all $p \in (f \circ \Phi_1 \circ \Phi_2)^{-1}(q)$ for which $p$ is a 3-point or $p$ is a 2-point of type (9) of Definition 3.1 for $u, v, w$.

Let $f_1 = f \circ \Phi_1$. Let $\nu$ be a 0-dimensional valuation of $k(X)$. Let $p$ be the center of $\nu$ on $X_1$. Say that $\nu$ is resolved on $X_1$ if

1. $f_1(p) \neq q$, or
2. $p$ is not a 2-point of type (9) of Definition 3.1, or
3. $p$ is a 2-point of type (9) and $u, v, w$ are super parameters at $p$.

We construct an infinite sequence of morphisms

$$\cdots \to X_n \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_3} X_2 \xrightarrow{\Phi_2} X_1$$

as follows. Order the 2-curves $C$ of $X_1$ such that $q \in (f \circ \Phi_1)(C) \subset \Sigma(Y)$. Let $\Phi_2 : X_2 \to X_1$ be the blow up of the 2-curve $C_1$ on $X_1$ of smallest order. Order the 2-curves $C'$ of $X_2$ such that $q \in (f \circ \Phi_1 \circ \Phi_2)(C') \subset \Sigma(Y)$ so that the 2-curves contained in the exceptional divisor of $\Phi_2$ have order larger than the order of the (strict
transform of the) 2-curves $C$ of $X$ such that $q \in f(C) \subset \Sigma(Y)$. Let $\Phi_3 : Y_3 \rightarrow Y_2$ be the blow ups of the 2-curve $C_3$ on $Y_3$ of smallest order, and repeat to inductively construct the morphisms $\Phi_n : X_n \rightarrow X_{n-1}$. Let $\Phi_n = \Phi_2 \circ \cdots \Phi_n : X_n \rightarrow X_1$. The morphisms $f \circ \Phi_n$ are prepared by Remark 5.3.

Suppose that $p \in f^{-1}(q)$ is a 2-point satisfying (9) of Definition 3.1, so there exist permissible parameters $x, y, z$ at $p$ for $u, v, w$ such that

$$u = x^a y^b$$
$$v = x^c y^d$$
$$w = g(x, y) + x^e y^f$$

with $ad - bc \neq 0$. There exist regular parameters $\pi, \eta, \gamma$ in $O_{X_1, p}$ and unit series $\lambda_1, \lambda_2 \in \hat{O}_{X_1, p}$ such that

$$x = \pi \lambda_1, y = \eta \lambda_2.$$

$\eta y = 0$ is a local equation of $D_{X_1}$ at $p$.

If $\nu(\pi), \nu(\eta)$ are rationally independent, then the center of $\nu$ on $X_n$ is a 2-point for all $n$, there exists an $n$ such that the center of $\nu$ on $X_n$ is a 2-point satisfying (9) of Definition 3.1 and $u, v, w$ are super parameters at $p_1$, by embedded resolution of plane curve singularities (cf. Section 3.4 and Exercise 3.3 [C3]) applied to $g(x, y) = 0$. If $\nu(\pi), \nu(\eta)$ are rationally dependent, then there exists an $n$ such that the center $p_1$ of $\nu$ on $X_n$ is a 1-point.

By compactness of the Zariski-Riemann manifold of $k(X)$ [Z1], there exists an $n$ such that all points of $X_n$ are resolved. Thus there exists a sequence of blow ups of 2-curves $\Phi_2 : X_2 \rightarrow X_1$ such that the conclusions of Step 2 hold.

**Step 3.** We will show that there exists a diagram (79) satisfying the conclusions of the lemma.

After replacing $f$ with $f \circ \Phi_1 \circ \Phi_2$, we can assume that $f$ satisfies the conclusions of Step 2. We construct a sequence of diagrams (79) satisfying 1, 2, 4 and 5 of the conclusions of the lemma as follows. Let $\Psi_1 : Y_1 \rightarrow Y$ be the blow up of the 2-curve $C$ containing $q$. We order the two 2-curves in $Y_1$ which dominate $C$. Let $\Psi_2 : Y_2 \rightarrow Y_1$ be the blow up of the 2-curve of smallest order. Now extend the ordering to the 2-curves of $Y_2$ which dominate $C$, by requiring that the two 2-curves on the exceptional divisor of $\Psi_2$ which dominate $C$ have larger order than the order of the (strict transform of the) 2-curve on $Y_1$ dominating $C$ (which was not blown up by $\Psi_2$). Now let $\Psi_3 : Y_3 \rightarrow Y_2$ be the blow up of the 2-curve of smallest order. We continue this process to construct a sequence of blow ups of 2-curves

$$\cdots \rightarrow Y_n \overset{\Psi_n}{\rightarrow} Y_{n-1} \rightarrow \cdots \rightarrow Y_1 \overset{\Psi_1}{\rightarrow} Y.$$

Let $\Psi_n = \Psi_1 \circ \cdots \Psi_{n-1} \circ \Psi_n$. Let

$$U = \left\{ p \in f^{-1}(q) \text{ such that } u, v, w \text{ have a form (8) or (10) of Definition 3.1 or of 2 (b) or 2 (c) of Definition 3.4 at } p \right\}.$$

$U$ is an open subset of $f^{-1}(q)$. For each $p \in U$, there exists $n(p)$ such that $n \geq n(p)$ implies the rational map $\overline{\Psi}_n^{-1} \circ f$ is defined at $p$, and $(\overline{\Psi}_n^{-1} \circ f)(p_1)$ is a 1-point, for $p_1$ in some neighborhood $U_p$ of $p$ in $U$. $\{U_p\}$ is an open cover of $U$, so there exists a finite subcover $\{U_{p_1}, \ldots, U_{p_m}\}$ of $U$. Let $n = \max\{n(p_1), \ldots, n(p_m)\}$. We have that $\overline{\Psi}_n^{-1} \circ f(p)$ is a 1-point if $p \in U$. 

By Lemma 5.2, we can now construct a commutative diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\Phi_n & \downarrow & \Psi_n \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that 1, 2, 4 and 5 of the conclusions of the lemma hold.

If \( p_1 \in X_n \) is such that \( f_n(p_1) = q_1 \in \Psi_n^{-1}(q) \) is a 2-point, then \( \Phi_n(p_1) \in Y - U \), and thus \( u, v, w \) have a form 2 or 4 of Definition 5.5 at \( p \). As observed at the beginning of the proof, \( p_1 \) must have one of the forms 1 – 4 of Definition 5.5 with respect to the permissible parameters \( u_1, v_1, w_1 \) at \( q_1 \) defined by 3 of the statement of Lemma 5.6. Thus \( f_n \) satisfies 3 of the statement of Lemma 5.6. □

**Theorem 5.7.** Suppose that \( f : X \to Y \) is prepared, \( \tau = \tau_f(X) \geq 1 \) and all 3-points \( p \) of \( X \) such that \( \tau_f(p) = \tau \) map to 2-points of \( Y \).

Let

\[
\Omega = \left\{ q \in Y \text{ such that } q \text{ is a 2-point and } \text{there exists a 3-point } p \in f^{-1}(q) \right\}
\]

Then there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that the following conditions hold:

1. \( \Phi \) is a product of blow ups of 2-curves and 3-points such that \( \Phi \) is an isomorphism above \( f_1^{-1}(Y - \Sigma(Y)) \). \( \Psi \) is a product of blow ups of 2-curves.
2. \( f_1 \) is prepared.
3. All 3-points \( p_1 \) of \( X_1 \) such that \( \tau_{f_1}(p_1) = \tau \) map to 2-points of \( Y_1 \), and if \( p_1 \in X_1 \) is a 3-point, then \( \tau_{f_1}(p_1) = \tau_f(\Phi(p_1)) \).
4. Let

\[
\Omega_1 = \left\{ q \in Y_1 \text{ such that } q \text{ is a 2-point, } q \in \Psi^{-1}(\Omega) \text{ and there exists a 3-point } p \in f_1^{-1}(q) \right\}.
\]

If \( q \in \Omega_1 \) is a 2-point and \( p_1, \ldots, p_r \in f_1^{-1}(q) \) are the 3 points in \( f_1^{-1}(q) \), then there exist \( u, v \in \mathcal{O}_{Y_1,q} \) and \( w_i \in \hat{\mathcal{O}}_{Y_1,q} \) for \( 1 \leq i \leq r \) such that \( u, v, w_i \) are (formal) permissible parameters at \( q \) for \( 1 \leq i \leq r \) and at the point \( p_1 \) we have permissible parameters \( x, y, z \) for \( u, v, w_i \) such that we have an expression

(a)

\[
\begin{align*}
u &= x^a y^b z^c \\
v &= x^d y^e z^f \\
w_i &= M \gamma
\end{align*}
\]

where \( \gamma \) is a unit series, \( M \) is a monomial in \( x, y, z \) and there is a relation

\[
M^{e_i} = u^{a_i} v^{b_i}
\]

with \( a_i, b_i, e_i \in \mathbb{Z}, \ e_i > 1 \) and \( \gcd(a_i, b_i, e_i) = 1 \) or

(b)

\[
\begin{align*}
u &= x^a y^b z^c \\
v &= x^d y^e z^f \\
w_i &= x^g y^h z^i
\end{align*}
\]
where

\[
\begin{pmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{pmatrix}
\]

has rank 3.

Further, there exists \( w \in \mathcal{O}_{Y_1, q} \) and \( \lambda_i(u, v) \in k[[u, v]] \) for \( 1 \leq i \leq r \) such that

\[ w_i = w - \lambda_i(u, v) \]

for \( 1 \leq i \leq r \).

5. For \( q \in \Omega_1 \) and for \( u, v, w_i \) with \( 1 \leq i \leq r \) in 4 above, \( u, v, w_i \) are super parameters at \( q \).

6. Suppose that \( D_X \) is cuspidal for \( f \). Then \( D_{X_1} \) is cuspidal for \( f_1 \).

7. Suppose that \( p_1 \in X_1 \) is a 3-point. Then \( p = \Phi_1(p) \) is a 3-point and \( \tau_{f_1}(p_1) = \tau_f(p) \).

Remark 5.8. The proof actually produces expressions 4 (a) with

\[ w_i = M\gamma = g_i(u^\frac{a}{2}, v^\frac{b}{2}) + N \]

where \( l \in \mathbb{N} \) and \( g_i \) is a series, \( N \) is a monomial in \( x, y, z \) and \( \text{rank}(u, v, N) = 3 \).

Proof. By Lemma 5.4, there exists a diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

(84)

where \( \Phi, \Psi \) are products of blow ups of 2-curves such that for all 2-points \( q \in \Psi^{-1}(\Omega) \), there exist (algebraic) permissible parameters \( u_q, v_q, w_q \) at \( q \) such that if \( p \in X_1 \) is a 3-point such that \( f(p) = q \), then there are permissible parameters \( x, y, z \) at \( p \) such that there is an expression

\[
\begin{align*}
  u_q &= x^a y^b z^c \\
  v_q &= x^d y^e z^f \\
  w_q &= \gamma + N
\end{align*}
\]

(85)

where \( \text{rank}(u_q, v_q) = 2 \), \( \gamma \) is a (possibly trivial if \( \tau_{f_j}(p) = 0 \)) series \( \gamma = \sum \alpha_i M_i \) in monomials \( M_i \) in \( x, y, z \) such that \( \text{rank}(u_q, v_q, M_i) = 2 \) for all \( i \), and \( N \) is a monomial in \( x, y, z \) such that \( \text{rank}(u_q, v_q, N) = 3 \). Further, there are relations

\[
M_i^{e_i} = u_q^{a_i} v_q^{b_i}
\]

(86)

for all \( M_i \), with \( a_i, b_i, c_i \in \mathbb{N} \) and \( e_i > 0 \). Let \( h_p(u_q, v_q) \) be the series in the monomials \( M_i \) of \( \gamma \) such that \( e_i = 1 \) in (86). Let \( w_p = u_q - h_p \). Let

\[
\Omega' = \left\{ q \in Y_1 \text{ such that } q \text{ is a 2-point, } q \in \Psi^{-1}(\Omega) \text{ and there exists a 3-point } p \in f^{-1}(q) \right\}
\]

After performing a further sequence of blowups of 2-curves above \( X_1 \), we then achieve (by Lemma 3.11) that 4 of the conclusions of Theorem 5.7 hold. Now the proof follows from Lemma 5.6, applied successively to all \( q \in \Omega' \) and permissible parameters \( u, v, w_p \), for \( p \in f_1^{-1}(q) \) a 3-point.

□
6. Relations

In this section, we suppose that $Y$ is a nonsingular projective 3-fold with toroidal structure $D_Y$, and $f : X \to Y$ is a birational morphism of nonsingular projective 3-folds, with toroidal structures $D_Y$ and $D_X = f^{-1}(D_Y)$, such that $D_Y$ contains the fundamental locus of $f$.

**Definition 6.1.** Suppose that $Y$ is a nonsingular 3-fold with SNC divisor $D_Y$. A 2-point pre-relation $R$ on $Y$ is an association from a finite set $U(R)$ of 2-points of $Y$. If $q \in U(R)$, then

$$R(q) = \left( S = S_R(q), E_1 = E_{R,1}(q), E_2 = E_{R,2}(q), w = w_R(q), u = u_R(q), v = v_R(q), c = c_R(q), a = a_R(q), b = b_R(q), \lambda = \lambda_R(q) \right)$$

where $E_1, E_2$ are the components of $D_Y$ containing $q$, $a, b, c, v \in \mathbb{Z}$, $\gcd(a, b, c) = 1$ and $e > 1$. $u, v, w \in \mathcal{O}_{Y,q}$ are such that $u, v, w$ are (formal) permissible parameters at $q$, $u = 0$ is a local equation of $E_1$, $v = 0$ is a local equation of $E_2$.

$S = \text{spec}(\mathcal{O}_{Y,q}(w))$, $0 \neq \lambda \in k$.

We will also allow 2-point pre-relations with $a = b = -\infty$, $e = 1$ and $\lambda = 1$.

Observe that if $R$ is a 2-point pre-relation (and $a, b \neq -\infty$) then $R(q)$ is determined by the expression

$$w^e - \lambda u^a v^b.$$  \hfill (88)

Depending on the signs of $a$ and $b$, this expression determines a (formal) germ of an (irreducible) surface singularity

$$F = F_R(q) = 0$$  \hfill (89)

of one of the following forms:

$$F = w^e - \lambda u^a v^b = 0$$

if $a, b \geq 0$ and $a + b > 0$,

$$F = w^e u^{-a} - \lambda v^b = 0$$

if $a < 0$, $b > 0$,

$$F = w^e v^{-b} - \lambda u^a = 0$$

if $b < 0$, $a > 0$.

In the remaining case, $a, b \leq 0$,

$$F = w^e u^{-a} v^{-b} - \lambda$$

is a unit in $\mathcal{O}_{Y,q}$ and $F(q) \neq 0$.

If $a, b = -\infty$, then $R(q)$ is determined by

$$F_R(q) = w_R(q) = 0.$$  \hfill (90)

**Definition 6.2.** A 2-point pre-relation $R$ on $Y$ is algebraic if there exists a nonsingular irreducible locally closed surface $\Omega(R) \subset Y$ such that $\Omega(R)$ makes SNCs with $D_Y$, $U(R) \subset \Omega(R)$ and $S_R(q)$ is the (formal) germ of $\Omega(R)$ at $q$ for all $q \in U(R)$.

**Definition 6.3.** A 3-point pre-relation $R$ on $Y$ is an association from a finite set $U(R)$ of 3-points of $Y$. If $q \in U(R)$ then

$$R(q) = \left( E_1 = E_{R,1}(q), E_2 = E_{R,2}(q), E_3 = E_{R,3}(q), u = u_R(q), v = v_R(q), w = w_R(q), a = a_R(q), b = b_R(q), c = c_R(q), \lambda = \lambda_R(q) \right)$$

where $E_1, E_2, E_3$ are the components of $D_Y$ containing $q$, $a, b, c, v \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, $\min\{a, b, c\} < 0 < \max\{a, b, c\}$. $u, v, w \in \mathcal{O}_{Y,q}$ are permissible parameters at $q$ such
that \( u = 0 \) is a local equation of \( E_1 \), \( v = 0 \) is a local equation of \( E_2 \), \( w = 0 \) is a local equation of \( E_3 \), and \( 0 \neq \lambda \in k \).

Observe that if \( R \) is a 3-point pre-relation then \( R(q) \) is uniquely determined by the expression

\[
u^a v^b w^c = \lambda. \tag{92}\]

Depending on the signs of \( a, b \) and \( c \), this expression determines a germ of an (irreducible) surface singularity

\[
F = F_R(q) = 0 \tag{93}
\]
of one of the following forms:

\[
\begin{align*}
F &= w^c - \lambda u^{-a} v^{-b} = 0 \text{ if } a, b \leq 0, c > 0 \\
F &= v^b - \lambda u^{-a} w^{-c} = 0 \text{ if } a, c \leq 0, b > 0 \\
F &= u^a - \lambda v^{-b} w^{-c} = 0 \text{ if } b, c \leq 0, a > 0 \\
F &= w^{-c} - \frac{1}{\lambda} u^a v^b \text{ if } a, b > 0, c < 0 \\
F &= v^{-b} - \frac{1}{\lambda} u^a w^c \text{ if } a, c > 0, b < 0 \\
F &= u^{-a} - \frac{1}{\lambda} v^b w^c \text{ if } b, c > 0, a < 0
\end{align*}
\]  

(94)

A pre-relation \( R \) is resolved if \( F_R(q) \) is a unit in \( \hat{O}_{Y,q} \) for all \( q \in U(R) \) (This includes the case \( U(R) = \emptyset \)).

**Definition 6.4.** A subvariety \( G \) of \( Y \) is an admissible center for a 2-point pre-relation \( R \) on \( Y \) if one of the following holds:

1. \( G \) is a 2-point.
2. \( G \) is a 2-curve of \( Y \).
3. \( G \subset D_Y \) is a nonsingular curve which contains a 1-point and makes SNCs with \( D_Y \). If \( q \in U(R) \cap G \) then \( S_R(q) \) contains the germ of \( G \) at \( q \). If \( R \) is algebraic, then \( G \) makes SNCs with \( \Omega(R) \).

**Definition 6.5.** A curve \( C \subset Y \) is an admissible center for a 3-point pre-relation \( R \) on \( Y \) if \( C \) is a 2-curve.

Observe that admissible centers are possible centers.

Suppose that \( R \) is a 2-point (or a 3-point) pre-relation on \( Y \), \( G \) is an admissible center for \( R \), and \( \Psi : Y_1 \rightarrow Y \) is the blow up of \( G \).

If \( R \) is a 2-point pre-relation then the transform \( R^1 \) of \( R \) on \( Y_1 \) is the 2-point pre-relation on \( Y_1 \) defined by the condition that \( U(R^1) \) is the union over \( q \in U(R) \) of 2-points \( q_1 \) in \( \Psi^{-1}(q) \) such that \( q_1 \) is on the strict transform of \( w_R(q) = 0 \). For such \( q_1 \), \( R^1(q_1) \) is determined by the strict transform of the form \( F_R(q) = 0 \) (89) (or (90)) on \( Y_1 \) at \( q_1 \).

If \( q \in U(R) \cap G \), and

\[
u = u_R(q), v = v_R(q), w = w_R(q),
\]

then \( G \) has local equations of one of the following forms at \( q \):

1. \( u = v = w = 0 \),
2. \( u = v = 0 \),
3. \( u = w = 0 \) or \( v = w = 0 \).

If \( q_1 \in U(R^1) \cap \Psi^{-1}(q) \), then after possibly interchanging \( u \) and \( v \),

\[
u_1 = u_{R^1}(q_1), v_1 = v_{R^1}(q_1), w_1 = w_{R^1}(q_1)
\]

are defined, respectively, by

1. \( u = u_1, v = u_1 v_1, w = u_1 w_1 \),
2. \( u = u_1, v = u_1v_1, w = w_1 \),
3. \( u = u_1, v = v_1, w = w_1 \).

If \( R \) is algebraic, then the transform \( R^1 \) of \( R \) is algebraic, where \( \Omega(R^1) \) is the strict transform of \( \Omega(R) \) by \( \Psi \).

If \( R \) is a 3-point pre-relation then the transform \( R^1 \) of \( R \) on \( Y_1 \) is the 3-point pre-relation on \( Y_1 \) defined by the condition that \( U(R^1) \) is the union over \( q \in U(R) \) of 3-points \( q_1 \) in \( \Psi^{-1}(q) \) such that \( q_1 \) is on the strict transform of the form \( F_R(q) = 0 \) of (93) on \( Y_1 \). For such \( q_1 \), \( R^1(q_1) \) is determined by the strict transform of the form \( F_R(q) = 0 \) at \( q_1 \).

After possibly interchanging \( u = u_R(q), v = v_R(q) \) and \( w = w_R(q) \),
\[
   u_1 = u_R(q_1), v_1 = v_R(q_1), w_1 = w_R(q_1)
\]
are defined by
\[
   u = u_1, v = u_1v_1, w = w_1.
\]

**Definition 6.6.** Suppose that \( f : X \to Y \) is a prepared morphism. A primitive 2-point relation \( R \) for \( f \) is
1. A 2-point pre-relation \( \overline{R} \) on \( Y \),
2. A set of 3-points \( T = T(R) \subset \bigcup_{q \in U(\overline{R})} f^{-1}(q) \) such that if \( p \in T(R) \) and
\[
   \overline{R}(f(p)) = (S, E_1, E_2, w, u, v, e, a, b, \lambda_p)
\]
then there exist permissible parameters \( x, y, z \) at \( p \) for \( u, v, w \) such that
\[
   w^e = u^a v^b \overline{X}(x, y, z)
\]
where \( \overline{X} \) is a unit series such that \( \overline{X}(0, 0, 0) = \lambda_p \) if \( a, b \neq -\infty \), and \( u, v, w \) have a monomial form at \( p \) if \( a = b = -\infty \).

We define \( R(p) = \overline{R}(f(p)) \) if \( p \in T(R) \), and denote
\[
   R(p) = \left( S = S_R(p), E_1(p), E_2(p), w = w_R(p), u = u_R(p), v = v_R(p), e = e_R(p), a = a_R(p), b = b_R(p), \lambda_p = \lambda_R(p) \right).
\]

A 2-point relation \( R \) for \( f \) is a finite set of 2-point pre-relations \( \{ \overline{R_i} \} \) on \( Y \) with associated primitive 2-points relations \( R_i \) for \( f \) such that the sets \( T(R_i) \) are pairwise disjoint. We denote \( U(R) = \bigcup_i U(\overline{R}_i) \) and \( T(R) = \bigcup_i T(R_i) \), and define
\[
   R(p) = R_i(p)
\]
if \( p \in T(R_i) \).

We further require that
\[
   u_{R_i}(q) = u_{\overline{R}_i}(q), v_{R_i}(q) = v_{\overline{R}_i}(q)
\]
if \( q \in U(\overline{R}_i) \cap U(\overline{R}_j) \). We will call \( \{ \overline{R}_i \} \) the 2-point pre-relations associated to \( R \).

We will say that \( R \) is algebraic if each \( \overline{R}_i \) is algebraic and
\[
   \Omega(\overline{R}_i) \cap U(R) = U(\overline{R}_i)
\]
for all \( i \). For \( p \in T(R_i) \subset T(R) \), we denote
\[
   R(p) = \left( S = S_R(p), E_1(p), E_2(p), w = w_R(p), u = u_R(p), v = v_R(p), e = e_R(p), a = a_R(p), b = b_R(p), \lambda_p = \lambda_R(p) \right).
\]

**Definition 6.7.** Suppose that \( f : X \to Y \) is a prepared morphism. A primitive 3-point relation \( R \) for \( f \) is
1. A 3-point pre-relation \( \overline{R} \) on \( Y \),
2. A set of 3-points $T(R) \subset \bigcup_{q \in U(\overline{R}_t)} f^{-1}(q)$ such that if $p \in T(R)$ and

$$\overline{R}(f(p)) = (E_1, E_2, E_3, u, v, w, a, b, c, \lambda_p),$$

then there exist permissible parameters $x, y, z$ at $p$ for $u, v, w$ such that

$$u^aw^bw^c = \Lambda(x, y, z)$$

where $\Lambda$ is a unit series such that $\Lambda(0, 0, 0) = \lambda_p$.

We define $R(p) = \overline{R}(f(p))$ if $p \in T(R)$, and denote

$$R(p) = \left( \begin{array}{c} E_1 = E_1(p), E_2 = E_2(p), E_3 = E_3(p), u = u_R(p), v = v_R(p), \phi = \phi_R(p) \\
\phi = \phi_R(p), a = a_R(p), b = b_R(p), c = c_R(p), \lambda_p = \lambda_R(p) \end{array} \right).$$

A 3-point relation $R$ for $f$ is a finite set of 3-point pre-relations $\{R_t\}$ on $Y$ with associated primitive 3-points relations $R_t$ such that the sets $T(R_t)$ are pairwise disjoint. We denote $U(R) = \bigcup U(\overline{R}_t)$ and $T(R) = \bigcup T(R_t)$, and define

$$R(p) = R_t(p)$$

if $p \in T(R_t)$.

We further require that we have equalities

$$u_{R_t}(q) = u_{R_t}(q), v_{R_t}(q) = v_{R_t}(q), w_{R_t}(q) = w_{R_t}(q)$$

if $q \in U(\overline{R}_t) \cap U(\overline{R}_t)$. We will say that $\{R_t\}$ are the 3-point pre-relations associated to $R$. For $p \in T(R_t) \subset T(R)$, we denote

$$R(p) = \left( \begin{array}{c} E_1 = E_1(p), E_2 = E_2(p), E_3 = E_3(p), u = u_R(p), v = v_R(p), \phi = \phi_R(p) \\
\phi = \phi_R(p), a = a_R(p), b = b_R(p), c = c_R(p), \lambda_p = \lambda_R(p) \end{array} \right).$$

A (2-point or 3-point) relation $R$ is resolved if $T(R) = \emptyset$.

**Definition 6.8.** Suppose that $f : X \to Y$ is prepared, $R$ is a 2-point (respectively 3-point) relation for $f$ and

$$\xymatrix{ X_1 \ar[r]^{f_1} \ar[d]_{\Phi} & Y_1 \ar[d]^\Psi \\
X \ar[r]^f & Y }$$

is a commutative diagram such that

1. $\Phi$ is a product of blow ups which are admissible for all of the pre-relations $\overline{R}_t$ associated to $R$ (and their transforms) and $\Phi$ is a product of blow ups of possible centers
2. $f_1$ is prepared.
3. Let $\overline{R}_t$ be the transforms of the $\overline{R}_t$ on $Y_1$ and let

$$T_t = \{p \in X_1 \mid p \text{ is a 3-point and } p \in \Phi^{-1}(T(R_t)) \cap f_1^{-1}(U(\overline{R}_t)) \}.$$ 

Suppose that the condition 2 of Definition 6.6 (respectively 2 of Definition 6.7) are satisfied for $f_1 : X_1 \to Y_1$, and all $\overline{R}_t$ and $T_t$.

Then the transform $R^1$ of $R$ for $f_1$ is the 2-point (respectively 3-point) relation for $f_1$ defined by Definition 6.6 (respectively Definition 6.7) as

$$T(R^1) = \bigcup T_t,$n

$$R^1(p) = \overline{R}_t(f_1(p))$$

for $p \in T_t$. 
**Theorem 6.9.** Suppose that \( R \) is a 3-point pre-relation on \( Y \). Then there exists a sequence of blow ups of 2-curves \( Y_1 \to Y \), such that if \( R^1 \) is the transform of \( R \) on \( Y_1 \), then \( R^1 \) is resolved.

*Proof.* Suppose that \( \Phi_1 : Y_1 \to Y \) is a sequence of blow ups of 2-curves. Let \( R^1 \) be the transform of \( R \) on \( Y_1 \). We will say that a 0-dimensional valuation \( w \) of \( k(Y) \) is resolved on \( Y_1 \) if the center \( q_1 \) of \( w \) on \( Y_1 \) is not in \( U(R^1) \).

Observe that if \( \omega \) is resolved on \( Y_1 \), then there exists an open neighborhood \( \Sigma \) of \( q_1 \) in \( Y_1 \) such that all 0-dimensional valuations \( \nu \) of \( k(Y) \) whose center on \( Y_1 \) is in \( \Sigma \) are resolved on \( Y_1 \). Further, if \( \Phi_2 : Y_2 \to Y \) is a sequence of blow ups of 2-curves, which factors through \( Y_1 \), and \( \omega \) is resolved on \( Y_1 \), then \( \omega \) is resolved on \( Y_2 \).

We will show that for each 0-dimensional valuation \( \nu \) of \( k(Y) \), there exists a sequence of blow ups of 2-curves \( Y_\nu \to Y \) such that the center of \( \nu \) is resolved on \( Y_\nu \).

Let \( \nu \) be a 0-dimensional valuation of \( k(Y) \), and suppose that the center of \( \nu \) on \( Y \) is \( q \in U(R) \). Let \( F = F_R(q) \) (with the notation of (94)). The sequence of blow ups of 2-curves \( Y_\nu \to Y \) such that \( \nu \) is resolved on \( Y_\nu \) is constructed as follows.

