1. Tensor Products

**Theorem 1.1.** Suppose that $V$, $W$ are finite dimensional vector spaces over a field $F$. Then there exists a vector space $T$ over $F$, and a bilinear map $\varphi : V \times W \to T$ such that $T$ satisfies the following “universal property”:

If $U$ is a vector space over $F$ and $g : V \times W \to U$ is a bilinear map then there exists a unique linear map $g_\ast : T \to U$ such that for all $(v, w) \in V \times W$ we have $g(v, w) = g_\ast(\varphi(v, w))$.

**Lemma 1.2.** Suppose that $T_1$, with bilinear map $\varphi_1 : V \times W \to T_1$, satisfies the universal property of theorem 1.1, and $T_2$, with bilinear map $\varphi_2 : V \times W \to T_2$, satisfies the universal property of theorem 1.1. Then there exists a (unique) linear isomorphism $\psi : T_1 \to T_2$ such that $\psi \varphi_1 = \varphi_2$.

Thus the vector space $T$ of the conclusions of Theorem 1.1 is uniquely determined up to isomorphism by the universal property. We denote $T$ by $V \otimes W$ and call $T$ the “tensor product of $V$ and $W$ over $F$”. We denote $\varphi(v, w) = v \otimes w$ for $v \in V$ and $w \in W$, so that $g(v, w) = g_\ast(v \otimes w)$.

Since $\varphi : V \times W$ is bilinear, we have that

$$
\begin{align*}
(x + x') \otimes y &= (x \otimes y) + (x' \otimes y), \\
(cx) \otimes y &= c(x \otimes y), \\
x \otimes (y + y') &= (x \otimes y) + (x \otimes y'), \\
x \otimes (cy) &= c(x \otimes y)
\end{align*}
$$

for $x, x' \in V$, $y, y' \in W$ and $c \in F$.

Theorem 1.1 states that there is a “commutative diagram”

\[ V \times W \xrightarrow{\varphi} V \otimes W \xrightarrow{g_\ast} U \]

That is, $g_\ast \varphi = g$.

To prove Theorem 1.1, we will use the following theorem.

**Theorem 1.3.** Let $S$ be a set of objects, and $F$ be a field. Then there exists a vector space $V(S)$ over $F$ which contains $S$, and $S$ is a basis of $V(S)$ over $F$.

The elements of $V(S)$ are “formal” linear combinations $c_1v_1 + \cdots + c_nv_n$ with $v_1, \ldots, v_n \in S$ and $c_1, \ldots, c_n \in F$. Two sums $c_1v_1 + \cdots + c_nv_n$ and $d_1v_1 + \cdots + d_nv_n$ in $V(S)$ are equal precisely when $c_i = d_i$ for $1 \leq i \leq n$.

We now prove Theorem 1.1. Let $\{v_1, \ldots, v_m\}$ be a basis of $V$ and let $\{w_1, \ldots, w_n\}$ be a basis of $W$. Let $t_{i,j}$ be a symbol for $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $T = V(\{t_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\})$ be the vector space over $F$ consisting of formal linear combinations of
the $t_{i,j}$, so that $\{t_{i,j} \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ is a basis of $T$. The elements of $T$ are linear combinations

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} t_{i,j}$$

with $c_{ij} \in F$. Define a bilinear map $\varphi : V \times W \to T$ by

$$\varphi(v, w) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j t_{i,j}$$

for $v = \sum_{i=1}^{m} x_i v_i \in V$ and $w = \sum_{j=1}^{n} y_j w_j \in W$.

We now prove that $T$ with the bilinear map $\varphi$ has the desired universal property. Suppose that $g : V \times W \to U$ is a bilinear map. Since $\{t_{ij}\}$ is a basis of $T$, there exists a unique linear map $g_* : T \to U$ such that $g_*(t_{ij}) = g(v_i, w_j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Since $\varphi(v_i, w_j) = t_{ij}$, we see that the map $g_*$ of the conclusions of the Theorem exists and is uniquely determined.