After possibly permuting \( u, v, w \) in (94), possibly replacing \( \lambda \) with \( \frac{1}{\lambda} \), and observing that \( \gcd(a, b, c) = 1 \), we have an expression

\[
F = w_1^\sigma - \lambda w_1^\bar{\sigma} v^\bar{\nu} = 0
\]

(with \( \sigma = \pm a, \bar{\sigma} = \pm b \geq 0, \bar{\nu} = \pm c > 0 \)) and such that if \( \sigma = 0 \) then \( \bar{\sigma} \leq \bar{\nu} \) and if \( \bar{\nu} = 0 \) then \( \bar{\sigma} \leq \sigma \).

Suppose that \( \sigma + \bar{\nu} < \bar{\sigma} \) in (96), so that \( \sigma > 0 \) (and \( \bar{\nu} > 0 \)). Let \( \Phi_1 : Y_1 \to Y \) be the blow up of the 2-curve with local equations \( u = w = 0 \) at \( q \). Let \( R^1 \) be the transform of \( R \) on \( Y_1 \). There are two 3-points \( q_1, q_2 \in \Phi_1^{-1}(q) \). \( q_1 \) has regular parameters \( u_1, v_1, w_1 \) defined by \( u = u_1, v = v_1, w = w_1 \). The strict transform of \( F = 0 \) has the local equation

\[
F^1 = v_1^\bar{\nu} - \frac{1}{\lambda} w_1^{\bar{\sigma} - \bar{\nu}} v_1^{\bar{\nu}} = 0
\]

at \( q_1 \). We have a form (96) for \( F^1 = F_{R^1}(q_1) \) with a reduction in \( \bar{\sigma} \) to \( \bar{\nu} \). \( q_2 \) has regular parameters \( u_1, v_1, w_1 \) defined by \( u = u_1 w_1, v = v_1, w = w_1 \). The strict transform of \( F = 0 \) has the local equation

\[
F^1 = w_1^{\bar{\sigma} - \bar{\nu}} - \lambda u_1^{\bar{\sigma}} v_1^\bar{\nu} = 0
\]

at \( q_2 \). Thus \( q_2 \notin U(R^1) \) and we have a form (96) for \( F^1 = F_{R^1}(q_2) \) with a reduction in \( \bar{\sigma} \) to \( \bar{\nu} - \frac{1}{\lambda} \).

Suppose that \( \sigma \geq \bar{\sigma} \) in (96). Let \( \Phi_1 : Y_1 \to Y \) be the blow up of the 2-curve with local equation \( u = w = 0 \) at \( q \). Let \( R^1 \) be the transform of \( R \) on \( Y_1 \). There are two 3-points \( q_1, q_2 \in \Phi_1^{-1}(q) \). \( q_1 \) has regular parameters \( u_1, v_1, w_1 \) defined by \( u = u_1, v = v_1, w = u_1 w_1 \). The strict transform of \( F = 0 \) has the local equation

\[
F^1 = u_1^\sigma - \lambda v_1^{\bar{\sigma} - \bar{\nu}} w_1^{\bar{\nu}} = 0
\]

at \( q_1 \). Suppose that \( q_1 \in U(R^1) \) (which holds if \( \bar{\sigma} + \bar{\nu} - \bar{\sigma} > 0 \)). If \( \bar{\sigma} = \sigma \) and \( \bar{\nu} < \bar{\sigma} \) we have a reduction in \( \bar{\sigma} \) in the expression of the form (96). If \( \bar{\nu} = 0 \) and \( \bar{\sigma} - \bar{\nu} < \bar{\sigma} \) we have a reduction in \( \bar{\sigma} \). Otherwise, \( \bar{\sigma} \) stays the same, but we have a reduction in \( \bar{\sigma} + \bar{\nu} \) in \( F^1 = F_{R^1}(q_1) \). \( q_2 \) has regular parameters \( u_1, v_1, w_1 \) defined by \( u = u_1 w_1, v = v_1, w = w_1 \). The strict transform of \( F = 0 \) has the local equation

\[
F^1 = u_1^{\bar{\sigma}} v_1^{\bar{\nu}} - \frac{1}{\lambda} = 0
\]

at \( q_2 \). Thus \( q_2 \notin U(R^1) \).
We have a similar analysis to the above paragraph if \( b \geq c \). In this case we blow up the 2-curve with local equations \( v = w = 0 \) at \( q \). There is at most a single point \( q_1 \in \Phi_{-1}^{-1}(q) \cap U(R^1) \). We have a reduction in a local equation \( F^1 = F_{R^1}^{-1}(q_1) = 0 \) of the strict transform of \( F = 0 \) at \( q_1 \) of the form \( (96) \) of \( \bar{\tau} \), or \( \bar{\tau} \) stays the same but we have a reduction in \( a + b \).

The final case which we must consider is when \( a + b \geq c \) and \( a, b < c \). Let \( \Phi: Y_1 \to Y \) be the blow up of the 2-curve with local equation \( u = w = 0 \) at \( q \). There are two 3-points \( q_1, q_2 \in \Phi_1^{-1}(q) \). \( q_1 \) has regular parameters \( u_1, v_1, w_1 \) defined by \( u = u_1, v = v_1, w = u_1w_1 \). The strict transform of \( F = 0 \) has the local equation

\[
F^1 = v_1^\pi - \frac{1}{\lambda}u_1^\pi w_1^{\pi - \pi} = 0
\]

at \( q_1 \), so that \( q_1 \in U(R^1) \) and we have a reduction in \( \bar{\tau} \) in the expression of \( F^1 = F_{R^1}(q_1) \) of the form of \( (96) \). \( q_2 \) has regular parameters \( u_1, v_1, w_1 \) defined by \( u = u_1w_1, v = v_1, w = w_1 \). The strict transform of \( F = 0 \) has the local equation

\[
F^1 = u_1^\pi - \lambda u_1^\pi b_1 = 0
\]

at \( q_2 \). Thus \( q_2 \in U(R^1) \) and we have a drop in \( \bar{\tau} \) in \( F^1 = F_{R^1}(q_2) \) in \( (96) \).

By descending induction on the above invariants, always performing one of the above blow ups if \( \nu \) is not resolved, we can construct the desired morphism \( Y_\nu \to Y \) such that \( \nu \) is resolved on \( Y_\nu \).

Now by compactness of the Zariski-Riemann manifold \([Z1]\), there exist finitely many

\[
Y_1, \ldots, Y_n \in \{Y_\nu \mid \nu \text{ is a 0-dimensional valuation of } K(Y)\}
\]

such that if \( R^1, \ldots, R^n \) are the respective transforms of \( R \) on \( Y_1, \ldots, Y_n \) and \( \nu \) is a 0-dimensional valuation of \( K(Y) \), then the center of \( \nu \) on some \( Y_i \) is not in \( U(R^i) \). By Lemma 3.11, there exists a sequence of blow ups of 2-curves \( \Phi: Y_\nu \to Y \) such that there exist morphisms \( \Phi_i: Y_i \to Y_i \) for all \( i \) which factor \( \Phi \). In particular, the transform of \( R \) on \( Y_\nu \) is resolved. \( \square \)

**Theorem 6.10.** Suppose that \( f: X \to Y \) is prepared and that \( R \) is a 3-point relation for \( f \). Then there exists a commutative diagram,

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are products of blow ups of 2-curves, \( f_1 \) is prepared and the transform \( R^1 \) of \( R \) for \( f_1 \) is defined and is resolved. In particular, all 3-points of \( X_1 \) in \( \Phi^{-1}(T(R)) \) must map to 2-points of \( Y_1 \). Furthermore, \( \tau_{f_1}(p_1) = \tau_f(\Phi(p_1)) \) if \( p_1 \in X_1 \) is a 3-point. If \( D_X \) is cuspidal for \( f \) then \( D_{X_1} \) is cuspidal for \( f_1 \).

**Proof.** Let \( \{R_i\} \) be the pre-relations on \( Y \) associated to \( R \). By Theorem 6.9 there exists a sequence of blow ups of 2-curves \( \Phi: Y_1 \to Y \) such that the pre-relations \( \{R_i\} \) which are the transforms of \( \{R_i\} \) on \( Y_1 \) are resolved on \( Y_1 \). By Lemma 5.2 there exists a sequence of blow ups of 2-curves \( \Phi: X_1 \to X \) such that \( f_1 = \Psi^{-1} \circ f \circ \Phi: X_1 \to Y_1 \) is a prepared morphism, and if \( D_X \) is cuspidal for \( f \), then \( D_{X_1} \) is cuspidal for \( f_1 \). Furthermore, \( \tau_{f_1}(p_1) = \tau_f(\Phi(p_1)) \) if \( p_1 \in X_1 \) is a 3-point.

Suppose that \( p_1 \in X_1 \) is a 3-point. Then \( p = \Phi(p_1) \) is a 3-point. Suppose that \( u, v, w \) are permissible parameters at \( q = f(p) \) and \( x, y, z \) are permissible parameters for \( u, v, w \) at \( p \). Then there exist permissible parameters \( x_1, y_1, z_1 \) for \( u, v, w \) at \( p_1 \).
such that

\[
\begin{align*}
x & = x_1^{b_{11}} y_1^{b_{12}} z_1^{b_{13}} \\
y & = x_1^{b_{21}} y_1^{b_{22}} z_1^{b_{23}} \\
z & = x_1^{b_{31}} y_1^{b_{32}} z_1^{b_{33}}
\end{align*}
\]

(97)

with \(\text{Det}(b_{ij}) = \pm 1\). We have (after possibly exchanging \(u, v, w\)) an expression of the form (23) at \(p\). On substitution of (97) into (23) we see that we have an expression

\[
\begin{align*}
u & = x_1^{\tau} y_1^{\tau} z_1^{\tau} \\
v & = x_1^{\tau} y_1^{\tau} z_1^{\tau} \\
w & = \sum \alpha_i M_i + N
\end{align*}
\]

(98)
at \(p_1\) of the form of (23).

We have a factorization

\[
Y_1 = Y' \xrightarrow{\Psi_1} Y'_{1-1} \to \cdots \to Y' \xrightarrow{\Psi_1} Y'_{0} = Y
\]

where each \(\Psi_i\) is the blow up of a 2-curve \(C_i\).

Let \(f_i : X_1 \to Y'_1\) be the resulting maps.

Let \(R_i\) be the primitive relations associated to \(R\), and let \((R'_i)^j\) be the transform of the pre-relation \(R_i\) on \(Y'_j\). By induction on \(j\), we will show that the transform \((R_i')^j\) of \(R\) for \(f_j\) is defined. It suffices to verify this for \((R'_1)^1\). Suppose that (with the notation of Definition 6.7), \(p_1 \in \Phi^{-1}(T(R_i))\) is a 3-point. Let \(p = \Phi(p_1), q = f(p), \bar{q} = f(p_1) \in T(R_i),\) so that \(q \in U(R_i)\) is a 3-point. Then \(\tau_f(p) \geq 1,\) and

\[
w = x_1^{\bar{\tau}} y_1^{\bar{\tau}} z_1^{\bar{\tau}} \Lambda
\]

(99)

where \(\Lambda\) is a unit series in (98).

Let \(F = F_{\bar{R}_i}(q) = 0\) be the expression of (94) which determines \(\bar{R}_i(q)\). Then we have (with the notation of (92), (98), (99) and Definition 6.7) that

\[
a(\bar{\sigma}, \bar{\tau}, \bar{\tau}) + b(\bar{\rho}, \bar{\tau}, \bar{\tau}) + c(\bar{\gamma}, \bar{\tau}, \bar{\tau}) = (0, 0, 0)
\]

and \(\Lambda(0, 0, 0) = \lambda = \lambda_{\bar{p}}.\) After possibly interchanging \(u\) and \(v,\) permissible parameters at \(\bar{q}\) are \(u_1, v_1, w_1\) with

\[
u = u_1, v = u_1(v_1 + \alpha), w = w_1
\]

(100)

for some \(\alpha \in k,\) or

\[
u = u_1, v = v_1, w = u_1(w_1 + \alpha)
\]

(101)

for some \(\alpha \in k\) or

\[
u = u_1 w_1, v = v_1, w = w_1.
\]

(102)

Substituting the expressions (100), (101) or (102) into the expression \(F = 0\) of (94) and computing the strict transform of \(F = 0\) at \(\bar{q},\) we see that if \(\bar{q}\) is a 3-point, then \(\bar{\sigma} = 0\) in (100), \(\bar{q}\) is on the strict transform of \(F = 0, q \in U((R'_1)^1),\) and the transform \((R'_1)^1\) of \(R\) for \(f_1\) is defined. By induction on the number of 2-curves blown up by \(\Psi,\) we see that the transform \(R_1^1\) of \(R\) for \(f_1\) is defined. Since \(U(R_1^1) = \emptyset, R_1^1\) is resolved.\]
7. WELL PREPARED MORPHISMS

Suppose that \( f : X \to Y \) is a birational, prepared morphism of nonsingular projective 3-folds with toroidal structures \( D_Y \) and \( D_X = f^{-1}(D_Y) \). Further suppose that the fundamental locus of \( f \) is contained in \( D_Y \). If \( R \) is a 2-point relation for \( f \) with associated 2-point pre-relations \( \{ \overline{R}_i \} \), then for \( p \in T(\overline{R}_i) \) we have that

\[
R(p) = \left( \begin{array}{l}
S_1 = S_R(p), E_1 = E_{R,1}(p), E_2 = E_{R,2}(p), w_i = w_R(p), u = u_R(p), \\
v = v_R(p), e_i = e_R(p), a_i = a_R(p), b_i = b_R(p), \overline{\lambda}_i = \lambda_R(p)
\end{array} \right)
\]

which we will abbreviate (as in (88)) as

\[
R(p) = w_i^{e_i} - \overline{\lambda}_i u^{a_i} v^{b_i},
\]

with \( e_i > 1 \), if \( a_i, b_i \neq -\infty \), or (as in (90))

\[
R(p) = w_i
\]

if \( a_i, b_i = -\infty \). In this case, \( u, v, w_i \) have a monomial form at \( p \). Recall that if \( p' \in T(R) \) is such that \( f(p') = f(p) \), then \( u_R(p') = u_R(p) = u \) and \( v_R(p') = v_R(p) = v \).

Let \( I \) be an index set for the \( \{ \overline{R}_i \} \) associated to \( R \).

**Definition 7.1.** Suppose that \( \tau \geq 1 \). A prepared morphism \( f : X \to Y \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \) if:

1. \( p \in X \) a 3-point implies \( \tau_f(p) \leq \tau \).
2. \( T(R) \) is the set of 3-points \( p \) on \( X \) such that \( \tau_f(p) = \tau \).
3. Suppose that \( p \in T(R) \). Then \( \tau > 1 \) implies \( R(p) \) has a form (104), \( \tau = 1 \) implies \( R(p) \) has a form (105).
4. If \( q \in U(\overline{R}_i) \) \( \cap U(\overline{R}_j) \), then there exists \( \lambda_{ij}(u,v) \in k[[u,v]] \), with

\[
\begin{align*}
\lambda_{ij}(u,v) &= w_{\overline{R}_i}(q) = v_{\overline{R}_j}(q), \\
v &= v_{\overline{R}_j}(q), w_i = w_{\overline{R}_i}(q),
\end{align*}
\]

such that

\[
w_j = w_i + \lambda_{ij}(u,v).
\]

where \( w_i = w_{\overline{R}_i}(q), w_j = w_{\overline{R}_j}(q) \).

5. Suppose that \( q \in U(\overline{R}_i) \), where \( \overline{R}_i \) is a 2-point relation associated to \( R \). Then \( u = v_{\overline{R}_i}(q), v = v_{\overline{R}_j}(q), w_i = w_{\overline{R}_i}(q) \) are super parameters at \( q \) (Definition 5.5).

**Definition 7.2.** \( f : X \to Y \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \) and pre-algebraic structure if \( f \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \) and \( w_{\overline{R}_i}(q) \in \mathcal{O}_{Y,q} \) for all \( \overline{R}_i \) associated to \( R \), and \( q \in U(\overline{R}_i) \).

**Definition 7.3.** \( f : X \to Y \) is \( \tau \)-well prepared with 2-point relation \( R \) if

1. \( f \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \) and pre-algebraic structure.
2. The primitive pre-relations \( \{ \overline{R}_i \} \) associated to \( R \) are algebraic, and \( R \) is algebraic (Definition 6.6).
3. Suppose that \( q \in U(\overline{R}_i) \) \( \cap U(\overline{R}_j) \). Let \( w_i = w_{\overline{R}_i}(q) \) and \( w_j = w_{\overline{R}_j}(q), u = v_{\overline{R}_i}(q) = v_{\overline{R}_j}(q) = v_{\overline{R}_i}(q) \). Then there exists a unit series \( \phi_{ij} \in k[[u,v]] \) and \( a_{ij}, b_{ij} \in \mathbb{N} \) (or \( \phi_{ij} = 0 \) with \( a_{ij} = b_{ij} = \infty \)) with

\[
w_j = w_i + u^{a_{ij}} v^{b_{ij}} \phi_{ij},
\]

4. For \( q \in U(R) \), set \( I_q = \{ i \mid q \in U(\overline{R}_i) \} \). Then the set

\[
\{(a_{ij}, b_{ij}) \mid i, j \in I_q\}
\]

from equation (106) is totally ordered.
Definition 7.4. Suppose that \( f : X \rightarrow Y \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \).

1. A 2-point \( q \in U(R) \) is prepared for \( R \).
2. A 2-point \( q \in Y \) such that \( q \notin U(R) \) is prepared for \( R \) if there exist super parameters \( u,v,w \) at \( q \) (where \( w \) could be formal).
3. A 2-curve \( C \subset Y \) is prepared for \( R \).

If \( E \) is a component of \( D_Y \), \( \mathcal{R}_i \) is pre-algebraic, and \( q \in U(\mathcal{R}_i) \), we will denote \( \overline{E \cdot S_{\mathcal{R}_i}(q)} \) for the Zariski closure in \( Y \) of the curve germ \( u = w = 0 \) at \( q \), where \( u = 0 \) is a local equation of \( E \), \( w = 0 \) is an (algebraic) local equation of \( S_{\mathcal{R}_i}(q) \) at \( q \).

Definition 7.5. Suppose that \( f : X \rightarrow Y \) is \( \tau \)-well prepared with 2-point relation \( R \) for \( f \). A nonsingular curve \( C \subset D_Y \) which makes SNCs with \( D_Y \) is prepared for \( R \) of type 4 if

1. \( C = \overline{E_\alpha \cdot S_{\mathcal{R}_i}(q_\beta)} \) for some component \( E_\alpha \) of \( D_Y \), pre-relation \( \mathcal{R}_i \) associated to \( R \) and \( q_\beta \in U(\mathcal{R}_i) \).
2. \( \Omega(\mathcal{R}_i) \) contains \( C \).
3. If \( C' = \overline{E_\gamma \cdot S_{\mathcal{R}_i}(q_\delta)} \) is such that \( C' \subset \Omega(\mathcal{R}_j) \), \( C \neq C' \), and \( q \in C \cap C' \), then \( q \in U(\mathcal{R}_i) \cap U(\mathcal{R}_j) \) and \( C' = \overline{E_\gamma \cdot S_{\mathcal{R}_j}(q)} \).
4. If \( j \neq i \) and \( C = \overline{E_\gamma \cdot S_{\mathcal{R}_j}(q_\delta)} \) then \( C \) satisfies 1 and 2 and 3 of this definition (for \( \mathcal{R}_j \)). (In this case we have by (95) that \( U(\mathcal{R}_j) \cap C = U(\mathcal{R}_i) \cap C \).
5. Let \( I_C = \{ j \in I \mid C = \overline{E_\gamma \cdot S_{\mathcal{R}_j}(q_\delta)} \) for some \( \mathcal{R}_j, E_\gamma, q_\delta \in U(\mathcal{R}_j) \} \).

Suppose that \( q \in C \) is a 1-point or a 2-point such that \( q \notin U(R) \). Then there exist \( u,v \in O_{Y,q} \) such that for \( j \in I_C \), there exists \( \hat{w}_j \in O_{Y,q} \) such that

(a) \( \hat{w}_j = 0 \) is a local equation of \( \Omega(\mathcal{R}_j) \) and \( u,v,\hat{w}_j \) are permissible parameters at \( q \) such that \( u = \hat{w}_j = 0 \) are local equations of \( C \) at \( q \).
(b) If \( q \) is a 1-point and \( p \in f^{-1}(q) \), then there exists a relation of one of the following forms for \( u,v,\hat{w}_j \) at \( p \).

(i) \( p \) a 1-point

\[
\begin{align*}
u &= x^a \\
v &= y \\
\hat{w}_j &= x^\gamma(x,y) + x^dz
\end{align*}
\]

where \( \gamma \) is a unit series (or zero),

(ii) \( p \) a 2-point

\[
\begin{align*}
u &= (x^a y^b)^k \\
v &= z \\
\hat{w}_j &= (x^a y^b)^l \gamma(x^a y^b, z) + x^dz
\end{align*}
\]

where \( \gamma \) is a unit series (or zero) and \( ad - bc \neq 0 \).
(c) If \( q \) is a 2-point, then \( u,v,\hat{w}_j \) are super parameters at \( q \).
(d) If \( i,j \in I_C \) and \( q \) is a 1-point, there exist relations

\[
\hat{w}_i - \hat{w}_j = u^{\phi_{ij}}(u,v)
\]

where \( \phi_{ij} \) is a unit series (or \( \phi_{ij} = 0 \) and \( c_{ij} = \infty \)).
(e) If \( i,j \in I_C \) and \( q \) is a 2-point (with \( q \notin U(R) \)) then there exist relations

\[
\hat{w}_i - \hat{w}_j = u^{\phi_{ij}}(u,v)
\]
where \( \phi_{ij} \) is a unit series (or \( \phi_{ij} = 0 \) and \( c_{ij} = d_{ij} = \infty \)), and the set \( \{(c_{ij}, d_{ij})\} \) is totally ordered.

If \( f : X \to Y \) is \( \tau \)-well prepared with 2-point relation \( R \), and \( \overline{R}_i \) is a pre-relation associated to \( R \), we will feel free to replace \( \Omega(\overline{R}_i) \) with an open subset of \( \Omega(\overline{R}_i) \) containing \( U(\overline{R}_i) \), and all curves \( C = \overline{E \cdot S_{\overline{R}_i}(q)} \) such that \( E \) is a component of \( D_Y \), \( q \in U(\overline{R}_i) \) and \( C \) is prepared for \( R \) of type 4. This convention will allow some simplification of the statements of the theorems and proofs.

**Definition 7.6.** \( f : X \to Y \) is \( \tau \)-very-well prepared with 2-point relation \( R \) if

1. \( f \) is \( \tau \)-well prepared with 2-point relation \( R \).
2. If \( E \) is a component of \( D_Y \) and \( q \in U(\overline{R}_i) \cap E \), then \( C = \overline{E \cdot S_{\overline{R}_i}(q)} \) is prepared for \( R \) of type 4 (Definition 7.5).
3. For all \( \overline{R}_i \) associated to \( R \), let
   \[
   V_i(Y) = \left\{ \gamma = \overline{E_{\alpha} \cdot S_{\overline{R}_i}(q_\gamma)} \mid q_\gamma \in U(\overline{R}_i), E_{\alpha} \text{ is a component of } D_Y \right\}.
   \]
   Then
   \[
   F_i = \sum_{\gamma \in V_i(Y)} \gamma
   \]
   is a SNC divisor on \( \Omega(\overline{R}_i) \) whose intersection graph is a tree.

If \( f : X \to Y \) is \( \tau \)-very-well prepared, we will feel free to replace \( \Omega(\overline{R}_i) \) with an open neighborhood of \( F_i \) in \( \Omega(\overline{R}_i) \). This will allow some simplification of the proofs.

**Remark 7.7.** Suppose that \( f : X \to Y \) is \( \tau \)-very well prepared. Then it follows from Definition 7.6 and (95) that \( F_i \cap U(R) = U(\overline{R}_i) \) for all \( \overline{R}_i \) associated to \( R \).

**Definition 7.8.** Suppose that \( f : X \to Y \) is \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared or \( \tau \)-very-well prepared) with 2-point relation \( R \). Let \( \{\overline{R}_i\} \) be the pre-relations associated to \( R \). Suppose that \( G \) is a point or nonsingular curve in \( Y \) which is an admissible center for all of the \( \overline{R}_i \). Then \( G \) is called a permissible center for \( R \) if there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]  

(110)

where \( \Psi \) is the blow up of \( G \) and \( \Phi \) is a sequence of blow ups

\[
X_1 = X_n \to \cdots \to X_1 \to X
\]

of nonsingular curves and 3-points \( \gamma_i \) which are possible centers such that

1. \( f_1 \) is prepared and the assumptions of Definition 6.8 hold so that the transform \( R^1 \) of \( R \) for \( f_1 \) is defined.
2. \( f_1 : X_1 \to Y_1 \) is \( \tau \)-quasi-well prepared, (or \( \tau \)-well prepared or \( \tau \)-very-well prepared) with 2-point relation \( R^1 \).

(110) is called a \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared or \( \tau \)-very-well prepared) diagram of \( R \) (and \( \Psi \)).
**Definition 7.9.** Suppose that \( f : X \to Y \) is \( \tau \)-well prepared (or \( \tau \)-very-well prepared) with 2-point relation \( R \) and \( C \subset Y \) is prepared for \( R \) of type 4. Then \( C \) is a \( * \)-permissible center for \( R \) if there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y \\
\end{array}
\tag{111}
\]

such that

1. \( f_1 \) is prepared and the assumptions of Definition 6.8 hold so that the transform \( R^3 \) of \( R \) for \( f_1 \) is defined.
2. \( f_1 : X_1 \to Y_1 \) is \( \tau \)-well prepared (or \( \tau \)-very-well prepared).
3. (111) has a factorization

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y \\
\end{array} \tag{112}
\]

where \( \Psi_1 \) is the blow up of \( C \),

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 & \downarrow & \Psi_1 \\
X & \xrightarrow{f} & Y \\
\end{array} \tag{113}
\]

is a \( \tau \)-well prepared diagram of \( R \) and \( \Psi_1 \) of the form (110), each \( \Psi_{i+1} : Y_{i+1} \to Y_i \) for \( i \geq 1 \) is the blow up of a 2-point \( q \in Y_i \) which is prepared for the transform \( R^i \) of \( R \) on \( X_i \) of type 2 of Definition 7.4, and

\[
\begin{array}{ccc}
X_{i+1} & \xrightarrow{f_{i+1}} & Y_{i+1} \\
\Phi_{i+1} & \downarrow & \Psi_{i+1} \\
X_i & \xrightarrow{f_i} & Y_i \\
\end{array}
\]

is a \( \tau \)-well prepared diagram of \( R^i \) and \( \Psi_{i+1} \) of the form of (110).

**Definition 7.10.** Suppose that \( f : X \to Y \) is \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared or \( \tau \)-very-well prepared) with 2-point relation \( R \). Suppose that

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y \\
\end{array} \tag{114}
\]
is a commutative diagram such that there is a factorization

\[
\begin{array}{c}
\begin{array}{ccc}
X_1 &=& \overline{X}_m \\
\downarrow f_1 &=& \downarrow \overline{f}_m \quad = \quad Y_1 \\
\vdots &=& \downarrow \\
X_2 &=& \overline{X}_2 \\
\downarrow \\
X_1 &=& \overline{X}_1 \\
\downarrow \\
X &=& \overline{X}
\end{array}
\end{array}
\]  

where each commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
X_{i+1} &=& Y_{i+1} \\
\Phi_{i+1} &=& \downarrow \Psi_{i+1} \\
X_i &=& Y_i
\end{array}
\end{array}
\]

is either of the form (110) or of the form (111). Then (114) is called a \(\tau\)-quasi-well prepared (or \(\tau\)-well prepared or \(\tau\)-very-well prepared) diagram of \(R\) (and \(\Psi\)).

**Lemma 7.11.** Suppose that \(f : X \to Y\) is \(\tau\)-quasi-well prepared (or \(\tau\)-well prepared or \(\tau\)-very-well prepared) and \(C \subset Y\) is a 2-curve. Then \(C\) is a permissible center for \(R\), and there exists a \(\tau\)-quasi-well-prepared (or \(\tau\)-well prepared or \(\tau\)-very-well prepared) diagram (110) of \(R\) and the blow up \(\Psi : Y_1 \to Y\) of \(C\) such that \(\Phi\) is a product of blow ups of 2-curves. Furthermore,

1. If \(D_X\) is cuspidal for \(f\) then \(D_{X_1}\) is cuspidal for \(f_1\).
2. Further suppose that \(f\) is \(\tau\)-well prepared. Then
   (a) Let \(E\) be the exceptional divisor for \(\Psi\). Suppose that \(q \in U(\overline{R}_i) \cap E\) for some \(\overline{R}_i\) associated to \(R\). Let \(\gamma_i = \frac{1}{\overline{R}_i(q)} : E\). Then \(\gamma_i = \Psi^{-1}(\Psi(q))\) is a prepared curve for \(R^1\) of type 4.
   (b) If \(\gamma\) is a prepared curve for \(R\), then the strict transform of \(\gamma\) on \(Y_1\) is a prepared curve for \(R^1\).
3. \(\Phi\) is an isomorphism over \(f^{-1}(Y - \Sigma(Y))\)

**Proof.** By Lemma 5.2, there exists a commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
X_1 &=& Y_1 \\
\Phi &=& \downarrow \Psi \\
X &=& Y
\end{array}
\end{array}
\]

where \(\Phi\) is a product of blow ups of 2-curves and \(f_1\) is prepared, with the property that \(\tau_{f_1}(p_1) = \tau_f(\Phi(p_1))\) if \(p_1 \in X_1\) is a 3-point. We further have that \(D_{X_1}\) is cuspidal for \(f_1\) if \(D_X\) is cuspidal for \(f\), and \(\Phi\) is an isomorphism over \(f^{-1}(Y - \Sigma(Y))\).