Proof of Lemma 1.2: Since $T_1$ with bilinear map $\varphi_1 : V \times W \to T_1$ satisfies the universal property, we have a unique linear map $(\varphi_2)_* : T_1 \to T_2$ such that $(\varphi_2)_* \varphi_1 = \varphi_2$. These maps form a commutative diagram

$$V \times W \xrightarrow{\varphi_1} T_1 \xrightarrow{\varphi_2} (\varphi_2)_* : T_2 \quad (2)$$

Since $T_2$ with bilinear map $\varphi_2 : V \times W \to T_2$ satisfies the universal property, we have a unique linear map $(\varphi_1)_* : T_2 \to T_1$ such that $(\varphi_1)_* \varphi_2 = \varphi_1$. These maps form a commutative diagram

$$V \times W \xrightarrow{\varphi_2} T_2 \xleftarrow{(\varphi_1)_*} T_1 \quad (3)$$

Thus $(\varphi_1)_*(\varphi_2)_* : T_1 \to T_1$ satisfies $(\varphi_1)_*(\varphi_2)_* \varphi_1 = \varphi_1$. That is, combining the commutative diagrams (2) and (3), we obtain a commutative diagram

$$V \times W \xrightarrow{\varphi_1} T_1 \xleftarrow{(\varphi_1)_*(\varphi_2)_*} T_1 \quad (4)$$

Now we have that the identity map $I_{T_1} : T_1 \to T_1$ satisfies the condition that $I_{T_1} \varphi_1 = \varphi_1$, so by uniqueness in the universal property of $T_1$, we have that $I_{T_1} = (\varphi_1)_*(\varphi_2)_*$.

Combining (3) and (2), and applying the the universal property of $T_2$, we obtain that the identity map of $T_2$ is $I_{T_2} = (\varphi_2)_*(\varphi_1)_*$. Thus $(\varphi_1)_*$ is an isomorphism of $T_1$ and $T_2$ with inverse $(\varphi_2)_*$.

**Lemma 1.4.** If $V$ and $W$ are finite dimensional vector spaces, of respective dimensions $m$ and $n$, then $V \otimes W$ has dimension $mn$.

**Proof.** This follows from the construction of $T$ in the proof of Theorem 1.1. $\square$

**Lemma 1.5.** Suppose that $\{\alpha_i \mid 1 \leq i \leq m\}$ is a basis of $V$ and $\{\beta_j \mid 1 \leq j \leq n\}$ is a basis of $W$. Then $\{\alpha_i \otimes \beta_j \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ is a basis of $V \otimes W$. 

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Proof. One proof of this follows from our construction of $V \otimes W$, which is applicable for any bases of $V$ and $W$ (and the uniqueness of this construction up to isomorphism by Lemma 1.2).

A direct proof is obtained as follows. Let $\{v_1, \ldots, v_m\}$ be the basis of $V$ and $\{w_1, \ldots, w_n\}$ be the basis of $V \otimes W$ used in our construction of $V \otimes W$, so that $\{v_i \otimes w_j \mid 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ is a basis of $V \otimes W$. Expand

$$v_i = \sum_{k=1}^{m} x_{ik} \alpha_k$$

and

$$w_j = \sum_{l=1}^{n} y_{jl} \beta_l.$$ 

We obtain

$$v_i \otimes w_j = \sum_{k=1}^{m} \sum_{l=1}^{n} x_{ik} y_{jl} \alpha_k \otimes \beta_l$$

for $1 \leq i \leq m$ and $1 \leq j \leq n$. Thus the $mn$ vectors $\{\alpha_i \otimes \beta_j\}$ span $V \otimes W$, as $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$. Since the $mn$ vectors $\{v_i \otimes w_j\}$ are a basis of $V$, we have that $V \otimes W$ has dimension $mn$, so that $\{\alpha_i \otimes \beta_j\}$ is a basis of $V \otimes W$. □

Warning: Although the vector space $V \otimes W$ is spanned by elements of the form $v \otimes w$ with $v \in V$, $w \in W$, not every element of $V \otimes W$ can be written in this form (unless one of $V$, $W$ has dimension 1).

The following Proposition follows from the construction of Theorem 1.1 and from Lemma 1.2.

Lemma 1.6. Suppose that $V$ and $W$ are finite dimensional vector spaces. Then there is a unique isomorphism $V \otimes W \to W \otimes V$ such that $x \otimes y \mapsto y \otimes x$ for $x \otimes y \in V \otimes W$.

Proof. The map $V \times W \to W \otimes V$ such that $(x, y) \mapsto y \otimes x$ is bilinear, and thus factors through the tensor product $V \otimes W$, sending $x \otimes y$ to $y \otimes x$. Since this last map has an inverse (by symmetry) we obtain the desired isomorphism. □

Warning: If $v \in V$ and $w \in V$ are distinct vectors in a vector space $V$, then $v \otimes w \neq w \otimes v$ in $V \otimes V$. The above Lemma only says that the map $v \otimes w \mapsto w \otimes v$ is a linear isomorphism of $V \otimes V$, which is not the identity.

Lemma 1.7. Suppose that $U$, $V$ and $W$ are finite dimensional vector spaces. Then there is a unique isomorphism $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$, such that $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$ for $x \in U$, $y \in V$ and $z \in W$.

Proof. Let $\{u_i\}$ be a basis of $U$, $\{v_j\}$ be a basis of $V$, and $\{w_k\}$ be a basis of $W$. Then the elements $(u_i \otimes v_j) \otimes w_k$ form a basis of $(U \otimes V) \otimes W$, and the elements $u_i \otimes (v_j \otimes w_k)$ form a basis of $U \otimes (V \otimes W)$. By the general theorem on the existence and uniqueness of linear maps, there exists a unique linear map

$$F : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$
which maps \((u_i \otimes v_j) \otimes w_k\) on \(u_i \otimes (v_j \otimes w_k)\). Now by taking linear expansions in terms of the basis, we see that \(F((u \otimes v) \otimes w) = u \otimes (v \otimes w)\) for \(u \in U, v \in V, w \in W\). It follows that \(F\) has the desired effect. Since \(F\) maps a basis of \((U \otimes V) \otimes W\) on a basis of \(U \otimes (V \otimes W)\), it follows that \(F\) is an isomorphism. \(\square\)

This Lemma tells us that we can unambiguously write \(U_1 \otimes U_2 \otimes \cdots \otimes U_n\) for the tensor product of vector spaces \(U_1, \ldots, U_n\), and \(u_1 \otimes \cdots \otimes u_n\) for the the tensor product of elements of \(U_1, \ldots, U_n\). If \(\{u_1^1, \ldots, u^1_n\}\) are any bases of \(U_1, \ldots, U_n\), then \(\{u^1_1 \otimes \cdots \otimes u^n_n\}\) is a basis of \(U_1 \otimes \cdots \otimes U_n\).

This product is universal for multilinear maps, analogous to the way that \(U \otimes V\) of the basis, we see that \(F\) is an isomorphism.

Theorem 1.8. Suppose that \(V_1, \ldots, V_n\) are finite dimensional vector spaces over a field \(F\). Then there exists a vector space \(T\) over \(F\), and a multilinear map \(\varphi : V_1 \times V_2 \times \cdots \times V_n \rightarrow T\) such that \(T\) satisfies the following “universal property”:

If \(U\) is a vector space over \(F\) and \(g : V_1 \times V_2 \times \cdots \times V_n \rightarrow U\) is a multilinear map then there exists a unique linear map \(g_* : T \rightarrow U\) such that for all \((v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n\) we have \(g(v_1, \ldots, v_n) = g_*(\varphi(v_1, \ldots, v_n))\).

\(T = V_1 \otimes \cdots \otimes V_n\), as defined above, satisfies the universal property of Theorem 1.8.

Lemma 1.9. Suppose that \(T_1\), with multilinear map \(\varphi_1 : V_1 \times \cdots \times V_n \rightarrow T_1\), satisfies the universal property of theorem 1.1, and \(T_2\), with multilinear map \(\varphi_2 : V_1 \times \cdots \times V_n \rightarrow T_2\), satisfies the universal property of Theorem 1.1. Then there exists a (unique) linear isomorphism \(\psi : T_1 \rightarrow T_2\) such that \(\psi \varphi_1 = \varphi_2\).

Lemma 1.10. Let \(V\) be a finite dimensional vector space over \(F\). Let \(V^*\) be the dual space, and \(\mathcal{L}(V,V)\) the space of linear maps of \(V\) into itself. There exists a unique isomorphism

\[V^* \otimes V \rightarrow \mathcal{L}(V,V),\]

which to each element \(\varphi \otimes v\) (with \(\varphi \in V^*\) and \(v \in V\)) associates the linear map \(L_{\varphi \otimes v}\) such that

\[L_{\varphi \otimes v}(w) = \varphi(w)v\]

for \(w \in V\).

Proof. To each pair \((\varphi, v) \in V^* \times V\) associate the linear map \(L_{\varphi \otimes v}\) such that

\[L_{\varphi \otimes v}(w) = \varphi(w)v\]

for \(w \in V\). This association \((\varphi, v) \mapsto L_{\varphi \otimes v}\) is a bilinear map \(V^* \times V \rightarrow \mathcal{L}(V,V)\). Thus, by Theorem 1.1, there exists a unique linear map \(V^* \otimes V \rightarrow \mathcal{L}(V,V)\) which to \(\varphi \otimes v\) associates our linear map \(L_{\varphi \otimes v}\).