Let \(\{\overline{R}_i\}\) be the 2-point pre-relations on \(Y\) associated to \(R\). \(C\) is an admissible center for the \(\{\overline{R}_i\}\) (Definition 6.4). Let \(\{\overline{R}_i^1\}\) be the transforms of the \(\{\overline{R}_i\}\) on \(Y_1\).

We will show that the conditions of Definition 6.8 hold so that we can define the transform \(R^1\) of \(R\) for \(f_1\). Suppose that \(q_1 \in U(\overline{R}_i^1)\), and \(p_1 \in f_1^{-1}(q_1) \cap \Phi^{-1}(T(\overline{R}_i))\) is a 3-point. Let \(q = \Psi(q_1), p = \Phi(p_1)\). There exist permissible parameters \(u = u_R(p), v = v_R(p), w = w_R(p)\) at \(q\) such that \(R(p) = \overline{R}_i(q)\) is determined by

\[w^c - \lambda u^a v^b\]
if \( a = a_R(p), b = b_R(p) \neq -\infty \), and by
\[
w = 0
\]
if \( a = a_R(p) = b = b_R(p) = -\infty \). There exist permissible parameters \( x, y, z \) for \( u, v, w \) at \( p \) such that an expression (23) of Definition 3.9 holds for \( u, v, w \) and we have a relation
\[
w^e = u^a v^b \Lambda(x, y, z)
\]
where \( \Lambda(x, y, z) \) is a unit series with \( \Lambda(0, 0, 0) = \lambda \) if \( a, b \neq -\infty \) and \( u, v, w \) have a monomial form in \( x, y, z \) if \( a = b = -\infty \). After possibly interchanging \( u \) and \( v \), we may assume (since \( q_1 \) is a 2-point) that \( q_1 \) has permissible parameters \( \overline{u}, \overline{v}, w \) such that
\[
u = \overline{u}, v = \overline{v}.
\]
Since \( p_1 \) is a 3-point, \( \hat{O}_{X_1, p_1} \) has regular parameters \( x_1, y_1, z_1 \) such that
\[
x = x_1^{a_{11}} y_1^{a_{12}} z_1^{a_{13}}
\]
\[
y = x_1^{b_{11}} y_1^{b_{12}} z_1^{b_{13}}
\]
\[
z = x_1^{c_{11}} y_1^{c_{12}} z_1^{c_{13}}
\]
with \( \text{Det}(a_{ij}) = \pm 1 \). Thus \( \overline{u}, \overline{v}, w \) has an expression of the form of (23) in \( x_1, y_1, z_1 \). If \( a, b \neq -\infty \) we have the relation
\[
w^e = \overline{u}^{a+b} \overline{v}^{b} \Lambda,
\]
and \( R^l_1(q_1) \) is determined by
\[
w^e = \overline{u}^{a+b} \overline{v}^{b} \lambda.
\]
If \( a = b = -\infty \), \( \overline{u}, \overline{v}, w \) have a monomial expression in \( x_1, y_1, z_1 \). The transform \( R^l \) of \( R \) for \( f_1 \) is thus defined.

Now we will verify that \( f_1 \) is \( \tau \)-quasi-well prepared. From the above calculations, we see that \( f_1 \) satisfies 1, 2, 3 and 4 of Definition 7.1. It remains to verify that 5 of Definition 7.1 holds.

Suppose that \( q_1 \in U(R^l_1) \) and \( p_1 \in f^{-1}_1(q_1) \). Then
\[
u = u_{R^l_1}(q), v = v_{R^l_1}(q), w = w_{R^l_1}(q)
\]
are super parameters at \( q = \Psi(q_1) \), and \( p = \Phi(p_1) \in f^{-1}(q) \) has permissible parameters \( x, y, z \) for \( u, v, w \) such that one of the forms of Definition 5.5 hold for \( u, v, w \) and \( x, y, z \).

After possibly interchanging \( u \) and \( v \), we have
\[
u_{R^l_1}(q_1) = \overline{u} = u
\]
\[
u_{R^l_1}(q_1) = \overline{v} = \frac{v}{u}
\]
\[
u_{R^l_1}(q_1) = w.
\]
We can verify that there exist permissible parameters \( x_1, y_1, z_1 \) at \( p_1 \) such that \( \overline{u}, \overline{v}, w \) have one of the forms of Definition 5.5 in \( x_1, y_1, z_1 \). The most difficult case to verify is when \( p_1 \) is a 1-point and \( p \) is a 3-point. Then \( \hat{O}_{X_1, p_1} \) has regular parameters \( \overline{x}_1, \overline{y}_1, \overline{z}_1 \) defined by
\[
x = \overline{x}_1(\overline{y}_1 + \overline{z}_1)^{\overline{p}}(\overline{y}_1 + \overline{z}_1)^{\overline{q}}
\]
\[
y = \overline{x}_1(\overline{y}_1 + \overline{z}_1)^{\overline{p}}(\overline{y}_1 + \overline{z}_1)^{\overline{q}}
\]
\[
w = \overline{x}_1^0(\overline{y}_1 + \overline{z}_1)^{\overline{p}}(\overline{y}_1 + \overline{z}_1)^{\overline{q}}
\]
where $\pi, \beta, \gamma \in k$ are nonzero and

$$\text{Det} \begin{pmatrix} \pi & \beta & \gamma \\ d & e & f \\ j & k & l \end{pmatrix} = \pm 1.$$ 

We substitute into

$$u = x^a y^b z^c,$$

$$v = x^d y^f z^j,$$

$$w = x^g y^h z^i \gamma + x^j y^k z^l$$

of 4 of Definition 5.5. Using the fact that

$$\text{rank} \begin{pmatrix} a & b & c \\ d & e & f \\ j & k & l \end{pmatrix} = 3,$$

we can make a change of variables in $x_1, y_1, z_1$ to get permissible parameters $x_1, y_1, z_1$ at $p_1$ satisfying

$$u = x_1^\pi,$$

$$v = x_1^\gamma + y_1,$$

$$x_1^j y_1^k z_1^l = x_1^\varepsilon + z_1$$

with $\varepsilon > \pi$ and $0 \neq \varepsilon, \gamma \in k$.

For each of the monomials $M$ in the series $x^g y^h z^i \gamma$ we have a relation

$$M^e = u^a v^b$$

with $e, a, b \in Z$. On substitution of (117) and (118) into (119) we see that

$$M = x_1^e \phi_1(y_1)$$

where $\phi_1$ is a unit series. Thus $\pi, \pi, w$ have an expansion of the form 1 of Definition 5.5 in terms of $x_1, y_1, z_1$.

The other cases can be verified by a similar but simpler argument to show that $\pi, \pi, w$ are super parameters at $q_1$. Thus 5 of Definition 7.1 holds for $f_1$, so that $f_1$ is $\tau$-quasi-well prepared.

Suppose that $f$ is $\tau$-well prepared. We will verify that $f_1$ is $\tau$-well prepared. 1 and 2 of Definition 7.3 are immediate. We must verify that 3 and 4 of Definition 7.3 hold for $f_1$.

Suppose that $q_1 \in U(R_i) \cap U(R_j)$. Let $q = \Phi(q_1) \in U(R_i) \cap U(R_j)$, and

$$u = u_{R_i}(q) = u_{R_j}(q),$$

$$v = v_{R_i}(q) = v_{R_j}(q),$$

$$w_i = w_{R_i}(q), w_j = w_{R_j}(q).$$

We have a relation

$$w_j = w_i + u^{a_{ij}} v^{b_{ij}} \phi_{ij}(u, v),$$

from (106) for $f$. After possibly interchanging $u$ and $v$ we have permissible parameters

$$\pi = u_{R_i}^{-1}(q_1) = u_{R_j}^{-1}(q_1),$$

$$\varpi = v_{R_i}^{-1}(q_1) = v_{R_j}^{-1}(q_1),$$

$$w_i = w_{R_i}^{-1}(q_1), w_j = w_{R_j}^{-1}(q_1)$$

at $q_1$, where $u = \pi, v = \varpi$. We have

$$w_j = w_i + \overline{w}^{a_{ij}} + b_{ij} \overline{v}^{b_{ij}} \phi_{ij}(\overline{w}, \overline{v})(121)$$
so 3 of Definition 7.3 holds for $f_1$.

Since the set (107) of Definition 7.3 is totally ordered for $q \in U(\mathcal{R})$, it follows from (121) that the corresponding set (107) for $q_1 \in U(\mathcal{R}_1)$ is totally ordered. Thus 4 of Definition 7.3 holds for $f_1$ and $\mathcal{R}_1$ and $f_1$ is $\tau$-well prepared.

We now verify 2 of Lemma 7.11. Suppose that $f_1$ is $\tau$-well prepared and (110) is a $\tau$-well prepared diagram of $R$ and $\Psi$. Let $E = \Psi^{-1}(C)$ be the exceptional divisor of $\Psi$. Suppose that $q_1 \in U(\mathcal{R}_1) \cap E$. Let $q = \Psi(q_1)$. There exist permissible parameters

$$u = v_{\mathcal{R}_1}(q), v = v_{\mathcal{R}_1}(q), w_i = w_{\mathcal{R}_1}(q)$$

at $q$ such that $u = v = 0$ are local equations of $C$, and after possibly interchanging $u$ and $v$,

$$\overline{u} = u_{\mathcal{R}_1}(q_1) = u, \overline{v} = v_{\mathcal{R}_1}(q_1) = \frac{v}{u}, \overline{w_i} = w_{\mathcal{R}_1}(q_1) = w_i$$

are permissible parameters at $q_1$. Since $\overline{u}$ is a local equation of $E$ at $q_1$, $\gamma_1 = \overline{\gamma}_1$. Let $\gamma = \gamma_1$.

We will verify that $\gamma_1$ is prepared for $R_1$ of type 4. Since $q \in \Omega(\mathcal{R}_1)$, $C$ intersects $\Omega(\mathcal{R}_1)$ transversally at $q$ (and possibly a finite number of other points), and $\Omega(\mathcal{R}_1)$ is the strict transform of $\Omega(\mathcal{R}_1)$ by $\Psi$, we have that $\gamma_1 \in \Omega(\mathcal{R}_1)$. Thus 2 of Definition 7.5 holds. Suppose that for some $j$, component $E_\alpha$ of $D_{Y_j}$ and $q_\beta \in U(\mathcal{R}_j)$,

$$\gamma' = E_\gamma \cdot \mathcal{S}_{\mathcal{R}_1}(q_\beta) \subset \Omega(\mathcal{R}_j),$$

$\gamma \neq \gamma'$ and there exists $\overline{q} \in \gamma \cap \gamma'$. Let $E_1 = \Psi(E_\alpha)$, a component of $D_Y$. Then $\overline{q} = \Psi(\gamma')$ is a curve on $D_Y$ through $q$. Since $\gamma' \subset \Omega(\mathcal{R}_j)$, we must have $\overline{q} \subset \Omega(\mathcal{R}_j) \cap E_1$, so that $q \in U(R) \cap \overline{q} = U(\mathcal{R}_j) \cap \overline{q}$, and thus $\overline{q} = E_1 \cdot \mathcal{S}_{\mathcal{R}_1}(q)$. We thus have that $E_\alpha$ is the strict transform of $E_1$, $\overline{q} = E_\alpha \cdot \gamma \in U(\mathcal{R}_j) \cap U(\mathcal{R}_1)$ and $\gamma' = E_\alpha \cdot \mathcal{S}_{\mathcal{R}_1}(q)$. Thus 3 of Definition 7.5 holds.

Suppose that $\gamma' = E_\gamma \cdot \mathcal{S}_{\mathcal{R}_1}(q_\beta)$ and $\gamma = \gamma'$. Then we must have $\gamma' = E \cdot \mathcal{S}_{\mathcal{R}_1}(q_1)$ and 4 of Definition 7.5 holds.

Since for $q \in U(\mathcal{R}_j)$, $U(\mathcal{R}_j)$ contains both 2-points above $q$ in $Y_1$, we need only verify 5 of Definition 7.5 at 1-points $\overline{q} \in \gamma$. Let

$$I_\gamma = \{j \mid \gamma = E \cdot \mathcal{S}_{\mathcal{R}_1}(q_1)\}.$$ 

At $\overline{q}$ there exist regular parameters $\tilde{u}, \tilde{v}, \tilde{w}_j$ (for all $j \in I_\gamma$) such that

$$u = \tilde{u}, v = \tilde{v}(\tilde{v} + \alpha), w_j = \tilde{w}_j$$

(122)

where $u = u_{\mathcal{R}_1}(q) = u_{\mathcal{R}_1}(q), v = v_{\mathcal{R}_1}(q) = v_{\mathcal{R}_1}(q), w_j = w_{\mathcal{R}_1}(q)$ and $0 \neq \alpha \in \mathbb{k}$.

Thus 5 (a) of Definition 7.5 holds for $u, \tilde{v}, \tilde{w}_j$ at $\overline{q}$.

As in our verification that $f_1$ is $\tau$-quasi-well prepared, we see that if $p \in f_1^{-1}(\overline{q})$, then there exist permissible parameters $x, y, z$ for $u, v, w_j$ at $p$ such that one of the forms of Definition 5.5 hold for $u, v, w_j$, and $x, y, z$. Substituting in (122), we see that (since $\alpha \neq 0$) $u, v, w_j$ must satisfy a form 1 or 3 of Definition 5.5 at $p$, and $\tilde{u}, \tilde{v}, \tilde{w}_j$ must satisfy one of the forms (i) or (ii) of 5 (b) of Definition 7.5. Thus 5 (b) of Definition 7.5 holds at $p$. Substituting (122) into (120), we see that 5 (d) of Definition 7.5 holds at $\overline{q}$. Thus $\gamma = \gamma_1$ is prepared for $R^1$ of type 4.

We now verify that if $\gamma$ is a prepared curve for $R$ on $Y$ then the strict transform $\gamma'$ of $\gamma$ on $Y_1$ is a prepared curve for $R^1$. We may assume that $\gamma_1$ is prepared of type 4 and $\gamma \cap C \neq \emptyset$. The verification that $\gamma'$ is prepared for $f_1$ now follows from a local calculation at points $q \in C \cap \gamma$. 


Suppose that \( f \) is \( \tau \)-very-well prepared. We have seen that \( f_1 \) is \( \tau \)-well prepared, so that 1 of Definition 7.6 holds for \( f_1 \), and 2 of Definition 7.6 holds for \( f_1 \) by our verification of 2 of this lemma. Let \( \Psi_i : \Omega(\mathcal{R}_i^1) \to \Omega(\mathcal{R}_i) \) be the restriction of \( \Psi \) to \( \Omega(\mathcal{R}_i^1) \). Then \( \Psi_i \) is the blow up of the union of nonsingular points \( C \cdot \Omega(\mathcal{R}_i) \) on the nonsingular surface \( \Omega(\mathcal{R}_i) \). Thus since

\[
F_i = \sum_{\gamma \in V_i(Y)} \gamma
\]

is a SNC divisor on \( \Omega(\mathcal{R}_i) \) whose intersection graph is a tree,

\[
\Psi^{-1}(F_i) = \sum_{\gamma' \in V_i(Y_1)} \gamma'
\]

is a SNC divisor on \( \Omega(\mathcal{R}_i^1) \) whose intersection graph is a tree and 3 of Definition 7.6 holds for \( f_1 \). Thus \( f_1 \) is \( \tau \)-very-well prepared. \( \square \)

**Remark 7.12.** The proof of Lemma 7.11 shows that if \( f : X \to Y \) is \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared or \( \tau \)-very-well prepared), \( C \subset Y \) is a 2-curve, \( \Psi : Y_1 \to Y \) is the blow up of \( C \) and \( \Phi : X_1 \to X \) is a sequence of blow ups of 2-curves and 3-points such that the rational map \( f_1 : X_1 \to Y_1 \) is a morphism, then

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

is \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared or \( \tau \)-very-well prepared) for \( R \) and \( \Psi \). If \( D_X \) is cuspidal for \( f \), then \( D_X \) is cuspidal for \( f_1 \). In fact, with the above notation, if \( f \) satisfies 1 - 4 of Definition 7.1, then \( f_1 \) satisfies 1 - 4 of Definition 7.1.

**Lemma 7.13.** Suppose that \( f : X \to Y \) is \( \tau \)-quasi-well-prepared (or \( \tau \)-well-prepared or \( \tau \)-very-well-prepared) and \( q \in U(R) \) is a 2-point (prepared of type 1 in Definition 7.4). Then \( q \) is a permissible center for \( R \), and there exists a \( \tau \)-quasi-well-prepared (or \( \tau \)-well-prepared or \( \tau \)-very-well-prepared) diagram (110) of \( R \) and the blow up \( \Psi : Y_1 \to Y \) of \( q \) such that:

1. Suppose that \( D_X \) is cuspidal for \( f \). Then \( D_X \) is cuspidal for \( f_1 \).
2. Suppose that \( f \) is \( \tau \)-well-prepared. Then
   (a) Let \( E \) be the exceptional divisor of \( \Psi \). Suppose that \( q_1 \in U(\mathcal{R}_{i}) \cap E \). Let \( \gamma_i = \sum_{R_{i}}(q_1) \cdot E \). Then \( \gamma_i \) is a prepared curve for \( R_1 \) of type 4. Suppose that \( q' \in U(\mathcal{R}_{i}^1) \cap E \). Let \( \gamma_j = \sum_{R_j}(q') \cdot E \). Then either
      (i) \( \gamma_i = \gamma_j \) or
      (ii) \( \gamma_i, \gamma_j \) intersect transversally at a 2 point on \( E \) (their tangent spaces have distinct directions at this point and are otherwise disjoint).
   (b) If \( \gamma \) is a prepared curve on \( Y \) then the strict transform of \( \gamma \) is a prepared curve on \( Y_1 \).
3. \( \Phi \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \).

**Proof:** There exists a pre-relation \( \mathcal{R}_i \) associated to \( R \) such that \( q \in U(\mathcal{R}_i) \subset U(R) \). Fix such an \( i \). Let

\[
u = u_{\mathcal{R}_i}(q), v = v_{\mathcal{R}_i}(q), w_i = w_{\mathcal{R}_i}(q).
\]

\( u, v, w_i \) are super parameters at \( q \), and \( w_i = 0 \) is a local equation of \( S_{\mathcal{R}_i}(q) \). Let \( m_q \subset \mathcal{O}_{Y,q} \) be the maximal ideal.
By Lemma 3.11, there exists a morphism $\Phi_0 : X_0 \rightarrow X$ which is a sequence of blow ups of 2-curves such that $(u, v)\mathcal{O}_{X_0, p}$ is invertible for all $p \in (f \circ \Phi_0)^{-1}(q)$, $\Phi_0$ is an isomorphism over $f^{-1}(Y - \Sigma(Y))$, $f \circ \Phi_0$ is prepared and $u, v, w_i$ are super parameters for $f \circ \Phi_0$.

We will next show that there exists a sequence of blow ups of 2-curves and 3-points $\Phi_1 : X_1 \rightarrow X_0$ such that $(u, v)\mathcal{O}_{X_1, p}$ is invertible at all $p \in (f \circ \Phi_0 \circ \Phi_1)^{-1}(q)$ and if $m_q\mathcal{O}_{X_1, p}$ is not invertible, then we have permissible parameters $x, y, z$ for $u, v, w$ at $p$ of one of the following forms:

$p$ is a 1-point

$$u = x^a, \quad v = x^{b}(\alpha + y), \quad w_i = x^d z$$

with $\alpha \neq 0$ and $d < \min\{a, b\}$ or,

$p$ is a 2-point of the type of (9) of Definition 3.1

$$u = x^a y^b, \quad v = x^c y^d, \quad w_i = x^g y^h (z + \tau)$$

with $\tau \in k, \quad ad - bc \neq 0$, and $(g, h) \leq \min\{(a, b), (c, d)\}$.

In fact, we will construct $\Phi_1 : X_1 \rightarrow X_0$ such that only (123) or (124) occur at points $p_1$ above $q$ such that $m_q\mathcal{O}_{X_1, p_1}$ is not invertible.

Suppose that $p \in (f \circ \Phi_0)^{-1}(q)$ is a 3-point, so that $p$ has a form 4 of Definition 5.5 at $p$. There exist $\pi, \eta, \tau \in \mathcal{O}_{X_0, p}$ and series $\lambda_1, \lambda_2, \lambda_3 \in \hat{\mathcal{O}}_{X_0, p}$ such that $x = \pi \lambda_1$, $y = \eta \lambda_2$, $z = \tau \lambda_3$. Let $I^p \subset \mathcal{O}_{X_0, p}$ be the ideal

$$I^p = (u, v, x^a y^b z^c, x^d y^g z^h).$$

By Lemma 3.13, there exists a sequence of blow ups of 2-curves and 3-points $\Phi_1 : X_1 \rightarrow X_0$ such that $\Phi_0 \circ \Phi_1$ is an isomorphism above $f^{-1}(Y - \Sigma(Y))$ and $I^p\mathcal{O}_{X_1, p}$ is invertible for all 3-points $p \in (f \circ \Phi_0)^{-1}(q)$ and $p_1 \in \Phi_1^{-1}(p)$. Thus if $p \in (f \circ \Phi_0)^{-1}(q)$ is a 3-point and $p_1 \in (f \circ \Phi_0 \circ \Phi_1)^{-1}(p)$, then $m_q\mathcal{O}_{X_1, p_1}$ is invertible or $p_1$ has a form (123) or (124). $f \circ \Phi_0 \circ \Phi_1$ is prepared, and $u, v, w_i$ are super parameters for $f \circ \Phi_0 \circ \Phi_1$.

We construct an infinite sequence of morphisms

$$\cdots \rightarrow X_n \xrightarrow{\Phi_3} \cdots \xrightarrow{\Phi_2} X_2 \xrightarrow{\Phi_1} X_1$$

as follows. Order the 2-curves $C$ of $X_1$ such that $q \in (f \circ \Phi_0 \circ \Phi_1)(C) \subset \Sigma(Y)$. Let $\Phi_2 : X_2 \rightarrow X_1$ be the blow up of the 2-curve $C_1$ on $X_1$ of smallest order. Order the 2-curves $C'$ of $X_2$ such that $q \in (f \circ \Phi_0 \circ \Phi_1 \circ \Phi_2)(C') \subset \Sigma(Y)$ so that the 2-curves contained in the exceptional divisor of $\Phi_2$ have order larger than the order of the (strict transform of the) 2-curves $C$ of $X$ such that $q \in f(C) \subset \Sigma(Y)$. Let $\Phi_3 : Y_3 \rightarrow Y_2$ be the blow ups of the 2-curve $C_2$ on $Y_3$ of smaller order. Let $\Phi_n = \Phi_2 \circ \cdots \circ \Phi_n : Y_n \rightarrow X_1$. The morphisms $f \circ \Phi_0 \circ \Phi_1 \circ \Phi_n$ are prepared, and $u, v, w_i$ are super parameters for $f \circ \Phi_0 \circ \Phi_1 \circ \Phi_n$.

Suppose that $\nu$ is a 0-dimensional valuation of $k(X)$. Let $p_n$ be the center of $\nu$ on $X_n$.

Say that $\nu$ is resolved on $X_n$ if (at least) one of the following holds:

1. $m_q\mathcal{O}_{X_n, p_n}$ is invertible.
2. $p_n$ is a 1-point of the form (123).
3. $p_n$ is a 2-point of the form (124).

If $\nu$ is resolved on $X_n$, with center $p_n$, then there exists an open neighborhood $U$ of $p_n$ in $X_n$ such that a 0-dimensional valuation $\mathcal{V}$ of $k(X)$ is resolved on $X_n$ if the center of $\mathcal{V}$ is in $U$, since 1, 2 or 3 is an open condition on $X_n$. Further, if $n' > n$, then $X_{n'}$ is also resolved at all 0-dimensional valuations $\mathcal{V}$ which are resolved on $X_n$.

If the center of $\nu$ on $X_n$ is a 3-point, then the center of $\nu$ on $X_1$ is also a 3-point, so $m_q\mathcal{O}_{X_n, p_n}$ is invertible.
Suppose that the center of $\nu$ on $X_n$ is a 1-point. Then $u, v, w_1$ have a form 1 of Definition 5.5 at $p_n$, and thus have a form (123) at $p_n$ if $m_q\mathcal{O}_{X_n,p_n}$ is not invertible.

Suppose that the center $p$ of $\nu$ on $X_1$ is a 2-point such that $u, v, w_1$ have a form 2 of Definition 5.5 at $p$. There exist $x, y \in \mathcal{O}_{X_1,p}$ and series $\lambda_1, \lambda_2 \in \mathcal{O}_{X_1,p}$ such that $x = \bar{x}\lambda_1$, $y = \bar{y}\lambda_2$. Let $I^p \subset \mathcal{O}_{X_1,p}$ be the ideal $(u, v, \bar{x}\bar{y}, \bar{x}^2\bar{y}^2)$. There exists an $n$ such that $I^p\mathcal{O}_{X_n,p_1}$ is invertible for all $p_1 \in \mathcal{F}_n^{-1}(p)$. Let $p_1$ be the center of $\nu$ on $X_n$. $u, v, w_1$ are super parameters for $f \circ \Phi_0 \circ \Phi_1 \circ \mathcal{F}_n$ at $q$. Then $p_1$ is either a 1-point (so that $\nu$ is resolved on $X_n$) or a 2-point of the form 2 of Definition 5.5. If $p_1$ is a 2-point and $m_q\mathcal{O}_{X_n,p_1}$ is not invertible, then $u, v, w_1$ have a form (124) (with $\gamma = 0$). Thus $\nu$ is resolved on $X_n$.

Suppose that the center $p$ of $\nu$ on $X_1$ is a 2-point such that $u, v, w_1$ have a form 3 of Definition 5.5 at $p$. Then there exist $x, y \in \mathcal{O}_{X,p}$ and series $\lambda_1, \lambda_2 \in \mathcal{O}_{X,p}$ such that $x = \bar{x}\lambda_1$, $y = \bar{y}\lambda_2$. Let $I^p \subset \mathcal{O}_{X,p}$ be the ideal

\[ I^p = (u, v, (\bar{x}\bar{y})^i, \bar{x}^2\bar{y}^2). \]

There exists an $n$ such that $I^p\mathcal{O}_{X_n,p_1}$ is invertible for all $p_1 \in \mathcal{F}_n^{-1}(p)$. Let $p_1$ be the center of $\nu$ on $X_n$. Then $m_q\mathcal{O}_{X_n,p_1}$ is invertible or $p_1$ is a 1-point of the form (123). Thus $\nu$ is resolved on $X_n$.

By compactness of the Zariski-Riemann manifold $[Z1]$, there exists an $n$ such that every valuation $\nu$ of $k(X)$ is resolved on $X_n$. Let $X_1 = X_n$ and $f_1 = f \circ \Phi_0 \circ \Phi_1 \circ \mathcal{F}_n : X_1 \to Y$. If $m_q\mathcal{O}_{X_1,p_1}$ is not invertible at some $p_1 \in f_1^{-1}(q)$, then one of the forms (123) or (124) must hold at $p_1$.

The locus of points $p$ on $X_1$ where $m_q\mathcal{O}_{X_1,p}$ is not invertible is a (possibly not irreducible) curve $\mathcal{F}$ which makes SNCs with the toroidal structure of $X$. $\mathcal{F}$ is supported at points of the form (123) (with $d < \min(a, b)$) and (124) (with $\gamma = 0$ and $(g, h) < \min\{(a, b), (c, d)\}$). $x = z = 0$ is a local equation of $\mathcal{F}$ in (123). $x = z = 0$, $y = z = 0$ or $xy = z = 0$ are the possible local equations of $\mathcal{F}$ in (124).

For an irreducible component $C$ of $\mathcal{F}$, define an invariant

\[ A(C) = \min\{a, b\} - d > 0 \]

computed at a 1-point $p \in C$ (which has an expression (123)). Let $C$ be a component of $\mathcal{F}$ such that $A(C) = \max_{C \subset \mathcal{F}} A(C)$. $C$ is nonsingular and makes SNCs with $D_{X_1}$. Let $\Phi_2 : X_2 \to X_1$ be the blow up of $C$.

Suppose that $p \in C$, so that $u, v, w_1$ have the form (123) or (124) at $p$. We may assume that $x = z = 0$ are local equations of $C$ at $p$. Suppose that $p_1 \in \Phi_2^{-1}(p)$. $p_1$ has (formal) regular parameters $x_1, y_1, z_1$ defined by

\[ x = x_1, y = y_1, z = x_1(z_1 + \beta) \]

(126)

with $\beta \in k$ or

\[ x = x_1z_1, z = z_1. \]

(127)

Suppose that $p \in C$ is a 1-point, so that (123) holds for $u, v, w_1$ at $p$. Under (126) we have $m_q\mathcal{O}_{X_2,p_1}$ and thus $m_q\mathcal{O}_{X_2,p_1}$ is invertible except possibly if $\beta = 0$. If $m_q\mathcal{O}_{X_2,p_1}$ is not invertible we have

\[ u = x_1^a, v = x_1^b(\alpha + y), w_1 = x_1^{d+1}z_1 \]

with

\[ d + 1 < \min\{a, b\}. \]
Then the curve $C_1$ with local equations $x_1 = z_1$ is a component of the locus where $m_qO_{X_1}$ is not invertible. We have

$$A(C_1) = \min\{a, b\} - (d + 1) < A(C).$$

Under (127) we have a 2-point

$$u = (x_1 z_1)^a, v = (x_1 z_1)^b(\alpha + y), w_i = x_1^{d_i} z_1^{d_i+1}$$

and a local equation of the toric structure $D_{X_2}$ is $x_1 z_1 = 0$. Since

$$d + 1 \leq \min\{a, b\},$$

$m_qO_{X_1, p_1}$ is invertible.

Suppose that $p \in C$ is a 2-point, so that (124) holds at $p$ (with $\tau = 0$). We may assume that $x = z = 0$ is a local equation of $C$ at $p$. Then $g < \min\{a, c\}$. Let $p_1 \in \Phi_2^{-1}(p)$. $p_1$ has regular parameters $x_1, y_1, z_1$ defined by (126) or (127).

Under the substitution (126) $u, v, w$ have a form (124) at $p_1$. If $m_qO_{X_2, p_1}$ is not invertible, we have $\beta = 0$ and

$$u = x_1^a y_1^b, v = x_1^c y_1^d, w_i = x_1^{g_i+1} y_1^{h_i} z_1$$

which is back in the form (124) with $\tau = 0$.

Under the substitution (127),

$$u = x_1^a y_1^b z_1, v = x_1^c y_1^d z_1, w_i = x_1^{g_i} y_1^{h_i} z_1$$

so that $p_1$ is a 3-point, and since

$$(g + 1, h) \leq \min\{(a, b), (c, d)\},$$

$m_qO_{X_2, p_1}$ is invertible.

Observe that $u, v, w_i$ are super parameters for $f_1 \circ \Phi_2$.

By descending induction on $\max_{\tau \in E}\{A(C)\}$, we construct a sequence of blow ups $\Phi_4 : X_4 \to X_2$ such that for $f_4 = f_1 \circ \Phi_2 \circ \Phi_4 : X_4 \to Y$, $u, v, w_i$ are super parameters at $q$ for $f_4$, and $m_qO_{X_4}$ is invertible. Thus $f_4 : X_4 \to Y$ factors through the blow up $\Psi : Y_1 \to Y$ of $q$. Let $f_3 : X_3 \to Y_1$, $\Phi : X_4 \to X$ be the resulting maps.

Let $q_1 \in \Psi^{-1}(q)$. We obtain permissible parameters $\bar{u}, \bar{v}, \bar{w}$ at $q_1$ of one of the following forms:

1. $q_1$ a 1-point

$$u = \bar{u}, v = \bar{u}(\bar{v} + \alpha), w_i = \bar{u}(\bar{w} + \beta)$$

with $\alpha, \beta \in k$, $\alpha \neq 0$. In this case there are no 3-points in $f_4^{-1}(q_1)$ and $\bar{u}, \bar{v}$ are toroidal forms at all points $p \in f_4^{-1}(q_1)$.

2. $q_1$ a 2-point

$$u = \bar{u}, v = \bar{u} \bar{v}, w_i = \bar{u}(\bar{w} + \alpha)$$

with $\alpha \in k$, or

$$u = \bar{u} \bar{v}, v = \bar{v}, w_i = \bar{v}(\bar{w} + \alpha)$$

with $\alpha \in k$, or

3. $q_1$ a 3-point

$$u = \bar{u} \bar{w}, v = \bar{u} \bar{w}, w_i = \bar{w}. $$

If $q_1$ has the form (129) or (130) and $p \in f_4^{-1}(q_1)$ then $\bar{u}, \bar{v}$ are toroidal forms at $p$.

Suppose that $q_1$ has the form (131). Let $p \in f_4^{-1}(q_1)$. $u, v, w_i$ have one of the forms 1 - 4 of Definition 5.5 at $p$. Since $w_i$ must divide $u$ and $v$, we certainly have that $\bar{u}, \bar{v}, \bar{w}$ are monomials in the local equations of the toroidal structure at $p$, times unit series.
Suppose that \( u, v, w_i \) have a form 1 of Definition 5.5 at \( p \). We have an expression
\[
\begin{align*}
u &= x^a \\
v &= x^b(\alpha + y) \\
w_i &= x^c(\tilde{\gamma}(x, y) + x^d z)
\end{align*}
\]
where \( 0 \neq \alpha \in k, c < a, c < b, \tilde{\gamma} \) is a unit series and \( \tilde{d} \geq 0 \). Set \( \pi = x(\tilde{\gamma} + x^d z)^{\tilde{\gamma}} \). We have expansions
\[
\begin{align*}
w_i &= \pi^c \\
u &= \pi^a(\tilde{\gamma} + x^d z)^{-\tilde{\gamma}} \\
v &= \pi^b(\tilde{\gamma} + x^d z)^{-\tilde{\gamma}}(\alpha + y)
\end{align*}
\]
at \( p \).

If \( \tilde{d} > 0 \) and \( \frac{\partial \tilde{\gamma}}{\partial y}(0, 0) = 0 \), then there exist \( \overline{y}, \overline{z} \in \hat{\mathcal{O}}_{X_4, p} \) such that \( \overline{x}, \overline{y}, \overline{z} \) are regular parameters in \( \hat{\mathcal{O}}_{X_4, p} \) and
\[
\begin{align*}
w_i &= \pi^c \\
u &= \pi^a(\pi + \overline{y}) \\
u &= \pi^b(\pi + \overline{y})
\end{align*}
\]
and \( w_i, \pi \) are toroidal forms at \( p \).

If \( \tilde{d} = 0 \) or \( \frac{\partial \tilde{\gamma}}{\partial y}(0, 0) \neq 0 \), then there exist \( \overline{y}, \overline{z} \in \hat{\mathcal{O}}_{X_4, p} \) such that \( \overline{x}, \overline{y}, \overline{z} \) are regular parameters in \( \hat{\mathcal{O}}_{X_4, p} \) and
\[
\begin{align*}
w_i &= \pi^c \\
u &= \pi^a(\pi + \overline{y}) \\
u &= \pi^b(\pi + \overline{y})
\end{align*}
\]
where \( 0 \neq \pi \in k \) and \( \tilde{\gamma} \) is a unit series. Then
\[
\begin{align*}
w_i &= \pi^c \\
u &= \pi^a(\pi + \overline{y}) \\
u &= \pi^b(\pi + \overline{y})
\end{align*}
\]
and \( w_i, \pi \) are toroidal forms at \( p \).

Suppose that \( u, v, w_i \) have a form 3 of Definition 5.5 at \( p \). There are two cases. Either
\[
\begin{align*}
u &= (x^a y^b)^k \\
v &= (x^a y^b)^t(\alpha + z) \\
w_i &= x^c y^d
\end{align*}
\]
with \( 0 \neq \alpha \in k \) and \( ad - bc \neq 0 \), or
\[
\begin{align*}
u &= (x^a y^b)^k \\
v &= (x^a y^b)^t(\alpha + z) \\
w_i &= (x^a y^b)^{\tilde{\gamma}}
\end{align*}
\]
with \( 0 \neq \alpha \in k, l \leq \min\{k, t\} \) and \( \tilde{\gamma} \) is a unit series.

If (132) holds then \( \overline{x}, \overline{w}_i \) are toroidal forms at \( p \) of the form of (9) of Definition 3.1. Assume that (133) holds. We then have
\[
\begin{align*}
\overline{x} &= (x^a y^b)^{k-l\tilde{\gamma}^{-1}} \\
\overline{v} &= (x^a y^b)^{t-l\tilde{\gamma}^{-1}}(\alpha + z) \\
\overline{w}_i &= (x^a y^b)^{\tilde{\gamma}}.
\end{align*}
\]
If \( \frac{\partial^2}{\partial x^2}(0,0,0) \neq 0 \), then there exist regular parameters \( \pi, \eta, \tau \) at \( p \) such that
\[
\begin{align*}
\pi &= (x^a y^b)^{k-l} \\
\eta_i &= (x^a y^b)^l (\beta + \tau) \\
\tau &= (x^a y^b)^{t-l} \hat{\gamma}(\pi, \eta, \tau)
\end{align*}
\]
where \( 0 \neq \beta \in k \) and \( \hat{\gamma} \) is a unit series. Thus \( \pi, \eta, \tau \) are toroidal forms at \( p \).

If \( \frac{\partial^2}{\partial x^2}(0,0,0) = 0 \) then there exist regular parameters \( \pi, \eta, \tau \) at \( p \) such that
\[
\begin{align*}
\pi &= (x^a y^b)^{k-l} \\
\eta_i &= (x^a y^b)^l (\beta + \tau) \\
\eta_i &= (x^a y^b)^{t-l} \hat{\gamma}(\pi, \eta, \tau)
\end{align*}
\]
where \( 0 \neq \beta \in k \) and \( \hat{\gamma} \) is a unit series. Thus \( \pi, \eta, \tau \) are toroidal forms at \( p \).

For all \( \gamma \), let \( \Phi_0 \circ \Phi_1 : X_1 \to X \) which we constructed is a product of blow ups of 2-curves and 3-points. Then we have
\[
\begin{align*}
\pi, \eta_i, \eta, \tau &= \text{super parameters for } f, u, v, w_j\text{ are super parameters for } f \circ \Phi_0 \circ \Phi_1.
\end{align*}
\]

First assume that \( p \in C \) is a 1-point. Then, we have
\[
\begin{align*}
u &= x^a, v = x^b(\alpha + y), w_i = x^d z
\end{align*}
\]
with \( 0 \neq \alpha \in k \) and \( d < \min\{a, b\} \). Thus, since
\[
\begin{align*}
w_j &= w_i + \lambda_{ij}(u, v), \\
w_j &= x^d \tau \text{ where } \tau = z + x \Omega(x, y) \text{ for some series } \Omega. \text{ It follows that } x = \tau = 0 \text{ are local equations of } C \text{ at } p, \text{ and } u, v, w_j \text{ are super parameters for } f \circ \Phi_1 \circ \Phi_2.
\end{align*}
\]

Now assume that \( p \in C \) is a 2-point. Then we have
\[
\begin{align*}
u &= x^a y^b, v = x^c y^d, w_i = x^d y^h z
\end{align*}
\]
where after possibly interchanging \( u, v \), we have \( (g, h) < (a, b) \leq (c, d) \) (recall that \( (u, v) \mathcal{O}_{X_1, p} \) is invertible). Since \( w_j - w_i \in k[[u, v]] \), we have an expression
\[
w_j = x^d y^h \tau
\]
at \( p \) where \( x = \tau = 0 \) are local equations of \( C \) at \( p \). It follows that \( u, v, w_j \) are super parameters for \( f \circ \Phi_0 \circ \Phi_1 \circ \Phi_2 \).

By induction, \( u, v, w_j \) are super parameters for \( \mathcal{F}_4 = f \circ \Phi_0 \circ \Phi_1 \circ \Phi_2 \circ \Phi_4 \).

\( q \) is an admissible center for all 2-point relations \( \mathcal{R}_j \) associated to \( R \) (Definition 6.4).

For all \( j \), let \( \mathcal{R}_j \) be the transform of \( \mathcal{R}_j \) on \( Y_1 \). Suppose that \( p_1 \in \mathcal{F}_4^{-1}(q) \cap \mathcal{R}_j^{-1}(T(\mathcal{R}_j)) \)
is a 3-point. \( p = \Phi(p_1) \in T(R_j) \) is a 3-point with permissible parameters \( x, y, z \) such that

\[
\begin{align*}
u &= x^a y^b z^c \\
v &= x^d y^e z^f \\
w_j &= M_0 \gamma
\end{align*}
\]

(135)

where \( \text{rank}(u, v) = 2 \), \( \gamma(x, y, z) \) is a unit series, \( M_0 \) is a monomial in \( x, y, z \) and \( w_j = w_{R_j}(q) \). Let \( w_j^{e_j} = \lambda_j u^{a_j} v^{b_j} \) define \( R_j(q) \) if \( \tau > 1 \), \( w_j = 0 \) define \( R_j(q) \) if \( \tau = 0 \). If \( \tau > 1 \), then

\[
M_j^{e_j} = u^{a_j} v^{b_j} \text{ and } \gamma^{e_j}(0, 0, 0) = \lambda_j,
\]

(136)

and \( \text{rank}(u, v, M_0) = 3 \) if \( \tau = 1 \). By its construction, \( \Phi \) is a sequence of blow ups of 2-curves and 3-points above \( p \).

Thus since \( p_1 \) is a 3-point, we have permissible parameters \( x_1, y_1, z_1 \) at \( p_1 \) such that

\[
\begin{align*}x &= x_1^{a_{11}} y_1^{a_{12}} z_1^{a_{13}} \\
y &= y_1^{a_{21}} y_1^{a_{22}} z_1^{a_{23}} \\
z &= z_1^{a_{31}} y_1^{a_{32}} z_1^{a_{33}}
\end{align*}
\]

and \( \text{Det}(a_{ij}) = \pm 1 \). On substitution into (135) we see that an expression

\[
\begin{align*}u &= x_1^{a_{11}} y_1^{a_{12}} z_1^{a_{13}} \\
v &= x_1^{a_{21}} y_1^{a_{22}} z_1^{a_{23}} \\
w_j &= M_0 \gamma
\end{align*}
\]

(137)

where \( \text{rank}(x_1, y_1, z_1) = 2 \), holds at \( p_1 \) for \( u, v, w_j \), and the relation (136) holds at \( p_1 \) if \( \tau > 1 \), and \( \text{rank}(x_1, y_1, z_1) = 3 \) if \( \tau = 1 \).

Suppose that \( q_1 \in U(R_j^1) \cap \Psi^{-1}(q) \). After possibly interchanging \( u \) and \( v \), \( q_1 \) has permissible parameters \( \overline{u}, \overline{v}, \overline{w}_j \) with

\[
\begin{align*}u &= \overline{u}, v = \overline{v}, w_j = \overline{w}_j
\end{align*}
\]

(138)

The pre-relation \( R_j^1(q_1) \) is then defined if \( \tau > 1 \) by

\[
\overline{w}_j^{e_j} = \lambda_j \overline{u}^{a_j} + b_j - e_j \overline{v}^{b_j},
\]

(139)

and \( \overline{R}_j^1(q_1) \) is defined by \( \overline{w}_j = 0 \) if \( \tau = 1 \). We have seen that if \( p_1 \in f_4^{-1}(q) \cap \Phi^{-1}(T(R_j)) \) is a 3-point, then there are permissible parameters \( x_1, y_1, z_1 \) at \( p_1 \) such that an expansion of the form (137) holds for \( u, v, w_j \), and (136) holds if \( \tau > 1 \). If \( \tau = 1 \), then \( u, v, w_j \) is a monomial form at \( p_1 \). If we also have that \( p_1 \in f_4^{-1}(q_1) \) then we have the expression

\[
\begin{align*}u &= x_1^{a_{11}} y_1^{a_{12}} z_1^{a_{13}} \\
v &= x_1^{a_{21}} y_1^{a_{22}} z_1^{a_{23}} \\
w_j &= M_0 \gamma
\end{align*}
\]

at \( p_1 \), and (if \( \tau > 1 \) (136) becomes

\[
(M_0 \overline{u})^{e_j} = \overline{w}_j^{a_j} + b_j - e_j \overline{v}^{b_j}.
\]

Thus the transform \( R^1 \) of \( R \) for \( f_4 \) (Definition 6.8) is defined. We further have

\[
\tau f_4(p_1) = \tau f(\Phi(p_1)) = \tau.
\]
If \( p_1 \in \mathcal{F}_4^{-1}(q) \cap \Phi^{-1}(T(R)) \) is a 3-point and \( p_1 \not\in f_4^{-1}(U(R)) \) then \( f_4(p_1) = q_1 \) where (after possibly interchanging \( u \) and \( v \)) \( q_1 \) has permissible parameters

\[
\begin{align*}
  u &= \overline{u} \\
  v &= \overline{v}(\tau + \alpha) \\
  w_j &= \overline{w}(\tau + \beta)
\end{align*}
\]  

(140)

\( \alpha, \beta \in k \) and and at least one of \( \alpha, \beta \) is non zero, or

\[
\begin{align*}
  u &= \underline{u} w_j \\
  v &= \underline{v} w_j \\
  w_j &= \underline{w}_j.
\end{align*}
\]  

(141)

Suppose that (140) holds at \( q_1 \). Since \( \text{rank}(u, v) = 2 \) in (137), we must have \( \alpha = 0 \) and \( 0 \neq \beta \). But then \( M_0 = u \), a contradiction to the assumption that \( e_j > 1 \) (and \( \gcd(a_j, b_j, e_j) = 1 \)) in (136) if \( \tau > 1 \), or to the assumption that \( \text{rank}(u, v, M_0) = 3 \) if \( \tau = 1 \).

Suppose that (141) holds at \( q_1 \). Then \( q_1 = f_4(p) \) is a 3-point. From equations (141) and (137) we have (in the notation of Definition 3.9) that \( \tau_{f_4}(p) = -\infty \) if \( \tau = 1 \) and if \( \tau > 1 \) then

\[
H_{f_4,p} = H_{f_4,p} = H_{f,\Phi(p)},
\]

\[
A_{f_4,p} = A_{f,\Phi(p)} + ZM_0
\]

since \( q = f_4(p) = f(\Phi(p)) \) is a 2-point. Thus, since \( e_j > 1 \) in (136), we have

\[
\tau_{f_4}(p) = \left| H_{f_4,p}/A_{f_4,p} \right| < \left| H_{f,\Phi(p)}/A_{f,\Phi(p)} \right| = \tau.
\]

Finally, suppose that \( p \in \mathcal{F}_4^{-1}(q) \cap \Phi^{-1}(T(R)) \) is a 3-point. Suppose that \( \tilde{p} = \Phi_2 \circ \Phi_4(p) \) is a 2-point. Then \( u, v, w_i \) have a form (124) at \( \tilde{p} \), and (with \( \tau = 0 \)) \( u, v, w_i \) have a form (128) at \( p \), with \( (g + 1, h) \leq \min\{(a, b), (c, d)\} \). We see (since \( (u, v) \mathcal{O}_{X_4,p} \) is invertible) that \( w_i \mid u, w_i \mid v \) at \( p \), and thus \( f_4(p) \) is a 3-point with permissible parameters \( \overline{u}, \overline{v}, \overline{w}_i \) defined by

\[
\begin{align*}
  u &= \underline{u} w_i \\
  v &= \underline{v} w_i \\
  w_i &= \underline{w}_i
\end{align*}
\]

Thus, \( \overline{u}, \overline{v}, \overline{w}_i \) have a toroidal form at \( p \), so that \( \tau_{f_4}(p) = -\infty < \tau \).

Suppose that \( \tilde{p} = \Phi_2 \circ \Phi_4(p) \) is a 3-point. Then \( \Phi_2 \circ \Phi_4 \) is the identity near \( p \), and thus \( \Phi(p) \) is a 3-point and \( \overline{u} \) factors as a sequence of blow ups of 2-curves and 3-points near \( p \). Thus \( \tau_{f_4}(p) \leq \tau_f(\Phi(p)) < \tau \). Thus 1, 2 and 3 of Definition 7.1 hold for \( f_4 \).

Now we verify 4 of Definition 7.1. Suppose that \( q_1 \in U(R^1) \). Let \( q = \Psi(q_1) \in U(R) \). Since \( f \) is \( \tau \)-quasi-well prepared, there exists \( w_q \in \mathcal{O}_{Y,q} \) satisfying 4 of Definition 7.1 for \( f \). If \( q_1 \in U(R^1) \cap U(R) \) then (after possibly interchanging \( u \) and \( v \)) we have

\[
\begin{align*}
  u &= \overline{u} \\
  v &= \overline{v} \\
  w_i &= \overline{w}_i \\
  w_j &= \overline{w}_j
\end{align*}
\]

where

\[
\begin{align*}
  u &= u_R(q) = u_R(q), v = v_R(q) = v_R(q), w_i = w_R(q), w_j = w_R(q)
\end{align*}
\]

and

\[
\begin{align*}
  \overline{u} &= u_R(q_1) = u_R(q_1), \overline{v} = v_R(q_1) = v_R(q_1), \overline{w}_i = w_R(q_1), \overline{w}_j = w_R(q_1).
\end{align*}
\]

Since \( f \) is \( \tau \)-quasi-well prepared, there exists a series \( \lambda_{ij}(u, v) \) such that

\[
\lambda_{ij}(u, v) = w_j = w_i + \lambda_{ij}(u, v).
\]
Since $\lambda_{ij}(0,0) = 0$, $\nu | \lambda_{ij}(\nu,\nu\nu)$, and
\[\nu_j = \nu_i + \frac{\lambda_{ij}(\nu,\nu\nu)}{\nu} .\]
Thus 4 of Definition 7.1 holds for $f_4$.

Earlier in the proof we verified that if $q \in U(\overline{R}_j)$, for some $\overline{R}_j$ associated to $R$, then
\[u = u_{\overline{R}_j}(q), v = v_{\overline{R}_j}(q), w_j = w_{\overline{R}_j}(q)\]
are super parameters for $\overline{f}_4$. If $q_1 \in U(\overline{R}_j) \cap \Psi^{-1}(q)$, then (after possibly interchanging $u$ and $v$) we have permissible parameters
\[\nu = u_{\overline{R}_j}(q_1), \nu = v_{\overline{R}_j}(q_1), \nu_j = w_{\overline{R}_j}(q_1)\]
such that
\[u = \nu, v = \nu, w_j = \nu_j .\]
Substituting into the forms of Definition 5.5, we see that $\nu, \nu, \nu_j$ are super parameters for $f_4$ at $q_1$. Thus 5 of Definition 7.1 holds for $f_4$ and $f_4$ is $\tau$-quasi-well prepared.

We now verify that $D_{X_4}$ is cuspidal for $f_4$ if $D_X$ is cuspidal for $f$ (this is 1 of the conclusions of the lemma). Since the property of being cuspidal is stable under blow ups of 2-curves and 3-points, it suffices to show that if $C$ is a component of $E$ on $D_{X_4}$, such that $A(C) > 0$, $C$ contains no 2-points, and $p \in C$ then there exists a Zariski open neighborhood $U$ of $p$ in $X_1$ such that $f_4$ is toroidal on $(\Phi_2 \circ \Phi_4)^{-1}(U)$.

There exist permissible parameters $x, y, z$ at $p$ such that (after possibly interchanging $u$ and $v$) we have expressions
\[u = u_{\overline{R}_j}(q) = x^a, v = v_{\overline{R}_j}(q) = x^b(\alpha + y), w_j = w_{\overline{R}_j}(q) = x^d z\]
with $0 \neq \alpha, d < a \leq b$ and $x = z = 0$ are local equations of $C$ at $p$.

We consider the effect of the blow up of $C$, $\Phi_2 : X_2 \to X_1$. If $p_1 \in \Phi_2^{-1}(p)$ then $p_1$ has regular parameters $x_1, y_1, z_1$ of one of the forms
\[x = x_1, y = y_1, z = x_1(z_1 + \gamma)\]
with $\gamma \in k$, or
\[x = x_1z_1, y = y_1, z = z_1 .\]
Under (144) we have a local factorization of the rational map $X_1 \to Y_1$, at $p$, by
\[\nu_{w_1} = x_1^{a-d}z_1^{a-d-1}, w_j = x_1^{d+1}, v_{w_i} = x_1^{b-d}z_1^{b-d-1}(\alpha + y_1)\]
which is toroidal (of the form 2 of Definition 3.7).

Under (143) we obtain a form
\[u = x_1^a, v = x_1^b(\alpha + y), w_i = x_1^{d+1}(z_1 + \gamma)\]
which gives a toroidal factorization of $X_1 \to Y_1$, at $p$ (of the form 3 or 5 following Definition 3.7), except if $\gamma = 0$ and $x_1 = z_1 = 0$ are local equations of a curve $C_1$ with $0 < A(C_1) < A(C)$, and such that $C_1$ contains no 2-points (or 3-points). By successive blowing up of curves in the construction of $\Phi_2 : X_4 \to X_2$, we see that $X_4 \to Y_1$ is a toroidal morphism on $(\Phi_2 \circ \Phi_4)^{-1}(U)$ for some Zariski open neighborhood $U$ of $p$.

Now suppose that $f$ is $\tau$-well prepared. We will verify that $f_4$ is $\tau$-well prepared. 1 and 2 of Definition 7.3 are immediate.
We will verify that 3 of Definition 7.3 holds for $f_4$. Suppose that $q_1 \in U(R^{1}_i) \cap U(R^{1}_j)$ and
\[
\begin{align*}
\overline{u} &= u_{R^{1}_i}^{-1}(q_1) = u_{R^{1}_j}^{-1}(q_1), \\
\overline{v} &= v_{R^{1}_i}^{-1}(q_1) = v_{R^{1}_j}^{-1}(q_1), \\
\overline{w}_i &= w_{R^{1}_i}^{-1}(q_1), \\
\overline{w}_j &= w_{R^{1}_j}^{-1}(q_1).
\end{align*}
\]
Let $q = \Psi(q_1)$,
\[
\begin{align*}
u &= u_{R^{1}_i}(q), \\
v &= v_{R^{1}_i}(q), \\
w_i &= w_{R^{1}_i}(q), \\
w_j &= w_{R^{1}_j}(q).
\end{align*}
\]
Since $f$ is $\tau$-well prepared, there exist unit series $\phi_{ij}(u, v)$ such that
\[
w_j = w_i + u^{a_{ij} + b_{ij}} \phi_{ij}
\]
(or $\phi_{ij} = 0$ and $a_{ij} = b_{ij} = \infty$). After possibly interchanging $u$ and $v$, we may assume that
\[
u = \overline{u}, v = \overline{w}, w_i = \overline{w}_i, w_j = \overline{w}_j.
\]
Let
\[
\overline{\phi}_{ij}(\overline{u}, \overline{v}) = \phi_{ij}(\overline{u}, \overline{w}).
\]
Then we have
\[
\overline{w}_j = \overline{w}_i + \overline{w}_j \overline{w}_i \overline{\phi}_{ij}
\]
where
\[
\phi_{ij} = a_{ij} + b_{ij} - 1, \quad \overline{\phi}_{ij} = b_{ij}.
\]
Thus 3 of Definition 7.3 holds for $f_4$.

Since the set (107) associated to $q$ and $R$ is totally ordered, the set (107) associated to $q_1$ and $R^3$ is also totally ordered. Thus 4 of Definition 7.3 holds for $f_4$, and we see that $f_4$ is $\tau$-well prepared.

We now verify 2 of Lemma 7.13. Suppose that $q_1 \in U(R^{1}_i) \cap E$ for some $R^{1}_i$ associated to $R^3$. Continuing with the notation we used in the verification that $f_4$ is $\tau$-well prepared, let $\gamma_i = \overline{S}_{R^{1}_i}(q_1) \cdot E$. $\gamma_i$ is covered by two affine charts, with uniformizing parameters $\overline{u}, \overline{v}, \overline{w}$ defined by
\[
u = \overline{u}, v = \overline{w}, w_i = \overline{w}_i \overline{w}_j
\]
and
\[
u = \tilde{u} \overline{w}, v = \tilde{v}, w_i = \tilde{w}_i \overline{w}_j.
\]
In the chart (147), $\overline{u} = 0$ is a local equation for $E$, $\overline{w}_i = w_{R^{1}_i}(q_1) = 0$ is a local equation for $S_{R^{1}_i}(q_1)$ and $\overline{w}_j = 0$ is a local equation for the strict transform of the component $E_2$ of $D_Y$ with local equation $v = 0$ at $q$. In (148), $\tilde{v} = 0$ is a local equation of $E$ and $\tilde{u} = 0$ is a local equation of the strict transform of the component $E_1$ of $D_Y$ with local equation $u = 0$ at $q$. In the chart defined by (147) $\overline{w}_i = \overline{w}_j = 0$ are local equations of $\gamma_i$ and in the chart defined by (148) $\tilde{w}_i = \tilde{w}_j = 0$ are local equations of $\gamma_i$. Thus $\gamma_i$ makes SNCs with $D_{Y_{ij}}$, and $\gamma_i$ is a line on $E \cong \mathbb{P}^2$.