We must now prove that this map

\[\varphi \otimes v \mapsto L_{\varphi \otimes v}\]

gives an isomorphism of \(V^* \otimes V\) and \(\mathcal{L}(V,V)\). Let \(\{v_1, \ldots, v_n\}\) be a basis of \(V\), and let \(\{v_1^*, \ldots, v_n^*\}\) be the dual basis. Then \(v_i^*(v_k) = \delta_{ik}\) is the Kronecker delta. Let

\[L_{ij} = L_{v_i^* \otimes v_j}\]

to simplify notation. We will show that the \(L_{ij}\) for \(1 \leq i \leq n\) and \(1 \leq j \leq n\) are linearly independent. Suppose that we have a relation

\[\sum_{i} \sum_{j} c_{ij} L_{ij} = 0\]
with $c_{ij} \in F$. Apply the left hand side to a $v_k$. We obtain

$$0 = \sum_j \sum_i c_{ij}L_{ij}(v_k).$$

In the sum, $L_{ij}(v_k) = 0$ unless $i = k$, in which case it is equal to $v_j$. Hence

$$0 = \sum_j c_{k}v_j.$$

From the linear dependence of $v_1, \ldots, v_n$, we conclude that $c_{kj} = 0$ for all $j$ and $k$, proving that the linear maps $L_{ij}$ are linearly independent. There are $n^2$ such maps, which is the dimension of $L(V, V)$. Hence these maps form a basis of $L(V, V)$. Since our map $V^* \otimes V \to L(V, V)$ sends a basis $\{v_i^* \otimes v_j\}$ onto the basis $\{L_{ij}\}$ of $L(V, V)$, it follows that this map is an isomorphism.

**Lemma 1.11.** Suppose that $V, W$ are finite dimensional vector spaces over $F$. Then there is a unique isomorphism $V^* \otimes W^* \to (V \otimes W)^*$ which to each $\varphi \otimes \psi$ (with $\varphi \in V^*$ and $\psi \in W^*$) associates the functional $L_{\varphi, \psi}$ having the property that

$$L_{\varphi, \psi}(v \otimes w) = \varphi(v)\psi(w)$$

for $v \in V$ and $w \in W$.

By the universal property, Theorem 1.1, we may identify $(V \otimes W)^*$ with the vector space of bilinear forms on $V$ (the bilinear forms are the bilinear maps from $V \times V$ to $F$). The proceeding Lemma thus allows us to identify $V^* \otimes V^*$ with the bilinear forms on a finite dimensional vector space $V$.

We have a multilinear analogue of the above.

**Lemma 1.12.** Suppose that $V_1, \ldots, V_n$ are finite dimensional vector spaces over $F$. Then there is a unique isomorphism $V_1^* \otimes \cdots \otimes V_n^* \to (V_1 \otimes \cdots \otimes V_n)^*$ which to each $\varphi_1 \otimes \cdots \otimes \varphi_n$ (with $\varphi_i \in V_i^*$) associates a functional $L_{\varphi_1, \ldots, \varphi_n}$ having the property that

$$L_{\varphi_1, \ldots, \varphi_n}(v_1 \otimes \cdots \otimes v_n) = \varphi_1(v_1) \cdots \varphi_n(v_n).$$

By the universal property, Theorem 1.8, we may thus identify $(V^*)^n = V^* \otimes \cdots \otimes V^*$ with the vector space of multilinear forms on a finite dimensional vector space $V$.

**Theorem 1.13.** Suppose that $L_1 : U_1 \to V_1$ and $L_2 : U_2 \to V_2$ are linear maps. Then there is a unique linear map $L_1 \otimes L_2 : U_1 \otimes U_2 \to V_1 \otimes V_2$ such that $u_1 \otimes u_2 \mapsto L_1(u_1) \otimes L_2(u_2)$ for $u_1 \in U_1$ and $u_2 \in U_2$. This map is functorial; that is if $M_1 : V_1 \to W_1$ and $M_2 : V_2 \to W_2$ are linear, then $(M_1 \otimes M_2) \circ (L_1 \otimes L_2) = (M_1 \circ L_1) \otimes (M_2 \circ L_2)$ are the same maps from $U_1 \otimes U_2$ to $W_1 \otimes W_2$. In particular, if $L_1$ and $L_2$ are isomorphisms, with inverses $M_1$ and $M_2$, then $L_1 \otimes L_2$ is an isomorphism, with inverse $M_1 \otimes M_2$.