If $q \in U(R^{1}_j)$ for some $j \neq i$, then we see from (145) and (146) that $\gamma_j = \overline{S}_{R^{1}_j}(q') \cdot E$ (where $q' \in \Psi^{-1}(q) \cap U(R^{1}_j)$) has local equations $\overline{w} = \overline{w}_j = 0$ in the chart (147) where
\[
\overline{w}_j = \overline{w}_i + \overline{w}_j \overline{\phi}_{ij}.
\]
In the chart (148), $\gamma_j$ has local equations $\tilde{v} = \tilde{w}_j = 0$ where $\tilde{w}_j = \tilde{w}_i + \tilde{w}_j \tilde{w}_i \tilde{\phi}_{ij}$. If $a_{ij} + b_{ij} > 1$, then $\gamma_i = \gamma_j$ so that $2$ (a) (i) holds in the statement of Lemma 7.13. If $a_{ij} + b_{ij} = 1$ then $2$ (a) (ii) of the statement of Lemma 7.13 holds.
We will now verify that $\gamma_i$ is a prepared curve for $R^1$ of type 4. 1 – 4 of Definition 7.5 are immediate from the above calculation.

We now verify that 5 of Definition 7.5 holds for $\gamma_i$. Let

$I_{\gamma_i} = \{ j \mid q \in U(\mathcal{R}_j) \text{ and } \gamma_j = \gamma_i \}.$

We have permissible parameters $u,v,w_j = w_{\mathcal{R}_j}(q)$ at $q$ for $j \in I_{\gamma_i}$. By construction, $\gamma_i \cap U(\mathcal{R}_j)$ is the set of 2-points in $\gamma_i$ for all $j \in I_{\gamma_i}$. Suppose that $q_1 \in \gamma_i$ is a 1-point. Then we have (after possibly interchanging $u$ and $v$) permissible parameters $\overline{u}, \overline{v}, \overline{w}_j$ at $q_1$ for $j \in I_{\gamma_i}$ defined by

$$u = \overline{u}, v = \overline{v}(\overline{v} + \alpha), w_j = \overline{w}_j$$

for some $0 \neq \alpha \in k$, where $\overline{w}_j = 0$ is a local equation of $\Omega(\mathcal{R}_j^1)$ at $q_1$. Thus 5 (a) of Definition 7.5 holds. From the relation

$$w_j - w_i = u^{a_{ij}+b_{ij}} \phi_{ij}(u,v),$$

we have

$$\overline{w}_j - \overline{w}_i = \overline{w}^{a_{ij}+b_{ij}}(\overline{v} + \alpha)^{b_{ij}} \phi_{ij}(\overline{u}, \overline{v}(\overline{v} + \alpha)).$$

Thus 5 (d) of Definition 7.5 holds.

1 or 3 of Definition 5.5 hold at all

$$p \in f_4^{-1}(q_1) \subset \mathcal{T}^1_4(q)$$

for $u,v,w_j$, and after substitution of (149) into this form, we see that 5 (b) of Definition 7.5 holds. Thus $\gamma_i$ is prepared for $R^1$ of type 4. We have completed the verification of 2 (a) of Lemma 7.13.

We now verify 2 (b) of Lemma 7.13. Suppose that $\gamma$ is prepared for $R$, $q \in \gamma$, and $\gamma$ is prepared for $R$ of type 4. Then $\gamma = E_2 \cdot S_{\mathcal{R}_i}(q)$ for some $\mathcal{R}_i$ and component $E_2$ of $D_Y$ containing $q$ and $\gamma \subset \Omega(\mathcal{R}_i)$. Let $u = u_{\mathcal{R}_i}(q), v = v_{\mathcal{R}_i}(q), w_i = w_{\mathcal{R}_i}(q)$. We may assume that $v = 0$ is a local equation of $E_2$. Then $v = w_i = 0$ are local equations at $q$ of $\gamma$. Let $\gamma'$ be the strict transform of $\gamma$ on $Y_1$. $q_1 = \gamma' \cdot E$ has permissible parameters

$$\overline{u} = u_{\mathcal{R}_i^1}(q_1), \overline{v} = v_{\mathcal{R}_i^1}(q_1), \overline{w}_i = w_{\mathcal{R}_i^1}(q_1),$$

where

$$u = \overline{u}, v = \overline{v}, w_i = \overline{w}_i,$$

and $\overline{v} = w_i = 0$ are local equations of $\gamma'$. Let $\overline{E}_2$ be the strict transform of $E_2$. Since $q_1 \in U(\overline{\mathcal{R}}_i)$, we have $\gamma' = \overline{E}_2 \cdot S_{\overline{\mathcal{R}}_i}(q_1)$ and $\gamma' \subset \Omega(\overline{R}^1_i)$. Thus the conditions of Definition 7.5 hold for $\gamma'$, and $\gamma'$ is prepared for $R^1$. If $\gamma$ is a 2-curve, then the strict transform $\gamma'$ of $\gamma$ on $Y_1$ is a 2-curve so $\gamma'$ is prepared for $R^1$.

Finally, suppose that $f$ is $\tau$-very-well prepared. We have shown that 1 and 2 of Definition 7.6 hold for $f_4$. Since whenever $\overline{\mathcal{R}}_i$ is a pre-relation associated to $R$ containing $q$, $\Omega(\overline{\mathcal{R}}_i)$ is the blow up of a point on a nonsingular surface, and $V_1(Y) = 3$ of Definition 7.6, 3 of Definition 7.6 holds for $V_1(Y_1)$. Thus $f_4$ is $\tau$-very-well prepared.

\[ \square \]

\textbf{Lemma 7.14.} Suppose that $f : X \to Y$ is $\tau$-quasi-well prepared (or $\tau$-well prepared or $\tau$-very-well prepared) with 2-point relation $R$. Suppose that $q \in Y$ is a 2-point such that $q \notin U(R)$ and $q$ is prepared (of type 2 of Definition 7.4) for $R$. Then $q$ is a permissible center for $R$ and there exists a $\tau$-quasi-well prepared (or $\tau$-well prepared or $\tau$-very-well prepared) diagram $(110)$ of $R$ and the blow up $\Psi : Y_1 \to Y$ of $q$ such that:

1. Suppose that $D_X$ is cuspidal for $f$. Then $D_X$ is cuspidal for $f_1$. 

76 STEVEN DALE CUTKOSKY
2. Suppose that \( f \) is \( \tau \)-well prepared. If \( \gamma \) is a prepared curve on \( Y \) then the strict transform of \( \gamma \) is a prepared curve on \( Y_1 \).

3. \( \Phi \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \)

\[ \begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array} \]

of \( R \) of the form of (111). If \( D_X \) is cuspidal for \( f \) then \( D_{X_1} \) is cuspidal for \( f_1 \). \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y) \cup C) \). If \( C \) is contained in the fundamental locus of \( f \) then \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \).

\[ \begin{align*}
\Phi \quad & : \quad E \\
\Phi \quad & : \quad E \\
\Phi \quad & : \quad E
\end{align*} \]

\[ \begin{align*}
u, v, \tilde{w}_i
\end{align*} \]

such that \( u = \tilde{w}_i = 0 \) are local equations at \( \varpi \) for \( C \), with the notation of 5 of Definition 7.5, if \( \varpi \notin U(R_1) \), and \( u = v_{\varpi} (\varpi), \varpi = v_{\varpi} (\varpi), \tilde{w}_i = v_{\varpi} (\varpi) \) if \( \varpi \in U(R_1) \). As in the proof of Lemma 7.13, after blowing up 2-curves and 3-points above \( X \), by a morphism \( \Phi_0 \circ \Phi_1 : X_1 \to X \), with associated morphism \( f_1 = f \circ \Phi_0 \circ \Phi_1 : X_1 \to Y \), we have that the following holds: Suppose that \( \varpi \in C \) is a 2-point, and \( \varpi \in f_1^{-1}(\varpi) \) and \( \mathcal{L}_C \mathcal{O}_{X_1, \varpi} \) is not invertible, then one of the forms (123) or (124) hold at \( \varpi \in f_1^{-1}(\varpi) \) (with \( d < a \) in (123), \( (g, h) < (a, b) \) and \( \varpi = 0 \) in (124)). We have that \( \Phi_0 \circ \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y) \cup C) \) and if \( C \) is contained in the fundamental locus of \( f \), then \( \Phi_0 \circ \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \).

If \( \varpi \in C \) is a 1 point then a form (151) below holds at \( \varpi \in f_1^{-1}(\varpi) \) if \( \mathcal{L}_C \mathcal{O}_{X_1, \varpi} \) is not invertible.

\[ \begin{align*}
u & = x^a \\
v & = y \\
\tilde{w}_i & = x^d z
\end{align*} \]

where \( d < a \) and \( \varpi \) is a 1-point.

As in the proof of Lemma 7.13, the locus of points in \( X_1 \) where \( \mathcal{L}_C \mathcal{O}_{X_1} \) is not invertible is a (possibly reducible) curve \( \mathcal{C} \) which makes SNCs with the toroidal structure of \( X_1 \). As in the proof of Lemma 7.13, we can construct a sequence of blow ups of sections over components of \( \mathcal{C}, X_4 \to X_1 \), such that the resulting map \( \overline{f}_4 : X_4 \to Y \) factors through the blow up \( \Psi_1 : Y_1 \to Y \) of \( C \). Let \( \overline{f} : X_4 \to X \) be the composite map. By construction, if \( u, v, \tilde{w}_i \) are our permissible parameters at \( \varpi \in C \) of (150), and \( \varpi \in \overline{f}_4^{-1}(\varpi) \), we have permissible parameters at \( \varpi \) such that a form 5 (b) of Definition 7.5 holds for \( u, v, \tilde{w}_i \) at \( \varpi \) if \( \varpi \) is a 1-point and \( u, v, \tilde{w}_i \) are super parameters for \( \overline{f}_4 \) if \( \varpi \) is a 2-point.

Let \( f_4 : X_4 \to Y_1 \) be the resulting morphism. As in the proof of Lemma 7.13, we see that \( f_4 \) is prepared.

Suppose that \( \varpi \in C \cap U(R_1) \) for some \( j \). Then \( \varpi \in U(R_1) \), since \( C \) is prepared of type 4. By the hypothesis of this lemma, the germ of \( C \) at \( \varpi \) is contained in \( S_{R_1}(\varpi) \).
Since $C \subset E_\alpha$, the germ of $C$ at $\bar{\eta}$ is $E_\alpha \cdot \mathcal{S}_{\mathcal{R}_j}(\bar{\eta})$. Thus in the forms of (106) of Definition 7.3 for $R$, we have $a_{jk} > 0$ for all $j, k \in I_{\bar{\eta}}$.

$\Psi_1^{-1}(\bar{\eta})$ is covered by 2 affine charts. The first chart has uniformizing parameters $\bar{u}, \bar{v}, \bar{w}_i$ defined by

$$u = \bar{u}, v = \bar{v}, \bar{w}_i = \bar{w}_i, \quad (152)$$

The second chart has uniformizing parameters $u', v', w'_i$ defined by

$$u = u'w'_i, v = v', \bar{w}_i = w'_i. \quad (153)$$

For $j \in I_{\bar{\eta}}$ we have a relation

$$\bar{w}_j = \bar{w}_i + u^{a_{ij}}v^{b_{ij}}\phi_{ij}(u, v)$$

with $a_{ij} > 0$ (where $\bar{w}_j = w_{\mathcal{R}_j}(\bar{\eta})$). As in the proof of Lemma 7.13, it follows that the transform $R^1$ of $R$ for $f_4$ is defined, $f_4$ is $\tau$-quasi-well prepared, and $D_X$ is cuspidal for $f_4$ if $D_X$ is cuspidal for $f$.

We now verify that $f_4$ is $\tau$-well prepared. 1 of Definition 7.3 is immediate.

Suppose that $C \cap U(\mathcal{R}_j) \neq \emptyset$ for some $j$. Then $C = E_\alpha \cdot \mathcal{R}_j(q_j) \subset \Omega(\mathcal{R}_j)$. Since $\Omega(\mathcal{R}_j)$ is nonsingular and makes SNCs with $D_{Y_j}$, $\Omega(\mathcal{R}_j^1)$ is nonsingular, makes SNCs with $D_{Y_j}$, contains $U(\mathcal{R}_j^1)$, and $\Omega(\mathcal{R}_j^1) \cap U(R^1) = U(\mathcal{R}_j^1)$. If $C \cap U(\mathcal{R}_j) = \emptyset$, then after possibly replacing $\Omega(\mathcal{R}_j)$ with a neighborhood of $F_j$ in $\Omega(\mathcal{R}_j)$ (with the notation of Definition 7.6, and following our convention on $\Omega(\mathcal{R}_j)$ stated after Definition 7.6), we have that $\Omega(\mathcal{R}_j) \cap C = \emptyset$. Thus 2 of Definition 7.3 holds.

If $\bar{\eta} \in C \cap U(\mathcal{R}_j)$, then $q_j = U(\mathcal{R}_j^1) \cap f_4^{-1}(\bar{\eta})$ is a 2-point in the chart (152), and

$$w_{\mathcal{R}_j^1}(q_j) = \bar{w}_j = \bar{w}_i + \bar{w}_i^{a_{ij}}v^{b_{ij}}\phi_{ij}(\bar{u}, \bar{v}). \quad (154)$$

Thus $q_j \in U(\mathcal{R}_j^1)$ if and only if $a_{ij} - 1 + b_{ij} > 0$. Let $q_i = U(\mathcal{R}_i^1) \cap f_4^{-1}(\bar{\eta})$, $I_{q_i} = \{ j \mid q_i \in U(\mathcal{R}_j) \}$.

$j \in I_{q_i}$, if and only if $j \in I_{\bar{\eta}}$ and $a_{ij} + b_{ij} > 1$. Suppose that $\tau > 1$, and $j \in I_{q_i}$. If $\mathcal{R}_j(q_j)$ is defined by an expression

$$\bar{w}_j^{c_{ij}} = \lambda_j u^{\alpha_j}v^{\beta_j},$$

then $\mathcal{R}_j^1(q_i)$ is defined by the expression

$$\bar{w}_j^{c_{ij}} = \lambda_j \bar{w}_j^{\alpha_j - c_{ij} - \beta_j}. \quad (155)$$

From equation (154) we see that the set (107) of Definition 7.3 corresponding to $q_i$ and $R^1$ is totally ordered, since the set (107) of Definition 7.3 corresponding to $\bar{\eta}$ and $R$ is totally ordered. In particular, we see that 3 and 4 of Definition 7.3 hold for $f_4$.

We have completed the verification that $f_4$ is $\tau$-well prepared.

Let $E = \Psi_1^{-1}(C)$ and if $C \cap U(\mathcal{R}_j) \neq \emptyset$, let $\gamma_j = \Omega(\mathcal{R}_j^1) \cap E$. $\gamma_j$ is also nonsingular and makes SNCs with $D_{Y_j}$. We have that $\gamma_j = E \cdot S_{\mathcal{R}_j^1}(\bar{q}_j)$ for all $q_j \in E \cap U(\mathcal{R}_j^1)$, and $\gamma_j$ is a section of $E$ over $C$. In particular, 1 and 2 of Definition 7.5 hold for $\gamma_j$.

Suppose that for some $\mathcal{R}_k$ associated to $R$, there exists $\gamma' \subset \Omega(\mathcal{R}_k^1)$, with $\gamma' = E_{E_{\beta_j}} \cdot S_{\mathcal{R}_k^1}(\bar{q}_k)$, and $\gamma' \cap \gamma_j \neq \emptyset$.

First suppose that $\gamma' \subset E$. Then $U(\mathcal{R}_k^1) \cap E \neq \emptyset$ and $\gamma' = \gamma_k = E \cdot S_{\mathcal{R}_k^1}(\bar{q}_2)$ for all $q_2 \in U(\mathcal{R}_k^1) \cap E$. Let $F = E_{\alpha_j}$ be the component of $D_Y$ containing $C$. Since $C$
is prepared and $\Psi_1(\gamma_j) = \Psi_1(\gamma_k) = C$, we have that $C \cap U(\mathcal{R}_j) = C \cap U(\mathcal{R}_k)$ and $C = \mathcal{F} \cdot S_{\mathcal{R}_j}(\eta) = \mathcal{F} \cdot S_{\mathcal{R}_k}(\eta)$ for $\eta \in C \cap U(\mathcal{R}_j)$.

Suppose that $q_2 \in \gamma_j \cap \gamma_k$. Let $\eta = \Psi_1(q_2) \in C$. Let $u, v, \tilde{w}_j$ and $u, v, \tilde{w}_k$ be the permissible parameters at $\eta$ of (150).

Suppose that $\eta$ is a 1-point. We have

$$\hat{w}_j = \tilde{w}_k + u^{c_{jk}} \phi_{jk}(u, v),$$

where $\phi_{jk}$ is a unit series (or $\phi_{jk} = 0$) by 5 (d) of Definition 7.5, since $C$ is prepared for $R$ of type 4. We have that $q_2$ has permissible parameters $\bar{u}, \bar{v}, \bar{w}_j$ where

$$u = \bar{u}, v = \bar{v}, \hat{w}_j = \bar{w}_j \bar{u}. \tag{156}$$

Define $\bar{w}_k$ by $\bar{w}_k = \bar{w}_k \bar{u}$. We have that

$$\bar{w}_j = \bar{w}_k + \bar{u}^{c_{jk}} - 1 \phi_{jk}(\bar{u}, \bar{v}).$$

$\bar{u} = \bar{w}_j = 0$ are local equations of $\gamma_j$ at $q_2$ and $\bar{u} = \bar{w}_k = 0$ are local equations of $\gamma_k$ at $q_2$. Thus, $q_2$ is a 1-point with $c_{jk} - 1 > 0$ and $\gamma_j = \gamma_k$.

Now suppose that $q_2 \in \gamma_j \cap \gamma_k$ and $\eta = \Psi_1(q_2) \in C$ is a 2-point. First suppose that $\eta \notin C \cap U(\mathcal{R}_j) = C \cap U(\mathcal{R}_k)$. We have a relation

$$\hat{w}_j = \tilde{w}_k + u^{c_{jk}} \phi_{jk}(u, v)$$

where $\phi_{ij}$ is a unit series (or $\phi_{ij} = 0$) by 5 (e) of Definition 7.5 for $C$. We have $c_{jk} \geq 1$. $q_2$ has permissible parameters $\bar{u}, \bar{v}, \bar{w}_j$ where

$$u = \bar{u}, v = \bar{v}, \hat{w}_j = \bar{w}_j \bar{u}. \tag{157}$$

Define $\bar{w}_k$ by $\bar{w}_k = \bar{w}_k \bar{u}$. We then have

$$\bar{w}_j = \bar{w}_k + \bar{u}^{c_{jk}} - 1 \phi_{jk}(\bar{u}, \bar{v}). \tag{158}$$

Thus, $q_2$ is a 2-point (and $q_2 \notin U(R^1)$).

Finally, suppose that $q_2 \in \gamma_j \cap \gamma_k$ and $\eta = \Psi_1(q_2) \in C \cap U(\mathcal{R}_j) = C \cap U(\mathcal{R}_k)$. Then $q_2 \in U(\mathcal{R}_j) \cap U(\mathcal{R}_k)$, and we have equations (157) and (158).

Now suppose that $\gamma_j' = E_{\beta} \cdot S_{\mathcal{R}_j}(q_4) \notin E$, and $\gamma_j' \cap \gamma_j \neq \emptyset$. Then there exists a component $G$ of $D_Y$ such that $\eta = \Psi_1(\gamma_j') \subset G$, and $E_{\beta}$ is the strict transform of $G$. Suppose that $q_2 \in \gamma_j \cap \gamma_j'$. $\eta = \Psi_1(q_2) \in \eta \cap C$ implies $\eta \in U(\mathcal{R}_j) \cap U(\mathcal{R}_k)$ and $\eta = G \cdot S_{\mathcal{R}_j}(\eta)$ by 3 of Definition 7.5 for $R$. Thus, $q_2 \in U(\mathcal{R}_j) \cap U(\mathcal{R}_k)$.

Suppose that $q_2 \in \gamma_j$, $\eta = \Psi_1(q_2) \in C$ and $p \in f_4^{-1}(q_2)$. Let $u, v, \hat{w}_j$ be the permissible parameters at $\eta$ of (150). Further suppose that $q_2$ is a 1-point. Let $\bar{u}, \bar{v}, \bar{w}_j$ be the permissible parameters at $q_2$ of (156). $u, v, \hat{w}_j$ satisfy a form 5 (b) of Definition 7.5 at $p$. Substituting (156) into these forms, we see that $\bar{u}, \bar{v}, \bar{w}_j$ satisfy a form of 5 (b) of Definition 7.5 at $p$. Now suppose that $q_2 \in \gamma_j$ is a 2-point, but $q_2 \notin U(R^1)$. Let $\bar{u}, \bar{v}, \bar{w}_j$ be the permissible parameters at $q_2$ of (157). $u, v, \hat{w}_j$ satisfy a form 5 (c) of Definition 7.5 at $p$. Substituting (157) into these forms, we see that $\bar{u}, \bar{v}, \bar{w}_j$ satisfy a form 5 (c) of Definition 7.5 at $p$. Thus 5 of Definition 7.5 holds for $\gamma_j$.

We have seen that the curves $\gamma_j$ only fail to be prepared of type 4 for $f_4$ at a finite set of 2-points $T_1 \subset E$, where condition 3 of Definition 7.5 fails. If $q \in T_1$, then $q \notin U(R^1)$ and there exist $\gamma_j$ and $\gamma_k$ such that $\gamma_j \neq \gamma_k$ and $q \in \gamma_j \cap \gamma_k$. For $q \in T_1$, let

$$J_q = \{ j \mid q \in \gamma_j = \mathcal{E} \cdot S_{\mathcal{R}_j}^{-1}(q_3) \text{ for some } q_3 \in U(\mathcal{R}_j) \}. \{ q \in U(\mathcal{R}_j) \}. \{ q \in U(\mathcal{R}_j) \}.$$ 

Observe that:
1. For \( q \in T_1 \), \( j \in J_q \) and \( p \in f^{-1}_4(q) \), the permissible parameters \( \overline{w}_i, \overline{w}_j \) at \( q \) defined by (152) have a form 5 (c) of Definition 7.5.

2. For \( i, j \in J_q \) there exists a relation of the form 5 (c) of Definition 7.5 for \( \overline{w}_i \) and \( \overline{w}_j \), and the set \( \{(a_{ij}, b_{ij})\} \) is totally ordered.

In particular, the points in \( T_1 \) are prepared for \( R^1 \) of type 2 of Definition 7.4. Let \( \Psi_2 : Y_2 \to Y_1 \) be the blow up of \( T_1 \), and let

\[
\begin{array}{ccc}
X_5 & \xrightarrow{f_5} & Y_2 \\
\downarrow & & \downarrow \Psi_2 \\
X_4 & \xrightarrow{f_4} & Y_1
\end{array}
\]

be the \( \tau \)-well prepared diagram of Lemma 7.14. Let

\[
T_2 = \left\{ q \in \Psi_2^{-1}(T_1) \text{ such that } q \in \gamma_1^2 \cap \gamma_2^2 \text{ where } \gamma_1^2, \gamma_2^2 \text{ are the strict transforms of some } \gamma_i, \gamma_k \text{ such that } \gamma_i \neq \gamma_k \text{ and } \gamma_i \cap \gamma_k \neq \emptyset \right\}.
\]

The points of \( T_2 \) must again be prepared for the transform \( R^2 \) of \( R \) on \( X_5 \) of type 2 of Definition 7.4, and the points of \( T_2 \) satisfy the corresponding statements 1 and 2 above that the points of \( T_1 \) and \( J_q \) satisfy.

We can iterate this process a finite number of times to produce a \( \tau \)-well prepared diagram

\[
\begin{array}{ccc}
\overline{X}_m & \xrightarrow{\overline{f}_m} & \overline{Y}_m \\
\downarrow & & \downarrow \\
\overline{X}_1 = X_4 & \xrightarrow{f_1} & \overline{Y}_1 = Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

of the form of (112) such that the strict transform of the \( \gamma_i \) are disjoint on \( \overline{Y}_m \) above \( T_1 \). It follows that the strict transforms of the \( \gamma_i \) are prepared of type 4 for the transform \( R^m \) of \( R \) on \( \overline{X}_m \).

To show that \( \overline{f}_m \) is \( \tau \)-very-well prepared, it only remains to verify that the strict transform \( \gamma' \) of a curve \( \gamma = \overline{E}_\beta \cdot S_{\overline{R}_i}(q) \) on \( Y \) (with \( \gamma \neq C \) and \( \gamma \cap C \neq \emptyset \)) is prepared on \( \overline{Y}_m \). Since \( \gamma \) is prepared of type 4, we have that \( \gamma \cap C \subset U(R_i) \). By our previous analysis, we then know that \( \gamma' \cap E \subset U(R^1) \), so that \( \overline{Y}_m \to \overline{Y}_1 \) is an isomorphism in a neighborhood of \( \gamma' \). It thus suffices to check that \( \gamma' \) is prepared of type 4 on \( Y_1 \). This follows by a local analysis.

\[\square\]

**Remark 7.16.**

1. Suppose that \( f : X \to Y \) is \( \tau \)-quasi-well prepared and \( C \subset D_Y \) is a nonsingular (integral) curve which makes SNCs with \( D_Y \) and contains a 1-point such that

   a. \( q \in C \cap U(R_i) \) for some 2-point pre-relation \( \overline{R}_i \) associated to \( R \) implies the (formal) germ of \( C \) at \( q \) is contained in \( S_{\overline{R}_i}(q) \), and

   b. Suppose that \( q \in C - U(R) \). Then there exist permissible parameters \( u, v, w \) at \( q \) such that \( u = w = 0 \) are local equations of \( C \) at \( q \), and

   i. If \( q \) is a 1-point, \( C \) is not a component of the fundamental locus of \( f \), and \( p \in f^{-1}(q) \), then there exists a relation of one of the following forms for \( u, v, w \) at \( p \).
(A) p a 1-point
\begin{align*}
    u &= x^a \\
    v &= y \\
    w &= x^c \gamma(x, y) + x^d z
\end{align*}
(159)
where \( \gamma \) is a unit series (or zero),

(B) p a 2-point
\begin{align*}
    u &= (x^a y^b)^k \\
    v &= z \\
    w &= (x^a y^b)^l \gamma(x^a y^b, z) + x^c y^d
\end{align*}
(160)
where \( \gamma \) is a unit (or zero) and \( ad - bc \neq 0 \).

(ii) If \( q \) is a 2-point, \( u, v, w \) are super parameters for \( f \) at \( q \).

Then there exists a \( \tau \)-quasi-well prepared diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{\tau_1} & Y_1 \\
\Phi_1 & \downarrow & \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]
where \( \Phi_1 \) is the blow up of \( C \). \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - (\Sigma(Y) \cup C)) \).

If \( C \) is contained in the fundamental locus of \( f \), then \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \).

2. Further suppose that \( f : X \to Y \) is \( \tau \)-well prepared, and if \( \gamma = E \cdot R_k(q_{\alpha}) \) is prepared for \( R \) of type 4, then either \( C = \gamma \) or \( q \in C \cap \gamma \) implies \( q \in U(R_k) \) and the germ of \( C \) at \( q \) is contained in \( S_{R_k}(q) \). Then there exists a \( \tau \)-well prepared diagram
\[
\begin{array}{ccc}
X_1 & \xrightarrow{\Phi_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]
where \( \Psi \) is the blow up \( \Phi_1 \) of \( C \), possibly followed by blow ups of 2-points which are prepared for the transform of \( R \) (of type 2 of Definition 7.4) if \( C \) is prepared of type 4 for \( R \), such that

(a) If \( \gamma \subset Y \) is prepared for \( f \), (and \( \gamma \neq C \)) then the strict transform of \( \gamma \) is prepared for \( f_1 \).

(b) If \( C \subset Y \) is prepared for \( f \) (of type 4) and \( q \in U(R_i) \cap C \) for some \( i \), then \( E \cdot S_{R_i}^1(q') \) is prepared for \( f_1 \) (of type 4) for all \( q' \in E \cap U(R_i^1) \), where \( E \) is the component of \( D_{Y_1} \) dominating \( C \).

(c) Suppose that \( F \) is a component of \( D_{Y_1} \) such that \( \Psi(F) \) is a point. If \( q \in F \cap U(R_i^1) \) for some \( i \), then \( F \cdot S_{R_i}^1(q) \) is a prepared curve of type 4 for \( R^1 \).

(d) If \( D_X \) is cuspidal for \( f \), then \( D_{X_1} \) is cuspidal for \( f_1 \).

(e) \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - (\Sigma(Y) \cup C)) \). If \( C \) is contained in the fundamental locus of \( f \), then \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \).

The proof of Remark 7.16 is a variation of the proof of Lemma 7.15, using Lemma 5.1 if \( C \) is a component of the fundamental locus of \( f \).

8. Existence of a \( \tau \)-very-well prepared morphism

Suppose that \( f : X \to Y \) is a birational morphism of nonsingular projective 3-folds, with toroidal structures \( D_Y \) and \( D_X = f^{-1}(D_Y) \).
Theorem 8.1. Suppose that \( f : X \to Y \) is prepared. Let \( \tau = \tau_f(X) \). Suppose that \( \tau \geq 1 \) and if \( p \in X \) is a 3-point and \( \tau_f(p) = \tau \) then \( f(p) \) is a 2-point on \( Y \). Further suppose that \( D_X \) is cuspidal for \( f \). Then there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( \Phi, \Psi \) are products of blow ups of 2-curves, and there exists a 2-point relation \( R^1 \) for \( f_1 \) such that \( f_1 \) is \( \tau \)-quasi-well prepared with 2-point relation \( R^1 \). Further, \( D_{X_1} \) is cuspidal for \( f_1 \).