$L_1$ and $L_2$ induce a bilinear map $L_1 \times L_2 : U_1 \times U_2 \to V_1 \otimes V_2$, which induces a unique linear map $L_1 \otimes L_2 : U_1 \otimes U_2 \to V_1 \otimes V_2$ by the universal property Theorem 1.1, giving a commutative diagram

$$
\begin{array}{ccc}
U_1 \times U_2 & \to & U_1 \otimes U_2 \\
L_1 \times L_2 \downarrow & & \downarrow L_1 \otimes L_2 \\
V_1 \times V_2 & \to & V_1 \otimes V_2
\end{array}
$$

By the universal property 1.1, $L_1 \otimes L_2$ is the unique linear map making the above diagram commutative.
Theorem 1.14. Suppose that $L_i : U_i \to V_i$ are linear maps for $1 \leq i \leq n$. Then there is a unique linear map $L_1 \otimes \cdots \otimes L_n : U_1 \otimes \cdots \otimes U_n \to V_1 \otimes \cdots \otimes V_n$ such that $u_1 \otimes \cdots \otimes u_n \mapsto L_1(u_1) \otimes \cdots \otimes L_n(u_n)$ for $u_1 \in U_1, \ldots, u_n \in U_n$. This map is functorial; that is if $M_i : V_i \to W_i$ are linear for $1 \leq i \leq n$, then $(M_1 \otimes \cdots \otimes M_n) \circ (L_1 \otimes \cdots \otimes L_n) = (M_1 \circ L_1) \otimes \cdots \otimes (M_n \circ L_n)$ are the same maps from $U_1 \otimes \cdots \otimes U_n$ to $W_1 \otimes \cdots \otimes W_n$. In particular, if $L_i$ are isomorphisms, with inverses $M_i$, then $L_1 \otimes \cdots \otimes L_n$ is an isomorphism, with inverse $M_1 \otimes \cdots \otimes M_n$.

2. Alternating Products

Let $V$ be a vector space, and $U$ be a vector space, over a field $F$. Let $V(r)$ denote the product of $V$ with itself $r$ times. A multilinear map $f : V(r) \to U$ is alternating if $f(w_1, w_2, \ldots, w_r) = 0$ whenever two adjacent entries are equal; that is, whenever there exists an index $j < r$ such that $w_j = w_{j+1}$.

An example of an alternating multilinear map is the determinant, $\text{Det} : F^n \to F$, viewed as a function of the columns of a matrix.

Suppose that $f(w_1, \ldots, w_r)$ is an alternating multilinear map. As in the proof for determinants, we see that if two adjacent components $w_j$ and $w_{j+1}$ are interchanged, then $f$ changes signs. If two distinct components $w_i$ and $w_j$ are equal, then $f(w_1, \ldots, w_r) = 0$.

We now introduce some more notation. Let $A = (a_{ij})$ be an $r \times n$ matrix, with $1 \leq r \leq n$. Let $S$ be a subset of the integers $\{1, \ldots, n\}$, consisting of $r$ elements. The elements of such a set can be ordered, so that if $i_1, \ldots, i_r$ are these elements, then $i_1 < \cdots < i_r$. Let $\sigma : \{1, \ldots, r\} \to S$ be a 1-1 map. We may then view $\sigma$ as a permutation of $S$. If $i_1, \ldots, i_r$ are the elements of $S$, and are ordered so that $i_1 < \cdots < i_r$, then $\sigma$ gives rise to the permutation

$$i_1 \mapsto \sigma(1), \ i_2 \mapsto \sigma(2), \ \ldots, \ i_r \mapsto \sigma(r).$$

The sign of this permutation is denoted by $\varepsilon_S(\sigma)$.

Let $P(S)$ denote the set of 1-1 maps $\sigma : \{1, \ldots, r\} \to S$, which we have seen can be identified with the permutations of $S$.

Let $A = (a_{ij})$ be an $r \times n$ matrix, with $r \leq n$. For each subset $S$ of $\{1, \ldots, n\}$ consisting of precisely $r$ elements, we take the $r \times r$ submatrix of $A$ consisting of those elements $a_{ij}$ such that $j \in S$. We denote by $\text{Det}_S(A)$ the determinant of this submatrix. We have that

$$\text{Det}_S(A) = \sum_{\sigma \in P(S)} \varepsilon_S(\sigma)a_{1,\sigma(1)} \cdots a_{r,\sigma(r)}. $$

Now let $v_1, \ldots, v_r$ be elements of $V$. For each $S$ as above, we let

$$v_S = (v_{i_1}, \ldots, v_{i_r}),$$

where $i_1, \ldots, i_r$ are elements of $S$ so ordered that $i_1 < \cdots < i_r$.

Theorem 2.1. Let $V$, $U$ be vector spaces over $F$. Let

$$f : V(r) \to U$$

be an $r$-multilinear map. Let $v_1, \ldots, v_r$ be elements of $V$, and let $A = (a_{ij})$ be an $r \times n$ matrix with coefficients in $F$. Let

$$u_1 = a_{11}v_1 + \cdots + a_{1n}v_n$$

$$\vdots$$

$$u_r = a_{r1}v_1 + \cdots + a_{rn}v_n.$$

Then
\[ f(u_1,\ldots,u_r) = \sum_S \text{Det}_S(A)f(v_S) \]
where the sum is taken over all subsets \( S \) of \( \{1,\ldots,n\} \) consisting of precisely \( r \) elements.

**Proof.** If \( r > n \) the only alternating multilinear map \( f : V^{(r)} \rightarrow U \) to a vector space \( U \) is the zero map (for any \( w_1,\ldots,w_r \in V \), with \( r > n \), one of the \( w_i \) must be expressable as a linear combination of the others. Thus \( f(w_1,\ldots,w_r) = 0 \)). Further we have that there are no subsets \( S \) of \( \{1,\ldots,n\} \) consisting of \( r > n \) elements. We may interpret the empty sum in the formula as being 0, to obtain the (trivial) validity of the Theorem in the case when \( r > n \).

Now assume that \( r \leq n \). We have
\[ f(u_1,\ldots,u_r) = f(a_{11}v_1 + \cdots + a_{1n}v_n,\ldots,a_{r1}v_1 + \cdots + a_{rn}v_n). \]
Expanding out by multilinearity, we obtain a sum
\[ \sum_\sigma a_{1,\sigma(1)} \cdots a_{r,\sigma(r)}f(v_{\sigma(1)},\ldots,v_{\sigma(r)}) \]
over all possible choices \( \sigma \), assigning to each integer from 1 to \( r \) an integer from 1 to \( n \). Thus the sum is over all maps \( \sigma : \{a,\ldots,r\} \rightarrow \{1,\ldots,n\} \).

We observe that if \( \sigma(i) = \sigma(j) \) for some \( i \neq j \), then the corresponding term is 0 because \( f \) is alternating. Thus in our sum we may restrict to those \( \sigma \) which are 1-1.

Our sum can be decomposed into a double sum, by grouping together all maps \( \sigma \) which send \( \{1,\ldots,r\} \) into a given set \( S \), and then taking the sum over all such sets \( S \). Thus symbolically, we can write
\[ \sum_\sigma = \sum_S \sum_{\sigma \in P(S)} \sum_\sigma. \]
In each inner sum
\[ \sum_{\sigma \in P(S)} a_{1,\sigma(1)} \cdots a_{r,\sigma(r)}f(v_{\sigma(1)},\ldots,v_{\sigma(r)}) \]
we shuffle back the \( r \)-tuple \( (v_{\sigma(1)},\ldots,v_{\sigma(r)}) \) to the standard position \( (v_{i_1},\ldots,v_{i_r}) \), where \( i_1,\ldots,i_r \) are the elements of \( S \) ordered so that \( i_1 < \cdots < i_r \). Then \( f \) changes by the sign \( \varepsilon_S(\sigma) \) and consequently each inner sum is equal to
\[ \sum_{\sigma \in P(S)} \varepsilon_S(\sigma)a_{1,\sigma(1)} \cdots a_{r,\sigma(r)}f(v_S). \]
Taking the sum over all possible sets \( S \), we obtain the formula stated in the Theorem. \( \square \)

**Theorem 2.2.** Let \( V \) be a finite dimensional vector space over \( F \), of dimension \( n \). Let \( r \) be an integer. There exists a finite dimensional vector space \( T \) over \( F \), and an \( r \)-multilinear alternating map \( \varphi : V^{(r)} \rightarrow T \) satisfying the following universal property.

If \( U \) is a vector space over \( F \) and \( g : V^{(r)} \rightarrow U \) is an \( r \)-multilinear alternating map, then there exists a unique linear map \( g_* : T \rightarrow U \) such that for all \( u_1,\ldots,u_r \in V \) we have
\[ g(u_1,\ldots,u_r) = g_*(\varphi(u_1,\ldots,u_r)). \]
Proof. If $r > n$ the only alternating multilinear map $g : V^r \rightarrow U$ to a vector space $U$ is the zero map (for any $w_1, \ldots, w_r \in V$, with $r > n$, one of the $w_i$ must be expressable as a linear combination of the others. Thus $g(w_1, \ldots, w_r) = 0$). Thus $T = \{0\}$ satisfies the universal property for $r > n$.

Now assume that $r \leq n$ (actually, the following construction works “vacuously” for the case when $r > n$ also).