Proof. Let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

be a diagram satisfying the conclusions of Theorem 5.7.

Let \( T \) be the 3-points \( p \in X_1 \) with \( \tau_{f_1}(p) = \tau \). Let \( I \) be an index set of \( T \), and let \( U = \{ f_1(p) \mid p \in T \} \). We necessarily have that \( U \) consists of 2-points. Suppose that \( q \in U \) and \( p_i \in T \cap f_1^{-1}(q) \). We will define a 2-point pre-relation \( R^1_{p_i}(q) \) on \( Y_1 \) which has the property that \( U(R^1_{p_i}(q)) = \{ q \} \). Let \( u, v, w_i \) (with \( w_i \in \mathcal{O}_{Y_1,q} \)) be the permissible parameters at \( q \) of 4 of Theorem 5.7, which have the property that 4 (a) or (b) of Theorem 5.7 holds for permissible parameters \( x, y, z \) at \( p_i \). If \( \tau > 1 \), define \( R^1_{p_i}(p_i) \) from the expression

\[ w_i^{e_i} = \bar{x}_i u^{a_i} v^{b_i} \]

of 4 (a) of Theorem 5.7 where \( \bar{x}_i = \gamma(0,0,0)^{e_i} \). We have \( \gcd(a_i, b_i, e_i) = 1 \) and \( e_i > 1 \). If \( \tau = 1 \), then an expression as a monomial form 4 (b) of Theorem 5.7 holds at \( p_i \), and we define \( R^1_{p_i}(p_i) \) by \( a_i = b_i = -\infty \) and the expression \( w_i = 0 \). We now define a primitive 2-point relation \( R^1_{p_i} \) on \( X_1 \) by \( T(R^1_{p_i}) = \{ p_i \} \) and \( R^1_{p_i}(p_i) = \mathcal{R}_{p_i}(p_i) \). We can thus define a 2-point relation \( R^1 \) on \( X_1 \) with \( T(R^1) = T, \ U(R^1) = U \), and where the \( R^1 \) defined above are the primitive 2-point relations associated to \( \mathcal{R} \). It follows from Theorem 5.7 that \( f_1 \) is \( \tau \)-quasi-well prepared for \( R^1 \), and \( D_{X_1} \) is cuspidal for \( f_1 \).

Lemma 8.2. Suppose that \( Y \) is a nonsingular projective 3-fold with toroidal structure \( D_Y \) and 2-point pre-relations \( R_1, \ldots, R_n \) and \( \gamma \subset D_Y \) is a reduced (but possibly not irreducible) curve such that \( \gamma \) has no components which are 2-curves and \( \gamma \) is nonsingular at 1-points. If the \( R_i \) are algebraic, let \( \Omega(R_i) \) be the locally closed subset of Definition 6.2, and assume that \( \mathcal{N} \cap \Omega(R_i) \) is a union of 2-points, if \( \mathcal{N} \) is an irreducible component of \( \gamma \) such that \( \mathcal{N} \cap \Omega(R_i) \) is a finite set. Consider the following algorithm.

1. Perform a sequence of blow ups of 2-curves \( Y_1 \to Y \) so that the strict transform \( \mathcal{N}_1 \) of \( \gamma \) on \( Y_1 \) contains no 3-points.
2. Perform an arbitrary sequence of blow ups \( Y_2 \to Y_1 \) of 2-curves. Let \( \mathcal{N}_2 \) be the strict transform of \( \gamma \) on \( Y_2 \). Let \( \mathcal{R}^2_{1}, \ldots, \mathcal{R}^2_{n} \) be the transforms of \( R_1, \ldots, R_n \) on \( Y_2 \).
3. Blow up all 2-points \( q \in \mathcal{N}_2 \) such that \( \mathcal{N}_2 \) does not make SNCs with \( D_Y \) at \( q \) or \( q \in \Omega(R^2_i) \) for some \( i \) and the germ of \( \mathcal{N}_2 \) at \( q \) is not contained in \( S_{\mathcal{R}^2_i}(q) \), or \( q \in \Omega(R^2_i) \) for some \( i \), and the germ of \( \mathcal{N}_2 \) at \( q \) is not contained in the germ of \( \Omega(R^2_i) \) at \( q \).
4. Iterate steps 1.-3.

Then, after finitely many iterations, we produce a sequence of admissible blow ups \( \Psi_1 : Y_1 \to Y \) such that the strict transform \( \gamma_1 \) of \( \gamma \) on \( Y_1 \) is nonsingular and makes SNCs with \( D_{Y_1} \). Further, if \( R_i^1 \) is the transform of \( R_i \) on \( Y_1 \) for \( 1 \leq i \leq n \) and \( q \in U(R_i^1) \cap \gamma_1 \) for some \( i \), then the germ of \( \gamma_1 \) at \( q \) is contained in \( S_{R_i^1}(q) \). If \( q \in \Omega(R_i^1) \cap \gamma_1 \) for some \( i \), then the germ of \( \gamma_1 \) at \( q \) is contained in the germ of \( \Omega(R_i^1) \) at \( q \).

**Proof.** This follows from embedded resolution of plane curve singularities (cf. Section 3.4 and Exercise 3.13 [C3]). \( \square \)

**Lemma 8.3.** Suppose that \( f : X \to Y \) is prepared, and \( C \) is a reduced (but possibly not irreducible) curve, consisting of components of the fundamental locus of \( f \) which contain a 1-point. Then

1. \( C \) is nonsingular at 1-points of \( Y \).
2. If \( f \) is \( \tau \)-well prepared, and \( \gamma \subset Y \) is prepared for \( R \) of type 4 then either \( \gamma \) is a component of \( C \) or \( C \cap \gamma \) contains no 1-points of \( Y \).

**Proof.** 1 follows from Lemma 3.5. 2 follows from Lemma 3.5 and 5 (b) of Definition 7.5. \( \square \)

**Theorem 8.4.** Suppose that \( \tau \geq 1 \), \( f : X \to Y \) is \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) with 2-point relation \( R \), and \( C \subset Y \) is a reduced (but possibly not irreducible) curve consisting of components of the fundamental locus of \( f \) which contain a 1-point of \( Y \). Assume that \( D_X \) is cuspidal for \( f \). Then there exists a sequence of blow ups of 2-curves and 2-points \( \Psi_1 : Y_1 \to Y \) and a \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) diagram of \( R \) and \( \Psi_1 \)

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \Psi_1 \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]  

such that

1. The strict transform \( \overline{C} \) of \( C \) on \( Y_1 \) is nonsingular and makes SNCs with \( D_{Y_1} \).
2. If \( C_j \) is an irreducible component of \( \overline{C} \) and \( q \in U(R_i^1) \) for some \( R_i^1 \) associated to \( R_i \) such that \( q \in C_j \) then the germ of \( C_j \) at \( q \) is contained in \( S_{R_i^1}(q) \).
3. If \( f \) is \( \tau \)-well prepared, \( C_j \) is an irreducible component of \( \overline{C} \) and \( \gamma = \pi^{-1}(R_k^1(q_o)) \) is prepared for \( R_i \) of type 4, then either \( C_j = \gamma \) or \( q \in C_j \cap \gamma \) implies \( q \in U(R_k^1) \) (and thus the germ of \( C_j \) at \( q \) is contained in \( S_{R_k}(q) \)).
4. \( D_{X_1} \) is cuspidal for \( f_1 \).
5. \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \).

**Proof.** \( C \subset D_Y \) and \( C \) is nonsingular at 1-points by Lemma 8.3. We now follow the algorithm of Lemma 8.2. If \( f \) is \( \tau \)-well prepared, then by our convention on the \( \Omega(R_i) \) (following Definition 7.5), we may replace \( \Omega(R_i) \) with an open subset, so that if \( C_j \) is a component of \( C \), then either \( C_j \) is disjoint from \( \Omega(R_i) \), or intersects \( \Omega(R_i) \) in a finite set of 2-points which are in \( U(R_i) \), or on a curve \( \gamma \subset \Omega(R_i) \) which is prepared for \( R \) of type 4. By Lemma 8.3, if \( C_j \) is a component of \( C \) and \( \gamma \subset \Omega(R_i) \) is prepared of type 4 for \( R \), such that \( C_j \neq \gamma \), then \( \gamma \cap C_j \) is a set of 2-points.
We first construct a sequence of blow ups of 2-curves, \( \Psi_1 : Y_1 \to Y \) so that the strict transform \( C_1 \) of \( C \) on \( Y_1 \) contains no 3-points. Let 

\[
\begin{align*}
X_1 & \xrightarrow{f_1} Y_1 \\
\Phi_1 & \downarrow \Psi_1 \\
X & \xrightarrow{f} Y
\end{align*}
\]

be a \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) diagram of \( R \) and \( \Psi_1 \) such that \( \Phi_1 \) is an isomorphism above \( f^{-1}(Y - \Sigma(Y)) \) (this exists by Lemma 7.11).

If \( f \) (and thus also \( f_1 \)) is \( \tau \)-well prepared, and \( q \) is a 2-point on a curve \( \gamma \subset Y_1 \) which is prepared for \( R^1 \) of type 4, then \( q \) is prepared for \( f_1 \) (of type 1 or 2 of Definition 7.4).

Consider the 2-points 

\[ \Omega = \left\{ \text{2-points } q \in C_1 \text{ which are not in } U(R^1) \text{ and are not on a curve} \right\} \]

which is prepared of type 4 for \( f_1 \) (if \( f \) is \( \tau \)-well prepared).

By Lemma 5.6 and Remark 7.12, there exists a \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) diagram

\[
\begin{align*}
X_2 & \xrightarrow{f_2} Y_2 \\
\Phi_2 & \downarrow \Psi_2 \\
X_1 & \xrightarrow{f_1} Y_1
\end{align*}
\]

where \( \Phi_2 \) is a product of blow ups of 2-curves and 3-points and \( \Psi_2 \) is a product of blow ups of 2-curves such that \( \Phi_2 \) is an isomorphism above \( f_2^{-1}(Y_1 - \Sigma(Y_1)) \) and at all 2-points \( \overline{q}_1 \in \Psi_2^{-1}(\Omega) \), there exist permissible parameters \( \overline{\pi}, \overline{\tau}, w, \) at \( \overline{q}_1 \), such that \( \overline{\pi}, \overline{\tau}, w \) are super parameters for \( f_2 \) at \( \overline{q}_1 \), and thus \( \overline{q}_1 \) is prepared for \( R^2 \) with respect to \( \overline{\pi}, \overline{\tau}, w \). In particular, all 2-points on the strict transform \( C_2 \) of \( C \) on \( Y_2 \) are prepared for \( R^2 \) (of type 1 or 2 of Definition 7.4). The map \( Y_2 \to Y_1 \) is Step 2 of the algorithm of Lemma 8.2.

Now perform Step 3 of the algorithm, blowing up all 2-points \( q \) on the strict transform \( C_2 \) of \( C \) on \( Y_2 \) where \( C_2 \) is singular, or \( C_2 \) does not make SNCs with \( D_{Y_2} \) at \( q \), or \( q \in \Omega(Q_i^2) \) for some \( Q_i^2 \) associated to \( R^2 \) and an irreducible component \( C' \) of \( C_2 \) contains \( q \) but the germ of \( C' \) at \( q \) is not contained in the germ of \( \Omega(Q_i^2) \) at \( q \). Let \( \Psi_3 : Y_3 \to Y_2 \) be the resulting map. By Lemmas 7.13 and 7.14, there exists a \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) diagram

\[
\begin{align*}
X_3 & \xrightarrow{f_3} Y_3 \\
\Phi_3 & \downarrow \Psi_3 \\
X_2 & \xrightarrow{f_2} Y_2
\end{align*}
\]

such that \( \Phi_3 \) is an isomorphism above \( f_2^{-1}(Y_2 - \Sigma(Y_2)) \).

Now by Step 4 of the algorithm of Lemma 8.2, we can iterate this process to construct a \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) diagram (161) such that the conclusions of Theorem 8.4 hold. By our construction, and Lemmas 7.11, 7.13 and 7.14 and Remark 7.12, \( D_{X_1} \) is cuspidal for \( f_1 \). \( \square \)

**Theorem 8.5.** Suppose that \( f : X \to Y \), \( C \) and \( f_1 : X_1 \to Y_1 \) are as in the assumptions and conclusions of Theorem 8.4. Then there exists a sequence of blow ups of 2-curves \( \Psi_2 : Y_2 \to Y_1 \), and a \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) diagram

\[
\begin{align*}
X_2 & \xrightarrow{f_2} Y_2 \\
\Phi_2 & \downarrow \Psi_2 \\
X_1 & \xrightarrow{f_1} Y_1
\end{align*}
\]
such that

1. The conclusions of 1, 2 and 3 of Theorem 8.4 hold for the strict transform $C_2$ of $C$ on $Y_2$.
2. $D_{X_2}$ is cuspidal for $f_2$.
3. The components $\gamma$ of $C_2$ are permissible centers (or $^*$-permissible centers if $\gamma$ is prepared of type $4$) for $R^2$.
4. $\Phi_2$ is an isomorphism over $f_1^{-1}(Y_1 - \Sigma(Y_1))$.

In the resulting $\tau$-quasi-well prepared (or $\tau$-well prepared) diagram

\[
\begin{array}{ccc}
X_3 & \xrightarrow{f_3} & Y_3 \\
\Phi_3 & & \Phi_3 \\
X_2 & \xrightarrow{f_2} & Y_2 \\
\end{array}
\]

$\Psi_3$ is the blow up of $C_2$, possibly followed by blow ups of 2-points which are prepared for the transform of $R$ if $f$ is $\tau$-well prepared, and $C$ contains a component which is prepared of type $4$ for $R$. Further, $D_{X_3}$ is cuspidal for $f_3$ and $\Phi_3$ is an isomorphism above $f_2^{-1}(Y_2 - \Sigma(Y_2))$.

**Proof.** For all $q \in C$ a 2-point such that $q \notin U(R^1)$, let $u_q, v_q, w_q$ be permissible parameters at $q$ such that $u_q = w_q = 0$ are local equations for $C$ at $q$. By Lemma 5.6 and Remark 7.12, there exists a $\tau$-quasi-well prepared (or $\tau$-well prepared) diagram

\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & Y_2 \\
\Phi_2 & & \Phi_2 \\
X_1 & \xrightarrow{f_1} & Y_1 \\
\end{array}
\]

where $\Phi_2$ is a product of blow ups of 2-curves and 3-points, $\Psi_2$ is a product of blow ups of 2-curves, such that $\Phi_2$ is an isomorphism over $f_1^{-1}(Y_1 - \Sigma(Y_1))$ and for all 2-points $q \in C$, which are not in $U(R^1)$, at 2-points $q_1 \in \Psi_2^{-1}(q)$ we have permissible parameters $\bar{u}, \bar{v}, \bar{w}$ at $q_1$ such that $q_1$ is prepared (of type 2 of Definition 7.4) for $R^2$ with respect to the parameters $\bar{u}, \bar{v}, \bar{w}$. In particular, this is true for the point $q_1 \in \Psi_2^{-1}(q)$ on the strict transform $C_2$ of $C$ on $Y_2$. At this $q_1$, $\bar{u}, \bar{v}, \bar{w}$ satisfy

\[u_q = \bar{u}^n, v_q = \bar{v}, w_q = \bar{w}\]

for some $n, \bar{u} = \bar{v} = \bar{w} = 0$ are local equations of $C_2$ at $q_1$, and $\bar{u}, \bar{v}, \bar{w}$ are super parameters at $q_1$.

Thus, the hypothesis of Remark 7.16 are satisfied, and the conclusions of Theorem 8.5 hold.

**Lemma 8.6.** Suppose that $\tau \geq 1$, $f : X \to Y$ is $\tau$-quasi-well prepared with 2-point relation $R$ and $D_X$ is cuspidal for $f$. Further suppose there exists a $\tau$-quasi-well prepared diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
\tilde{\Phi} & & \tilde{\Psi} \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

where $\tilde{R}$ is the transform of $R$ on $\tilde{X}$, such that if $q_1 \in U(\tilde{R})$ is on a component $E$ of $D_\tilde{X}$ such that $\tilde{\Psi}(E)$ is not a point, then $T(\tilde{R}) \cap f^{-1}(q_1) = \emptyset$. Further suppose that
$D_X$ is cuspidal for $\tilde{f}$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
X_1 & f_1 & Y_1 \\
\Phi & \downarrow & \Psi \\
\tilde{X} & \tilde{f} & \tilde{Y}
\end{array}
$$

such that $\Phi$, $\Psi$ are products of blow ups of possible centers, and $f_1$ is $\tau$-quasi-well prepared with 2-point relation $R^1$ and pre-algebraic structure. ($R^1$ will in general not be the transform of $\tilde{R}$.) Further, $D_{X_1}$ is cuspidal for $f_1$.

Proof. Given a diagram (162), we will define a new 2-point relation $\tilde{\Psi}$ on $\tilde{X}$ for $\tilde{f}$. This is accomplished as follows. Suppose that $q_1 \in U(\tilde{R})$ is such that $\tilde{f}^{-1}(q_1) \cap T(\tilde{R}) \neq \emptyset$.

Let $J_{q_1} = \{ i \mid T(\tilde{R}_i) \cap \tilde{f}^{-1}(q_1) \neq \emptyset \}$.

For $j \in J_{q_1}$, let $q = \tilde{\Psi}(q_1)$,

$$u = u_{\tilde{R}_j}(q), v = v_{\tilde{R}_j}(q), w_j = w_{\tilde{R}_j}(q).$$

Let

$$u_1 = u_{\tilde{R}_j}(q_1), v_1 = v_{\tilde{R}_j}(q_1), w_{j,1} = w_{\tilde{R}_j}(q_1).$$

Since $\tilde{\Psi}$ is a composition of admissible blow ups for the transforms of the pre-relations $\tilde{R}_i$ on $Y$, we have relations

$$u = u_1^q v_1^q, v = u_1^q v_1^q, w_j = u_1^q v_1^q w_{j,1} \tag{163}$$

with $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = \pm 1$. Since if $q_1$ is on a component $E$ of $D_Y$ we must have $\tilde{\Psi}(E)$ is a point, we have $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ all nonzero.

For $p_1 \in T(\tilde{R}_j) \cap \tilde{f}^{-1}(q_1)$, there exist permissible parameters $x_1, y_1, z_1$ for $u_1, v_1, w_{j,1}$ (and $u, v, w_j$) at $p_1$ and an expression

$$u = x_1^q y_1^q z_1^q$$

$$v = x_1^q y_1^q z_1^q$$

$$w_j = x_1^q y_1^q z_1^q \gamma_j \tag{164}$$

where $\gamma_j$ is a unit series in $\tilde{O}_{\tilde{X},p_1}$. Since $\tilde{a}\tilde{d} - \tilde{b}\tilde{c} = \pm 1$, there exist (after possibly interchanging $u$ and $v$) $m > 0$ and a factorization of the expression of $u$ and $v$ in (163) by the three successive substitutions:

$$u = \overline{\pi}, v = \overline{\pi}^m \overline{\pi},$$

$$\overline{\pi} = \tilde{a}\tilde{v}, \overline{\pi} = \tilde{v}$$

$$\tilde{u} = u_1^q v_1^q, \tilde{v} = u_1^q v_1^q \tilde{d} \tag{165}$$

for some $\overline{\pi}, \overline{\tau}, \overline{d} \in \mathbb{N}$, with $\overline{\pi} \overline{d} - \overline{\tau} \overline{c} = \pm 1$. Substituting (165) into (164), we see that

$$(a, b, c) > (d, e, f) - m(a, b, c) > 0.$$}

Thus

$$m(a, b, c) < (d, e, f) < (m + 1)(a, b, c). \tag{166}$$

If there does not exist a natural number $r$ such that $w_j | v^r$ and $w_j | v^r$ in $\tilde{O}_{\tilde{X},p_1}$, then there exists an expression by (166) (after possibly interchanging $u$ and $v$, and $x_1, y_1, z_1$)

$$u = x_1^q y_1^b$$

$$v = x_1^q y_1^b$$

$$w_j = x_1^q y_1^b z_1^q \gamma_j \tag{167}$$
with \( i \neq 0 \). But we now obtain a contradiction, since \( uw = 0 \) is a local equation of \( D_\mathcal{X} \) at \( p_1 \).

In conclusion, there exists a natural number \( r \) such that \( w_j \) divides \( u^r \) and \( v^r \) in \( \hat{\mathcal{O}}_{X,p_1} \).

Set \( \eta(p_1) = \max\{2r, \hat{e}, \hat{f}\} \), and \( \eta = \max\{\eta(p_1) \mid p_1 \in T(\hat{R}) \cap \hat{f}^{-1}(q_i)\} \). Fix \( j \in J_{q_i} \) and \( p_1 \in T(R_j) \cap \hat{f}^{-1}(q_i) \). There exists \( \sigma(u,v,w_j) \in \mathbb{k}[u,v,w_j] = \mathcal{O}_{Y,q} \) such that the order of the series \( \sigma \) is greater than \( \eta \) and \( w_j + \sigma \in \mathcal{O}_{Y,q} \). Let

\[
\hat{w}_{p_1}^* = w_j + \sigma(u,v,w_j).
\]

In \( \hat{\mathcal{O}}_{X,p_1} \), we have

\[
\hat{w}_{p_1}^* = w_j \gamma_{p_1}'' = x_j y_1^{\hat{e}'} y_1^{\hat{f}'} \gamma_{p_1}'
\]

where \( \gamma_{p_1}', \gamma_{p_1}'' \) are unit series. Set

\[
w_{p_1} = \frac{w_{p_1}^*}{u_j^* v_j^*} = w_{j,1} + \frac{\sigma(u_j^* v_j^*, w_{j,1})}{u_j^* v_j^*} \in \hat{\mathcal{O}}_{Y,q_1} \cap \mathbb{k}(Y) = \mathcal{O}_{Y,q_1}.
\]

We further have

\[
w_{p_1} = w_{j,1} \gamma_{p_1}''.
\]

For \( k \in J_{q_i} \), there exists \( \lambda_{j,k}(u,v) \in \mathbb{k}[u,v] \) such that \( w_k = w_j + \lambda_{j,k}(u,v) \). Suppose that \( p_2 \in T(\hat{R}_k) \cap \hat{f}^{-1}(q_1) \). Write

\[
\lambda_{j,k}(u,v) = \alpha_{p_2}(u,v) + h_{p_2}(u,v)
\]

where \( \alpha_{p_2}(u,v) \) is a polynomial, and \( h_{p_2}(u,v) \) is a series of order greater than \( \eta \). Set

\[
w_{p_2} = w_j + \sigma(u,v,w_j) + \lambda_{j,k}(u,v) - h_{p_2}(u,v) \in \mathcal{O}_{Y,q}
\]

\[
w_{p_2}^* = w_k + \sigma(u,v,w_k) = w_{p_2} + \sigma_{p_2}(u,v,w_k)
\]

where \( \sigma_{p_2} \) is a series of order greater than \( \eta \). Set

\[
w_{p_2} = \frac{w_{p_2}^*}{u_j^* v_j^*}.
\]

From (163) we see that \( u_1, v_1, w_{p_2} \) are permissible parameters at \( q_1 \) and \( w_{p_2} \in \mathcal{O}_{Y,q_1} \).

We further have that \( w_{p_2} = w_{k,1} \gamma_{p_2}'' \) for some unit series \( \gamma_{p_2}'' \in \hat{\mathcal{O}}_{X,p_2} \).

We further have

\[
w_{p_2} - w_{p_1} = \frac{w_{p_2}^* - w_{p_1}^*}{u_j^* v_j^*} = \lambda_{j,k}(u,v) - h_{p_2}(u,v) \in \mathbb{k}((u_1, v_1)) \cap \mathbb{k}[u_1, v_1, w_{j,1}] = \mathbb{k}[u_1, v_1].
\]

We now define the new 2-point relation \( \hat{R}' \) on \( \hat{X} \) for \( \hat{f} \). Set \( T = T(\hat{R}) \), \( U = \hat{f}(T(\hat{R})) \).

For \( p_1 \in T \) we define a primitive 2-point relation \( R_{p_1} \) by \( U(R_{p_1}) = \{q_1 = \hat{f}(p_1)\} \), \( T(R_{p_1}) = \{p_1\} \),

\[
u_{R_{p_1}}(p_1) = u_{\hat{R}}(p_1), v_{R_{p_1}}(p_1) = v_{\hat{R}}(p_1), w_{R_{p_1}}(p_1) = w_{p_1}.
\]

If \( \tau > 1 \), we define

\[
a_{R_{p_1}}(p_1) = a_{\hat{R}}(p_1), b_{R_{p_1}}(p_1) = b_{\hat{R}}(p_1), e_{R_{p_1}}(p_1) = e_{\hat{R}}(p_1)
\]

and

\[
\lambda_{R_{p_1}}(p_1) = \lambda_{\hat{R}}(p_1) \gamma_{p_1}''(0,0,0)^{\hat{f}_0(p_1)}.
\]

If \( \tau = 1 \), we define

\[
a_{R_{p_1}}(p_1) = b_{R_{p_1}}(p_1) = -\infty.
\]
Let \( \tilde{R}' \) be the 2-point relation associated to the \( R_{p_1} \) for all \( p_1 \in T \). We have \( T(\tilde{R}') = T(\tilde{R}) \) and \( U(\tilde{R}') = \tilde{f}(T(\tilde{R})) \). From the above calculations, we see that \( f : \tilde{X} \to \tilde{Y} \) with the relation \( \tilde{R}' \) satisfies 1 - 4 of the conditions of Definition 7.1 of a \( \tau \)-quasi-well prepared morphism. Finally, by Lemma 5.6 and Remark 7.12, there exists a commutative diagram

\[
\begin{array}{ccc}
\tilde{X}_1 & \xrightarrow{\tilde{f}_1} & \tilde{Y}_1 \\
\Phi_1 & \downarrow & \Psi_1 \\
\tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y}
\end{array}
\]

where \( \Phi_1 \) is a product of blow ups of 2-curves and 3-points and \( \Psi_1 \) is a product of blow ups of 2-curves such that the transform \( \tilde{R}_i^1 \) of \( \tilde{R}' \) for \( \tilde{f}_1 \) is defined and \( \tilde{R}_i^1 \) satisfies 5 of the conditions of Definition 7.1 (as well as 1 – 4). Then \( \tilde{f}_1 \) is \( \tau \)-quasi-well prepared with pre-algebraic structure. Thus the conclusions of Lemma 8.6 hold. \( \square \)

**Theorem 8.7.** Suppose that \( \tau \geq 1 \), \( f : X \to Y \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \) and \( D_X \) is cuspidal for \( f \). Then there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi \) and \( \Psi \) are products of blow ups of possible centers and \( f_1 \) is \( \tau \)-quasi-well prepared with 2-point relation \( R_1 \) and pre-algebraic structure. (\( R_1 \) will in general not be the transform of \( R \).) Further, \( D_{X_1} \) is cuspidal for \( f_1 \).

**Proof.** We will show that there exists a \( \tau \)-quasi-well prepared diagram (162) as in the hypothesis of Lemma 8.6. Then Lemma 8.6 implies that the conclusions of Theorem 8.7 hold.

**Step 1.** Let \( A_0 \) be the set of 2-points \( q \in Y \) such that \( q \in U(\overline{R}_i) \) for some \( \overline{R}_i \) associated to \( R \) and \( f^{-1}(q) \cap T(R_i) \neq \emptyset \).

For \( q \in A_0 \cap U(\overline{R}_i) \), set

\[
u = u_{\overline{R}_i}(q), v = v_{\overline{R}_i}(q), w_i = w_{\overline{R}_i}(q).
\]

(168)

Let \( \Psi_1 : Y_1 \to Y \) be the blowup of all \( q \in A_0 \), and let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 & \downarrow & \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]

be a \( \tau \)-quasi-well prepared diagram of \( R \) and \( \Psi_1 \) (such a diagram exists by Lemma 7.13). Let \( A_1 \) be the set of all 2-points \( q_1 \in Y_1 \) such that for some \( i \), \( q_1 \in U(\overline{R}_i^1) \), \( f_i^{-1}(q_1) \cap T(\overline{R}_i^1) \neq \emptyset \) and \( q_1 \) is on a component \( E \) of \( D_{Y_1} \) such that \( \Psi_1(E) \) is not a 2-point. We have \( A_1 \subset \Psi_1^{-1}(A_0) \).