For each subset $S$ of $\{1, \ldots, n\}$ consisting of precisely $r$ elements, select a letter $t_S$. Let $T = V(\{t_S\})$ be the vector space over $F$ consisting of formal linear combinations of the $\{t_S\}$. A basis of $V(\{t_S\})$ is $\{t_S\}$. The dimension of $T$ is $\binom{n}{r}$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$. Let $u_1, \ldots, u_r$ be elements of $V$. Let $A = (a_{ij})$ be the matrix with coefficients in $F$ such that

$$u_1 = a_{11}v_1 + \cdots + a_{1n}v_n,$$

$$u_r = a_{r1}v_1 + \cdots + a_{rn}v_n.$$ Define $\varphi : V^r \rightarrow T$ by

$$\varphi(u_1, u_2, \ldots, u_r) = \sum_S \det_S(A)t_S.$$

$\varphi$ is multilinear since each $\det_S$ is multilinear in the rows of $A$. $\varphi$ is alternating since if $u_i = u_{i+1}$ for some $i$, then two adjacent rows of $A$ are equal. Hence for each $S$, two adjacent rows of the submatrix of $A$ corresponding to the set $S$ are equal, and hence $\det_S(A) = 0$.

Observe that $t_S = \varphi(v_{i_1}, v_{i_2}, \cdots, v_{i_r})$ if $i_1, \ldots, i_r$ are the elements of $S$, ordered so that $i_1 < \cdots < i_r$.

From the theorem on existence and uniqueness of linear maps having prescribed values on basis elements, we conclude that if $g : V^r \rightarrow U$ is a multilinear alternating map, then there exists a unique linear map $g_* : T \rightarrow U$ such that for each set $S$, we have

$$g_*(t_S) = g(v_S) = g(v_{i_1}, \ldots, v_{i_r})$$

if $i_1, \ldots, i_r$ are as above. By Theorem 2.1, it follows that

$$g(u_1, \ldots, u_r) = g_*(\varphi(u_1, \ldots, u_r))$$

for all elements $u_1, \ldots, u_r$ of $V$.

Theorem 2.3. Suppose that $T_1$, with alternating multilinear map $\varphi_1 : V^r \rightarrow T_1$, satisfies the universal property of Theorem 2.2, and $T_2$, with alternating multilinear map $\varphi_2 : V^r \rightarrow T_2$, satisfies the universal property of Theorem 2.2. Then there exists a (unique) linear isomorphism $\psi : T_1 \rightarrow T_2$ such that $\psi \varphi_1 = \varphi_2$.

We denote the vector space satisfying the universal property of Theorem 2.2 by $\bigwedge^r V$, and denote $\varphi(u_1, \ldots, u_r)$ by $u_1 \wedge \cdots \wedge u_r$.

Proposition 2.4. Suppose that $\{w_1, \ldots, w_n\}$ is a basis of $V$. Then the set of elements $\{w_{i_1} \wedge \cdots \wedge w_{i_r}\}$ ($1 \leq i_1 < \cdots < i_r \leq n$) is a basis of $\bigwedge^r V$. Hence $\bigwedge^r V$ has dimension $\binom{n}{r}$.

Theorem 2.5. Let $V$ be a finite dimensional vector space, and let $r \geq 1$ be an integer. Then there is a unique linear map

$$\bigwedge^r \left(V^*\right) \rightarrow \left(\bigwedge^r V\right)^*$$
given by

\[ \varphi \wedge \cdots \wedge \varphi_r \mapsto L_{\varphi_1 \wedge \cdots \wedge \varphi_r} \]

for \( \varphi_1, \ldots, \varphi_r \in V^* \), where

\[ L_{\varphi_1 \wedge \cdots \wedge \varphi_r}(v_1 \wedge \cdots \wedge v_r) = \operatorname{Det}(\varphi_i(v_j)) \]

for \( v_1 \wedge \cdots \wedge v_r \in \bigwedge^r V \). This map is an isomorphism.

**Proof.** We first show that the desired linear map exists, and is unique. For fixed \( \varphi_1, \ldots, \varphi_r \in V^* \), the map \( V^{(r)} \to F \) defined by \((v_1, \ldots, v_r) \mapsto \operatorname{Det}(\varphi_i(v_j))\) is multilinear and alternating, so there exists a unique linear map \( L(\varphi_1, \ldots, \varphi_r) : \bigwedge^r V \to F \) defined by \( v_1 \wedge \cdots \wedge v_r \mapsto \operatorname{Det}(\varphi_i(v_j)) \).

Now the map \((V^*)^{(r)} \to (\bigwedge^r V)^*\) given by \((\varphi_1, \ldots, \varphi_r) \mapsto L(\varphi_1, \ldots, \varphi_r)\) is multilinear and alternating, so there is a unique linear map \( \bigwedge^r (V^*) \to (\bigwedge^r V)^*\) given by \( \varphi_1 \wedge \cdots \wedge \varphi_r \mapsto L(\varphi_1, \ldots, \varphi_r) \). This is the desired linear map.

It remains to show that this map is an isomorphism. Since \( V \) and \( V^* \) have the same dimension, \( \bigwedge^r (V^*) \) and \( (\bigwedge^r V)^* \) have the same dimension. It thus suffices to show that the map is 1-1.

Let \( \{w_1, \ldots, w_n\} \) be a basis of \( V \), with dual basis \( \{w_1^*, \ldots, w_n^*\} \) of \( V^* \).

\[ \{w_{\beta_1} \wedge \cdots \wedge w_{\beta_r} \mid 1 \leq \beta_1 < \beta_2 < \cdots < \beta_r \leq n\} \]

is a basis of \( \bigwedge^r V \) and

\[ \{w_{\alpha_1}^* \wedge \cdots \wedge w_{\alpha_r}^* \mid 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq n\} \]

is a basis of \( \bigwedge^r (V^*) \). We compute that for \( 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq n \) and \( 1 \leq \beta_1 < \beta_2 < \cdots < \beta_r \leq n \),

\[ \operatorname{Det}(w_{\alpha_i}^*(w_{\beta_j})) = \begin{cases} 1 & \text{if } \alpha_i = \beta_i \text{ for } 1 \leq i \leq r \\ 0 & \text{otherwise.} \end{cases} \]

Suppose that

\[ \varphi = \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_r \leq n} c_{\alpha_1, \ldots, \alpha_r} w_{\alpha_1}^* \wedge \cdots \wedge w_{\alpha_r}^* \in \bigwedge^r (V^*) , \]

with \( c_{\alpha_1, \ldots, \alpha_r} \in F \), and the corresponding map \( L \in (\bigwedge^r V)^* \) is zero. Then \( L(w_{\beta_1} \wedge \cdots \wedge w_{\beta_r}) = 0 \) for all \( 1 \leq \beta_1 < \beta_2 < \cdots < \beta_r \leq n \). Since \( L(w_{\beta_1} \wedge \cdots \wedge w_{\beta_r}) = c_{\beta_1, \ldots, \beta_r} \), we have that \( \varphi = 0 \). Thus the map is 1-1 and an isomorphism.

By the universal property, we have an isomorphism of \( (\bigwedge^r (V))^* \) with multilinear alternating forms on \( V^{(r)} \). The above theorem gives an identification of multilinear alternating forms on \( V^{(r)} \) with \( \bigwedge^r (V^*) \).

**Theorem 2.6.** Let \( V \) be a finite dimensional vector space over \( F \). For each pair of integers \( r, s \geq 1 \) there exists a unique bilinear map

\[ \bigwedge^r V \times \bigwedge^s V \to \bigwedge^{r+s} V \]

such that if \( u_1, \ldots, u_r \) and \( w_1, \ldots, w_s \) are elements of \( V \), then under this map we have

\[ (u_1 \wedge \cdots \wedge u_r) \times (w_1 \wedge \cdots \wedge w_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s. \]
Proof. Given \( u_1, \ldots, u_r \in V \), the map \( V^s \to \bigwedge^{r+s} V \) given by

\[
(w_1, \ldots, w_s) \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s
\]

is \( s \)-multilinear and alternating. Hence there exists a unique linear map

\[
L_{(u_1, \ldots, u_r)} : \bigwedge^s V \to \bigwedge^{r+s} V
\]

such that

\[
w_1 \wedge \cdots \wedge w_s \mapsto u_1 \wedge \cdots \wedge u_r \wedge w_1 \wedge \cdots \wedge w_s.
\]

The association

\[
(u_1, \ldots, u_r) \mapsto L_{(u_1, \ldots, u_r)}
\]

is an \( r \)-multilinear map

\[
V^r \to \mathcal{L}(\bigwedge^s V, \bigwedge^{r+s} V)
\]

which is alternating, so there exists a unique linear map

\[
\bigwedge^r V \to \mathcal{L}(\bigwedge^s V, \bigwedge^{r+s} V)
\]

denoted by \( \omega \mapsto L_\omega \) such that

\[
u_1 \wedge \cdots \wedge u_r \mapsto L_{(u_1, \ldots, u_r)}.
\]

The association

\[
(\omega, \eta) \mapsto L_\omega(\eta)
\]

is the desired linear map. \( \square \)