Let \( \Psi_2 : Y_2 \to Y_1 \) be the blowup of all \( q_1 \in A_1 \), and let

\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & Y_2 \\
\Phi_2 & \downarrow & \Psi_2 \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]

be a \( \tau \)-quasi-well prepared diagram of \( \overline{R}_i^1 \) and \( \Psi_2 \). Continue in this way to construct a sequence of \( n \) blow ups of 2-points \( \Phi_{k+1} : X_{k+1} \to Y_{k+1} \) for \( 0 \leq k \leq n - 1 \) with
\[ \tau \text{-quasi-well prepared diagrams} \]

\[
\begin{array}{ccc}
X_{k+1} & \overset{f_{k+1}}{\rightarrow} & Y_{k+1} \\
\Phi_{k+1} & \downarrow & \Psi_{k+1} \\
X_k & \overset{f_k}{\rightarrow} & Y_k \\
\end{array}
\]

of \( R^k \) and \( \Psi_{k+1} \). We have a resulting \( \tau \text{-quasi-well prepared diagram of} \) \( R \)

\[
\begin{array}{ccc}
X_n & \overset{f_n}{\rightarrow} & Y_n \\
\Phi & \downarrow & \Psi \\
X & \overset{f}{\rightarrow} & Y. \\
\end{array}
\]  

(169)

Suppose that \( q_n \in Y_n \) is a 2-point such that \( q_n \) is on a component \( E \) of \( D_{Y_n} \) such that \( \Psi(E) \) is not a point, and for some \( i, q_n \in U(R^n_i) \) and \( f_n^{-1}(q_n) \cap T(R^n_i) \neq \emptyset \). We have permissible parameters

\[
u_1 = v_{R^n_i}(q_n), v_1 = v_{R^n_i}(q_n), w_{i,1} = w_{R^n_i}(q_n)
\]

(170)

at \( q_n \) such that for \( \Psi(q_n) = q \) and with notation of (168),

\[
\begin{align*}
u &= u_1 \\
v &= u_1^n v_1 \\
w_i &= u_1^n w_{i,1}
\end{align*}
\]

(171)
or

\[
\begin{align*}
u &= u_1 v_1^n \\
v &= v_1 \\
w_i &= v_1^n w_{i,1}.
\end{align*}
\]

Suppose that \( p \in X \) is a 3-point such that \( p \in T(R_i) \cap f^{-1}(q) \). Then there are permissible parameters \( x, y, z \) for \( u, v, w_i \) at \( p \) such that

\[
\begin{align*}
u &= x^n y^b z^e \\
v &= x^d y^f z^i \\
w_i &= x^g y^i z^i \gamma
\end{align*}
\]

(172)

where \( \gamma \) is a unit series.

We will show that we can choose \( n \) sufficiently large in the diagram (169), so that if \( p_n \in X_n \) is a 3-point such that \( p_n \in \Phi^{-1}(p) \cap T(R^n) \) and \( q_n = f_n(p_n) \) is on a component \( E \) of \( D_{Y_n} \) such that \( \Psi(E) \) is not a point, then (172) must have one of the following forms (after possibly interchanging \( u, v \) and \( x, y, z \)):

\[
\begin{align*}
u &= x^n y^b \\
v &= x^d y^f z^i \\
w_i &= x^g y^i z^i \gamma
\end{align*}
\]

(173)

where \( b \neq 0, f \neq 0 \) and \( i \neq 0 \) or

\[
\begin{align*}
u &= x^n \\
v &= x^d y^f z^i \\
w_i &= x^g y^i z^i \gamma
\end{align*}
\]

(174)

with \( e \) and \( f \neq 0 \), and \( h \) or \( i \neq 0 \).

We will now prove this statement. Since \( p \) is a 3-point, there exist regular parameters \( \overline{x}, \overline{y}, \overline{z} \) in \( \mathcal{O}_{X,p} \) and unit series \( \lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}_{X,p} \) such that \( \overline{x} = x \lambda_1, \overline{y} = y \lambda_2, \overline{z} = z \lambda_3 \). Let \( \nu \) be any valuation of \( k(X) \) which has center \( p_n \) on \( X_n \). \( q_n \) has permissible parameters (170).
After possibly interchanging $u$ and $v$, we have a relation (171), so that $\nu(v) > n\nu(u)$. We can reindex $x, y, z$ so that $0 < \nu(x) \leq \nu(y) \leq \nu(z)$. Then

$$(f + e + d - nc)\nu(z) \geq (f - nc)\nu(x) + (e - nb)\nu(y) + (d - na)\nu(z) > 0.$$ 

Thus if $c \neq 0$, and $n > f + e + d$ for all $d, e, f$ in local forms (172) for 3-points $p \in T(R)$, we achieve that $c = 0$ in all local forms (172) which are the images of 3-points $p_n \in T(R^n)$ which map to a point $q_n$ of $Y_n$ which is on a component $E$ of $D_{Y_n}$ such that $\Psi(E)$ is not a point.

If $i = 0$ (and $c = 0$) in (172) we have

$$(h + g - nb)\nu(y) \geq (h - nb)\nu(y) + (g - na)\nu(x) > 0$$

so that if $b \neq 0$ and $n > h + g$ we have a contradiction. Thus, by taking $n \gg 0$ in (169), we see that if $b \neq 0$, then a form (173) must hold at $p$ (since $uv = 0$ is a local equation of $D_X$ at $p$). If $b = c = 0$ in (172), then a similar calculation shows that a form (174) must hold at $p$ (for $n \gg 0$).

We observe that in (173) we have

$$(z) \cap \hat{O}_{Y,q} = (v, w_i). \quad (175)$$

Suppose that (174) holds. If $i \neq 0$ then

$$(z) \cap \hat{O}_{Y,q} = (v, w_i). \quad (176)$$

If $h \neq 0$, then

$$(y) \cap \hat{O}_{Y,q} = (v, w_i). \quad (177)$$

We will show that in (173), $v = w_i = 0$ is a formal branch of an algebraic curve $C$ in the fundamental locus of $f : X \to Y$. Let $R = \mathcal{O}_{Y,q}$, $S = \mathcal{O}_{X,p}$. $\nu = 0$ is a local equation for a component of $D_X$. We have that $v \in (\nu) \cap R$ and $u \notin (\nu) \cap R$ so that $(\nu) \cap R = (v)$ or $(\nu) \cap R = a$ where $a$ is a height two prime containing $v$. We have $(zS) \cap R = (v, w_i)$. Suppose that $(\nu) \cap R = (v)$. We then have an induced morphism

$$\tilde{R}/(v) \to \tilde{S}/(z)$$

which is an inclusion by the Zariski Subspace Theorem (Theorem 10.14 [Ab]). This is impossible, so that $a$ is a height 2 prime in $R$, and defines a curve $C$, which is necessarily in the fundamental locus of $f$ since $\nu = 0$ is a local equation at $p$ of a component of $D_X$ which dominates $C$. A similar argument shows that in (174), $v = w_i = 0$ is a formal branch of an algebraic curve $C$ in the fundamental locus of $f$.

**Step 2.** Let $C$ be the reduced curve in $Y$ whose components are the curves in the fundamental locus of $f$ which are not 2-curves. Let $\tilde{C}$ be the reduced curve in $Y_n$ which is the strict transform of $C$. The components of $\tilde{C}$ are then in the fundamental locus of $f_n$. By Theorems 8.4 and 8.5, we can perform a sequence of blow ups of 2-points and 2-curves $\Psi' : Y' \to Y_n$ so that we can construct a $r$-quasi-well prepared diagram of $\Psi'$ and $R^n$.

$$(X' \xrightarrow{f'} Y') \xrightarrow{\Psi'} (X_n \to Y_n) \quad (178)$$

where $R'$ is the transform of $R^n$ on $X'$, such that the strict transform $\mathcal{C}$ of $\mathcal{C}$ on $Y'$ is nonsingular, if $q' \in U(R'_i) \cap \mathcal{C}$ for some $i$ then the germ at $q'$ of $\mathcal{C}$ is contained in $S_{R'_i}(q')$, and the (disjoint) components of $\mathcal{C}$ are permissible centers for $R'$. 
Let $\Psi(1) : Y(1) \to Y'$ be the blow up of $\bar{C}$. By Theorem 8.5, we have a $\tau$-quasi-well prepared diagram of $\Psi(1)$ and $R'$

$$
\begin{align*}
X(1) & \xrightarrow{f(1)} Y(1) \\
\Phi(1) & \downarrow \quad \downarrow \Psi(1) \\
X' & \xleftarrow{f'} Y'.
\end{align*}
$$

(179)

Let $R(1)$ be the transform of $R'$ on $X(1)$. Suppose that $\bar{q} \in U(R_i(1)) \subset Y(1)$ is a 2-point on the exceptional divisor of $\Psi(1)$, $\bar{q}$ is on a component $E$ of $D_Y(1)$ such that $(\Psi \circ \Psi' \circ \Psi(1))(E)$ is not a point of $Y$, and there exists a 3-point $\bar{p}_i \in (f(1))^{-1}(\bar{q}) \cap T(R_i(1)) \subset X(1)$. Let $q = \Psi \circ \Psi' \circ \Psi(1)(\bar{q})$, $\bar{q} = \Psi(1)(\bar{q})$. Let

$$
\begin{align*}
\bar{u} &= u_{\overline{\Pi}_i}(\bar{q}), \bar{v} = v_{\overline{\Pi}_i}(\bar{q}), \bar{w}_i = w_{\overline{\Pi}_i}(\bar{q}), \\
u &= u_{\overline{\Pi}_i}(q), v = v_{\overline{\Pi}_i}(q), w_i = w_{\overline{\Pi}_i}(q).
\end{align*}
$$

We have an expression, after possibly interchanging $u$ and $v$,

$$
u = \bar{u}, v = \bar{v}^e \bar{w}, w_i = \bar{w}_i^f \bar{w}_i, u = \bar{u}^f \bar{w}_i 
$$

(180)

for some $e, f > 0$. $\bar{v} = 0$ is a local equation of the strict transform of $D_Y$ at $\bar{q}$, and $\bar{v} = \bar{w}_i = 0$ are local equations of $\bar{C}$ at $\bar{q}$ (by (175) (176) or (177)). $\Psi(1)$ is the blow up of $(\bar{v}, \bar{w}_i)$ above $\bar{q}$. Since $\bar{q} \in U(R_i(1))$, we must have

$$
\begin{align*}
\bar{u} &= u_{\overline{\Pi}_i}(\bar{q}), \bar{v} = v_{\overline{\Pi}_i}(\bar{q}), \bar{w}_i = v_{\overline{\Pi}_i}(\bar{q})w_{\overline{\Pi}_i}(\bar{q}).
\end{align*}
$$

Substituting into (180), we have

$$
\begin{align*}
u &= u_{\overline{\Pi}_i}(\bar{q}), v = u_{\overline{\Pi}_i}(\bar{q})^e v_{\overline{\Pi}_i}(\bar{q}), w_i = u_{\overline{\Pi}_i}(\bar{q})^f v_{\overline{\Pi}_i}(\bar{q})w_{\overline{\Pi}_i}(\bar{q})
\end{align*}
$$

(181)

with $e, f > 0$.

We now apply steps 1 and 2 of the proof to $f(1) : X(1) \to Y(1)$ and $R(1)$. We construct a $\tau$-quasi-well prepared diagram

$$
\begin{align*}
X(2) & \xrightarrow{f(2)} Y(2) \\
\overline{\Phi}(2) & \downarrow \quad \downarrow \overline{\Psi}(2) \\
X(1) & \to Y(1),
\end{align*}
$$

where $R(2)$ is the transform of $R(1)$ on $X(2)$ such that if $q_2 \in U(R_i(2)) \subset Y(2)$ is a 2-point such that $q_2$ is on a component $E$ of $D_Y(2)$ such that $\overline{\Psi}(2))(E)$ is not a point of $Y(1)$ and there exists a 3-point $p_2 \in f(2)^{-1}(q_2) \cap T(R_i(2)) \subset X(2)$ and $q_1 = \overline{\Psi}(2)(q_2)$ then we have an expression:

$$
\begin{align*}
u_{\overline{\Pi}_i}(1)(q_1) &= u_{\overline{\Pi}_i}(2)(q_2) \\
v_{\overline{\Pi}_i}(1)(q_1) &= u_{\overline{\Pi}_i}(2)(q_2)^e v_{\overline{\Pi}_i}(2)(q_2) \\
w_{\overline{\Pi}_i}(1)(q_1) &= u_{\overline{\Pi}_i}(2)(q_2)^f v_{\overline{\Pi}_i}(2)(q_2)w_{\overline{\Pi}_i}(2)(q_2)
\end{align*}
$$

(182)

or

$$
\begin{align*}
u_{\overline{\Pi}_i}(1)(q_1) &= u_{\overline{\Pi}_i}(2)(q_2)v_{\overline{\Pi}_i}(2)(q_2)^e \\
v_{\overline{\Pi}_i}(1)(q_1) &= v_{\overline{\Pi}_i}(2)(q_2) \\
w_{\overline{\Pi}_i}(1)(q_1) &= u_{\overline{\Pi}_i}(2)(q_2)v_{\overline{\Pi}_i}(2)(q_2)^f w_{\overline{\Pi}_i}(2)(q_2)
\end{align*}
$$

(183)

with $e, f > 0$.

Let $q = (\Psi \circ \Psi' \circ \Psi(1) \circ \overline{\Psi}(2))(q_2)$. Substituting (182) or (183) into (181), we see that if $q_2$ is on a component $E$ of $D_Y(2)$ which does not contract to $q$, then we have
We have expressions
\[ u = u_{R_i}(q) = u_{\mathfrak{P}_i(n)}(q_n) \]
\[ v = v_{R_i}(q) = u_{\mathfrak{P}_i(n)}(q_n) \]
\[ w = w_{R_i}(q) = u_{\mathfrak{P}_i(n)}(q_n) \]
\[ (184) \]
with \( e_2, f_2 \geq 2. \)

Iterating steps 1 and 2, we construct a sequence of \( \tau \)-quasi-well prepared diagrams
\[
\begin{array}{c c c}
X(n) & f(n) & Y(n) \\
\mathfrak{P}(n) & \downarrow & \mathfrak{P}(n) \\
X(n-1) & f(n-1) & Y(n-1) \end{array}
\]
\[ (185) \]

This algorithm continues as long as there exists \( q_n \in U(R_i(n)) \) for some \( i \) such that \( q_n \) is on a component \( E \) of \( D_Y(n) \) which does not contract to a point of \( Y \), and \( f(n)^{-1}(q_n) \cap T(R_i(n)) \neq \emptyset. \)

Suppose that the algorithm never terminates.

Let \( \nu \) be a 0-dimensional valuation of \( k(X) \). We will say that \( \nu \) is resolved on \( X(n) \) if the center of \( \nu \) on \( X(n) \) is at a point \( p_n \) of \( X(n) \) such that either \( p_n \notin T(R_i(n)) \) or \( p_n \in T(R_i(n)) \) and all components \( E \) of \( D_Y(n) \) containing \( q_n = f(n)(p_n) \) contract to a point of \( Y \).

By our construction, if \( \nu \) is resolved on \( X(n) \), then \( \nu \) is resolved on \( X(m) \) for all \( m \geq n \). Further, the set of \( \nu \) in the Zariski-Riemann manifold \( \Omega(X) \) of \( X \) which are resolved on \( X \) is an open set.

Suppose that \( \nu \) is a 0-dimensional valuation of \( k(X) \) such that \( \nu \) is not resolved on \( X(n) \) for all \( n \). Let \( p_n \) be the center of \( \nu \) on \( X(n) \), \( q_n \) be the center of \( \nu \) on \( Y(n) \).

There exists an \( i \) such that for all \( n, q_n \in U(R_i(n)) \) and \( p_n \in f(n)^{-1}(q_n) \cap T(R_i(n)) \). We have expressions
\[ u = u_{R_i}(q) = u_{\mathfrak{P}_i(n)}(q_n) \]
\[ v = v_{R_i}(q) = u_{\mathfrak{P}_i(n)}(q_n) \]
\[ w = w_{R_i}(q) = u_{\mathfrak{P}_i(n)}(q_n) \]
\[ (186) \]
with \( e_n, f_n \geq n \) for all \( n \).

From (186), we see that \( \nu(v) > n\nu(u) \) for all \( n \in N \). Thus \( \nu \) is a composite valuation, and there exists a prime ideal \( P \) of the valuation ring \( V \) of \( \nu \) such that \( v \in P, u \notin P. \)

Let \( \nu_1 \) be a valuation whose valuation ring is \( V_P \). We have \( \nu_1(u) = 0, \nu_1(v) > 0. \)

From (186) we see that
\[ \nu_1(w) - n\nu_1(v) > 0 \]
\[ (187) \]
for all \( n \in N \).

At \( p = p_0 \in X \), we have a form (173) or (174). In (173) we have \( \nu_1(y) = 0. \)

\[ uw = 0 \] is a local equation of \( D_X \) at \( p. \) Thus either \( a > 0 \) or \( d > 0. \) If \( a > 0 \) then...
There is a contradiction to (187) since \( d > 0 \), we again have a contradiction to (187). In (174) we have \( \nu_1(x) = 0 \) and \( \nu_1(y), \nu_1(z) \geq 0 \), a contradiction to (187), since \( e, f > 0 \).

We have shown that for all 0-dimensional valuations \( \nu \) of \( k(X) \), there exists \( n \) such that \( \nu \) is resolved on \( X(n) \).

By compactness of the Zariski-Riemann manifold [Z1] there exists \( N \) such that all \( \nu \in \Omega(X) \) are resolved on \( X(N) \), a contradiction to our assumption that (185) is of infinite length. The diagram

\[
\begin{array}{ccc}
X(N) & \rightarrow & Y(N) \\
\downarrow & & \downarrow \\
X & \rightarrow & Y
\end{array}
\]

thus satisfies the hypothesis of (162) of Lemma 8.6, so that the conclusions of Theorem 8.7 hold.

\[ \square \]

**Lemma 8.8.** Suppose that \( \tau \geq 1, f : X \rightarrow Y \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \) and pre-algebraic structure (or \( \tau \)-well prepared with 2-point relation \( R \)), \( q \in U(R_i) \subset U(R) \) and \( p \in f^{-1}(q) \cap T(R_i) \) is a 3-point. Suppose that \( E \) is a component of \( D_\gamma \) containing \( q \). Let \( C = E \cdot S_{R_i}(q) \).

Let \( \Psi : Y_n \rightarrow Y \) be obtained by blowing up \( q \), then blowing up the point \( q_1 \) which is the intersection of the exceptional divisor over \( q \) and the strict transform of \( C \) on \( Y_1 \), and iterating this procedure \( n \) times, blowing up the intersection point of the last exceptional divisor with the strict transform of \( C \). Let

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

be the \( \tau \)-quasi-well prepared (or \( \tau \)-well prepared) diagram of \( R \) and \( \Psi \) obtained from Lemma 7.13 (so that \( \Phi \) is an isomorphism above \( f^{-1}(Y - \Sigma(Y)) \)).

Suppose that for all \( n >> 0 \) there exists a 3-point \( p_n \in \Phi^{-1}(p) \cap T(R^n_i) \) such that \( f_n(p_n) = q_n = \Psi^{-1}(q) \cap C_n \), where \( C_n \) is the strict transform of \( C \) on \( Y_n \). Then \( C \) is a component of the fundamental locus of \( f \).

**Proof.** At \( p \) there is an expression

\[
\begin{align*}
u & = x^ay^bz^c \\
v & = x^dy^ez^f \\
w_i & = x^gy^hz^i\gamma
\end{align*}
\]

where

\[
u = v_{R_i}(p), v = v_{R_i}(p), w_i = w_{R_i}(p), \]

\( x, y, z \) are permissible parameters at \( p \) for \( u, v, w \), \( \gamma \) is a unit series, and \( v = 0 \) is a local equation of \( E \) at \( q \). Then \( v = w_i = 0 \) are local equations of \( C \) at \( q \).

By our construction of \( \Psi \), we have that

\[
\begin{align*}
u_1 & = v_1 = v_{R_i}(p_n), w_{i,1} = w_{R_i}(p_n)
\end{align*}
\]

are defined by

\[
u_1 = u, v_1 = u^nv_1, w_{i,1} = u^nw_i.
\]

Now the conclusions of the lemma follow from the argument from (172) to the end of Step 1 in the proof of Theorem 8.7. \( \square \)
Theorem 8.9. Suppose that \( \tau \geq 1 \), \( f : X \to Y \) is \( \tau \)-quasi-well prepared with 2-point relation \( R \) and pre-algebraic structure, and \( D_X \) is cuspidal for \( f \). Then there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that \( \Phi, \Psi \) are products of blow ups of possible centers and \( f_1 \) is \( \tau \)-very-well prepared with 2-point relation \( R^1 \). (In general, \( R^1 \) is not the transform of \( R \)). Further, \( D_{X_1} \) is cuspidal for \( f_1 \).

Proof. After modifying \( R \) by replacing the primitive 2-point relations \( \{ \overline{R}_i \} \) associated to \( R \) with pre-relations such that each \( U(\overline{R}_i) \) is a single point, we may assume that each \( \overline{R}_i \) is algebraic (Definition 6.2), and \( R \) is algebraic (Definition 6.6).

There exists a sequence of blow ups of 2-curves \( \Psi_1 : Y_1 \to Y \) such that 3 and 4 of Definition 7.3 hold for the transforms \( \{ \overline{R}_1^i \} \) of the \( \{ \overline{R}_i \} \) on \( Y_1 \), by embedded resolution of plane curve singularities (cf. Section 3.4, Exercise 3.13 [C3]) and by Lemma 5.14 [C3]). By Lemma 7.11, there exists a \( \tau \)-quasi-well prepared diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]

of \( R \) and \( \Psi_1 \) where \( \Phi_1, \Psi_1 \) are products of blow ups of 2-curves. Thus \( f_1 \) is \( \tau \)-well prepared (with respect to the transform \( R^1 \) of \( R \)). We may thus assume that \( f \) is \( \tau \)-well prepared.

Let

\[
\mathcal{V}_0 = \left\{ \gamma_i = E \cdot S_R(p_i) \text{ such that } E \text{ is a component of } D_Y, p_i \in T(R), \gamma_i \text{ is a component of the fundamental locus of } f \right\}
\]

Suppose that \( \gamma_i \in \mathcal{V}_0 \). Let \( \eta_i \) be a general point of \( \gamma_i \). In a neighborhood of \( \eta_i \), \( f : X \to Y \) can be factored by blowing up finitely many curves which dominate \( \gamma_i \) ([Ab1] or [D]). Let \( r(0) \) be the total number of components of \( D_X \) which dominate the curves \( \gamma_i \in \mathcal{V}_0 \).

**Step 1.** By Lemma 8.8, there exists a \( \tau \)-well prepared diagram of \( R \) and \( \Psi_1 \)

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( \Psi_1 \) is a product of blow ups of prepared 2-points (of type 1 of Definition 7.4) such that if

\[
\gamma_i(1) \in \mathcal{V}_1 = \left\{ \gamma_i(1) = E \cdot S_{R^1}(p_i) \text{ such that } E \text{ is a component of } D_{Y_1}, p_i \in T(R^1) \text{ and } \Psi_1(\gamma_i(1)) \text{ is not a point} \right\}
\]

then \( \gamma_i(1) \) dominates a component \( \gamma_i \) of \( \mathcal{V}_0 \), and \( \Phi_1 \) is an isomorphism over \( f^{-1}(Y - \Sigma(Y)) \) and thus is an isomorphism over the preimage by \( f \) of a general point \( \eta_j \) of \( \gamma_j \) for all \( \gamma_j \in \mathcal{V}_0 \). Let \( r(1) \) be the number of components of \( D_{X_1} \) which dominate some \( \gamma_i \in \mathcal{V}_0 \) and are exceptional for \( f_1 \). We have \( r(1) = r(0) \).
Step 2. By Theorems 8.4 and 8.5 there exists a $\tau$-well prepared diagram of $R^1$

\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & Y_2 \\
\Phi_2 \downarrow & & \downarrow \Psi_2 \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]

where $\Psi_2$ is a product of blow ups of permissible 2-curves and 2-points, followed by the blow ups of the strict transforms of the $\gamma_i$ in $\overline{V}_0$, and possibly by blow ups of more 2-points above the $\gamma_i$. $\Phi_1 \circ \Phi_2$ is an isomorphism over $f^{-1}(Y - \Sigma(Y))$, and thus is an isomorphism over the preimage by $f$ of a general point $\eta_j$ of all $\gamma_j \in \overline{V}_0$. $\Psi_2$ is the blow up of the strict transform $\gamma_j(1)$ of $\gamma_j$ over a general point $\eta_j$ of $\gamma_j$ for all $\gamma_j \in \overline{V}_0$. Let $r(2)$ be the number of components of $D_{X_2}$ which dominate some $\gamma_i \in \overline{V}_0$, and are exceptional for $f_2$. We have $r(2) < r(0)$.

Let

\[
V_2 = \left\{ \gamma_i(2) = E \cdot S_{R^2}(p_i) \text{ such that } E \text{ is a component of } D_{Y_2}, \begin{array}{c} p_i \in T(R^2) \text{ and } (\Phi_1 \circ \Phi_2)(\gamma_i(1)) \text{ is not a point} \end{array} \right\}
\]

By construction, if $\gamma_i(2) \in V_2$, then $\gamma_i(2)$ dominates a component of $\overline{V}_0$.

Step 3. We now repeat Step 1. As in the construction of $f_1$, there exists a $\tau$-well-prepared diagram

\[
\begin{array}{ccc}
X_3 & \xrightarrow{f_3} & Y_3 \\
\Phi_3 \downarrow & & \downarrow \Psi_3 \\
X_2 & \xrightarrow{f_2} & Y_2
\end{array}
\]

where $\Psi_3$ is a product of blow ups of 2-points such that if

\[
\gamma_i(3) \in V_3 = \left\{ \gamma_i(3) = E \cdot S_{R^3}(p_i) \mid E \text{ is a component of } D_{Y_3}, p_i \in T(R^3) \text{ and } (\Phi_1 \circ \Phi_2 \circ \Phi_3)(\gamma_i(3)) \text{ is not a point} \right\}
\]

then $\gamma_i(3)$ is in the fundamental locus of $f_3$, $\gamma_i(3)$ dominates a component $\gamma_i$ of $\overline{V}_0$ and $\Phi_3$ is an isomorphism over $f_{x_2}^{-1}(Y - \Sigma(Y_2))$.

The total number $r(3)$ of components of $D_{X_3}$ which dominate some $\gamma_i \in \overline{V}_0$ is equal to $r(2)$. We now repeat Step 2 to get a reduction $r(4) < r(3)$, where $r(4)$ is the number of components of $D_{X_4}$ which dominate some $\gamma_i \in \overline{V}_0$ and are exceptional for $f_4$. We see that by induction on $r(2n)$, iterating Step 3, we can construct a $\tau$-well-prepared diagram for $R$

\[
\begin{array}{ccc}
X_4 & \xrightarrow{f_4} & Y_4 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that if $E$ is a component of $D_{Y_4}$ and $S = S_{R^4}(p)$ for some $p \in T(R^4)$ intersects $E$, then $\gamma = E \cdot S$ is either contracted to a point by $\Psi$ or $f_4$ is an isomorphism over the generic point of $\gamma$.

By Lemma 8.8, there exists a $\tau$-well prepared diagram for $R^4$

\[
\begin{array}{ccc}
X_5 & \xrightarrow{f_5} & Y_5 \\
\Phi_5 \downarrow & & \downarrow \Psi_5 \\
X_4 & \xrightarrow{f_4} & Y_4
\end{array}
\]

such that if $E$ is a component of $D_{Y_5}$ and $S = S_{R^5}(p)$ for some $p \in T(R^5)$ intersects $E$, then $\gamma = E \cdot S$ is exceptional for $\Psi \circ \Psi_5$. 

Let

\[ W_5 = \left\{ \gamma = \frac{S_{R_5}}{R_5}(q) \cdot E \mid \text{E is a component of } D_{Y_5}, \right. \]
\[ \left. R_5 \text{ is associated to } R^5 \text{ and } q \in f_5(T(R_5^5)) \right\} \]

and let

\[ Z_5 = \left\{ q \in U(R^5) - f_5(T(R_5^5)) \text{ such that there exist } \gamma_i, \gamma_j \in W_5 \right. \]
\[ \left. \text{such that } \gamma_i \neq \gamma_j \text{ and } q \in \gamma_i \cap \gamma_j. \right\} \]

The points of \( Z_5 \) are prepared 2-points for \( R^5 \) (of type 1 of Definition 7.4). Let \( \Psi_6 : Y_6 \to Y_5 \) be the blow up of \( Z_5 \). By Lemma 7.13, there exists a \( \tau \)-well prepared diagram

\[
\begin{array}{ccc}
X_6 & f_6 & Y_6 \\
\Phi_6 | & \downarrow & \downarrow \Psi_6 \\
X_5 & f_5 & Y_5
\end{array}
\]

of \( R^5 \) and \( \Psi_6 \).

Define

\[ W_6 = \left\{ \gamma = \frac{S_{R_6}}{R_6}(q) \cdot E \mid \text{E is a component of } D_{Y_6}, \right. \]
\[ \left. R_6 \text{ is associated to } R^6 \text{ and } q \in \Psi_6^{-1}(f_5(T(R_5^5))) \cap U(R_6^6) \right\} \]

\[ Z_6 = \left\{ q \in U(R^6) - \Psi_6^{-1}(f_5(T(R_5^5))) \cap U(R^6) \text{ such that there exist } \gamma_i, \gamma_j \in W_6 \right. \]
\[ \left. \text{such that } \gamma_i \neq \gamma_j \text{ and } q \in \gamma_i \cap \gamma_j. \right\} \]

We necessarily have that the curves in \( W_6 \) are strict transforms of curves in \( W_5 \). We can iterate, blowing up \( Z_6 \), and constructing a \( \tau \)-well prepared diagram, and repeating until we eventually construct a \( \tau \)-well prepared diagram

\[
\begin{array}{ccc}
X_7 & f_7 & Y_7 \\
\Phi_7 | & \downarrow & \downarrow \Psi_7 \\
X_6 & f_6 & Y_6
\end{array}
\]

such that \( \Psi_7 \) is a sequence of blow ups of prepared 2-points (of type 1 of Definition 7.4) and if \( \gamma_1 = \frac{S_{R_7}}{R_7}(p_1) \cdot E_1, \gamma_2 = \frac{S_{R_7}}{R_7}(p_2) \cdot E_2 \) for \( p_1, p_2 \in T(R_7^7) \) and \( E_1, E_2 \) components of \( D_{Y_7} \), are such that \( \gamma_1 \neq \gamma_2 \), then \( \gamma_1 \cap \gamma_2 \subset U(R_7^7) \cap (\Psi_6 \circ \Psi_7)^{-1}(f_5(T(R_5^5))) \).

Let \( \beta = \Psi \circ \Psi_5 \circ \Psi_6 \circ \Psi_7 \). We now construct pre-relations \( \overline{R}_i \) on \( Y_7 \) with associated primitive relations \( R_i^7 \) for \( f_7 \).

If \( \overline{R}_i \) is a 2-point pre-relation associated to \( R \), let \( T(R_i^7) = T(\overline{R}_i) \) and let

\[ U(\overline{R}_i^7) = \left\{ q \in U(R^7) \cap (\Psi_6 \circ \Psi_7)^{-1}(f_5(T(R^5))) \text{ such that } q \in E \cdot S_{R_7}(p) \right\} \]

(188)

For \( q' \in U(\overline{R}_i^7) \), define \( \overline{R}_i(q') = \overline{R}_i^7(q') \). For \( p \in T(R_i^7) \) define \( R_i(p) = \overline{R}_i^7(f_7(p)) \).

Let \( R' \) be the 2-point relation for \( f_7 \) defined by the \( \{ R_i^7 \} \). Let \( \Omega(\overline{R}_i^7) \) be an open subset of \( \Omega(\overline{R}_i) \) which contains all \( E \cdot S_{R_i}(q) \) for \( q \in U(\overline{R}_i^7) \). Recall that these curves are all exceptional for \( \beta \) and we can take \( \Omega(\overline{R}_i) \) so that \( \Omega(\overline{R}_i) \cap U(\overline{R}) = U(\overline{R}_i) \).

\( f_7 \) is \( \tau \)-well prepared with 2-point relation \( R' \). For all \( \overline{R}_i^7 \), let

\[ V_i(Y_7) = \left\{ \gamma = \frac{E_\alpha}{R_7}(q) \mid q \in U(\overline{R}_i^7), E_\alpha \text{ is a component of } D_{Y_7} \right\} \]

By our construction, Lemmas 7.11, 7.13 and 7.14, and 2 of Remark 7.16, every curve \( \gamma \in V_i(Y_7) \) is prepared for \( R^5 \) of type 4. By (188), we now conclude that every curve \( \gamma \in V_i(Y_7) \) is prepared for \( R' \) of type 4. 1 and 2 of Definition 7.6 thus hold for \( f_7 \) and \( R' \). 3 of Definition 7.6 holds for \( f_7 \) and \( R' \) since for all \( \overline{R}_i^7, V_i(Y_7) \) consists of
exceptional curves of $\Omega(R')$ contracting to a nonsingular point $q_i \in \Omega(R_i)$. Thus $f_7$ is $\tau$-very-well prepared with 2-point relation $R'$.

\[ \square \]

**Theorem 8.10.** Suppose that $f : X \to Y$ is prepared, $\tau f(X) = \tau \geq 1$ and $D_X$ is cuspidal for $f$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

such that $f_1$ is prepared, $\Phi, \Psi$ are products of blowups of 2-curves, $\tau f_1(X_1) = \tau$, $D_{X_1}$ is cuspidal for $f_1$ and all 3-points $p \in X_1$ such that $\tau f_1(p) = \tau \mapsto 2$-points of $Y_1$.

**Proof.** We first define a 3-point relation $R$ on $X$. Let

\[ T = \{ p \in X \mid p \text{ is a 3-point, } q = f(p) \text{ is a 3-point and } \tau f(p) = \tau \} . \]

For $p \in T$, since $\tau \geq 1$, there exist permissible parameters $u, v, w$ at $q = f(p)$ and permissible parameters $x, y, z$ at $p$ such that

\[
\begin{align*}
  u &= x^5 y^5 z^5 \\
  v &= x^5 y^5 z^5 \\
  w &= x^5 y^5 z^5 
\end{align*}
\]

where $\gamma$ is a unit series, $\text{rank}(u,v) = 2$ and $\text{rank}(u,v,w) = 2$. Thus there exist $a, b, c \in \mathbb{Z}$ such that

\[
(x^5 y^5 z^5)^a (x^5 y^5 z^5)^b (x^5 y^5 z^5)^c = 1, \quad (189)
\]

with $\gcd(a, b, c) = 1$ and $\min\{a, b, c\} < 0 < \max\{a, b, c\}$.

For $p \in T$, define 3-point pre-relations $R_p$ on $Y$ by $U(R_p) = \{ q = f(p) \}$, and $R(q)$ is defined (with the notation of (91)) so that $u = 0$ is a local equation of $E_1$, $v = 0$ is a local equation of $E_2$, $w = 0$ is a local equation of $E_3$; $a, b, c$ are defined by (189) and $\lambda = \lambda_p = \gamma(0,0,0)^c$.

For $p \in T$, we now define primitive 3-point relations $R_p$ for $f$ by $T(R_p) = \{ p \}$, with associated 3-point pre-relation $R_p$. We define $R$ to be the associated 3-point relation for $f$ with $T(R) = \bigcup_{p \in T} T(R_p) = T$.

By Theorem 6.10, there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi \downarrow & & \downarrow \Psi \\
X & \xrightarrow{f} & Y
\end{array}
\]

where $\Phi, \Psi$ are products of blowups of 2-curves such that $f_1$ is prepared, $\tau f_1(X_1) = \tau$, the transform $R^1$ of $R$ for $f_1$ is resolved, and $D_{X_1}$ is cuspidal for $f_1$. Thus all 3-points $p \in X_1$ with $\tau f_1(p) = \tau \mapsto 2$-points of $Y_1$.

\[ \square \]

**Theorem 8.11.** Suppose that $f : X \to Y$ is prepared, $\tau f(X) = \tau \geq 1$ and $D_X$ is cuspidal for $f$. Then there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]
such that $\Phi_1$ and $\Psi_1$ are products of blow ups of possible centers, $f_1$ is $\tau$-very-well prepared with 2-point relation $R^1$, and $D_{X_1}$ is cuspidal for $f_1$.

Proof. By Theorem 8.10, there exists a commutative diagram

$$
\begin{array}{c}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi & \downarrow & \Psi \\
X & \xrightarrow{f} & Y
\end{array}
$$

such that $\Phi$ and $\Psi$ are products of 2-curves, $f_1$ is prepared, $\tau f_1(X_1) \leq \tau f(X)$, $D_{X_1}$ is cuspidal for $f_1$ and all 3-points $p$ of $X_1$ such that $\tau f_1(p) = \tau$ map to 2-points of $Y_1$.

Now by Theorems 8.1, 8.7 and 8.9 there exists a commutative diagram

$$
\begin{array}{c}
X_2 & \xrightarrow{f_2} & Y_2 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
$$

where the vertical arrows are products of blow ups of possible centers such that $f_2$ is $\tau$-very-well prepared and $D_{X_2}$ is cuspidal for $f_2$.

\[ \square \]

9. Toroidalization

Suppose that $f : X \to Y$ is a birational morphism of nonsingular projective 3-folds with toroidal structures $D_Y$ and $D_X = f^{-1}(D_Y)$, such that $D_X$ contains the singular locus of $f$.

Theorem 9.1. Suppose that $\tau \geq 1$ and $f : X \to Y$ is $\tau$-very-well prepared with 2-point relation $R$. Further suppose that $D_X$ is cuspidal for $f$. Then there exists a $\tau$-very-well prepared diagram

$$
\begin{array}{c}
X_1 & \xrightarrow{f_1} & Y_1 \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
$$

such that the transform $R^1$ of $R$ is resolved and $D_{X_1}$ is cuspidal for $f_1$. In particular, $f_1$ is prepared, $\tau f_1(X_1) < \tau$, and $D_{X_1}$ is cuspidal for $f_1$.

Proof. Fix a pre-relation $\mathcal{R}_i$ associated to $R$ on $Y$, with associated primitive relation $R_i$. By induction on the number of pre-relations associated to $R_i$ it suffices to resolve $R_i$ by a $\tau$-very-well prepared diagram (for $R$).

Recall (Definition 7.6)

$$
V_i(Y) = \left\{ \frac{E \cdot S}{S = S_{\mathcal{R}_i}(q) \text{ for some } q \in U(\mathcal{R}_i)} \mid \text{E is a component of } D_Y \right\}.
$$

$F_i = \sum_{\gamma \in V_i(Y)} \gamma$ is a SNC divisor on $\Omega(\mathcal{R}_i)$ whose intersection graph is a tree.

If $\gamma_1 = E_1 \cdot S_{\mathcal{R}_i}(q_i) \in V_i(Y)$ and $q \in \gamma_1$, say that $\gamma_1$ is good at $q$ if whenever $q \in U(\mathcal{R}_i)$ for some $i$, then $S_{\mathcal{R}_i}(q)$ contains the germ of $\gamma_1$ at $q$ (so that $\gamma_1 = E_1 \cdot S_{\mathcal{R}_i}(q) \subset \Omega(\mathcal{R}_i)$). Otherwise, say that $\gamma_1$ is bad at $q$. Say that $\gamma_1$ is good if $\gamma_1$ is good at $q$ for all $q \in \gamma_1$. 
Let $Y_0 = Y$, $X_0 = X$, $f_0 = f$. We will show that there exists a sequence of $\tau$-very-well prepared diagrams

$$
\begin{align*}
X_i & \xrightarrow{\Phi_{i+1}} Y_{i+1} \\
& \Phi_{i+1} \downarrow \\
& \Psi_i \downarrow \\
Y_i & \xrightarrow{\Psi_{i+1}} Y_{i+1}
\end{align*}
$$

for $0 \leq i \leq m - 1$ such that the transform $R'_{i+1}$ of $R_i$ on $X_{i+1}$ is resolved.

Suppose that $\gamma_1 \in V_i(Y)$ and $q \in \gamma_1$ is a bad point. By Remark 7.7, we have that $q \in U(R_i)$. Suppose that $E_1, E_2$ are the two components of $D_Y$ containing $q$, and $\gamma_1 = \frac{E_1}{S_{R_i}(q)}$. Let $\gamma_2 = \frac{E_2}{S_{R_i}(q)}$.

We will show that $q$ is a good point of $\gamma_2$.

$\gamma_1$ not good at $q$ implies there exists $j \neq t$ such that $q \in U(R_j)$ and the germ of $\gamma_1$ at $q$ is not contained in $S_{R_j}(q)$. Let

$$
u = \nu_{R_i}(q), \nu = \nu_{R_i}(q), w_t = w_{R_i}(q).$$

After possibly interchanging $u$ and $v$ we have that $u = w_1 = 0$ are local equations of $\gamma_1$, $v = w_t = 0$ are local equations of $\gamma_2$ at $q$. Let $w_j = w_{R_j}(q)$. In the equation

$$w_j = w_t + u^{a_{1j}} v^{b_{1j}} \phi_{1j},$$

we thus have $b_{1j} = 0$. But we must have

$$(0, b_{1j}) \leq (a_{1j}, 0) \text{ or } (a_{1k}, 0) \leq (0, b_{1k})$$

by 4 of Definition 7.3, which is impossible. Thus $q$ is a good point for $\gamma_2$.

Suppose that all $\gamma \in V_i(Y)$ are bad. Pick $\gamma_1 \in V(Y)$. Since $\gamma_1$ is bad there exists $\gamma_2 \in V_i(Y) - \{\gamma_1\}$ such that $\gamma_2$ is good at $q_1 = \gamma_1 \cap \gamma_2$ (as shown above), $\gamma_1 \cap \gamma_2$ is a single point since $V_i(Y)$ is a tree. Since $\gamma_2$ is bad and $V_i(Y)$ is a tree, there exists $\gamma_3 \in V_i(Y)$ which intersects $\gamma_2$ at a single point $q_2$ and is disjoint from $\gamma_1$ such that $\gamma_3$ is good at $q_2$. Since $V_i(Y)$ is a finite set, we must eventually find a curve which is good, a contradiction.

Let $\gamma \in V_i(Y)$ be a good curve, so that it is prepared for $R$ of type 4, and is an *-permissible center (Lemma 7.15) and let $\Psi_1 : Y_1' \rightarrow Y$ be the blow up of $\gamma$.

By Lemma 7.15 we can construct a $\tau$-very-well prepared diagram of the form of (111) of Definition 7.9

$$
\begin{align*}
X_1 & \xrightarrow{f_2} Y_1 \\
& \downarrow \downarrow \\
& Y_1' \xrightarrow{\Psi'_1} Y' \\
& \downarrow \downarrow \\
X & \rightarrow Y.
\end{align*}
$$

where $Y_1 \rightarrow Y_1'$ is a sequence of blow ups of 2-points which are prepared for the transform of $R$ of type 2 of Definition 7.4. Observe that if $\gamma_1 \in V_i(Y)$ is a good curve, with $\gamma_1 \neq \gamma$, then the strict transform of $\gamma_1$ is a good curve in $V_i(Y_1)$.

We now iterate this process. We order the curves in $V_i(Y)$, and choose $\gamma = \frac{E}{S_{R_i}(q)} \in V_i(Y)$ in the construction of the diagram (191) so that it is the minimum good curve in $V_i(Y)$. 
We inductively define a sequence of \( \tau \)-very well prepared diagrams (190) by blowing up the good curve in \( V_t(Y_t) \) with smallest order, and then constructing a very well prepared diagram (190) of the form of (191). Then we define the total ordering on \( V_t(Y_{t+1}) \) so that the ordering of strict transforms of elements of \( V_t(Y_t) \) is preserved, and these strict transforms have smaller order than the element of \( V_t(Y_{t+1}) \) which is not a strict transform of an element of \( V_t(Y_t) \). We repeat, as long as \( R_t^m \) is not resolved (\( T(R_t^m) \neq \emptyset \)).

Suppose that the algorithm does not converge in the construction of \( f_m : X_m \to Y_m \) such that the transform \( R_t^m \) of \( R_t \) is resolved. Then there exists a diagram

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_n} & Y_n \\
\Phi_n & \downarrow & \downarrow \\
X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \\
\vdots & \ddots & \ddots \\
X_0 &=& f_0 = f \\
\end{array}
\]

(192)

constructed by infinitely many iterations of the algorithm such that \( T(R_t^m) \neq \emptyset \) for all \( n \).

Suppose that \( q_n \in U(R_t^n) \) is an infinite sequence of points such that \( \Psi_n(q_n) = q_{n-1} \) for all \( n \) and \( \Psi_n \) is not an isomorphism for infinitely many \( n \).

By construction, the restriction of \( \Psi_n \) to \( S_{R_t^n}(q_n) \) is an isomorphism onto \( S_{R_t^{n-1}}(q_{n-1}) \) for all \( n \). Thus the restriction

\[
\overline{\Psi}_n = \Psi_1 \circ \cdots \circ \Psi_n : S_{R_t^n}(q_n) \to S_{R_t^n}(q)
\]

is an isomorphism, where \( q = q_0 = \Psi_1 \circ \cdots \circ \Psi_n(q_n) \). Without loss of generality, we may assume that no \( \Psi_n \) is an isomorphism (on \( Y_n \)) at \( q_n \). We have permissible parameters \( u_i = w_{R_t^n}(q_i), v_i = v_{R_t^n}(q_i), w_{t,i} = w_{R_t^n}(q_i) \) at \( q_i \) for all \( i \) such that either

\[
u_i = u_{i+1}, v_i = v_{i+1}, w_{t,i} = u_{i+1}w_{t,i+1}
\]  

(193)

or

\[
u_i = u_{i+1}, v_i = v_{i+1}, w_{t,i} = v_{i+1}w_{t,i+1}.
\]  

(194)

Suppose there exists \( k \neq t \) such that \( q_n \in U(R_k^n) \) for all \( n \).

Let \( w_{k,i} = w_{R_k^n}(q_i) \) for \( i \geq 0 \).

The relations

\[
w_{k,i} - w_{t,i} = u_i a_{tk} v_i^b_{tk} \phi_{t,k}
\]

of 3 of Definition 7.3 transform to

\[
w_{k,i+1} - w_{t,i+1} = u_i a_{tk}^{-1} v_i^b_{tk} \phi_{t,k}
\]

under (193), and transform to

\[
w_{k,i+1} - w_{t,i+1} = u_i a_{tk} v_i^b_{tk} \phi_{t,k}
\]

under (194). But we see that after a finite number of iterations \( q_n \notin U(R_k^n) \), unless \( a_{tk} = b_{tk} = \infty \). Thus there exists \( n_0 \), such that whenever \( n \geq n_0, q_n \notin U(R_k^n) \) if \( k \neq t \) and \( a_{kt}, b_{kt} \neq \infty \).
Theorem 9.2. Suppose that \( f : X \to Y \) is prepared, \( \tau_f(X) = -\infty \) and \( D_X \) is cuspidal for \( f \). Then \( f \) is toroidal.
Proof. Suppose that $E$ is a component of $D_X$ and $E$ contains a 3-point $p$. Let $f(p) = q$. $q$ is a 3-point, and if $\pi, \tau, \omega$ are permissible parameters at $q$, then there exists an expression

$$
\begin{align*}
\pi & = \pi^{a_{11}} \pi^{a_{12}} \pi^{a_{13}} \\
\tau & = \pi^{b_{21}} \pi^{b_{22}} \pi^{b_{23}} \\
\omega & = \pi^{c_{31}} \pi^{c_{32}} \pi^{c_{33}}
\end{align*}
$$

(197)

at $p$, where $\pi, \tau, \omega$ are permissible parameters at $\pi, \tau, \omega$ and $x = 0$ is a local equation of $E$. In particular, $f$ has a toroidal form of type 1 following Definition 3.7 at $p$. Thus $f(E) = D$ is a component of $D_Y$, $f(E) = C$ is a 2-curve or $f(E) = \emptyset$. Since $D_X$ is cuspidal for $f$, all components $E$ of $D_X$ map to a 3-point, a 2-curve or a component of $D_Y$.

Suppose that $E$ is a component of $D_X$, $p \in E$, $q = f(p)$, $u, v, w$ are permissible parameters at $q$, and $x, y, z$ are permissible parameters for $u, v, w$ at $p$, such that $x = 0$ is a local equation of $E$. For $g \in \hat{O}_{X,p}$, let $\ord_E g$ be the largest power of $x$ which divides $g$ in $\hat{O}_{X,p}$. Let

$$
\text{Jac}(f) = \det \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\
\frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}
\end{pmatrix}
$$

Then we define

$$
\lambda(E) = \ord_E uvw - \ord_E \text{Jac}(f).
$$

$\lambda(E)$ is an invariant of $E$, since $f(E)$ is a component of $D_Y$, a 2-curve or a 3-point. Notice that

$$
\lambda(E) = \ord_E uv - \ord_E \text{Jac}(f)
$$

if $q \in f(E)$ is a 2-point, and

$$
\lambda(E) = \ord_E u - \ord_E \text{Jac}(f)
$$

if $q \in f(E)$ is a 1-point. We compute $\lambda(E)$ in (197) or in a toroidal form following Definition 3.7 if $E$ does not contain a 3-point (recall that $D_X$ is cuspidal for $f$), to see that

$$
\lambda(E) = 1
$$

for all components $E$ of $D_X$.

We will now verify that if $p \in D_X$, then $f$ is toroidal at $p$, from which the conclusions of the theorem follow.

Case 1. Suppose that $f(p) = q$ is a 3-point. Let $u, v, w$ be permissible parameters at $q$, and $x, y, z$ be permissible parameters for $u, v, w$ at $p$.

First suppose that $p$ is a 1-point. Then (since $f$ is prepared), we have an expression (after possibly interchanging $u, v, w$)

$$
\begin{align*}
u & = x^a \\
v & = x^b (\alpha + y) \\
w & = x^c (\gamma(x, y) + x^d z)
\end{align*}
$$

(198)

with $0 \neq \alpha, a, b, c > 0$, $\gamma$ a unit series, $d \geq 0$.

Let $E$ be the component of $D_X$ containing $p$. Computing $\lambda(E)$ at $p$ from (198), we get

$$
1 = (a + b + c) - (a + b + c + d - 1) = 1 - d.
$$

Thus $d = 0$ and $u, v, w$ have the toroidal form 3 following Definition 3.7 at $p$. 

Suppose that \( p \) is a 2-point. Let \( E \) and \( E' \) be the components of \( D_X \) containing \( p \). Then we have (after possibly interchanging \( u,v,w \) and \( x,y,z \)), a form

\[
    u = x^a y^b, \quad v = x^c y^d, \quad w = x^m y^n (g(x,y) + x^e y^f z)
\]

where \( g(x,y) \) is a unit series, \( x = 0 \) is a local equation of \( E \), and \( y = 0 \) is a local equation of \( E' \), or

\[
    u = (x^a y^b)^k, \quad v = (x^a y^b)^m (g(x^a y^b, z) + x^c y^d)
\]

where \( g \) is a unit series, \( \alpha \neq 0 \), \( ad - bc \neq 0 \), \( x = 0 \) is a local equation of \( E \), and \( y = 0 \) is a local equation of \( E' \).

In (199), we compute,

\[
    1 = \lambda(E) = a + c + m - [a + c + e + m - 1] = 1 - e
\]

implies \( e = 0 \). We also compute

\[
    1 = \lambda(E') = b + d + n - [b + d + n + f - 1] = 1 - f
\]

which implies \( f = 0 \). Thus we have a toroidal form \( 2 \) following Definition 3.7 at \( p \).

Suppose that \( p \) satisfies (200). Then

\[
    1 = \lambda(E) = a(k + t + m) - [a(k + t + m) + c - 1] = 1 - c
\]

implies \( c = 0 \).

\[
    1 = \lambda(E') = b(k + t + m) - [b(k + t + m) + d - 1] = 1 - d
\]

implies \( d = 0 \). This is impossible, so (200) cannot occur.

**Case 2. Suppose that \( f(p) = q \) is a 2-point.** Let \( u,v,w \) be permissible parameters at \( q \), and \( x,y,z \) be permissible parameters for \( u,v,w \) at \( p \).

Suppose that \( p \) is a 1-point, and let \( E \) be the component of \( D_X \) containing \( p \). We have an expression

\[
    u = x^a, \quad v = x^b (\alpha + y), \quad w = g(x,y) + x^c z
\]

(201)

where \( x = 0 \) is a local equation of \( E \), and \( 0 \neq \alpha \), or

\[
    u = x^a, \quad v = x^c (\gamma(x,y) + x^d z), \quad w = y
\]

(202)

where \( \gamma \) is a unit series.

Computing \( \lambda(E) \) in (201), we have

\[
    1 = \lambda(E) = a + b - (a + b + c - 1) = 1 - c
\]

implies \( c = 0 \). Thus after making an appropriate change of variables, we have a toroidal form \( 5 \) following Definition 3.7 at \( p \).

Computing \( \lambda(E) \) in (202), we have

\[
    1 = \lambda(E) = (a + c) - [a + c + d - 1] = 1 - d
\]

implies \( d = 0 \). Thus \( f \) has a toroidal form \( 5 \) following Definition 3.7 at \( p \).

Suppose that \( p \) is a 2-point. Let \( E \) and \( E' \) be the components of \( D_X \) containing \( p \). Then we have a form

\[
    u = x^a y^b, \quad v = x^c y^d, \quad w = g(x,y) + x^e y^f z
\]

(203)
where \( x = 0 \) is a local equation of \( E \), \( y = 0 \) is a local equation of \( E' \), and \( ad - bc \neq 0 \), or
\[
u = (x^ay^b)^k, v = (x^ay^b)^l(\alpha + z), w = g(x^ay^b, z) + x^cy^d \tag{204}\]
where \( x = 0 \) is a local equation of \( E \), \( y = 0 \) is a local equation of \( E' \), \( 0 \neq \alpha \in \mathbb{k} \) and \( ad - bc \neq 0 \), or
\[
u = (x^ay^b)^k, v = (x^ay^b)^l(\gamma(x^ay^b, z) + x^cy^d), w = z \tag{205}\]
where \( \gamma \) is a unit series, \( x = 0 \) is a local equation of \( E \), \( y = 0 \) is a local equation of \( E' \), \( a, b > 0 \) and \( ad - bc \neq 0 \).

Computing \( \lambda(E), \lambda(E') \) in (203), we get
\[
1 = \lambda(E) = a + c - (a + c + e - 1) = 1 - e
\]
which implies \( e = 0 \).
\[
1 = \lambda(E') = b + d - (b + d + f - 1) = 1 - f
\]
implies \( f = 0 \). Thus after an appropriate change of variables, we have a toroidal form 4 following Definition 3.7 at \( p \).

Computing \( \lambda(E), \lambda(E') \) in (204), we get
\[
1 = \lambda(E) = a(k + t) - [a(k + t) + c - 1] = 1 - c
\]
which implies \( c = 0 \), and
\[
1 = \lambda(E') = b(k + t) - [b(k + t) + d - 1] = 1 - d
\]
which implies \( d = 0 \). Since \( c = d = 0 \) is impossible in (204), we see that this form cannot occur.

Computing \( \lambda(E), \lambda(E') \) in (205), we get
\[
1 = \lambda(E) = a(k + l) - [a(k + l) + c - 1] = 1 - c
\]
implies \( c = 0 \),
\[
1 = \lambda(E') = b(k + l) - [b(k + l) + d - 1] = 1 - d
\]
implies \( d = 0 \). Thus (205) cannot hold.

**Case 3. Suppose that \( f(p) = q \) is a 1-point.** Let \( u, v, w \) be permissible parameters at \( q \) satisfying 3 of Definition 3.4. Suppose that \( p \) is a 1-point, and let \( E \) be the component of \( D_X \) containing \( p \). We have an expression
\[
u = x^a, v = y, w = g(x, y) + x^cz \tag{206}\]
at \( p \).
\[
1 = \lambda(E) = a - [a + c - 1] = 1 - c
\]
implies \( c = 0 \) and thus \( f \) has a toroidal form 6 following Definition 3.7 at \( p \).

Suppose that \( p \) is a 2-point, and let \( E, E' \) be the components of \( D_X \) containing \( p \). We have an expression
\[
u = (x^ay^b)^k, v = z, w = g(x^ay^b, z) + x^cy^d \tag{207}\]
at \( p \), where \( x = 0 \) is a local equation of \( E \), \( y = 0 \) is a local equation of \( E' \), \( a, b > 0 \) and \( ad - bc \neq 0 \).
\[
1 = \lambda(E) = ak - [ak + c - 1] = 1 - c
\]
implies \( c = 0 \).
\[
1 = \lambda(E') = bk - [bk + d - 1] = 1 - d
\]
implies \( d = 0 \). Thus (207) cannot occur. \( \square \)
Proof of Theorem 0.1 By resolution of singularities and resolution of indeterminacy [H] (cf. Section 6.8 [C3]), and by [M], there exists a commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & Y_1 \\
\Phi_1 \downarrow & & \downarrow \Psi_1 \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \(\Phi_1, \Psi_1\) are products of blow ups of points and nonsingular curves, such that \(X_1\) and \(Y_1\) are nonsingular and projective. By Lemma 4.3, Remark 4.4 and Theorem 4.5, we can construct a commutative diagram

\[
\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & Y_2 \\
\Phi_2 \downarrow & & \downarrow \Psi_2 \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]

such that \(\Phi_2, \Psi_2\) are products of blow ups of points and nonsingular curves, such that \(f_2\) is prepared and \(D_{X_2}\) is cuspidal for \(f_2\).

Now by descending induction on \(\tau(X_2)\) and Theorems 8.11 and 9.1, there exists a commutative diagram

\[
\begin{array}{ccc}
X_3 & \xrightarrow{f_3} & Y_3 \\
\Phi_3 \downarrow & & \downarrow \Psi_3 \\
X_2 & \xrightarrow{f_2} & Y_2
\end{array}
\]

such that \(\Phi_2, \Psi_3\) are products of blow ups of possible centers, \(f_3\) is prepared, \(D_{X_3}\) is cuspidal for \(f_3\) and \(\tau_{f_3}(X_3) = -\infty\).

By Theorem 9.2, \(f_3\) is toroidal, and the conclusions of the theorem follow.

References


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Danilov, V., Birational geometry of toric 3-folds, Math USSR Izv. 21 (83), 269 – 280.


